Module 3

The Dirac Equation

Lecture Notes

2017-2018

PHASM(G)-442

Probability density, current and continuity equation in non relativistic QM from Schrodinger equation.

$$E = T + V = \frac{\vec{p}^{\,2}}{2m} + V$$

RECALL:

$$\vec{p} \rightarrow -i\vec{\nabla}, \quad E \rightarrow i\frac{\partial}{\partial t}$$

Assume $V = 0$

$$-\frac{1}{2m}\vec{\nabla}^2\psi = i\frac{\partial \psi}{\partial t} \qquad (S1)$$

with plane wave solutions

$$\psi = N e^{i(\vec{p}\cdot\vec{r} - Et)}$$

$$\begin{cases} -i\nabla\psi = \vec{p}\psi \\ i\frac{\partial\psi}{\partial t} = E\psi \end{cases}$$

Derive prob. density / current

$$(S1)^* \Rightarrow -\frac{1}{2m}\vec{\nabla}^2\psi^* = -i\frac{\partial\psi^*}{\partial t} \qquad (S2)$$

$$\psi^* \times (S1) - \psi \times (S2)$$

$$\Downarrow$$

$$-\frac{1}{2m}\left(\psi^*\nabla^2\psi - \psi\nabla^2\psi^*\right) = i\left(\psi^*\frac{\partial\psi}{\partial t} + \psi\frac{\partial\psi^*}{\partial t}\right)$$

OR

$$-\frac{1}{2m}\vec{\nabla}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right) = i\frac{\partial}{\partial t}\left(\psi^*\psi\right)$$

to be compared with $\boxed{\vec{\nabla}.\vec{j} + \frac{\partial\rho}{\partial t} = 0}$ RECALL EM!

Continuity Equation

Leading to

$$\rho = \psi\psi^* = |\psi|^2 \qquad \vec{j} = \frac{1}{2mi}\left(\psi^*\vec{\nabla}\psi - \psi\vec{\nabla}\psi^*\right)$$

FOR a plane wave $\psi = N e^{i(\vec{p}\vec{r} - Et)}$

$$\rho = |N|^2 \qquad \underbrace{\vec{j} = |N|^2\frac{\vec{p}}{m} = |N|^2\vec{v}}_{\text{flux}}$$

Probability density, current and continuity equation in relativistic QM from Klein-Gordon equation.

$E^2 = |\vec{p}|^2 + m^2$ gives K-G:

$$\frac{\partial^2 \psi}{\partial t^2} = \vec{\nabla}^2 \psi - m^2 \psi \qquad (KG1)$$

RECALL: $\partial^\mu \partial_\mu = \frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$

OR

$$(\partial^\mu \partial_\mu + m^2)\psi = 0 \longleftrightarrow \text{K-G. equation}$$

FOR $\psi = N e^{i(\vec{p}\cdot\vec{r} - Et)}$ $\quad -E^2\psi = -|\vec{p}|^2\psi - m^2\psi$

$$E = \pm\sqrt{|\vec{p}|^2 + m^2} \longleftarrow \text{negative energy solutions}$$

SAME Recepy for prob. and current densities

$$\frac{\partial^2 \psi^*}{\partial t^2} = \vec{\nabla}^2 \psi^* - m^2 \psi^* \qquad (KG2)$$

$$\psi^* (KG1) - \psi\, (KG2)$$

$$\psi^* \frac{\partial^2 \psi}{\partial t^2} - \psi \frac{\partial^2 \psi^*}{\partial t^2} = \psi^*(\vec{\nabla}^2 \psi - m^2 \psi) - \psi(\vec{\nabla}^2 \psi^* - m^2 \psi^*)$$

$$\frac{\partial}{\partial t}\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right) = \vec{\nabla}\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)$$

Recall continuity equation

$$\rho = i\left(\psi^* \frac{\partial \psi}{\partial t} - \psi \frac{\partial \psi^*}{\partial t}\right)$$

$$\vec{j} = -i\left(\psi^* \vec{\nabla}\psi - \psi \vec{\nabla}\psi^*\right)$$

For $\psi = N e^{i(\vec{p}\vec{r} - Et)}$

$$\rho = 2E|N|^2 \qquad \vec{j} = |N|^2 \vec{p}$$

negative particle densities!

The Dirac Equation :

Negative particle densities come from $\underline{2^{nd}}$ order terms:

$$-\frac{1}{2}\vec{\nabla}^2\psi = i\frac{\partial\psi}{\partial t} \qquad (Sch)$$

$$(\underline{\partial^\mu\partial_\mu} + m^2)\psi = 0 \qquad (K-G)$$

Dirac looked for alternative 1st order e ation

$$\hat{H}\psi = (\vec{\alpha}\cdot\hat{\vec{p}} + \beta m)\psi = i\frac{\partial\psi}{\partial t}$$

As usual $\hat{\vec{p}} = -i\vec{\nabla}$

$$\left(-i\alpha_x\frac{\partial}{\partial x} - i\alpha_y\frac{\partial}{\partial y} - i\alpha_z\frac{\partial}{\partial z} + \beta m\right) = \left(i\frac{\partial}{\partial t}\right)\psi$$

$$(D1)$$

"Squaring" (D1)

$$\left(-i\alpha_x\frac{\partial}{\partial x}-i\alpha_y\frac{\partial}{\partial y}-i\alpha_z\frac{\partial}{\partial z}+\beta m\right)\left(-i\alpha_x\frac{\partial}{\partial x}-i\alpha_y\frac{\partial}{\partial y}-i\alpha_z\frac{\partial}{\partial z}+\beta m\right)\psi=-\frac{\partial^2\psi}{\partial t^2}$$

and expanding

$$\begin{aligned}
-\frac{\partial^2\psi}{\partial t^2} &= -\alpha_x^2\frac{\partial^2\psi}{\partial x^2}-\alpha_y^2\frac{\partial^2\psi}{\partial y^2}-\alpha_z^2\frac{\partial^2\psi}{\partial z^2}+\beta^2m^2\psi \\
&\quad-(\alpha_x\alpha_y+\alpha_y\alpha_x)\frac{\partial^2\psi}{\partial x\partial y}-(\alpha_y\alpha_z+\alpha_z\alpha_y)\frac{\partial^2\psi}{\partial y\partial z}-(\alpha_z\alpha_x+\alpha_x\alpha_z)\frac{\partial^2\psi}{\partial z\partial x} \\
&\quad-(\alpha_x\beta+\beta\alpha_x)m\frac{\partial\psi}{\partial x}-(\alpha_y\beta+\beta\alpha_y)m\frac{\partial\psi}{\partial y}-(\alpha_z\beta+\beta\alpha_z)m\frac{\partial\psi}{\partial z}
\end{aligned}$$

still must satisfy K-G

$$-\frac{\partial^2\psi}{\partial t^2}=-\frac{\partial^2\psi}{\partial x^2}-\frac{\partial^2\psi}{\partial y^2}-\frac{\partial^2\psi}{\partial z^2}+m^2\psi$$

$$\alpha_x^2=\alpha_y^2\overset{!!}{=}\alpha_z^2=\beta^2=1$$

$$\alpha_j\beta+\beta\alpha_j=0$$

$$\alpha_j\alpha_k+\alpha_k\alpha_j=0 \quad \text{if } j\neq k$$

α_j, β can <u>not</u> be numbers

4x4 anti-commuting matrices (minimal)

Thus ψ must be four-component

Dirac Spinor

$\hat{H}$ must be hermitian

Hence:

$\alpha_x = \alpha_x^\dagger \quad \alpha_y = \alpha_y^\dagger \quad \alpha_z = \alpha_z^\dagger \quad \beta = \beta^\dagger$

Explicit representation based on Pauli matrices

$$\beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \quad \alpha_j = \begin{pmatrix} 0 & \sigma_j \\ \sigma_j & 0 \end{pmatrix}$$

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Probability Density and Current with Dirac Equation

$(D1)^\dagger$:

$$+i\frac{\partial\psi^\dagger}{\partial x}\alpha_x^\dagger + i\frac{\partial\psi^\dagger}{\partial y}\alpha_y^\dagger + i\frac{\partial\psi^\dagger}{\partial z}\alpha_z^\dagger +$$

$(D2)$:

$$+m\psi^\dagger\beta^\dagger = -i\frac{\partial\psi^\dagger}{\partial t}$$

As usual: $\psi^\dagger \times (D1) - (D2)\times\psi$

$$\psi^\dagger\left(-i\alpha_x\frac{\partial\psi}{\partial x} - i\alpha_y\frac{\partial\psi}{\partial y} - i\alpha_z\frac{\partial\psi}{\partial z} + \beta m\psi\right) - \left(i\frac{\partial\psi^\dagger}{\partial x}\alpha_x + i\frac{\partial\psi^\dagger}{\partial y}\alpha_y + i\frac{\partial\psi^\dagger}{\partial z}\alpha_z + m\psi^\dagger\beta\right)\psi = i\psi^\dagger\frac{\partial\psi}{\partial t} + i\frac{\partial\psi^\dagger}{\partial t}\psi$$

$\times i$ and

$$\underbrace{\psi^\dagger\left(\alpha_x\frac{\partial\psi}{\partial x}\right.}+\alpha_y\frac{\partial\psi}{\partial y} + \alpha_z\frac{\partial\psi}{\partial z}\Bigg) + \Bigg(\underbrace{\frac{\partial\psi^\dagger}{\partial x}\alpha_x}+\frac{\partial\psi^\dagger}{\partial y}\alpha_y + \frac{\partial\psi^\dagger}{\partial z}\alpha_z\Bigg)\psi + \frac{\partial(\psi^\dagger\psi)}{\partial t} = 0$$

$$\frac{\partial(\psi^\dagger\alpha_x\psi)}{\partial x} = \psi^\dagger\alpha_x\frac{\partial\psi}{\partial x} + \frac{\partial\psi^\dagger}{\partial x}\alpha_x\psi$$

$$\frac{\partial(\psi^\dagger \alpha_x \psi)}{\partial x} + \frac{\partial(\psi^\dagger \alpha_y \psi)}{\partial y} + \frac{\partial(\psi^\dagger \alpha_z \psi)}{\partial z}$$

$$+ \frac{\partial(\psi^\dagger \psi)}{\partial t} = 0$$

$$\vec{\nabla} \underbrace{(\psi^\dagger \vec{\alpha} \psi)}_{\vec{j}} + \frac{\partial \overbrace{(\psi^\dagger \psi)}^{\rho}}{\partial t} = 0$$

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix} \qquad \psi^\dagger = (\psi_1^*, \psi_2^*, \psi_3^*, \psi_4^*)$$

$$\rho = \psi^\dagger \psi = \overbrace{|\psi_1|^2}^{\psi_1 \psi_1^*} + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2$$

$$\boxed{\rho > 0}$$

Extra degrees of freedom in ψ from Dirac Equation:

fermion + anti-fermions with intrinsic angular momentum

$$\boxed{\vec{\mu} = \frac{q}{m}\vec{S}}$$

SPIN-$\frac{1}{2}$!

Dirac Equation in Covariant Notation.
Dirac gamma matrices.

$$\gamma^0 \equiv \beta; \;\; \gamma^1 \equiv \beta\alpha_x; \;\; \gamma^2 \equiv \beta\alpha_y; \;\; \gamma^3 \equiv \beta\alpha_z$$

Take (D 1)

$$\left(-i\alpha_x \frac{\partial}{\partial x} - i\alpha_y \frac{\partial}{\partial y} - i\alpha_z \frac{\partial}{\partial z} + \beta m\right)\psi = \left(i\frac{\partial}{\partial t}\right)\psi$$

$\times(-\beta)$

$$\underbrace{i\beta\alpha_x}_{\gamma^1} \frac{\partial\psi}{\partial x} + \underbrace{i\beta\alpha_y}_{\gamma^2} \frac{\partial\psi}{\partial y} + \underbrace{i\beta\alpha_z}_{\gamma^3} \frac{\partial\psi}{\partial z} - \underbrace{\beta^2}_{(\gamma^0)^2=1} m\psi = -i\beta\frac{\partial\psi}{\partial t}$$

$$i\gamma^1\frac{\partial\psi}{\partial x} + i\gamma^2\frac{\partial\psi}{\partial y} + i\gamma^3\frac{\partial\psi}{\partial z} - m\psi = -i\gamma^0\frac{\partial\psi}{\partial t}$$

use $\partial_\mu \equiv \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$$\boxed{(i\gamma^\mu\partial_\mu - m)\psi = 0}$$

$\boxed{\text{Properties of } \gamma\text{-matrices}}$

Can be summarised:

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}$$

Fully defines the algebra

$(\gamma^0)^2 - ?$

$$\gamma^0\gamma^0 + \gamma^0\gamma^0 = 2g^{00} = 2$$

$$2 \equiv 2 \times I$$

$(\gamma^k)^2$, $k \neq 0$

common notation

$\mu(\nu) = 0, 1, 2, 3$

$k = 1, 2, 3$

$$\gamma^k\gamma^k + \gamma^k\gamma^k = 2g^{kk} = -2$$

$$(\gamma^k)^2 = -1 \qquad \gamma^k\gamma^j = -\gamma^j\gamma^k$$

$$\boxed{\gamma^{0\dagger} = \gamma^0,\ \ \gamma^{1\dagger} = -\gamma^1,\ \ \gamma^{2\dagger} = -\gamma^2,\ \ \gamma^{3\dagger} = -\gamma^3}$$

check!

Pauli-Dirac representation. Adjoint spinor

$$\gamma^0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix};\ \gamma^1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix};\ \gamma^2 = \begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix};\ \gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\gamma^0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} \qquad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix}$$

$$\rho = \psi^\dagger \psi \qquad \vec{j} = \psi^\dagger \vec{\alpha}\, \psi$$

$$j_x = \psi^\dagger \alpha_x \psi = \psi^\dagger \gamma^0 \gamma^1 \psi$$

$$j_y = \psi^\dagger \gamma^0 \gamma^2 \psi \qquad \gamma^0\gamma^1 = \alpha_x$$

$$j_z = \psi^\dagger \gamma^0 \gamma^3 \psi \qquad \gamma^0\gamma^2 = \alpha_y$$

$$\gamma^0\gamma^3 = \alpha_z$$

Can write as

$$j^\mu = (\rho, \vec{j})$$

$$j^0 = \psi^\dagger \underbrace{\gamma^0 \gamma^0}_{1} \psi = \psi^\dagger \psi = \rho$$

$$j^x = \psi^\dagger \gamma^0 \gamma^1 \psi \quad \ldots\ldots \quad j^y, j^z$$

$$\partial_\mu j^\mu = \frac{\partial \rho}{\partial t} + \frac{\partial j_x}{\partial x} + \frac{\partial j_y}{\partial y} + \frac{\partial j_z}{\partial z}$$

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \vec{j} = 0 \qquad \boxed{\partial_\mu j^\mu = 0}$$

$$j^\mu = \underbrace{\psi^\dagger \gamma^0}_{\bar{\psi} = \psi^\dagger \gamma^0} \gamma^\mu \psi$$

$$\boxed{j^\mu = \bar{\psi} \gamma^\mu \psi}$$

Recap

(1) K-G. $(\partial^\mu \partial_\mu + m^2)\psi = 0 \Rightarrow$ -ve particle densities

(2) $\hat{H}_D = \vec{\alpha} \cdot \hat{\vec{p}} + \beta m \Rightarrow$ D.E. $\quad \gamma^0 \equiv \beta$

$(i\gamma^\mu \partial_\mu - m)\psi = 0 \qquad \gamma^0 \vec{\gamma} = \vec{\alpha}$

$\Downarrow$

+ve part. densities $\quad \psi \rightarrow \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \\ \psi_4 \end{pmatrix}$

(3) $j^\mu = \bar{\psi}\gamma^\mu\psi$ — 4-vector current

Spin and Dirac Hamiltonian

$$\hat{H}_{S.E.} = \frac{\hat{\vec{p}}^2}{2m} \quad (V=0) \quad [\hat{H}_{S.E.}, \hat{\vec{L}}] = [\hat{H}_{S.E.}, \hat{\vec{r}} \times \hat{\vec{p}}] = 0$$

$$[\hat{H}_D, \hat{\vec{L}}] =$$

$$= [\vec{\alpha}\cdot\hat{\vec{p}} + \beta m, \hat{\vec{r}} \times \hat{\vec{p}}] = [\vec{\alpha}\cdot\hat{\vec{p}}, \hat{\vec{r}} \times \hat{\vec{p}}] = -i\gamma^0(\vec{\gamma} \times \hat{\vec{p}}) \sim i(\vec{\alpha} \times \hat{\vec{p}})$$

$$\hat{\vec{S}} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \quad [\hat{H}_D, \hat{\vec{S}}] = i\gamma^0(\vec{\gamma} \times \hat{\vec{p}})$$

$$[\hat{H}_D, \hat{\vec{J}}] = 0, \text{ where } \vec{J} = \vec{L} + \vec{S}$$

$\Rightarrow$ $\vec{J}$ is <u>conserved</u>

Solutions to the Dirac Equation. Free particle at rest.

Look for solutions in the form

$$\psi = u(E, \vec{p})\, e^{i(\vec{p}\cdot\vec{r} - Et)}$$

satisfy $\quad (i\gamma^\mu \partial_\mu - m)\psi = 0$

First, let's derive D.E. in "momentum" form

$$\partial_0 \psi = \frac{\partial \psi}{\partial t} = (-iE)\psi \; ; \quad \partial_1 \psi = \frac{\partial \psi}{\partial x} = (ip_x)\psi$$

$$\partial_2 \psi = (ip_y)\psi \qquad \partial_3 \psi = (ip_z)\psi$$

$$i\gamma^0 \partial_0 \psi = i\gamma^0(-iE)\psi = \gamma^0 E \psi$$

$$i\gamma^1 \partial_1 \psi = i\gamma^1(ip_x)\psi = -\gamma^1 p_x \psi$$

$$(\gamma^0 E - \gamma^1 p_x - \gamma^2 p_y - \gamma^3 p_z - m)u = 0$$

$$\boxed{(\gamma^\mu p_\mu - m)u = 0}$$

D.E. in "momentum" form.

FOR particle at rest, i.e. $\vec{p}=0$

$$\psi = u(E,0)\, e^{-iEt}$$

$$E\gamma^0 u - m u = 0$$

OR

$$E\gamma^0 u = m u$$

$$E\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix} = m\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{pmatrix}$$

Four orthogonal solutions

$$\underbrace{u_1(m,0)=\begin{pmatrix}1\\0\\0\\0\end{pmatrix};\quad u_2(m,0)=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}}_{E>0 \quad E=m};\quad \underbrace{u_3(m,0)=\begin{pmatrix}0\\0\\1\\0\end{pmatrix};\quad u_4(m,0)=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}}_{E=-m\,;\quad E<0}$$

$$\underbrace{\psi_1=\begin{pmatrix}1\\0\\0\\0\end{pmatrix}e^{-imt};\quad \psi_2=\begin{pmatrix}0\\1\\0\\0\end{pmatrix}e^{-imt}}_{\text{Two spin states with } E>0};\quad \underbrace{\psi_3=\begin{pmatrix}0\\0\\1\\0\end{pmatrix}e^{+imt};\ \text{and}\ \psi_4=\begin{pmatrix}0\\0\\0\\1\end{pmatrix}e^{+imt}}_{\text{Two spin states with } E<0}$$

General Free Particle Solutions to the Dirac Equation.

$$\psi = u(E, \vec{p})\, e^{i(\vec{p}\cdot\vec{r} - Et)}$$

$$(\gamma^\mu p_\mu - m)u = 0$$

$$\gamma^\mu p_\mu - m = E\gamma^0 - p_x\gamma^1 - p_x\gamma^2 - p_x\gamma^3 - m$$

$$= \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} E - \begin{pmatrix} 0 & \vec{\sigma} \\ -\vec{\sigma} & 0 \end{pmatrix} \vec{p} - m \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} =$$

$$= \begin{pmatrix} (E-m) I & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -(E+m) I \end{pmatrix}$$

$$\begin{pmatrix} (E-m)I & -\vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & (-E-m)I \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad (\gamma^\mu p_\mu - m)u = 0$$

$$\left.\begin{aligned} (\vec{\sigma}\cdot\vec{p})\, u_B &= (E-m)\, u_A \\ (\vec{\sigma}\cdot\vec{p})\, u_A &= (E+m)\, u_B \end{aligned}\right\}$$

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} p_x + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} p_y + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} p_z$$

$$\vec{\sigma}\cdot\vec{p} = \begin{pmatrix} p_z & p_x - i p_y \\ p_x + i p_y & -p_z \end{pmatrix}$$

$$u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\, u_A = \frac{1}{E+m}\begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} u_A$$

$$u_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{OR} \quad u_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\Downarrow \qquad\qquad\qquad\qquad \Downarrow$$

$$u_1 = \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix} \quad \text{OR} \quad u_2 = \begin{pmatrix} 0 \\ 1 \\ \frac{p_x - ip_y}{E+m} \\ -\frac{p_z}{E+m} \end{pmatrix}$$

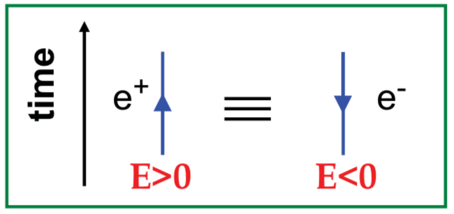
Now $\quad u_B = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad$ OR $\quad u_B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

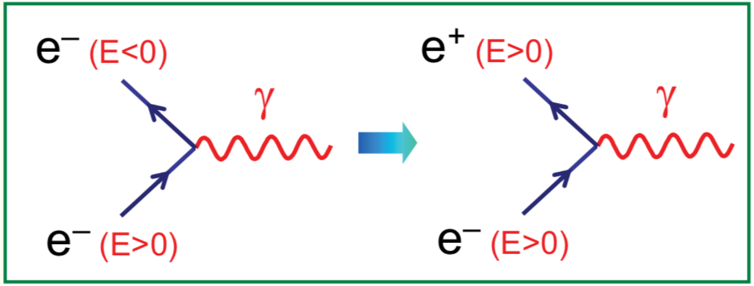
Four solutions $\psi_i = u_i(E,\vec{p})\, e^{i(\vec{p}\vec{r}-Et)}$

$$\underbrace{u_1 = N_1 \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}; \quad u_2 = N_2 \begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}}_{E>0}; \quad \underbrace{u_3 = N_3 \begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix}; \quad u_4 = N_4 \begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}}_{E<0}$$

Interpretation of -ve solutions

Feynman-Stückelberg





$$e^{-iEt} \equiv e^{-i(-E)(-t)}$$

-ve Energy solutions of D.E.

<u>Anti-</u> particles with $E>0$, propagating forward in time

Anti-Particle Solutions

$$\psi = v(E,\vec{p})\, e^{-i(\vec{p}\cdot\vec{r} - Et)} \longrightarrow (i\gamma^\mu \partial_\mu - m)\psi = 0$$

$$\boxed{(\gamma^\mu p_\mu + m)\, v = 0}$$

D.E. for anti-particle.

$$\hat{H}\psi = i\frac{\partial \psi}{\partial t} = -E\psi \qquad \hat{H}^{(v)} = -i\frac{\partial}{\partial t}$$

$$\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}} \Rightarrow \hat{\vec{L}}^{(v)} \longrightarrow -\hat{\vec{L}}^{(u)} \qquad \hat{\vec{P}}^{(v)} = i\vec{\nabla}$$

$$[\hat{H}, \hat{\vec{L}} + \hat{\vec{S}}] = 0 \qquad \hat{\vec{S}}^{(v)} \longrightarrow -\hat{\vec{S}}^{(u)}$$

Particle and Anti-Particle Spinors

$$\psi = u\, e^{i(\vec{p}\cdot\vec{r} - Et)} \quad \text{particle}$$

$$\underbrace{u_1 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix};\quad u_2 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}}_{E>0};\quad \underbrace{\cancel{u_3 = N\begin{pmatrix} \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \\ 1 \\ 0 \end{pmatrix};\quad u_4 = N\begin{pmatrix} \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \\ 0 \\ 1 \end{pmatrix}}}_{E<0}$$

$$\psi = V\, e^{-i(\vec{p}\cdot\vec{r} - Et)} \quad \text{anti-particle}$$

$$\underbrace{v_1 = N\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix};\quad v_2 = N\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}}_{E>0};\quad \underbrace{\cancel{v_3 = N\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E-m} \\ \frac{p_x+ip_y}{E-m} \end{pmatrix};\quad v_4 = N\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E-m} \\ \frac{-p_z}{E-m} \end{pmatrix}}}_{E<0}$$

choice

Wave-Function Normalisation

$$\psi_1 = u_1 e^{i(\vec{p}\cdot\vec{r} - Et)}$$

$$u_1 = N \begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + ip_y}{E+m} \end{pmatrix}$$

$$\rho = \psi^\dagger \psi = 2E$$

$$\rho = u_1^\dagger u = |N|^2 \left(1 + \frac{p_z^2}{(E+m)^2} + \frac{p_x^2 + p_y^2}{(E+m)^2} \right) =$$

$$= |N|^2 \left(\frac{(E+m)^2 + |\vec{p}|^2}{(E+m)^2} \right) = |N|^2 \left(\frac{E^2 + \cancel{m^2} + 2Em + E^2 - \cancel{m^2}}{(E+m)^2} \right)$$

$$= |N|^2 \left(\frac{2E^2 + 2Em}{(E+m)^2} \right) = |N|^2 \frac{2E}{E+m} = 2E$$

$$\boxed{N = \sqrt{E+m}}$$

Charge Conjugation

$$\psi' = \hat{C}\psi = i\gamma^2\psi^*$$

$$\psi = u_1 e^{i(\vec{p}\vec{r} - Et)} \qquad \psi' = i\gamma^2\psi^* = \underbrace{i\gamma^2 u_1^*}\, e^{-i(\vec{p}\vec{r} - Et)}$$

$$i\gamma^2 u_1^* = i\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x + i p_y}{E+m} \end{pmatrix}^* =$$

$$= \sqrt{E+m}\begin{pmatrix} \frac{p_x - i p_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix} = v_1$$

Summary of solutions to the Dirac Equation

$$\psi = u(E,\vec{p})e^{+i(\vec{p}.\vec{r}-Et)}$$ **satisfy** $$\boxed{(\gamma^\mu p_\mu - m)u = 0}$$

with

$$u_1 = \sqrt{E+m}\begin{pmatrix} 1 \\ 0 \\ \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \end{pmatrix}; \qquad u_2 = \sqrt{E+m}\begin{pmatrix} 0 \\ 1 \\ \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \end{pmatrix}$$

$$\psi = v(E,\vec{p})e^{-i(\vec{p}.\vec{r}-Et)}$$ **satisfy** $$\boxed{(\gamma^\mu p_\mu + m)v = 0}$$

with

$$v_1 = \sqrt{E+m}\begin{pmatrix} \frac{p_x-ip_y}{E+m} \\ \frac{-p_z}{E+m} \\ 0 \\ 1 \end{pmatrix}; \qquad v_2 = \sqrt{E+m}\begin{pmatrix} \frac{p_z}{E+m} \\ \frac{p_x+ip_y}{E+m} \\ 1 \\ 0 \end{pmatrix}$$

Spin States

$$\vec{p} = 0$$

$$u_1 = N\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$$

$$\hat{S}_z u_1 = +u_1$$

$$\hat{S}_z = \tfrac{1}{2}\Sigma_z = \tfrac{1}{2}\begin{pmatrix}\sigma_z & 0\\ 0 & \sigma_z\end{pmatrix} = \tfrac{1}{2}\begin{pmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & -1\end{pmatrix}$$

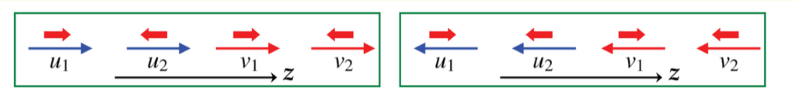
General u_1, u_2, v_1, v_2 - solutions are <u>not</u> eigenstates of $\hat{S}_z$

Choose particle travelling in z-direction ($p_x = 0$, $p_y = 0$)

$$u_1 = N\begin{pmatrix}1\\0\\ \frac{\pm|\vec{p}|}{E+m}\\0\end{pmatrix} \quad u_2 = N\begin{pmatrix}0\\1\\0\\ \frac{\pm|\vec{p}|}{E+m}\end{pmatrix} ; \quad v_1 \ldots \quad v_2 \ldots$$

$$\hat{S}_z u_1 = +\tfrac{1}{2}u_1 \quad \hat{S}_z u_2 = -\tfrac{1}{2}u_2 ; \quad S_z^{(v)} v_1 = +\tfrac{1}{2}v_1$$

$$S_z^{(v)} v_2 = -\tfrac{1}{2}v_2$$



Helicity

$$\vec{S} = \frac{1}{2}\vec{\Sigma} = \frac{1}{2}\begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$$

$$h \equiv \frac{\vec{S}\cdot\vec{p}}{|\vec{S}||\vec{p}|} = \frac{2\vec{S}\cdot\vec{p}}{|\vec{p}|} = \vec{\Sigma}\,\frac{\vec{p}}{|\vec{p}|} = \vec{\Sigma}\,\hat{p}$$

$\hat{p}$ unit vector

$u\uparrow$ $\qquad\qquad$ $u\downarrow$

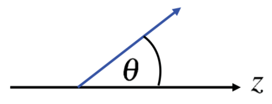
$$[\hat{H}, \hat{\vec{S}}\cdot\hat{p}] = 0$$

$$(\vec{\Sigma}\cdot\hat{p})\,u\uparrow = +u\uparrow$$

$$(\vec{\Sigma}\cdot\hat{p})\,u\downarrow = -u\downarrow$$

Helicity Eigenstates

$\hat{p} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$



$$\begin{pmatrix} \vec{\sigma}\cdot\hat{p} & 0 \\ 0 & \vec{\sigma}\cdot\hat{p} \end{pmatrix} \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \pm \begin{pmatrix} u_A \\ u_B \end{pmatrix}$$

$$\left.\begin{aligned} (\vec{\sigma}\cdot\hat{p}) u_A &= \pm u_A \\ (\vec{\sigma}\cdot\hat{p}) u_B &= \pm u_B \end{aligned}\right\}$$

$$\vec{\sigma}.\hat{p} = \begin{pmatrix} p_z & p_x - ip_y \\ p_x + ip_y & -p_z \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta\cos\phi - i\sin\theta\sin\phi \\ \sin\theta\cos\phi + i\sin\theta\sin\phi & -\cos\theta \end{pmatrix}$$

$$\vec{\sigma}\cdot\hat{p} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix}$$

$$u_A = \begin{pmatrix} a \\ b \end{pmatrix} \quad \text{OR} \quad u_B = \begin{pmatrix} a \\ b \end{pmatrix}$$

$$\begin{pmatrix} \cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & -\cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \pm \begin{pmatrix} a \\ b \end{pmatrix}$$

$$a \cdot \cos\theta + b \sin\theta\, e^{-i\phi} = \pm a \quad / a$$

$$\cos\theta + \frac{b}{a} \sin\theta\, e^{-i\phi} = \pm 1$$

$$\frac{b}{a} = \frac{\pm 1 - \cos\theta}{\sin\theta} e^{i\phi}$$

FOR $n = +1$ $$\frac{b}{a} = \frac{1 - \cos\theta}{\sin\theta} e^{i\phi} = \frac{2\sin^2\frac{\theta}{2}}{2\sin\frac{\theta}{2} \cdot \cos\frac{\theta}{2}} e^{i\phi}$$

$$\Longrightarrow$$

$$\frac{b}{a} = e^{i\phi} \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$u_A \uparrow \propto \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix} \quad ; \quad u_B \uparrow \propto \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$u\uparrow = \begin{pmatrix} u_A \\ u_B \end{pmatrix} = \begin{pmatrix} K_1 \cos\frac{\theta}{2} \\ K_1 e^{i\phi}\sin\frac{\theta}{2} \\ K_2 \cos\frac{\theta}{2} \\ K_2 e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}$$

$$(\vec{\sigma}\cdot\vec{p})\,u_A = (E+m)\,u_B \qquad \vec{p}$$

$$u_B = \frac{\vec{\sigma}\cdot\vec{p}}{E+m}\,u_A = \frac{|\vec{p}|}{(E+m)}\underbrace{(\vec{\sigma}\cdot\hat{p})}_{\text{Helicity}}\,u_A$$

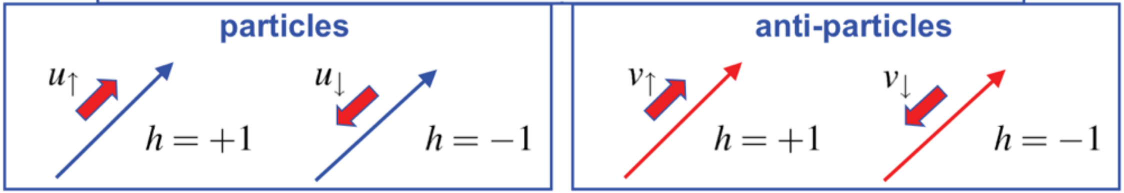
$$u_B = \pm\frac{|\vec{p}|}{(E+m)}\,u_A$$

$$u\uparrow = N\begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \\ \frac{|\vec{p}|}{E+m}\cos\frac{\theta}{2} \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\cdot\sin\frac{\theta}{2} \end{pmatrix} \qquad N=\sqrt{E+m}$$

Particle and Anti-Particle helicity eigenstates

$$u_{\uparrow} = N \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad u_{\downarrow} = N \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$v_{\uparrow} = N \begin{pmatrix} \frac{|\vec{p}|}{E+m}\sin\left(\frac{\theta}{2}\right) \\ -\frac{|\vec{p}|}{E+m}e^{i\phi}\cos\left(\frac{\theta}{2}\right) \\ -\sin\left(\frac{\theta}{2}\right) \\ e^{i\phi}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \qquad v_{\downarrow} = N \begin{pmatrix} \frac{|\vec{p}|}{E+m}\cos\left(\frac{\theta}{2}\right) \\ \frac{|\vec{p}|}{E+m}e^{i\phi}\sin\left(\frac{\theta}{2}\right) \\ \cos\left(\frac{\theta}{2}\right) \\ e^{i\phi}\sin\left(\frac{\theta}{2}\right) \end{pmatrix}$$



$$N = \sqrt{E+m}$$

Parity Operator

$$\psi \longrightarrow \hat{P}\psi = \pm\gamma^0\psi$$

$$x' \rightarrow -x$$
$$y' \rightarrow -y$$
$$z' \rightarrow -z$$

$$\hat{P}\,u_1 = \pm u_1 \qquad \hat{P}\,v_1 = \mp v_1$$

$$\hat{P}\,u_2 = \pm u_2 \qquad \hat{P}\,v_2 = \mp v_2$$

Choice! $\hat{P} = +\gamma^0$ for particles.

SUMMARY

(1) $\hat{H}\psi = (\vec{\alpha}\cdot\vec{p} + \beta m)\psi = i\dfrac{\partial \psi}{\partial t} \Rightarrow \psi = \begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\\ \psi_4\end{pmatrix}$

(2) $(i\gamma^\mu \partial_\mu - m)\psi = 0 \rightarrow$

(3) Solutions to D.E. describe free spin-$\frac{1}{2}$ fermions

(4) Give anti-particle solutions, and spin-$\frac{1}{2}$ framework.