PHASM426 / PHAS4426
Advanced Quantum Theory Problem Sheet 1

To be handed in by 5pm on Tuesday 31rst October 2017.

Please hand in your completed work at the **end** of the lecture on that day. If you are unable to attend the lecture, you may scan your work, save it as a single PDF file and email it to me **prior** to this lecture. You may also bring the work to me in my office (B12) before the lecture. **Make sure your completed work is clearly labelled with your name and college**. Please note that UCL places severe penalties on late-submitted work.

1. Consider the vector space of real-valued polynomials of the power not larger than 3:

$$P_3(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 \,.$$

(a) Write down a set of functions that form a basis of this vector space. **[1]**

(b) What is the dimension of this vector space? **[1]**

Model answer: $P_3(x)$ is already written as a linear combination of the functions $f_0(x) = x^0, f_1(x) = x^1, f_2(x) = x^2$ and $f_3(x) = x^3$ which are linearly independent. The dimension of this basis is 4.
Marks: 1 mark for a correct answer for (a) or (b)

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2. Consider the basis of vectors$|\phi_j\rangle$ where j spans from 1 to n. Show that if the basis of vectors $\{|\phi_j\rangle\}$ is linearly independent, then for any vector $|\Psi\rangle$ the coefficients c_j of the expansion

$$|\Psi\rangle = \sum_{j=1}^{n} c_j |\phi_j\rangle$$

are unique. *Hint:* Two prove uniqueness you need to assume that there is another set of coefficients that will expand the state $|\Psi\rangle = \sum_{j=1}^{n} a_j |\phi_j\rangle$ and then prove that $a_j = c_j$. **[2]**

Model answer: Let's assume $|\Psi\rangle = \sum_{j=1}^{n} a_j |\phi_j\rangle$. Since $|\Psi\rangle$ belongs to a vector space, then its additive inverse exists, that is, there is a $|-\Psi\rangle$ such that $|\Psi\rangle + |-\Psi\rangle = 0$. Hence

$$\sum_{j=1}^{n} (c_j - a_j)|\phi_j\rangle = 0.$$

Since the basis $\{|\phi_j\rangle\}$ is linearly independent, then $c_j - a_j$ must equal zero for each j. Hence $c_j = a_j$.
Marks: 1 mark for correctly justifying the existence of the inverse additive, 1

marks for using correctly the argument of linear independence.

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3. The Hamiltonian of a quantum system is written in its spectral decomposition as $H = \sum_{n=1}^{d} \lambda_n |\phi_n\rangle\langle\phi_n|$, with $\langle\phi_m|\phi_n\rangle = \delta_{m,n}$ where the closure relationship is satisfied i.e. $\mathbb{1} = \sum_{n=1}^{d} |\phi_n\rangle\langle\phi_n|$. Prove that the exponential of H takes the form $e^H = \sum_{n=1}^{d} e^{\lambda_n} |\phi_n\rangle\langle\phi_n|$. **[3]**

   Model answer: In analogy of the series expansion of $e^x = \sum_{j=0}^{\infty} x^j/j!$ we expand e^H as

$$e^H = \mathbb{1} + H + \frac{H^2}{2} + \frac{H^3}{3!} + \cdots .$$

   Let us find out H^j for $j \geq 1$.

$$H^J = (\sum_{n=1}^{d} \lambda^n |\psi_n\rangle\langle\phi_n|)^j = \sum_{n=1}^{d} \lambda_n^j |\psi_n\rangle\langle\phi_n|$$

   because $\langle\phi_n|\phi_m\rangle = \delta_{m,m}$. Hence

$$\begin{aligned}
e^H =&\mathbb{1} + \sum_{n=1}^{d} \lambda_n |\phi_n\rangle\langle\phi_n| + \sum_{n=1}^{d} (\lambda_n^2/2) |\psi_n\rangle\langle\phi_n| + \cdots \\
=&\mathbb{1} + \sum_{n=1}^{d} (\lambda_n + \lambda_n^2/2 + \lambda_n^3/3! + \cdots) |\phi_n\rangle\langle\phi_n| \\
=&\mathbb{1} + \sum_{n=1}^{d} (e^{\lambda_n} - 1) |\phi_n\rangle\langle\phi_n| \\
=&\sum_{n=1}^{d} e^{\lambda_n} |\phi_n\rangle\langle\phi_n| + \mathbb{1} - \sum_{n=1}^{d} |\phi_n\rangle\langle\phi_n| \\
=&\sum_{n=1}^{d} e^{\lambda_n} |\phi_n\rangle\langle\phi_n|,
\end{aligned}$$

   where we have used the closure relationship $\mathbb{1} = \sum_{n=1}^{d} |\phi_n\rangle\langle\phi_n|$.
   Marks: 1 Mark for correctly expanding e^H, 1 mark for correctly finding H^j and 1 mark for using the closure relationship. Partial makrs for partial answers.

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4. Given two arbitrary vectors $|\phi_1\rangle$ and $|\phi_2\rangle$ belonging to the inner product space $\mathcal{H}$, the Cauchy-Schwartz inequality states that

$$|\langle\phi_1|\phi_2\rangle|^2 \leq \langle\phi_1|\phi_1\rangle\langle\phi_2|\phi_2\rangle. \tag{1}$$

The purpose of this problem is to use the properties of inner product to prove this inequality. To proceed with the proof consider the vector $|\Psi\rangle$ defined as:

$$|\Psi\rangle = |\phi_1\rangle + \lambda|\phi_2\rangle$$

where λ is a complex number that can be written as $\lambda = a + ib$.

(a) Write an expression for the inequality $\langle\Psi|\Psi\rangle \geq 0$ as a function of λ i.e. $f(\lambda)$. Then, re-write this expression as a function of a and b i.e. $f(a, b)$. **[2]**

Model answer: As a function of a and b the the inner product $\langle\Psi|\Psi\rangle$ becomes

$$\begin{aligned}
\langle\Psi|\Psi\rangle &= \langle\phi_1|\phi_1\rangle + \lambda\langle\phi_1|\phi_2\rangle + \lambda^*\langle\phi_2|\phi_1\rangle + |\lambda|^2\langle\phi_2|\phi_2\rangle \\
&= \langle\phi_1|\phi_1\rangle + a(\langle\phi_1|\phi_2\rangle + \langle\phi_2|\phi_1\rangle) + ib(\langle\phi_1|\phi_2\rangle - \langle\phi_2|\phi_1\rangle) \\
&\quad +(a^2+b^2)\langle\phi_2|\phi_2\rangle \\
&= \langle\phi_1|\phi_1\rangle + 2a\mathrm{Re}(\langle\phi_2|\phi_1\rangle) + 2b\mathrm{Im}(\langle\phi_2|\phi_1\rangle) + (a^2+b^2)\langle\phi_2|\phi_2\rangle \\
&=: f(a,b) \geq 0
\end{aligned}$$

In writing the last line we have used the fact that $\langle\phi_2|\phi_1\rangle = \langle\phi_1|\phi_2\rangle^*$.
Marks: 1 mark for the correct expression as a function of λ, 1 for a correct expression as a function of a and b.

(b) Show that the value of λ that minimises $\langle\Psi|\Psi\rangle$ is

$$\lambda_{min} = -\frac{\langle\phi_2|\phi_1\rangle}{\langle\phi_2|\phi_2\rangle} \qquad (2)$$

*Hint:*Compute the derivates of the function $f(a, b)$ obtained in (a) with respect to a and b. Solve these equations to obtain a_{min} and b_{min} and then compute λ_{min}. **[2]**

Model answer: To minimise $f(a, b)$ we need to calculate:

$$\frac{\partial f(a,b)}{\partial a} = 2\mathrm{Re}(\langle\phi_2|\phi_1\rangle) + 2a\langle\phi_2|\phi_2\rangle = 0$$
$$\frac{\partial f(a,b)}{\partial b} = 2\mathrm{Im}(\langle\phi_2|\phi_1\rangle) + 2b\langle\phi_2|\phi_2\rangle = 0$$

which leads to

$$a_{min} = -\frac{\mathrm{Re}(\langle\phi_2|\phi_1\rangle)}{\langle\phi_2|\phi_2\rangle} \text{ and } b_{min} = -\frac{\mathrm{Im}(\langle\phi_2|\phi_1\rangle)}{\langle\phi_2|\phi_2\rangle}$$

such that

$$\lambda_{min} = a_{min} + ib_{min} = -\frac{\mathrm{Re}(\langle\phi_2|\phi_1\rangle) + \mathrm{Im}(\langle\phi_2|\phi_1\rangle)}{\langle\phi_2|\phi_2\rangle} = -\frac{\langle\phi_2|\phi_1\rangle}{\langle\phi_2|\phi_2\rangle}.$$

Since $\langle\phi_2|\phi_2\rangle > 0$ the matrix of second derivatives is a positive definite and we therefore have found a minimum.

Marks: 1 Marks for obtaining a_{min}, b_{min} and writing correctly λ_{min} and 1 mark for justifying why we have indeed a minimum. Partial marks for partial answers.

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(c) Substitute Eq. (2) in the expression of $f(\lambda)$ derived in (a) and show that it reduces to the expression for the Cauchy-Schwuartz inequality (Eq. (1)). **[2]**

Model answer: Inserting Eq. (2) in the expression of $f(\lambda)$ we obtain

$$\begin{aligned} 0 &\leq \langle\phi_1|\phi_1\rangle - \frac{\langle\phi_2|\phi_1\rangle}{\langle\phi_2|\phi_2\rangle}\langle\phi_1|\phi_2\rangle - \frac{\langle\phi_1|\phi_2\rangle}{\langle\phi_2|\phi_2\rangle}\langle\phi_2|\phi_1\rangle + \left|\frac{\langle\phi_2|\phi_1\rangle}{\langle\phi_2|\phi_2\rangle}\right|^2 \langle\phi_2|\phi_2\rangle \\ &\leq \langle\phi_1|\phi_1\rangle - \frac{|\langle\phi_2|\phi_1\rangle|^2}{\langle\phi_2|\phi_2\rangle} \ , \end{aligned}$$

which is equivalent to Eq. (1).

Marks: 2 marks for deriving the correct expression.

(d) Which relation do $|\phi_1\rangle$ and $|\phi_2\rangle$ satisfy such that the equality in Eq. (1) is realised? **[1]**

Model answer: Eq. (1) becomes an equality when $|\phi_1\rangle = c|\phi_2\rangle$ i.e. the vectors are linearly dependent. Marks: 1 mark for stating that $|\phi_1\rangle$ need to be linearly dependent.

(e) Discuss in which cases the Cauchy-Schwartz inequality is important in quantum mechanics **[1]**

Model answer: The Cauchy-Schwartz inequality is important to demonstrate the generalised uncertainty principle for two arbitrary operators A and B.

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5. Consider a Hermitian operator A with eigenvalues $\{\lambda_1, \lambda_2, \cdots \lambda_n\}$ and eigenvectors $\{|\psi_1\rangle, |\psi_2\rangle \cdots |\psi_n\rangle\}$. Show that A can be written in terms of a unitary transformation U as $A = UDU^\dagger$, where D is a diagonal matrix. **[2]**

Model answer: We write A in its spectral decomposition, that is: $A = \sum_i \lambda_i|\psi_i\rangle\langle\psi_i|$. Since the states $\{|\psi_i\rangle\}$ form an orthonormal basis there is a unitatry transformation U such that $|\psi_i\rangle = U|\phi_i\rangle$ where $\{|\psi_i\rangle\}$ also form an orthonormal basis.

Hence we can re-write A as

$$A = \sum_i \lambda_i U |\phi_i\rangle\langle\phi_i| U^\dagger = U \left(\sum_i \lambda_i |\phi_i\rangle\langle\phi_i| \right) U^\dagger = UDU^\dagger.$$

Notice that $D = \sum_i \lambda_i |\phi_i\rangle\langle\phi_i|$ defines a diagonal matrix.
Marks: 1 for using the unitary transformation between basis and 1 for finding the diagonoal matrix.

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6. Consider a quantum system with Hamiltonian H and consider the measurement of an observable with a non-degenerate spectral decomposition $A = \sum_n a_n |\psi_n\rangle\langle\psi_n|$. The system is initially in the eigenstate $|\psi_n\rangle$ of A, with eigenvalue a_n. A series of ideal measurements on the observable A are carried out. The first measurement is carried out at time $t = \theta$. Then subsequent measurements are made at $t = 2\theta$, $t = 3\theta$ and so on. Here θ is very small.

   (a) Expand the state of the system to second order in time t and show that the probability of obtaining the eigenvalue a_n at $t = \theta$ is given by

   $$w_{nn}(\theta) \simeq 1 - (\Delta E)_n^2 \theta^2,$$

   where $(\Delta E)_n^2 = \langle\psi_n|H^2|\psi_n\rangle - \langle\psi_n|H|\psi_n\rangle^2$. Notice that $w_{nn}(\theta)$ is the probability that the system is still in the initial state. **[3]**

   Model answer: We consider $\hbar = 1$ and the time-evolution of the initial state $|\psi_n\rangle$ given by $|\Psi(t)\rangle = e^{iHt}|\psi_n\rangle$. Expanding the state to second order means to expand the exponential as $e^{iHt} = 1 - iHt - H^2t^2/2$, hence $|\Psi(t)\rangle = (1 - iHt - H^2t^2/2)\,|\psi_n\rangle$. For $t = \theta$ we have

   $$|\Psi(\theta)\rangle = \left(1 - iH\theta - H^2\theta^2/2\right)|\psi_n\rangle.$$

   Then the probability of obtaining the eigenvalue a_n is $w_{nn}(\theta) = |\langle\psi_n|\Psi(\theta)\rangle|^2$. We have $\langle\psi_n|\Psi(\theta)\rangle = 1 - i\theta\langle\psi_n|H|\psi_n\rangle - (1/2)\theta^2\langle\psi_n|H^2|\psi_n\rangle$ and therefore

   $$\begin{aligned} w_{nn}(\theta) =& (1 - i\theta\langle\psi_n|H|\psi_n\rangle - (1/2)\theta^2\langle\psi_n|H^2|\psi_n\rangle)\times \\ & (1 + i\theta\langle\psi_n|H|\psi_n\rangle - (1/2)\theta^2\langle\psi_n|H^2|\psi_n\rangle) \\ =& 1 - \theta^2\langle\psi_n|H|\psi_n\rangle^2 - \theta^2\langle\psi_n|H^2|\psi_n\rangle + \text{terms of higher order} \\ \simeq& 1 - (\Delta E)_n^2\theta^2 \end{aligned}$$

   Above we discarded the higher order terms as they depend on powers of θ greater than 2. Since θ is small, such terms can be neglected.

   Marks: 1 mark for expanding the state correctly, 1 mark for writting the correct definition of the probability and 1 mark for justifying discanring the higher order terms

(b) Show that after k measurements i.e. at $\tau = k\theta$, the probability $w_{nn}(\tau)$ becomes

$$w_{nn}(\tau) \simeq [1 - (\Delta E)^2_n \theta^2]^k.$$

[3]

Model answer:To compute the probability after k measurements we need to know the state after k measurements.

$$\tau = \theta \rightarrow |\Psi'(\theta)\rangle = \frac{|\psi_n\rangle\langle\psi_n|\Psi(\theta)\rangle}{\sqrt{w_{nn}(\theta)}} = |\psi_n\rangle$$

$\tau = 2\theta \rightarrow |\Psi(2\theta)\rangle = U(\theta)|\psi_n\rangle$ and probability of finding a_n is approximatly $w_{nn}(\theta)$.

Hence after two measurements the probability of finding the system in state $|\psi_n\rangle$ is $w_{nn}(2\theta) \simeq w_{nn}(\theta) \times w_{nn}(\theta)$. We can then genralize this to k mesurements or $\tau = k\theta$ such that

$$w_{nn}(k\theta) \simeq \underbrace{w_{nn}(\theta) \times w_{nn}(\theta) \cdots \times w_{nn}(\theta)}_{k \text{ times}} = w_{nn}(\theta)^k.$$

Marks: 1 for shwoing the state after measurement is $|psi_n\rangle$ and 2 marks for correct argument of multimplying probabilities after k measurements. Partial marks for partial answers.

(c) Assume k is large and τ is fixed such that $\theta/k \rightarrow 0$. Show that in this limit

$$w_{nn}(\tau) \simeq \exp[-(\Delta E)^2_n \tau\theta] \rightarrow 1.$$

You may use without proof the fact that **[2]**

$$e^x = \lim_{n\rightarrow\infty} \left(1 + \frac{x}{n}\right)^n.$$

Model answer: Write $\theta = \tau/k$. Hence $w_{nn}(\tau) \simeq [1-(\Delta E)^2_n \theta(\theta/k)]^k$. Set $x = -(\Delta E)^2_n \theta$ so that $w_{nn}(\tau) \simeq [1 + x/k]^k$. Consider the limit of $k \rightarrow \infty$ and use the limit given so $w_{nn}(\tau) \simeq e^x = e^{-(\Delta E)^2_n \theta}$. When $k \rightarrow \infty$ then $\theta \rightarrow 0$ and $w_{nn}(\tau) \rightarrow e^0 = 1$

Marks: 1 mark for replacing $\theta = \tau/k$ and 1 mark for using correctly the limit given. Partial marks for partial answers.