Problem set 3

Question 1

$$(p_1 \cdot p_2)^2 - m_1^2 m_2^2 = (E_1 E_2 - \underline{p_1} \cdot \underline{p_2})^2 - (E_1^2 - \underline{p_1}^2)(E_2^2 - \underline{p_2}^2)$$

$$= E_1^2 E_2^2 + (\underline{p_1} \cdot \underline{p_2})^2 - 2E_1 E_2\, \underline{p_1} \cdot \underline{p_2} - E_1^2 E_2^2 - \underline{p_1}^2 \underline{p_2}^2 + E_1^2 \underline{p_2}^2 + E_2^2 \underline{p_1}^2$$

But $\underline{p_1} \cdot \underline{p_2} = |\underline{p_1}||\underline{p_2}|$ if the momenta are collinear.

$$\rightarrow \quad E_1^2 \underline{p_2}^2 + E_2^2 \underline{p_1}^2 - 2E_1 E_2 |\underline{p_1}||\underline{p_2}|$$

But $\underline{v} = \underline{p}/E$

$$\rightarrow (p_1 \cdot p_2)^2 - m_1^2 m_2^2 = E_1^2 E_2^2 (|\underline{v_1}|^2 + |\underline{v_2}|^2 - 2|\underline{v_1}||\underline{v_2}|$$

$$= E_1^2 E_2^2 \, |\underline{v_1} - \underline{v_2}|^2 = \underline{\underline{|E_2 \underline{p_1} - E_1 \underline{p_2}|^2}}.$$

$$\sigma = \iint \frac{d^3 \underline{p_c}}{(2\pi)^3} \frac{d^3 \underline{p_d}}{(2\pi)^3} \frac{1}{4E_c E_d} |m_{fi}|^2 (2\pi)^4 \frac{\delta(p_a + p_b + p_c + p_d)}{4\sqrt{(p_a \cdot p_b)^2 - m^4}}$$

Let $p = p_a + p_b$ and work in centre of mass frame.

$$s = p^2 = 2m^2 + 2\, p_a \cdot p_b$$

$$\therefore \quad p_a \cdot p_b = s/2 - m^2$$

$$\therefore \quad \sqrt{(p_a \cdot p_b)^2 - m^4} = \frac{s}{2} \sqrt{1 - 4m^2/s}$$

$E_c = E_d$

$$\sigma = \frac{1}{8s} \frac{1}{(1-4m^2/s)^{1/2}} \iint \frac{d^3p_c \, d^3p_d}{(2\pi)^2 E_c^2} \delta^4(p_a + p_b + p_c + p_d) |M_{fi}|^2$$

$$= \frac{1}{8s \, (1-4m^2/s)^{1/2}} \int \frac{d^3p_c}{(2\pi)^2 E_c^2} \delta(E_a + E_b - E_c - E_d) |M_{fi}|^2$$

But $E_a = E_b$ and $E_c = E_d \rightarrow \delta(E_a + E_b - E_c - E_d) = \delta(2E_a - 2E_c)$

$$= \frac{1}{2}\delta(E_a - E_c)$$

$$\therefore \sigma = \frac{1}{16s} \frac{1}{(1-4m^2/s)^{1/2}} \frac{1}{(2\pi)^2} \int \frac{d^3p_c}{E_c^2} \delta(E_a - E_c) |M_{fi}|^2$$

But $\frac{d^3p_c}{E_c} = p_c^2 \frac{dp_c}{E_c} d\Omega$

$$E_c^2 = m^2 + p_c^2$$

$$\therefore dE_c = \frac{p_c}{E_c} dp_c \quad \text{and} \quad p_c \, dE_c = \frac{p_c^2}{E_c} dp_c$$

$$\therefore \sigma = \frac{1}{16s} \frac{1}{(1-4m^2/s)^{1/2}} \frac{1}{(2\pi)^2} \underbrace{\int \frac{p_c}{E_c} dE_c \, \delta(E_a - E_c) |M_{fi}|^2 d\Omega}$$

$$= \int \frac{\sqrt{E_c^2 - m^2}}{E_c} dE_c \, \delta(E_a - E_c) |M_{fi}|^2 d\Omega$$

$$\therefore \sigma = \frac{1}{16s} \frac{1}{(1-4m^2/s)^{1/2}} \frac{1}{(2\pi)^2} \frac{(E_a^2 - M^2)^{1/2}}{E_a} \int |M_{fi}|^2 d\Omega.$$

But $\quad E_a = \sqrt{s}/2$

$$\therefore \quad \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\,\frac{1}{s}\,\frac{(1-4M^2/s)^{1/2}}{(1-4m^2/s)^{1/2}}\,|M_{fi}|^2$$

Question 2

$\underline{\Sigma}$ is defined by $\underline{\Sigma} = -i\,\alpha^1\alpha^2\alpha^3\,\underline{\alpha}$

so $\Sigma_i = -i\,\alpha^1\alpha^2\alpha^3\alpha_i$

But $\alpha_1\alpha_2\alpha_3 \equiv \frac{1}{6}\epsilon_{ijk}\alpha_i\alpha_j\alpha_k$ since

$\alpha_l\alpha_m = -\alpha_m\alpha_l \quad l \neq m.$

$\therefore \quad \alpha_1\alpha_2\alpha_3\alpha_i = \frac{1}{6}\epsilon_{lmn}\alpha_l\alpha_m\alpha_n\alpha_i$

For any i, i will be equal to one of l, m or n in $\epsilon_{lmn}\alpha_l\alpha_m\alpha_n$, and $\alpha_i\alpha_k = \delta_{ik}$ if $i = k$.

e.g. if $i = n$, $\frac{1}{6}\epsilon_{lmn}\alpha_l\alpha_m\alpha_n\alpha_i = \frac{1}{6}\epsilon_{lmn}\alpha_l\alpha_m\delta_{in}$

$$= \frac{1}{6}\epsilon_{lmi}\alpha_l\alpha_m$$

$$= \frac{1}{6}\epsilon_{ilm}\alpha_l\alpha_m$$

But 3 possibilities $\rightarrow \frac{1}{6}\epsilon_{lmn}\alpha_l\alpha_m\alpha_n\alpha_i = \frac{1}{2}\epsilon_{ijk}\alpha_j\alpha_k$

$\therefore \quad \Sigma_i = -\frac{i}{2}\epsilon_{ijk}\alpha_j\alpha_k.$

$\beta\alpha_i = -\alpha_i\beta$

$\therefore \quad \beta\Sigma_i = \Sigma_i\beta$ since β commutes through 2 αs.

$\therefore \quad [\underline{\Sigma}, \beta] = 0.$

$$[\Sigma_i, \alpha_j] = -\frac{i}{2}[\epsilon_{ijm}\alpha_j\alpha_m, \alpha_j]$$

$$= -\frac{i}{2}\epsilon_{ilm}[\alpha_l\alpha_m, \alpha_j]$$

$$= -\frac{i}{2}\epsilon_{ilm}\left(\alpha_l\alpha_m\alpha_j - \alpha_j\alpha_l\alpha_m\right)$$

$$= -\frac{i}{2}\epsilon_{ilm}\left(\alpha_l\{\alpha_m,\alpha_j\} - \{\alpha_l,\alpha_j\}\alpha_m\right)$$

$$= -\frac{i}{2}\epsilon_{ilm}\left(\alpha_l 2\delta_{mj} - 2\delta_{lj}\alpha_m\right)$$

$$= -i\epsilon_{ilj}\alpha_l + i\epsilon_{ijm}\alpha_m$$

$$= \underline{\underline{2i\epsilon_{ijk}\alpha_k.}}$$

$$\therefore \quad [\Sigma_i, H] = [\Sigma_i, p_j\alpha_j + \beta m]$$

$$= [\Sigma_i, \alpha_j]p_j$$

$$= 2i\epsilon_{ijk}p_j\alpha_k = -2i\epsilon_{ikj}\alpha_k p_j$$

$$\therefore \quad \underline{\underline{[\underline{\Sigma}, H] = -2i\,\underline{\alpha}\times\underline{p}}}$$

Question 3

The Dirac Equation is

$$(i\partial_\mu \gamma^\mu - m)\psi = 0$$

$$\therefore \quad i\partial_\nu \gamma^\nu (i\partial_\mu \gamma^\mu - m)\psi = 0$$

$\partial_\nu \gamma^\nu \psi = -im\psi$

$$\therefore (- \partial_\mu \partial_\nu \gamma^\nu \gamma^\mu + im\partial_\nu \gamma^\nu)\psi = 0$$

$\partial_\mu \partial_\nu$ unchanged under permutation

$$\therefore \quad \left(\tfrac{1}{2}\partial_\mu \partial_\nu \{\gamma^\mu, \gamma^\nu\} + m^2\right)\psi$$

But ψ also satisfies the Klein-Gordon equation

$$(\partial^\mu \partial_\mu + m^2)\psi = 0 \quad = (\partial_\nu g^{\mu\nu} \partial_\mu + m^2)\psi = 0$$

We have $\left(\tfrac{1}{2}\partial_\mu \partial_\nu \{\gamma^\mu, \gamma^\nu\} + m^2\right)\psi = 0$

$$\therefore \quad \{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$$

Question 4

$$S(\Lambda) = 1 + i\epsilon_{\rho\sigma} S^{\rho\sigma} \quad \text{where } S^{\rho\sigma} = \frac{1}{2}\sigma^{\rho\sigma} = \frac{i}{4}[\gamma^\rho, \gamma^\sigma]$$

$$\text{So } S(\Lambda) = 1 + \frac{i}{2}\epsilon_{\alpha\beta}\sigma^{\alpha\beta}$$

$$\text{But } S^{-1}(\Lambda)\gamma^\sigma S(\Lambda) = \left(1 - \frac{i}{2}\epsilon_{\alpha\beta}\sigma^{\alpha\beta} + \dots\right)\gamma^\sigma\left(1 + \frac{i}{2}\epsilon_{\gamma\delta}\sigma^{\gamma\epsilon} + \dots\right)$$

$$= \gamma^\sigma - \frac{i}{2}\epsilon_{\alpha\beta}\left[\sigma^{\alpha\beta}, \gamma^\sigma\right] + \dots$$

$$= \gamma^\sigma + \frac{1}{4}\epsilon_{\alpha\beta}\left[[\gamma^\alpha, \gamma^\beta], \gamma^\sigma\right]$$

$$= \gamma^\sigma + \frac{1}{4}\epsilon_{\alpha\beta}\left[(\gamma^\alpha\gamma^\beta - \gamma^\beta\gamma^\alpha), \gamma^\sigma\right]$$

$$= \gamma^\sigma + \frac{1}{4}\epsilon_{\alpha\beta}\left(\gamma^\alpha\gamma^\beta\gamma^\sigma - \gamma^\beta\gamma^\alpha\gamma^\sigma - \gamma^\sigma\gamma^\alpha\gamma^\beta + \gamma^\sigma\gamma^\beta\gamma^\alpha\right)$$

$$= \gamma^\sigma + \frac{1}{4}\epsilon_{\alpha\beta}\left(\{\{\gamma^\sigma, \gamma^\beta\}, \gamma^\alpha\} - \{\{\gamma^\sigma, \gamma^\alpha\}, \gamma^\beta\}\right)$$

$$= \gamma^\sigma + \frac{1}{4}\epsilon_{\alpha\beta}\left(\{2g^{\sigma\beta}, \gamma^\alpha\} - \{2g^{\sigma\alpha}, \gamma^\beta\}\right)$$

$$= \gamma^\sigma + \frac{1}{2}\epsilon_{\alpha\beta}\left(2g^{\sigma\beta}\gamma^\alpha - 2g^{\sigma\alpha}\gamma^\beta\right)$$

$$= \left(\delta^\sigma_\mu + \epsilon_{\alpha\beta}\left(g^{\sigma\beta}\delta^\alpha_\mu - g^{\sigma\alpha}\delta^\beta_\mu\right)\right)\gamma^\mu$$

$$= \underline{\Lambda^\sigma_\mu \gamma^\mu}$$

$$\gamma^0 = \beta, \quad \gamma^i = \beta\alpha^i$$

$$\text{For } \mu = 0 \quad \gamma^0\gamma^0\gamma^0 = \gamma^0\beta^2 = \gamma^0 = \beta = \beta^\dagger$$

$$\therefore \gamma^0\gamma^0\gamma^0 = \gamma^{0\dagger}$$

$\mu = i \quad \gamma^0 \gamma^i \gamma^0 = \beta\beta\alpha_i\beta = \alpha_i\beta = \alpha_i^+ \beta^+ = (\beta\alpha_i)^+ = \gamma^{i+}$

$$\therefore \quad \underline{\underline{\gamma^{\mu +} = \gamma^0 \gamma^\mu \gamma^0}}$$

$S(\Lambda) = 1 - \frac{1}{4}\epsilon_{\rho\sigma}[\gamma^\rho, \gamma^\sigma] = 1 - \frac{1}{4}\epsilon_{\rho\sigma}(\gamma^\rho\gamma^\sigma - \gamma^\sigma\gamma^\rho)$

$\therefore S^+(\Lambda) = 1 - \frac{1}{4}\epsilon_{\rho\sigma}\left(\gamma^{\sigma +}\gamma^{\rho +} - \gamma^{\rho +}\gamma^{\sigma +}\right)$

$= 1 + \frac{1}{4}\epsilon_{\rho\sigma}[\gamma^{\rho +}, \gamma^{\sigma +}]$

$S^{-1}(\Lambda) = 1 + \frac{1}{4}\epsilon_{\rho\sigma}[\gamma^\rho, \gamma^\sigma]$

$\therefore \gamma^0 S^{-1}(\Lambda)\gamma^0 = 1 + \frac{1}{4}\gamma^0[\gamma^\rho \underset{\uparrow \gamma^{0 2}}{} \gamma^\sigma - \gamma^\sigma \underset{\uparrow \gamma^{0 2}}{} \gamma^\rho]\gamma^0$

$= 1 + \frac{1}{4}(\gamma^{\rho +}\gamma^{\sigma +} - \gamma^{\sigma +}\gamma^{\rho +}) = S^+(\Lambda)$

$$\therefore \quad \underline{\underline{S^+\gamma^0 = \gamma^0 S^{-1}}}$$

Under a Lorentz transformation

$\psi \to \psi' = S(\Lambda)\psi$

$\therefore \psi^+ \to \psi'^+ = \psi^+ S^+(\Lambda)$

$\therefore \bar{\psi} \to \bar{\psi}' = \psi^+ S^+(\Lambda)\gamma^0$

$= \psi^+\gamma^0(\gamma^0 S^+(\Lambda)\gamma^0) = \bar{\psi} S^{-1}(\Lambda)$

$$\therefore \quad \underline{\underline{\bar{\psi}' = \bar{\psi} S^{-1}(\Lambda)}}$$

Question 5

$$(i\not{\partial}-m)\psi=0$$

$$\therefore (-i\not{\partial}+m)\psi=0$$

Take Hermitian conjugate

$$\psi^{+}(i\overleftarrow{\partial_a}\gamma^{a+}+m)=0$$

$$\therefore \psi^{+}(i\overleftarrow{\partial_0}\gamma_0^{+}+i\gamma^{i+}\overleftarrow{\nabla_i}+m)=0$$

$$\therefore \psi^{+}\gamma^0(i\gamma^0\gamma^{0+}\overleftarrow{\partial_0}+i\gamma^0\gamma^{i+}\overleftarrow{\nabla_i}+m)\gamma^0=0$$

$$\therefore \bar{\psi}(i\gamma^0\overleftarrow{\partial_0}+i\gamma^i\overleftarrow{\nabla_i}+m)=0$$

$$\therefore \underline{\bar{\psi}(i\overleftarrow{\not{\partial}}+m)=0}$$

Question 6

$$\rho = \psi^{+}\psi \qquad \underline{J} = \psi^{+}\underline{\alpha}\,\psi$$

$$\therefore \rho = \bar{\psi}\gamma^{0}\psi \qquad \underline{\gamma} = \beta\underline{\alpha} = \gamma^{0}\underline{\alpha} \rightarrow \gamma^{0}\underline{\gamma} = \underline{\alpha}$$

$$\therefore \underline{J} = \bar{\psi}\,\underline{\gamma}\,\psi$$

$$\therefore J^{\mu} = (\rho, \underline{J}) = \bar{\psi}\gamma^{\mu}\psi.$$

Under Lorentz transformation

$$\bar{\psi}\gamma^{\mu}\psi \rightarrow \bar{\psi}S^{-1}(\Lambda)\gamma^{\mu}S(\Lambda)\psi$$

$$= \bar{\psi}\Lambda^{\mu}{}_{\nu}\gamma^{\nu}\psi = \Lambda^{\mu}{}_{\nu}\,\bar{\psi}\gamma^{\nu}\psi.$$

$\therefore$ components of J^{μ} transform as 4-vector.

Question 7

$$\frac{1}{2}\gamma^5\underline{\alpha} = \frac{1}{2}i\gamma^0\gamma^1\gamma^2\gamma^3\underline{\alpha}$$

$$\gamma^0 = \beta \quad , \quad \gamma^i = \beta\alpha^i$$

$$\therefore \frac{1}{2}\gamma^5\underline{\alpha} = \frac{1}{2}i\beta^2\alpha^1\beta\alpha^2\beta\alpha^3\underline{\alpha}$$

$$= \frac{1}{2}i\alpha^1\beta\alpha^2\beta\alpha^3\underline{\alpha} \qquad \text{using } \beta^2=1$$

$$\alpha^i\beta = -\beta\alpha^i$$

$$\therefore \frac{1}{2}\gamma^5\underline{\alpha} = \frac{1}{2}i(-\beta\alpha^1\alpha^2\beta\alpha^3\underline{\alpha})$$

$$= \frac{1}{2}i(-\beta^2\alpha^1\alpha^2\alpha^3\underline{\alpha})$$

$$= -\frac{1}{2}i\alpha^1\alpha^2\alpha^3\underline{\alpha}$$

$$= \frac{1}{2}\Sigma$$

$$\therefore \frac{1}{2}\gamma^5\underline{\alpha} = \frac{1}{2}\underline{\Sigma} = \underline{S}$$