6 Interacting Field Theories

Up to now we have considered the scalar field theory with the Lagrangian density

$$\mathcal{L}_0 = \tfrac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - \tfrac{1}{2}m^2\phi^2, \tag{6.1}$$

which resulted in a non-interacting theory – the solutions are plane waves with the standard relation $E^2(\mathbf{p}) - \mathbf{p}^2 = m^2$ for a relativistic particle of mass m. We now consider the possibility of adding an interaction term $-\mathcal{L}_{int}(\phi)$ to the Lagrangian density, which using the rule for converting from the Lagrangian to Hamiltonian formalism will result in an interaction term $\mathcal{H}_{int}(\phi) = \mathcal{L}_{int}(\phi)$. In principle we could consider adding any term to the Lagrangian which is a function of $\partial^\mu\phi$ or ϕ. However, the theory of renormalisation ultimately tells us that we only need consider those terms which have canonical dimension equal to 4, the number of space-time dimensions. Since both the field ϕ and the derivative ∂^μ have dimension 1, the same as mass m, the only new term which can be added to the free Lagrangian density, and thus the only relevant interaction term is proportional to ϕ^4.

We will consider the scattering of particles within the framework of an interacting theory. In order to do this we assume that the interaction term becomes negligible at $t = \pm\infty$. Hence, at $t = -\infty$ we have well separated free particles which we bring together to interact, producing some scattering process, and then we detect them again as free particles at $t = +\infty$. In order to work within this framework we have to define a new quantum picture, called the Interaction picture. In this, the operators carry the time dependence as the Heisenberg operators in a free theory, i.e.

$$Q_I = e^{iH_0 t} Q_S e^{-iH_0 t}, \tag{6.2}$$

where H_0 is the Hamiltonian for the free theory, and hence they satisfy the simple evolution equation

$$i\frac{dQ_I}{dt} = [Q_I, H_0]. \tag{6.3}$$

However, there is now an interaction part to the Hamiltonian

$$H_S = H_{0,S} + H_{int,S}. \tag{6.4}$$

Therefore, remembering that

$$|\Psi, t\rangle_S = e^{-iH_S t}|\Psi, t_0\rangle, \tag{6.5}$$

in the Interaction picture the picture dependence of expectation values leads to

$$\begin{aligned} |\Psi, t\rangle_I &= e^{iH_0 t}e^{-iH_S t}|\Psi, t_0\rangle_H \\ &\equiv U(t, t_0)|\Psi, t_0\rangle_H, \end{aligned} \tag{6.6}$$

and the corresponding equation for operators

$$Q_I(t) = U(t, t_0) Q_H(t) U^{-1}(t, t_0), \tag{6.7}$$

or equivalently

$$Q_H(t) = U^{-1}(t, t_0) Q_I(t) U(t, t_0). \tag{6.8}$$

6.1 The S-matrix

We may now apply this picture to scattering in an interacting theory. From our above assumptions, as $t \to -\infty$, $\phi_H(x) = \phi_I(x)$, i.e. the Heisenberg field and the interaction picture field are identical, and $\phi(x)$ is a free field whose evolution is governed by $H_0(\phi_I(x), \pi_I(x))$. As $H_{int}(\phi(x))$ starts to play a role the Heisenberg field ϕ_H will start to deviate from ϕ_I, the difference being governed by $Q_H(t) = U^{-1}(t,t_0)Q_I(t)U(t,t_0)$. However, using the definition $U(t,t_0) = e^{iH_0t}e^{-iH_St}$ we are relating the Heisenberg field to the Interaction picture field via an operator depending on Schrödinger picture fields ($H_{0,S} = H_{0,I}$ since H_0 commutes with itself, so does not depend on time in the Interaction picture). However, we would like to define the transformation $U(t,t_0)$ in terms of fields in the Interaction picture, in which case we could define the Heisenberg fields, which contain the full time dependence of the system, in terms of the free Interaction picture fields which we know how to deal with. This may be done by considering the equation for the time evolution of $U(t,t_0)$:

$$\begin{aligned} i\frac{dU(t,t_0)}{dt} &= -H_0e^{iH_0t}e^{-iH_St} + e^{iH_0t}H_Se^{-iH_St} \\ &= e^{iH_0t}(H_S - H_0)e^{-iH_st} \\ &= e^{iH_0t}H_{int,S}e^{-iH_0t} \times e^{iH_0t}e^{-iH_St} \\ &= H_{int,I}(\phi_I, \pi_I)U(t,t_0). \end{aligned} \tag{6.9}$$

Also, we know that we have the boundary condition that $U(t) \to 1$ as $t \to -\infty$, and hence this equation may be solved. Integrating both sides

$$i\int_{-\infty}^{t} \frac{dU(t_1)}{dt_1}\,dt_1 = \int_{-\infty}^{t} H_{int,I}(\phi_I(t_1), \pi(t_1))U(t_1)\,dt_1, \tag{6.10}$$

and therefore,

$$U(t) = 1 - i\int_{-\infty}^{t} H_{int,I}(\phi_I(t_1), \pi(t_1))U(t_1)\,dt_1. \tag{6.11}$$

Substituting the expression for $U(t)$ into the right-hand side

$$U(t) = 1 - i\int_{-\infty}^{t} H_{int,I}(\phi_I(t_1), \pi(t_1))\left(1 - i\int_{-\infty}^{t_1} H_{int,I}(\phi_I(t_2), \pi(t_2))U(t_2)\,dt_2\right)dt_1. \tag{6.12}$$

Repeating this procedure leads to a formal power series with n_{th} term

$$(-i)^n\int_{-\infty}^{t} dt_1 \int_{-\infty}^{t_2} dt_2 \cdots \int_{-\infty}^{t_{n-1}} dt_n\, H_{int,I}(t_1)\cdots H_{int,I}(t_n), \tag{6.13}$$

where by definition $t > t_1 > t_2 \cdots > t_n$.

In order to progress further it is useful to define what we call a time-ordered product of operators A and B:

$$\begin{aligned} T(A(t_1)B(t_2)) &= A(t_1)B(t_2) \qquad \text{if} \quad t_1 > t_2, \\ &= B(t_2)A(t_1) \qquad \text{if} \quad t_2 > t_1, \\ &= \Theta(t_1 - t_2)A(t_1)B(t_2) + \Theta(t_2 - t_1)B(t_2)A(t_1). \end{aligned} \tag{6.14}$$

This has the obvious generalisation to an arbitrary number of operators and times. Using this notation eq.(6.13) may be written as

$$\frac{(-i)^n}{n!}\prod_{i=1}^{n}\Big(\int_{-\infty}^{t} dt_i\Big)\, T\big(H_{int,I}(t_1)\cdots H_{int,I}(t_n)\big), \tag{6.15}$$

and hence $U(t)$ may be written as

$$U(t) = T\exp\Big(-i\int_{-\infty}^{t} H_{int,I}(t')dt'\Big). \tag{6.16}$$

The asymptotic form of the operator

$$U(+\infty) \equiv S = T\exp\Big(-i\int_{-\infty}^{\infty} H_{int,I}(t')dt'\Big) \tag{6.17}$$

is known as the S-matrix operator, and in the Heisenberg picture evolves the free "in" fields from initial time $-\infty$ to final time $+\infty$. Assuming the interaction vanishes again at $t = +\infty$, $\phi_H(t = +\infty)$ will become a free field again, and will exist in the same basis as the "in" fields.

The quantity we actually wish to calculate is called the scattering matrix element, which for a given set of incoming particles with particular momentum, represented by the state $|p, -\infty : in\rangle$, gives us the amplitude for producing a given set of outgoing particles with momentum, represented by the state $|q, \infty : out\rangle$. The S-matrix element is given by

$$S_{qp} = \langle q, \infty : out|p, \infty : in\rangle. \tag{6.18}$$

In the interaction picture the operators have all evolved as in a free field theory, and hence the set of "in" eigenstates is the same as the set of "out" eigenstates, i.e. $|p : in\rangle = |p : out\rangle$. However, in this picture the state vector representing the system does evolve in time, $|\Psi, t_1\rangle = U(t_1, t_2)|\Psi, t_2\rangle$. Hence,

$$|p, \infty : in\rangle = S|p, -\infty : in\rangle, \tag{6.19}$$

and the S-Matrix operator is

$$S_{qp} = \langle q, \infty : out|S|p, -\infty : in\rangle = \langle q, \infty : in|S|p, -\infty : in\rangle. \tag{6.20}$$

In the Heisenberg picture there is no evolution of state vectors and $|p, \infty : in\rangle = |p, -\infty : in\rangle$. However in this case the creation and annihilation operators for momentum eigenstates do undergo a nontrivial evolution

$$a^{\dagger}_{\mathbf{k},out} = S^{-1}a^{\dagger}_{\mathbf{k},in}S. \tag{6.21}$$

Hence, in order that the expectation value of a creation or annihilation operator remains the same in both the in and out state, i.e.

$$\langle p : out|a^{\dagger}_{\mathbf{k},out}|p : out\rangle = \langle p : in|a^{\dagger}_{\mathbf{k},in}|p : in\rangle \tag{6.22}$$

the eigenstates for particles of a given momentum must satisfy

$$|p : out\rangle = S^{-1}|p : in\rangle, \qquad \langle q : out| = \langle q : in|S. \tag{6.23}$$

So in this picture the S-matrix elements are given by

$$S_{qp} = \langle q, \infty : out | p, \infty : in \rangle = \langle q, \infty : in | S | p, -\infty : in \rangle. \tag{6.24}$$

Hence, as we would hope the ultimate expression for the S-matrix is the same in both pictures, but the interpretation is different – in the Interaction picture the operator S arises due to time evolution of the state vector, whereas in the Heisenberg picture it arises due to time evolution of operators and hence eigenstates.

We wish to calculate matrix elements, and in order to obtain the rules for doing this we will begin in the Heisenberg picture. In this case the matrix elements may be written in the form

$$S_{qp} = \langle q_1, \ldots, q_m : out | p_1, \ldots, p_n : in \rangle. \tag{6.25}$$

representing the amplitude of n incoming particles with momentum p scattering to provide m outgoing particles of momentum p. This may trivially be rewritten as

$$S_{qp} = \langle q_1, \ldots, q_m : out | a^\dagger_{\mathbf{p}_1, in} | p_2, \ldots, p_n : in \rangle. \tag{6.26}$$

where we remember that we are currently working with Heisenberg operators. In order to progress it is useful to re-express $a^\dagger_{\mathbf{p}}$ in terms of $\phi(x)$. Consider

$$\int d^3\mathbf{x} \big((-\partial_0 e^{-ip\cdot x})\phi(x) + e^{-ip\cdot x}(\partial_0 \phi(x)) \big). \tag{6.27}$$

Using the explicit expression for a free field $\phi(x)$ in terms of creation and annihilation operators, i.e.

$$\phi(x) = \int \frac{d^3\mathbf{q}}{2E(\mathbf{q})(2\pi)^3} \big(a_{\mathbf{q}} e^{-iq\cdot x} + a^\dagger_{\mathbf{q}} e^{iq\cdot x} \big), \tag{6.28}$$

with a little algebra one may show that

$$\int d^3\mathbf{x} \big((-\partial_0 e^{-ip\cdot x})\phi(x) + e^{-ip\cdot x}(\partial_0 \phi(x)) \big) = i a^\dagger_{\mathbf{p}}. \tag{6.29}$$

Similarly one finds that

$$\int d^3\mathbf{x} \big((-\partial_0 e^{+ip\cdot x})\phi(x) + e^{+ip\cdot x}(\partial_0 \phi(x)) \big) = -i a_{\mathbf{p}}. \tag{6.30}$$

Inserting eq. (6.29) into eq. (6.26) we obtain

$$S_{qp} = -i \langle q_1, \ldots, q_m : out | \int d^3\mathbf{x} \big((-\partial_0 e^{-ip_1\cdot x_1})\phi(x) + e^{-ip_1\cdot x_1}(\partial_0 \phi(x_1)) \big) | p_2, \ldots, p_n : in \rangle \quad t_1 \to -\infty, \tag{6.31}$$

where the time shows that it is an incoming particle, and our assumption that the field is non-interacting as $t_1 \to -\infty$ allow us to use the relationship between the creation operator and a free field. This may be simplified by using the relationship

$$f(t)_{t\to+\infty} - f(t)_{t\to-\infty} = \int_{-\infty}^{\infty} \frac{df}{dt} dt, \tag{6.32}$$

leading to

$$\begin{aligned}
S_{qp} &= -i\langle q_1,\ldots,q_m : out|\int d^3\mathbf{x}\big((-\partial_0 e^{-ip_1\cdot x_1})\phi(x_1)+e^{-ip_1\cdot x_1}(\partial_0\phi(x_1))\big)|p_2,\ldots,p_n : in\rangle \quad t_1\to+\infty \\
&+ i\langle q_1,\ldots,q_m : out|\int d^4x_1\frac{d}{dt_1}\big((-\partial_0 e^{-ip_1\cdot x_1})\phi(x_1)+e^{-ip_1\cdot x_1}(\partial_0\phi(x_1))\big)|p_2,\ldots,p_n : in\rangle \\
&= \langle q_1,\ldots,q_m : out|a^\dagger_{\mathbf{p}_1,out}|p_2,\ldots,p_n : in\rangle \\
&+ i\langle q_1,\ldots,q_m : out|\int d^4x_1\frac{d}{dt_1}\big((-\partial_0 e^{-ip_1\cdot x_1})\phi(x_1)+e^{-ip_1\cdot x_1}(\partial_0\phi(x_1))\big)|p_2,\ldots,p_n : in\rangle. \qquad (6.33)
\end{aligned}$$

In the first of these terms the operator acts as an annihilation operator on the out states, and therefore only produces a non-zero result if one of the particles in the out state has momentum $\mathbf{p}_1$. however, this would mean that one of the outgoing particles has identical momentum to one of the incoming particles, and thus that one particle as not scattered at all. We neglect any such state as being one where not all incoming particles scatter, and hence obtain

$$S_{qp} = i\langle q_1,\ldots,q_m : out|\int d^4x\frac{d}{dt_1}\big((-\partial_0 e^{-ip_1\cdot x_1})\phi(x_1)+e^{-ip_1\cdot x_1}(\partial_0\phi(x_1))\big)|p_2,\ldots,p_n : in\rangle. \qquad (6.34)$$

We can simplify this by performing the time derivatives

$$\begin{aligned}
\frac{d}{dt_1}\big(e^{-ip_1\cdot x_1}(\partial_0\phi(x_1))+(-\partial_0 e^{-ip_1\cdot x_1})\phi(x_1)\big) &= \frac{d}{dt_1}\big(e^{-ip_1\cdot x_1}\partial_0\phi(x_1)+iE(\mathbf{p}_1)\phi(x_1)e^{-ip_1\cdot x_1}\big) \\
&= e^{-ip_1\cdot x_1}\partial_0^2\phi(x_1)+E^2(\mathbf{p}_1)e^{-ip_1\cdot x_1}\phi(x_1). \qquad (6.35)
\end{aligned}$$

Using the fact that $E^2(\mathbf{p}_1)e^{-ip\cdot x} = (-\boldsymbol{\nabla}^2+m^2)e^{-ip\cdot x}$ we obtain

$$S_{qp} = i\langle q_1,\ldots,q_m : out|\int d^4x\big(e^{-ip_1\cdot x_1}\partial_0^2\phi(x_1)+\phi(x_1)(-\boldsymbol{\nabla}^2+m^2)e^{-ip_1\cdot x_1}\big)|p_2,\ldots,p_n : in\rangle, \qquad (6.36)$$

and integrating the derivatives with respect to space coordinates by parts

$$S_{qp} = i\langle q_1,\ldots,q_m : out|\int d^4x\big(e^{-ip_1\cdot x_1}(\partial_0^2-\boldsymbol{\nabla}^2+m^2)\phi(x_1)\big)|p_2,\ldots,p_n : in\rangle, \qquad (6.37)$$

and finally denoting $(\partial_0^2-\boldsymbol{\nabla}^2)$ by $\partial_{x_1}^2$ we reach the result

$$S_{qp} = i\int d^4x e^{-ip_1\cdot x_1}(\partial_{x_1}^2+m^2)\langle q_1,\ldots,q_m : out|\phi(x_1)|p_2,\ldots,p_n : in\rangle. \qquad (6.38)$$

Now let us consider

$$S'_{qp} = \langle q_1,\ldots,q_m : out|\phi(x_1)|p_2,\ldots,p_n : in\rangle. \qquad (6.39)$$

In the same manner as above this may also be trivially rewritten as

$$S'_{qp} = \langle q_2,\ldots,q_m : out|a_{\mathbf{q}_1,out}\phi(x_1)|p_2,\ldots,p_n : in\rangle. \qquad (6.40)$$

Using the expression for $a_{\mathbf{q}}$ in terms of the free asymptotic field this becomes

$$S'_{qp} = i\int d^3\mathbf{x}_{n+1}\langle q_2,\ldots,q_m : out|T((-\partial_0 e^{+iq_1\cdot x_{n+1}})\phi(x_{n+1})$$
$$+e^{+iq_1\cdot x_{n+1}}(\partial_0\phi(x_{n+1})))\phi(x_1)\big)|p_2,\ldots,p_n : in\rangle. \quad t_{n+1}\to+\infty, \qquad (6.41)$$

where we can use the time ordering notation because $t_{n+1}\to+\infty$. Repeating the same sort of steps as above for p_1 we obtain

$$S_{qp} = (+i)^2\int d^4x_1\int d^4x_{n+1}e^{+iq_1\cdot x_{n+1}}e^{-ip_1\cdot x_1}(\partial^2_{x_{n+1}}+m^2)(\partial^2_{x_1}+m^2)$$
$$\langle q_2,\ldots,q_m : out|T\big(\phi(x_1)\phi(x_{n+1})\big)|p_2,\ldots,p_n : in\rangle. \qquad (6.42)$$

This can be repeated for every incoming and outgoing particle, ultimately resulting in

$$S_{qp} = (+i)^{n+m}\int d^4x_1\cdots\int d^4x_n e^{-ip_1\cdot x_1\cdots-ip_n\cdot x_n}\int d^4x_{n+1}\cdots\int d^4x_{n+m}e^{+iq_{n+1}\cdot x_{n+1}\cdots+iq_{n+m}\cdot x_{n+m}}$$
$$(\partial^2_{x_1}+m^2)\cdots(\partial^2_{x_{n+m}}+m^2)\langle 0 : out|T\big(\phi_H(x_1)\ldots\phi_H(x_{n+m})\big)|0 : in\rangle, \qquad (6.43)$$

where now we write the picture in which we are working explicitly. (The vacuum state $|0\rangle$ is the same in all pictures.)

The problem with eq. (6.43) is that it is expressed in terms of the Heisenberg picture fields which have a time evolution which depends on the full Hamiltonian and for which we do not know how to solve. In order to perform any meaningful calculations we have to go back to the Interaction picture where we can work in terms of the free fields which we can deal with. One may do this by using the relation

$$\phi_H(t) = U^{-1}(t)\phi_I(t)U(t). \qquad (6.44)$$

Choosing the ordering of the Heisenberg fields so that they are correctly time ordered we can therefore write the vacuum expectation value of the product of fields as

$$\langle 0|U^{-1}(t_{n+m})\phi_I(t_{n+m})U(t_{n+m})U^{-1}(t_{n+m-1})\ldots U(t_2)U^{-1}(t_1)\phi_I(t_1)U(t_1)|0\rangle, \qquad (6.45)$$

where $t_{n+m} > t_{n+m-1} > \cdots > t_1$. We then introduce some time t such that $t > t_{n+m}$ and $-t < t_1$, and the notation $U(t_2,t_1) = U(t_2)U^{-1}(t_1)$. From the definition of the $U(t)$ operators this means that

$$U(t_2,t_1) = T\exp\bigg(-i\int_{t_1}^{t_2} H_{int,I}(t')dt'\bigg). \qquad (6.46)$$

Inserting $1 = U^{-1}(t)U(t)$ at the beginning and $1 = U^{-1}(-t)U(-t)$ at the end of the vacuum expectation value of fields we then obtain

$$\langle 0|U^{-1}(t)U(t,t_{n+m})\phi_I(t_{n+m})U(t_{n+m},t_{n+m-1})\ldots U(t_2,t_1)\phi_I(t_1)U(t_1,-t)U(-t)|0\rangle. \qquad (6.47)$$

All terms between $U(t,t_{n+m})$ and $U(t_1,-t)$ are now automatically time ordered correctly, so we may write

$$\langle 0|U^{-1}(t)T\big(U(t,t_{n+m})\phi_I(t_{n+m})U(t_{n+m},t_{n+m-1})\ldots U(t_2,t_1)\phi_I(t_1)U(t_1,-t)\big)U(-t)|0\rangle. \qquad (6.48)$$

The product of all the U operators within the time ordering is then $U(t,-t)$. We may now use the facts that as $t \to \infty, U(t,-t) \to S$ and $U(-t) \to 1$ to obtain

$$\langle 0|U^{-1}(t)T\big(\phi_I(t_1)\ldots\phi_I(t_{n+m})S\big)|0\rangle. \tag{6.49}$$

Finally, we know that we are describing particle scattering in this formalism, and thus if we start with no particles we should finish with no particles, so

$$\langle 0|U^{-1}(t) = K\langle 0|, \tag{6.50}$$

where K is just a phase factor. In fact

$$K = \langle 0|U^{-1}(t)|0\rangle, \tag{6.51}$$

and it can be shown that

$$K = \langle 0|U^{-1}(t)|0\rangle = 1/\langle 0|U(t)|0\rangle. \tag{6.52}$$

So since $t \to \infty$ the vacuum expectation value of the original Heisenberg fields is

$$\frac{\langle 0|T\big(\phi_I(t_1)\ldots\phi_I(t_{n+m})S\big)|0\rangle}{\langle 0|S|0\rangle}. \tag{6.53}$$

Since the S-matrix elements themselves are related to the vacuum expectation values, or Green's functions, in eq. (6.43) then calculating eq. (6.53) provides us with the S-matrix elements directly. Also, since the S-Matrix operator S may be described entirely in terms of the Interaction picture operators we have some hope of being able to calculate eq. (6.53) explicitly. This will be the topic of the next section.