Quantum Field Theory

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1 Introduction

1.1 Course Content

The purpose of this series of lectures is to provide you with an introduction to the fundamentals of quantum field theory. Ultimately the aim is to teach you how to calculate amplitudes, cross sections and decay rates, particularly for Quantum Electrodynamics (QED), though other field theories will be discussed. By the end of the course you should be able to appreciate and understand the origin of a Feynman diagram, and relate this to a number for the cross section. The rules for Quantum Chromodynamics (QCD) and the electroweak sector of the Standard Model are slightly more complicated. However, the basic techniques for the calculation are very similar.

- Our starting point is ordinary Quantum Mechanics and our first goal will be to write down a 'relativistic version' of Quantum Mechanics. Hence, in section 2 we will look at relativistic wave equations, in particular to begin with the Klein-Gordon equation, which describes scalar particles. The interpretation of these wave equations as one-particle theories will lead to inconsistencies. Instead, we will have to construct a field theory. This step from a one-particle theory to a field theory (sometimes referred to as 'second quantization') is one of the central topics of this course. In the first case I will work entirely within the context of a neutral scalar field in order to avoid complications.

- All quantum field theories are described in terms of their field (and hence, particle) content, and the Lagrangian describing the properties of the particular theory. In order to explain these concepts section 3 will be a brief introduction to the Lagrangian and Hamiltonian formulation of classical mechanics and classical field theory. Essentially the Lagrangian and Hamiltonian formalism are the way in which we describe the fundamental degrees of freedom defining the quantum field theory, and when it becomes a quantum theory these degrees of freedom are quantised following the sort of rules you are already familiar with.

- This quantisation, and the introduction of operators representing the creation and annihilation of particles will be covered for non-interacting particles in section 4.

- In section 5 we introduce time-dependence of the operators for quantum fields into the theory.

- In section 6 we then consider the case of interacting field theories and derive an expression for the matrix elements for scattering processes in terms of the vacuum exception values of time-ordered products of quantum fields.

- Within the context of an interaction which can be parameterised by a small coupling constant we derive the coordinate space expressions for the scattering-, or S-matrix elements in section 7, finding that the rules for calculating these elements may easily be thought of in terms of Feynman rules for diagrams representing the scattering process.

- In section 8 we convert these expressions into momentum space, obtaining the Feynman rules for the scattering of particles with momentum p_i into particles with momentum q_j. Hence we will have discovered where these rules come from. We will then work out how to go from quantum mechanical probability amplitudes to cross sections and decay rates.

- In Section 9 we will then widen our scope and start to look at particles with spin. We will return briefly to relativistic quantum mechanics and the Dirac equation, which describes particles with spin 1/2, showing that we now need four-component spinors to describe fermions, rather than a single function.

- Section 10 will sketch the construction of a quantum field theory of fermions interacting via the electromagnetic force, i.e. QED, including the Feynman rules. This will rely heavily on the field theory formalism already derived for the scalar field in earlier sections.

- In section 11, with these tools in hand we will look at some examples of tree level QED processes. Here you will get hands-on experience of calculating transition amplitudes and getting from them to cross sections. We will initially restrict ourselves to calculations at lowest order, i.e. *tree level*.

- At the end of the course, in section 12, we will also look very quickly at higher order *loop* effects, which, amongst other things, are responsible for the running of the couplings. For QCD, this running means that the coupling appears weaker when measured at higher energy scales and is the reason why we can sometimes do perturbative QCD calculations. However, in higher order calculations divergences appear and we have to understand — at least in principle — how these divergences can be removed.

There is, of course, much more to quantum field theory than the rules for scattering matrix elements in terms of powers of a small coupling constant, and this set of lectures does not attempt to address the issue of nonperturbative quantum field theory at all. Also, I note that it is often remarked that there are two approaches to quantum field theory, the canonical quantisation approach and the method of path integrals. In so much that this is true I follow the canonical approach. However, it is only when the expression for the matrix elements in terms of vacuum expectation values of quantum fields has been obtained that the two approaches diverge, and the path integral approach provides an alternative way of obtaining (exactly the same) Feynman rules. Hence, the vast bulk of this course is independent of the approach taken. The path integral approach, while having some advantages, would take far more explanation, and is in some senses less physically intuitive. I will introduce the topic at the end of the course if time permits.

1.2 Units and Conventions

I will use natural units, $c = 1$, $\hbar = 1$, so mass, energy, inverse length and inverse time all have the same dimensions. Generally think of energy E as the basic unit, e.g mass m has units of GeV and distance x has unit GeV^{-1}.

Very often, we will work with 4-vectors. A 4-vector is by definition an object that transforms in a specific way under Lorentz transformations (LT). Let $\Lambda^{\mu}{}_{\nu}$ represent a LT:

A familiar example of a LT is a boost along the z-axis, for which

$$\Lambda^{\mu}{}_{\nu} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix},$$

with, as usual, $\beta = v$ (in units of c) and $\gamma = (1-\beta^2)^{-1/2}$. LT's can be thought of as generalized rotations, and rotations in three spatial dimension are a subgroup.

x^μ is then a (contravariant) 4-vector if under a LT

$$x^\mu \to x'^\mu = \Lambda^{\mu}{}_{\nu} x^\nu \tag{1.1}$$

The Greek labels $\mu, \nu \ldots \in \{0, 1, 2, 3\}$ denote Lorentz indices and the summation convention is used. The Minkowski scalar product of two 4-vectors p and q is defined as

$$p \cdot q = p^\mu q^\nu g_{\mu\nu} = p^\mu q_\mu \tag{1.2}$$

where the metric

$$g^{\mu\nu} = g_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1), \qquad g^{\mu\lambda} g_{\lambda\nu} = \delta^\mu_\nu = \begin{cases} 1 & \text{if } \mu = \nu \\ 0 & \text{if } \mu \neq \nu \end{cases} \tag{1.3}$$

has been introduced. The last step in eq. (1.2) is nothing but the definition of a covariant 4-vector (sometimes referred to as co-vector)

$$p_\mu \equiv g_{\mu\nu} p^\nu \tag{1.4}$$

Writing 4-vectors in components we have

$$p^\mu = (p^0, \mathbf{p}) \qquad p_\mu = (p_0, -\mathbf{p}) \tag{1.5}$$

with $p^0 = p_0$. Note that the invariance of the scalar product implies

$$\Lambda^T g \Lambda = g \to g \Lambda^T g = \Lambda^{-1}, \tag{1.6}$$

i.e. a generalization of the orthogonality property of the rotation matrix $R^T = R^{-1}$. For the rotation matrices acting only in three dimensions the orthogonality is preserved.

Note that ∂_μ is a covariant 4-vector,

$$\partial_\mu \equiv \frac{\partial}{\partial x^\mu}, \quad \partial_\mu x^\nu = \delta^\nu_\mu, \tag{1.7}$$

so $\partial_\mu = (\partial^0, \boldsymbol{\nabla})$ and $\partial^\mu = (\partial^0, -\boldsymbol{\nabla})$. The convention for the totally antisymmetric Levi-Civita tensor is

$$\epsilon^{\mu\nu\lambda\sigma} = \begin{cases} +1 & \text{if } \{\mu, \nu, \lambda, \sigma\} \text{ an even permutation of } \{0, 1, 2, 3\} \\ -1 & \text{if an odd permutation} \\ 0 & \text{otherwise} \end{cases} \tag{1.8}$$

Note that $\epsilon^{\mu\nu\lambda\sigma} = -\epsilon_{\mu\nu\lambda\sigma}$, and $\epsilon^{\mu\nu\lambda\sigma} p_\mu q_\nu r_\lambda s_\sigma$ changes sign under a parity transformation since it contains an odd number of spatial components.

2 Relativistic Wave Equations

The starting point for this course is the Schrödinger equation which can be written quite generally as

$$H\psi = i\frac{\partial\psi}{\partial t} \tag{2.1}$$

where H is the Hamiltonian (i.e. the energy operator). In this equation ψ is the wave function describing the single particle probability amplitude. I shall usually reserve the Greek symbol ψ for spin 1/2 fermions and ϕ for spin 0 bosons. So for pions and the like I shall write

$$H\phi = i\frac{\partial\phi}{\partial t}. \tag{2.2}$$

In this course we want to extend the non-relativistic quantum mechanics, familiar to undergraduates, into the relativistic domain. For example, in non-relativistic quantum mechanics you are used to writing

$$H = T + V \tag{2.3}$$

where T is the kinetic energy and V is the potential energy. A particle of mass m and momentum $\mathbf{p}$ has non-relativistic kinetic energy,

$$T = \frac{\mathbf{p}^2}{2m} \tag{2.4}$$

For a slow moving particle $v \ll c$ (e.g. an electron in a Hydrogen atom) this is adequate, but for relativistic systems ($v \sim c$) the Hamiltonian above breaks down. For a free relativistic particle the total energy E is given by the Einstein equation

$$E^2 = \mathbf{p}^2 + m^2. \tag{2.5}$$

Thus the square of the relativistic Hamiltonian H^2 is simply given by promoting the momentum to operator status:

$$H^2 = \mathbf{p}^2 + m^2. \tag{2.6}$$

So far, so good, but now the question arises of how to implement the Schrödinger equation, which is expressed in terms of H rather than H^2. Naively the relativistic Schrödinger equation looks like

$$\sqrt{\mathbf{p}^2 + m^2}\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{2.7}$$

but this is difficult to interpret because of the square root. There are two ways forward:

1. Work with H^2. By iterating the Schrödinger equation we have

$$H^2\phi(t) = -\frac{\partial^2\phi(t)}{\partial t^2} \quad \left[\text{or} \quad \left(\frac{i\partial}{\partial t} - V\right)^2\phi(t)\right] \tag{2.8}$$

   This is known as the Klein-Gordon (KG) equation. In this case the wave function describes spinless bosons.

2. Invent a new Hamiltonian H_D that is linear in momentum, and whose square is equal to H^2 given above, $H_D^2 = \mathbf{p}^2 + m^2$. In this case we have

$$H_D\psi(t) = i\frac{\partial\psi(t)}{\partial t} \tag{2.9}$$

which is known as the Dirac equation, with H_D being the Dirac Hamiltonian. In this case the wave function describes spin 1/2 fermions, as we shall see. In this case ψ is no longer a simple function but must contain at least four components.

2.1 The Klein-Gordon Equation

Let us now take a more detailed look at the KG equation (2.8). In position space we write the momentum operator as

$$\mathbf{p} \to -i\boldsymbol{\nabla}, \tag{2.10}$$

so that the KG equation (for zero potential V) becomes

$$(\Box + m^2)\,\phi(x) = 0 \tag{2.11}$$

where we have introduced the box notation,

$$\Box = \partial_\mu\partial^\mu = \partial^2/\partial t^2 - \nabla^2 \tag{2.12}$$

and x is the 4-vector $(t, \mathbf{x})$.

The operator $\Box$ is Lorentz invariant, so the Klein-Gordon equation is relativistically covariant (that is, transforms into an equation of the same form) if ϕ is a scalar function. That is to say, under a Lorentz transformation $(t, \mathbf{x}) \to (t', \mathbf{x}')$,

$$\phi(t, \mathbf{x}) \to \phi'(t', \mathbf{x}') = \phi(t, \mathbf{x})$$

so ϕ is invariant. In particular ϕ is then invariant under spatial rotations so it represents a spin-zero particle (more on spin when we come to the Dirac equation); there being no preferred direction which could carry information on a spin orientation.

The Klein-Gordon equation has plane wave solutions:

$$\phi(x) = Ne^{-i(Et - \mathbf{p}\cdot\mathbf{x})} \tag{2.13}$$

where N is a normalization constant and $E = \pm\sqrt{\mathbf{p}^2 + m^2}$. Thus, there are both positive and negative energy solutions. The negative energy solutions pose a severe problem if you try to interpret ϕ as a wave function (as indeed we are trying to do). The spectrum is no longer bounded from below, and you can extract arbitrarily large amounts of energy from the system by driving it into ever more negative energy states. Any external perturbation capable of pushing a particle across the energy gap of $2m$ between the positive and negative energy continuum of states can uncover this difficulty. Furthermore, we cannot just throw away these solutions as unphysical since they appear as Fourier modes in any realistic solution of (2.11). Note that if one interprets ϕ as a quantum field there is no problem, as you will see in the

field theory course. The positive and negative energy modes are just associated with operators which create or destroy particles.

A second problem with the wave function interpretation arises when trying to find a probability density. Since ϕ is Lorentz invariant, $|\phi|^2$ does not transform like a density so we will not have a Lorentz covariant continuity equation $\partial\rho/\partial t + \boldsymbol{\nabla}\cdot\mathbf{J} = 0$. To search for a candidate we derive such a continuity equation. Defining ρ and $\mathbf{J}$ by

$$\rho \equiv i\left(\phi^*\frac{\partial\phi}{\partial t} - \phi\frac{\partial\phi^*}{\partial t}\right), \qquad \left[\text{or} \quad \phi^*\left(i\frac{\partial}{\partial t} - V\right)\phi + \phi\left(-i\frac{\partial}{\partial t} - V\right)\phi^*\right] \tag{2.14}$$

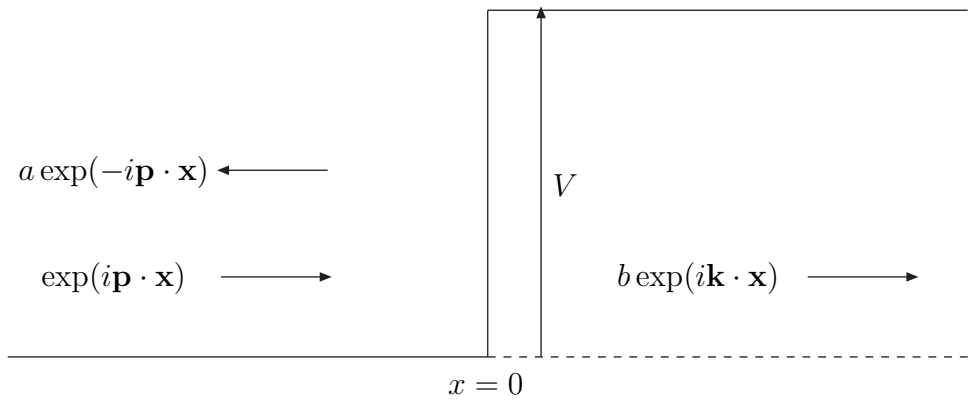
$$\mathbf{J} \equiv -i\left(\phi^*\boldsymbol{\nabla}\phi - \phi\boldsymbol{\nabla}\phi^*\right) \tag{2.15}$$

you obtain a covariant conservation equation

$$\partial_\mu J^\mu = 0 \tag{2.16}$$

where J is the 4-vector $(\rho, \mathbf{J})$. It is thus natural to interpret ρ as a probability density and $\mathbf{J}$ as a probability current. However, for a plane wave solution (2.13), $\rho = 2|N|^2E$, so ρ is not positive definite since we've already found E can be negative.

In order to illustrate the types of problem we can encounter we consider the wave incident on a potential step shown below.



One can show that if $V > m + E_p$, where $E_p = \sqrt{p^2 + m^2}$ then one has a travelling rather than decaying wave for $x > 0$ but cannot avoid using the negative square root $k = -\sqrt{(E_p - V)^2 - m^2}$ and getting negative currents and densities. In order to do this we use the fact that $\phi(x)$ and $\partial\phi(x)/\partial x$ are continuous at $x = 0$, and ensure that the group velocity $v_g = \partial E/\partial k$ is positive for $x > 0$. To be explicit, the solution for $x < 0$ is

$$\phi = \exp(-iE_pt + ipx) + a\exp(-iE_pt - ipx), \tag{2.17}$$

and for $x > 0$ is

$$\phi = b\exp(-iE_pt + ikx), \tag{2.18}$$

where a and b are the reflection and transmission coefficients respectively, and $p = \pm\sqrt{E_p^2 - m^2}$ and $k = \pm\sqrt{(E_p - V)^2 - m^2}$. Imposing the continuity of ϕ at $x = 0$, and of $\partial\phi/\partial x$ gives the two conditions

$$1 + a = b, \tag{2.19}$$

$$p(1 - a) = kb. \tag{2.20}$$

one can quickly rearrange these to show that

$$a = \frac{p - k}{p + k}, \qquad \text{and} \qquad b = \frac{2p}{p + k}. \tag{2.21}$$

For a forward travelling incident wave we must take $p = +\sqrt{E_p^2 - m^2}$. However, for $x > 0$ the group velocity is given by

$$v_g = \frac{\partial E_p}{\partial k} = \frac{k}{E_p - V}. \tag{2.22}$$

Hence, since $V > E_p$ in order to have a forward travelling transmitted wave, i.e. $v_g > 0, x > 0$, we must take the solution $k = -\sqrt{(E_p - V)^2 - m^2}$, so that $k = -|k|$. This means we can write the reflection coefficient as

$$a = \frac{p + |k|}{p - |k|}, \tag{2.23}$$

and so $|a| > 1$. It is now a straightforward matter to show that for $x < 0$ there is a negative current density and for $x > 0$ the density ρ is negative.

Thus, ρ may well be considered as the density of a conserved quantity (such as electric charge), but we cannot use it for a probability density. To Dirac, this and the existence of negative energy solutions seemed so overwhelming that he was led to introduce another equation, first order in time derivatives but still Lorentz covariant, hoping that the similarity to Schrödinger's equation would allow a probability interpretation. Dirac's original hopes were unfounded because his new equation turned out to admit negative energy solutions too! Even so, he did find the equation for spin-1/2 particles and predicted the existence of antiparticles.

Before turning to discuss what Dirac did, let us put things in context. We have found that the Klein-Gordon equation, a candidate for describing the quantum mechanics of spinless particles, admits unacceptable negative energy states when ϕ is interpreted as the single particle wave function. We could solve all our problems here and now, and restore our faith in the Klein-Gordon equation, by simply re-interpreting ϕ as a quantum field. However we will not do that. There is another way forward (this is the way followed in the textbook of Halzen & Martin) due to Stueckelberg and Feynman. Causality forces us to ensure that positive energy states propagate forwards in time. But if we force the negative energy states only to propagate backwards in time then we find a theory that is consistent with the requirements of causality and that has none of the aforementioned problems. In fact, the negative energy states cause us problems only so long as we think of them as real physical states propagating forwards in time. Therefore, we should interpret the emission (absorption) of a negative energy particle with momentum p^μ as the absorption (emission) of a positive energy antiparticle with momentum $-p^\mu$.

In order to get more familiar with this picture, consider a process with a π^+ and a photon in the initial state and final state. In figure 1(a) the π^+ starts from the point A and at a later time t_1 emits a photon at the point $\mathbf{x}_1$. If the energy of the π^+ is still positive, it travels on forwards in time and eventually will absorb the initial state photon at t_2 at the point $\mathbf{x}_2$. The final state is then again a photon and a (positive energy) π^+.

There is another process however, with the same initial and final state, shown in figure 1(b). Again, the π^+ starts from the point A and at a later time t_2 emits a photon at the point $\mathbf{x}_1$. But this time, the energy of the photon emitted is bigger than the energy of the initial π^+. Thus, the energy of the π^+ becomes negative and it is forced to travel backwards in time. Then at an *earlier* time t_1 it absorbs the initial state photon at the point $\mathbf{x}_2$, thereby rendering its energy positive again. From there, it travels forward in time and the final state is the same as in figure 1(a), namely a photon and a (positive energy) π^+.

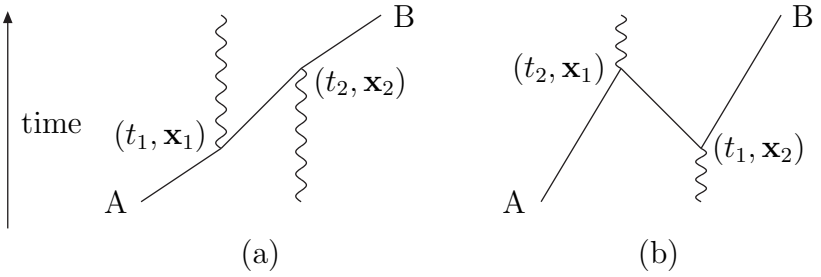


Figure 1: Interpretation of negative energy states

In today's language, the process in figure 1(b) would be described as follows: in the initial state we have an π^+ and a photon. At time t_1 and at the point $\mathbf{x}_2$ the photon creates an π^+-π^- pair. Both propagate forwards in time. The π^+ ends up in the final state, whereas the π^- is annihilated at (a later) time t_2 at the point $\mathbf{x}_1$ by the initial state π^+, thereby producing the final state photon. To someone observing in real time, the negative energy state moving backwards in time looks to all intents and purposes like a negatively charged pion with positive energy moving forwards in time. However, once we start interpreting processes in this manner, i.e. in terms of particle creation and annihilation, it is more appropriate to start working with quantum field theory.