Answer THREE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

The following formulae may be used if required:

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!}$$

$$\sin(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!}$$

$$\cos(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!}$$

Probability current

$$J = \frac{\hbar}{2im}\left(\psi(x)^* \frac{\partial \psi(x)}{\partial x} - \psi(x)\left[\frac{\partial \psi(x)}{\partial x}\right]^*\right).$$

WKB Connection formulae

Right-hand barrier - with classical turning point at $x = a$:

$$\frac{2}{\sqrt{p(x)}}\cos\left(\int_x^a p(x')dx'/\hbar - \frac{\pi}{4}\right) \quad \leftarrow \quad \frac{1}{\sqrt{q(x)}}\exp\left[-\int_a^x q(x')dx'/\hbar\right]$$

$$-\frac{1}{\sqrt{p(x)}}\sin\left(\int_x^a p(x')dx'/\hbar - \frac{\pi}{4}\right) \quad \rightarrow \quad \frac{1}{\sqrt{q(x)}}\exp\left[+\int_a^x q(x')dx'/\hbar\right]$$

Left-hand barrier - with classical turning point at $x = a$:

$$\frac{1}{\sqrt{q(x)}}\exp\left[-\int_x^a q(x')dx'/\hbar\right] \quad \rightarrow \quad \frac{2}{\sqrt{p(x)}}\cos\left(\int_a^x p(x')dx'/\hbar - \frac{\pi}{4}\right)$$

$$\frac{1}{\sqrt{q(x)}}\exp\left[+\int_x^a q(x')dx'/\hbar\right] \quad \leftarrow \quad -\frac{1}{\sqrt{p(x)}}\sin\left(\int_a^x p(x')dx'/\hbar - \frac{\pi}{4}\right)$$

1. (a) In this question, you will study a one-dimensional tunnelling problem. Name one other application of the WKB approximation. Under what conditions is the WKB approximation valid? **[3]**

(b) In a classically allowed region, the WKB wave-function for a particle of energy E in potential $V(x)$ has the following form:

$$\psi(x) = \frac{A}{\sqrt{p(x)}} \exp\left[\pm i \int p(x')dx'/\hbar\right]$$

where $p(x) = \sqrt{2m(E - V(x))}$. Why can this wave-function not be valid at a classical turning point? **[2]**

(c) In a one-dimensional tunnelling problem, a particle of mass m is incident from the left (i.e. from $x = -\infty$) onto a barrier where the potential $V(x)$ takes the form: $V(x) = 0$ when $|x| > L/2$ and $V(x) = V_0(1 - (2x/L)^2)$ when $|x| \leq L/2$.
Obtain the classical turning points t_1 and t_2, where $t_2 > 0$, as a function of E. **[2]**

(d) In the classically allowed region to the right of the barrier, the wave-function of the tunnelled particle has the form,

$$\psi(x) = \frac{A}{\sqrt{p(x)}} \exp\left[i\left(\int_{t_2}^{x} p(x')dx'/\hbar - \frac{\pi}{4}\right)\right]$$

Using the expression on page one of this script, show that the probability current for the particle in this region is $J_t = |A|^2/m$. **[3]**

(e) Show that the wave-function within the barrier is

$$\psi(x) = \frac{A}{\sqrt{q(x)}}\left(\frac{1}{2r}\exp\left[\int_{t_1}^{x} q(x')dx'/\hbar\right] - ir\exp\left[-\int_{t_1}^{x} q(x')dx'/\hbar\right]\right)$$

where $q(x) = \sqrt{2m(V(x) - E)}$ and $r = \exp[\int_{t_1}^{t_2} q(x')dx'/\hbar]$. You may use the connection formulae on the first page of this exam script. **[4]**

(f) By applying connection formulae across the left-hand barrier, and identifying incident and reflected waves, one can derive that the incident probability current is

$$J_i = \left(\frac{1}{4r} + r\right)^2 \frac{|A|^2}{m}.$$

Using this expression, show that, in the limit that $r \gg 1$, the tunnelling probability takes the form

$$P \approx r^{-2}$$

[2]

(g) For the barrier given above, calculate the tunnelling probability as a function of energy. In solving the integral, you may find the substitution $x = (L/2)\sqrt{1 - E/V_0}\, y$ and the standard integral $\int_{-1}^{+1} \sqrt{1 - y^2}dy = \pi/2$ helpful. **[4]**

2. (a) What is an "open quantum system"? Why do we typically use the density matrix formalism to study the dynamics of open quantum systems? **[3]**

(b) A super-operator $S[\rho]$ is written in Kraus form

$$S[\rho] = \sum_j K_j \rho K_j^\dagger$$

where $\sum_j K_j^\dagger K_j = \mathbb{1}$.

Given that ρ is a physical density operator, show that $\rho' = S[\rho]$ is Hermitian and that $\mathrm{Tr}[\rho'] = 1$. **[4]**

(c) What do we mean when we say that the evolution of an open quantum system is Markovian? In your answer, you may refer to the relationship between $\rho(t + dt)$ and $\rho(t)$, or information flow between system and environment. **[2]**

(d) One can show that a Markovian evolution equation (called a master equation) for an open quantum system can be written in Lindblad form

$$\frac{\partial \rho}{\partial t} = \frac{-i}{\hbar}[H_{\text{eff}}\rho - \rho H_{\text{eff}}^\dagger] + \sum_j L_j \rho L_j^\dagger$$

where $H_{\text{eff}} = H_0 - (i\hbar/2)\sum_j L_j^\dagger L_j$. Explain the role of the operators H_0 and L_j in this equation. If all L_j operators are set to zero, what equation do we recover? **[3]**

(e) A three-level atom has energy levels $|1\rangle$ with energy $E_1 = 0$, and degenerate levels $|2\rangle$ and $|3\rangle$ with energy $E_2 = E_3 = \hbar\omega$. The system undergoes spontaneous emission due to its interaction with its environment.

We describe the dynamics of the atom by a Master equation in Lindblad form. Spontaneous emission from state $|2\rangle$ to state $|1\rangle$ is described by jump operator $\gamma_2|1\rangle\langle 2|$ and from state $|3\rangle$ to state $|1\rangle$ by jump operator $\gamma_3|1\rangle\langle 3|$. Spontaneous emission between states $|2\rangle$ and $|3\rangle$ is negligible. Show that H_{eff} for this system has the form

$$H_{\text{eff}} = \alpha\,|2\rangle\,\langle 2| + \beta\,|3\rangle\,\langle 3|$$

and identify α and β. Hence write down the Lindblad form master equation for this evolution. **[3]**

(f) The atom is prepared in the initial state $\rho(0) = (1/2)(|2\rangle\,\langle 2| + |3\rangle\,\langle 3|)$. Given that the state of the atom evolving in time will have the general form:

$$\rho(t) = \rho_1(t)\,|1\rangle\,\langle 1| + \rho_2(t)\,|2\rangle\,\langle 2| + \rho_3(t)\,|3\rangle\,\langle 3|$$

derive, from the master equation, evolution equations for $\rho_1(t)$, $\rho_2(t)$ and $\rho_3(t)$. Solve these evolution equations to find $\rho(t)$ given the initial conditions. **[5]**

3. (a) The Pauli operators act on the spin-half basis states as follows:
$\sigma_x |\uparrow\rangle = |\downarrow\rangle$, $\sigma_x |\downarrow\rangle = |\uparrow\rangle$, $\sigma_z |\uparrow\rangle = |\uparrow\rangle$, $\sigma_z |\downarrow\rangle = -|\downarrow\rangle$.
Construct a matrix representation of σ_x and σ_z. **[2]**

(b) We say a pair of operators A and B commute if $AB - BA = 0$. If $AB + BA = 0$ we say the operators anti-commute.

i. Show that σ_x and σ_z anti-commute.
ii. Show that $\sigma_z \otimes \sigma_z$ and $\sigma_x \otimes \sigma_x$ commute. **[3]**

(c) Show that $|\psi(t)\rangle = U(t)\,|\psi(0)\rangle$ is a solution to the time-dependent Schrödinger equation,

$$\frac{\partial\,|\psi(t)\rangle}{\partial t} = \frac{-i}{\hbar} H\,|\psi(t)\rangle$$

where H is a (time-invariant) Hamiltonian and $U(t) = \exp[-iHt/\hbar]$. **[4]**

(d) If Hermitian operator $\hat{A}$ is self-inverse, show that

$$\exp[iAt] = \cos(t)\mathbb{1} + i\sin(t)\hat{A}.$$

[3]

(e) If operators A and B commute, the following identity holds,

$$\exp[i(A+B)t] = \exp[iAt]\exp[iBt]. \qquad (1)$$

When operators A and B do not commute, the identity in equation (1) is usually no longer valid. In this case, we can use a Trotter decomposition such as

$$\exp[i(A+B)t] \approx \exp[iAt/2]\exp[iBt]\exp[iAt/2].$$

to approximate the evolution operator. Show that this approximation is correct to the 2nd order in t. **[4]**

(f) Two-spin half particles are interacting under the following Hamiltonian

$$H = \hbar\omega(\sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z).$$

Without employing an approximation, compute the time evolved state $\psi(t)$ given that the system is initially in state $\psi(0) = |\downarrow\rangle \otimes |\uparrow\rangle$. **[4]**

4. (a) The interaction picture is often employed to describe the dynamics of a system whose Hamiltonian has the form $\hat{H} = \hat{H}_0 + \hat{V}$, where $\hat{H}_0$ is solved, i.e. its eigenstates $|\phi_j\rangle$ and eigenenergies E_j are known.
Given that the state $|\psi_I(t)\rangle$ in the interaction picture is related to the state $|\psi_S(t)\rangle$ in the standard Schrödinger picture via $|\psi_I(t)\rangle = U_0(t)^\dagger |\psi_S(t)\rangle$, where $U_0(t) = \exp[-i(t/\hbar)H_0]$, show that the $|\psi_I(t)\rangle$ will evolve according to the following equation:

$$i\hbar\frac{\partial}{\partial t}|\psi_I(t)\rangle = U_0(t)^\dagger V U_0(t)|\psi_I(t)\rangle .$$

You may use without proof the operator identity $(\partial/\partial t)\exp[\hat{A}t] = \hat{A}\exp[\hat{A}t]$. **[5]**

(b) A system subjected to a time-dependent perturbation is described by a Hamiltonian: $\hat{H} = \hat{H}_0 + \lambda\hat{V}(t)$, where the eigenvalues and eigenstates of H_0 are $|\phi_j\rangle$ and E_j. By describing the state of the system in the interaction picture as, $|\psi_I(t)\rangle = \sum_j c_j(t)|\phi_j\rangle$, show that the coefficients $c_j(t)$ satisfy the following equations of motion,

$$\dot{c}_j(t) = \frac{\lambda}{i\hbar}\sum_k c_k(t)\exp\left[i\omega_{jk}t\right]V_{jk}(t)$$

defining all the symbols in this expression. **[4]**

(c) In perturbation theory, we expand $c_j(t)$ as a power series in λ, $c_j(t) = \sum_{m=0}^{\infty}\lambda^m c_j^{(m)}$. Show that the mth order terms in this expansion satisfy the following expressions:

$$\dot{c}_j^{(0)}(t) = 0 \qquad\qquad \dot{c}_j^{(m)}(t) = \frac{1}{i\hbar}\sum_k \exp[i\omega_{jk}t]V_{jk}(t)c_k^{(m-1)}(t)$$

for $m = 1, 2, 3, \ldots$. **[3]**

(d) A quantum Harmonic oscillator is affected by a perturbation. The Hamiltonian is

$$\hat{H} = \hbar\omega\left(\hat{a}^\dagger\hat{a} + \frac{1}{2}\right) + f(t)(a^\dagger + a)$$

where $f(t) = 0$ for $t \leq 0$. If the system is initially in the ground state $|0\rangle$, into which state or states can transitions be induced to first order? Recall that $\hat{a}|n\rangle = \sqrt{n}|n-1\rangle$. **[3]**

(e) Into which state(s) would transitions be expected, up to *second* order, and why? **[2]**

(f) In an experiment, this perturbation is switched on at a constant strength between time $t = 0$ and $t = 10000/\omega$. In lectures, we derived that, for a constant perturbation of duration τ the first order transition probability between initial state $|\phi_j\rangle$ and final state $|\phi_k\rangle$ is proportional to $F(\omega_{jk}) = \tau^2\text{sinc}^2(\omega_{jk}\tau/2)$ where $\hbar\omega_{jk}$ is the difference in energy between $|\phi_j\rangle$ and final state $|\phi_k\rangle$. Without detailed calculation, explain why this tells us that we may expect the transition probability observed in this experiment to be small.
You may use without proof that the central peak of the function $F(\omega)$ has a width of approximately $2\pi/\tau$. **[3]**

5. (a) The states $|\phi_j\rangle$, for $j = 1$ to d, are vectors in a d-dimensional space. What conditions must the states satisfy to be an orthonormal basis for that space? **[2]**

(b) If $|\phi_j\rangle$ is an orthonormal basis for a space, prove that

$$\left(\sum_j |\phi_j\rangle\langle\phi_j|\right)|\psi\rangle = |\psi\rangle$$

for any state $|\psi\rangle$ in the space and hence derive the closure relation

$$\sum_j |\phi_j\rangle\langle\phi_j| = \mathbb{1}.$$

[3]

(c) Let $|\psi_j\rangle$, for $j = 1$ to d, be the orthonormal eigenstates of Hermitian operator $\hat{A}$ with corresponding eigenvalues λ_j. Show that

$$\hat{A} = \sum_j \lambda_j |\psi_j\rangle\langle\psi_j|$$

[4]

(d) Consider a three-level atom, with energy eigenstates $|1\rangle$, $|2\rangle$ and $|3\rangle$. The following observable is measured

$$M_1 = |a\rangle\langle a| - |b\rangle\langle b| - |c\rangle\langle c|$$

where $|a\rangle = (1/\sqrt{3})(|1\rangle + |2\rangle + |3\rangle)$, $|b\rangle = (1/\sqrt{3})(|1\rangle + \omega|2\rangle + \omega^*|3\rangle)$, $|c\rangle = (1/\sqrt{3})(|1\rangle + \omega^*|2\rangle + \omega|3\rangle)$ and $\omega = \exp[i2\pi/3]$.

i. Show that $|a\rangle$, $|b\rangle$ and $|c\rangle$ are orthogonal. **[3]**

ii. Identify the projectors corresponding to the two outcomes of observable M_1 and write them in terms of the basis $|1\rangle$, $|2\rangle$ and $|3\rangle$. Simplify your expressions as much as you can. You may find it convenient to use a matrix representation. **[4]**

iii. The atom is prepared in state $(1/\sqrt{2})(|1\rangle+|2\rangle)$ and observable M_1 is measured. What is the probability that the -1 outcome is observed and what is the (normalised) state of the system following that outcome? **[4]**

END OF PAPER