Module 2

Symmetries and Conservation Laws

Lecture Notes

PHASM/G442

2017 - 2018

For physical predictions to be unchanged by a symmetry transformation:

$$\hat{U}^{\dagger}\hat{U} = 1$$ → unitary

$$[\hat{H}, \hat{U}] = 0$$ $\hat{U}$ - commutes with $\hat{H}$

Consider $\hat{U} = 1 + i\varepsilon \hat{G}$ ← generator of transformation

$\varepsilon \ll 1$

$$\hat{U}\hat{U}^{+} = (1 + i\varepsilon\hat{G})(1 - i\varepsilon\hat{G}^{+}) =$$

$$= 1 - i\varepsilon\hat{G}^{+} + i\varepsilon\hat{G} + \underbrace{\varepsilon^2 \hat{G}\hat{G}^{+}}_{} \simeq$$ neglect ε^2

$$\simeq 1 + i\varepsilon(\hat{G} - \hat{G}^{+})$$

Since $\hat{U}\hat{U} = 1$ $\boxed{\hat{G} = \hat{G}^{+}}$ - hermitian observable

Corresponds to observable quantity

Now, $[\hat{H}, \hat{U}] = [\hat{H}, 1 + i\varepsilon\hat{G}] = 0$

Hence, $[\hat{H}, \hat{G}] = 0$

From QM

$$\frac{d}{dt}\langle \hat{G} \rangle = i \langle [\hat{H}, \hat{G}] \rangle = 0$$

Requirements of symmetry (invariance)

$\Downarrow$

unitarity and $[\hat{H}, \hat{U}] = 0$

$\Downarrow$

conserved observable quantity

Noether's Theorem

Example : small spatial translation

$x \to x+\varepsilon, \quad \varepsilon \ll 1$

$$\psi(x) \to \psi' = \psi(x+\varepsilon)$$

Taylor's expansion

$$\psi'(x) = \psi(x+\varepsilon) = \psi(x) + \frac{\partial \psi}{\partial x}\varepsilon \ldots$$

neglect ε^2 etc terms

$$\psi' = \left(1 + \varepsilon \frac{\partial}{\partial x}\right)\psi(x)$$

Recall: $\hat{P}_x = -i\frac{\partial}{\partial x}$

$$\psi'(x) = \left(1 + i\varepsilon\hat{P}_x\right)\psi(x)$$

c.f.

$$\underbrace{\hat{U} = \left(1 + i\varepsilon\hat{G}\right) \quad \hat{P}_x\psi = P_x\psi}_{\text{momentum conservation}}$$

Doesn't matter what
a, b or $\cdot$ are
Important: unique solution

Group Examples

$$a \cdot x = b$$

a, b - group elements

$\cdot$ - group "operation"

E.g. Integer elements, $+$ → "operation"

1) Closure: If $a, b \in G$, $a \cdot b \in G$

$5, 7 \in I$, $5 + 7 \in I$

2) Associativity: If $a, b, c \in G$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

$$(5+7)+4 = 5+(7+4)$$

3) Identity: Element e, so that

$$a \cdot e = a$$

$e = 0$ $\quad 5 + 0 = 5$

4) Inverses: For any a, there's a^{-1}

$$a \cdot a^{-1} = e$$

For 5, → -5 $\quad 5 + (-5) = 0$