PHASM426 / PHASG426
Advanced Quantum Theory Problem Sheet 2

Deadline: Tuesday 14 November 2017

Please hand in your completed work at the **end** of the lecture on that day. Attache the coversheet. If you are unable to attend the lecture, you may scan your work, *save it as a single PDF file* and email it to me **prior** to this lecture. You may also bring the work to me in my office (B12) before the lecture. **Make sure your completed work is clearly labelled with your name and college**. Please note that UCL places severe penalties on late-submitted work.

1. Two quantum systems a and b are prepared in a joint state $|\Psi\rangle = |\psi_a\rangle|\psi_b\rangle$. Let the set $\{|\phi_k\rangle\}$ be a basis of states for system a, $\{|\nu_j\rangle\}$ a basis for system b and A an operator on the Hilbert space of a with the following spectral decomposition:

$$A = \lambda_1 |\phi_1\rangle\langle\phi_1| + \lambda_2 \sum_{n=2}^{N} |\phi_n\rangle\langle\phi_n|.$$

   (a) Write down an expression for the projector acting on the total Hilbert space and which is associated to the measurement outcome λ_2. Derive an expression for the probability of obtaining λ_2 in a measurement of A on the state $|\Psi\rangle$.

   (b) Derive an expression for the state $|\Psi'\rangle$ of the global system after the measurement. Has the state of the system b changed?

   (c) Discuss whether the predictions in (i) and (ii) would change depending of the choice of basis for the system b. Will the results of measurements on the systems a and b be correlated? Justify your answer.

[4]

2. In this question, we shall consider experiments which generates pure states probabilistically. Such a probability distribution over pure states can be represented by a (single) mixed state in the density matrix formalism. Write down a matrix representation for the density operator in each of the following cases. You may express them in bra and ket notation or use a matrix representation, but in either case simplify your expressions as much as possible. Note that $|\phi_1\rangle$, $|\phi_2\rangle$ and $|\phi_3\rangle$ are an orthonormal set of states.

   (a) The state $|\phi_1\rangle$ is prepared with probability $1/4$, and the state $(|\phi_1\rangle + i|\phi_2\rangle)/\sqrt{2}$ is prepared with probability $3/4$.

   (b) The following states are each prepared with probability $1/3$,

      - $(|\phi_1\rangle + |\phi_2\rangle)/\sqrt{2}$,

- $(|\phi_1\rangle + \exp(i2\pi/3)|\phi_2\rangle)/\sqrt{2}$,
- $(|\phi_1\rangle + \exp(-i2\pi/3)|\phi_2\rangle)/\sqrt{2}$.

[4]

3. The combined state of a pair of two-level atoms, A and B, is given by the density matrix:
$$\rho = \frac{1}{2}|g_A, g_B\rangle\langle g_A, g_B| + \frac{1}{2}|g_A, e_B\rangle\langle g_A, e_B|.$$
   (a) Calculate the reduced density matrix operator for the each of the two-level systems.

   (b) Calculate the purity of system A and indicate whether the state of the two atoms is a product or an entangled state. Justify your answer.

[3]

4. A physicist runs two experiments A and B to prepare quantum systems in a variety of initial states. In experiment A she uses a probabilistic machine that can prepare a single quantum system in one of n possible pure states $\{|\Psi_1\rangle, |\Psi_2\rangle, \cdots, |\Psi_n\rangle\}$ with corresponding probabilities $\{p_1, p_2, \cdots, p_n\}$. In experiment B, instead, she generates m non-interacting quantum systems, each of them prepared in its corresponding lower energy state $\{|\Phi_1\rangle, |\Phi_2\rangle, \cdots, |\Phi_m\rangle\}$. Let ρ_A and ρ_B be the density matrix operators for the quantum states prepared in experiments A and B, respectively.

   (a) Let P_j and P_k be the projectors associated to the states $|\Psi_j\rangle$ and $|\Psi_k\rangle$ produced by A. Discuss whether the product P_jP_k vanishes. Justify your answer. **[1]**

   (b) Derive expressions for ρ_A and ρ_B and justify your answers in detail. **[3]**

   (c) Let O_A be a Hermitian operator describing an observable of the system produced by A. Derive an expression for the expected value of O_A. **[1]**

   (d) Let O_B^k be a Hermitian operator describing an observable of the kth system produced by B. Derive an expression for the expected value of O_B^k. **[1]**

5. Similar to the position basis, we can consider the continuous momentum basis $|p\rangle$. This is a Dirac delta-normalised basis which represents eigenstates of the momentum operator, $\hat{p}|p\rangle = p|p\rangle$, where the eigenvalues p take any real value. The closure relationship $\mathbb{1} = \int_{-\infty}^{\infty} dp|p\rangle\langle p|$ is satisfied. One can show that the momentum basis states satisfy the following inner product with respect to the position basis:
$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}}e^{ipx/\hbar}$$

In analogy to the position wave-function, we can define a momentum wave-function, $\Psi(p) = \langle p|\psi\rangle$. Show that $\Psi(p)$ is the Fourier transform of the position wave-function $\psi(x)$, i.e.

$$\Psi(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} e^{-ipx/\hbar} \psi(x) dx$$

[2]

6. In this question we will explore why we can neglect term $S_2(x)$ in the expansion of $S(x)$ in the derivation of the WKB wavefunction, when the potential $V(x)$ is slowly varying with respect to the local de Broglie wavelength.

   In the lectures we showed that the TISE may be written, using the Ansatz $\psi(x) = \exp[iS(x)]$ as:

   $$-\frac{i\hbar}{2m} S''(x) + \frac{1}{2m} S'(x)^2 + V(x) - E = 0.$$

   where $'$ denotes the derivative with respect to x, and $''$ the second derivative with respect to x.

   We then replaced $S(x)$ with the series expansion $S(x) = \sum_j \hbar^j S_j(x)$ and derived zeroth and first order equations from which we derived expressions:

   $$S_0'(x) = p(x) \qquad S_1(x) = \frac{i}{2} \log p(x)$$

   where for simplicity I have neglected the $\pm$ signs.

   (a) By considering terms in the above equation proportional to $\hbar^2$ derive the second order equation:

   $$S_2'(x) = \frac{iS_1''(x) - S_1'(x)^2}{2S_0'(x)}$$

   (b) Using the definitions for $S_0'(x)$ and $S_1(x)$ above show that:

   $$S_2'(x) = \frac{1}{8} \left(\frac{3p'(x)^2}{p(x)^3} - \frac{2p''(x)}{p(x)^2} \right)$$

   (c) And verify that the following expression is a solution to this differential equation:

   $$S_2(x) = -\frac{1}{4} \frac{p'(x)}{p(x)^2} - \frac{1}{8} \int^x \frac{p'(y)^2}{p(y)^3} dy,$$

   where y is a dummy integration variable.

(d) To be able to neglect the term containing $S_2(x)$ in the series expansion for $S(x)$ we need that $|\hbar S_2(x)| \ll 1$. We will focus solely on the first term in $S_2(x)$ (a similar analysis can be performed on the second term, but is more complicated due to the integral). If the local de Broglie wavelength is defined $\lambda(x) = h/p(x)$ show that

$$\left|\hbar \frac{p'(x)}{p(x)^2}\right| \ll 1$$

implies

$$\left|\frac{1}{2\pi}\lambda'(x)\right| \ll 1.$$

[4]

7. Consider a particle of mass m in a potential such that $V(x) = 0$ for $|x| > \kappa$ and $V(x) = -V_0\left(1 - \frac{|x|}{\kappa}\right)$ for $|x| \leq \kappa$. Given that $mV_0\kappa^2/\hbar^2 = 18$ calculate the number of bound states and their energies expressed in terms of V_0.

You may use the integral

$$\int (1-y)^{1/2} dy = \frac{-2}{3}(1-y)^{3/2} + c$$

and use the WKB quantisation condition for a smooth well derived in lectures:

$$\int_a^b p(x')dx'/\hbar = \left(n + \frac{1}{2}\right)\pi$$

where a and b are the positions of the classical turning points on the left and right of the well, respectively, and x' is a dummy integration variable (x' does not mean dx/dx).

Hint: By sketching the form of the potential, consider whether the energy of a bound particle will be positive or negative. Think about how the energy of the particle affects the classical turning points in this problem, and how that affects, in turn, the integral in the quantisation condition.

[3]