# Chapter 4

# Quantum Light, Atoms and the Interaction Picture

## 4.1 The Quantum Harmonic Oscillator

Light was one of the first systems to be studied in the context of the emerging theory of quantum mechanics. The idea that light should be quantised into packets of energy, "photons", stemming from Einstein's reasoning about the photo-electric effect, was one of the earliest clues that quantum mechanics would have a radically different structure to classical theories. The quantum description of light is very closely related to a system which you will have studied previously, the quantum harmonic oscillator (QHO). For that reason, before we quantise light we will revise the properties of the QHO.

A QHO has the following Hamiltonian,

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2. \tag{4.1}$$

The eigenstates of the Hamiltonian are a set of energy levels labelled $|n\rangle$, where $n$ is an integer, with equally spaced energies $E_n = (n + 1/2)\hbar\omega$. We can think of the energy being quantised into equal packets of energy each of energy $\hbar\omega$. It is convenient to study QHO using ladder operators

$$\hat{a} = \sqrt{\frac{m\omega}{2\hbar}}(\hat{x} + i\hat{p}) \qquad \hat{a}^\dagger = (\hat{a})^\dagger \tag{4.2}$$

These operators are called lowering and raising operators (or annihilation and

creation operators) since they act on the states $|n\rangle$ as follows:

$$\hat{a}|n\rangle = \sqrt{n}|n-1\rangle \qquad \hat{a}^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{4.3}$$

and satisfy the commutation relation $[\hat{a}, \hat{a}^\dagger] = \mathbb{1}$. Combining these expressions we find:

$$\hat{a}^\dagger\hat{a}|n\rangle = n|n\rangle \tag{4.4}$$

The operator $\hat{a}^\dagger\hat{a}$ is called the number operator since its eigenvalue is $n$ the number of excitations in state $|n\rangle$.

Since

$$\begin{aligned} \hat{x} &= \sqrt{\frac{\hbar}{2m\omega}}(\hat{a}^\dagger + \hat{a}) \\ \hat{p} &= \sqrt{\frac{\hbar}{2m\omega}}i(\hat{a}^\dagger - \hat{a}) \end{aligned} \tag{4.5}$$

we can rewrite the Hamiltonian in the ladder operator picture as

$$H = \hbar\omega(\hat{a}^\dagger\hat{a} + 1/2) \tag{4.6}$$

## 4.2 Quantum Light

Quantum theory is not derived from classical theory, however one can often obtain a quantum description of a system for which a classical description is known by a method known as quantisation, where classical observables e.g. $x$ and $p$ are replaced by quantum operators $\hat{x}$ and $\hat{p}$ satisfying an appropriate commutation relation. There are standard methods which can be used to do this (e.g. a method known as "canonical quantisation") but all such methods are to a certain extent ad hoc. The replacement of classical values by quantum operators is essentially good guesswork, which must then be verified by experiment.

In this section we will see how a quantum description of light may be obtained starting with the classical Maxwell equations, casting them in a form in which simple harmonic motion can be identified, and then replacing the classical oscillators with quantum oscillators as described in the previous section.

The Maxwell equations in vacuum

$$\begin{aligned} \nabla \times E = -\frac{\partial B}{\partial t} \qquad & \nabla \times B = \mu_0\epsilon_0\frac{\partial E}{\partial t} \\ \nabla \cdot E = 0 \qquad & \nabla \cdot B = 0 \end{aligned} \tag{4.7}$$

can be combined to derive the electric field wave-equation

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E = 0 \tag{4.8}$$

where $1/c^2 = \mu_0\epsilon_0$.

We can solve this equation using the separation of variables technique, writing $E(r,t) = u(r)T(t)$ where $r$ is the vector position and $t$ is time,

$$\begin{aligned} \frac{(\nabla^2 u(r))T(t)}{u(r)T(t)} &= \frac{u(r)}{c^2}\frac{\partial^2 T}{\partial t^2}\frac{1}{u(r)T(t)} = -\frac{\omega^2}{c^2} \\ \frac{(\nabla^2 u(r))}{u(r)} &= \frac{1}{c^2}\frac{\partial^2 T}{\partial t^2}\frac{1}{T(t)} = -\frac{\omega^2}{c^2} \end{aligned} \tag{4.9}$$

which gives us two $\omega$ dependent equations ($-\omega/c^2$ is chosen to be the separation constant for convenience).

The spatial equation, called the Helmholtz equation is

$$\nabla^2 u(r) = -\frac{\omega^2}{c^2}u(r). \tag{4.10}$$

This is a vector equation, and has many solutions depending on $\omega$ and boundary conditions (i.e. the presence of conducting surfaces such as mirrors).

The temporal equation takes the form of a simple Harmonic oscillator equation

$$\frac{\partial^2 T(t)}{\partial t^2} = -\omega^2 T(t) \tag{4.11}$$

the solutions of which will have the form $\exp[\pm i\omega t]$.

Labelling the solutions to these equations $T_j(t)$ and $u_j(r)$, where $\omega_j$ is the associated frequency, to obtain a general solution we would compute all the solutions to these equations for all values of $\omega$ and combine them linearly

$$E(r,t) = \sum_j \alpha_j u_j(r) T_j(t) \tag{4.12}$$

where $E_{0j}$ is a unit vector representing the polarisation of the mode and the $\alpha_j$ coefficients depend on the initial conditions. We can think of this classical solution as representing a sum of independent oscillators, each associated with a particular

spatial function $u_j(r)$ and oscillating (according to $T_j(t)$ as an oscillator with a frequency which we will label $\omega_j$[1].

Each of these oscillators is called a **mode**, and we call $u_j(r)$ for a particular mode its mode function and $\omega_j$ its frequency. Analogous equations occur in acoustics. There the modes are often called resonant modes, each of which vibrates as an independent oscillator with a certain frequency.

The quantisation of the EM-field proceeds by replacing each of these classical oscillators with a quantum oscillator. In other words, each mode will be represented by a separate quantum harmonic oscillator. The energy eigenstates of each oscillator $|n\rangle$, represents a state of $n$ photons in the mode. We shall label the states and operators for mode $j$ with a subscript $j$ $|n\rangle_j$, $a_j$, etc.

Thus each mode has a Hamiltonian

$$H_j = \hbar\omega(a_j^\dagger a_j + 1/2) \tag{4.13}$$

and the Hamiltonian for the full field is $H = \sum_j H_j$.

The electric field is now represented by a quantum operator

$$\hat{E} = \sum_j u_j(r) E_{0j}(a_j + a_j^\dagger) \tag{4.14}$$

where $E_{0j}$ has the units of electric field and other quantities are dimensionless.

## Optical Cavities

In free space, light propagates at speed $c$, however, it is possible to trap light inside a set of mirrors, called a **cavity**. The quantum properties of light are more easily probed and, as we shall see below, interactions with atoms can be enhanced or suppressed. In particular, it is possible to cause the atom to interact with a single mode alone. This single mode can be studied as an ideal quantum oscillator, and such interactions have been an important experimental testing ground for the predictions of quantum mechanics.

Below we shall consider the interaction between an atom and a single cavity mode. We shall write the eigenstates of the mode $|n\rangle$ and the mode operators $a$ and $a^\dagger$. The Hamiltonian for the mode is written

$$H = \hbar\omega(a^\dagger a + 1/2) \tag{4.15}$$

[1] In some cases (e.g. in free space) we would have an integral rather than a sum, but, apart from some technical subtleties, the interpretation is the same.

and the electric field operator is

$$\hat{E} = u(r)E_0(\hat{a} + \hat{a}^\dagger) \tag{4.16}$$

## 4.3 Interaction between light and an atom

The full theory of the interaction between light and matter would go beyond the scope of this course. However, in the energy scales of quantum optics experiments, this interaction can be very well approximated by the **dipole approximation**. This provides us with a Hamiltonian (known as the Jaynes-Cummings Hamiltonian) which can be solved to study the interaction of atoms and light. We shall derive this Hamiltonian in this chapter.

The starting point for the dipole approximation is to picture the atom as a dipole, consisting of an outer electron of charge $-e$ and the remainder of the atom, charge $+e$. A classical dipole has dipole moment $\vec{D} = q\vec{r}$ where $\vec{r}$ is the vector describing the displacement between positive and negative charges. A classical dipole in an electric field has interaction energy

$$U = -\vec{D} \cdot \vec{E} \tag{4.17}$$

The Jaynes-Cummings interaction is derived from the quantised version of this interaction energy, which gives an additional term which we add to our Hamiltonian (recall that the Hamiltonian represents the energies of the systems under study).

We have already encountered a quantum mechanical electric field operator above, so to derive this we require a quantum operator for the dipole moment.

### The two-level atom

To simplify our task, it is common to make a further approximation. Atoms typically have a complicated level structure, and you will recall from past courses how even the level structure of a Hydrogen atom is relatively complex. Here we shall, instead, take a toy model of an atom, the two-level atom, and assume that the atom's states consist only of the ground state $|g\rangle$ and a single excited state $|e\rangle$. The energy of the ground state will be zero, and the energy of $|e\rangle$ will be $\hbar\omega_a$. The Hamiltonian for this atom is therefore

$$H = 0|g\rangle\langle g| + \hbar\omega_a|e\rangle\langle e| = \hbar\omega_a|e\rangle\langle e| \tag{4.18}$$

Although very simple, this model is a good approximation for an atom that is interacting with a light mode at resonance, e.g. when the atomic transition energy $\hbar\omega_a$ is equal to the energy of a photon in the mode $\hbar\omega_j$, i.e. when $\omega_a = \omega_j$, but all transitions to other states are off resonance.

For a two-level atom it is straight-forward to derive a general form for the dipole moment operator. The dipole operator $\hat{D}$ for a two-level system can only have 4 possible matrix elements and we analyse those in turn.

The dipole moment operator is proportional to the displacement operator $\hat{r}$ for the displacement between electron and positive atomic core. The diagonal elements of $\hat{D}$ are thus proportional to expectations of $\hat{r}$,

$$\begin{aligned} \langle g|\hat{D}|g\rangle &= -e\langle g|\hat{r}|g\rangle \\ \langle e|\hat{D}|e\rangle &= -e\langle e|\hat{r}|e\rangle \end{aligned} \tag{4.19}$$

For most atomic states these elements are zero since the electron wave-function is symmetric about the nucleus (and thus the mean displacement is zero). We will therefore assume that $\langle g|\hat{D}|g\rangle = \langle e|\hat{D}|e\rangle = 0$.

This leaves the off-diagonal elements $\langle g|\hat{D}|e\rangle = \langle e|\hat{D}|g\rangle^*$. We shall leave these general, except to assume that they are real (this assumption does not change the physics and simplifies our analysis). The dipole moment is a 3-D vector in space, and thus the matrix elements have three components. We thus have

$$\langle g|\hat{D}|e\rangle = \langle e|\hat{D}|g\rangle = \vec{d} \tag{4.20}$$

where $\vec{d}$ is a real 3-D vector, which depends upon the atomic states in question.

Thus we obtain a simple form for $\hat{D}$:

$$\hat{D} = \vec{d}(|g\rangle\langle e| + |e\rangle\langle g|). \tag{4.21}$$

## Dipole interaction Hamiltonian

We can now write down a Hamiltonian to describe the interaction between atom and light in the dipole approximation. The classical expression for the interaction energy between dipole and field was $U = -\vec{E}\cdot\vec{D}$. To obtain the quantised Hamiltonian we simply replace each of these quantities with the quantum operators in equation (4.16) and (4.21) to achieve

$$\begin{aligned} H &= (-\vec{d}\cdot u(r)E_0)(\hat{a} + \hat{a}^\dagger)(|g\rangle\langle e| + |e\rangle\langle g|) \\ &= \hbar g(r)(\hat{a} + \hat{a}^\dagger)(|g\rangle\langle e| + |e\rangle\langle g|) \end{aligned} \tag{4.22}$$

where $\hbar g(r) = (-\vec{d}\cdot u(r)E_0)$ represents the strength of the coupling between light and atom.

### Suppressed tensor product notation

In constructing the above operator, we have implicitly combined the state space of the atom with the state space of the light. The joint space has a tensor product structure, with basis states

$$|g,n\rangle = |g\rangle \otimes |n\rangle \qquad |e,n\rangle = |e\rangle \otimes |n\rangle \tag{4.23}$$

However, to save space, and since notationally atom states and operators look very different to light states and operators, it is conventional not to write the tensor products.

When we write operators $\hat{a}$ and $\hat{a}^\dagger$, we are abbreviating $\mathbb{1} \otimes \hat{a}$ and $\mathbb{1} \otimes \hat{a}^\dagger$ respectively. Likewise we write $|g\rangle\langle e|$ for $|g\rangle\langle e| \otimes \mathbb{1}$ etc.

These operators thus act on the joint states as in the following examples:

$$\begin{aligned} \hat{a}|g,n\rangle &= \sqrt{n}|g,n-1\rangle \qquad & \hat{a}^\dagger|g,n\rangle &= \sqrt{n+1}|g,n+1\rangle \\ |g\rangle\langle e||e,n\rangle &= |g,n\rangle \qquad & |g\rangle\langle e||g,n\rangle &= 0 \end{aligned} \tag{4.24}$$

We are now able to write down the full Hamiltonian for the atom and light mode, including their interaction:

$$H = \hbar\omega_j(\hat{a}^\dagger\hat{a} + 1/2) + \hbar\omega_a|e\rangle\langle e| + \hbar g(r)(\hat{a} + \hat{a}^\dagger)(|g\rangle\langle e| + |e\rangle\langle g|) \tag{4.25}$$

The dynamics of this system can be solved in the Heisenberg or Schrödinger pictures, however, as we shall see in the next sections, this can be simplified significantly by adopting the Interaction picture.

## 4.4 Interaction Picture

If we know the eigenstates, $|\phi_j\rangle$, of a constant Hamiltonian $H$, we can compute the evolution of a system via the following formula

$$|\psi(t)\rangle = \sum_j \alpha_j e^{-iE_j t/\hbar}|\phi_j\rangle \tag{4.26}$$

where the constants $\alpha_j$ are determined by the initial conditions.

In many examples in quantum physics, in particular when we are studying the interactions between systems, the Hamiltonian will be of the form

$$H = H_0 + V \tag{4.27}$$

where $H_0$ is a solved Hamiltonian (i.e. whose eigenstates and energies are known). Typically $H_0$ will represent the Hamiltonian for the system if it were not interacting, and $V$ will represent extra terms introduced due to the interaction. We can see that equation (4.25) is of this form, with

$$H_0 = \hbar\omega_j(\hat{a}^\dagger\hat{a} + 1/2) + \hbar\omega_a|e\rangle\langle e| \tag{4.28}$$

$$V = \hbar g(r)(\hat{a} + \hat{a}^\dagger)(|g\rangle\langle e| + |e\rangle\langle g|) \tag{4.29}$$

The eigenstates of $H_0$ are known ($|g, n\rangle$ and $|e, n\rangle$), and if there were no interaction, we could write down the evolution, without further calculation, via equation (4.26). The **Interaction picture** (IP) allows us to exploit the fact that part of the Hamiltonian is solved to simplify our calculations. In the Heisenberg picture, all dynamics are carried by evolution of observables, and in the Schrödinger picture, all dynamics are carried by the state. In the Interaction picture, we map the dynamics associated with the solved part of the Hamiltonian ($H_0$) onto operators, and thus the evolution of the states reflects the interaction alone. This approach usually simplifies the calculations, and is analogous to working in an appropriate rotating frame when studying classical mechanics problems.

Our starting point is to define the evolution operator $U_0(t)$

$$U_0(t) = \exp[-iH_0t/\hbar]. \tag{4.30}$$

This is sometimes called the free-evolution operator, and represents the dynamics we would see if no interaction were present.

Let $|\psi\rangle_S(t)$ represent the time-evolving state in the Schrödinger picture. In the interaction picture, we "remove" from this state the dynamics due to $H_0$, by defining the interaction picture state $|\psi\rangle_I(t)$ as follows:

$$|\psi\rangle_I(t) = U_0(t)^\dagger|\psi\rangle_S(t) \tag{4.31}$$

Since $U_0(t)^\dagger$ is the inverse of $U_0(t)$, this is equivalent to "unwinding" the evolution in the state just due to $H_0$ alone. It is usually easier to calculate $|\psi\rangle_I(t)$ (we shall see below how to do this), and then apply $U_0(t)$ to recover the Schrödinger picture.

Expectation values must be the same in all pictures, which mean that observables $\hat{O}_I(t)$ in the interaction picture must evolve as follows

$$\hat{O}_I(t) = U_0^\dagger(t)\hat{O}_S U_0(t) \tag{4.32}$$

where $\hat{O}_S$ is the Schrödinger picture observable. You can see that in the Interaction picture, similar to the Heisenberg picture, observables evolve in time, however, they evolve under $U_0(t)$ rather than the full evolution operator $U(t)$. The following table summarises the similarities and differences between all three pictures.

| Picture | Observables | States |
|---|---|---|
| Schrödinger | $\hat{O}_S$ | $\|\psi(t)\rangle_S = U(t)\|\psi(0)\rangle_S$ |
| Heisenberg | $\hat{O}_H = U^\dagger(t)\hat{O}_S U(t)$ | $\|\psi\rangle_H = \|\psi(0)\rangle_S$ |
| Interaction | $\hat{O}_I = U_0^\dagger(t)\hat{O}_S U_0(t)$ | $\|\psi(t)\rangle_I = U_0^\dagger(t)\|\psi(t)\rangle_S$ |

All three pictures coincide at $t = 0$.

As an indication of why the Interaction picture will simplify calculations, consider the following example. Let $|\phi_k\rangle$ represent an orthonormal basis. We can therefore represent an Interaction picture state as

$$|\psi(t)\rangle_I = \sum_j c_j(t)|\phi_k\rangle \tag{4.33}$$

Transforming back to the Schrödinger picture, the state will be

$$|\psi(t)\rangle_S = U_0(t)|\psi(t)\rangle_I = \sum_j c_j(t)\exp[-iE_j t/\hbar]|\phi_k\rangle \tag{4.34}$$

From this we see that the interaction picture focuses on the evolution of coefficients $c_j(t)$, ignoring the phases $\exp[-iE_j t/\hbar]$ which are due to the non-interaction part of the Hamiltonian.

## Interaction Picture Schrödinger picture

The main advantage of the Interaction picture is that it simplifies the form of the Schrödinger equation. To derive the Schrödinger equation in the Interaction picture let us start with the standard Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}|\psi_S(t)\rangle = H|\psi_S(t)\rangle \tag{4.35}$$

and substitute in the defining identity $|\psi(t)\rangle_S = U_0(t)|\psi(t)\rangle_I$. We then use the product rule to transform the left-hand-side of the above equation:

$$\begin{aligned} i\hbar\frac{\partial}{\partial t}\left(U_0(t)|\psi(t)\rangle_I\right) &= i\hbar\left(U_0(t)\frac{\partial|\psi(t)\rangle_I}{\partial t} + \frac{\partial U_0(t)}{\partial t}|\psi(t)\rangle_I\right) \\ &= i\hbar\left(U_0(t)\frac{\partial|\psi(t)\rangle_I}{\partial t} + \frac{-iH_0}{\hbar}U_0(t)|\psi(t)\rangle_I\right) \end{aligned} \tag{4.36}$$

Hence,

$$i\hbar U_0(t)\frac{\partial|\psi(t)\rangle_I}{\partial t} + H_0U_0(t)|\psi(t)\rangle_I = HU_0(t)|\psi(t)\rangle_I \tag{4.37}$$

and multiplying from the left by $U_0^\dagger(t)$ and rearranging, we obtain

$$i\hbar\frac{\partial|\psi(t)\rangle_I}{\partial t} = U_0^\dagger(t)(H - H_0)U_0(t)|\psi(t)\rangle_I = U_0^\dagger(t)VU_0(t)|\psi(t)\rangle_I \tag{4.38}$$

This has a very similar form to the standard Schrödinger equation, and indeed if we define an Interaction Picture Hamiltonian $H_I = U_0^\dagger(t)VU_0(t)$, we have derived the Interaction Picture Schrödinger equation,

$$i\hbar\frac{\partial|\psi(t)\rangle_I}{\partial t} = H_I|\psi(t)\rangle_I \tag{4.39}$$

This has exactly the same form as the standard SE and we can use the same techniques to solve it. The key difference is that the Interaction Picture Hamiltonian has few terms in it, simplifying the calculations.

## Example: Atom-cavity coupling

In the past section, we derived a Hamiltonian for the interaction between an atom and a single mode of light (e.g. inside an optical cavity). This Hamiltonian was of the form $H = H_0 + V$ with

$$H_0 = \hbar\omega_j(\hat{a}^\dagger\hat{a} + 1/2) + \hbar\omega_a|e\rangle\langle e| \tag{4.40}$$

and

$$V = \hbar g(r)(\hat{a} + \hat{a}^\dagger)(|g\rangle\langle e| + |e\rangle\langle g|) \tag{4.41}$$

We will study this evolution in the interaction picture. To do so we need to compute the Interaction Picture Hamiltonian $H_I = U_0^\dagger(t)VU_0(t)$. This seems

daunting at first sight, but we can break the calculation down. The first observation is that $\hat{a}^\dagger\hat{a}$ and $|e\rangle\langle e|$ commute, so we can write

$$U_0(t) = \exp[-i\omega_j(\hat{a}^\dagger\hat{a} + 1/2)t] \exp[-i\omega_a|e\rangle\langle e|t] \tag{4.42}$$

Since the mode operators $\hat{a}$ and $\hat{a}^\dagger$ commute with the atomic operators $|g\rangle\langle e|$, etc., we can group the two different types of operator in $H_I$ together

$$\begin{aligned} H_I &= U_0^\dagger(t) V U_0(t) \\ &= \hbar g(r) \exp[+i\omega_j(\hat{a}^\dagger\hat{a} + 1/2)t](\hat{a} + \hat{a}^\dagger) \exp[-i\omega_j(\hat{a}^\dagger\hat{a} + 1/2)t] \\ &\quad \exp[+i\omega_a|e\rangle\langle e|t](|g\rangle\langle e| + |e\rangle\langle g|) \exp[-i\omega_a|e\rangle\langle e|t] \end{aligned} \tag{4.43}$$

We then proceed term by term

$$\begin{aligned} &\exp[+i\omega_j(\hat{a}^\dagger\hat{a} + 1/2)t]\hat{a} \exp[-i\omega_j(\hat{a}^\dagger\hat{a} + 1/2)t] \\ &\quad = \exp[+i\omega_j\hat{a}^\dagger\hat{a}t]\hat{a} \exp[-i\omega_j\hat{a}^\dagger\hat{a}t] \end{aligned} \tag{4.44}$$

We can simplify this by using

$$\exp[+i\omega_j\hat{a}^\dagger\hat{a}t]|n\rangle = \exp[+i\omega_j nt]|n\rangle \tag{4.45}$$

which holds because $|n\rangle$ is an eigenstate of $\hat{a}^\dagger\hat{a}$ with eigenvalue $n$, and can be verified using the power series for the exponential, and also

$$\hat{a} = \sum_n \sqrt{n+1}|n\rangle\langle n+1| \tag{4.46}$$

to derive

$$\begin{aligned} &\exp[+i\omega_j\hat{a}^\dagger\hat{a}t]\hat{a} \exp[-i\omega_j\hat{a}^\dagger\hat{a}t] \\ &\quad = \sum_n \sqrt{n+1} \exp[+i\omega_j nt]|n\rangle\langle n+1| \exp[-i\omega_j(n+1)t] \\ &\quad = \exp[-i\omega_j t] \sum_n \sqrt{n+1}|n\rangle\langle n+1| = \exp[-i\omega_j t]\hat{a} \end{aligned} \tag{4.47}$$

and taking the Hermitian conjugate of this we obtain:

$$\exp[+i\omega_j\hat{a}^\dagger\hat{a}t]\hat{a}^\dagger \exp[-i\omega_j\hat{a}^\dagger\hat{a}t] = \exp[+i\omega_j t]\hat{a}^\dagger. \tag{4.48}$$

Hence

$$\exp[+i\omega_j(\hat{a}^\dagger\hat{a}+1/2)t](\hat{a}+\hat{a}^\dagger) \exp[-i\omega_j(\hat{a}^\dagger\hat{a}+1/2)t] = \exp[-i\omega_j t]\hat{a}+\exp[+i\omega_j t]\hat{a}^\dagger \tag{4.49}$$

Similarly, we can show that (this is left as an exercise)

$$\exp[+i\omega_a|e\rangle\langle e|t](|g\rangle\langle e|+|e\rangle\langle g|)\exp[-i\omega_a|e\rangle\langle e|t] = \exp[-i\omega_a t]|g\rangle\langle e|+\exp[+i\omega_a t]|e\rangle\langle g| \tag{4.50}$$

Thus, the IP Hamiltonian is

$$\begin{aligned} H_I &= \hbar g(r)(\exp[-i\omega_j t]\hat{a} + \exp[+i\omega_j t]\hat{a}^\dagger)(\exp[-i\omega_a t]|g\rangle\langle e| + \exp[+i\omega_a t]|e\rangle\langle g|) \\ &= \hbar g(r)\big(\exp[-i(\omega_j-\omega_a)t]\hat{a}|e\rangle\langle g| + \exp[i(\omega_j-\omega_a)t]\hat{a}^\dagger|g\rangle\langle e| \\ &\quad + \exp[-i(\omega_j+\omega_a)t]\hat{a}|g\rangle\langle e| + \exp[i(\omega_j+\omega_a)t]\hat{a}^\dagger|e\rangle\langle g|\big) \end{aligned} \tag{4.51}$$

We mentioned above that an atom and cavity mode have a particularly strong interaction when they are at resonance, i.e. when the transition energy $\hbar\omega_a$ is close to photon energy $\hbar\omega_j$. Here we shall assume exact resonance $\omega_a = \omega_j = \omega$ where we shall use $\omega$ to stand for either quantity.

At resonance, the interaction picture Hamiltonian is

$$\begin{aligned} H_I = \hbar g(r)\big(&\hat{a}|e\rangle\langle g| + \hat{a}^\dagger|g\rangle\langle e| \\ &+ \exp[-i2\omega t]\hat{a}|g\rangle\langle e| + \exp[i2\omega t]\hat{a}^\dagger|e\rangle\langle g|\big) \end{aligned} \tag{4.52}$$

## Rotating wave approximation

In equation (4.52) we see that 2 of the terms are time-independent, while 2 terms are oscillating at angular frequency $2\omega$. In the Rotating Wave Approximation, we neglect those fast oscillating terms. This is justified when the period of oscillation is much shorter than the time-scales of the most significant physical processes. Loosely speaking, the fast oscillating terms have little effect on the time-evolution at longer time-scales since they average to zero. The RWA can be shown to be a very good approximation in cavity QED experiments. Making the RWA, and dropping the final two terms we obtain a very simple form for $H_I$

$$H_I = \hbar g\big(\hat{a}|e\rangle\langle g| + \hat{a}^\dagger|g\rangle\langle e|\big) \tag{4.53}$$

A good way to remember which terms are retained in the RWA is to note that the terms we kept correspond to an energy conserving process (absorbtion of a photon + excitation of atom, or emission of photon plus deexcitation of the atom), whereas the terms we dropped describe a non-energy conserving process (photon absorbed plus atom excited, etc.).

## Example: Single photon in a cavity

We shall now solve the IP Schrödinger equation for the case where initially, there is a single photon in the cavity and the atom is in its ground state, e.g. $|\psi(0)\rangle_I = |g,1\rangle$.

Notice that

$$H_I|g,1\rangle = \hbar g|e,0\rangle \quad H_I|e,0\rangle = \hbar g|g,1\rangle \tag{4.54}$$

This means that $H_I$ only couples $|e,0\rangle$ and $|g,1\rangle$, which means that the general solution to the IP Schrödinger equation with this initial condition will have the form

$$|\psi(t)\rangle_I = \alpha(t)|g,1\rangle + \beta(t)|e,0\rangle \tag{4.55}$$

To solve it, we need to compute $\alpha(t)$ and $\beta(t)$. Substituting equation (4.55) into the IP Schrödinger equation, and multiplying from the left with $\langle g,1|$ or $\langle e,0|$ respectively we obtain two coupled differential equations

$$\begin{aligned} \frac{\partial\alpha(t)}{\partial t} &= -ig\beta(t) \\ \frac{\partial\beta(t)}{\partial t} &= -ig\alpha(t) \end{aligned} \tag{4.56}$$

Applying the boundary conditions $\alpha(0) = 1$ and $\beta(0) = 0$ we can solve these by standard methods to obtain the solution

$$\alpha(t) = \cos(gt) \qquad \beta(t) = -i\sin(gt) \tag{4.57}$$

and hence

$$|\psi(t)\rangle_I = \cos(gt)|g,1\rangle - i\sin(gt)|e,0\rangle. \tag{4.58}$$

We see that the system oscillates back and forward between these two states. This phenomenon is a signature of a strong atom cavity coupling and is called **Rabi oscillation**.

To transform the state back to the Schrödinger picture, we apply the operator $U_0(t)$, which adds the phases $\exp[-iE_jt/\hbar]$ to each term, where $E_j$ is the energy of that eigenstate. Both states above have the same energy $(3/2)\hbar\omega$, and hence the Schrödinger depiction of the state is

$$|\psi(t)\rangle_S = U_0(t)|\psi(t)\rangle_I = \exp[-i3\omega t/2](\cos(gt)|g,1\rangle - i\sin(gt)|e,0\rangle). \tag{4.59}$$

Notice that the interaction picture contained all the important dynamics of the interaction, and transforming back has just added (less important) phases to the evolution.

Rabi oscillation occurs when the cavity is prepared in any Fock state $|n\rangle$, and via similar methods to those used here one can show that a similar oscillation occurs when the system is prepared in $|g, n\rangle$. Indeed we find

$$|\psi(t)\rangle_I = \cos(\sqrt{n}gt)|g, n\rangle - i\sin(\sqrt{n}gt)|e, n-1\rangle. \tag{4.60}$$