PHASM426 / PHASG426
Advanced Quantum Theory Problem Sheet 4

Deadline: Tuesday 9th January 2018 at 13:00 pm

Please bring the work to me in my office (B12) on Tuesday 9th January 2018 between 12:00 and 13:00, leave it in my pigeon hole (**at your own risk**) or scan your work and email it to me as **a single PDF file** that does not exceed 5 MB. In any case, please make sure your completed work is clearly labelled with your name and college, and **stapled** if you are handing in a paper version.

1. Consider H is the total Hamiltonian descrbing a quantum system interacting with an environment. The total state of the system and environment $\rho(t)$ is given by the unitary evolution $U(t)$ associated to the total H and at time $t = 0$ we have $\rho(t = 0) = \rho_s(0) \otimes \rho_B(0)$, where the inital state of the environment is $\rho_B(0) = |B_0\rangle\langle B_0|$. Here the basis set $\{B_k\}$ spans the enviornment states.

   (a) Show that the reduced density matrix operator for the system $\rho_s(t)$ takes the form
   $$\rho_s(t) = \sum_k S_k \rho_s(0) S_k^\dagger$$
   where $S_k = \langle B_k|U(t)|B_0\rangle$. [5]

   (b) Discuss whether S_k is an operator acting on the system or on the environment, or wether it is an expected value. [2]

   (c) Show that $\sum_k S_k^\dagger S_k = \mathbb{1}$ and discuss the physical meaning of this result. [3]

2. Consider a two-level atom with excited state $|e\rangle$ and ground state $|g\rangle$ such that its Hamiltonian is $H = \hbar\omega|e\rangle\langle e|$. The action of the environment interacting with the atom is described by the jump operators $L_1 = \Gamma|e\rangle\langle g|$ and $L_2 = \gamma|g\rangle\langle e|$.

   (a) Assuming that at $t = 0$ the state of the atom is $\rho(0) = |g\rangle\langle g|$, show that the probability of finding the atom in the excited state at time t , $\rho_{ee}(t) = \langle e|\rho(t)|e\rangle$, is given by
   $$\rho_{ee}(t) = \frac{|\Gamma|^2}{|\gamma|^2 + |\Gamma|^2}\Big(1 - e^{-(|\gamma|^2+|\Gamma|^2)t}\Big).$$
   You may use the fact that a linear differential equation of the form $\frac{dy}{dx} + ay = b$ with a and b real numbers, has the general solution
   $$y(x) = e^{-ax}\Big(\frac{b}{a}e^{ax} + \kappa\Big),$$
   where κ is to be determined by initial conditions.

   [10]

(b) Find a relation between Γ and γ such that in the long-time limit $\rho_{ee}(\infty)$ equals the probability of finding the atom in its excited state when it is in thermal equilibrium at temperature T. Recall that in thermal equilibrium, a system with Hamiltonian H is described by the density matrix operator $\rho_{eq} = \frac{\exp(-H/k_BT)}{\mathrm{Tr}[\exp(-H/k_BT)]}$. Express your answer as

$$|\Gamma|^2 = C|\gamma|^2$$

and specify the value of C as a function of ω and k_BT where k_B is the Boltzman constant. [5]