PS2. Model Answers

AY 2017-18

PHASM/G 442

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PHASM/G 442. 2017 : Problem Sheet 2.
Model Answers

1. **[Total 4 marks]**

   Using the identities

   $$\gamma^{0\dagger} = \gamma^0 \quad \gamma^{k\dagger} = -\gamma^k \quad \gamma^0\gamma^0 = I \quad \gamma^0\gamma^k = -\gamma^k\gamma^0$$

   for $\mu = 0$ we have,

   $$\gamma^{0\dagger} = \gamma^0 = \gamma^0\gamma^0\gamma^0$$

   And for $\mu = k \neq 0$,

   $$\gamma^{k\dagger} = -\gamma^k = -\gamma^0\gamma^0\gamma^k = \gamma^0\gamma^k\gamma^0$$

   Therefore for $\mu = 0, 1, 2, 3$,

   $$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0$$

   [4]

2. **[Total 6 marks]**
   Taking the Hermitian conjugate of the Dirac equation:

   $$[(\gamma^\mu p_\mu - m)\, u]^\dagger = u^\dagger \left(\gamma^{\mu\dagger} p_\mu - m\right) = 0$$

   This can be rewritten as

   $$u^\dagger\gamma^0\gamma^0 \left(\gamma^{\mu\dagger} p_\mu - m\right) = u^\dagger\gamma^0 \left(\gamma^0\gamma^{\mu\dagger} p_\mu - \gamma^0 m\right) = 0.$$

   [2]

   Using the identity $\gamma^0\gamma^{\mu\dagger} = \gamma^\mu\gamma^0$ that follows from:

   $$\gamma^0\gamma^{0\dagger} = \gamma^0\gamma^0; \qquad \gamma^0\gamma^{k\dagger} = -\gamma^0\gamma^k = \gamma^k\gamma^0 \quad \text{where} \quad k = 1, 2, 3,$$

   [2]

   we have:

   $$\bar{u}\left(\gamma^\mu\gamma^0 p_\mu - m\gamma^0\right) = \bar{u}\,(\gamma^\mu p_\mu - m)\,\gamma^0 = 0.$$

   Finally, multiplying both parts by γ^0 we obtain the required:

   $$\bar{u}\,(\gamma^\mu p_\mu - m) = 0.$$

   [2]

3. **[Total 5 marks]**

For the particle solution $\psi = ue^{+i(\vec{p}\cdot\vec{r}-Et)}$:

$$\hat{H}\psi = i\frac{\partial}{\partial t}\left[ue^{+i(\vec{p}\cdot\vec{r}-Et)}\right] = i^2(-E)ue^{+i(\vec{p}\cdot\vec{r}-Et)} = E\psi,$$

Therefore E is the real eigenvalue of $\hat{H}$ and represent the physical energy of the particle. **[2]**

For the *anti*-particle solution $\psi = ve^{-i(\vec{p}\cdot\vec{r}-Et)}$ the same argument as above will lead to a negative energy solution.

By swapping the sign of the $\hat{H}$ operator we obtain

$$\hat{H}^{(v)}\psi = -i\frac{\partial}{\partial t}ve^{-i(\vec{p}\cdot\vec{r}-Et)} = (-i)(-i)(-E)ve^{-i(\vec{p}\cdot\vec{r}-Et)} = E\psi,$$

i.e. positive energy solutions for *anti*-particles. **[3]**

4. **[Total 5 marks]**

The Dirac Hamiltonian given in the question can be rewritten as:

$$\hat{H}_D = \gamma^0\left(\vec{\gamma}\cdot\vec{p}\right) + m\gamma^0$$

Using the gamma matrices definition in terms of Pauli matrices,

$$\vec{\gamma}\cdot\vec{p} = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ -\vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix}$$

Therefore **[2]**

$$\gamma^0\left(\vec{\gamma}\cdot\vec{p}\right) = \begin{pmatrix} 0 & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & 0 \end{pmatrix} \qquad m\gamma^0 = \begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}$$

and hence

$$\hat{H}_D = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}$$

[3]

5. **[Total 13 marks]**

   (a) This result means that the Dirac Hamiltonian, $\hat{H}_D$, does *NOT* commute with the operator of orbital angular momentum, $\hat{L}$. This in turn means that the orbital angular momentum is *NOT* a conserved quantity. **[2]**

(b) Using the expression for $\hat{H}_D$ in matrix form obtained in Q4 we have:

$$\left[\hat{H}_D, \hat{S}\right] = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} m\vec{\sigma} & (\vec{\sigma}\cdot\vec{p})\,\vec{\sigma} \\ (\vec{\sigma}\cdot\vec{p})\,\vec{\sigma} & -m\vec{\sigma} \end{pmatrix} - \frac{1}{2} \begin{pmatrix} m\vec{\sigma} & \vec{\sigma}\,(\vec{\sigma}\cdot\vec{p}) \\ \vec{\sigma}\,(\vec{\sigma}\cdot\vec{p}) & -m\vec{\sigma} \end{pmatrix} =$$

$$= \frac{1}{2} \begin{pmatrix} 0 & \left[(\vec{\sigma}\cdot\vec{p})\,,\vec{\sigma}\right] \\ \left[(\vec{\sigma}\cdot\vec{p})\,,\vec{\sigma}\right] & 0 \end{pmatrix}$$

Taking the x-component of $\vec{\sigma} = \sigma_1\hat{i} + \sigma_2\hat{j} + \sigma_3\hat{h}$: **[2]**

$$\left[(\vec{\sigma}\cdot\vec{p})\,,\sigma_1\right] = (\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z)\,\sigma_1 - \sigma_1\,(\sigma_1 p_x + \sigma_2 p_y + \sigma_3 p_z) = p_y\left[\sigma_2, \sigma_1\right] + p_z\left[\sigma_3, \sigma_1\right] =$$

$$-2ip_y\sigma_3 + 2ip_z\sigma_2 = 2i\,(\vec{\sigma}\times\vec{p})_i$$

By symmetry arguments,

$$\left[(\vec{\sigma}\cdot\vec{p})\,,\sigma\right] = 2i\,(\vec{\sigma}\times\vec{p})$$

and therefore, **[3]**

$$\left[\hat{H}_D, \hat{S}\right] = i \begin{pmatrix} 0 & \vec{\sigma}\times\vec{p} \\ \vec{\sigma}\times\vec{p} & 0 \end{pmatrix}$$

which can be rewritten as

$$\left[\hat{H}_D, \hat{S}\right] = i \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & (\vec{\sigma}\times\vec{p}) \\ -(\vec{\sigma}\times\vec{p}) & 0 \end{pmatrix}$$

The first matrix in the above expression is γ^0, and the second is $(\vec{\gamma}\times\vec{p})$. Therefore,

$$\left[\hat{H}_D, \hat{S}\right] = i\gamma^0\,(\vec{\gamma}\times\vec{p})$$

[3]

(c)

$$\left[\hat{H}_D, \hat{J}\right] = \left[\hat{H}_D, \hat{L}\right] + \left[\hat{H}_D, \hat{S}\right]$$

And given the results obtained in (a) and (b),

$$\left[\hat{H}_D, \hat{J}\right] = 0$$

Therefore the total angular momentum is conserved! **[3]**

6. **[Total 6 marks]**

Consider a Dirac spinor, $\psi(x,y,z,t)$, which satisfies the Dirac equation

$$i\gamma^1\frac{\partial\psi}{\partial x}+i\gamma^2\frac{\partial\psi}{\partial y}+i\gamma^3\frac{\partial\psi}{\partial z}-m\psi=-i\gamma^0\frac{\partial\psi}{\partial t}$$

Under the parity transformation with $\hat{P}=\gamma^0$, $\psi'(x',y',z',t')=\gamma^0\psi(x,y,z,t)$. Since $(\gamma^0)^2=1$, then $\psi(x,y,z,t)=\gamma^0\psi'(x',y',z',t')$. We can then write

$$i\gamma^1\gamma^0\frac{\partial\psi'}{\partial x}+i\gamma^2\gamma^0\frac{\partial\psi'}{\partial y}+i\gamma^3\gamma^0\frac{\partial\psi'}{\partial z}-m\gamma^0\psi'=-i\gamma^0\gamma^0\frac{\partial\psi'}{\partial t}$$

Taking into account that the parity operator flips the sign of the spatial coordinates we can express the derivatives in terms of the primed system: **[3]**

$$-i\gamma^1\gamma^0\frac{\partial\psi'}{\partial x'}-i\gamma^2\gamma^0\frac{\partial\psi'}{\partial y'}-i\gamma^3\gamma^0\frac{\partial\psi'}{\partial z'}-m\gamma^0\psi'=-i\gamma^0\gamma^0\frac{\partial\psi'}{\partial t'}$$

Since γ^0 anti-commutes with γ^1, γ^2, γ^3,

$$i\gamma^0\gamma^1\frac{\partial\psi'}{\partial x'}+i\gamma^0\gamma^2\frac{\partial\psi'}{\partial y'}+i\gamma^0\gamma^3\frac{\partial\psi'}{\partial z'}-m\gamma^0\psi'=-i\frac{\partial\psi'}{\partial t'}$$

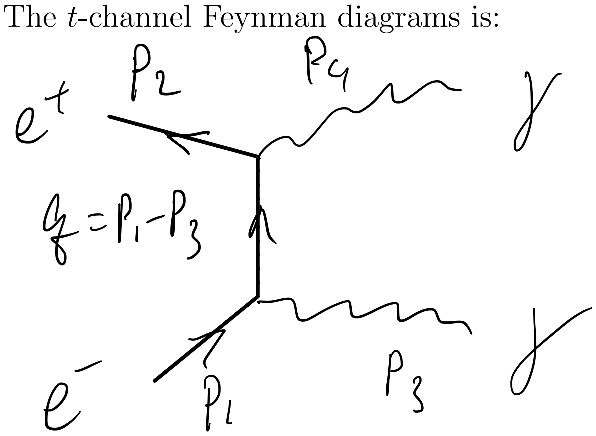
Pre-multiplying by γ^0,

$$i\gamma^1\frac{\partial\psi'}{\partial x'}+i\gamma^2\frac{\partial\psi'}{\partial y'}+i\gamma^3\frac{\partial\psi'}{\partial z'}-m\psi'=-i\gamma^0\frac{\partial\psi'}{\partial t'}$$

which is the Dirac equation in the new coordinates. **[3]**

7. **[Total 5 marks]**

The t-channel Feynman diagrams is:



[2]

The matrix element is:

$$-iM_t=\left[\varepsilon^*_\mu(p_3)ie\gamma^\mu u(p_1)\right]\cdot\left[-\frac{i\left(\gamma^\rho q_\rho+m_e\right)}{q^2-m_e^2}\right]\cdot\left[\bar{v}(p_2)ie\gamma^\nu\varepsilon^*_\nu(p_4)\right]$$

[3]

8. **[Total 6 marks]**

(a) Under the transformation $\psi(x) \to \psi'(x) = e^{iq\chi(x)}\psi(x)$ the Lagrangian becomes

$$\mathcal{L}' = i\overline{\psi'}\gamma^\mu\partial_\mu\psi' - m\overline{\psi'}\psi'$$

$$= ie^{-iq\chi}\overline{\psi}\gamma^\mu\left[e^{iq\chi(x)}\partial_\mu\psi + iq(\partial_\mu\chi)e^{iq\chi(x)}\psi\right]\psi - me^{-iq\chi(x)}\overline{\psi}e^{iq\chi(x)}\psi$$

$$= \mathcal{L} - q\overline{\psi}\gamma^\mu(\partial_\mu\chi)\psi$$

If χ is constant the term $(\partial_\mu\chi)$ disappears and the Lagrangian is restored (invariant). However if χ is a function of x the Lagrangian is *not* invariant under the local phase transformation. **[3]**

(b) To restore the Lagrangian invariance we need to cancel the term $q\overline{\psi}\gamma^\mu(\partial_\mu\chi)\psi$. The cancellation is achieved by introducing a new field which transforms as

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu\chi$$

Therefore the gauge-invariant Lagrangian for a spin-half fermion becomes:

$$\mathcal{L} = \overline{\psi}\left(i\gamma^\mu\partial_\mu - m\right)\psi - q\overline{\psi}\gamma^\mu A_\mu\psi$$

The term $q\overline{\psi}\gamma^\mu A_\mu\psi$ describes the interaction of the fermion with the new field A_μ, **[2]** which can be identified as the photon. Therefore the requirement of gauge invariance introduces the interaction between fermions in QED via exchange of gauge bosons (photons). **[1]**

[Total for paper: 50 marks]