Remark on the application of the connection formulae of the WKB approximation

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During class we discussed the significance of the arrows in the connecting formulae of the WKB approach. When connecting the wave-function from one region to another, careful attention need also to be taken in carrying the correct pre-factor. This means the following.

In the Tunelling example we considered in class (section 2.4 on your lecture notes) we have that the wave-function in region 3 is given by

$$\psi_3(x) = \frac{A}{\sqrt{p(x)}}\left(\cos\left[\int_b^x p(x')dx'/\hbar - \frac{\pi}{4}\right] + i\sin\left[\int_b^x p(x')dx'/\hbar - \frac{\pi}{4}\right]\right) \tag{1}$$

We discussed that in this case it is correct to apply the connection formulae (for the left-barrier) in the opposite direction of the arrow. So, we can use both:

$$\begin{aligned} &\frac{1}{\sqrt{q(x)}}\exp\left[-\int_x^b q(x')dx'/\hbar\right] \\ &\qquad\qquad \rightarrow \quad \frac{2}{\sqrt{p(x)}}\cos\left(\int_b^x p(x')dx'/\hbar - \frac{\pi}{4}\right) \end{aligned} \tag{2}$$

$$\begin{aligned} &\frac{1}{\sqrt{q(x)}}\exp\left[+\int_x^b q(x')dx'/\hbar\right] \\ &\qquad\qquad \leftarrow \quad -\frac{1}{\sqrt{p(x)}}\sin\left(\int_b^x p(x')dx'/\hbar - \frac{\pi}{4}\right) \end{aligned} \tag{3}$$

Notice that in Eq. (2) the function $\cos(...)$ has a pre-factor 2 and in Eq. (3) $\sin(...)$ has a pre-factor -1. This implies that in Eq (1) the term

$$\frac{1}{\sqrt{p(x)}}\cos\left[\int_b^x p(x')dx'/\hbar - \frac{\pi}{4}\right]$$

will connect to

$$\frac{1}{2}\frac{1}{\sqrt{q(x)}}\exp\left[-\int_x^b q(x')dx'/\hbar\right]$$

in region 2 (here we have used the connection formulae in the opposite direction). Similarly, the term

$$\frac{1}{\sqrt{p(x)}} \sin\left[\int_b^x p(x')dx'/\hbar - \frac{\pi}{4}\right]$$

connects to

$$\frac{-1}{\sqrt{q(x)}} \exp\left[+\int_x^b q(x')dx'/\hbar\right].$$

This is the reason why the wave-function in region 2 has pre-factors $1/2$ and -1 in the exponentially decreasing and growing functions respectively, i.e.

$$\psi_2(x) = \frac{A}{\sqrt{q(x)}}\left(\frac{1}{2}\exp\left[-\int_x^b q(x')dx'/\hbar\right] - i\exp\left[\int_x^b q(x')dx'/\hbar\right]\right)$$

as it is correctly indicated on your lecture notes on Eq. 2.55.