PHASM426 / PHAS4426
Advanced Quantum Theory Problem Sheet 1

To be handed in by 5pm on Tuesday 31rst October 2017.

Please hand in your completed work at the **end** of the lecture on that day. If you are unable to attend the lecture, you may scan your work, save it as a single PDF file and email it to me **prior** to this lecture. You may also bring the work to me in my office (B12) before the lecture. **Make sure your completed work is clearly labelled with your name and college**. Please note that UCL places severe penalties on late-submitted work.

1. Consider the vector space of real-valued polynomials of the power not larger than 3:
$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3\,.$$
   (a) Write down a set of functions that form a basis of this vector space. **[1]**
   (b) What is the dimension of this vector space? **[1]**

2. Consider the basis of vectors$|\phi_j\rangle$ where j spans from 1 to n. Show that if the basis of vectors $\{|\phi_j\rangle\}$ is linearly independent, then for any vector $|\Psi\rangle$ the coefficients c_j of the expansion
$$|\Psi\rangle = \sum_{j=1}^{n} c_j |\phi_j\rangle$$
are unique. *Hint:* Two prove uniqueness you need to assume that there is another set of coefficients that will expand the state $|\Psi\rangle = \sum_{j=1}^{n} a_j|\phi_j\rangle$ and then prove that $a_j = c_j$. **[2]**

3. The Hamiltonian of a quantum system is written in its spectral decomposition as $H = \sum_{n=1}^{d} \lambda_n |\phi_n\rangle\langle\phi_n|$, with $\langle\phi_m|\phi_n\rangle = \delta_{m,n}$ where the closure relationship is satisfied i.e. $\mathbb{1} = \sum_{n=1}^{d} |\phi_n\rangle\langle\phi_n|$. Prove that the exponential of H takes the form $e^H = \sum_{n=1}^{d} e^{\lambda_n}|\phi_n\rangle\langle\phi_n|$. **[3]**

4. Given two arbitrary vectors $|\phi_1\rangle$ and $|\phi_2\rangle$ belonging to the inner product space $\mathcal{H}$, the Cauchy-Schwartz inequality states that
$$|\langle\phi_1|\phi_2\rangle|^2 \leq \langle\phi_1|\phi_1\rangle\langle\phi_2|\phi_2\rangle. \quad (1)$$
The purpose of this problem is to use the properties of inner product to prove this inequality. To proceed with the proof consider the vector $|\Psi\rangle$ defined as:
$$|\Psi\rangle = |\phi_1\rangle + \lambda|\phi_2\rangle$$
where λ is a complex number that can be written as $\lambda = a + ib$.

(a) Write an expression for the inequality $\langle\Psi|\Psi\rangle \geq 0$ as a function of λ i.e. $f(\lambda)$. Then, re-write this expression as a function of a and b i.e. $f(a, b)$. **[2]**

(b) Show that the value of λ that minimises $\langle\Psi|\Psi\rangle$ is

$$\lambda_{min} = -\frac{\langle\phi_2|\phi_1\rangle}{\langle\phi_2|\phi_2\rangle} \tag{2}$$

Hint: Compute the derivates of the function $f(a, b)$ obtained in (a) with respect to a and b. Solve these equations to obtain a_{min} and b_{min} and then compute λ_{min}. **[2]**

(c) Substitute Eq. (2) in the expression of $f(\lambda)$ derived in (a) and show that it reduces to the expression for the Cauchy-Schwuartz inequality (Eq. (1)). **[2]**

(d) Which relation do $|\phi_1\rangle$ and $|\phi_2\rangle$ satisfy such that the equality in Eq. (1) is realised? **[1]**

(e) Discuss in which cases the Cauchy-Schwartz inequality is important in quantum mechanics **[1]**

5. Consider a Hermitian operator A with eigenvalues $\{\lambda_1, \lambda_2, \cdots \lambda_n\}$ and eigenvectors $\{|\psi_1\rangle, |\psi_2\rangle \cdots |\psi_n\rangle\}$. Show that A can be written in terms of a unitary transformation U as $A = UDU^\dagger$, where D is a diagonal matrix. **[2]**

6. Consider a quantum system with Hamiltonian H and consider the measurement of an observable with a non-degenerate spectral decomposition $A = \sum_n a_n|\psi_n\rangle\langle\psi_n|$. H and A do not commute. The system is initially in the eigenstate $|\psi_n\rangle$ of A, with eigenvalue a_n. A series of ideal measurements on the observable A are carried out. The first measurement is carried out at time $t = \theta$. Then subsequent measurements are made at $t = 2\theta$, $t = 3\theta$ and so on. Here θ is very small.

   (a) Expand the state of the system to second order in time t and show that the probability of obtaining the eigenvalue a_n at $t = \theta$ is given by

   $$w_{nn}(\theta) \simeq 1 - (\Delta E)_n^2\theta^2,$$

   where $(\Delta E)_n^2 = \langle\psi_n|H^2|\psi_n\rangle - \langle\psi_n|H|\psi_n\rangle^2$. Notice that $w_{nn}(\theta)$ is the probability that the system is still in the initial state. **[3]**

   (b) Show that after k measurements i.e. at $\tau = k\theta$, the probability $w_{nn}(\tau)$ becomes

   $$w_{nn}(\tau) \simeq [1 - (\Delta E)_n^2\theta^2]^k.$$

   [3]

   (c) Assume k is large and τ is fixed such that $\theta/k \to 0$. Show that in this limit

   $$w_{nn}(\tau) \simeq \exp[-(\Delta E)_n^2\tau\theta] \to 1.$$

   You may use without proof the fact that **[2]**

   $$e^x = \lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n.$$

A Olaya-Castro, October 2017