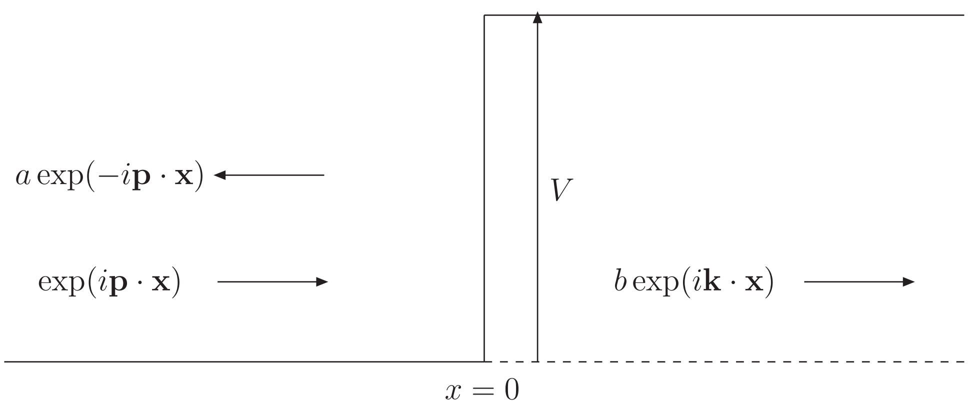
PHAS445 Quantum Field Theory R S Thorne

Problem Set 1

To be handed in by 5pm, February 8th

1. Derive the continuity equation $\partial_\mu J^\mu = 0$ of section 1. Start with the Klein-Gordon equation multiplied by ϕ^* and subtract the complex conjugate of the KG equation multiplied by ϕ. [6]

2. Consider the wave incident on a potential step shown below.



As shown in lectures, if $V > m + E_p$, where $E_p = \sqrt{\mathbf{p}^2 + m^2}$ then one cannot avoid using the negative square root $\mathbf{k} = -\sqrt{(E_p - V)^2 - m^2}$. Show that this gives a negative current for $x < 0$ and a negative density for $x > 0$. Give an interpretation of these solutions. [6]

3. Write down the Lagrangian for the motion of a particle of mass m in spherically symmetric potential $V(r)$. Show that the radial equation of motion is

$$0 = m\ddot{r} - mr(\dot{\theta}^2 + \sin^2\theta\dot{\phi}^2) + \frac{\partial V}{\partial r}.$$

Also show that in spherical polars the Hamiltonian of the particle is

$$H = \frac{1}{2m}\left(p_r^2 + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta}\right) + V(r).$$

What conserved quantity does the independence of the Hamiltonian on the coordinate ϕ give? [8]

4. Show that when using the definition of the Hamiltonian operator for the quantum mechanical simple harmonic oscillator in terms of $\hat{a}$ and $\hat{a}^\dagger$ that the definition

$$|n\rangle = \frac{1}{\sqrt{n!}}(\hat{a}^\dagger)^n|0\rangle,$$

has the correct normalisation for a quantum state. (If $|n\rangle$ is correctly normalised check the normalisation of the state $\hat{a}^\dagger|n\rangle \propto |n+1\rangle$.) [5]

5. Verify the commutation relation

$$[\tilde{\phi}(\mathbf{p}), \tilde{\pi}(\mathbf{q})] = i(2\pi)^3\delta^3(\mathbf{p}+\mathbf{q})$$

for the momentum-space quantum fields, and use these to verify that

$$[a_\mathbf{q}, a_\mathbf{p}^\dagger] = (2\pi)^3 2E(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{q}). \qquad [10]$$

6. Show that $|\mathbf{p}\rangle = a_\mathbf{p}^\dagger|0\rangle$ is a correctly normalised state vector, i.e. that the integral of $\langle\mathbf{q}|\mathbf{p}\rangle$ over the full Lorentz-invariant phase space gives unity. [5]

7. Using the expression for the unequal time commutator for $\phi(x)$ and $\phi(y)$ in Section 5 show explicitly that $[\phi(\mathbf{x}), \phi(\mathbf{y})] = 0$ for space-like separations and prove the equal time commutation relation $[\phi(\mathbf{x}), \pi(\mathbf{y})] = i\delta^3(\mathbf{x}-\mathbf{y})$. [10]