10 Quantum Electrodynamics

10.1 Quantizing the Dirac Field

In this section we use the Dirac equation as the basis for a a field theory, which allows the creation and annihilation of particles naturally. Quantizing a field basically means that the wave function becomes an operator. The space in which this operator acts is called the Fock space, which contains states with an arbitrary number of particles and therefore we will be able to describe processes where the number of states changes.

Dirac field theory is defined to be the theory whose field equations correspond to the Dirac equation. We regard the two Dirac fields $\psi(x)$ and $\bar{\psi}(x)$ as being dynamically independent fields and postulate the Dirac Lagrangian density:

$$\mathcal{L} = \bar{\psi}(x)(i\gamma^\mu \partial_\mu - m)\psi(x). \tag{10.1}$$

Then the Euler-Lagrange equation

$$\frac{\partial}{\partial x^\mu}\frac{\partial \mathcal{L}}{\partial(\partial_\mu \psi)} - \frac{\partial \mathcal{L}}{\partial \psi} = 0 \tag{10.2}$$

and the same with $\psi \to \bar{\psi}$ leads to the Dirac equation. The canonical momentum is

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\psi}(x)} = i\psi^\dagger(x) \tag{10.3}$$

so, using the fact that at the solutions to the equation of motion the Lagrangian density is zero, the value of the Hamiltonian density for the physical solutions is

$$\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = \psi^\dagger i\frac{\partial \psi}{\partial t}. \tag{10.4}$$

Now we want to regard ψ as a quantum field rather than as a wave function. In order to quantize this field, naively we would try to impose the usual equal time commutation relations, i.e.

$$\begin{aligned}
[\psi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)] &= i\delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}), \\
[\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{y},t)] &= 0, \\
[\pi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)] &= 0,
\end{aligned} \tag{10.5}$$

where α and β label the spinor components of ψ and π. Without proving it for the moment we note that this would lead to a disaster. In particular, the Hamiltonian is unbounded from below - there is no ground state. The only way to cure the problem is to impose anti-commutation relations (we will soon see that this leads to the desired properties for spin-1/2):

$$\begin{aligned}
\{\psi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} &= i\delta_{\alpha\beta}\delta^3(\mathbf{x}-\mathbf{y}). \\
\{\psi_\alpha(\mathbf{x},t), \psi_\beta(\mathbf{y},t)\} &= 0, \\
\{\pi_\alpha(\mathbf{x},t), \pi_\beta(\mathbf{y},t)\} &= 0.
\end{aligned} \tag{10.6}$$

The Heisenberg equations of motion for the field operators have the solution

$$\psi_\alpha(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E}\sum_{s=1,2}[b(s,\mathbf{k})u_\alpha(s,\mathbf{k})e^{-ik\cdot x} + d^\dagger(s,\mathbf{k})v_\alpha(s,\mathbf{k})e^{ik\cdot x}] \tag{10.7}$$

$$\bar{\psi_\alpha}(\mathbf{x},t) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E}\sum_{s=1,2}[b^\dagger(s,\mathbf{k})\bar{u}_\alpha(s,\mathbf{k})e^{ik\cdot x} + d(s,\mathbf{k})\bar{v}_\alpha(s,\mathbf{k})e^{-ik\cdot x}] \tag{10.8}$$

Since ψ is now an operator, so are the expansion coefficients $b^\dagger, d^\dagger, b$ and d. They are interpreted as creation and annihilation operators for electrons and positrons respectively. The anti-commutation relations for the fields, eq. (10.6), imply that

$$\begin{aligned}
\left\{b(r,\mathbf{k}), b^\dagger(s,\mathbf{k}')\right\} &= (2\pi)^3\, 2E\, \delta^3(\mathbf{k}-\mathbf{k}')\delta_{sr} \\
\left\{d(r,\mathbf{k}), d^\dagger(s,\mathbf{k}')\right\} &= (2\pi)^3\, 2E\, \delta^3(\mathbf{k}-\mathbf{k}')\delta_{sr} \\
\{b(r,\mathbf{k}), b(s,\mathbf{k}')\} &= \left\{b^\dagger(r,\mathbf{k}), b^\dagger(s,\mathbf{k}')\right\} = 0 \\
\{d(r,\mathbf{k}), d(s,\mathbf{k}')\} &= \left\{d^\dagger(r,\mathbf{k}), d^\dagger(s,\mathbf{k}')\right\} = 0
\end{aligned} \tag{10.9}$$

There is a useful result for the sum of spins of spinor combinations, i.e.

$$\begin{aligned}
\sum_s u(s,p)\,\bar{u}(s,p) &= \not{p} + m, \\
\sum_s v(s,p)\,\bar{v}(s,p) &= \not{p} - m.
\end{aligned} \tag{10.10}$$

This can be verified by acting with either $\sum_s u(s,p)\,\bar{u}(s,p)$ or $\sum_s v(s,p)\,\bar{v}(s,p)$ on spinors (either from the left or the right), using the Dirac equations $(\not{p}-m)u = 0$ or $(\not{p}+m)v = 0$ and the normalisation conditions in eq. (9.48). For example,

$$\Big(\sum_s u(s,p)\,\bar{u}(s,p)\Big)u(1,p) = 2mu(1,p) = (\not{p}+m)u(1,p), \tag{10.11}$$

or

$$\Big(\sum_s u(s,p)\,\bar{u}(s,p)\Big)v(1,p) = 0 = (\not{p}+m)v(1,p), \tag{10.12}$$

and where we can repeat for all possibilities. Using these spin sum results one can see that the anti-commutation relations in eq. (10.9) lead to the correct anti-commutation relations for the fields $\psi_\alpha(\mathbf{x},t)$ and $\pi_\beta(\mathbf{x},t)$.

The total Hamiltonian is

$$H = \int d^3\mathbf{x}\ :\mathcal{H}: \tag{10.13}$$

The symbols $:\ :$ denote normal ordering of the operator inside, i.e. we put all creation operators to the left of all annihilation operators so that $H|0\rangle = 0$ by definition, and is the way we remove the ambiguity associated with the order of operators. Note that if we move an anti-commuting

(fermion) operator through another such operator then we pick up a minus sign. Using eq. (10.4) after some algebra we get

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E}E\sum_{s=1,2}[b^\dagger(s,\mathbf{k})b(s,\mathbf{k}) + d^\dagger(s,\mathbf{k})d(s,\mathbf{k})]. \tag{10.14}$$

The formula in eq. (10.14) has a very nice interpretation. The operator $b^\dagger b$ is nothing but the number operator for electrons and $d^\dagger d$ that for positrons. Thus, to get the total Hamiltonian, we have to count all electrons and positrons for all spin states s and momenta $\mathbf{k}$ and multiply this number by the corresponding energy E.

If we had tried to impose commutation relations, the $d^\dagger d$ term would have entered with a minus sign in front, which would signal that something has gone wrong. In particular, it would mean that $d^\dagger$ creates particles of negative energy, and so to the Hamiltonian potentially being unbounded from below. This is not supposed to happen in the quantized field theory. (We could try to fix the problem by simply re-labelling $d \leftrightarrow d^\dagger$ but it may be shown that this leads to acausal propagation.)

So, in order to quantize the Dirac field we are necessarily led to the introduction of anti-commutation relations. Remarkably we find that we have automatically taken into account the Pauli exclusion principle! For example,

$$\left\{b^\dagger(r,\mathbf{k}), b^\dagger(s,\mathbf{k}')\right\} = 0$$

implies that it is not possible to create two quanta in the same state, i.e.

$$b^\dagger(s,\mathbf{k})b^\dagger(s,\mathbf{k})|0\rangle = 0.$$

This intimate connection between spin and statistics is a direct consequence of desiring our theory to be consistent with the laws of relativity and quantum mechanics.

Finally consider the charge operator

$$Q = \int d^3\mathbf{x}\ :j_0(x): \ = \int d^3\mathbf{x}\ :\psi^\dagger\psi:$$

which, in terms of the creation and annihilation operators, is

$$Q = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E}\sum_{s=1,2}[b^\dagger(s,\mathbf{k})b(s,\mathbf{k}) - d^\dagger(s,\mathbf{k})d(s,\mathbf{k})] \tag{10.15}$$

This shows again that $b^\dagger$ creates fermions while $d^\dagger$ creates the associated anti-fermions of opposite charge.

10.2 Quantizing the Electromagnetic Field

The Maxwell equations can be derived from the Lagrangian density

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j_\mu A^\mu \tag{10.16}$$

where the field strength tensor is

$$F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{10.17}$$

and j_μ is an external source for the field. Maxwell's equations do not change under the gauge transformation

$$A_\mu(x) \to A_\mu(x) + \partial_\mu \Lambda(x) \tag{10.18}$$

where $\Lambda(x)$ is some scalar field. This shows that there is some redundancy, and the 4 components of $A_\mu(x)$ are more than is required to describe the electromagnetic field (there are two transverse polarisations of e.m. radiation). This leads to a problem in quantization. To see this note that the canonically conjugate field to A_μ is

$$\Pi^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_0 A_\mu)} = F^{\mu 0} \tag{10.19}$$

and from this it follows that $\Pi^0 = 0$. This means there is no possibility of imposing a non-zero commutation relation between Π^0 and A^0, which we would need if we are to quantize the field.

To get around this problem we recognise that gauge invariance allows us to impose an extra condition, which we use to *fix* the gauge invariance, and effectively lower the degrees of freedom. For example, we can impose the Lorentz gauge condition, i.e.

$$\partial_\mu A^\mu = 0. \tag{10.20}$$

Note that, even after fixing the Lorentz gauge, we can perform another gauge transformation on A_μ, i.e. $A_\mu(x) \to A_\mu(x) + \partial_\mu \chi(x)$ where $\chi(x)$ must satisfy the wave equation, $\partial_\mu \partial^\mu \chi = 0$, i.e. we have two unphysical degrees of freedom and the two physical fields.

We impose the constraint by noting that since $\partial_\mu A^\mu = 0$, there is no harm in adding it to the Lagrangian density as

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - j_\mu A^\mu - \frac{1}{2\xi}(\partial_\mu A^\mu)^2. \tag{10.21}$$

Indeed what we are doing here is following the Lagrange multiplier method of imposing constraints ($\lambda = 1/2\xi$ being the Lagrange multiplier), and recognising that we should find the stationary points of $S = \int d^4x \mathcal{L}$ subject to the constraint $(\partial_\mu A^\mu)^2 = 0$, i.e. this comes from the "equation of motion" $\partial\mathcal{L}/\partial(\lambda) = 0$.

Using the gauge-fixed Lagrangian, the equations of motion are now

$$\partial^\mu F_{\mu\nu} - j_\nu + \frac{1}{\xi}\partial_\nu(\partial^\mu A_\mu) = 0.$$

If we require that these equations are satisfied *and then* also $\partial_\mu A^\mu = 0$, we have the original equations of motion but in a fixed gauge.

In the Feynman gauge $\xi = 1$, the Lagrangian is particularly simple (after some integration by parts under $\int d^4x$):

$$\mathcal{L} = -\frac{1}{2}\partial_\mu A_\nu \partial^\mu A^\nu - j_\mu A^\mu,$$

and quantization can now proceed: $\Pi^\mu = -\partial_0 A^\mu$ and thus

$$[A^\mu(\mathbf{x}, t), \Pi^\nu(\mathbf{y}, t)] = -ig^{\mu\nu}\delta^3(\mathbf{x} - \mathbf{y}) \tag{10.22}$$

with all other commutators vanishing. The Heisenberg operator corresponding to the photon field is

$$A_\mu(x) = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E}\sum_{\lambda=0}^{3}\left[\varepsilon_\mu(\lambda, \mathbf{k})a(\lambda, \mathbf{k})e^{-ik\cdot x} + \varepsilon^*_\mu(\lambda, \mathbf{k})a^\dagger(\lambda, \mathbf{k})e^{ik\cdot x}\right] \tag{10.23}$$

where $\varepsilon_\mu(\lambda, \mathbf{k})$ are a set of four linearly independent basis 4-vectors for polarization ($\lambda = 0, 1, 2, 3$). For example, if $k = (k_0, \mathbf{k})$, we might choose $\varepsilon^\mu(0) = (1, 0, 0, 0)$, $\varepsilon^\mu(3) = (0, \mathbf{k})/k_0$, $\varepsilon^\mu(1) = (0, \mathbf{n}_1)$ and $\varepsilon^\mu(2) = (0, \mathbf{n}_2)$, where $k_0^2 = \mathbf{k}^2$, $\mathbf{n}_1 \cdot \mathbf{k} = 0$, $\mathbf{n}_2 \cdot \mathbf{k} = 0$ and $\mathbf{n}_1 \cdot \mathbf{n}_2 = 0$. $\varepsilon^\mu(1)$ and $\varepsilon^\mu(2)$ are therefore polarization vectors for transverse polarisations whilst $\varepsilon^\mu(0)$ is referred to as the time-like polarization vector and $\varepsilon^\mu(3)$ is referred to as the longitudinal polarization vector. For example, if $k = (k_0, 0, 0, k_0)$, $\varepsilon^\mu(0) = (1, 0, 0, 0)$, $\varepsilon^\mu(3) = (0, 0, 0, 1)$, $\varepsilon^\mu(1) = (0, 1, 0, 0)$ and $\varepsilon^\mu(2) = (0, 0, 1, 0)$.

The commutation relation (10.22) implies that

$$\left[a(\lambda, \mathbf{k}), a^\dagger(\lambda', \mathbf{k}')\right] = -g_{\lambda\lambda'}\, 2E\, (2\pi)^3\delta^3(\mathbf{k} - \mathbf{k}'). \tag{10.24}$$

At a glance this looks fine, i.e. we interpret $a^\dagger(\lambda, \mathbf{k})$ as an operator that creates quanta of the electromagnetic field (photons) with polarization λ and momentum $\mathbf{k}$. However, for $\lambda = 0$ we have a problem since the sign on the RHS of (10.24) is opposite to that of the other 3 polarisations. This shows up in the fact that these time-like photons make a negative contribution to the energy:

$$H = \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2E}E\left(-a^\dagger(0, \mathbf{k})a(0, \mathbf{k}) + \sum_{i=1,3} a^\dagger(i, \mathbf{k})a(i, \mathbf{k})\right). \tag{10.25}$$

Fortunately we have already solved the problem. Recall that we still have to impose $\partial_\mu A^\mu = 0$. It turns out that it is impossible to do this at the operator level, but we can do it for all physical expectation values, i.e. we can impose the correct physics. It then turns out that contributions from the time-like and longitudinal photons always cancel. More explicitly, by demanding for any state $|\chi\rangle$ that

$$\langle\chi|\partial_\mu A^\mu|\chi\rangle = 0 \tag{10.26}$$

it follows that

$$\langle\chi|a^\dagger(3, \mathbf{k})a(3, \mathbf{k}) - a^\dagger(0, \mathbf{k})a(0, \mathbf{k})|\chi\rangle = 0. \tag{10.27}$$

and therefore $\langle\chi|H|\chi\rangle \geq 0$. This is nice because it is in accord with our knowledge that free photons are transversely polarized.

10.3 Interactions in QED

We are now ready to let our fermions and photons interact with each other. The interaction is described by the Lagrangian

$$\mathcal{L}_{\text{int}} = -e\,\bar{\psi}\gamma^\mu A_\mu \psi. \tag{10.28}$$

Such an interaction may be introduced by the concept of 'minimal substitution' familiar from classical electrodynamics:

$$\mathbf{p} \to \mathbf{p} - e\mathbf{A}, \tag{10.29}$$

$$E \to E - e\phi. \tag{10.30}$$

or, in 4-vector notation:

$$p^\mu \to p^\mu - eA^\mu. \tag{10.31}$$

Applying this classical concept of minimal substitution to the Dirac equation gives

$$(i\not{D} - m)\psi = 0 \tag{10.32}$$

where we have introduced the covariant derivative

$$D_\mu \equiv \partial_\mu + ieA_\mu.$$

The QED Lagrangian describing electrons, photons and their interactions is then given by

$$\begin{aligned}\mathcal{L}_{\text{QED}} &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\xi}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\not{D} - m)\psi \\ &= -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}(\partial_\mu A^\mu)^2 + \bar{\psi}(i\not{D} - m)\psi,\end{aligned} \tag{10.33}$$

where $(\partial \cdot A)^2/2$ is the gauge fixing term for the Feynman gauge.

There is a much nicer and theoretically much more appealing way to get the interaction Lagrangian eq. (10.28). Indeed, if we require the QED Lagrangian to be invariant under *local gauge symmetry*, we have to add the term eq. (10.28) to the free Lagrangian eqs. (10.1) and (10.21). Local gauge symmetry consists of the simultaneous gauge transformations of the photon field eq. (10.18) and a phase transformation on the electron field:

$$\psi(x) \to e^{-ie\Lambda(x)}\psi(x). \tag{10.34}$$

Under this transformation

$$\partial_\mu\psi(x) \to e^{-ie\Lambda(x)}\partial_\mu\psi(x) - ie\partial_\mu\Lambda(x)e^{-ie\Lambda(x)}\psi(x), \tag{10.35}$$

and the second term means $\bar{\psi}(x)\not{\partial}\psi(x)$ is not invariant, changing by $-ie\partial_\mu\Lambda(x)\bar{\psi}(x)\gamma^\mu\psi(x)$. However, under the same transformation the change in the gauge field in eq. (10.18) means that

$$ie\,\bar{\psi}\gamma^\mu A_\mu\psi \to ie\,\bar{\psi}\gamma^\mu A_\mu\psi + ie\partial_\mu\Lambda(x)\bar{\psi}\gamma^\mu\psi, \tag{10.36}$$

and the second term cancels with the variation in $\bar{\psi}(x)\not{\partial}\psi(x)$ meaning that $\bar{\psi}(x)\not{D}\psi(x)$, and hence the fermionic part of the Lagrangian, is invariant.

10.4 Feynman Rules of QED

Now we have obtained the QED Lagrangian density. From it we can figure out the Feynman rules. The precise manner in which the rules are obtained was demonstrated in the earlier lectures for the simpler case of a scalar field theory, and hopefully this enables you to see roughly where they come from in the more complicated case of QED. The Feynman rules for computing the amplitude $\mathcal{M}$ for an arbitrary process in QED are summarized in Table 1.

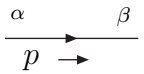
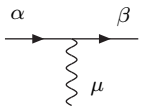
For every ...	draw ...	write ...
Internal photon line		$\frac{-ig^{\mu\nu}}{p^2+i\epsilon}$
Internal fermion line		$\frac{i(\not{p}+m)_{\alpha\beta}}{p^2-m^2+i\epsilon}$
Vertex		$-ie\gamma^{\mu}_{\alpha\beta}$
Outgoing electron		$\bar{u}_\alpha(s,p)$
Incoming electron		$u_\alpha(s,p)$
Outgoing positron		$v_\alpha(s,p)$
Incoming positron		$\bar{v}_\alpha(s,p)$
Outgoing photon		$\varepsilon^{*\mu}(\lambda,p)$
Incoming photon		$\varepsilon^{\mu}(\lambda,p)$

Table 1: Feynman rules for QED. μ, ν are Lorentz indices, α, β are spinor indices and s and λ fix the polarization of the electron and photon respectively.

First of all, we can read off the propagators from the bilinear terms in the Lagrangian. Consider e.g. the term eq. (10.1). In order to get the propagator for the fermion from this term we have to invert the operator sandwiched between the fields $\bar{\psi}$ and ψ (and multiply by a factor i). In momentum space, this leads to $i(\not{p}+m)/(p^2-m^2+i\epsilon)$ as given in Table 1.

Proceeding in a similar way with eq. (10.33) we get the propagator for the photon. The precise form of the propagator depends on the chosen gauge, i.e. on the value of ξ. However, scattering amplitudes are independent of ξ. Thus, we can pick any value to do the calculation. For most purposes, $\xi = 1$ (Feynman gauge) is a very convenient gauge since it leads to a very simple form for the propagator, and we will use this gauge throughout.

The rule for the vertex can be obtained directly from $\mathcal{L}_{\text{int}}$. The vertex factor is simply i times the factor that multiplies the term $\bar{\psi}A^\mu\psi$. Finally, the external line factors are derived by considering simple matrix elements in the operator formalism. They are what is left behind from the expansions of fields in terms of annihilation and creation operators, after the operators have all been (anti-)commuted until they annihilate the vacuum.

The spinor indices in the Feynman rules are such that matrix multiplication is performed in the opposite order to that defining the flow of fermion number. The arrow on the fermion line itself denotes the fermion number flow, *not* the direction of the momentum associated with the line: I will try always to indicate the momentum flow separately as in Table 1. This will become clear in the examples which follow. We have already met the Dirac spinors u and v. I will say more about the photon polarization vector ε when we need to use it.

To summarize, the procedure for calculating the amplitude for any process in QED is the following:

1. Draw all possible distinct diagrams
2. Associate a directed 4-momentum with all lines
3. Apply the Feynman rules for the propagators, vertices and external legs
4. Ensure 4-momentum conservation at each vertex by adding $(2\pi)^4\delta^4(k_i - k_f)$,where k_i and k_f are the total incoming and outgoing 4-momenta of the vertex respectively
5. Perform the integration over all internal momenta with the measure $\int d^4k/(2\pi)^4$

It is also part of the Feynman rules for QED that when diagrams differ by an interchange of two fermion lines, a relative minus sign must be included. This is because of the anti-commutation of the fermion operators in the original matrix element. Note, however, that you don't need to get the absolute sign of an amplitude right, just its sign relative to the other amplitudes, since it is the modulus of the amplitude squared that we need ultimately.

This sounds rather complicated. In particular there seem to be an awful lot of integrations to be done. However, at tree-level, i.e. if there are no loop diagrams, the delta functions attached to the vertices together with the integration over the internal momenta simply result in an overall 4-momentum conservation, i.e. in a factor $(2\pi)^4\delta^4(P_i - P_f)$, where P_i and P_f are the total incoming and outgoing 4-momenta of the process. Thus at tree-level, no 'real' integration has to be done. At one loop, however, there is one non trivial integration to be done. Generally, the calculation of an n-loop diagram involves n non trivial integrations. Even worse, these integrals very often are divergent. Still, we can get perfectly reasonable theoretical predictions at any order in QED. The procedure to get these results is called renormalization and will be the topic of section 12.

At this point, some remarks concerning step 1, i.e. drawing all possible distinct Feynman diagrams, might be useful. In order to establish whether two diagrams are distinct, we have to try to convert one into the other. If this is possible without cutting lines and without gluing lines – that is solely by twisting and stretching the lines and rotating the whole figure – then the two diagrams are identical. It should be noted that the external lines are labelled in this process. Therefore, the two diagrams shown in figure 4 are different. Finally, let me mention that the diagrams shown in figure 1 are not Feynman diagrams. When drawing Feynman diagrams we are only interested in what particles are incoming and which ones are outgoing and there is no time direction involved.