PHASM/G442 Particle Physics

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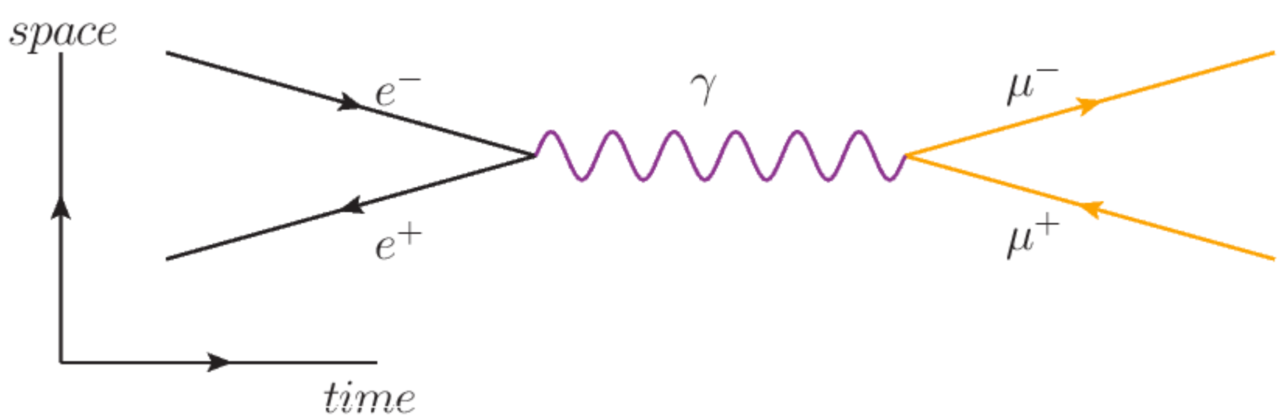
Module V

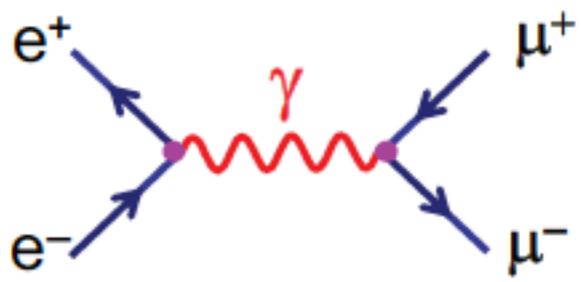
QED Calculations, e+e- annihilation



http://moodle.ucl.ac.uk/course/view.php?id=2589

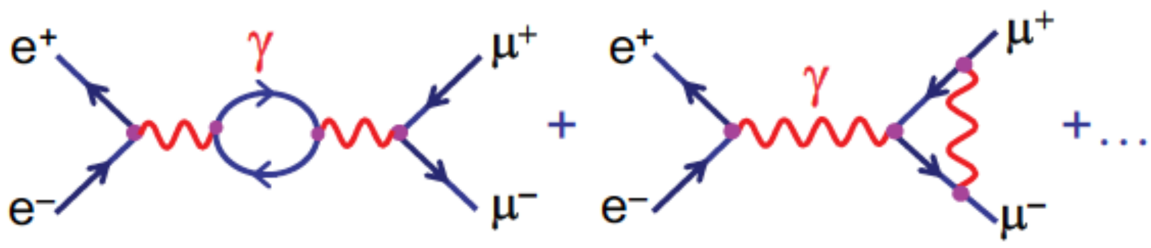
Calculation Algorithm

- Draw lowest order Feynman Diagram



$$M \propto e^2 \propto \alpha_{em}$$

- and second (and higher) order diagrams



$$M \propto e^4 \propto \alpha_{em}^2$$

- For each diagram calculate matrix element derived in Module IV

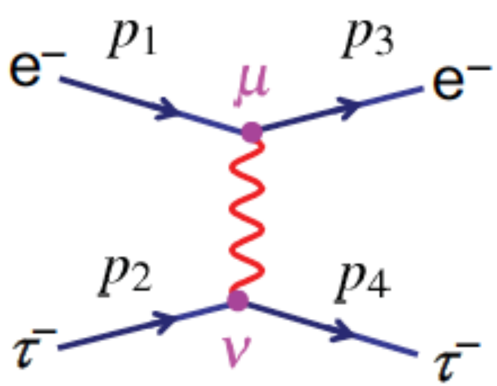
- Sum individual matrix elements

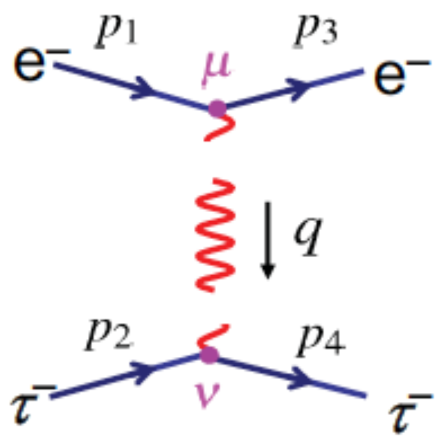
$$M_{fi} = M_1 + M_2 + M_3 + \ldots.$$

N.B. Can interfere positively or negatively

Reminder (Module 4)

- For t-channel:





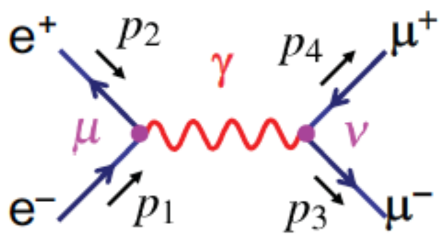
$$\bar{u}_e(p_3)[ie\gamma^\mu]u_e(p_1)$$

$$\frac{-ig_{\mu\nu}}{q^2}$$

$$\bar{u}_\tau(p_4)[ie\gamma^\nu]u_\tau(p_2)$$

$$-iM = [\bar{u}_e(p_3)ie\gamma^\mu u_e(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}_\tau(p_4)ie\gamma^\nu u_\tau(p_2)]$$

- For s-channel:



$$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}(p_3)ie\gamma^\nu v(p_4)]$$

NOTE:

- At each vertex the **adjoint spinor is written first**
- Each vertex has a different index
- $g_{\mu\nu}$ connects indices at vertices

- Then $$|M_{fi}|^2 = (M_1 + M_2 + M_3 + ...)(M_1^* + M_2^* + M_3^* + ...)$$

- For QED $\alpha_{em} \sim 1/137$ the lowest order diagram dominates. Can neglect higher orders for most practical cases

- Calculate decay rate/cross-section using formulae derived in Module I, e.g.

Decay: $$\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \, d\Omega$$

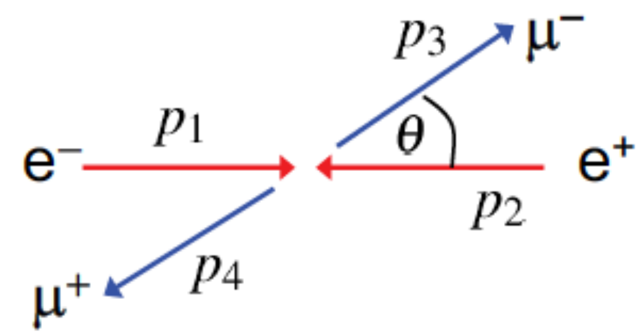
Scattering in CoM: $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega^*} = \frac{1}{64\pi^2 s} \frac{|\vec{p}_f^*|}{|\vec{p}_i^*|} |M_{fi}|^2$$

Scattering in Lab Frame: $$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2$$

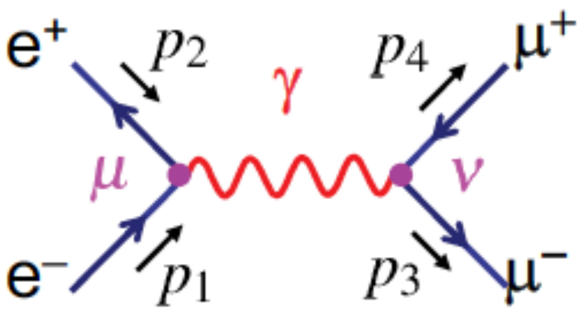
Electron-Positron annihilation

- Consider $e^+e^- \to \mu^+\mu^-$ in CoM

$p_1 = (E,0,0,p) \quad p_2 = (E,0,0,-p) \quad p_3 = (E,\vec{p}_f) \quad p_4 = (E,-\vec{p}_f)$



- We will consider only lowest order Feynman diagram



Feynman rules:

$$-iM = [\bar{v}(p_2)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\bar{u}(p_3)ie\gamma^\nu v(p_4)]$$

which can be written in terms of electron and muon currents

$$\boxed{M = -\frac{e^2}{s}j_e \cdot j_\mu}$$

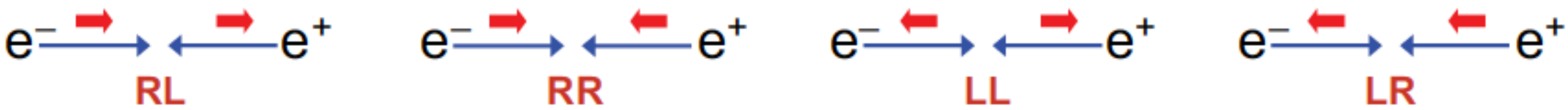
In CoM

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f|}{|\vec{p}_i|}|M_{fi}|^2$$

with $s = (p_1+p_2)^2 = (E+E)^2 = 4E^2$

Spin in e+e- annihilation

- In general, e- and e+ are produced unpolarised, 4 possible combinations in initial state



- Similarly, in the final states μ+μ- have 4 helicity combinations

- Therefore we need to **sum** over all **16 helicity combinations** and **average** over **4** combinations in initial state

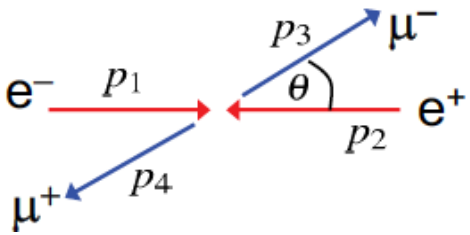
$$\langle |M|^2 \rangle = \frac{1}{4} \sum_{\text{spins}} |M_i|^2 = \frac{1}{4} \left(|M_{LL\to LL}|^2 + |M_{LL\to LR}|^2 + \ldots \right)$$

- Fortunately, we'll see that in **relativistic** limit only **4 combinations** will survive — an important feature of QED and QCD!

Helicity Spinors

- In CoM with $E >> m$

$$p_1 = (E,0,0,E);\;\; p_2 = (E,0,0,-E)$$
$$p_3 = (E, E\sin\theta, 0, E\cos\theta);$$
$$p_4 = (E, -\sin\theta, 0, -E\cos\theta)$$



- Helicity spinor with $E >> m$ (see Module 3)

$$\boxed{u_\uparrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi}\end{pmatrix};\; u_\downarrow = \sqrt{E}\begin{pmatrix} -s \\ ce^{i\phi} \\ s \\ -ce^{i\phi}\end{pmatrix};\;\; v_\uparrow = \sqrt{E}\begin{pmatrix} s \\ -ce^{i\phi} \\ -s \\ ce^{i\phi}\end{pmatrix};\; v_\downarrow = \sqrt{E}\begin{pmatrix} c \\ se^{i\phi} \\ c \\ se^{i\phi}\end{pmatrix}}$$

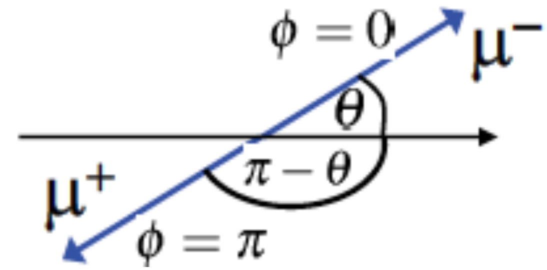
$$s = \sin\frac{\theta}{2}$$
$$c = \cos\frac{\theta}{2}$$

- The initial state spinors of electron and positron

$$u_\uparrow(p_1) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ 1 \\ 0\end{pmatrix};\; u_\downarrow(p_1) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1\end{pmatrix};\qquad v_\uparrow(p_2) = \sqrt{E}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 0\end{pmatrix};\; v_\downarrow(p_2) = \sqrt{E}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1\end{pmatrix}$$

e-   e+

- For the final state



$$u_\uparrow(p_3) = \sqrt{E}\begin{pmatrix} c \\ s \\ c \\ s \end{pmatrix};\ u_\downarrow(p_3) = \sqrt{E}\begin{pmatrix} -s \\ c \\ s \\ -c \end{pmatrix}$$

μ-

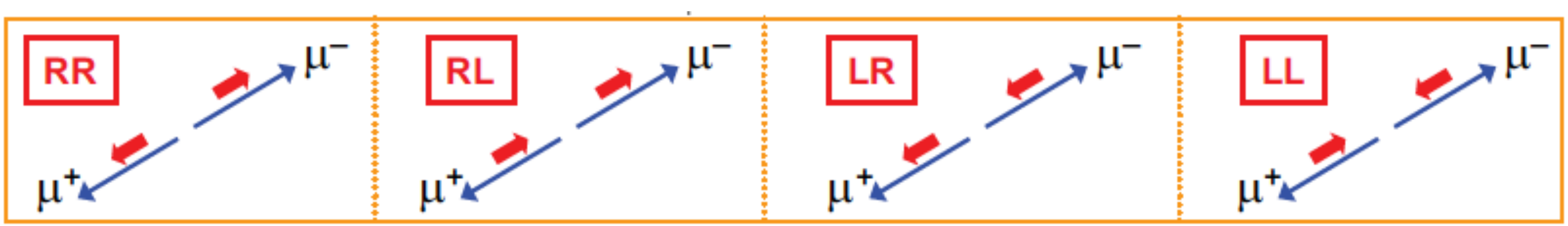
$$v_\uparrow(p_4) = \sqrt{E}\begin{pmatrix} c \\ s \\ -c \\ -s \end{pmatrix};\ v_\downarrow(p_4) = \sqrt{E}\begin{pmatrix} s \\ -c \\ s \\ -c \end{pmatrix}$$

μ+

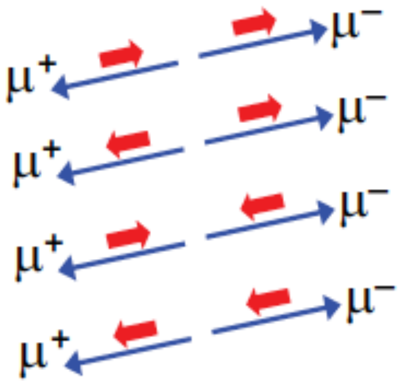
- Now we want to calculate

$$M = -\frac{e^2}{s} j_e \cdot j_\mu$$

- Will begin with the muon current, j_μ

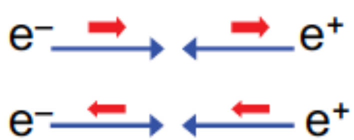


Muon and Electron Currents



$$\begin{array}{lcll}
\bar{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4) & = & 2E(0,-\cos\theta,i,\sin\theta) & \textbf{RL} \\
\bar{u}_{\uparrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4) & = & (0,0,0,0) & \textbf{RR} \\
\bar{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\downarrow}(p_4) & = & (0,0,0,0) & \textbf{LL} \\
\bar{u}_{\downarrow}(p_3)\gamma^{\nu}v_{\uparrow}(p_4) & = & 2E(0,-\cos\theta,-i,\sin\theta) & \textbf{LR}
\end{array}$$

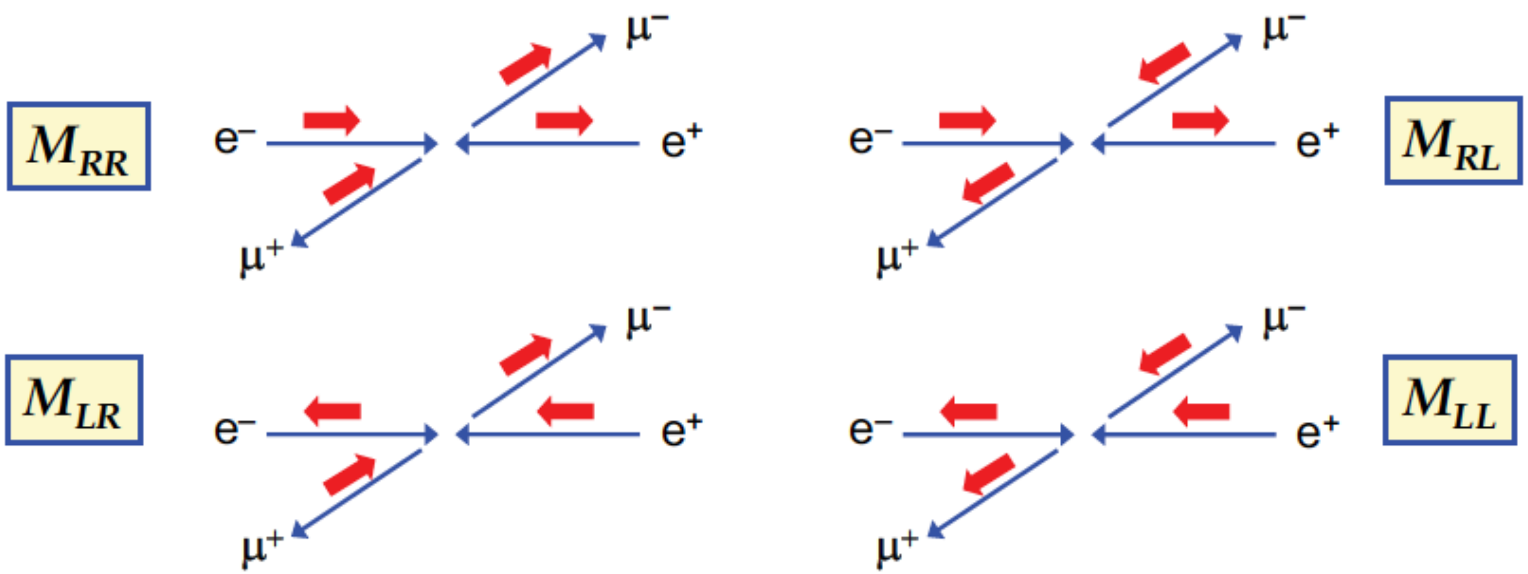
- In the limit $E >> m$ only **two helicity combinations** are non zero!
- This is an important feature of QED. The origin of it will be discussed late in this Module
- The same situation is with the electron (initial state) current



$$\begin{array}{lclcl}
e_R^- e_L^+ & : & \bar{v}_{\downarrow}(p_2)\gamma^{\nu}u_{\uparrow}(p_1) & = & 2E(0,-1,-i,0) \\
e_L^- e_R^+ & : & \bar{v}_{\uparrow}(p_2)\gamma^{\nu}u_{\downarrow}(p_1) & = & 2E(0,-1,i,0)
\end{array}$$

- As a result out of **16** possible helicity combinations only **4** give **non-zero** matrix element!

Only 4 helicity combinations have to be considered for $e^+e^- \to \mu^+\mu^-$



Subscripts refer to helicity of e- and μ-. No need to specify others due to **"helicity conservation"**

$$M = -\frac{e^2}{s} j_e . j_\mu$$

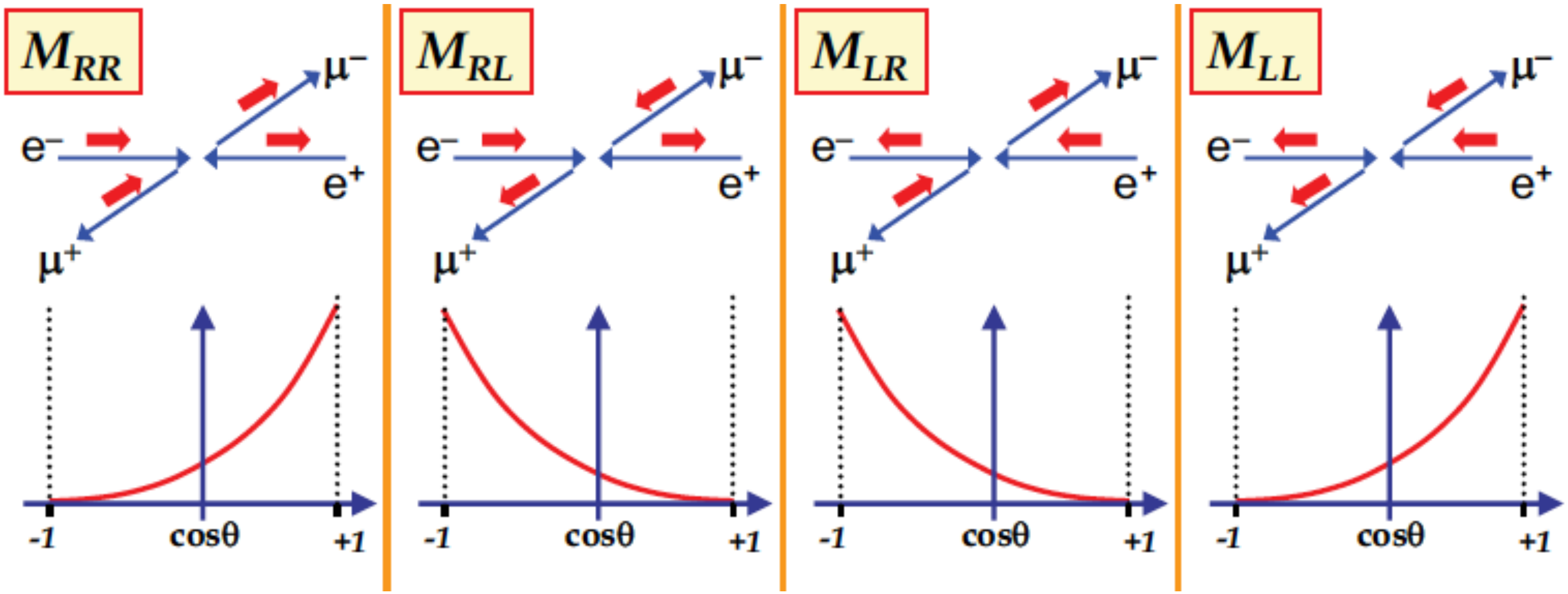
$$M_{RR} = -\frac{e^2}{s}[2E(0,-1,-i,0)].[2E(0,-\cos\theta,i,\sin\theta)]$$

$$= e^2(1+\cos\theta) = 4\pi\alpha(1+\cos\theta)$$ where $\alpha = e^2/4\pi \approx 1/137$

Same procedure for M_{RL}, M_{LR}, M_{LL}

Matrix element(s) of e+e- →μ+μ-

$$|M_{RR}|^2 = |M_{LL}|^2 = (4\pi\alpha)^2(1+\cos\theta)^2$$
$$|M_{RL}|^2 = |M_{LR}|^2 = (4\pi\alpha)^2(1-\cos\theta)^2$$



$e^4(1+\cos\theta)^2$ $\quad$ $e^4(1-\cos\theta)^2$ $\quad$ $e^4(1-\cos\theta)^2$ $\quad$ $e^4(1+\cos\theta)^2$

The final matrix element is obtained by **averaging** over the **initial** spin states

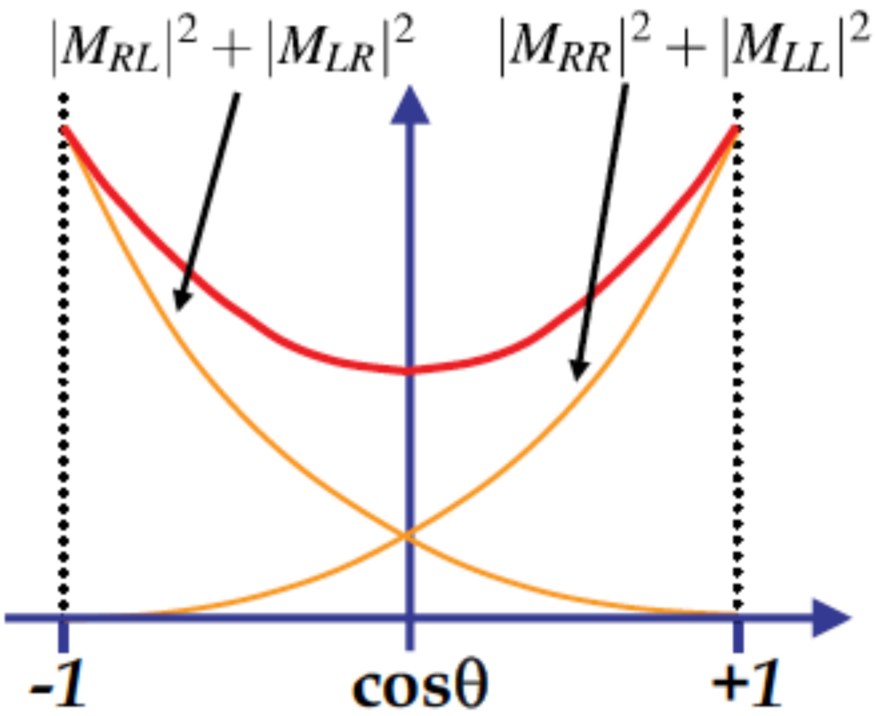
$$\langle|M_{fi}|^2\rangle \quad = \quad \frac{1}{4}\times(|M_{RR}|^2+|M_{RL}|^2+|M_{LR}|^2+|M_{LL}^2|) \quad = \quad e^4(1+\cos^2\theta)$$

which can be also written in Lorentz Invariant form: $\boxed{\langle|M_{fi}|^2\rangle = 2e^4\frac{(p_1.p_3)^2+(p_1.p_4)^2}{(p_1.p_2)^2}}$

- Plugging it in the cross-section formula,

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 s}\left\langle |M_{fi}|^2 \right\rangle = \frac{(4\pi\alpha)^2}{64\pi^2 s}\left(1+\cos^2\theta\right)$$

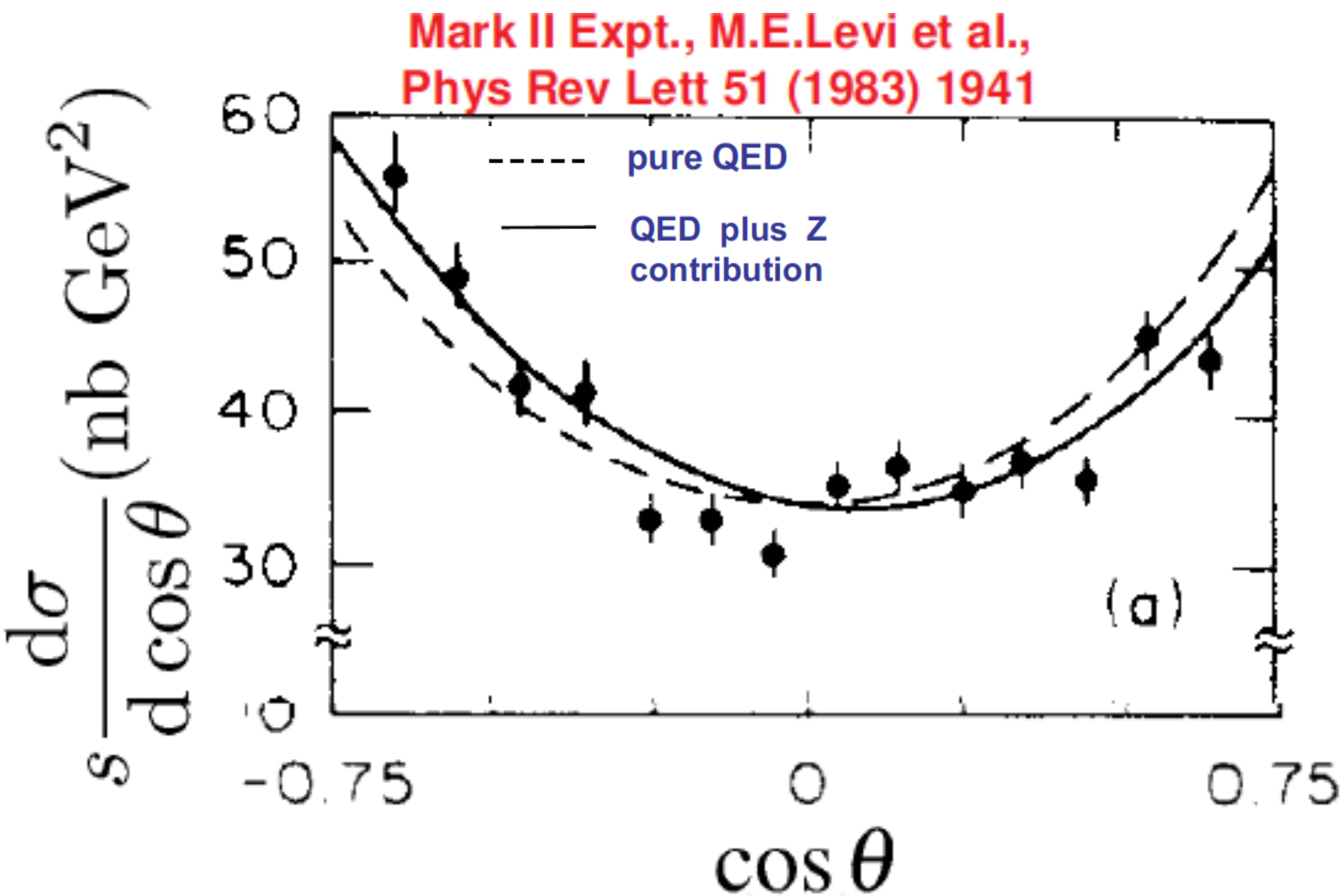
$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}$$



Theory vs Experiment

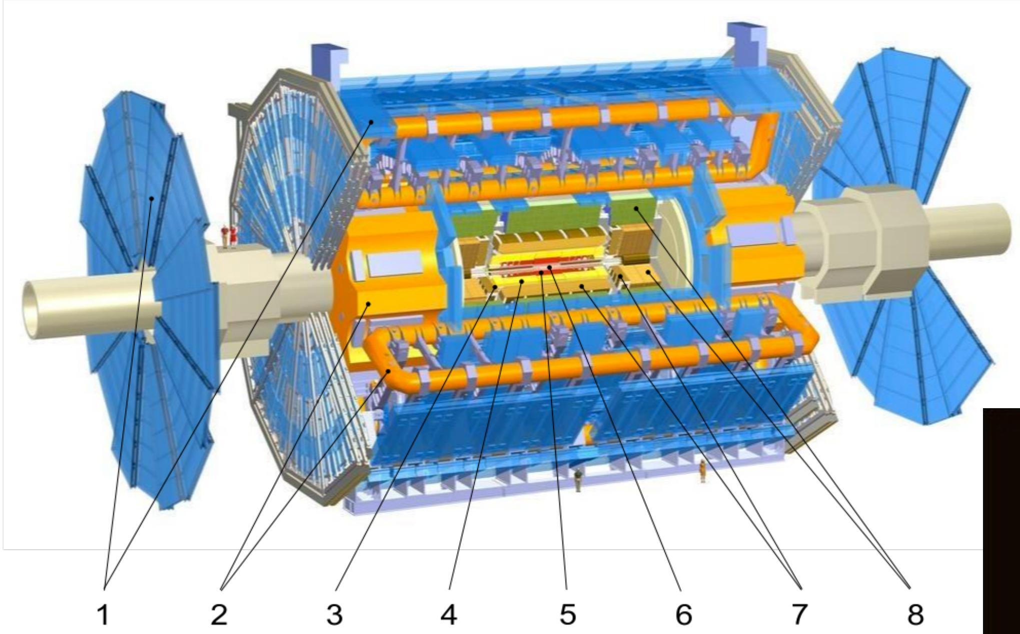
$e^+e^- \rightarrow \mu^+\mu^-$

$\sqrt{s} = 29\ \text{GeV}$



Angular distribution becomes slightly asymmetric when higher-order QED or Z-contribution is taken into account

How do we observe $\mu^+\mu^-$?



(1)Muon Detectors

(2) Toroid Magnets

(3) Solenoid Magnet

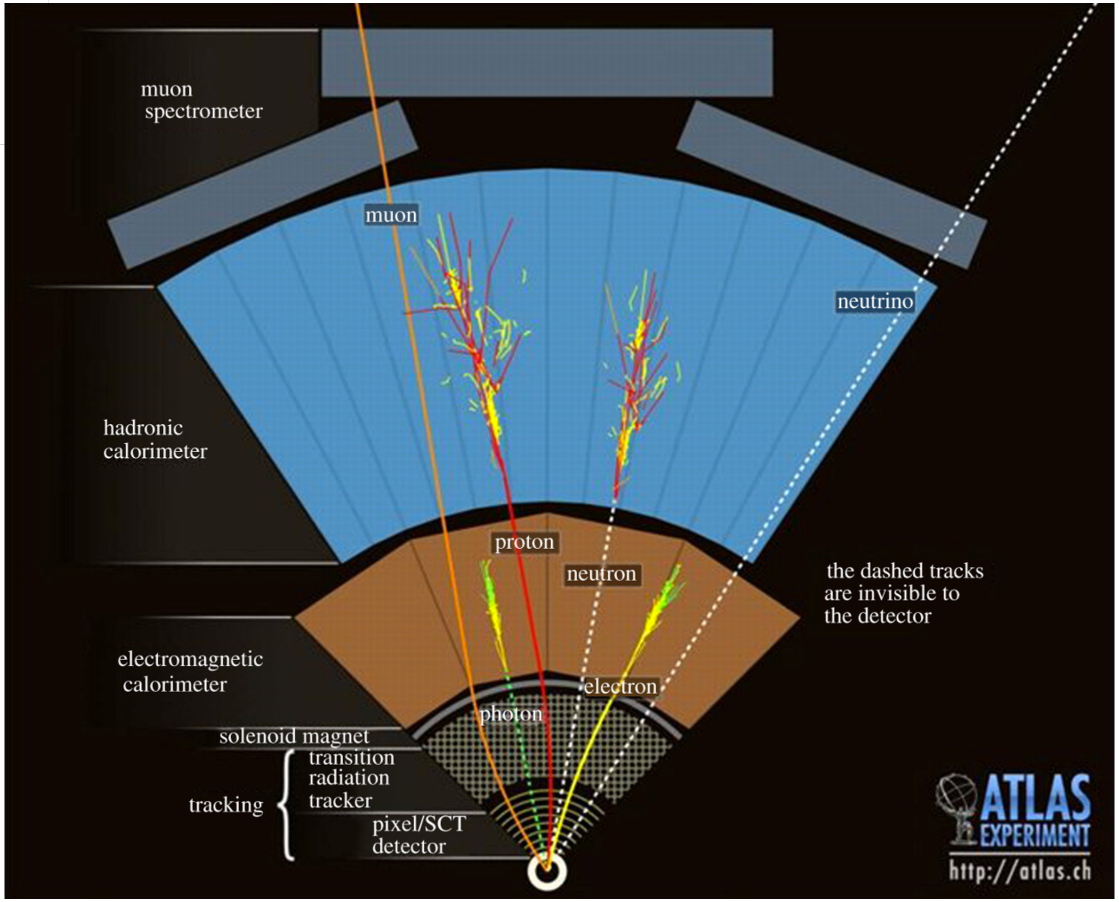
(4) Transition Radiation Tracker

(5) Semi-Conductor Tracker

(6) Pixel Detector Calo

(7) Liquid Argon Calorimeter

(8) Tile Calorimeter

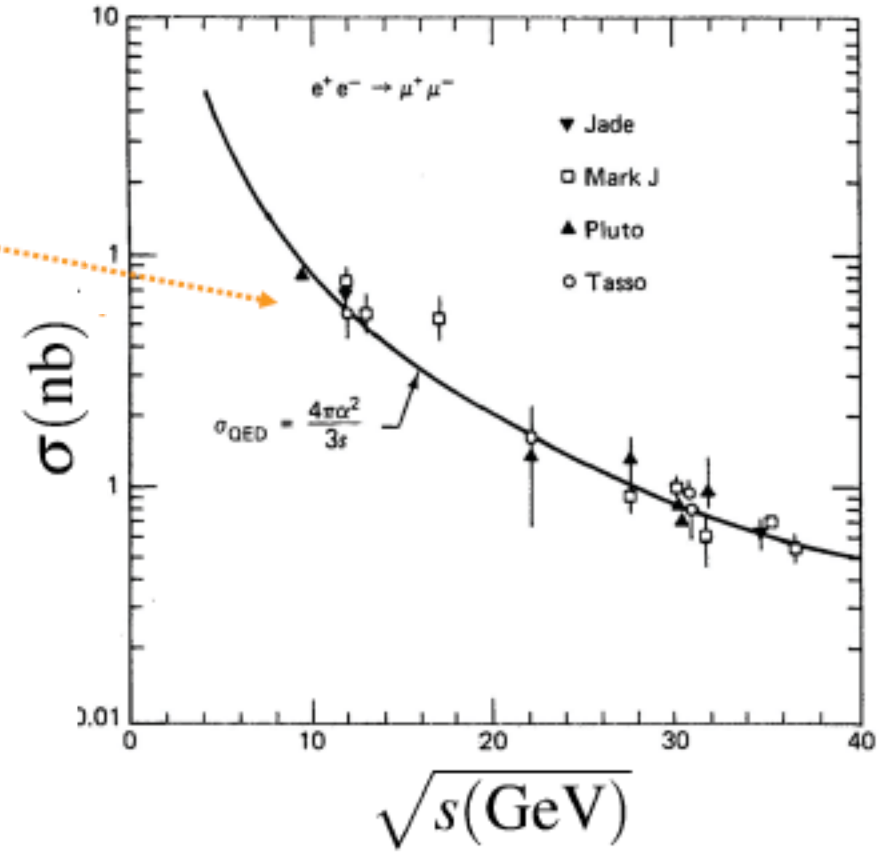


Total Cross-Section

- Integrating over angles obtain **total cross-section** for **e+e- → μ+μ-**

$$\boxed{\sigma = \frac{4\pi\alpha^2}{3s}}$$

Lowest order calculation provides a good description of data (good to **1%**!)



Chirality

- **Helicity** is not a fundamental concept (reference frame dependent)
- In ultra-relativistic limit, $E >> m$, helicity is the same as **chirality**
- Lagrangians are written in terms of **chiral states** (reference frame independent, i.e. fundamental)
- More formally, define matrix

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

- **In the limit** $E >> m$, **helicity** states are **eigenstates** of γ^5

$$\gamma^5 u_\uparrow = +u_\uparrow; \;\; \gamma^5 u_\downarrow = -u_\downarrow; \;\; \gamma^5 v_\uparrow = -v_\uparrow; \;\; \gamma^5 v_\downarrow = +v_\downarrow$$

(see slide 6 to check this)

- In general, define define the eigenstates of γ^5 as **left** and **right** handed **chiral** states

$$u_R; \quad u_L; \quad v_R; \quad v_L \qquad \gamma^5 u_R = +u_R; \;\; \gamma^5 u_L = -u_L; \;\; \gamma^5 v_R = -v_R; \;\; \gamma^5 v_L = +v_L$$

- **Only** in the limit $E >> m$

$$u_R \equiv u_\uparrow; \quad u_L \equiv u_\downarrow; \quad v_R \equiv v_\uparrow; \quad v_L \equiv v_\downarrow$$

Chirality

- Chirality is an important concept in the structure of QED, and any interaction of the form $\bar{u}\gamma^{\nu}u$

  - Define the **projection operators**

$$\boxed{P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)}$$

which project out chiral eigenstates:

$$\boxed{\begin{array}{llll} P_R u_R = u_R; & P_R u_L = 0; & P_L u_R = 0; & P_L u_L = u_L \\ P_R v_R = 0; & P_R v_L = v_L; & P_L v_R = v_R; & P_L v_L = 0 \end{array}}$$

E.g. P_R projects out right-handed particle states and left-handed antiparticle states

- Any spinor can be written in terms of its **left** and **right** chiral components

$$\psi = \psi_R + \psi_L = \tfrac{1}{2}(1+\gamma^5)\psi + \tfrac{1}{2}(1-\gamma^5)\psi$$

Chirality in QED and "Helicity Conservation"

- In QED the basic interaction between the fermion and the photon

$$ie\overline{\psi}\gamma^{\mu}\phi$$

- which can be decomposed into **Right** and **Left**-handed chiral components

$$\begin{aligned} ie\overline{\psi}\gamma^{\mu}\phi &= ie(\overline{\psi}_L+\overline{\psi}_R)\gamma^{\mu}(\phi_R+\phi_L) \\ &= ie(\overline{\psi}_R\gamma^{\mu}\phi_R+\overline{\psi}_R\gamma^{\mu}\phi_L+\overline{\psi}_L\gamma^{\mu}\phi_R+\overline{\psi}_L\gamma^{\mu}\phi_L) \end{aligned}$$

- One can show that

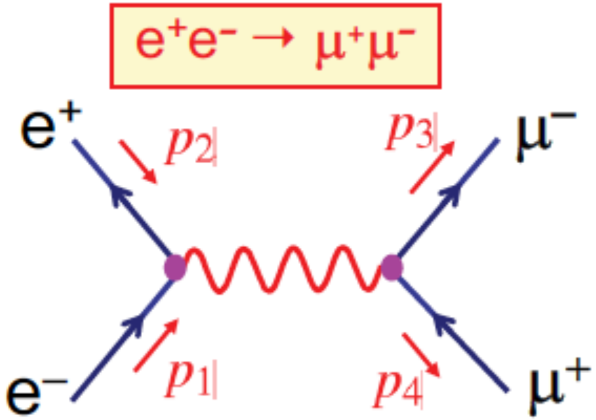
$$\overline{\psi}_R\gamma^{\mu}\phi_L=0; \quad \overline{\psi}_L\gamma^{\mu}\phi_R=0$$

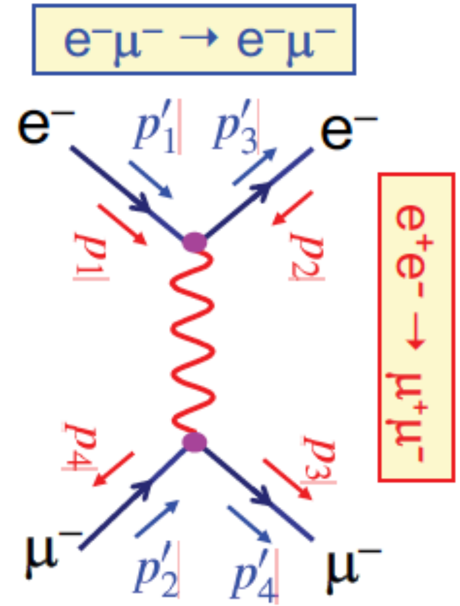
- Hence, only certain combinations of **chiral** states contribute to the interaction. This statement is **always** true

- For $E >> m$, **chiral** and **helicity** eigenstates are equivalent. Therefore **in this case** only certain helicity state combinations contribute to the interaction. This is why we previously found that only 4 helicity combinations contribute to the e+e- → μ+μ- matrix element — **"helicity conservation"** (but only if $E >> m$)

e-μ-⟶e-μ- Matrix Element

Elegant “cheating” to obtain e-μ-→e-μ- Matrix Element — **“Crossing Symmetry”**





$$\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p_1.p_3)^2 + (p_1.p_4)^2}{(p_1.p_2)^2}$$

$$\boxed{\langle |M_{fi}|^2 \rangle = 2e^4 \frac{(p'_1.p'_4)^2 + (p'_1.p'_2)^2}{(p'_1.p'_3)^2}}$$

(also a spin-averaged M.E.)

Summary

- Calculated the matrix element and cross-section for e+e- → μ+μ- in the ultra-relativistic case, $E >> m$

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)}$$

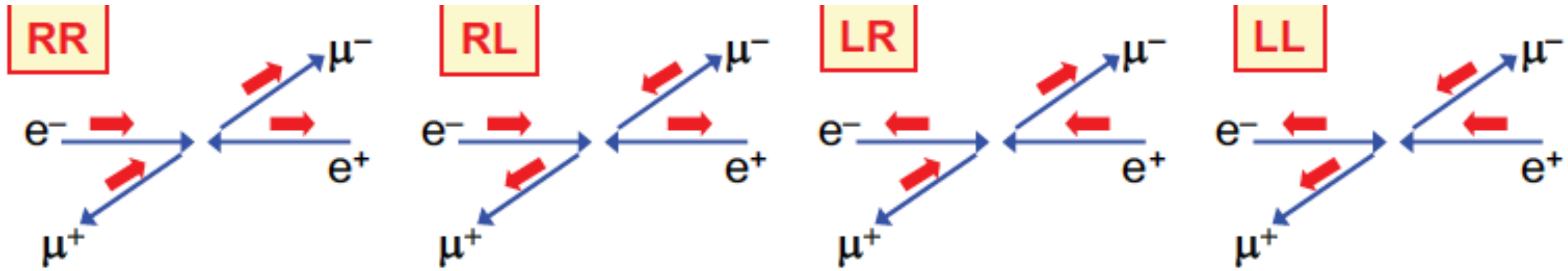
$$\boxed{\sigma = \frac{4\pi\alpha^2}{3s}}$$

- In QED only certain combinations of **Left** and **Right**-handed **chiral** states give non-zero contribution to matrix element

- Chiral states are defined by chiral projection operators

$$\boxed{P_R = \tfrac{1}{2}(1+\gamma^5); \qquad P_L = \tfrac{1}{2}(1-\gamma^5)}$$

- __**In the limit**__ $E >> m$ **chiral** and **helicity** eigenstates are equivalent and only certain helicity combinations contribute to the interaction — **“helicity conservation”**



Going beyond lowest order

Higher orders & renormalisation

- Lowest order A + A $\rightarrow$ B +B had $d\sigma/d\Omega \sim g^4$
- First assumption is that higher orders are suppressed since involve g^n ($n > 4$)

- But the calculation gives a divergent result at high energies !!

- This was a killer problem for 40 years and often plagues any new theories

- The fix is to ask the question - what is g (or equivalent “e” for QED processes) in the Feynman diagrams / rules

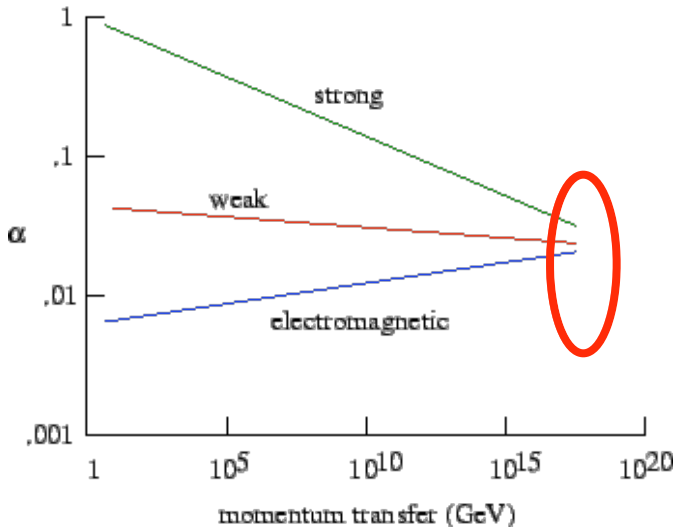
- If we use a “renormalised” value for “e” which actually corresponds to the one measured at a given momentum transfer (q) in the |M| calculation then this cancels the divergences. But it means our couplings are not fixed but “run”

Renormalisable theories & running couplings

- A renormalisable theory is one in which the “trick” of using renormalised quantities (masses, couplings) remove all infinities to all orders.

- It was shown that the class of theories known as gauge theories (of which QED and QCD are examples) are all renormalisable and so this is the type of theory people always start with, (Nobel Prize 1999).

- EM (QED) coupling constant increases with energy
- Strong (QCD) coupling constant decreases with energy (Nobel Prize 2004)



Don’t actually meet or unify unless new particles !