UNIVERSITY COLLEGE LONDON

EXAMINATION FOR INTERNAL STUDENTS

MODULE CODE : **PHASG426**

ASSESSMENT PATTERN : **PHASG426C**

MODULE NAME : **Advanced Quantum Theory**

DATE : **13-May-15**

TIME : **10:00**

TIME ALLOWED : **2 Hours 30 Minutes**

Answer THREE questions.

The numbers in square brackets in the right-hand margin indicate the provisional allocation of maximum marks per sub-section of a question.

The following formulae may be used if required:

$$e^x = \sum_{j=0}^{\infty} \frac{x^j}{j!} \quad \sin(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j+1}}{(2j+1)!} \quad \cos(x) = \sum_{j=0}^{\infty} (-1)^j \frac{x^{2j}}{(2j)!}$$

The Time-dependent Schrödinger Equation

$$i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle = H |\psi(t)\rangle$$

Pauli operators

$\sigma_z |\uparrow\rangle = |\uparrow\rangle$, $\sigma_z |\downarrow\rangle = -|\downarrow\rangle$; $\sigma_x |\uparrow\rangle = |\downarrow\rangle$, $\sigma_x |\downarrow\rangle = |\uparrow\rangle$; $\sigma_y |\uparrow\rangle = i|\downarrow\rangle$, $\sigma_y |\downarrow\rangle = -i|\uparrow\rangle$, where $\{|\uparrow\rangle, |\downarrow\rangle\}$ is the orthonormal basis for a spin-half quantum system.

WKB Connection formulae

Right-hand barrier - with classical turning point at $x = a$:

$$\frac{2}{\sqrt{p(x)}} \cos\left(\int_x^a p(x')dx'/\hbar - \frac{\pi}{4}\right) \quad \leftarrow \quad \frac{1}{\sqrt{q(x)}} \exp\left[-\int_a^x q(x')dx'/\hbar\right]$$

$$-\frac{1}{\sqrt{p(x)}} \sin\left(\int_x^a p(x')dx'/\hbar - \frac{\pi}{4}\right) \quad \rightarrow \quad \frac{1}{\sqrt{q(x)}} \exp\left[+\int_a^x q(x')dx'/\hbar\right]$$

Left-hand barrier - with classical turning point at $x = a$:

$$\frac{1}{\sqrt{q(x)}} \exp\left[-\int_x^a q(x')dx'/\hbar\right] \quad \rightarrow \quad \frac{2}{\sqrt{p(x)}} \cos\left(\int_a^x p(x')dx'/\hbar - \frac{\pi}{4}\right)$$

$$\frac{1}{\sqrt{q(x)}} \exp\left[+\int_x^a q(x')dx'/\hbar\right] \quad \leftarrow \quad -\frac{1}{\sqrt{p(x)}} \sin\left(\int_a^x p(x')dx'/\hbar - \frac{\pi}{4}\right)$$

1. In order to derive the WKB wave-function in a classically-allowed region where $V(x) < E$, we use the ansatz $\psi(x) = A\exp[iS(x)/\hbar]$ and expand $S(x)$ as a power series in $\hbar$. The zeroth and first order solutions of $S(x)$ read as:

$$S_0(x) = \pm \int^x p(x')dx' + C \quad \text{and} \qquad S_1(x) = \frac{i}{2}\Big(\ln(p(x)) + D\Big),$$

where $p(x) = \sqrt{2m(E - V(x))}$ and C and D are constants.

(a) Derive the WKB wave-function in the classically-allowed region clearly stating the approximations made and the conditions under which they hold. [3]

(b) One can show that the second order term in the ansatz function $\psi(x)$ can be neglected when

$$\left|\frac{\hbar m}{p(x)^3}\frac{dV(x)}{dx}\right| \ll 1.$$

Show that this condition may equally be written in the form

$$\left|\frac{1}{2\pi}\frac{d\lambda(x)}{dx}\right| \ll 1\,,$$

where $\lambda(x) = h/p(x)$ is the local de Broglie wavelength. [3]

Hint: Notice that $\lambda(x)$ is a function of $p(x)$. You may want to write $V(x)$ as a function of $p(x)$ and consider the corresponding derivative $dV(x)/dx$.

(c) Consider a particle of mass m and energy $E < 0$, subject to the potential $V(x) = -c/x$ for $x > 0$ and $V(x = 0) = \infty$ for $x = 0$. Here $c > 0$.

i. Sketch $V(x)$ as a function of x and indicate the classical turning points. [1]

ii. Write down the form of the eigenfunctions of the Hamiltonian for $x \geq 0$. Justify your answer in detail. You will find WKB connection formulae in the rubric at the begining of this paper. [3]

iii. Show that in this case the WKB approximation leads to the following quantization condition:

$$\int_0^a p(x')dx' = \hbar\pi\left(n + \frac{3}{4}\right)\,,$$

where a is a classical turning point. Write down an expression for the classical turning point in terms of E. [3]

iv. Using the quantization condition derived above, show that the allowed energies are given by

$$E = -\frac{c^2 m}{2\hbar^2(n + 3/4)^2}\,.$$

You may use the integral [7]

$$\int_0^1 dx\sqrt{-1 + \frac{1}{x}} = \frac{\pi}{2}.$$

2. (a) Let H be the total Hamiltonian of a quantum system and its environment and $U(t)$ the unitary evolution associated to H. At time $t = 0$ the total state of the system and environment is $\rho(t=0) = \rho_s(0) \otimes \rho_B(0)$, where the initial state of the environment is $\rho_B(0) = |B_0\rangle \langle B_0|$. Here the basis set $|B_k\rangle$ spans the environment states.

i. Show that the reduced density matrix operator for the system $\rho_s(t)$ takes the form

$$\rho_s(t) = \sum_k S_k \rho_s(0) S_k^\dagger,$$

where $S_k = \langle B_k | U(t) | B_0 \rangle$. [2]

ii. Justify whether S_k is an operator acting on the system or on the environment and show that $\sum_k S_k^\dagger S_k = \mathbb{1}$. Discuss the physical implications of this result. [3]

(b) A two-level atom, with energy eigenstates $|g\rangle$ with energy $E_g = 0$ and $|e\rangle$ with $E_e = \hbar\omega$, is interacting with an environment such that the probability of finding the atom in the excited state at time t, $\rho_{ee}(t) = \langle e| \rho(t) |e\rangle$, is given by

$$\rho_{ee}(t) = \frac{|\Gamma|^2}{|\gamma|^2 + |\Gamma|^2}\left(1 - e^{-(|\gamma|^2+|\Gamma|^2)t}\right).$$

Find a relation between Γ and γ such that in the long-time limit $\rho_{ee}(\infty)$ equals the probability of finding the atom in its excited state when it is in thermal equilibrium at temperature T. Express your answer as $|\Gamma|^2 = C|\gamma|^2$ and specify the value of C as a function of ω and k_BT, where k_B is the Boltzmann constant. Recall that in thermal equilibrium, a system with Hamiltonian H is described by the density matrix operator $\rho_{eq} = \frac{\exp(-H/k_BT)}{\mathrm{Tr}[\exp(-H/k_BT)]}$. [4]

(c) The Markovian master equation for the density matrix ρ describing an open quantum system can be written in the following Lindblad form as

$$\frac{\partial \rho(t)}{\partial t} = \frac{-i}{\hbar}\left(H_{\text{eff}}\rho(t) - \rho(t)H_{\text{eff}}^\dagger\right) + \sum_j L_j \rho(t) L_j^\dagger,$$

where $H_{\text{eff}} = H - (i\hbar/2)\sum_j L_j^\dagger L_j$. Suppose that in the master equation above you neglect the term $\sum_j L_j \rho(t) L_j^\dagger$. What unphysical features will the resultant density matrix operator exhibit? Justify your answer with specific calculations. [5]

(d) Consider a two-level atom, with energy eigenstates $|g\rangle$ with energy $E_g = 0$ and $|e\rangle$ with $E_e = \hbar\omega$. The atom is initially in the state $|\psi(0)\rangle = a\,|e\rangle + b\,|g\rangle$, where a and b are real numbers. The influence of an environment on this atom is described by the operator $L = \alpha\,|e\rangle\langle e|$. Show that the time-evolution of the coherence $\rho_{eg}(t) = \langle e|\rho(t)|g\rangle$ takes the form

$$\rho_{eg}(t) = ab\, e^{-|\alpha|^2 t/2} e^{-i\omega t}.$$

Draw a sketch of the real part of $\rho_{eg}(t)$. [6]

3. (a) The interaction picture is employed to describe the dynamics of a system whose Hamiltonian has the form $H = H_0 + V$, where the eigenstates and eigenenergies of H_0 are known. In this representation, operators take the form $O_I(t) = U_0^\dagger(t) O U_0(t)$ with $U_0(t) = \exp[-iH_0 t/\hbar]$. A Hamiltonian describing the interaction of a spin-$\frac{1}{2}$ particle and a quantum harmonic oscillator takes the form (for simplicity we omit the tensor product notation):

$$H = \frac{\hbar\epsilon}{2}\sigma_z + \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + \hbar g \sigma_z (a + a^\dagger).$$

i. Show that $V_I(t) = \hbar g \sigma_z \left(e^{-i\omega t} a + e^{i\omega t} a^\dagger\right)$. You may use the identity $a = \sum_{n=0} \sqrt{n+1}\,|n\rangle\langle n+1|$. [3]

ii. Assume that the joint initial state of the spin and the harmonic oscillator is $|\Psi\rangle = |\uparrow\rangle|0\rangle$. Under the action of $V(t)$, will the state of the spin change? Will the state of the harmonic oscillator change? Justify your answers with calculations. [2]

(b) For a quantum system subject to a weak time-dependent perturbation with associated Hamiltonian $H(t) = H_0 + \lambda V(t)$, its state in the interaction picture can be written as $|\Psi(t)\rangle_I = \sum_j c_j(t)\,|\psi_j\rangle$, where $|\psi_j\rangle$ are eigenstates of H_0. Using the expansion $c_j(t) = \sum_{n=0}^{\infty} \lambda^n c_j^{(n)}(t)$ one finds that the nth-order term satisfies the following equation:

$$\frac{\partial c_j^{(n)}(t)}{\partial t} = \frac{1}{i\hbar}\sum_k V_{jk}(t)\exp[i\omega_{jk}t]\,c_k^{(n-1)}(t)\,.$$

i. Explain what $V_{jk}(t)$ and ω_{jk} stand for. Assuming the system is initially in an eigenstate $|\psi_m\rangle$, write down general expressions for the first and second-order solutions $c_j^{(1)}(t)$ and $c_j^{(2)}(t)$. [4]

ii. Consider a case where $c_j^{(1)}(t) = 0$ for $j \neq m$. Does this imply $c_j^{(2)}(t)$ also vanishes? Justify your answer. [1]

iii. Show that the energy of the system described by $H(t)$ is not conserved. [3]

(c) A quantum harmonic oscillator exposed to a weak perturbation has a Hamiltonian

$$H = \hbar\omega\left(a^\dagger a + \frac{1}{2}\right) + f(t)(a + a^\dagger)\,,$$

where $f(t) = \hbar A$ for $0 \leq t < \tau$. At time $t = 0$ the state is $|\Psi(0)\rangle = |n = 0\rangle$. Show that the probability (to second order) of finding the system in the state $|n = 2\rangle$ at $\tau = \pi/2\omega$ is given by $P_{n=2}^{(2)}(\tau = \pi/2) = A^4/\omega^4$. [7]

4. (a) Two quantum systems a and b are prepared in a joint state $|\Psi\rangle = |\psi_a\rangle |\psi_b\rangle$. Let the set $\{|\phi_k\rangle\}$ be a basis of states for system a, $\{|\nu_j\rangle\}$ a basis for system b and A an operator on the Hilbert space of a with the following spectral decomposition:

$$A = \lambda_1 |\phi_1\rangle \langle\phi_1| + \lambda_2 \sum_{n=2}^{N} |\phi_n\rangle \langle\phi_n| .$$

i. Write down an expression for the projector acting on the total Hilbert space and which is associated to the measurement outcome λ_2. Derive an expression for the probability of obtaining λ_2 in a measurement of A on the state $|\Psi\rangle$. [3]

ii. Derive an expression for the state $|\Psi'\rangle$ of the global system after the measurement. Has the state of the system b changed? [2]

iii. Discuss whether the predictions in (i) and (ii) would change depending of the choice of basis for the system b. Will the results of measurements on the systems a and b be correlated? Justify your answer. [2]

(b) Consider a spin-$\frac{1}{2}$ system whose Hamiltonian changes discontinuously as follows: for $0 \leq t \leq t_1$ the dynamics is given by $H_1 = \hbar g \sigma_x$ and for $t > t_1$ the Hamiltonian is $H_2 = \hbar\epsilon\sigma_z/2$. The state of the system at time $t > t_1$ is given by

$$|\Psi(t)\rangle = U_2(t - t_1)U_1(t_1) |\Psi(0)\rangle ,$$

where $U_1(t) = \exp\left[-i\frac{H_1 t}{\hbar}\right]$ and $U_2(t) = \exp\left[-i\frac{H_2 t}{\hbar}\right]$. Write down the time-evolved state $|\Psi(t)\rangle$ for $t > t_1$, assuming $|\Psi(0)\rangle = \frac{1}{\sqrt{2}}\left(|\uparrow\rangle + |\downarrow\rangle\right)$. Simplify your answer as much as possible. You may use, without proof, the identity $\exp[iA\alpha] = \cos(\alpha)\mathbb{1} + i\sin(\alpha)A$, which holds for all self-inverse linear operators A and scalars α. [4]

(c) Let $\hat{A}$ and $\hat{B}$ be two observables which, in the Schrödinger picture, satisfy $[\hat{A}, \hat{B}] = \hat{C}$. Show that the equivalent operators in the Heisenberg picture $\hat{A}_H(t)$, $\hat{B}_H(t)$ and $\hat{C}_H(t)$ satisfy $[\hat{A}_H(t), \hat{B}_H(t)] = \hat{C}_H(t)$. Recall that in the Heisenberg picture $\hat{O}_H(t) = U(t)^\dagger \hat{O} U(t)$. [2]

(d) In the Heisenberg picture, the observable $\hat{O}_H(t)$ satisfies the equation of motion

$$\frac{\partial}{\partial t}\hat{O}_H(t) = \frac{i}{\hbar}[H_H(t), \hat{O}_H(t)] + \left(\frac{\partial}{\partial t}\hat{O}(t)\right)_H .$$

Consider a free particle of mass m described by the Hamiltonian $H = \hat{p}^2/2m$.

i. Using the above equation, show that the position operator in the Heisenberg picture becomes

$$\hat{x}_H(t) = \hat{x}_H(0) + \frac{t}{m}\hat{p}_H .$$

You may use the relationship $[\hat{x}, \hat{p}^2] = 2i\hbar\hat{p}$ and the result of part 4(c). [4]

ii. Show that the commutator $[\hat{x}_H(0), \hat{x}_H(t)] = i\hbar t/m$. You may use the relationship $[\hat{x}, \hat{p}] = i\hbar\mathbb{1}$. [3]

5. (a) Let K be the operator defined as $K = |\phi\rangle\langle\psi|$, where $|\phi\rangle$ and $|\psi\rangle$ are two vectors of the Hilbert space of a system.

   i. Show that K is a linear operator. [1]

   ii. Under which condition is K Hermitian? [1]

   iii. Under which condition is K a projector? [1]

   iv. Show that K can always be written in the form $K = \lambda P_\phi P_\psi$, where λ is a constant and P_ϕ and P_ψ are the projector operators associated to $|\phi\rangle$ and $|\psi\rangle$. [2]

(b) Consider two distinct sets of complete orthonormal basis vectors $\{|u_j\rangle\}$ and $\{|v_k\rangle\}$. By defining the trace of an operator A with respect to each of these basis sets, show that the trace of an operator is basis invariant. [2]

(c) The position states $|x\rangle$ and the momentum states $|p\rangle$ satisfy, respectively, the closure relationships $\mathbb{1} = \int_{-\infty}^{\infty} dx|x\rangle\langle x|$ and $\mathbb{1} = \int_{-\infty}^{\infty} dp|p\rangle\langle p|$.

   i. Let A be an operator on the Hilbert space of a quantum system and $\tilde{A}$ its representation in the position basis. Using the relationship $\langle x|p\rangle = (2\pi\hbar)^{-1/2}e^{ipx/\hbar}$, show that $\tilde{p} = \frac{\hbar}{i}\frac{\partial}{\partial x}$. [1]

   ii. Show that $\langle x|p|\psi\rangle = \tilde{p}\psi(x)$, where $\psi(x) = \langle x|\psi\rangle$. [2]

   iii. Derive an expression for $\langle x|xp|\psi\rangle$ in terms of $\psi(x)$. [2]

(d) i. Let A and B be two observables. Prove that the product AB can be written as $AB = X + iY$, where X and Y are Hermitian and are given by

$$X = \frac{1}{2}(AB + BA) \quad \text{and} \quad Y = \frac{1}{2i}[A, B].$$

[2]

   ii. Using the above result, show that the observables A and B satisfy the uncertainty relation

$$\Delta A \Delta B \geq \frac{|\langle [A, B] \rangle|}{2},$$

   where $\Delta M = \sqrt{\langle M^2\rangle - \langle M\rangle^2}$ and $\langle M\rangle = \langle\Psi|\, M\, |\Psi\rangle$.

   Hint: You may want to start your proof by defining two vectors $|\phi_A\rangle = \big(A - \langle A\rangle\big)\,|\Psi\rangle$ and $|\phi_B\rangle = \big(B - \langle B\rangle\big)\,|\Psi\rangle$ and considering the product $\langle\phi_A|\phi_A\rangle\langle\phi_B|\phi_B\rangle$. This is turn can be bound using, without proof, the Cauchy-Schwartz inequality $|\langle\phi_1|\phi_2\rangle|^2 \leq \langle\phi_1|\phi_1\rangle\langle\phi_2|\phi_2\rangle$. [6]

END OF PAPER

PHASM426 and PHASG426: Advanced Quantum Theory - Addendum

Correction to Q3 (c) The final part of this question should read as follows:

Show that the probability (to second order) of finding the system in the state $|n = 2\rangle$ at $\tau = \pi/2\omega$ is given by $P^{(2)}_{n=2}(\tau = \pi/2) = 2A^4/\omega^4$.