PHASM/G 442. 2017 : Problem Sheet 2

Please return to Prof. Saakyan by the end of the lecture on November 16th 2017.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of marks.

1. Show that

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

[4]

2. Starting from the Dirac equation

$$(\gamma^\mu p_\mu - m)\, u = 0$$

show that the corresponding equation for the adjoint spinor is

$$\bar{u}\,(\gamma^\mu p_\mu - m) = 0.$$

[6]

3. Verify that the operator $\hat{H} = i\partial/\partial t$ acting on a free-particle solution:

$$\psi = u(E, \vec{p})e^{+i(\vec{p}\cdot\vec{r}-Et)}$$

gives the physical energy E of the particle. Explain why the same operator giving the physical energy of a free *anti*-particle solution is defined with the opposite sign:

$$\hat{H}^{(v)} = -i\partial/\partial t.$$

[5]

4. Using the following form of the Dirac Hamiltonian:

$$\hat{H}_D = -i\gamma^0\left(\vec{\gamma}\cdot\vec{\nabla}\right) + \gamma^0 m$$

show that it can be written in a matrix form as:

$$\hat{H}_D = \begin{pmatrix} m & \vec{\sigma}\cdot\vec{p} \\ \vec{\sigma}\cdot\vec{p} & -m \end{pmatrix}$$

[5]

5. (a) The commutator of the Dirac Hamiltonian with the orbital angular momentum operator, $\hat{\vec{L}} = \hat{\vec{r}} \times \hat{\vec{p}}$, is given by

$$\left[\hat{H}_D, \hat{L}\right] = -i\gamma^0\left(\vec{\gamma}\times\vec{p}\right)$$

Comment on the significance of this result.

(b) Using the expression for $\hat{H}_D$ in matrix form given in Q.4 find the commutator of the Dirac Hamiltonian with the spin operator, $\hat{\vec{S}} = \frac{1}{2}\hat{\vec{\Sigma}}$, where $\hat{\vec{\Sigma}} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}$.

(c) Hence, find the commutator $\left[\hat{H}_D, \hat{\vec{J}}\right]$, where $\hat{\vec{J}} = \hat{\vec{L}} + \hat{\vec{S}}$ is the operator of *total* angular momentum. Comment on the significance of this result. **[13]**

6. Show that under the parity transformation the form of the Dirac equation is unchanged provided that Dirac spinors transform as

$$\psi \to \hat{P}\psi = \gamma^0\psi$$

[6]

7. Draw the lowest order t-channel Feynman diagrams for $e^+e^- \to \gamma\gamma$ and use the Feynman rules for QED to write down the corresponding matrix elements. **[5]**

8. The Dirac Lagrangian for a free spin-half particle is given by:

$$\mathcal{L} = i\overline{\psi}\gamma^\mu\partial_\mu\psi - m\overline{\psi}\psi$$

(a) Show that this Lagrangian is *not* invariant under a phase transformation: $\psi(x) \to \psi'(x) = e^{iq\chi(x)}\psi(x)$ only if χ is a function of a coordinate $x \equiv x^\mu$, and *not* a constant.

(b) The required gauge invariance can be restored by replacing the derivative ∂_μ with the covariant derivative D_μ, $\partial_\mu \to D_\mu = \partial_\mu + iqA_\mu$, where A_μ is a new field. What transformation properties must A_μ have to make this work? What is the physical interpretation of A_μ?

[6]

Total: 50 marks