11 Processes in QED

11.1 Electron–Muon Scattering

This is as simple a process as one can find since at lowest order in the electromagnetic coupling, just one diagram contributes. It is shown in figure 3. The amplitude obtained by applying the Feynman rules to this diagram is

$$i\mathcal{M}_{fi} = (-ie)\,\bar{u}(p_c)\gamma^\mu u(p_a)\left(\frac{-ig_{\mu\nu}}{q^2}\right)(-ie)\,\bar{u}(p_d)\gamma^\nu u(p_b), \tag{11.1}$$

where $q^2 = (p_a - p_c)^2$. Note that, for clarity, I have dropped the spin label on the spinors. I will restore it when I need to. In constructing this amplitude we have followed the fermion lines backwards with respect to fermion flow when working out the order of matrix multiplication (which makes sense if you think of an unbarred spinor as a column vector and a barred spinor as a row vector and remember that the amplitude carries no spinor indices).

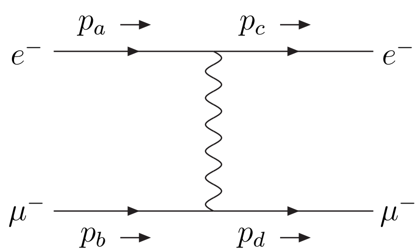

Figure 3: Lowest order Feynman diagram for $e^-\mu^- \to e^-\mu^-$ scattering.

The cross section involves the squared modulus of the amplitude, $|\mathcal{M}_{fi}|^2$. Let us see how we obtain a neat form for this. Consider

$$(\bar{u}(p_c)\gamma^\mu u(p_a))^* = (\bar{u}(p_c)\gamma^\mu u(p_a))^\dagger$$

since it is just a number. Using rules of matrix algebra we see that this is

$$\begin{aligned} (u(p_c)^\dagger\gamma^0\gamma^\mu u(p_a))^\dagger &= (u(p_a)^\dagger\gamma^{\mu\dagger}\gamma^{0\dagger}u(p_c)) \\ &= (u(p_a)^\dagger\gamma^{\mu\dagger}\gamma^0 u(p_c)). \end{aligned} \tag{11.2}$$

But in problem set 3 we saw that $\gamma^0\gamma^{\mu\dagger}\gamma^0 = \gamma^\mu$, and so this becomes

$$(u(p_a)^\dagger\gamma^0\gamma^\mu u(p_c)) = (\bar{u}(p_a)\gamma^\mu u(p_c)).$$

Using this general result in the expression for $|\mathcal{M}_{fi}|^2$ we obtain

$$\begin{aligned} |\mathcal{M}_{fi}|^2 &= \frac{e^4}{q^4}\,\bar{u}(p_c)\gamma^\mu u(p_a)\bar{u}(p_d)\gamma_\mu u(p_b)\bar{u}(p_a)\gamma^\nu u(p_c)\bar{u}(p_b)\gamma_\nu u(p_d) \\ &= \frac{e^4}{q^4}\,L^{\mu\nu}_{(e)}L_{(\mu)\,\mu\nu}, \end{aligned} \tag{11.3}$$

where the subscripts e and μ refer to the electron and muon respectively and

$$L^{\mu\nu}_{(e)} = \bar{u}(p_c)\gamma^\mu u(p_a)\bar{u}(p_a)\gamma^\nu u(p_c),$$

with a similar expression for $L^{\mu\nu}_{(\mu)}$.

Usually we have an unpolarized beam and target and do not measure the polarization of the outgoing particles. Thus we calculate the squared amplitudes for each possible spin combination, then average over initial spin states and sum over final spin states. Note that we square and then sum since the different spin configurations are in principle distinguishable. In contrast, if several Feynman diagrams contribute to the same process, you have to sum the amplitudes first. We will see examples of this below.

The spin sums are made easy by the following results which we already used in the last section:

$$\begin{aligned}\sum_s \big(u(s,p)\,\bar{u}(s,p)\big)_{\alpha,\beta} &= (\not{p}+m)_{\alpha,\beta},\\ \sum_s \big(v(s,p)\,\bar{v}(s,p)\big)_{\alpha,\beta} &= (\not{p}-m)_{\alpha,\beta}.\end{aligned} \tag{11.4}$$

Where do not forget that by m I really mean m times the unit 4×4 matrix.

Using the spin sums we find that

$$\begin{aligned}\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 &= \frac{e^4}{4q^4}\left[\gamma^\mu_{\alpha\beta}(\not{p}_a+m_e)_{\beta\zeta}\gamma^\nu_{\zeta\eta}(\not{p}_c+m_e)_{\eta\alpha}\right]\left[\gamma_{\mu,\alpha'\beta'}(\not{p}_b+m_\mu)_{\beta'\zeta'}\gamma_{\nu,\zeta'\eta'}(\not{p}_d+m_\mu)_{\eta'\alpha'}\right]\\ &= \frac{e^4}{4q^4}\operatorname{Tr}\big(\gamma^\mu(\not{p}_a+m_e)\gamma^\nu(\not{p}_c+m_e)\big)\operatorname{Tr}\big(\gamma_\mu(\not{p}_b+m_\mu)\gamma_\nu(\not{p}_d+m_\mu)\big),\end{aligned} \tag{11.5}$$

where in the first expression, I chose to make explicit the spinor indices in order that you can see how the trace that appears in the second expression emerges. Since all calculations of cross sections or decay rates in QED require the evaluation of traces of products of γ-matrices, you will generally find a table of "trace theorems" in any quantum field theory textbook. All these theorems can be derived from the fundamental anti-commutation relations of the γ-matrices in equation (9.31) together with the invariance of the trace under a cyclic change of its arguments. For now it suffices to use

$$\begin{aligned}\operatorname{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) &= 0 \quad \text{for } n \text{ odd}\\ \operatorname{Tr}(\gamma^{\mu_1}\ldots\gamma^{\mu_n}) &= g^{\mu_1\mu_2}\operatorname{Tr}(\gamma^{\mu_3}\ldots\gamma^{\mu_n}) - g^{\mu_1\mu_3}\operatorname{Tr}(\gamma^{\mu_2}\gamma^{\mu_4}\ldots\gamma^{\mu_n}) + \cdots\\ &\quad + g^{\mu_1\mu_n}\operatorname{Tr}(\gamma^{\mu_2}\ldots\gamma^{\mu_{n-1}})\\ &\quad = 4g^{\mu_1\mu_2} \quad n=2\\ \operatorname{Tr}(\not{a}\not{b}) &= 4\,a\cdot b,\\ \operatorname{Tr}(\not{a}\not{b}\not{c}\not{d}) &= 4(a\cdot b\; c\cdot d - a\cdot c\; b\cdot d + a\cdot d\; b\cdot c).\end{aligned} \tag{11.6}$$

Using these results,

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 &= \frac{e^4}{4q^4}\Big(\mathrm{Tr}\big(\gamma^\mu(\not{p}_a)\gamma^\nu(\not{p}_c)\big)+\mathrm{Tr}\big(\gamma^\mu\gamma^\nu\big)m_e^2\Big)\times\Big(\mathrm{Tr}\big(\gamma_\mu(\not{p}_b)\gamma_\nu(\not{p}_d)\big)+\mathrm{Tr}\big(\gamma_\mu\gamma_\nu\big)m_\mu^2\Big) \\
&= \frac{e^4}{4q^4}\Big(4\big(p_a^\mu p_c^\nu - g^{\mu\nu}p_a\cdot p_c + p_a^\nu p_c^\mu\big)+4g^{\mu\nu}m_e^2\Big)\times\Big(\cdots\Big) \\
&= \frac{e^4}{4q^4}\Big(4\big(p_a^\mu p_c^\nu + p_a^\nu p_c^\mu + g^{\mu\nu}(-p_a\cdot p_c + m_e^2)\big)\Big)\times\Big(\cdots\Big).
\end{aligned}
$$

But $t=(p_a-p_c)^2=-2p_a\cdot p_c+2m_e^2$ or $t=(p_b-p_d)^2=-2p_b\cdot p_d+2m_\mu^2$. So

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 &= \frac{4e^4}{t^2}\big(p_a^\mu p_c^\nu + p_a^\nu p_c^\mu + t/2g^{\mu\nu}\big)\times\big(p_{b\mu}p_{d\nu}+p_{b\nu}p_{d\mu}+t/2g_{\mu\nu}\big) \\
&= \frac{4e^4}{t^2}\big(2(p_a\cdot p_b)(p_c\cdot p_d)+2(p_a\cdot p_d)(p_b\cdot p_c)+t(p_a\cdot p_c)+t(p_b\cdot p_d)+t^2\big).
\end{aligned}
$$

Expressing the answer in terms of the Mandelstam variables of equation (8.33), we find

$$
\begin{aligned}
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 &= \frac{4e^4}{t^2}\Bigg[2\bigg(\frac{s}{2}-\frac{(m_e^2+m_\mu^2)}{2}\bigg)^2+2\bigg(\frac{u}{2}-\frac{(m_e^2+m_\mu^2)}{2}\bigg)^2 \\
&\quad +t\bigg(-\frac{t}{2}+m_e^2\bigg)+t\bigg(-\frac{t}{2}+m_\mu^2\bigg)+t^2\Bigg] \\
&= \frac{4e^4}{t^2}\bigg[\frac{s^2}{2}-s(m_e^2+m_\mu^2)+\frac{u^2}{2}-u(m_e^2+m_\mu^2)+(m_e^2+m_\mu^2)^2+t(m_e^2+m_\mu^2)\bigg] \\
&= \frac{2e^4}{t^2}\big(s^2+u^2-4(m_e^2+m_\mu^2)(s+u)+6(m_e^2+m_\mu^2)^2\big). \qquad (11.7)
\end{aligned}
$$

This can now be used in the $2\to 2$ cross section formula (8.32) to give, in the high energy limit $(s,|u|\gg m_e^2,m_\mu^2)$,

$$
\frac{d\sigma}{d\Omega^*}=\frac{e^4}{32\pi^2 s}\frac{s^2+u^2}{t^2} \qquad (11.8)
$$

for the differential cross section in the centre of mass frame.

Other calculations of cross sections or decay rates will follow the same steps we have used above. You draw the diagrams, write down the amplitude, square it and evaluate the traces (if you are using spin sum/averages). There are one or two more complications to be aware of, which we will illustrate below.

11.2 Electron–Electron Scattering

Since the two scattered particles are now identical, you cannot just replace m_μ by m_e in the calculation we did above. If you look at the diagram of figure 3 (with the muons replaced by

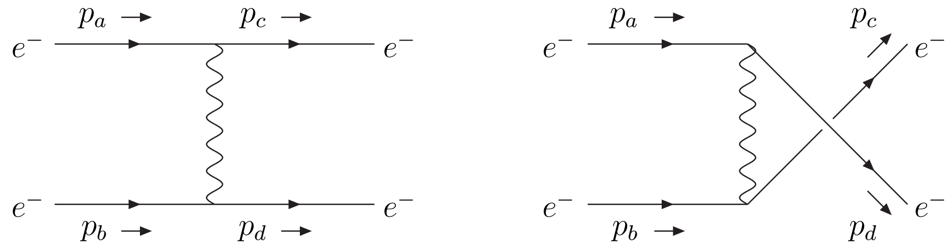

Figure 4: Lowest order Feynman diagrams for electron–electron scattering.

electrons) you will see that the outgoing legs can be labelled in two ways. Hence we get the two diagrams of figure 4.

The two diagrams give the amplitudes,

$$
\begin{aligned}
i\mathcal{M}_1 &= \frac{ie^2}{t}\bar{u}(p_c)\gamma^\mu u(p_a)\bar{u}(p_d)\gamma_\mu u(p_b), && (11.9)\\
i\mathcal{M}_2 &= -\frac{ie^2}{u}\bar{u}(p_d)\gamma^\mu u(p_a)\bar{u}(p_c)\gamma_\mu u(p_b). && (11.10)
\end{aligned}
$$

Notice the additional minus sign in the second amplitude, which comes from the anti-commuting nature of fermion fields. Remember that when diagrams differ by an interchange of two fermion lines, a relative minus sign must be included. This is important because

$$
\begin{aligned}
|\mathcal{M}_{fi}|^2 &= |\mathcal{M}_1 + \mathcal{M}_2|^2\\
&= |\mathcal{M}_1|^2 + |\mathcal{M}_2|^2 + 2\mathrm{Re}\mathcal{M}_1^*\mathcal{M}_2, && (11.11)
\end{aligned}
$$

so the interference term will have the wrong sign if you don't include the extra sign difference between the two diagrams. $|\mathcal{M}_1|^2$ and $|\mathcal{M}_2|^2$ are very similar to the previous calculation. The interference term is a little more complicated due to a different trace structure.

Performing the calculation explicitly by using

$$
\begin{aligned}
\gamma^\alpha\gamma^\mu\gamma_\alpha &= -2\gamma^\mu\\
\gamma^\alpha\gamma^\mu\gamma^\nu\gamma_\alpha &= 4g^{\mu\nu} && (11.12)\\
\gamma^\alpha\gamma^\mu\gamma^\nu\gamma^\rho\gamma_\alpha &= -2\gamma^\rho\gamma^\nu\gamma^\mu.
\end{aligned}
$$

yields (in the limit of negligible fermion masses),

$$
\frac{1}{4}\sum_{\text{spins}}|\mathcal{M}_{fi}|^2 = 2e^4\left(\frac{s^2+u^2}{t^2} + \frac{s^2+t^2}{u^2} + \frac{2s^2}{tu}\right). \qquad (11.13)
$$

11.3 Electron–Positron Annihilation

The two diagrams for this process are shown in figure 5, with the one on the right known as the annihilation diagram. They are just what you get from the diagrams for electron–electron scattering in figure 4 if you twist round the fermion lines, or equivalently swap p_b with $-p_d$. The fact that the diagrams are related in this way implies a relation between the amplitudes. The interchange of incoming particles/antiparticles with outgoing antiparticles/particles is called *crossing*. For our particular example, the squared amplitude for $e^+e^- \to e^+e^-$ is related to that for $e^-e^- \to e^-e^-$ by performing the interchange $s \leftrightarrow u$. Hence, squaring the amplitude and doing the traces yields (again neglecting fermion mass terms)

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = 2e^4 \left(\frac{s^2+u^2}{t^2} + \frac{u^2+t^2}{s^2} + \frac{2u^2}{ts} \right). \tag{11.14}$$

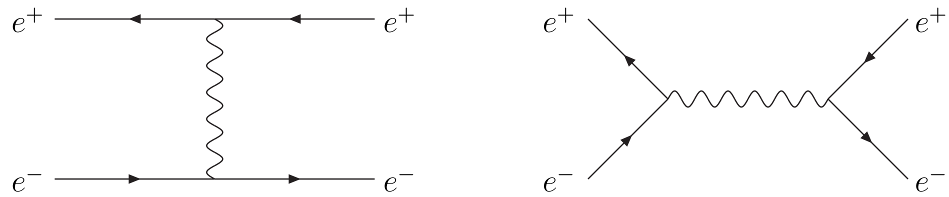

Figure 5: Lowest order Feynman diagrams for electron-positron scattering in QED.

Finally, we compute the total cross section for $e^+e^- \to \mu^+\mu^-$, neglecting the lepton masses. Here we only have the annihilation diagram, and for the amplitude, we get

$$\begin{aligned} \mathcal{M}_{fi} &= (-ie)^2 \bar{u}(p_d)\gamma^\mu v(p_c) \frac{-ig_{\mu\nu}}{s} \bar{v}(p_a)\gamma^\nu u(p_b) \\ &= \frac{ie^2}{s} \bar{u}_d \gamma^\mu v_c \bar{v}_a \gamma_\mu u_b. \end{aligned} \tag{11.15}$$

Summing over final state spins and averaging over initial spins gives,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{4s^2} \mathrm{Tr}(\gamma^\mu \not{p}_c \gamma^\nu \not{p}_d) \mathrm{Tr}(\gamma_\mu \not{p}_b \gamma_\nu \not{p}_a),$$

where we have neglected m_e and m_μ. Using the results in equation (11.6) to evaluate the traces gives,

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{8e^4}{s^2} (p_a \cdot p_d \, p_b \cdot p_c + p_a \cdot p_c \, p_b \cdot p_d).$$

Neglecting masses we have,

$$p_a \cdot p_c = p_b \cdot p_d = -t/2, \tag{11.16}$$

$$p_a \cdot p_d = p_b \cdot p_c = -u/2. \tag{11.17}$$

Hence $(1/4)\sum|\mathcal{M}_{fi}|^2 = 2e^4(t^2+u^2)/s^2$, which incidentally is what you get by applying crossing to the electron–muon amplitude of section 11.1. We can use this in eq. (8.32) to find the differential cross section in the CM frame,

$$\frac{d\sigma}{d\Omega^*} = \frac{e^4(t^2+u^2)}{32\pi^2 s^3}.$$

You could get straight to this point by noting that the appearance of v spinors instead of u spinors in $\mathcal{M}_{fi}$ does not change the answer since only quadratic terms in m_μ survive the Dirac algebra and we go on to neglect masses anyway. Hence you can use the result of equation (11.8) with appropriate changes.

Neglecting masses, the CM momenta are

$$p_a = \frac{1}{2}\sqrt{s}(1, \mathbf{e}) \qquad p_c = \tfrac{1}{2}\sqrt{s}(1, \mathbf{e}') \tag{11.18}$$

$$p_b = \frac{1}{2}\sqrt{s}(1, -\mathbf{e}) \qquad p_d = \tfrac{1}{2}\sqrt{s}(1, -\mathbf{e}'), \tag{11.19}$$

where $\mathbf{e}\cdot\mathbf{e}' = \cos\theta$. This gives $t = -s(1-\cos\theta)/2$ and $u = -s(1+\cos\theta)/2$. Hence, finally, defining $\alpha = e^2/4\pi$, the total cross section is,

$$\begin{aligned}
\sigma &= \int_{-1}^{1} \frac{d\sigma}{d\Omega^*} 2\pi d(\cos\theta) \\
&= \frac{\alpha^2}{2s} 2\pi . \int_{-1}^{1} \left[\frac{(1-\cos\theta)^2}{4} + \frac{(1+\cos\theta)^2}{4}\right] d(\cos\theta) \\
&= \frac{4\pi\alpha^2}{3s}.
\end{aligned} \tag{11.20}$$

If electrons and positrons collide and produce muon–antimuon or quark–antiquark pairs, then the annihilation diagram is the only one that contributes. At sufficiently high energies that the quark masses can be neglected, this immediately gives the lowest order QED prediction for the ratio of the annihilation cross section into hadrons to that into $\mu^+\mu^-$:

$$R \equiv \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3\sum_f Q_f^2, \tag{11.21}$$

where the sum is over quark flavours f and Q_f is the quark's charge in units of e. The 3 comes from the existence of three colours for each flavour of quark. Historically this was important: you could look for a step in the value of R as your e^+e^- collider's CM energy rose through a threshold for producing a new quark flavour. If you did not know about colour, the height of the step would seem too large. At the energies used at LEP you have to remember to include the diagram with a Z replacing the photon.

11.4 Compton Scattering

The diagrams which need to be evaluated to compute the Compton cross section for $\gamma e \to \gamma e$ are shown in figure 6. The matrix element for the left-hand diagram is

$$i\mathcal{M}_{fi} = (-ie)^2\, \varepsilon^{*\,\mu}(\lambda_d, p_d)\bar{u}(p_c)\gamma_\mu \frac{i(\not{p}_a + \not{p}_b + m)}{((p_a + p_b)^2 - m^2)} \varepsilon^\nu(\lambda_b, p_b)\gamma_\nu u(p_a), \tag{11.22}$$

and the right-hand diagram is the same with p_b and $-p_d$ swapped. For unpolarized initial and/or final states, the cross section calculation consequently involves terms of the form

$$\sum_\lambda \varepsilon^{*\,\mu}(\lambda, p)\, \varepsilon^\nu(\lambda, p), \tag{11.23}$$

where λ represents the polarization of the photon of momentum p. Since the photon is massless, the sum is over the two transverse polarization states, and must vanish when contracted with p_μ or p_ν. In principle eq. (11.23) is a complicated object. However, there is a simplification as far as the amplitude calculation is concerned. The photon is coupled to the electromagnetic current $J^\mu = \bar{\psi}\gamma^\mu\psi$ of equation (9.10). This is a conserved current, i.e. $\partial_\mu J^\mu = 0$, and in momentum space $p_\mu J^\mu = 0$. Hence, any term in the polarization sum (11.23) proportional to p^μ or p^ν does not contribute to the cross section. This means that in calculations one can make the replacement

$$\sum_\lambda \varepsilon^{*\,\mu}(\lambda, p)\, \varepsilon^\nu(\lambda, p) \to -g^{\mu\nu}, \tag{11.24}$$

and we have a simple, Lorentz-covariant prescription.

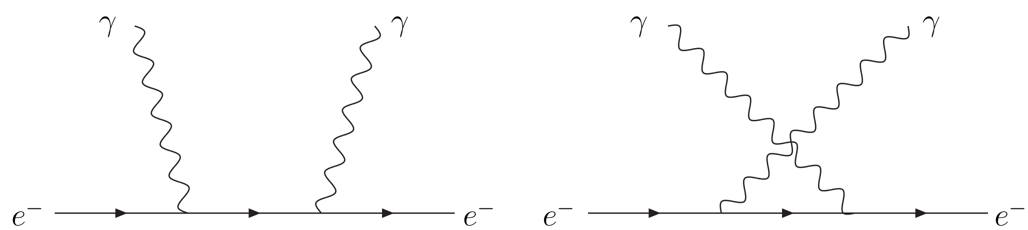

Figure 6: Lowest order Feynman diagrams for Compton scattering.

The spin summed/averaged squared matrix element for the left-hand diagram in the massless limit is therefore given by

$$\frac{1}{4}\sum_{\text{spins}} |\mathcal{M}_{fi}|^2 = \frac{e^4}{4s^2}\mathrm{Tr}(\not{p}_c\gamma^\mu(\not{p}_a + \not{p}_b)\gamma^\nu\not{p}_a\gamma_\nu(\not{p}_a + \not{p}_b)\gamma_\mu),$$

which using the standard trace theorems can quickly be shown to give a contribution $-2e^4 u/s$. Using the simple relationship between the right-hand and left-hand diagram it is straightforward to also calculate the contribution of the right-hand diagram and the interference term, giving a total of

$$|\mathcal{M}_{fi}|^2 = 2e^4\left(-\frac{u}{s} - \frac{s}{u}\right). \tag{11.25}$$