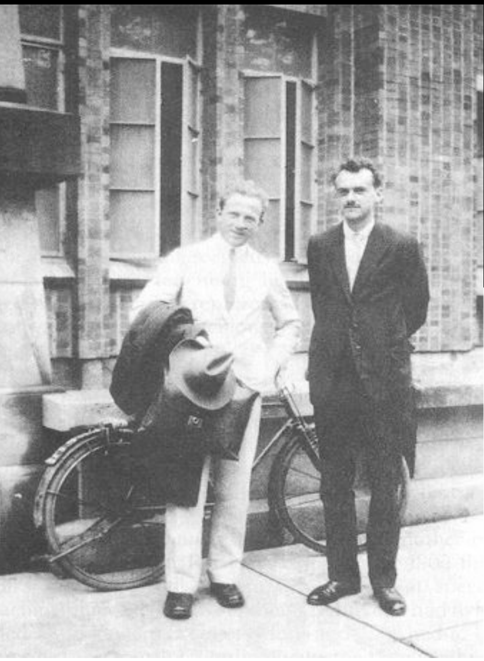
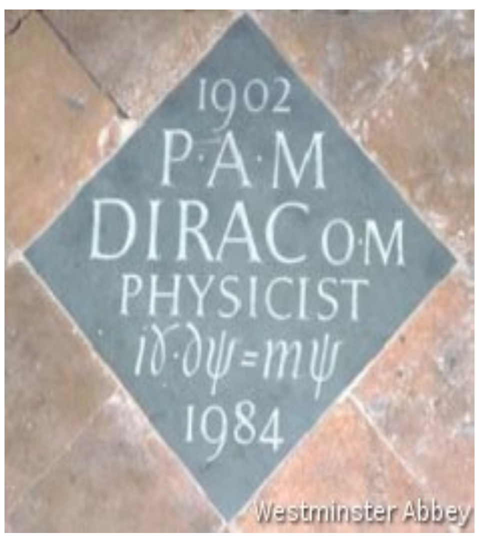


# PHASM/G442 Particle Physics

## Ruben Saakyan
## Module 3
## The Dirac Equation




$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

http://moodle.ucl.ac.uk/course/view.php?id=2589

# Introduction

- Need a proper theoretical description
  - Spin 1/2 matter particles (Fermions)
  - Interacting via spin 1 force particles (Bosons)
  - Must be relativistic

- Essentially need to find a replacement for the non-relativistic Schrödinger equation which originates from:

Schrödinger equation

$$\frac{p^2}{2m} + V = E \longrightarrow i\frac{\partial \Psi}{\partial t} = \hat{H}\Psi = \frac{-1}{2m}\nabla^2\Psi + V\Psi$$

**not** Lorentz invariant

using the standard quantum mechanics operators for energy and momentum

$$\hat{p} = -i\vec{\nabla}$$

$$\hat{E} = +i\frac{\partial}{\partial t}$$

$$\longrightarrow \hat{P}_\mu = i\partial_\mu$$

Probability density $\rho = \psi^* \psi = |\psi|^2$

# The Klein-Gordon Equation

$$E^2 = |\vec{p}|^2 + m^2 \xrightarrow[\hat{E} = +i\frac{\partial}{\partial t}]{\hat{p} = -i\vec{\nabla}} \partial^\mu \partial_\mu \Psi + m^2 \Psi = 0$$

For plane wave $\psi = N e^{i(\vec{p}.\vec{r} - Et)}$ it gives **<u>negative</u> energy solutions**

$$E = \pm\sqrt{|\vec{p}|^2 + m^2}$$

The **bigger** problem is **<u>negative</u> particle densities**

$\rho = 2E|N|^2$ (recall $\psi' = (2E)^{1/2}\psi$ for ***relativistic* invariant** w.function normalisation)

**<u>NOTE:</u>**

Schrödinger equation (with $V = 0$) $\boxed{-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial \psi}{\partial t}}$ 1st order in $\frac{\partial}{\partial t}$, 2nd order in $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$

Klein-Gordon Equation $\boxed{\left(\partial^\mu \partial_\mu + m^2\right)\psi = 0}$ 2nd order throghout

Dirac looked for an alternative which would be **1st order throughout**

$$\hat{H}\psi = (\vec{\alpha}.\vec{p} + \beta m)\psi = i\frac{\partial\psi}{\partial t} \qquad\qquad \vec{p} = -i\vec{\nabla}$$

Squaring it and expanding:

$$\begin{aligned} -\frac{\partial^2\psi}{\partial t^2} \quad = \quad & -\alpha_x^2\frac{\partial^2\psi}{\partial x^2} - \alpha_y^2\frac{\partial^2\psi}{\partial y^2} - \alpha_z^2\frac{\partial^2\psi}{\partial z^2} + \beta^2 m^2\psi \\ & -(\alpha_x\alpha_y + \alpha_y\alpha_x)\frac{\partial^2\psi}{\partial x\partial y} - (\alpha_y\alpha_z + \alpha_z\alpha_y)\frac{\partial^2\psi}{\partial y\partial z} - (\alpha_z\alpha_x + \alpha_x\alpha_z)\frac{\partial^2\psi}{\partial z\partial x} \\ & -(\alpha_x\beta + \beta\alpha_x)m\frac{\partial\psi}{\partial x} - (\alpha_y\beta + \beta\alpha_y)m\frac{\partial\psi}{\partial y} - (\alpha_z\beta + \beta\alpha_z)m\frac{\partial\psi}{\partial z} \end{aligned}$$

But it should still satisfy the Klein-Gordon Equation (since $E^2 = |\vec{p}|^2 + m^2$)

$$-\frac{\partial^2\psi}{\partial t^2} = -\frac{\partial^2\psi}{\partial x^2} - \frac{\partial^2\psi}{\partial y^2} - \frac{\partial^2\psi}{\partial z^2} + m^2\psi$$

- For Dirac to be consistent with Klein-Gordon:

$$\alpha_x^2 = \alpha_y^2 = \alpha_z^2 = \beta^2 = 1$$
$$\alpha_j \beta + \beta \alpha_j = 0$$
$$\alpha_j \alpha_k + \alpha_k \alpha_j = 0 \qquad (j \neq k)$$

- It is clear that $\alpha_j$ and $\beta$ **can not be numbers**

- Require **four mutually anti-commuting matrices**
- Must be (at least) **4x4** matrices

Consequently the wave function must be a **4-component Dirac spinor**

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$$

**A consequence of introducing an equation that is 1st order in time/space derivatives is that the wave-function has new degrees of freedom !**

- For the Hamiltonian to be Hermitian requires **4 anti-commuting Hermitian matrices**

$$\boxed{\alpha_x = \alpha_x^\dagger; \quad \alpha_y = \alpha_y^\dagger; \quad \alpha_z = \alpha_z^\dagger; \quad \beta = \beta^\dagger;}$$

- Physical results do not depend on the exact representation of the matrices. Everything in commutation relations.
- A convenient choice is based on the Pauli spin matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

# Aside (but important for what follows)

Analogy with the **continuity equation**:

$$\boxed{\vec{\nabla}.\vec{j}+\frac{\partial\rho}{\partial t}=0}$$

Non-relativistic case (Schrödinger)

$$-\frac{1}{2m}\vec{\nabla}.\left(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*\right) \quad = \quad i\frac{\partial}{\partial t}(\psi^*\psi) \implies$$

$$\rho=\psi^*\psi=|\psi|^2{}_{=|N|^2}$$

$$\vec{j}=\frac{1}{2mi}\left(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*\right)_{=|N|^2\frac{\vec{p}}{m}=|N|^2\vec{v}}$$

current
(important for Feynman diagram interpretation)

Relativistic case (Klein-Gordon)

$$\frac{\partial}{\partial t}\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right) \quad = \quad \vec{\nabla}.(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*) \implies$$

$$\rho=i\left(\psi^*\frac{\partial\psi}{\partial t}-\psi\frac{\partial\psi^*}{\partial t}\right)=2E|N|^2$$

$$\vec{j}=i(\psi^*\vec{\nabla}\psi-\psi\vec{\nabla}\psi^*)\ =|N|^2\vec{p}$$

Relativistic case (Dirac)

$$\vec{\nabla}.(\psi^\dagger\vec{\alpha}\psi)+\frac{\partial(\psi^\dagger\psi)}{\partial t}=0 \implies$$

$$\boxed{\rho=\psi^\dagger\psi}$$

$$\boxed{\vec{j}=\psi^\dagger\vec{\alpha}\psi}$$

- It can be shown that the probability density is now

$$\rho = \psi^\dagger \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2 > 0$$

- Therefore the Dirac equation has probability densities which are **always positive**
- In addition, the solution to the Dirac equation are the **four component Dirac spinors** which gives rise to the property of **intrinsic spin**.
- It can be shown that Dirac spinors represent spin-half particles with an intrinsic magnetic moment of

$$\boxed{\vec{\mu} = \frac{q}{m}\vec{S}}$$

# Covariant notation. Dirac Gamma matrices

The Dirac equation $\left(-\vec{\alpha}\cdot i\vec{\nabla}+\beta m\right)\psi = i\frac{\partial\psi}{\partial t}$

can be written more elegantly by introducing the following **gamma matrices**:

$$\gamma^0 \equiv \beta;\ \ \gamma^1 \equiv \beta\alpha_x;\ \ \gamma^2 \equiv \beta\alpha_y;\ \ \gamma^3 \equiv \beta\alpha_z$$

Remembering that $\partial_\mu = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

we arrive at the canonical **Dirac Equation**

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}$$

# Properties of gamma matrices

$$(\gamma^0)^2 = 1 \qquad (\gamma^1)^2 = (\gamma^2)^2 = (\gamma^3)^2 = -1$$

$$\begin{aligned} \gamma^0\gamma^j + \gamma^j\gamma^0 &= 0 \\ \gamma^j\gamma^k + \gamma^k\gamma^j &= 0 \quad (j \neq k) \end{aligned}$$

which can be compactly written as

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

$\gamma^0$ is Hermitian $\gamma^1, \gamma^2, \gamma^3$ are anti-Hermitian

$$\boxed{\gamma^{0\dagger} = \gamma^0, \;\; \gamma^{1\dagger} = -\gamma^1, \;\; \gamma^{2\dagger} = -\gamma^2, \;\; \gamma^{3\dagger} = -\gamma^3}$$

You will need to know your gamma matrix algebra (practise questions in PS2).

# Pauli-Dirac representation

- We will use the Pauli-Dirac representation of the gamma matrices

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}; \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

which can be written in full

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}; \; \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}$$

$$\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}; \; \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

# The Adjoint Spinor and Current

- Using gamma matrices we can writhe the **4-vector current**

$$j^\mu = (\rho, \vec{j}) = \psi^\dagger \gamma^0 \gamma^\mu \psi$$

- The continuity equation becomes

$$\partial_\mu j^\mu = 0$$

- The **adjoint spinor** is defined as

$$\boxed{\overline{\psi} = \psi^\dagger \gamma^0}$$

- In terms of the **adjoint spinor** the **4-vector current** can be written as

$$\boxed{j^\mu = \overline{\psi} \gamma^\mu \psi}$$

**Important in Feynman rules used to calculate Lorentz invariant matrix element for fundamental interactions**

# Solutions to the Dirac Equation. Free particle **at rest**.

- Look for **free particle** solutions to the D.E. of form

$$\boxed{\psi = u(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}}$$

- It will obey $(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$ which can be rewritten as

$$\boxed{(\gamma^{\mu}p_{\mu} - m)u = 0} \Leftarrow$$ the D.E. in "momentum" form

- For particle **at rest** ( $\psi = u(E,0)e^{-iEt}$): $E\gamma^{0}u - mu = 0$

- This equation has four orthogonal solutions

$$\underbrace{u_1(m,0) = \begin{pmatrix}1\\0\\0\\0\end{pmatrix};\ u_2(m,0) = \begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{\boxed{E = m}};\ \underbrace{u_3(m,0) = \begin{pmatrix}0\\0\\1\\0\end{pmatrix};\ u_4(m,0) = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{\boxed{E = -m}}$$

(still have negative energy solutions)

- Including the time dependence

$$\psi = u(E,0)e^{-iEt}$$

$$\underbrace{\psi_1 = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} e^{-imt}; \quad \psi_2 = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} e^{-imt};}_{\text{Two spin states with } E>0} \quad \underbrace{\psi_3 = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} e^{+imt}; \text{ and } \psi_4 = \begin{pmatrix}0\\0\\0\\1\end{pmatrix} e^{+imt}}_{\text{Two spin states with } E<0}$$

Four solutions, two of which with **negative** energy

# <u>General</u> free particle solutions to the Dirac Equation.

- Look for **general** solutions to the D.E. (plane wave)

$$\psi = u(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$$

- Starting from $(\gamma^\mu p_\mu - m)u = 0$

$$\begin{aligned} \gamma^\mu p_\mu - m &= E\gamma^0 - p_x\gamma^1 - p_y\gamma^2 - p_z\gamma^3 - m \\ &= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} .\vec{p} - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} \end{aligned}$$

N.B. 4x4 matrix is written in terms of 2x2 sub-matrices

$$(\gamma^\mu p_\mu - m)u = 0 \quad \Rightarrow \quad \begin{pmatrix} (E-m)I & -\vec{\sigma}.\vec{p} \\ \vec{\sigma}.\vec{p} & (-E-m)I \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

4-component spinor

Giving to coupled simultaneous equations:

$$\left.\begin{aligned}(\vec{\sigma}.\vec{p})u_B &= (E-m)u_A \\ (\vec{\sigma}.\vec{p})u_A &= (E+m)u_B\end{aligned}\right\}$$

Expanding this gives:

$$u_B = \frac{\vec{\sigma}.\vec{p}}{E+m}u_A = \frac{1}{E+m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A$$

Arbitrary (but **simplest**) choices for $u_A$ $\quad u_A = \begin{pmatrix}1\\0\end{pmatrix}$ OR $u_A = \begin{pmatrix}0\\1\end{pmatrix}$

giving $\quad u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$ and $u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$ where $N$ is the wave-function normalisation

Repeating it for $\quad u_B = \begin{pmatrix}1\\0\end{pmatrix}$ and $u_B = \begin{pmatrix}0\\1\end{pmatrix}$

gives the solutions for $u_3$ and $u_4$

**Therefore the four solutions are**

$$\psi_i = u_i(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$$

$$u_1 = N_1\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}; \quad u_2 = N_2\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}; \quad u_3 = N_3\begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}; \quad u_4 = N_4\begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}$$

★ So $u_1$, $u_2$ are the +ve energy solutions and $u_3$, $u_4$ are the -ve energy solutions

Anti-particle prediction!

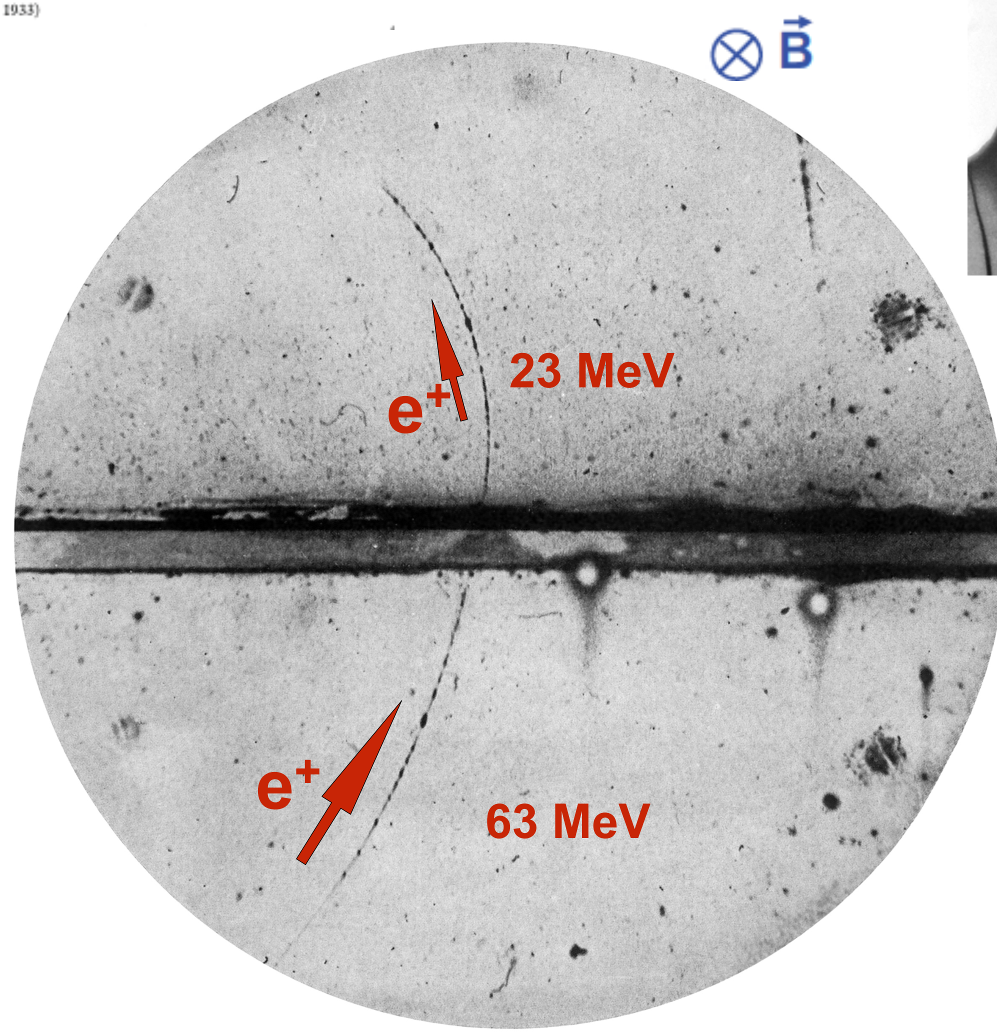
# The Positron

MARCH 15, 1933 PHYSICAL REVIEW VOLUME 43

**The Positive Electron**

CARL D. ANDERSON, *California Institute of Technology, Pasadena, California*
(Received February 28, 1933)

CARL D. ANDERSON



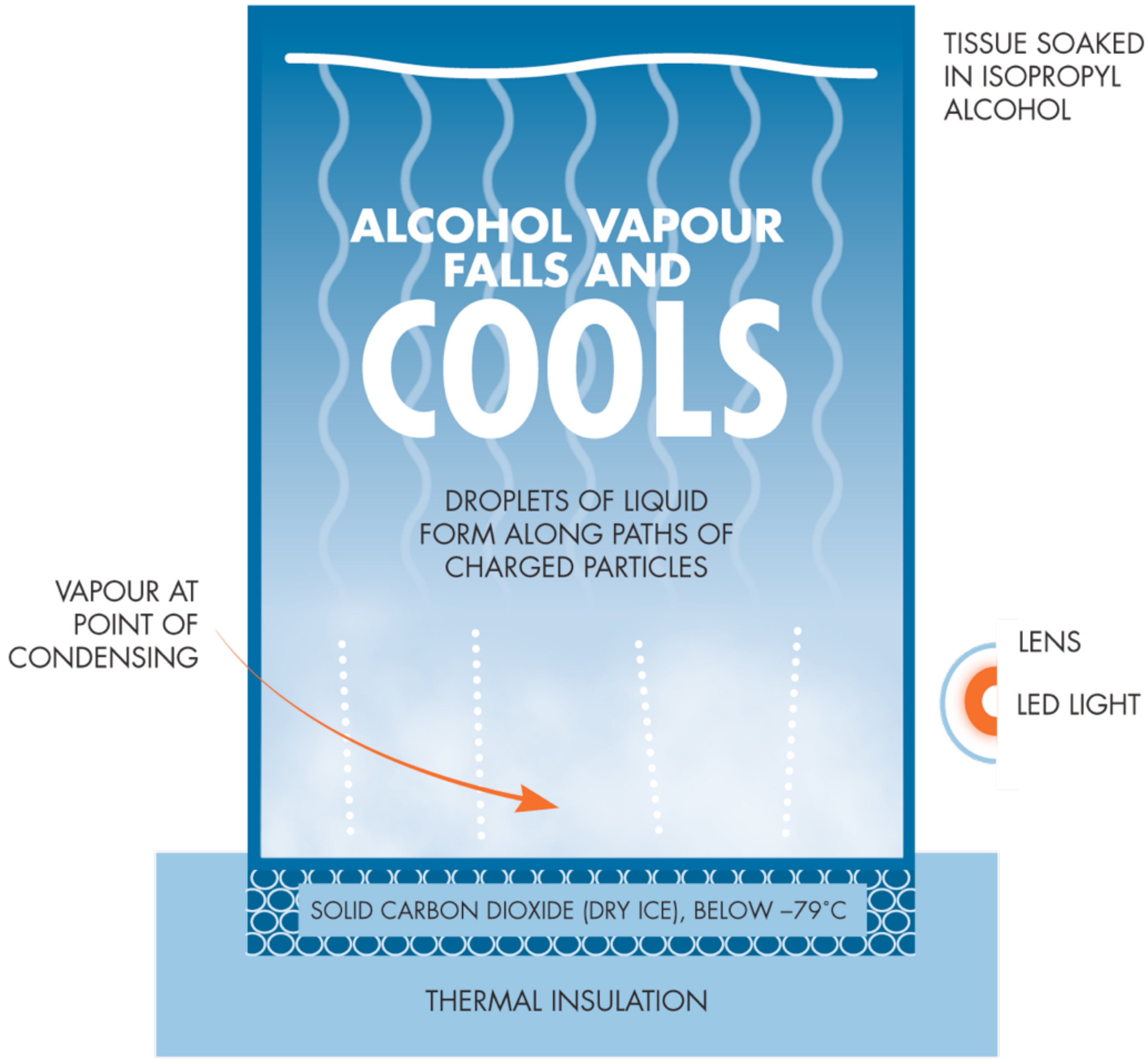

6mm
Lead plate

- In 1929 Dirac identified negative energy solutions to his equation — could be interpreted as positive energy positively charged particles (predicting **<u>antiparticles</u>**) — more on that later in the course.
- In 1933 Anderson measured one of these particles using a cloud chamber

*Q1.: How do we know it is not an electron?*
*Q2. How do we know it is not a proton?*

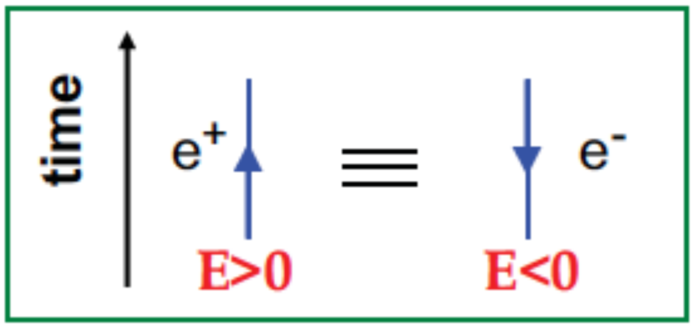
# The build your own version

- These days it is easy to build your own cloud chamber using dry ice and alcohol

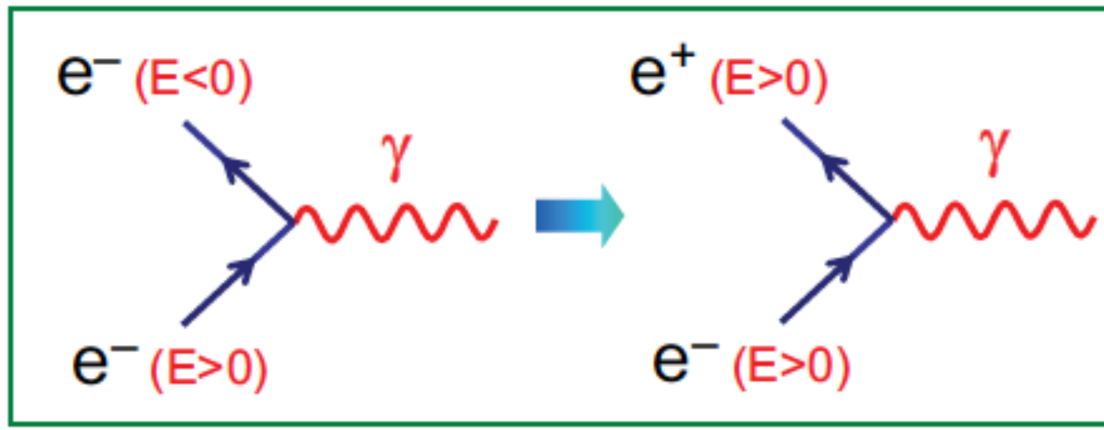


# Feynman-Stückelberg interpretation

Interpret a negative energy solution as a negative energy particle which propagates backwards in time or equivalently a positive energy anti-particle which propagates forwards in time




$$e^{-i(-E)(-t)} \rightarrow e^{-iEt}$$




**NOTE: in the Feynman diagram the arrow on the anti-particle remains in the backwards in time direction to label it an anti-particle solution.**

From now on use this interpretation to work with **anti-particle** wave-function with **positive** energy, $E = \sqrt{|\vec{p}|^2 + m^2}$

# Anti-particle spinors

- Find negative energy solutions to the D.E. of the form

$$\psi = v(E,\vec{p})e^{-i(\vec{p}\cdot\vec{r}-Et)} \quad \text{where} \quad E = |\sqrt{|\vec{p}|^2+m^2}|$$

- The D.E. in terms of momentum for $v$ spinors

$$\boxed{(\gamma^\mu p_\mu + m)v = 0}$$

- Solving it in the same manner as with particle ($u$ spinors) obtain:

$$v_1 = N_1' \begin{pmatrix} \frac{p_x - ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \qquad v_2 = N_2' \begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

# Particle and Anti-Particle Spinors

- Four solutions of the form $\psi_i = u_i(E,\vec{p})e^{i(\vec{p}.\vec{r}-Et)}$

$$\underbrace{u_1 = N\begin{pmatrix}1\\0\\\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\end{pmatrix};\; u_2 = N\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\end{pmatrix}}_{E=+\left|\sqrt{|\vec{p}|^2+m^2}\right|};\; \underbrace{u_3 = N\begin{pmatrix}\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\\1\\0\end{pmatrix};\; u_4 = N\begin{pmatrix}\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\\0\\1\end{pmatrix}}_{E=-\left|\sqrt{|\vec{p}|^2+m^2}\right|}$$

- Four solutions of the form $\psi_i = v_i(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$

$$\underbrace{v_1 = N\begin{pmatrix}\frac{p_x-ip_y}{E+m}\\\frac{-p_z}{E+m}\\0\\1\end{pmatrix};\; v_2 = N\begin{pmatrix}\frac{p_z}{E+m}\\\frac{p_x+ip_y}{E+m}\\1\\0\end{pmatrix}}_{E=+\left|\sqrt{|\vec{p}|^2+m^2}\right|};\; \underbrace{v_3 = N\begin{pmatrix}1\\0\\\frac{p_z}{E-m}\\\frac{p_x+ip_y}{E-m}\end{pmatrix};\; v_4 = N\begin{pmatrix}0\\1\\\frac{p_x-ip_y}{E-m}\\\frac{-p_z}{E-m}\end{pmatrix}}_{E=-\left|\sqrt{|\vec{p}|^2+m^2}\right|}$$

- Only four are linearly independent
- Natural to choose **positive energy solutions** $\boxed{\{u_1, u_2, v_1, v_2\}}$

# Wave-function normalisation and Charge Conjugation

- Wave-function normalisation $N$ is found from the relativistic probability density:

$$\rho = \psi^{\dagger}\psi = 2E$$

- It is easy to show that $\boxed{N = \sqrt{E+m}}$

Therefore, e.g. 1st spinor: $$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}$$

- Charge Conjugation is defined as $\boxed{\psi' = \hat{C}\psi = i\gamma^2\psi^*}$

$\hat{C}$ ➡ **particle spinor ↔ anti-particle spinor**

$$\psi = u_1 e^{i(\vec{p}.\vec{r}-Et)} \xrightarrow{\hat{C}} \psi' = v_1 e^{-i(\vec{p}.\vec{r}-Et)}$$
$$\psi = u_2 e^{i(\vec{p}.\vec{r}-Et)} \xrightarrow{\hat{C}} \psi' = v_2 e^{-i(\vec{p}.\vec{r}-Et)}$$

# Summary of the solutions to the Dirac Equation

- Free **particle** solution $\psi = u(E,\vec{p})e^{+i(\vec{p}.\vec{r}-Et)}$ satisfy $\boxed{(\gamma^\mu p_\mu - m)u = 0}$

with

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}; \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

- Free **anti**-**particle** solution $\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$ satisfy $\boxed{(\gamma^\mu p_\mu + m)v = 0}$

with

$$v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \qquad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

For **both** solutions: $E = \sqrt{|\vec{p}|^2 + m^2}$

# Spin States

- The Dirac equation solved the negative probability densities problem
- In addition, the additional degrees of freedom of four-component wave functions provide a natural description for the intrinsic angular momentum of spin-1/2 particles and antiparticles.
- The last statement is not obvious. It's proof is beyond this course but those interested (and not afraid) are encouraged to look at Section 4.4 of "Modern Particle Physics" by M. Thomson.

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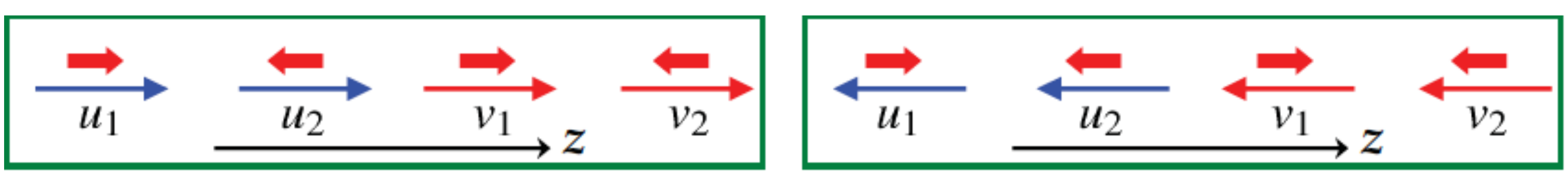
- $u_1, u_2, v_1, v_2$ are not eigenstates of the spin operator $\boldsymbol{S}$, or $S_z$
- However particle traveling in z-direction $\quad p_z = \pm|\vec{p}|$

$$u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{\pm|\vec{p}|}{E+m} \\ 0 \end{pmatrix}; \quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ 0 \\ \frac{\mp|\vec{p}|}{E+m} \end{pmatrix}; \quad v_1 = N\begin{pmatrix} 0 \\ \frac{\mp|\vec{p}|}{E+m} \\ 0 \\ 1 \end{pmatrix}; \quad v_2 = N\begin{pmatrix} \frac{\pm|\vec{p}|}{E+m} \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

are eigenstates of $S_z$

$$\hat{S}_z u_1 = +\tfrac{1}{2}u_1 \qquad \hat{S}_z^{(v)} v_1 = -\hat{S}_z v_1 = +\tfrac{1}{2}v_1$$

$$\hat{S}_z u_2 = -\tfrac{1}{2}u_2 \qquad \hat{S}_z^{(v)} v_2 = -\hat{S}_z v_2 = -\tfrac{1}{2}v_2$$



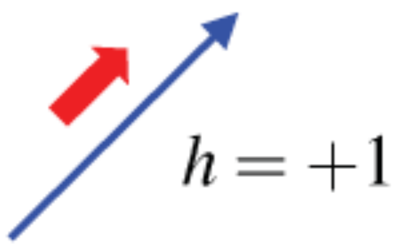
# HELICITY

- To have a more useful basis introduce a new concept — **HELICITY**

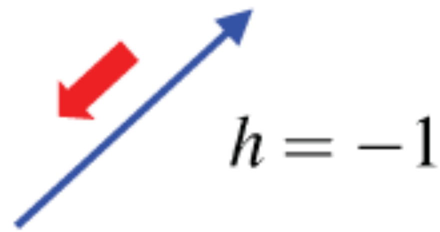
$$h \equiv \frac{\vec{S}.\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}.\vec{p}}{|\vec{p}|} = \frac{\vec{\Sigma}.\vec{p}}{|\vec{p}|}$$

Component of particle's spin on its direction of motion

- Good quantum number since $[\hat{H}, \hat{S}.\hat{p}] = 0$




**"right-handed"**




**"left-handed"**

Eigenvalues of helicity operator are ±1

N.B. We will later come across another important concept — **CHIRALITY**

It plays a key role in Lagrangian formalism of description of particles and their interactions

**Only** in the **relativistic** limit ($v \approx c$) **HELICITY** eigenstates are the same as **CHIRALITY** eigenstates

# Helicity Eigenstates

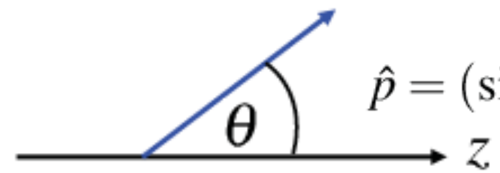
- Aim is to find solutions to the **D.E**. which are also eigenstates of Helicity

$$(\vec{\Sigma}.\hat{p})u_\uparrow = +u_\uparrow \qquad (\vec{\Sigma}.\hat{p})u_\downarrow = -u_\downarrow$$

$\hat{p}$ is the unit vector in particle's direction

$$\begin{pmatrix} \vec{\sigma}.\hat{p} & 0 \\ 0 & \vec{\sigma}.\hat{p} \end{pmatrix}\begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm\begin{pmatrix} u_A \\ u_B \end{pmatrix} \Rightarrow \begin{aligned} (\vec{\sigma}.\hat{p})u_A &= \pm u_A \\ (\vec{\sigma}.\hat{p})u_B &= \pm u_B \end{aligned}$$

For a particle propagating in $(\theta, \phi)$ direction




$$\hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

$$\vec{\sigma}.\hat{p} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

Writing either $u_A = \binom{a}{b}$ or $u_B = \binom{a}{b}$

$$\begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \pm\begin{pmatrix} a \\ b \end{pmatrix}$$

$$\frac{b}{a} = \frac{\boxed{\pm 1} - \cos\theta}{\sin\theta} e^{i\phi}$$

For the **right-handed** helicity (i.e. +1) $\frac{b}{a} = \frac{1-\cos\theta}{\sin\theta} e^{i\phi} = \frac{2\sin^2\left(\frac{\theta}{2}\right)}{2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)} e^{i\phi} = e^{i\phi}\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$

$$\Rightarrow \quad u_{A\uparrow} \propto \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_{B\uparrow} \propto \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

And therefore $$u_\uparrow = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} \kappa_1 \cos\left(\frac{\theta}{2}\right) \\ \kappa_1 e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \kappa_2 \cos\left(\frac{\theta}{2}\right) \\ \kappa_2 e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

We also know from the **D.E.**:

$$(\vec{\sigma}.\vec{p})u_A = (E+m)u_B$$

$$u_B = \frac{\vec{\sigma}.\vec{p}}{E+m}u_A = \frac{|\vec{p}|}{E+m}\underbrace{(\vec{\sigma}.\hat{p})}_{\boxed{\text{Helicity}}}u_A = \pm\frac{|\vec{p}|}{E+m}u_A \Rightarrow u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$

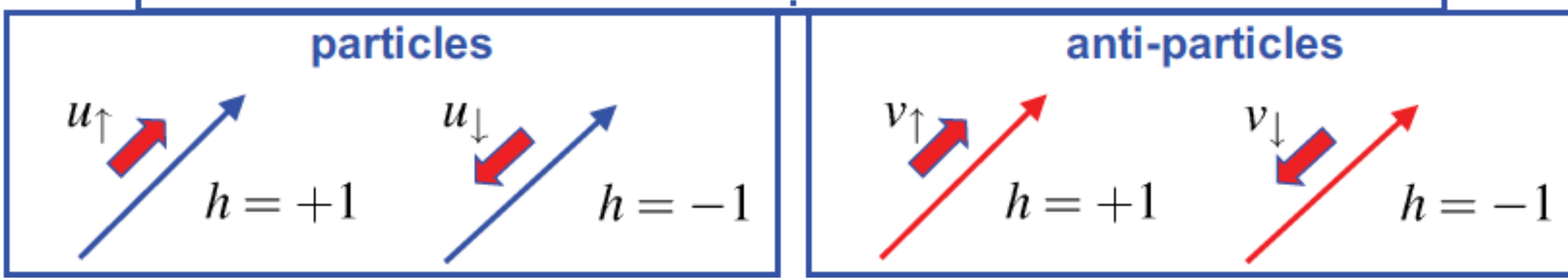
- The **negative helicity** (i.e. -1) particle state is obtained in the same way
- The **helicity** of **anti**-particle states is also obtained in the same way but taking into account:

$$\hat{S}^{(v)} = -\hat{S} \quad \Rightarrow \quad (\vec{\Sigma}.\hat{p})v_\uparrow = -v_\uparrow$$

# Summary of particle and anti-particle helicity eigenstates

$$u_\uparrow = N\begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_\downarrow = N\begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_\uparrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad v_\downarrow = N\begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$




As previously, $\boxed{N=\sqrt{E+m}}$

# Aside (but related). Parity operator.

- It can be shown (in fact you'll show it in PS2, q3) that under the parity transformation the form of the Dirac equation is unchanged provided that Dirac spinors transform as

$$\boxed{\psi \rightarrow \hat{P}\psi = \pm\gamma^0\psi}$$

$$\hat{P}u_1 = \pm \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \pm u_1 \quad \textbf{etc.} \quad \rightarrow \quad \boxed{\begin{array}{ll} \hat{P}u_1 = \pm u_1 & \hat{P}v_1 = \mp v_1 \\ \hat{P}u_2 = \pm u_2 & \hat{P}v_2 = \mp v_2 \end{array}}$$

- *Anti-particle* and *particle* have the **opposite** **intrinsic** parities

- The convention is that particle are chosen to have positive parity, i.e.

$$\boxed{\hat{P} = +\gamma^0}$$