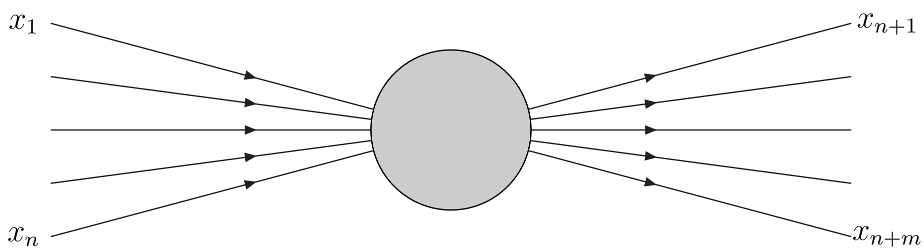
8 Momentum Space - Cross Sections and Decay Rates

8.1 Momentum-space Feynman Rules

We considering n incoming particles and m outgoing particles, i.e. connected diagrams of the form



Since we have an external leg for each incoming or outgoing particle we can make this explicit in the expression for the general Green's function, writing it in the form

$$G^{n+m}(x_1,\ldots,x_{n+m}) = \int d^4y_1 \cdots \int d^4y_l\, G_F(x_1-y_1)\cdots G_F(x_{n+m}-y_l) \times \bar{G}(y_1,\ldots y_l), \quad (8.1)$$

where $l \leq n+m$ since more than one external leg may couple to the same vertex. $\bar{G}(y_1,\ldots y_{n+m})$ is known as the truncated or amputated Green's function, since it is the full Green's function with all its legs removed.

Remember that ultimately we want to calculate the scattering matrix element

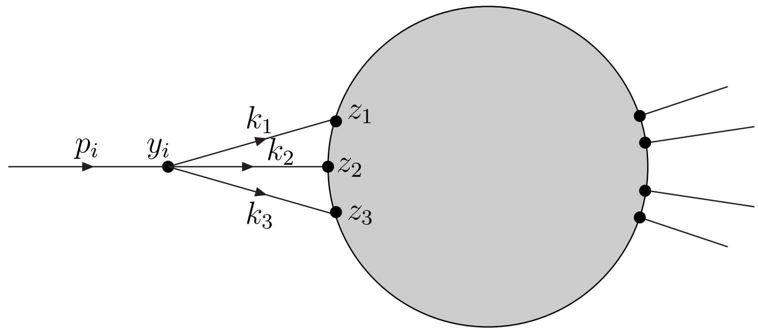
$$S_{qp} = (+i)^{n+m}\int d^4x_1\cdots\int d^4x_n e^{-ip_1\cdot x_1\cdots -ip_n\cdot x_n}\int d^4x_{n+1}\cdots\int d^4x_{n+m}e^{+iq_{n+1}\cdot x_{n+1}\cdots+iq_{n+m}\cdot x_{n+m}}$$
$$(\partial^2_{x_1}+m^2)\cdots(\partial^2_{x_{n+m}}+m^2)\int d^4y_1\cdots\int d^4y_l\, G_F(x_1-y_1)\cdots$$
$$G_F(x_{n+m}-y_l)\times\bar{G}(y_1,\ldots y_l). \quad (8.2)$$

But we know that $(\partial^2_{x_i}+m^2)G_F(x_i-y_j) = -i\delta^4(x_i-y_j)$, so carrying through this operation for each of the external legs, and performing the integrals over the x_i,

$$S_{qp} = \int d^4y_1\cdots\int d^4y_l\, e^{-ip_1\cdot y_1}\cdots e^{+iq_{n+m}\cdot y_l}\bar{G}(y_1,\ldots y_l). \quad (8.3)$$

So the S-matrix element is simply a type of Fourier transform of the amputated coordinate space Green's function, with a weighting of $e^{-ip_i\cdot y_j}$ for each incoming particle and of $e^{+iq_i\cdot y_j}$ for each outgoing particle.

Hence, we could obtain these matrix elements by using the x-space Feynman rules and performing this Fourier transform. However, it is straightforward to obtain a set of simple momentum-space Feynman rules. In order to do this let us pick out one of the vertices which couples to an external line, i.e. one of the y_i, and imagine for the moment that it has just one external line with associated momentum p_i. This can be represented as below



The part of the diagram represented by the circle is equal to the truncated Green's function $\bar{G}(z_1, z_2, z_3, y_1, \ldots y_{i-1}, y_{i+1} \ldots y_l)$, and hence

$$\bar{G}(y_1, \ldots y_l) = \frac{-i\lambda}{4!}\int d^4z_1 \int d^4z_2 \int d^4z_3\, G_F(z_1 - y_i)G_F(z_2 - y_i)G_F(z_3 - y_i)$$
$$\times \bar{G}(z_1, z_2, z_3, y_1, \ldots y_{i-1}, y_{i+1} \ldots y_l). \tag{8.4}$$

We know that

$$G_F(z_j - y_i) = \int i\frac{d^4k}{(2\pi)^4}\frac{e^{-ik\cdot(z_j - y_i)}}{k^2 - m^2 + i\epsilon}. \tag{8.5}$$

Hence, the total y_i dependence in S_{qp} is

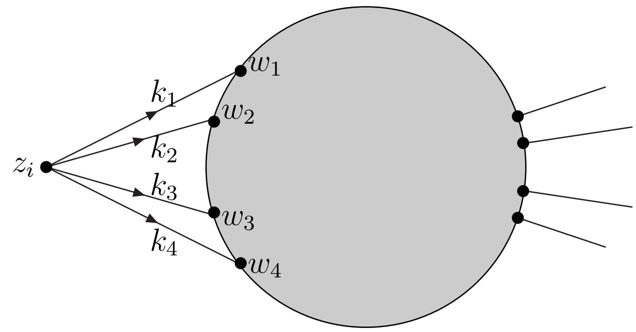
$$\int d^4y_i\, e^{-ip_i\cdot y_i + i(k_1+k_2+k_3)\cdot y_i} = (2\pi)^4\delta^4(-p_i + k_1 + k_2 + k_3), \tag{8.6}$$

where the $-p_i$ would be replaced by a $+q_i$ if it was an outgoing particle. Hence, we get a four-momentum conserving delta-function corresponding to this vertex. Similarly, for each internal line carrying momentum k_j we get a factor

$$\int \frac{d^4k_j}{(2\pi)^4}\frac{i}{k_j^2 - m^2 + i\epsilon}. \tag{8.7}$$

We also get a factor of $-i\lambda/4!$ for the vertex at y_i. This may be repeated for each of the external vertices in turn. It is easy to see that the results of of the same type if the vertex at y_i couples to more than one external line.

One can also do exactly the same for the internal vertices. Let us consider pulling one of these out of the diagram as below



In this case the total z_i dependence is

$$\int d^4 z_i \, e^{iz_i \cdot (k_1+k_2+k_3+k_4)} = (2\pi)^4 \delta^4(k_1+k_2+k_3+k_4), \tag{8.8}$$

and again we get a momentum-conserving delta-function at the vertex. We also get a factor of $-i\lambda/4!$ associated with the vertex, and factors of $\int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2-m^2+i\epsilon}$ for each internal line. We may repeat this process for all the internal and external vertices in the diagram, converting all the y and z dependence into p, q and k dependence. It is not to difficult to see that we finish up with the momentum-space Feynman rules for the truncated Green's functions which are identical to those for the S-matrix elements themselves. They are:

1. Draw all distinct connected diagrams with $n+m$ external lines and any number of 4-leg vertices. Assign momentum k_j to each internal line and p_i or q_i to each external line.

2. For each external line the propagator $\frac{i}{p_i^2-m^2+i\epsilon}$ is absent.

3. For each internal line we have $\int \frac{d^4k_j}{(2\pi)^4} \frac{i}{k_j^2-m^2+i\epsilon}$.

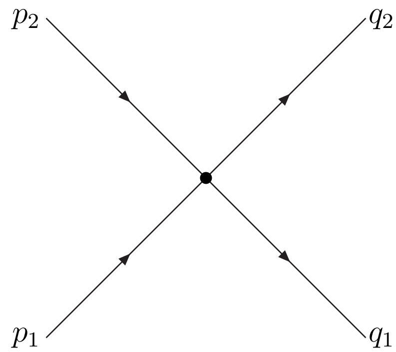
4. For each vertex we have $-i\lambda/4!(2\pi)^4\delta^4(\sum_i q_i)$, where the sum is over all lines exiting the vertex.

5. Introduce a factor C for the number of contractions leading to the same diagram.

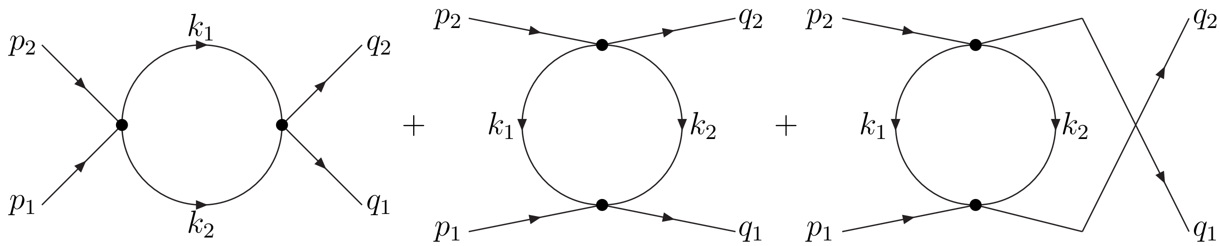
6. Remove the overall momentum conserving delta function, and one factor of $(2\pi)^4$ as these are usually already accounted for in the definition of the cross section or decay rate in terms of the matrix elements.

So again in momentum space we can think of the scattering process as being made up out of the propagation of free particles which undergo isolated $2 \to 2$ particle scatterings with strength λ and at which four-momentum is conserved. However, there is a break from the classical picture in so much that for the particles represented by the internal lines the four-momentum does not have to be on-shell, i.e. k^2 is not constrained to be equal to m^2 in general. So, as in x space the perturbative expansion in terms of Interaction picture fields results in diagrams which have a simple physical picture.

Let us look at the 4-point function as an example. At $\mathcal{O}(\lambda)$ we have a single diagram,



and hence $S_{2\to 2} = -i\lambda/4! \times 4! = -i\lambda$. At $\mathcal{O}(\lambda^2)$ we have three distinct diagrams as below



Evaluating the first of these leads to

$$C\int\int \frac{d^4k_1}{(2\pi)^4}\frac{d^4k_2}{(2\pi)^4}\frac{(-i\lambda)^2}{(4!)^2}\frac{(2\pi)^8(i)^2\delta^4(-p_1-p_2+k_1+k_2)\delta^4(-k_1-k_2+q_1+q_2)}{(k_1^2-m^2+i\epsilon)((k_2^2-m^2+i\epsilon)}$$

$$= C\frac{\lambda^2}{(4!)^2}\int \frac{d^4k_1}{(2\pi)^4}\frac{(2\pi)^4\delta^4(-p_1-p_2+q_1+q_2)}{(k_1^2-m^2+i\epsilon)((q_1+q_2-k_1)^2-m^2+i\epsilon)}. \tag{8.9}$$

The combinatorial factor $C = 4\times 3\times 4\times 3\times 2$ in this case, so extracting the overall $(2\pi)^4$ factor and the delta function we have

$$\frac{\lambda^2}{2}\int \frac{d^4k_1}{(2\pi)^4}\frac{1}{(k_1^2-m^2+i\epsilon)((q_1+q_2-k_1)^2-m^2+i\epsilon)}, \tag{8.10}$$

and obtaining the integral over k_1 we obtain the explicit expression.

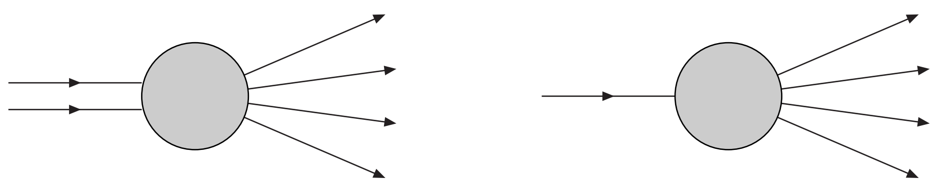
Performing the integrals corresponding to each of the diagrams is a very difficult matter, especially when one goes to diagrams which contain more than one loop. However, we have in principle now constructed the framework for calculating S-matrix elements for our scalar field theory, at least as a power series expansion in the hopefully small coupling λ, i.e. we have completely constructed the Feynman rules for this theory. The generalization of the Feynman rules for other, more realistic field quantum field theories is straightforward. In all cases there is a propagator corresponding to the Green's function of the operator representing the free theory (or being the inverse of the function of momentum in momentum space), and the vertices correspond to the interaction parts of the Lagrangian, with each field in the interaction term corresponding to a leg of the vertex. Also, for fermionic and vector particles there is a nontrivial factor associated with each of the external particles which must be included along with the propagators and vertices. These additional details are dealt with in the second half of the course where we consider the particular field theory QED, but the essentials of how one makes perturbative calculations with a quantum field theory are identical to the case covered so far.

8.2 Transition Rate.

Consider an arbitrary scattering process with an initial state i with total 4-momentum P_i and a final state f with total 4-momentum P_f. Let's assume we computed the scattering amplitude for this process in quantum field theory, i.e. we know the matrix element

$$i\mathcal{M}_{fi}(2\pi)^4\delta^4(P_f - P_i) \tag{8.11}$$

Our task in this section is to convert this into a scattering cross section (relevant if there is more than 1 particle in the initial state) or a decay rate (relevant if there is just 1 particle in the initial state), see below



The probability for the transition to occur is the square of the matrix element, i.e.

$$\text{Probability } = |i\mathcal{M}_{fi}(2\pi)^4\delta^4(P_f - P_i)|^2. \tag{8.12}$$

Attempting to take the squared modulus of the amplitude produces a meaningless square of a delta function. This is a technical problem because our amplitude is expressed between plane wave states. These states are states of definite momentum and so extend throughout all of space-time. In a real experiment the incoming and outgoing states are localized (e.g. they might leave tracks in a detector). To deal with this properly we would have to construct normalized wave packet states which do become well separated in the far past and the far future. Instead of doing this we will do a much simpler but rather sloppy derivation. First of all, we will put our system in a box of volume $V = L^3$. We also imagine that the interaction is restricted to act only over a time of order T. The final answers come out independent of V and T, reproducing the ones we would get if we worked with localized wave packets.

Using

$$(2\pi)^4\delta^4(P_f - P_i) = \int e^{i(P_f - P_i)x}\ d^4x \tag{8.13}$$

we get in our space-time box the result

$$\left|(2\pi)^4\delta^4(P_f - P_i)\right|^2 \simeq (2\pi)^4\delta^4(P_f - P_i)\int e^{i(P_f - P_i)x}\ d^4x \simeq VT\,(2\pi)^4\delta^4(P_f - P_i). \tag{8.14}$$

Now, there is a further point which needs to be addressed. If we have chosen to normalize the fields so as to correspond to $2E$ particles per unit volume (as we have implicitly done for the scalar fields earlier, i.e. $\rho \propto 2E$ in question 1, and will do explicitly for the fermionic fields), then we need to get rid of this factor by dividing the amplitude squared by $2EV$ per particle. Putting everything together, we find for the transition rate W, i.e. the probability per unit time

$$W = \frac{1}{T}|\mathcal{M}_{fi}|^2 VT(2\pi)^4\delta^4(P_f - P_i)\prod_{f=1}^{N}\left[\frac{1}{2E_f V}\right]\prod_{\text{in}}\left[\frac{1}{2E_i V}\right] . \tag{8.15}$$

As expected, the dependence on T cancels. Usually we are interested in much more detailed information than just the transition rate to a particular state. We want to know the differential transition rate dW, i.e. the transition rate into a particular element of the final state phase space. To get dW we have to multiply by the number of available states in the (small) part of phase space under consideration.

For a single particle final state, the number of available states dn in some momentum range $\mathbf{p}$ to $\mathbf{p} + d\mathbf{p}$ is, in the box normalization,

$$dn = V\frac{d^3\mathbf{p}}{(2\pi)^3} \tag{8.16}$$

This result is proved by recalling that the allowed momenta in the box have components that can only take on discrete values such as $p_x = 2\pi n_x/L$ where n_x is an integer. Thus $dn = dn_x dn_y dn_z$ and the result follows. For a two particle final state we have

$$dn = dn_1 dn_2 \qquad \text{where} \qquad dn_1 = V\frac{d^3\mathbf{p}_1}{(2\pi)^3}, \quad dn_2 = V\frac{d^3\mathbf{p}_2}{(2\pi)^3},$$

where dn is the number of final states in some momentum range $\mathbf{p}_1$ to $\mathbf{p}_1 + d\mathbf{p}_1$ for particle 1 and $\mathbf{p}_2$ to $\mathbf{p}_2 + d\mathbf{p}_2$ for particle 2. There is an obvious generalization to an N particle final state,

$$dn = \prod_{f=1}^{N}\frac{Vd^3\mathbf{p}_f}{(2\pi)^3}. \tag{8.17}$$

The transition rate for transitions into a particular element of final state phase space is thus given by, using equations (8.17) and (8.15),

$$\begin{aligned} dW &= |\mathcal{M}_{fi}|^2(2\pi)^4\delta^4(P_f - P_i)V\prod_{f=1}^{N}\left[\frac{1}{2E_f V}\right]\prod_{\text{in}}\left[\frac{1}{2E_i V}\right]\prod_{f=1}^{N}\frac{Vd^3\mathbf{p}_f}{(2\pi)^3} \\ &= |\mathcal{M}_{fi}|^2 V\prod_{\text{in}}\left[\frac{1}{2E_i V}\right] \times \text{LIPS}(N) , \end{aligned} \tag{8.18}$$

where in the second step we defined the Lorentz invariant phase space (LIPS) with N particles in the final state

$$\text{LIPS}(N) \equiv (2\pi)^4\delta^4(P_f - P_i)\prod_{f=1}^{N}\frac{d^3\mathbf{p}_f}{(2\pi)^3\, 2E_f}. \tag{8.19}$$

Observe that everything in the transition rate is Lorentz invariant save for the initial energy factor and the factors of V.

8.3 Decay Rates

We turn now to the special case where we have only one particle with mass m in the initial state i, i.e. we consider the decay of this particle into some final state f. In this case, the transition rate is called the partial decay rate and denoted by Γ_{if}. First of all, we observe that the dependence on V cancels. In the rest frame of the particle the partial decay rate is given by

$$\Gamma_{if} = \frac{1}{2m} \int |\mathcal{M}_{fi}|^2 \times \text{LIPS} \tag{8.20}$$

The important special case of two particles in the final state deserves further consideration, as it may be considerably simplified. Consider the partial decay rate for a particle i of mass m into two particles f_1 and f_2. The Lorentz-invariant phase space is

$$\text{LIPS}(N) = (2\pi)^4 \delta^4(p_i - p_1 - p_2) \frac{d^3\mathbf{p}_1}{(2\pi)^3\, 2E_1} \frac{d^3\mathbf{p}_2}{(2\pi)^3\, 2E_2}. \tag{8.21}$$

In the rest frame the four-vectors of each particle are

$$p_i = (m, 0), \qquad p_1 = (E_1, \mathbf{p}), \qquad p_2 = (E_2, -\mathbf{p}). \tag{8.22}$$

Therefore we can eliminate one three-momentum in the phase space

$$\text{LIPS}(N) = \frac{1}{(2\pi)^2} \delta(m - E_1 - E_2) \frac{d^3\mathbf{p}_2}{4E_1E_2}. \tag{8.23}$$

Hence the partial decay rate becomes

$$\Gamma_{if} = \frac{1}{8m(2\pi)^2} \int |\mathcal{M}_{fi}|^2 \delta(m - E_1 - E_2) \frac{|\mathbf{p}_f|^2 |d\mathbf{p}_f| d\Omega^*}{E_1E_2} \tag{8.24}$$

where $d\Omega^*$ is the solid angle element for the angle of one of the outgoing particles with respect to some fixed direction, and $\mathbf{p}_f$ is the momentum of one of the final state particles. But from the on-shell condition $E_1 = (\mathbf{p}_1^2 + m_1^2)^{1/2}$, we have $dE_1 = |\mathbf{p}_f|/E_1\, d|\mathbf{p}_f|$ and similarly for particle 2 and so

$$d(E_1 + E_2) = |\mathbf{p}_f| d|\mathbf{p}_f| \frac{E_1 + E_2}{E_1E_2},$$

therefore

$$|\mathbf{p}_f|^2 d|\mathbf{p}_f| \frac{1}{E_1E_2} = \frac{|\mathbf{p}_f|}{E_1 + E_2} d(E_1 + E_2). \tag{8.25}$$

Using this in eq. (8.24) and integrating over $(E_1 + E_2)$ we obtain the final result

$$\Gamma_{i \to f_1 f_2} = \frac{1}{32\pi^2 m^2} \int |\mathcal{M}_{fi}|^2 |\mathbf{p}_f| d\Omega^*. \tag{8.26}$$

The total decay rate of particle i is obtained by summation of the partial decay rates into all possible final states

$$\Gamma_{\text{tot}} = \sum_f \Gamma_{if} \tag{8.27}$$

The total decay rate is related to the mean life time τ via $(\Gamma_{\text{tot}})^{-1} = \tau$. For completeness I also give the definition of the branching ratio for the decay into a specific final state f

$$B_f \equiv \frac{\Gamma_{if}}{\Gamma_{\text{tot}}} \tag{8.28}$$

In an arbitrary frame we find, $W = (m/E)\Gamma_{\text{tot}}$, which has the expected Lorentz dilation factor. In the master formula (equation (8.18)) this is what the product of $1/2E_i$ factors for the initial particles does.

8.4 Cross Sections

The total cross section for a static target and a beam of incoming particles is defined as the total transition rate for a single target particle and a unit beam flux. The differential cross section is similarly related to the differential transition rate. We have calculated the differential transition rate with a choice of normalization corresponding to a single 'target' particle in the box, and a 'beam' corresponding also to one particle in the box. A beam consisting of one particle per volume V with a velocity v has a flux N_0 given by

$$N_0 = \frac{v}{V}$$

particles per unit area per unit time. Thus the differential cross section is related to the differential transition rate in equation (8.18) by

$$d\sigma = \frac{dW}{N_0} = dW \times \frac{V}{v}. \tag{8.29}$$

Now let us generalize to the case where in the frame in which you make the measurements, the 'beam' has a velocity v_1 but the 'target' particles are also moving with a velocity v_2. In a colliding beam experiment, for example, v_1 and v_2 will point in opposite directions in the laboratory. In this case the definition of the cross section is retained as above, but now the beam flux of particles N_0 is effectively increased by the fact that the target particles are moving towards it. The effective flux in the laboratory in this case is given by

$$N_0 = \frac{|\mathbf{v}_1 - \mathbf{v}_2|}{V}$$

which is just the total number of particles per unit area which run past each other per unit time. In the general case, then, the differential cross section is given by

$$d\sigma = \frac{dW}{N_0} = \frac{1}{|\mathbf{v}_1 - \mathbf{v}_2|}\,\frac{1}{4E_1E_2}|\mathcal{M}_{fi}|^2 \times \text{LIPS} \tag{8.30}$$

where we have used equation (8.18) for the transition rate, and the box volume V has again cancelled. The amplitude-squared and phase space factors are manifestly Lorentz invariant. What about the initial velocity and energy factors? Observe that

$$E_1E_2(\mathbf{v}_1 - \mathbf{v}_2) = E_2\mathbf{p}_1 - E_1\mathbf{p}_2.$$

In a frame where $\mathbf{p}_1$ and $\mathbf{p}_2$ are collinear,

$$|E_2\mathbf{p}_1 - E_1\mathbf{p}_2|^2 = (p_1 \cdot p_2)^2 - m_1^2 m_2^2,$$

and the last expression is manifestly Lorentz invariant.

Hence we can define a Lorentz invariant differential cross section. The total cross section is obtained by integrating over the final state phase space:

$$\sigma = \frac{1}{4\sqrt{(p_1 \cdot p_2)^2 - m_1^2 m_2^2}} \sum_{\text{final states}} \int |\mathcal{M}_{fi}|^2 \times \text{LIPS}. \tag{8.31}$$

A slight word of caution is needed in deciding on the limits of integration to get the total cross section. If there are identical particles in the final state then the phase space should be integrated so as not to double count.

An important special case is $2 \to 2$ scattering

$$a(p_a) + b(p_b) \to c(p_c) + d(p_d).$$

Following similar reasoning to our example for a two-particle decay we can show that in the centre of mass frame the differential cross section for two particles of mass m scattering to two of mass M is,

$$\frac{d\sigma}{d\Omega^*} = \frac{\sqrt{1 - 4M^2/s}}{64\pi^2 s\sqrt{1 - 4m^2/s}} |\mathcal{M}_{fi}|^2. \tag{8.32}$$

Invariant $2 \to 2$ scattering amplitudes are frequently expressed in terms of the *Mandelstam variables*, defined by

$$\begin{aligned} s &\equiv (p_a + p_b)^2 = (p_c + p_d)^2, \\ t &\equiv (p_a - p_c)^2 = (p_b - p_d)^2, \\ u &\equiv (p_a - p_d)^2 = (p_b - p_c)^2. \end{aligned} \tag{8.33}$$

In fact there are only two independent Lorentz invariant combinations of the available momenta in this case, so there must be some relation between s, t and u, and in the problems you showed that

$$s + t + u = m_a^2 + m_b^2 + m_c^2 + m_d^2,$$

and for two body scattering of particles of equal mass m,

$$s \geq 4m^2, \qquad t \leq 0, \qquad u \leq 0.$$