Quantum Field Theory — Problem Sheet 1

Problem 1.

The Klein-Gordon equation is

$$(\partial^\mu \partial_\mu - m^2)\phi = 0$$

$$\therefore \quad \phi^* (\partial^\mu \partial_\mu - m^2)\phi = 0 \qquad (1)$$

Taking complex conjugate

$$\phi\, (\partial^\mu \partial_\mu - m^2)\phi^* = 0 \qquad (2)$$

Subtracting (1) from (2)

$$\phi(\partial^\mu \partial_\mu)\phi^* - \phi^*(\partial^\mu \partial_\mu)\phi = 0$$

$$\therefore \quad \phi \frac{\partial^2}{\partial t^2}\phi^* - \phi^* \frac{\partial^2}{\partial t^2}\phi + \phi^* \underline{\nabla}^2 \phi - \phi \underline{\nabla}^2 \phi^* = 0$$

$$\therefore \quad \frac{\partial}{\partial t}\left(\phi \frac{\partial}{\partial t}\phi^* - \phi^* \frac{\partial}{\partial t}\phi\right) + \underline{\nabla}.\left(\phi^* \underline{\nabla}\phi - \phi \underline{\nabla}\phi^*\right) = 0 \qquad (3)$$

Since $\frac{\partial}{\partial t}\left(\phi \frac{\partial \phi^*}{\partial t} - \phi^* \frac{\partial \phi}{\partial t}\right) = \cancel{\frac{\partial \phi}{\partial t}\frac{\partial \phi^*}{\partial t}} + \phi \frac{\partial^2 \phi^*}{\partial t^2} - \cancel{\frac{\partial \phi^*}{\partial t}\frac{\partial \phi}{\partial t}} - \phi^* \frac{\partial^2 \phi}{\partial t^2}$

and similarly for $\underline{\nabla}.(\phi^* \underline{\nabla}\phi - \phi \underline{\nabla}\phi^*)$.

but $\quad J^\mu = i\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t},\ -(\phi^* \underline{\nabla}\phi - \phi \underline{\nabla}\phi^*)\right)$

$$\therefore \quad \partial_\mu J^\mu = i\left[\frac{\partial}{\partial t}\left(\phi^* \frac{\partial \phi}{\partial t} - \phi \frac{\partial \phi^*}{\partial t}\right) \mp \underline{\nabla}(\phi^* \underline{\nabla}\phi - \phi \underline{\nabla}\phi^*)\right]$$

$$\therefore \partial_\mu J^\mu = 0 \rightarrow -i\left[\frac{\partial}{\partial t}\left(\phi\frac{\partial\phi^*}{\partial t} - \phi^*\frac{\partial\phi}{\partial t}\right) + \underline{\nabla}\cdot(\phi^*\underline{\nabla}\phi - \phi\underline{\nabla}\phi^*)\right] = 0$$

$\therefore$ From (3) $\underline{\underline{\partial_\mu J^\mu = 0.}}$ 6

Problem 2

For $x < 0$, $\phi = e^{-iE_pt+ipx} + a\,e^{-iE_pt-ipx}$

$$\therefore \frac{\partial\phi}{\partial x} = ip\left(e^{-iE_pt+ipx} - a\,e^{-iE_pt-ipx}\right)$$

$$\phi^* = e^{+iE_pt-ipx} + a\,e^{+iE_pt+ipx}$$

$$\therefore \frac{\partial\phi^*}{\partial x} = -ip\left(e^{+iE_pt-ipx} - a\,e^{iE_pt+ipx}\right)$$

$$J_x = -i\left(\phi^*\frac{\partial\phi}{\partial x} - \phi\frac{\partial\phi^*}{\partial x}\right)$$

$$= p\left[1 + a\left(e^{iE_pt+ipx} - e^{-iE_p-ipx}\right) - a^2 + 1 - a\left(e^{iE_pt+ipx} - e^{-iE_p-ipx}\right) - a^2\right]$$

$$\therefore J_x = 2p(1-a^2).$$

But $|a| > 1$ $\therefore$ $1 - a^2 < 0$

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$\therefore$ $\underline{\underline{J_x < 0.}}$

$$\rho = \phi^*\left(i\frac{\partial}{\partial t} - V\right)\phi + \phi\left(-i\frac{\partial}{\partial t} - V\right)\phi^*$$

For $x > 0$ $\quad \phi = b\, e^{-iE_p t + ikx}$, $\phi^* = b\, e^{+iE_p t - ikx}$

$$\therefore \quad i\frac{\partial \phi}{\partial t} = E_p\, b\, e^{-iE_p t + ikx}, \quad -i\frac{\partial \phi^*}{\partial t} = E_p\, b\, e^{+iE_p t - ikx}$$

$$\therefore \quad \rho = b^2 (E_p - V) + b^2 (E_p - V)$$

$$\therefore \quad \rho = 2b^2 (E_p - V).$$

But $E_p < V$ $\quad \therefore \underline{\underline{\rho < 0.}}$ 2

$\therefore$ ρ cannot be a probability density and $J_x < 0$ implies more particles reflected than incident. $\therefore$ ρ must be a density of some quantum number, e.g. charge where antiparticles carry opposite to particles, and antiparticles must be created at the boundary. $\therefore$ antiparticles for $x > 0$ and both incident and new particles created at boundary reflected for $x < 0$. 2.

Question 3

In spherical polars kinetic energy

$\frac{1}{2} m \dot{r}^2$ — radial
$\frac{1}{2} m r^2 \dot{\theta}^2$ — motion in θ-direction
$\frac{1}{2} m r^2 \sin^2\theta\, \dot{\phi}^2$ — motion in ϕ direction

$$L = T - V = \frac{1}{2} m (\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\, \dot{\phi}^2) - V(r)$$ 1

Equation of motion $\frac{\partial L}{\partial r} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right)$

$\frac{\partial L}{\partial \dot{r}} = m\dot{r} \quad \therefore \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = m\ddot{r}$

$$\frac{\partial L}{\partial r} = mr\dot{\theta}^2 + mr\sin^2\theta\,\dot{\phi}^2 - \frac{\partial V}{\partial r}$$

$$\therefore \quad m\ddot{r} - r(\dot{\theta}^2 + \sin^2\theta\,\dot{\phi}^2) + \frac{\partial V}{\partial r} = 0$$ 2.

$$H = \sum_i p_i \dot{q}_i - L$$

$p_r = \frac{\partial L}{\partial \dot{r}} = m\dot{r}$, $p_\theta = \frac{\partial L}{\partial \dot{\theta}} = mr^2\dot{\theta}$, $p_\phi = \frac{\partial L}{\partial \phi} = mr^2\sin^2\theta\,\dot{\phi}$

$$\therefore H = m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2\sin^2\theta\,\dot{\phi}^2 - \frac{1}{2}(m\dot{r}^2 + mr^2\dot{\theta}^2$$
$$+ mr^2\sin^2\theta\,\dot{\phi}^2) + V(r)$$

$$= \frac{1}{2}(m\dot{r}^2 + mr^2\dot{\theta}^2 + mr^2\sin^2\theta\,\dot{\phi}^2) + V(r)$$

but writing in terms of p_r, p_θ, p_ϕ

$$H = \frac{p_r^2}{2m} + \frac{p_\theta^2}{r^2} + \frac{p_\phi^2}{r^2\sin^2\theta} + V(r).$$ 3

H independent of ϕ. $\therefore p_\phi = -\frac{\partial H}{\partial \phi} = 0$

But $p\phi$ = angular momentum about z-axis 2.

4. $a^+|n\rangle = c_n|n+1\rangle$, where c_n is a normalisation constant.

$\therefore \langle n|a = \langle n+1|c_n^*$

$\therefore \langle n|aa^+|n\rangle = \langle n+1|c_n^* c_n|n+1\rangle = |c_n|^2$.

$aa^+ = 1 + a^+a$

and $\hat{H} = \omega(a^+a + \frac{1}{2})$

$\therefore aa^+ = 1 + \left(\frac{\hat{H}}{\omega} - \frac{1}{2}\right)$

But $\hat{H}|n\rangle = \omega(n+\frac{1}{2})$.

$$\therefore \langle n|aa^+|n\rangle = \langle n|\left(\frac{1}{2} + \frac{\hat{H}}{\omega}\right)|n\rangle$$

$$= (n+1)\langle n|n\rangle = n+1$$

$\therefore |c_n|^2 = n+1 \quad \therefore c_n = \sqrt{n+1}$.

$\therefore$ if $|n\rangle = \frac{1}{\sqrt{n!}}(a^+)^n|0\rangle$ then $|n+1\rangle = \frac{1}{\sqrt{n+1!}}(a^+)^{n+1}|0\rangle$

Consider $|1\rangle = a^+|0\rangle$

$\therefore \langle 1|1\rangle = \langle 0|aa^+|0\rangle = \langle 0|0\rangle$ from above.

$\therefore$ true for $|1\rangle$ $\therefore$ true for all n by induction.

Question 5

The momentum space fields are defined by

$$\tilde{\phi}(\underline{p}) = \int d^3\underline{x} \; \phi(\underline{x}) e^{-i\underline{p}\cdot\underline{x}}$$

$$\tilde{\pi}(\underline{p}) = \int d^3\underline{x} \; \pi(\underline{x}) e^{-i\underline{p}\cdot\underline{x}}$$

$$\therefore \; [\tilde{\phi}(\underline{p}), \tilde{\pi}(\underline{q})] = \int d^3\underline{x}\, d^3\underline{y} \; [\phi(\underline{x}), \pi(\underline{y})] e^{-i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}}$$

Using $[\phi(\underline{x}), \pi(\underline{y})] = i\,\delta^3(\underline{x}-\underline{y})$

$$\Rightarrow [\tilde{\phi}(\underline{p}), \tilde{\pi}(\underline{q})] = i\int d^3\underline{x}\, d^3\underline{y} \; \delta^3(\underline{x}-\underline{y}) \; e^{-i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}}$$

$$= i\int d^3\underline{x} \; e^{-i\underline{p}\cdot\underline{x} - i\underline{q}\cdot\underline{y}}$$

$$= i\,(2\pi)^3 \; \delta(\underline{p}+\underline{q})$$

(using our convention for normalization of Fourier transforms).

$$a_{\underline{p}} = E(\underline{p})\,\tilde{\phi}(\underline{p}) + i\tilde{\pi}(\underline{p})$$
$$a_{\underline{p}}^{+} = E(-\underline{p})\,\tilde{\phi}(-\underline{p}) - i\tilde{\pi}(-\underline{p}) = E(\underline{p})\,\tilde{\phi}(\underline{p}) - i\tilde{\pi}(\underline{p})$$

Hence,

$$[a_{\underline{q}}, a_{\underline{p}}^{+}] = a_{\underline{q}} a_{\underline{p}}^{+} - a_{\underline{p}}^{+} a_{\underline{q}}$$

$$= (E(\underline{q})\,\tilde{\phi}(\underline{q}) + i\tilde{\pi}(\underline{q}))(E(\underline{p})\,\tilde{\phi}(\underline{p}) - i\tilde{\pi}(\underline{p})) - (E(\underline{p})\,\tilde{\phi}(\underline{p}) - i\tilde{\pi}(\underline{p}))(E(\underline{q})\,\tilde{\phi}(\underline{q}) + i\tilde{\pi}(\underline{q}))$$

$$= E(p)E(q)\Big(\tilde{\varphi}(p)\tilde{\varphi}(q) - \tilde{\varphi}(q)\tilde{\varphi}(p) + (\tilde{\pi}(q)\tilde{\pi}(p) - \tilde{\pi}(p)\tilde{\pi}(q)) - iE(q)\,(\tilde{\varphi}(q)\tilde{\pi}(p) - \tilde{\pi}(p)\tilde{\varphi}(q)) - iE(p)\,(\tilde{\varphi}(p)\tilde{\pi}(q) - \tilde{\pi}(q)\tilde{\varphi}(p))\Big)$$

But $(\tilde{\varphi}(p)\tilde{\varphi}(q) - \tilde{\varphi}(q)\tilde{\varphi}(p)) = (\tilde{\pi}(q)\tilde{\pi}(p) - \tilde{\pi}(p)\tilde{\pi}(q)) = 0$

$$\therefore [a_q, a_p^+] = -iE(q)\,[\tilde{\varphi}(q), \tilde{\pi}(p)] - iE(p)\,[\tilde{\varphi}(p), \tilde{\pi}(q)]$$

$$= -iE(q)\,(i(2\pi)^3\delta^3(p-q)) - iE(p)\,(i(2\pi)^3\delta^3(p-q))$$

The delta-function enforces $E(q) = E(p)$

$$\therefore \underline{[a_q, a_p^+] = (2\pi)^3 2E(p)\,\delta^3(p-q).}$$

Question 6

$$\langle q | p \rangle = \langle 0 | a_q a_p^+ | 0 \rangle$$

$$= \langle 0 | a_p^+ a_q | 0 \rangle + \langle 0 | [a_q, a_p^+] | 0 \rangle$$

$a_q |0\rangle = 0$ so first term $=0$.

$$\therefore \langle q | p \rangle = \langle 0 | (2\pi)^3 2E(p)\delta^3(p-q) | 0 \rangle$$

$$= (2\pi)^3 2E(p)\,\delta^3(p-q).$$

$$\therefore \int \frac{d^3p}{(2\pi)^3 2E(p)} \langle q | p \rangle = \int \frac{d^3p}{(2\pi)^3 2E(p)} (2\pi)^3 2E(p)\,\delta^3(p-q)$$

$$= \underline{\int d^3p\ \delta^3(p-q) = 1.}$$

Question 7

The unequal time commutator is given in the notes as

$$[\phi(x), \phi(y)] = \int \frac{d^3 p}{2E(p)(2\pi)^3} \left(e^{-ip\cdot(x-y)} - e^{ip(x-y)}\right).$$

For space-like separations we have $(x-y)^2 < 0$. All possible values can be obtained by setting $t_1 = t_2$ and varying $|(\underline{x}-\underline{y})|^2$. Since $[\phi(x), \phi(y)]$ is Lorentz invariant this will span all possible $(x-y)^2 < 0$ values.

$$\rightarrow \quad [\phi(x), \phi(y)] = \int \frac{d^3 \underline{p}}{2E(p)(2\pi)^3} \left(e^{i\underline{p}\cdot(\underline{x}-\underline{y})} - e^{-i\underline{p}\cdot(\underline{x}-\underline{y})}\right)$$

$$= \int \frac{d^3 \underline{p}}{(2\pi)^3\, 2E(p)} e^{i\underline{p}\cdot(\underline{x}-\underline{y})} - \int \frac{d^3 \underline{p}}{2E(p)(2\pi)^3} e^{-i\underline{p}\cdot(\underline{x}-\underline{y})}.$$

Letting $\underline{p} \rightarrow -\underline{p}$ in second term $\int d^3 \underline{p}$ unchanged

$$\rightarrow [\phi(x), \phi(y)] = \int \frac{d^3 \underline{p}}{2E(p)(2\pi)^3} e^{i\underline{p}\cdot(\underline{x}-\underline{y})} - \int \frac{d^3 \underline{p}}{(2E(p))(2\pi)^3} e^{i\underline{p}\cdot(\underline{x}-\underline{y})}$$

$$= 0.$$

$$\phi(x) = \int \frac{d^3 \underline{p}}{2E(p)} \frac{1}{(2\pi)^3} \left(a_{\underline{p}} e^{-ip\cdot x} + a_{\underline{p}}^{\dagger} e^{ip\cdot x}\right)$$

$$\pi(y) = i\int \frac{d^3 q}{2(2\pi)^3} \left(-a_{\underline{q}} e^{-iq\cdot x} + a_{\underline{q}}^{\dagger} e^{iq\cdot x}\right)$$

$$\therefore \; [\phi(x), \pi(y)] = i \iint \frac{d^3p}{2E(p)(2\pi)^3} \frac{d^3q}{2(2\pi)^3} [(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x}), (a_q e^{-iq\cdot x} + a_q^\dagger e^{iq\cdot x})]$$

$$= i \iint \frac{d^3p}{2E(p)(2\pi)^3} \frac{d^3q}{2(2\pi)^3} \left(-[a_p^\dagger, a_q] e^{ip\cdot x - iq\cdot y} + [a_p, a_q^\dagger] e^{iq\cdot y - ip\cdot x}\right)$$

Using the fact that $[a_p, a_q] = [a_p^\dagger, a_q^\dagger] = 0$

$$= i \iint \frac{d^3p}{2E(p)(2\pi)^3} \frac{d^3q}{2(2\pi)^3} \left((2\pi)^3 2E(p)\right)\left(e^{iq\cdot y - ip\cdot x} + e^{ip\cdot x - iq\cdot y}\right) \times \delta^3(p-q)$$

$$\therefore \; [\phi(x), \pi(y)] = i \int \frac{d^3p}{2(2\pi)^3} \left(e^{-ip(x-y)} + e^{ip\cdot(x-y)}\right)$$

Setting $t_1 = t_2$

$$[\phi(\underline{x}), \pi(y)] = i \int \frac{d^3p}{2(2\pi)^3} \left(e^{-ip\cdot(\underline{x}-y)} + e^{ip\cdot(\underline{x}-y)}\right)$$

Again letting $p \to -p$ in the second term

$$[\phi(\underline{x}), \pi(y)] = i \int \frac{d^3p}{2(2\pi)^3} \left[e^{-ip\cdot(\underline{x}-y)} + e^{-ip\cdot(\underline{x}-y)}\right]$$

$$= i \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(\underline{x}-y)}$$

$$= i (2\pi)^3 \delta(\underline{x}-y) / (2\pi)^3$$

$$\therefore \; \underline{\underline{[\phi(\underline{x}), \pi(y)] = i\delta(\underline{x}-y)}}$$