Problem Set 2 — Quantum Field Theory

Question 1

$$\phi(x) = \int \frac{d^3 q}{2E(q)(2\pi)^3} \left(a_q e^{-iq\cdot x} + a_q^\dagger e^{iq\cdot x} \right).$$

$$\therefore \quad \partial_0 \phi(x) = \int \frac{d^3 q}{(2\pi)^3 2E(q)} iE(q) \left(-a_q e^{-iq\cdot x} + a_q^\dagger e^{iq\cdot x} \right)$$

So
$$\int d^3 x \left(-\partial_0 e^{-ip\cdot x} \right) \phi(x) + e^{+ip\cdot x} \left(\partial_0 \phi(x) \right)$$

$$= \int d^3 x \; (iE(p)) e^{-ip\cdot x} \phi(x) + e^{+ip\cdot x} \left(\partial_0 \phi(x) \right)$$

$$= \iint \frac{d^3 x \, d^3 q}{2E(q)(2\pi)^3} \Big[iE(p) e^{-ip\cdot x} \left(a_q e^{-iq\cdot x} + a_q^\dagger e^{iq\cdot x} \right) + e^{ip\cdot x} iE(q) \left(-a_q e^{-iq\cdot x} + a_q^\dagger e^{+iq\cdot x} \right) \Big]$$

$$= i \int \frac{d^3 x \, d^3 q}{2E(q)(2\pi)^3} \Big[a_q \left(e^{-ip\cdot x - iq\cdot x} E(p) - e^{-ip\cdot x - iq\cdot x} E(q) \right) + a_q^\dagger \left(e^{iq\cdot x - ip\cdot x} E(p) + e^{iq\cdot x + ip\cdot x} E(q) \right) \Big]$$

$$= i \int \frac{d^3 q}{2E(q)} \Big[a_q \left(\delta^3(p+q) E(p) - \delta^3(p+q) E(q) \right) e^{-i(E(p)+E(q))t} + e^{+i(E(q)-E(p))t} a_q^\dagger \left(\delta(p-q) E(p) + \delta(p-q) E(q) \right) \Big]$$

$$= \frac{i a_{-p}}{2E(p)} \left(E(p) - E(-p) \right) e^{-i(E(p)+E(-p))t} + \frac{i a_p^\dagger}{2E(p)} \cdot 2E(p) \, e^{i(E(p)-E(p))t}$$

So using $E(p) = E(-p)$

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$$= \underline{\underline{i a_p^\dagger}}$$

Very similar for second case.

$$\int d^3\underline{x} \left(\left(-\partial_0 e^{+ip\cdot x}\right)\phi(x) + e^{+ip\cdot x}\left(\partial_0 \phi(x)\right)\right)$$

$$= \int d^3\underline{x} \left(\left(-iE(\underline{p})\, e^{+ip\cdot x}\phi(x)\right) + e^{+ip\cdot x}\,\partial_0\phi(x)\right)$$

$$= \iint \frac{d^3x\, d^3\underline{q}}{(2E(\underline{q}))(2\pi)^3}\Big[-iE(\underline{p})e^{+ip\cdot x}\left(a_{\underline{q}}\, e^{-iq\cdot x} + a_{\underline{q}}^{+} e^{+iq\cdot x}\right)$$
$$+ e^{+ip\cdot x}\, iE(\underline{q})\left(-a_{\underline{q}}\, e^{-iq\cdot x} + a_{\underline{q}}^{+} e^{+iq\cdot x}\right)\Big]$$

$$= i\iint \frac{d^3x\, d^3\underline{q}}{(2\pi)^3 2E(\underline{q})}\Big[-a_{\underline{q}}\left(E(\underline{p})\, e^{ip\cdot x - iq\cdot x} + E(\underline{q})e^{ip\cdot x - iq\cdot x}\right)$$
$$+ a_{\underline{q}}^{+}\left(-E(\underline{p})e^{+iq\cdot x + ip\cdot x} + E(\underline{q})\, e^{+ip\cdot x + iq\cdot x}\right)\Big]$$

$$= i\int \frac{d^3\underline{q}}{2E(\underline{q})}\Big[-a_{\underline{q}}\left(\delta^3(\underline{p}-\underline{q})E(\underline{p}) + \delta^3(\underline{p}-\underline{q})E(\underline{q})\right)e^{i(E(\underline{p})-E(\underline{q}))t}$$
$$+ a_{\underline{q}}^{+}\left(\delta^3(\underline{p}-\underline{q})\, E(\underline{q}) - \delta^3(\underline{p}-\underline{q})\, E(\underline{p})\right)e^{i(E(\underline{p})+E(\underline{q}))t}.$$

$$= -\frac{i a_{\underline{p}}}{2E(\underline{p})}\, 2E(\underline{p}) + \frac{i a_{-\underline{p}}^{+}}{2E(-\underline{p})}\left(E(-\underline{p}) - E(\underline{p})\right)e^{i(E(\underline{p})+E(-\underline{p}))t}$$

$$= \underline{\underline{-ia_{\underline{p}}}}$$

Question 2

Consider $T(A_1 A_2 A_3)$

where $A_1 = A_1^+ + A_1^-$, $A_2 = A_2^+ + A_2^-$, $A_3 = A_3^+ + A_3^-$

$$\therefore \quad A_1 A_2 A_3 = (A_1^+ + A_1^-)(A_2^+ A_3^+ + A_2^- A_3^+ + A_2^+ A_3^- + A_2^- A_3^-)$$

$$= (A_1^+ + A_1^-)(A_2^+ A_3^+ + A_3^+ A_2^- + [A_2^-, A_3^+] + A_2^+ A_3^- + A_2^- A_3^-)$$

$$= A_1^+ A_2^+ A_3^+ + A_1^+ A_3^+ A_2^- + A_1^+ [A_2^-, A_3^+] + A_1^+ A_2^+ A_3^- + A_1^+ A_2^- A_3^-$$
$$+ A_1^- A_2^+ A_3^+ + A_1^- A_3^+ A_2^- + A_1^- [A_2^-, A_3^+] + A_1^- A_2^+ A_3^- + A_1^- A_2^- A_3^-$$

$$= A_1^+ A_2^+ A_3^+ + A_1^+ A_3^+ A_2^- + A_1^+ A_2^+ A_3^- + A_1^+ A_2^- A_3^- + A_1^- A_2^- A_3^-$$
$$+ (A_1^+ + A_1^-)[A_2^-, A_3^+] + A_3^+ A_1^- A_2^- + [A_1^-, A_3^+] A_2^- + A_2^+ A_1^- A_3^- + [A_1^-, A_2^+] A_3^-$$
$$+ A_2^+ A_1^- A_3^+ + [A_1^-, A_2^+] A_3^+$$

$$\hookrightarrow = A_2^+ A_3^+ A_1^- + [A_1^-, A_3^+] A_2^+$$

$$\therefore \quad A_1 A_2 A_3 = A_1^+ A_2^+ A_3^+ + A_1^+ A_3^+ A_2^- + A_1^+ A_2^+ A_3^- + A_1^+ A_2^- A_3^- + A_1^- A_2^- A_3^-$$
$$+ A_3^+ A_1^- A_2^- + A_2^+ A_1^- A_3^- + A_2^+ A_3^+ A_1^-$$
$$+ (A_1^+ + A_1^-)[A_2^-, A_3^+] + (A_2^+ + A_2^-)[A_1^-, A_3^+] + (A_3^+ + A_3^-)[A_1^-, A_2^+]$$

$$\therefore A_1 A_2 A_3 = \,:A_1 A_2 A_3: + A_1 [A_2^-, A_3^+] + A_2 [A_1^-, A_3^+] + A_3 [A_1^-, A_2^+].$$

But commutators are numbers $\therefore$ e.g.

$$[A_2^-, A_3^+] = \langle 0 | [A_2^-, A_3^+] | 0 \rangle$$

~~But~~ $\cancel{[A_2, A_3] = A_2 A_3 - A_3 A_2 = (A_2^+ + A_2^-)(A_3^+ + A_3^-) - (A_3^+ + A_3^-)(A_2^+ + A_2^-)}$

and $A_2 A_3 = A_2^+ A_3^+ + A_2^+ A_3^- + A_2^- A_3^+ + A_2^- A_3^-$

and $\langle 0 | A^+ = A^- | 0 \rangle$

$\therefore \langle 0 | A_2 A_3 | 0 \rangle = \langle 0 | A_2^- A_3^+ | 0 \rangle = \langle 0 | (A_2^- A_3^+ - A_3^+ A_2^-) | 0 \rangle$

$\therefore \langle 0 | A_2^- A_3^+ | 0 \rangle = \langle 0 | A_2 A_3 | 0 \rangle$

But $A = A^+ + A^- \equiv \, :A:$

So $A_1 A_2 A_3 = \, :A_1 A_2 A_3: + :A_1: \langle 0 | A_2 A_3 | 0 \rangle$
$+ :A_2: \langle 0 | A_1 A_3 | 0 \rangle + :A_3: \langle 0 | A_1 A_2 | 0 \rangle$

$\therefore T(A_1 A_2 A_3) = \theta(t_1 - t_2)\theta(t_2 - t_3) :A_1 A_2 A_3:$
$+ \theta(t_1 - t_2)\theta(t_2 - t_3) \big[:A_1: \langle 0 | A_2 A_3 | 0 \rangle + :A_2: \langle 0 | A_1 A_3 | 0 \rangle$
$+ :A_3: \langle 0 | A_1 A_2 | 0 \rangle \big]$

with the same with permutations.

$\therefore$ Time ordering ensures terms in e.g. $\langle 0 | A_2 A_3 | 0 \rangle$ ordered consistently with time for each permutation, e.g. in this case get $\langle 0 | A_1 A_2 | 0 \rangle$ not $\langle 0 | A_2 A_1 | 0 \rangle$ when $t_1 > t_2$.

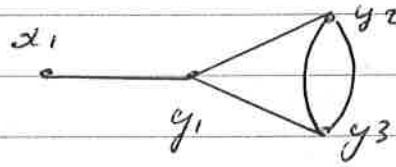
So adding permutations get

$T(A_1 A_2 A_3) = \, :A_1 A_2 A_3: + :A_1: \langle 0 | T(A_2 A_3) | 0 \rangle$
$+ :A_2: \langle 0 | T(A_1 A_3) | 0 \rangle + :A_3: \langle 0 | T(A_1 A_2) | 0 \rangle$

Question 3

The vertex now has 3 legs.

Consider 1 field, we have one $O(g^3)$ diagram

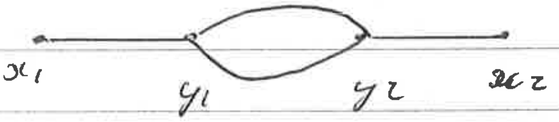


The factor C is $3 \times 3 \times 3 \times 2 = 3^2\, 3!$

$$\therefore \quad G(x_1) = 3^2\, 3! \left(\frac{-ig}{3!}\right)^3 \iiint d^4y_1\, d^4y_2\, d^4y_3\, G_F(x_1-y_1) \times G_F(y_1-y_2)\, G_F(y_1-y_3)\, G_F^2(y_2-y_3)$$

$$= +\frac{ig^3}{4} \iiint d^4y_1\, d^4y_2\, d^4y_3\, G_F(x_1-y_1)\, G_F(y_1-y_2)\, G_F(y_1-y_3)\, G_F^2(y_2-y_3)$$

For the 2-point diagram we have the zeroth order $x_1 \text{———} x_2 = G_F(x_1-x_2)$ and a diagram

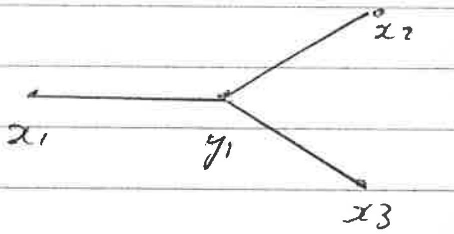


with $C = 3 \times 3 \times 2 = 3\, 3!$

$$\therefore \quad G(x_1, x_2) = G_F(x_1-x_2) + 3\, 3! \left(\frac{-ig}{3!}\right)^2 \iint d^4y_1\, d^4y_2\, G_F(x_1-y_1)\, G_F^2(y_1-y_2)\, G_F(y_2-x_2)$$

$$= G_F(x_1-x_2) - \frac{g^2}{2} \iint d^4y_1\, d^4y_2\, G_F(x_1-y_1)\, G_F^2(y_1-y_2)\, G_F(y_2-x_2)$$

For the 3-point functions there is the order g diagram

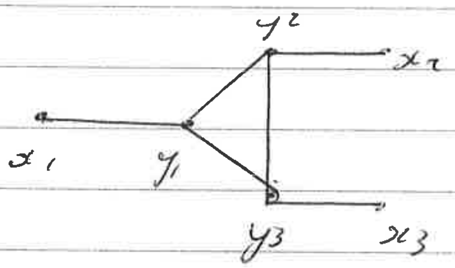


$C = 3 \times 2 = 3!$

and for this $G(x_1, x_2, x_3) = -\frac{ig}{3!} 3! \int d^4y_1 \, G_F(x_1 - y_1) \, G_F(x_2 - y_1) \, G_F(x_3 - y_3)$

$$\rightarrow -ig \int d^4y_1 \, G_F(x_1 - y_1) \, G_F(x_2 - y_1) \, G_F(x_3 - y_1)$$

There is also the order g^3 diagram.

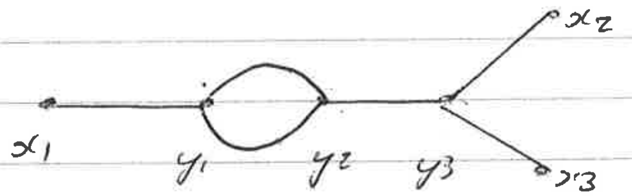


$C = 3 \times 3 \times 3 \times 2 \times 2 = (3!)^2 3$

$\therefore$ for this $G(x_1, x_2, x_3) = (3!)^2 3^2 \cdot \left(\frac{-ig}{3!}\right)^3 \iiint d^4y_1 \, d^4y_2 \, d^4y_3 \, G_F(x_1 - y_1)$
$G_F(y_1 - y_2) \, G_F(y_1 - y_3) \, G_F(y_2 - y_3) \, G_F(x_2 - y_2) \, G_F(x_3 - y_3)$

$$\rightarrow + \frac{ig^3}{2} \iiint d^4y_1 \, d^4y_2 \, d^4y_3 \, G_F(x_1 - y_1) \, G_F(x_2 - y_2) \, G_F(x_3 - y_3) \, G_F(y_1 - y_2) \, G_F(y_1 - y_3) \, G_F(y_2 - y_3)$$

There is also



$$C = 3\times3\times2\times3\times2 = (3!)^2\,3$$

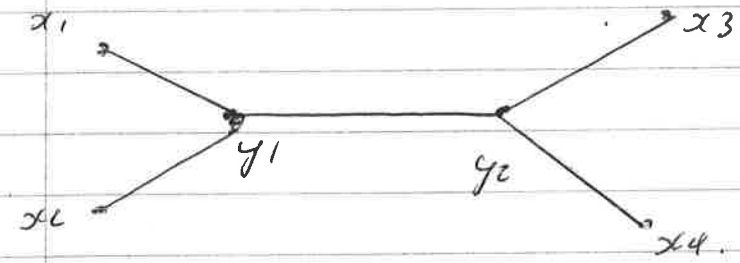
$$\therefore \text{ from this } G_1(x_1,x_2,x_3) = (3!)^2\,3\left(\frac{-ig}{3!}\right)^3 \int d^4y_1\,d^4y_2\,d^4y_3\; G_F(x_1-y_1)$$

$$\cdot\; G_F^2(y_1-y_2)\,G_F(y_2-y_3)\,G_F(y_3-x_2)\,G_F(y_3-x_3)$$

$$\rightarrow\; +\frac{ig^2}{2}\int d^4y_1\,d^4y_2\,d^4y_3\; G_F(x_1-y_1)\,G_F^2(y_1-y_2)\,G_F(y_2-y_3)\,G_F(y_3-x_2)\,G_F(y_3-x_3)$$

But there are two inequivalent permutations where it is x_2 or x_3 which are attached to the vertex with one external leg.

For 4-point diagrams we have



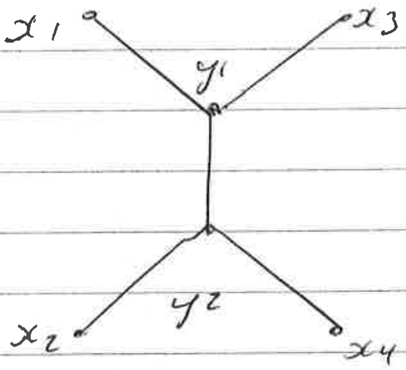
$$C = 3\times2\times3\times2 = (3!)^2$$

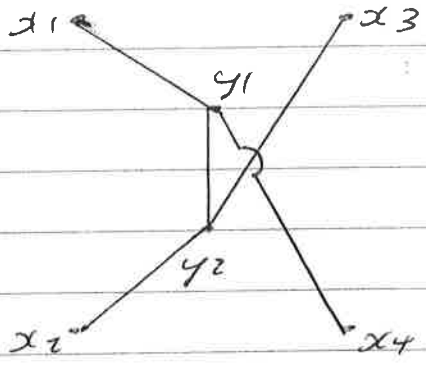
$$\therefore\; G(x_1,x_2,x_3,x_4) = (3!)^2\left(\frac{-ig}{3!}\right)^2 \iint d^4y_1\,d^4y_2\; G_F(x_1-y_1)\,G_F(x_2-y_1)$$

$$\cdot\; G_F(y_1-y_2)\,G_F(x_3-y_2)\,G_F(x_4-y_2)$$

$$\rightarrow\; -g^2 \iint d^4y_1\,d^4y_2\; G_F(x_1-y_1)\,G_F(x_2-y_1)\,G_F(y_1-y_2)\,G_F(x_3-y_2)\,G_F(x_4-y_2)$$

However, we also have.



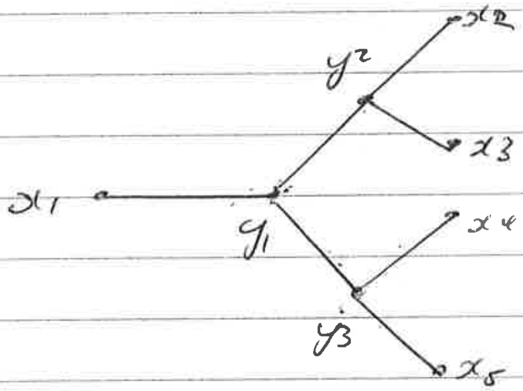


In the left-hand diagram we have the same expression as before but x_2 and x_3 swap. This then gives

$$-g^2\iint d^4y_1\, d^4y_2\, G_F(x_1-y_1)\, G_F(x_3-y_1)\, G_F(y_1-y_2)\, G_F(x_2-y_2)\, G_F(x_4-y_2)$$

which is not the same as before. The right-hand diagram gives a third expression which is the first with x_2 and x_4 swapped.

For the 5-point function we have



$$C = 3\times 3\times 3\times 2\times 2\times 2 = (3!)^3$$

$$\therefore\; G(x_1,x_2,x_3,x_4,x_5) = -i\left(\frac{ig}{3!}\right)^3 (3!)^3 \iiint d^4y_1\, d^4y_2\, d^4y_3\, G_F(x_1-y_1)$$

$$G_F(y_1-y_2)\, G_F(y_1-y_3)\, G_F(y_2-x_2)\, G_F(y_2-x_3)\, G_F(y_3-x_4)\, G_F(y_3-x_5).$$

$$\to + i g^3 \iiint d^4y_1\, d^4y_2\, d^4y_3\; G_F(x_1-y_1)\, G_F(y_1-y_2)\, G_F(y_1-y_3)\, G_F(y_2-x_2)\, G_F(y_2-x_3)\, G_F(y_3-x_4)\, G_F(y_3-x_5)$$

However, there are in equivalent permutations of the x_i. There are fives choices for x_i corresponding to the vertex with only one external leg, and for each 3 permutations left for the 4 remaining x_i, as in the 4-point diagram.

The $O(g^2)$ two-point function is a correction to the propagator. The $O(g^2)$ four-point function represents $2 \to 2$ particle scattering. if x_3, x_4 are at large times and x_1 and x_2 at early times then the first diagram represents annihilation of two particles into one which then "decays" to another two particles. The other two represent scattering with exchange of an intermediate particle.

For this theory the Hamiltonian contains an interaction term $g\phi^3$. This dominates for $|\phi| \to \infty$, but can be negative if $\phi \to -\infty$. Hence, the energy is unbounded from below in this theory i.e.

$$g\phi^3 \to -\infty \text{ as } \phi \to -\infty.$$

Question 4

In the first 1-loop four point function as $|k_1| \to \infty$ the external momentum and mass become negligible.

$$\therefore \text{ expression } \to \frac{\lambda^2}{2} \int \frac{d^4 k_1}{(2\pi)^4 k_1^4}$$

But $d^4 k_1 \propto k_1^3 \, d k_1$

$$\therefore \text{ expression } \propto \frac{\lambda}{2} \int \frac{k_1^3}{k_1^4} d k_1 = \frac{\lambda}{2} \int k_1^{-1} d k_1$$

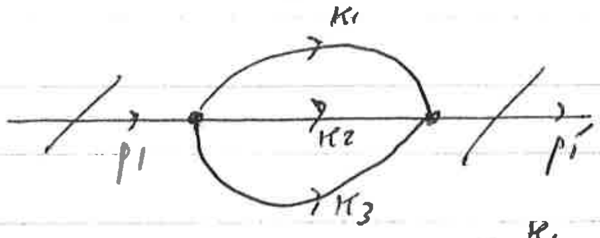
But upper limit of integral over k_1 has no limits. Expression proportional to

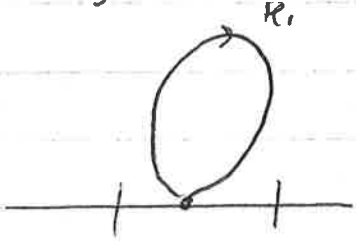
$$\int^{\infty} \frac{d k_1}{k_1} \to \left[\ln k_1 \right]^{\infty} \to \infty$$

$\therefore$ logarithmic divergence from high $|k_1|$ region.

Question 5

For the normal-ordered interaction the only $O(\lambda^2)$ two point diagram is



Since 

is forbidden by normal ordering. The momentum space expression is

$$c \times \left(\frac{-i\lambda}{4!}\right)^2 \iiint \frac{(i)^3\, d^4k_1\, d^4k_2\, d^4k_3}{(2\pi)^4(2\pi)^4(2\pi)^4} \frac{(2\pi)^8\,\delta^4(p_1-k_1-k_2-k_3)\,\delta^4(k_1+k_2+k_3-p_1')}{(k_1^2-m^2+i\epsilon)(k_2^2-m^2+i\epsilon)(k_3^2-m^2+i\epsilon)}$$

But $c = 4! \times 4$ (see lectures)

$$\therefore \to +\frac{i\lambda^2}{6} \iiint \frac{d^4k_1\, d^4k_2\, d^4k_3}{(2\pi)^4} \frac{\delta^4(p_1-k_1-k_2-k_3)\,\delta^4(k_1+k_2+k_3-p_1')}{(k_1^2-m^2+i\epsilon)(k_2^2-m^2+i\epsilon)(k_3^2-m^2+i\epsilon)}$$

Performing integral over e.g. k_3

$$\to +\frac{i\lambda^2}{6} \iint \frac{d^4k_1\, d^4k_2}{(2\pi)^3} \frac{(2\pi)^4\,\delta^4(p_1-p_1')}{(k_1^2-m^2+i\epsilon)(k_2^2-m^2+i\epsilon)((p_1-k_2-k_1)^2-m^2+i\epsilon)}$$

Removing $(2\pi)^4\delta(p_1'-p_1)$ gives final result

$$+\frac{i\lambda^2}{6} \iint \frac{d^4k_1\, d^4k_2}{(2\pi)^8 (k_1^2-m^2+i\epsilon)(k_2^2-m^2+i\epsilon)((p_1-k_1-k_2)^2-m^2+i\epsilon)}$$

Question 6.

$$s+t+u = (p_a+p_b)^2+(p_a-p_c)^2+(p_a-p_d)^2$$

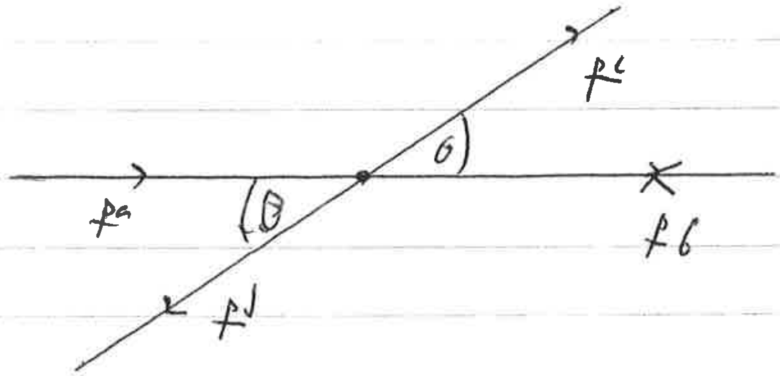
$$= 3p_a^2+p_b^2+p_c^2+p_d^2+2p_a\cdot p_b-2p_a\cdot p_c-2p_a\cdot p_d$$

$$= 3p_a^2+p_b^2+p_c^2+p_d^2+2p_a\cdot(p_b-p_c-p_d)$$

$$T = p_a \quad (p_a+p_b=p_c+p_d)$$

$$\rightarrow \quad p_a^2+p_b^2+p_c^2+p_d^2 = m_a^2+m_b^2+m_c^2+m_d^2.$$

Work in centre of mass frame, $\underline{p}_a = -\underline{p}_b$



$$s = (p_a+p_b)^2 = p_a^2+p_b^2+2p_a\cdot p_b$$

$$= 2m^2+2E^2+2p^2$$

$$= 4m^2+4p^2 = 4E^2 \geqslant 4m^2$$

$$t = (p_a-p_c)^2 = p_a^2+p_c^2-2p_a\cdot p_c$$

$$= 2m^2-2(E_aE_c-\underline{p}_a\cdot\underline{p}_c)$$

$$= 2m^2-2E_aE_c+2\underline{p}_a\cdot\underline{p}_c$$

B.6 $E_a = E_c$ $\quad |p_a| = |p_c|$ and $p_a \cdot p_c = |p_a|^2 \cos\theta$.

$$\therefore \quad t = 2m^2 - 2E_a^2 + 2|p_a|^2\cos\theta$$

$$= -2|p_a|^2 + 2|p_a|^2\cos\theta$$

$$= \underline{\underline{2|p_a|^2(\cos\theta - 1) \leq 0.}}$$

$$u = (p_a - p_d)^2 = p_a^2 + p_d^2 - p_a \cdot p_d$$

now $|p_d| = |p_a|$ and $p_a \cdot p_d = -|p_a||p_d|\cos\theta$.

$$\therefore \quad u = \underline{\underline{2|p_a|^2(-\cos\theta - 1) = -2|p_a|^2(1+\cos\theta) \leq 0}}$$