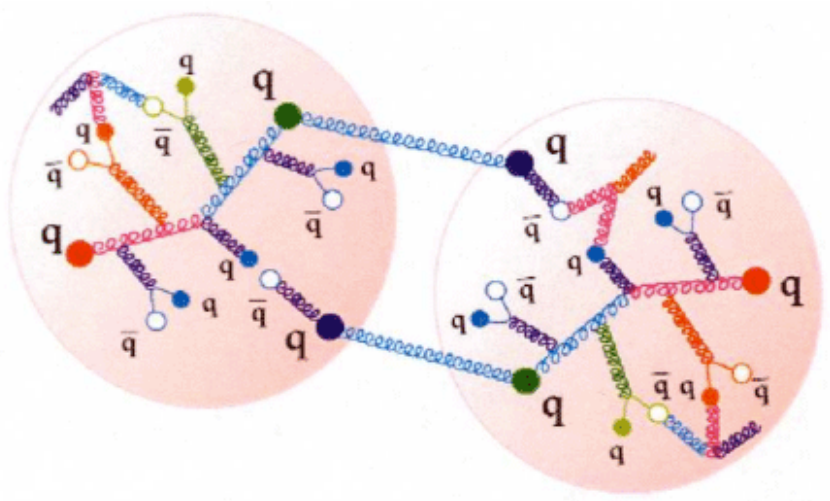
PHASM/G442 Particle Physics

Ruben Saakyan

Module VI

Proton Structure, quark properties and QCD



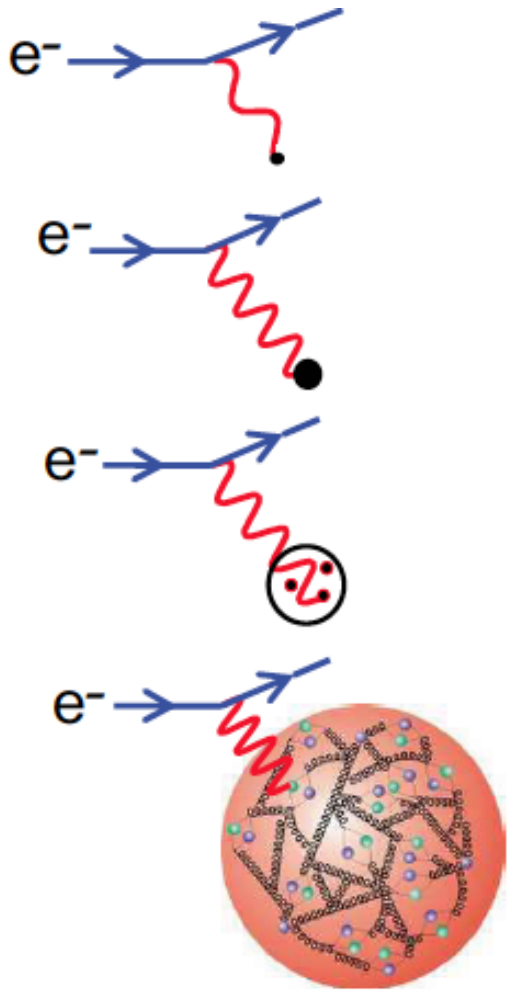
http://moodle.ucl.ac.uk/course/view.php?id=2589

Outline

- Probing proton structure with ep scattering (aka probing QCD with QED)
  - Non-relativistic elastic ep scattering calculation
  - Relativistic scattering
  - Taking finite proton size into account (form factors)
  - inelastic scattering
  - DIP — Deep Inelastic Scattering, probing proton structure
- Evidence for quarks and their properties
- Quantum Chromo-Dynamics (QCD)
  - Colour charge
  - Running coupling constant
  - QCD structure

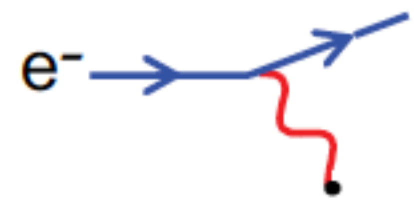
Probing proton structure with ep scattering

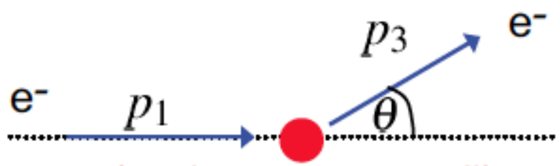
- At very low electron energies $\lambda \gg r_p$:
  the scattering is equivalent to that from a "point-like" **spin-less** object
- At low electron energies $\lambda \sim r_p$:
  the scattering is equivalent to that from a extended charged object
- At high electron energies $\lambda < r_p$:
  the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks
- At very high electron energies $\lambda \ll r_p$:
  the proton appears to be a sea of quarks and gluons.



Elastic scattering on point-like object

- At very low electron energies $\lambda \gg r_p$: the scattering is equivalent to that from a "point-like" **spin-less** object





(neglect proton recoil)

Initial and final state spinors

$$u_\uparrow(p_1) = N_e \begin{pmatrix} 1 \\ 0 \\ \alpha \\ 0 \end{pmatrix} \quad u_\downarrow(p_1) = N_e \begin{pmatrix} 0 \\ 1 \\ 0 \\ -\alpha \end{pmatrix} \quad u_\uparrow(p_3) = N_e \begin{pmatrix} c \\ s \\ \alpha c \\ \alpha s \end{pmatrix} \quad u_\downarrow(p_3) = N_e \begin{pmatrix} -s \\ c \\ \alpha s \\ -\alpha c \end{pmatrix}$$

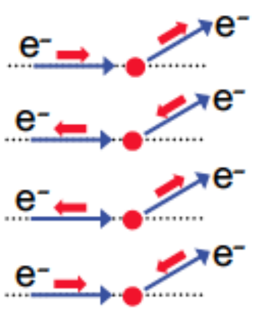
with

$$N = \sqrt{E+m}; \qquad s = \sin(\theta/2); \quad c = \cos(\theta/2) \qquad \alpha = \frac{|\vec{p}|}{E+m_e}$$

Non-relativistic limit: $\alpha \to 0$

Ultra-relativistic limit: $\alpha \to 1$

<u>Electron Current</u>



$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \qquad \mathbf{R \to R}$$

$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(\alpha^2+1)c, 2\alpha s, -2i\alpha s, 2\alpha c\right] \qquad \mathbf{L \to L}$$

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = (E+m_e)\left[(1-\alpha^2)s, 0, 0, 0\right] \qquad \mathbf{L \to R}$$

$$\bar{u}_\downarrow(p_3)\gamma^\mu u_\uparrow(p_1) = (E+m_e)\left[(\alpha^2-1)s, 0, 0, 0\right] \qquad \mathbf{R \to L}$$

Helicity $\not\equiv$ Chirality

All 4 helicity combinations give non-zero M.E. in non-relativistic case!

Proton is at rest, therefore $u_{\uparrow}(0) = \sqrt{2M_p}\begin{pmatrix}1\\0\\0\\0\end{pmatrix}$ $u_{\downarrow}(0) = \sqrt{2M_p}\begin{pmatrix}0\\1\\0\\0\end{pmatrix}$

giving the currents $j_{p\uparrow\uparrow} = j_{p\downarrow\downarrow} = 2M_p(1,0,0,0)$ $j_{p\uparrow\downarrow} = j_{p\downarrow\uparrow} = 0$

The spin-averaged **M.E**. summed over 8 helicity states:

$$\langle|M_{fi}^2|\rangle = \frac{1}{4}\frac{e^4}{q^4}(16M_p^2 m_e^2)(4c^2+4s^2) = \frac{16M_p^2 m_e^2 e^4}{q^4}$$

where $q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2 = -4|\vec{p}|^2 \sin^2(\theta/2)$ $\Rightarrow$ $\langle|M_{fi}^2|\rangle = \frac{M_p^2 m_e^2 e^4}{|\vec{p}|^4 \sin^4(\theta/2)}$

Using the Lab-Frame cross-section formula derived in Module I

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 M_p^2}|M_{fi}|^2 = \frac{m_e^2 e^4}{64\pi^2|\vec{p}|^4 \sin^4(\theta/2)}$$

$\alpha_{EM} = \frac{e^2}{4\pi} = \frac{1}{137}$

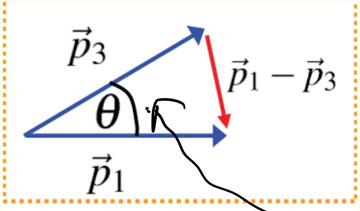
or

$$\boxed{\left(\frac{d\sigma}{d\Omega}\right)_{\text{Rutherford}} = \frac{\alpha_{EM}^2}{16E_K^2 \sin^4\theta/2}}$$

Rutherford cross-section, i.e. only interaction between **electric charges** matters

$$q^2 = (p_1 - p_3)^2 = (0, \vec{p}_1 - \vec{p}_3)^2$$

$E_1 = E_3$ in <u>this</u> case



$$q^2 = 0^2 - (\vec{p}_1 - \vec{p}_3)^2 = -[\vec{p}_1^{\,2} + \vec{p}_3^{\,2} - 2\vec{p}_1 \cdot \vec{p}_3 \cdot \cos\theta] =$$

$$= -[2|\vec{p}|^2 - 2|\vec{p}|^2 \cdot \cos\theta] = -2|\vec{p}|^2(1-\cos\theta) =$$

$$= -4|\vec{p}|^2 \sin^2\frac{\theta}{2} \qquad |\vec{p}_1| = |\vec{p}_3| \equiv |\vec{p}|$$

5a

The Mott cross-section formula

- If electron is relativistic, $E >> m$, only two helicity combinations contribute to the electron current

e.g for R→R and L→R:

$$\bar{u}_\uparrow(p_3)\gamma^\mu u_\uparrow(p_1) = 2E\,[c, s, -is, c] \qquad \bar{u}_\uparrow(p_3)\gamma^\mu u_\downarrow(p_1) = E\,[0,0,0,0]$$

Relativistic ➡ Electron "helicity conserved"

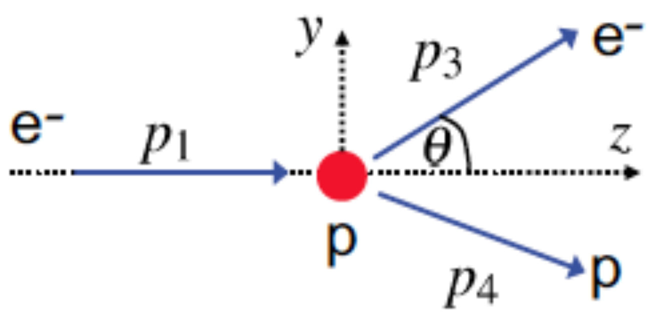
$$\Rightarrow \left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \underbrace{\frac{\alpha^2}{4E^2 \sin^4 \theta/2}}_{\text{Rutherford bit}} \underbrace{\cos^2 \frac{\theta}{2}}_{\text{Relativistic bit}}$$

- Interaction is still **electric** rather than magnetic (spin-spin)

- Proton recoil is **not** taken into account

Taking into account proton recoil

For $E_1 \gg m_e$ the general case is



$$\begin{aligned} p_1 &= (E_1, 0, 0, E_1) \\ p_2 &= (M, 0, 0, 0) \\ \boxed{p_3 = (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)} \\ p_4 &= (E_4, \vec{p}_4) \end{aligned}$$

neglecting electron mass

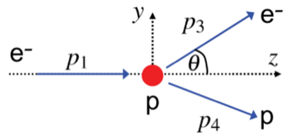
From QED if all helicity combinations contribute to M.E.

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1 . p_2)(p_3 . p_4) + (p_1 . p_4)(p_2 . p_3) - (p_1 . p_3) M^2 - (p_1 . p_4) m^2 + 2m^2 M^2 \right]$$

And neglecting electron mass

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} \left[(p_1 . p_2)(p_3 . p_4) + (p_1 . p_4)(p_2 . p_3) - (p_1 . p_3) M^2 \right]$$

$$p_1 \cdot p_2 = E_1 \cdot M$$



$$p_1 = (E_1, 0, 0, E_1)$$
$$p_2 = (M, 0, 0, 0)$$
$$\boxed{p_3 = (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)}$$
$$p_4 = (E_4, \vec{p}_4)$$

$$P_1 \cdot P_3 = E_1 \cdot E_3 (1 - \cos\theta)$$

$$P_2 \cdot P_3 = E_3 \cdot M$$

$$P_1 + P_2 = P_3 + P_4$$

$$P_4 = P_1 + P_2 - P_3$$

$$P_3 \cdot P_4 = P_3 \cdot P_1 + P_3 \cdot P_2 - \cancel{P_3 \cdot P_3}^{\,0} = E_1 E_3 (1 - \cos\theta) + E_3 \cdot M$$

$$P_1 \cdot P_4 = E_1 \cdot M - E_1 \cdot E_3 (1 - \cos\theta)$$

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Experimentally observe (usually) scattered electron only. So eliminating p_4

$$\langle |M_{fi}|^2 \rangle = \frac{8e^4}{(p_1 - p_3)^4} M E_1 E_3 \left[(E_1 - E_3)(1 - \cos\theta) + M(1 + \cos\theta) \right]$$
$$= \frac{8e^4}{(p_1 - p_3)^4} 2 M E_1 E_3 \left[(E_1 - E_3)\sin^2(\theta/2) + M\cos^2(\theta/2) \right]$$

$q^4 = (p_1 - p_3)^4$ and $(E_1 - E_3)$ can be rewritten

$$q^2 = (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = -2E_1E_3(1 - \cos\theta)$$
$$= -4E_1E_3 \sin^2\theta/2$$

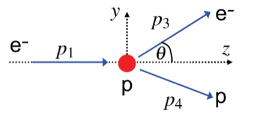
NOTE: $q^2 < 0$

$$\boxed{E_1 - E_3 = -\frac{q^2}{2M}}$$

➡ Energy transferred to proton → 8q

As q^2 is always negative ➡ $E_1 - E_3 > 0$

$$q^2 = (p_1 - p_3)^2 = \cancel{p_1^2}^{\to 0} + \cancel{p_3^2}^{\to 0} - 2p_1 \cdot p_3 =$$



$$p_1 = (E_1, 0, 0, E_1)$$
$$p_2 = (M, 0, 0, 0)$$
$$\boxed{p_3 = (E_3, 0, E_3 \sin\theta, E_3 \cos\theta)}$$
$$p_4 = (E_4, \vec{p}_4)$$

$$= -2E_1E_3(1-\cos\theta) = -4E_1 \cdot E_3 \cdot \sin^2\frac{\theta}{2}$$

$$q^2 < 0$$

$$q \cdot p_2 = (p_1 - p_3) \cdot p_2 = p_1 \cdot p_2 - p_3 \cdot p_2 = M(E_1 - E_3)$$

$(q + p_2)^2 = p_4^2$ because $q = p_1 - p_3 = p_4 - p_2$

$$q^2 + p_2^2 + 2qp_2 = p_4^2$$

$$q^2 + \cancel{M^2} + 2qp_2 = \cancel{M^2} \Rightarrow qp_2 = -\frac{q^2}{2}$$

$$(E_1 - E_3) = -\frac{q^2}{2M}$$

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$$
\begin{aligned}
\langle |M_{fi}|^2 \rangle &= \frac{8e^4}{16E_1^2E_3^2\sin^4\theta/2} 2ME_1E_3 \left[M\cos^2\theta/2 - \frac{q^2}{2M}\sin^2\theta/2 \right] \\
&= \frac{M^2e^4}{E_1E_3\sin^4\theta/2} \left[\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2 \right]
\end{aligned}
$$

Using the differential cross-section expression from Module-I

$$
\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \left(\frac{E_3}{ME_1} \right)^2 |M_{fi}|^2
$$

$$
\Rightarrow \boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2} \frac{E_3}{E_1} \left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2 \right)}
$$

$$
\boxed{\alpha = \frac{e^2}{4\pi} \approx \frac{1}{137}}
$$

Interpretation

Compare

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2 - \frac{q^2}{2M^2}\sin^2\theta/2\right)$$

with

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} = \frac{\alpha^2}{4E^2\sin^4\theta/2}\cos^2\frac{\theta}{2}$$

E_3/E_1 term is due to **proton recoil**

$$\frac{d\sigma}{d\Omega} = \underbrace{\frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\cos^2\theta/2\right.}_{\boxed{\text{Mott x-section with proton recoil}}} - \underbrace{\left.\frac{q^2}{2M^2}\sin^2\theta/2\right)}_{\boxed{\text{Magnetic interaction : due to the spin-spin interaction}}}$$

- But charge distribution of proton **not** taken into account

- The cross-section in the previous slide depends on a **single parameter**
- For an electron scattered at an angle θ , q^2 and E_3 are **fixed by kinematics**:

$$\frac{E_3}{E_1} = \frac{M}{M + E_1(1-\cos\theta)} \qquad \boxed{\text{PS3. Q3.}} \qquad q^2 = -\frac{2ME_1^2(1-\cos\theta)}{M+E_1(1-\cos\theta)}$$

- e.g. $e^-p \rightarrow e^-p$ at E_{beam} **= 529.5 MeV**, look at scattered electrons at θ **= 75°**

For elastic scattering expect:

$$E_3 = \frac{ME_1}{M+E_1(1-\cos\theta)}$$

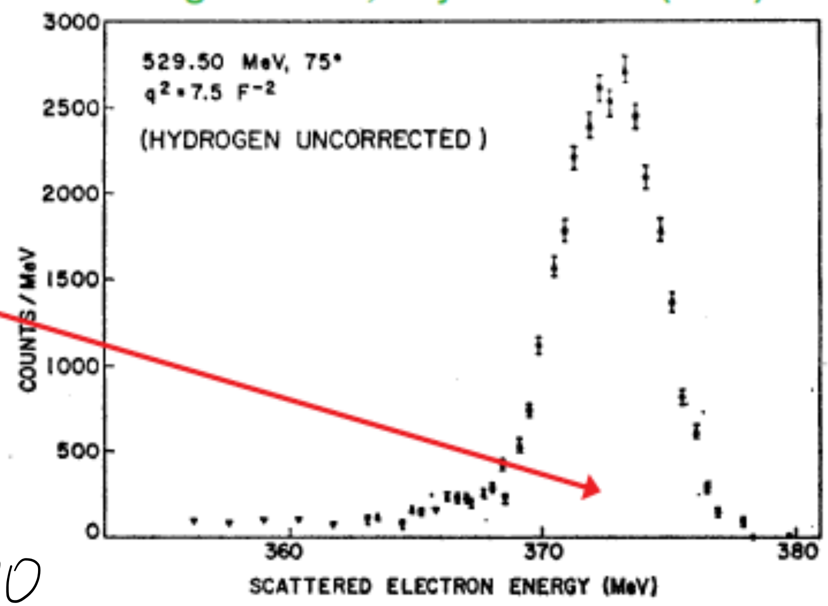
$$E_3 = \frac{938 \times 529}{938 + 529(1-\cos 75^\circ)} = 373 \text{ MeV}$$

The energy identifies the scatter as elastic.
Also know squared four-momentum transfer

$$|q^2| = \frac{2 \times 938 \times 529^2(1-\cos 75^\circ)}{938 + 529(1-\cos 75^\circ)} = \cancel{294}\,\text{MeV}^2$$

294 000

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



- Charge distribution of proton still **not** taken into account

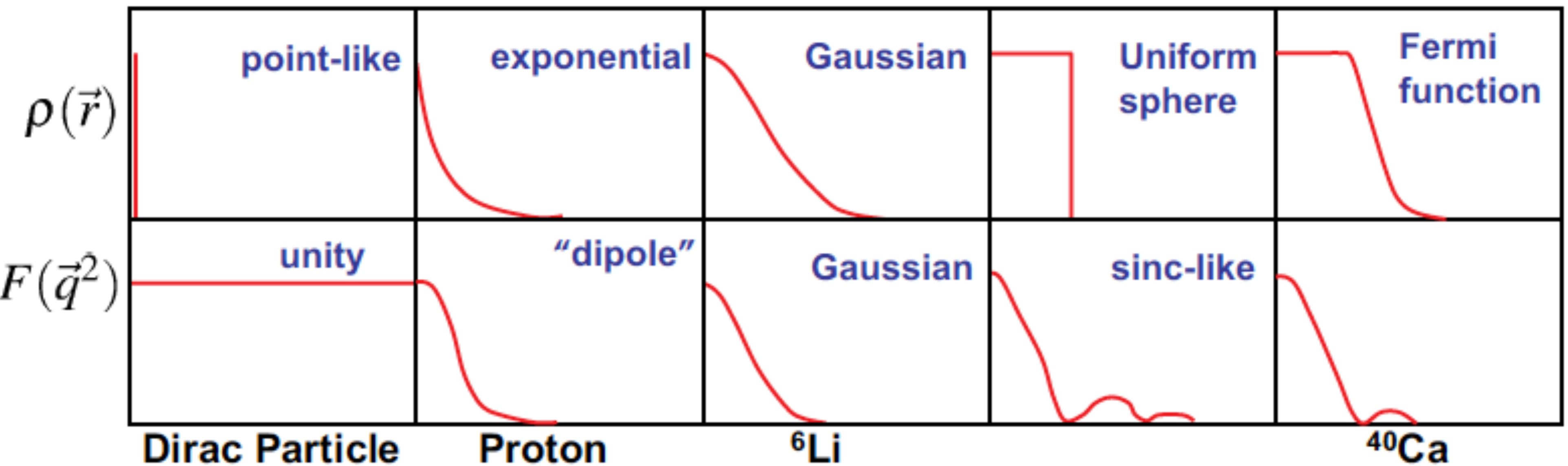
Form Factors

- Form Factors are used to take into account the spatial charge distribution

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{Mott}} \rightarrow \frac{\alpha^2}{4E^2 \sin^4 \theta/2} \cos^2 \frac{\theta}{2} \underbrace{|F(\vec{q}^2)|^2}_{\text{Form Factor}}$$

$$\boxed{F(\vec{q}^2) = \int \rho(\vec{r}) e^{i\vec{q}.\vec{r}} d^3\vec{r}}$$

— Fourier transform of charge distribution



Rosenbluth Formula. Elastic scattering from finite size proton.

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2}+2\tau G_M^2\sin^2\frac{\theta}{2}\right)}$$

with $\boxed{\tau = -\frac{q^2}{4M^2} > 0}$

$G_E(q^2)$ and $G_M(q^2)$ are two **structure functions** related to **charge** and **magnetic moment** distribution inside proton

Note $q^2 = (E_1 - E_3)^2 - \vec{q}^2$ $\rightarrow$ 4-vector At $q^2/4M^2 \ll 1$

$$G_E(q^2) \approx G_E(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}}\rho(\vec{r})d^3\vec{r}$$

$$G_M(q^2) \approx G_M(\vec{q}^2) = \int e^{i\vec{q}.\vec{r}}\mu(\vec{r})d^3\vec{r}$$

If proton is point-like spin-1/2 Dirac particle $\vec{\mu} = \frac{e}{M}\vec{S}$

However experimentally $\vec{\mu} = 2.79\frac{e}{M}\vec{S}$

Therefore for proton expect $G_E(0) = \int \rho(\vec{r})d^3\vec{r} = 1$

Hint that proton is **not point-like**

$$G_M(0) = \int \mu(\vec{r})d^3\vec{r} = \mu_p = +2.79$$

Measuring $G_E(q^2)$ and $G_M(q^2)$

$$\frac{d\sigma}{d\Omega} = \left(\frac{d\sigma}{d\Omega}\right)_0 \left(\frac{G_E^2 + \tau G_M^2}{(1+\tau)} + 2\tau G_M^2 \tan^2\frac{\theta}{2}\right)$$

where

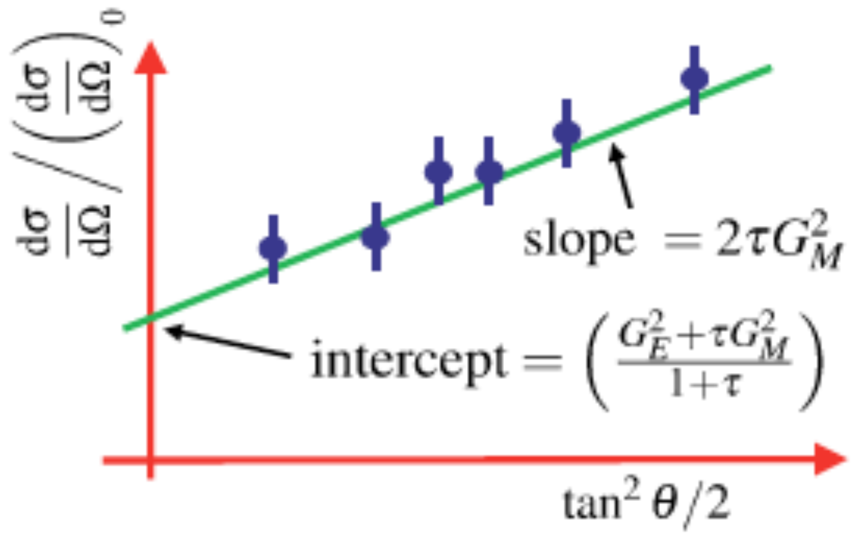
$$\left(\frac{d\sigma}{d\Omega}\right)_0 = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\cos^2\frac{\theta}{2}$$

- At very low q^2: $\tau = -q^2/4M^2 \approx 0$

$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx G_E^2(q^2)$$

- At high q^2: $\tau \gg 1$

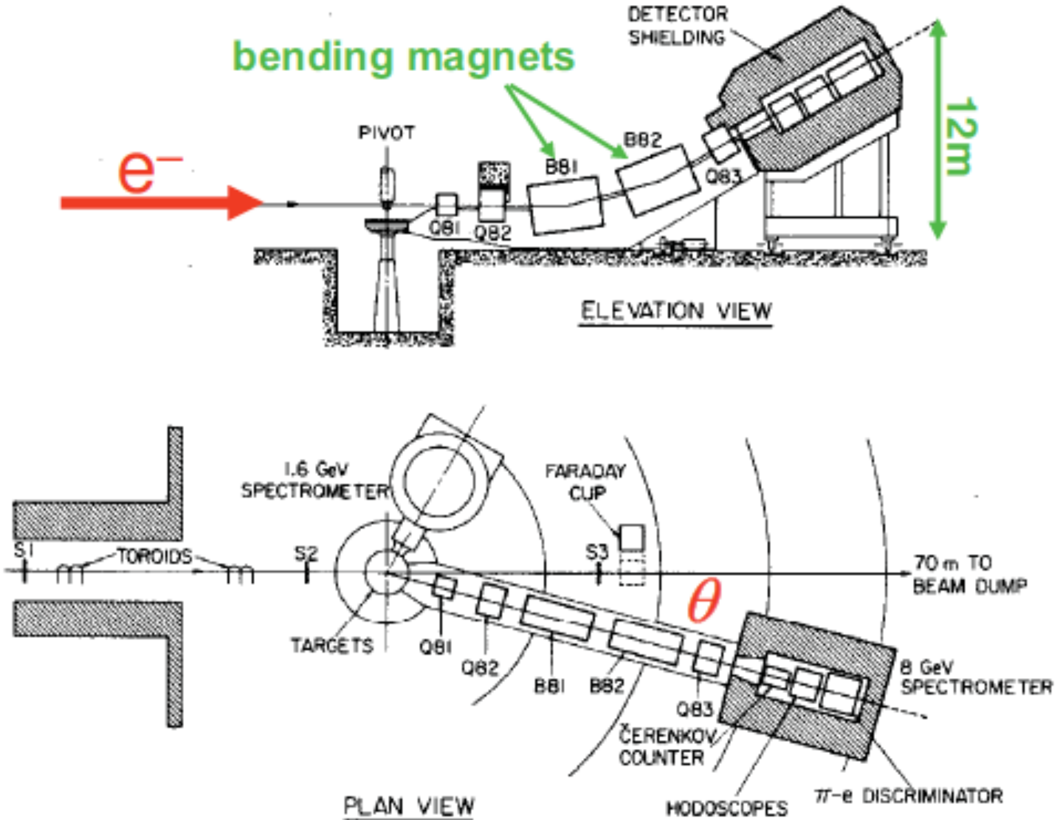
$$\frac{d\sigma}{d\Omega} \Big/ \left(\frac{d\sigma}{d\Omega}\right)_0 \approx \left(1 + 2\tau\tan^2\frac{\theta}{2}\right) G_M^2(q^2)$$



Sensitive to both structure functions. Resolve by measuring angular dependence of x-section at fixed q^2

★ **Use electron beam from SLAC LINAC:** $5 < E_{beam} < 20$ GeV

- **Detect scattered electrons using the "8 GeV Spectrometer"**



High q² → **Measure** $G_M(q^2)$

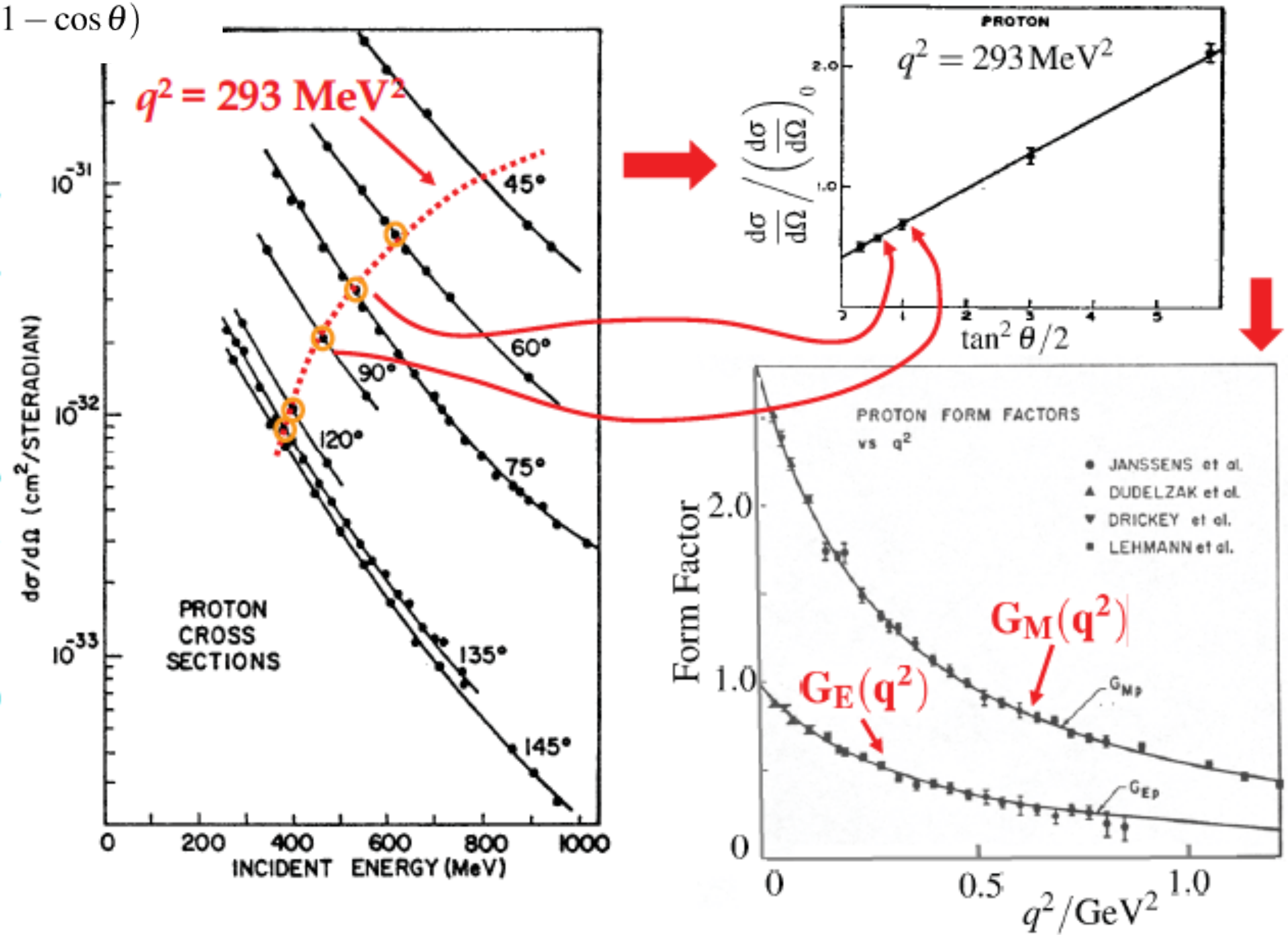
P.N.Kirk et al., Phys Rev D8 (1973) 63

EXAMPLE: $e^-p \rightarrow e^-p$ at E_{beam}= 529.5 MeV

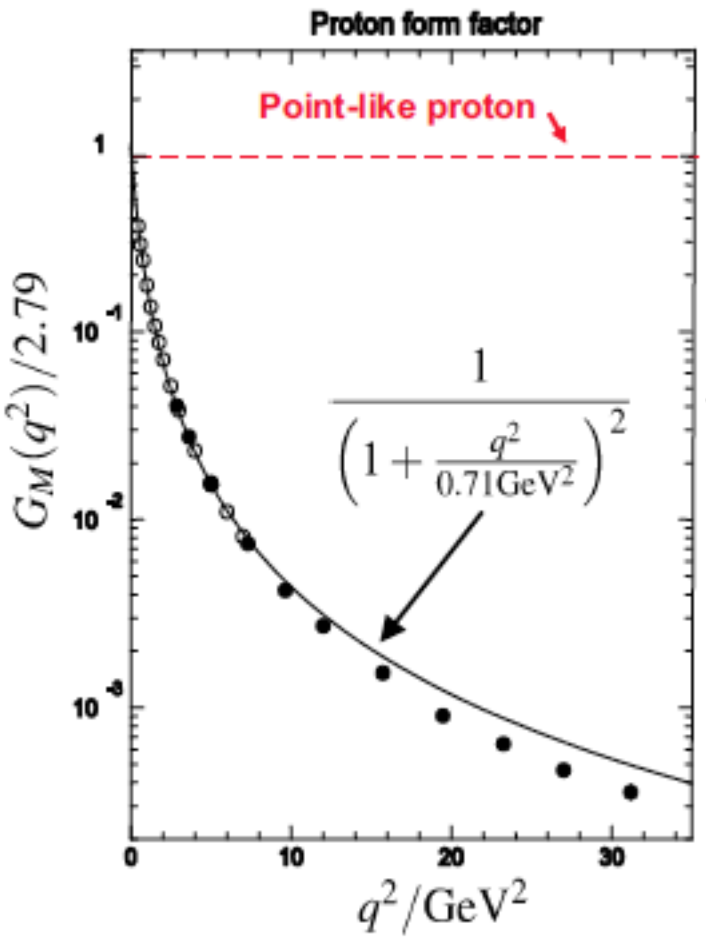
- Electron beam energies chosen to give certain values of q^2

$$q^2 = -\frac{2ME_1^2(1-\cos\theta)}{M+E_1(1-\cos\theta)}$$

E.B.Hughes et al., Phys. Rev. 139 (1965) B458



High q^2 results



R.C.Walker et al., Phys. Rev. D49 (1994) 5671
A.F.Sill et al., Phys. Rev. D48 (1993) 29

- Proton is not point-like!
- Corresponds to spatial charge and magnetic moment distribution:

$G_M \sim q^{-4}$

$$\boxed{\rho(r) \approx \rho_0 e^{-r/a}} \quad \text{where} \quad a \approx 0.24\ \mathrm{fm}$$

- Corresponds to a radius of $r_{rms} \approx 0.8\ \mathrm{fm}$
- Suggests but not proves **proton** is actually **composite**

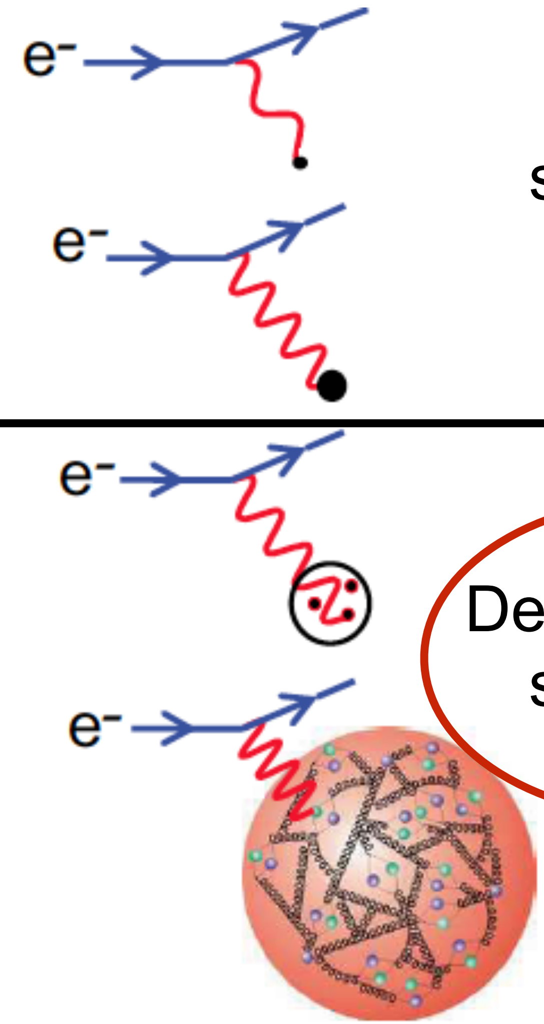
Need to up energy and break up proton to look inside —
Deep Inelastic Scattering

Deep Inelastic Scattering

- At very low electron energies $\lambda \gg r_p$:
  the scattering is equivalent to that from a "point-like" spin-less object
- At low electron energies $\lambda \sim r_p$:
  the scattering is equivalent to that from a extended charged object

---

- At high electron energies $\lambda < r_p$:
  the wavelength is sufficiently short to resolve sub-structure. Scattering from constituent quarks
- At very high electron energies $\lambda \ll r_p$:
  the proton appears to be a sea of quarks and gluons.



Elastic scattering

Deep Inelastic scattering (DIS)

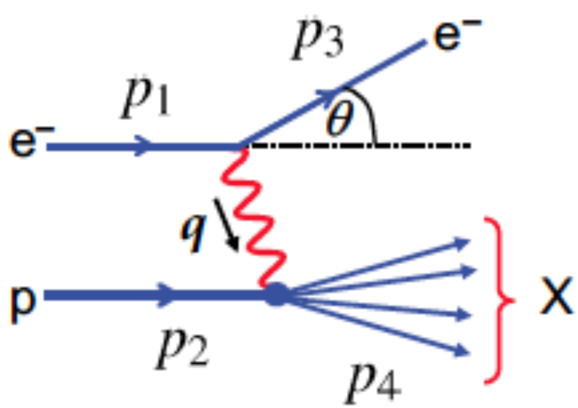
At **high q^2** the Rosenbluth x-section for **elastic** scattering becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{elastic} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2}\frac{E_3}{E_1}\left(\frac{q^2}{2M^2}G_M^2 \sin^2\frac{\theta}{2}\right) \qquad \tau = -\frac{q^2}{4M^2} \gg 1$$

From fit to experimental data (slide 17) $G_M(q^2) \propto q^{-4}$ $\Longrightarrow$ $\left(\frac{d\sigma}{d\Omega}\right)_{elastic} \propto q^{-6}$

Due to finite proton size elastic scattering at **high q^2** is unlikely
Proton breaks up and **inelastic scattering** starts to **dominate**

Inelastic Scattering



- Final state hadron**s** is no longer the proton mass, M
- Invariant mass of final state hadronic state ,

$$W^2 \equiv M_X^2 = p_4^2 = \left(E_4^2 - |\vec{p}_4|^2\right)$$

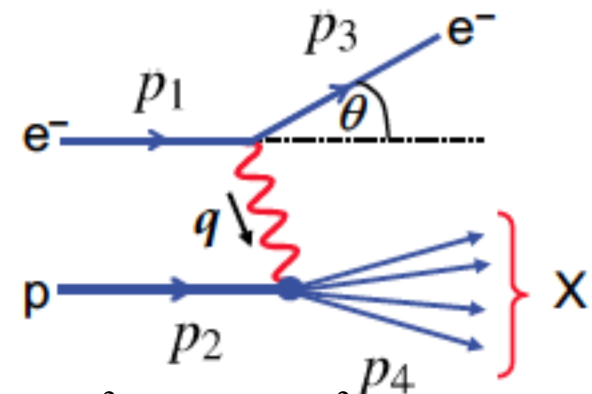
note:

$M_X > M$

Introduce new **Lorentz Invariant** Variables $\boxed{x, y, \nu, Q^2}$

DIS Kinematic Variables

$p_1 = (E_1, 0, 0, E_1) \quad p_2 = (M, 0, 0, 0) \quad q = (E_1 - E_3, \vec{p}_1 - \vec{p}_3)$



Define: $\boxed{x \equiv \frac{Q^2}{2p_2.q}}$ **Bjorken x**

where $\boxed{Q^2 \equiv -q^2}$ $\quad Q^2 > 0$

$$M_X^2 = p_4^2 = (q + p_2)^2 = -Q^2 + 2p_2.q + M^2$$

$$Q^2 = 2p_2.q + \underbrace{M^2 - M_X^2} \Rightarrow Q^2 \leq 2p_2.q$$

$M_X \geq M$

$\boxed{0 < x < 1 \text{ inelastic}}$ $\quad$ $\boxed{x = 1 \quad \text{elastic}}$ $\leftarrow$ **Proton intact** $M_X = M$

Define: $\boxed{y \equiv \frac{p_2.q}{p_2.p_1}}$ In the Lab. Frame $y = 1 - \frac{E_3}{E_1} \longrightarrow 0 < y < 1$

Fractional energy loss by the incoming particle

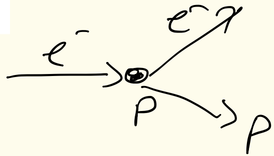
In CoM $y = \frac{1}{2}(1 - \cos\theta^*)$ (for $E>>M$)

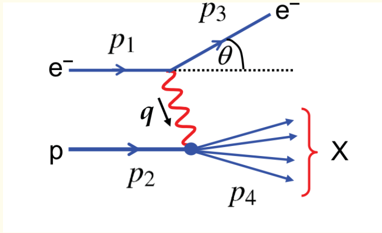
Define: $\boxed{\nu \equiv \frac{p_2.q}{M}}$ In the Lab. Frame $\nu = E_1 - E_3$

Energy lost by the incoming particle

$$p_1 = (E_1, 0, 0, E_1) \quad p_2 = (M, 0, 0, 0)$$
$$q = (E_1 - E_3, \vec{p}_1 - \vec{p}_3)$$

Lab Frame



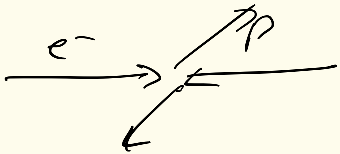


$$y = \frac{P_2 \cdot q}{P_2 \cdot P_1} = \frac{M(E_1 - E_3)}{M E_1}$$

$$y = 1 - \frac{E_3}{E_1}$$ fractional energy loss of e^-

$$p_1 = (E, 0, 0, E); \quad p_2 = (E, 0, 0, -E); \quad p_3 = (E, E\sin\theta^*, 0, E\cos\theta^*)$$

in the CoM



$$y = \frac{1}{2}(1 - \cos\theta^*)$$

Relationship between kinematic variables

$$e^- \xrightarrow{p_1} \quad \xleftarrow{p_2} p$$

$$s = (p_1 + p_2)^2$$

- Rewrite new kinematic variables in terms of *s*

$$2p_1 . p_2 = s - M^2$$

- For fixed *s*

$$Q^2 \equiv -q^2 \qquad x \equiv \frac{Q^2}{2p_2 . q} \qquad y \equiv \frac{p_2 . q}{p_2 . p_1} \qquad \nu \equiv \frac{p_2 . q}{M}$$

four kinematic variables are not independent

- x and y can be expressed as

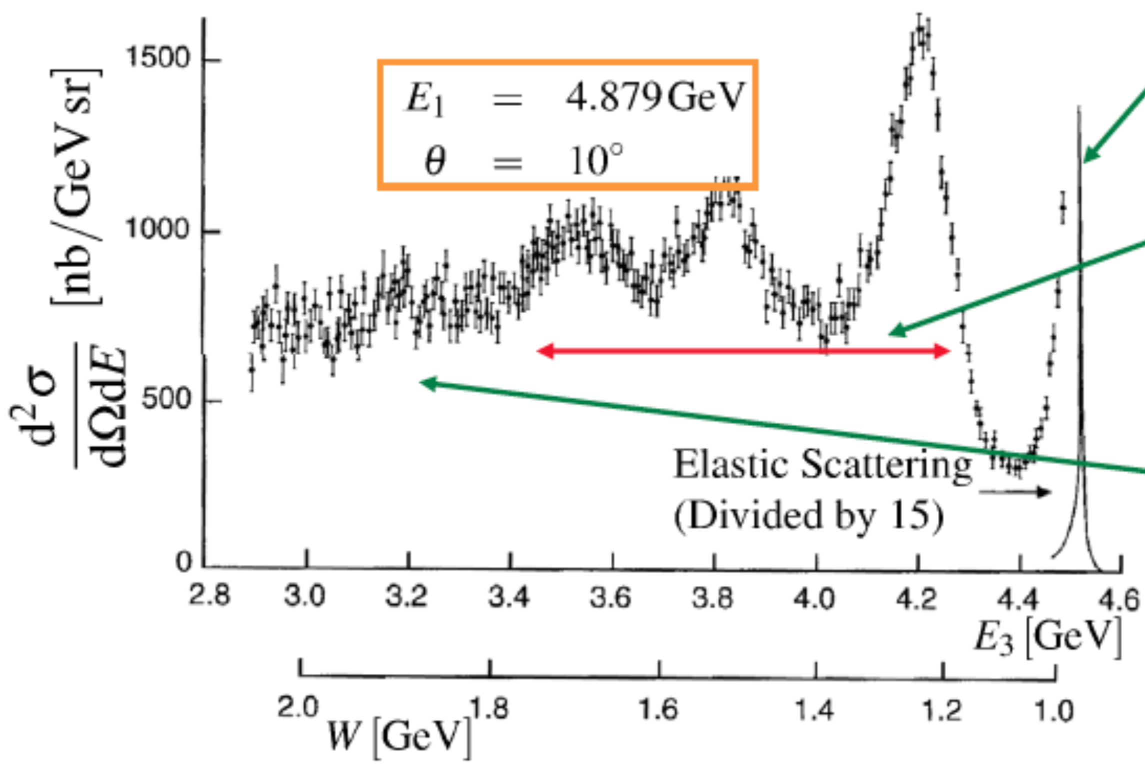
$$x = \frac{Q^2}{2M\nu} \qquad y = \frac{2M}{s - M^2}\nu$$

$$xy = \frac{Q^2}{s - M^2} \quad \Rightarrow \quad Q^2 = (s - M^2)xy$$

- For fixed C.o.M energy the interaction kinematics are completely defined by *any* **two** kinematic variables (except y and ν)

- For elastic scattering ($x=1$) - only **one** independent variable. If you measure electron scattering *angle* you know *everything else* (as before)

Example 4.9 GeV electrons scattered on protons at rest.
Detector placed at 10°



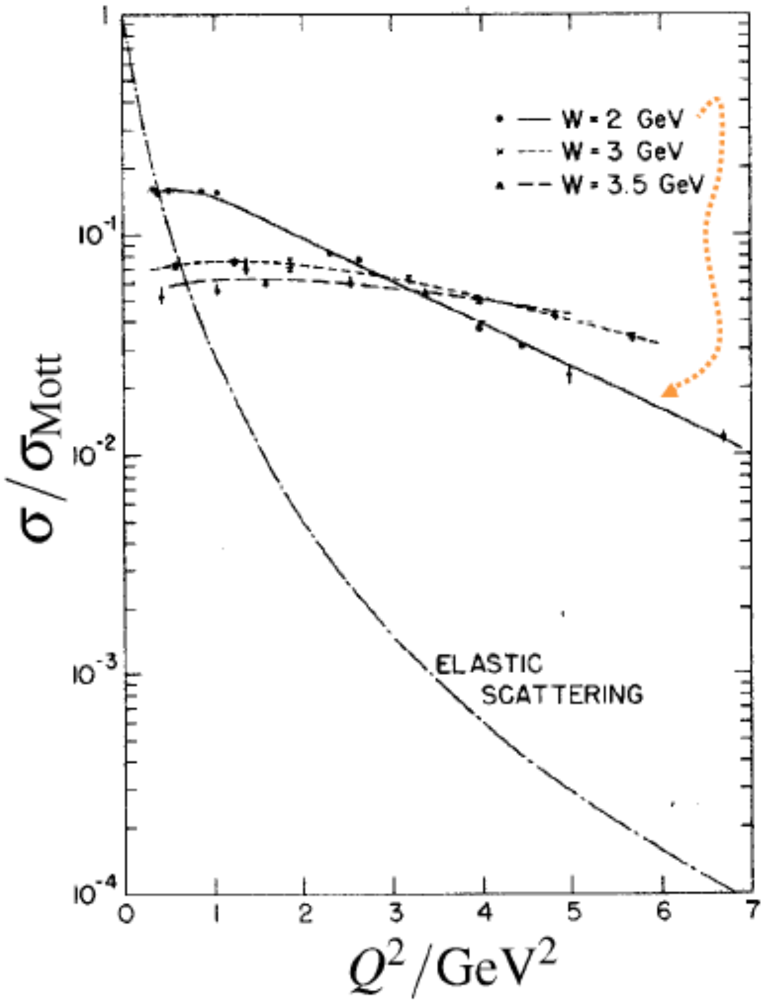
- **Elastic Scattering**
  proton remains intact
  $W = M$
- **Inelastic Scattering**
  produce "excited states" of proton e.g. $\Delta^+(1232)$
  $W = M_\Delta$
- **Deep Inelastic Scattering**
  proton breaks up resulting in a many particle final state

DIS = large W

Inelastic Cross-Section

- Repeat experiments at different angles/beam energies and determine q^2 dependence of elastic and inelastic cross-sections

M.Breidenbach et al., Phys. Rev. Lett. 23 (1969) 935



- Elastic scattering falls of rapidly with q^2 due to the proton not being point-like (i.e. form factors)
- Inelastic scattering cross sections only weakly dependent on q^2
- Deep Inelastic scattering cross sections almost independent of q^2!

i.e. "Form factor" → 1

→ **Scattering from point-like objects within the proton !**

From Elastic to Inelastic Scattering

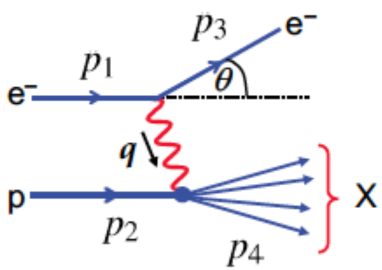
- Rosenbluth formula can be written in **Lorentz Invariant** form $\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4E_1^2 \sin^4\theta/2} \frac{E_3}{E_1}\left(\frac{G_E^2+\tau G_M^2}{(1+\tau)}\cos^2\frac{\theta}{2} + 2\tau G_M^2 \sin^2\frac{\theta}{2}\right)$

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\frac{G_E^2+\tau G_M^2}{(1+\tau)}\left(1-y-\frac{M^2y^2}{Q^2}\right)+\frac{1}{2}y^2G_M^2\right]$$

which can be written as:

$$\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[f_2(Q^2)\left(1-y-\frac{M^2y^2}{Q^2}\right)+\frac{1}{2}y^2 f_1(Q^2)\right]$$

- For DIS proton breaks up. Two independent variables are required



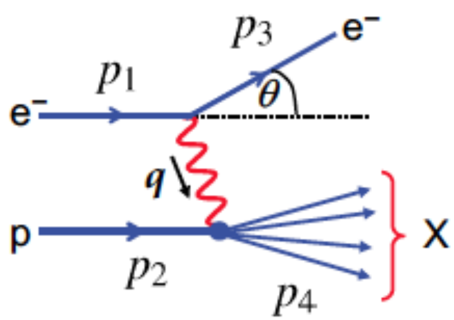
(1) $$\boxed{\frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[\left(1-y-\frac{M^2y^2}{Q^2}\right)\frac{F_2(x,Q^2)}{x}+y^2F_1(x,Q^2)\right]}$$

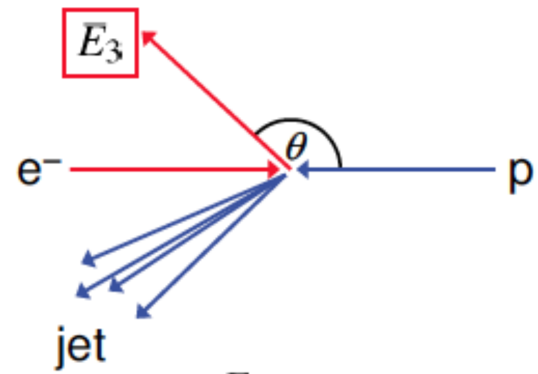
- Form factors $f_1(Q^2), f_2(Q^2)$ are replaced with **structure functions** $F_1(x,Q^2), F_2(x,Q^2)$

- In the high energy limit, i.e. $Q^2 \gg M^2y^2$ $\quad \frac{d^2\sigma}{dx\,dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x}+y^2F_1(x,Q^2)\right]$

DIS in Lab Frame

- In experiment we collide electron/positron and proton beams to study proton structure. Measure the scattered electron energy, E_3, and angle θ





$$Q^2 = 4E_1E_3\sin^2\theta/2; \quad x = \frac{Q^2}{2M(E_1 - E_3)}; \quad y = 1 - \frac{E_3}{E_1}; \quad \nu = E_1 - E_3$$

In the Lab frame formula (1) in the previous page becomes"

$$\boxed{\frac{d^2\sigma}{dE_3 d\Omega} = \frac{\alpha^2}{4E_1^2\sin^4\theta/2}\left[\frac{1}{\nu}F_2(x,Q^2)\cos^2\frac{\theta}{2} + \frac{2}{M}F_1(x,Q^2)\sin^2\frac{\theta}{2}\right]}$$

Electromagnetic Structure Function

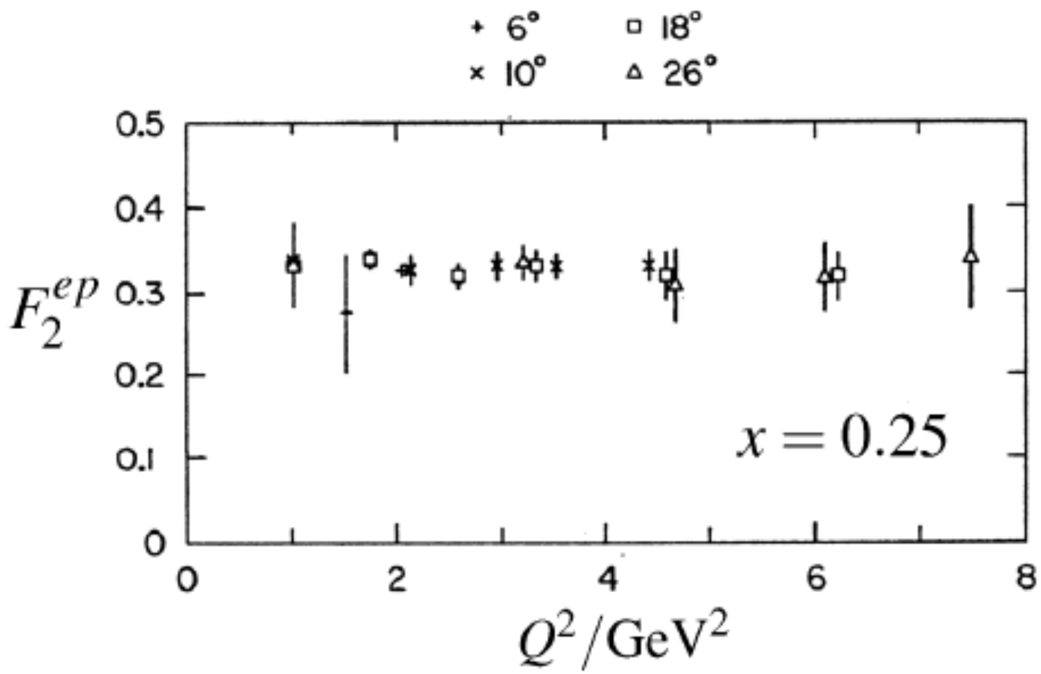
Pure Magnetic Structure Function

Measuring the Structure Functions

- To determine $F_1(x,Q^2)$ and $F_2(x,Q^2)$ for given x and Q^2

  need measurements of differential x-section at several angles and incoming beam energies

Example: **electron-proton scattering F_2 vs. Q^2 at fixed x**



J.T.Friedman + H.W.Kendall, Ann. Rev. Nucl. Sci. 22 (1972) 203

Experimentally it is observed that both F_1 and F_2 are (almost) independent of Q^2

Bjorken Scaling and Callan-Gross Relation

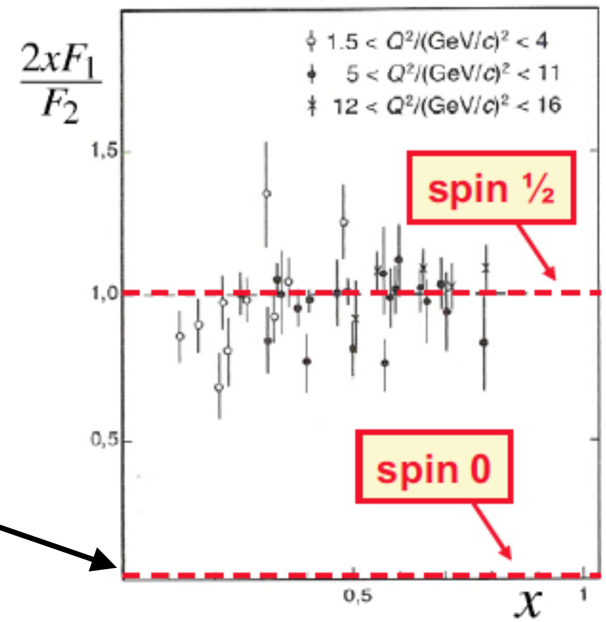
- The near independence of the Structure Functions on Q^2 is known as **Bjorken Scaling**

$$\boxed{F_1(x,Q^2) \to F_1(x) \qquad\qquad F_2(x,Q^2) \to F_2(x)}$$

- It strongly suggests that electrons are scattered from **point-like constituents** inside protons

- It is also observed that F_1 and F_2 are not independent but satisfy the **Callan-Gross relation**

$$\boxed{F_2(x) = 2xF_1(x)}$$

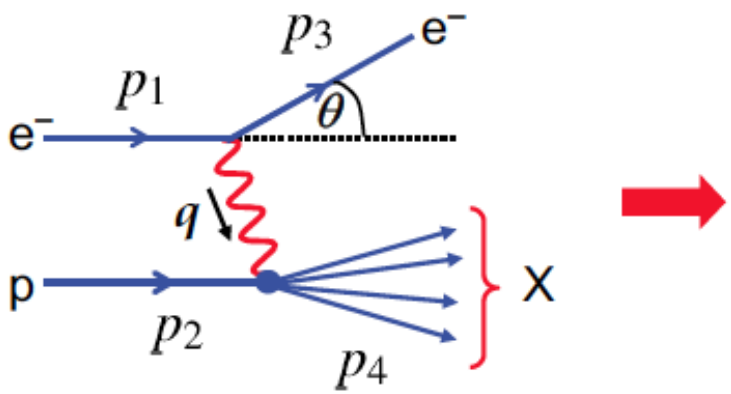


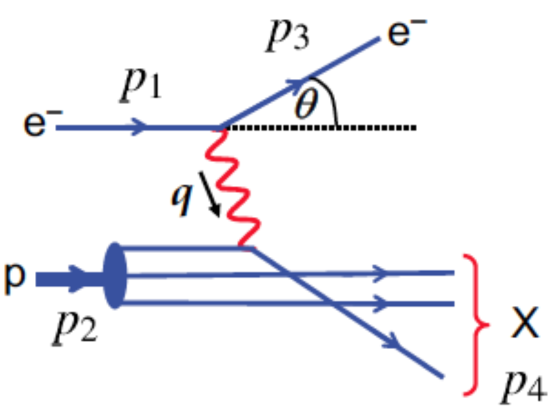
Q.: Why the C-G ratio must be zero if quarks were spin-0?

What is the physical meaning of the structure functions?

The Quark-Parton Model

- Feynman proposed that protons were made up of point-like **"partons"**
- Bjorken scaling and Callan-Gross relation can be explained by assuming that DIS is dominated by scattering on **point-like spin-1/2 constituents** inside the proton:



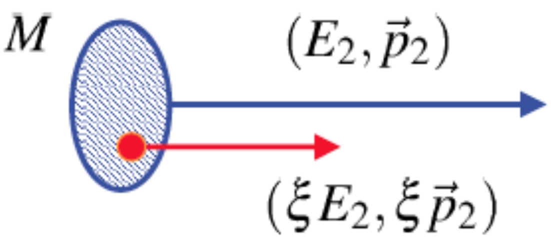


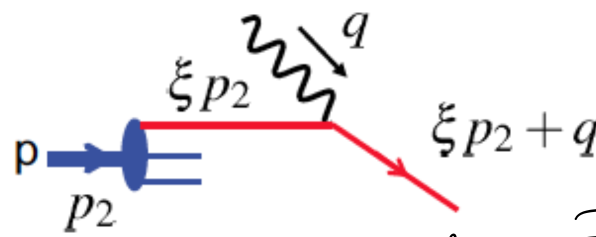
Scattering from a proton with structure functions → **Scattering from a point-like quark within the proton**

The Quark-Parton Model

- In the **parton** model the basic interaction is **elastic** scattering from a **quasi-free quark** inside the proton

- "Infinite momentum frame" - neglect the proton mass, quark mass and any momentum transverse to proton direction





$\Rightarrow P_2' = \xi P_2 + q$

$\xi = -\frac{q^2}{2P_2 \cdot q}$

$(P_2')^2 = m_q^2$

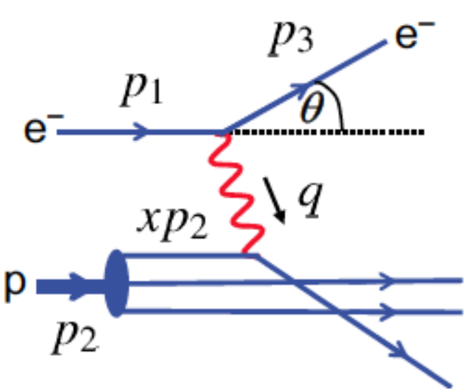
$$(\xi p_2 + q)^2 = m_q^2 \approx 0 \quad \rightarrow \quad \cancel{\xi^2 p_2^2} + q^2 + 2\xi p_2.q = 0 \qquad (\xi^2 p_2^2 = m_q^2 \approx 0)$$

$$\rightarrow \quad \xi = \frac{Q^2}{2p_2.q} = x$$

Bjorken x can be identified as the fraction of the proton momentum carried by the struck quark (in a frame where the proton has very high energy)

$m_1^2 \simeq m_2^2 \simeq 0$

$$s=(p_1+p_2)^2 \simeq 2p_1.p_2 \qquad y=\frac{p_2.q}{p_2.p_1} \qquad x=\frac{Q^2}{2p_2.q}$$



For underlying quark interaction

$$s^q=(p_1+xp_2)^2=2xp_1.p_2=xs$$

$$y_q=\frac{p_q.q}{p_q.p_1}=\frac{xp_2.q}{xp_2.p_1}=y \qquad x_q=1$$ elastic, i.e. quark does not break up

Cross-section for $e^-\mu^- \to e^-\mu^-$ derived in QED can be used for $e^-q \to e^-q$

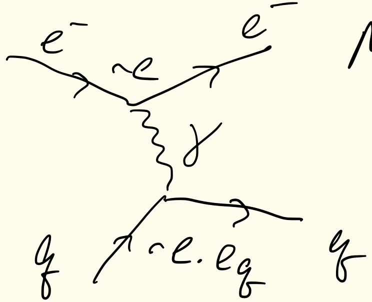
$$\frac{d\sigma}{dq^2}=\frac{2\pi\alpha^2 e_q^2}{q^4}\left[1+\left(1+\frac{q^2}{s_q}\right)^2\right]$$

e_q **is quark charge, i.e.** $e_u=+2/3; \quad e_d=-1/3$

$$-q^2=Q^2=(s_q-m^2)x_q y_q \quad \Rightarrow \quad \frac{q^2}{s_q}=-y_q=-y$$

$$\frac{d\sigma}{dQ^2}=\frac{2\pi\alpha^2 e_q^2}{Q^4}\left[1+(1-y)^2\right]$$

$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2 e_q^2}{q^4}\left[1+\left(1+\frac{q^2}{s_q}\right)^2\right]$$



$M \propto e \cdot e \cdot e_q = e^2 e_q$

$\sigma \propto |M|^2 = e^4 e_q^2 = \alpha^2 e_q^2$

RECALL $Q^2 = (s - M^2)xy$ (slide 21)

$$-q^2 = Q^2 = (s_q - m^2)\underbrace{x_q}_{=1}\underbrace{y_q}_{=y} = s_q y - \cancel{m^2 y}^{\to 0}$$

$$\frac{-q^2}{s_q} = y$$

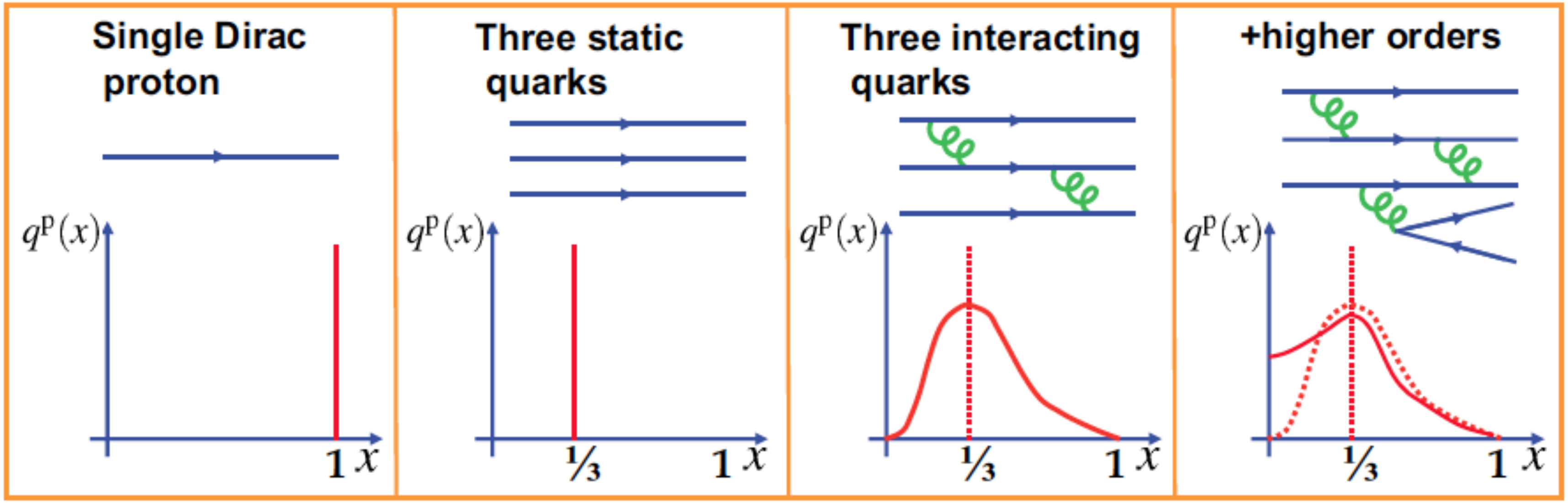
Parton Distribution Functions (PDF)

$$\boxed{\frac{d\sigma}{dQ^2} = \frac{4\pi\alpha^2 e_q^2}{Q^4}\left[(1-y)+\frac{y^2}{2}\right]}$$

differential x-section for elastic eq scattering for a quark carrying x-fraction of proton momentum

- Now need to account for quark momentum distribution inside proton
- Parton Distribution Functions (PDF) $q^{p}(x)dx$

  number of quarks of type q within a proton with momenta between $x \to x + dx$



PDF and Structure Functions

- Scattering from a particular quark type $\frac{d^2\sigma}{dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)+\frac{y^2}{2}\right] \times e_q^2 q^p(x)dx$

- ep scattering cross-section is obtained by summing over all quark types

$$\boxed{\frac{d^2\sigma^{ep}}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)+\frac{y^2}{2}\right]\sum_q e_q^2 q^p(x)}$$

to be compared with earlier obtained

$$\frac{d^2\sigma}{dx dQ^2} = \frac{4\pi\alpha^2}{Q^4}\left[(1-y)\frac{F_2(x,Q^2)}{x} + y^2 F_1(x,Q^2)\right] \quad \text{for} \quad Q^2 \gg M^2 y^2$$

$$\boxed{F_2^p(x,Q^2) = 2xF_1^p(x,Q^2) = x\sum_q e_q^2 q^p(x)} \Longrightarrow$$

Can relate measured structure functions to the underlying quark distributions

PDF and Structure Functions

The Parton Model predicts

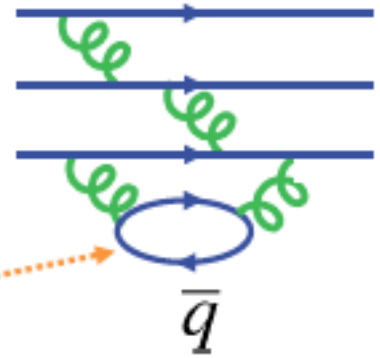
Bjorken Scaling $F_1(x,Q^2) \to F_1(x) \quad F_2(x,Q^2) \to F_2(x)$

Callan-Gross Relation $F_2(x) = 2xF_1(x)$

- Presently QCD cannot calculate parton distributions (strong interaction, perturbation theory does not work)

- Measure X-sections $\Longrightarrow$ extract Structure Functions $\Longrightarrow$ determine PDF

$$F_2^{\mathrm{p}}(x) = x\sum_q e_q^2 q^{\mathrm{p}}(x)$$



Due to higher orders need to include anti-up and anti-down quarks

(neglect contributions from heavier quarks at the moment)

PDF and Structure Functions

- Electron-proton scattering

$$F_2^{\mathrm{ep}}(x) = x\sum_q e_q^2 q^{\mathrm{p}}(x) = x\left(\frac{4}{9}u^{\mathrm{p}}(x) + \frac{1}{9}d^{\mathrm{p}}(x) + \frac{4}{9}\overline{u}^{\mathrm{p}}(x) + \frac{1}{9}\overline{d}^{\mathrm{p}}(x)\right)$$

- Electron-neutron scattering

$$F_2^{\mathrm{en}}(x) = x\sum_q e_q^2 q^{\mathrm{n}}(x) = x\left(\frac{4}{9}u^{\mathrm{n}}(x) + \frac{1}{9}d^{\mathrm{n}}(x) + \frac{4}{9}\overline{u}^{\mathrm{n}}(x) + \frac{1}{9}\overline{d}^{\mathrm{n}}(x)\right)$$

  - Assume "isospin symmetry". i.e. p and n is the same particle with isospin projection up and down: $d^{\mathrm{n}}(x) = u^{\mathrm{p}}(x); \quad u^{\mathrm{n}}(x) = d^{\mathrm{p}}(x)$

$p = uud$

$n = ddu$

$$u(x) \equiv u^{\mathrm{p}}(x) = d^{\mathrm{n}}(x); \qquad d(x) \equiv d^{\mathrm{p}}(x) = u^{\mathrm{n}}(x)$$

$$\overline{u}(x) \equiv \overline{u}^{\mathrm{p}}(x) = \overline{d}^{\mathrm{n}}(x); \qquad \overline{d}(x) \equiv \overline{d}^{\mathrm{p}}(x) = \overline{u}^{\mathrm{n}}(x)$$

Hence:

$$\boxed{F_2^{\mathrm{ep}}(x) = 2xF_1^{\mathrm{ep}}(x) = x\left(\frac{4}{9}u(x) + \frac{1}{9}d(x) + \frac{4}{9}\overline{u}(x) + \frac{1}{9}\overline{d}(x)\right)}$$

$$\boxed{F_2^{\mathrm{en}}(x) = 2xF_1^{\mathrm{en}}(x) = x\left(\frac{4}{9}d(x) + \frac{1}{9}u(x) + \frac{4}{9}\overline{d}(x) + \frac{1}{9}\overline{u}(x)\right)}$$

Integrating it

$$\int_0^1 F_2^{ep}(x)\mathrm{d}x = \int_0^1 x\left(\frac{4}{9}[u(x)+\overline{u}(x)] + \frac{1}{9}[d(x)+\overline{d}(x)]\right)\mathrm{d}x = \frac{4}{9}f_u + \frac{1}{9}f_d$$

$$\int_0^1 F_2^{en}(x)\mathrm{d}x = \int_0^1 x\left(\frac{4}{9}[d(x)+\overline{d}(x)] + \frac{1}{9}[u(x)+\overline{u}(x)]\right)\mathrm{d}x = \frac{4}{9}f_d + \frac{1}{9}f_u$$

35a

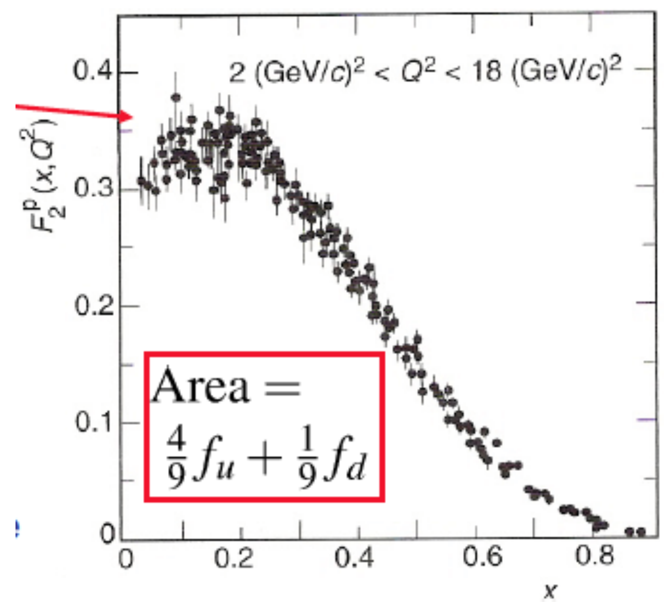
$$f_u = \int_0^1 [xu(x) + x\overline{u}(x)]\mathrm{d}x$$

the fraction of the proton momentum carried by the up and anti-up quarks

<u>**Experimentally**</u>

$$\int F_2^{ep}(x)\mathrm{d}x \approx 0.18$$

$$\int F_2^{en}(x)\mathrm{d}x \approx 0.12$$



- In proton u-quarks carry twice the momentum of d-quarks (as expected)
- The quarks carry just over 50% of the total proton momentum!

The rest is carried by **gluons!**

Experimentally

$$\int F_2^{ep}(x)\,\mathrm{d}x \approx 0.18$$
$$\int F_2^{en}(x)\,\mathrm{d}x \approx 0.12$$

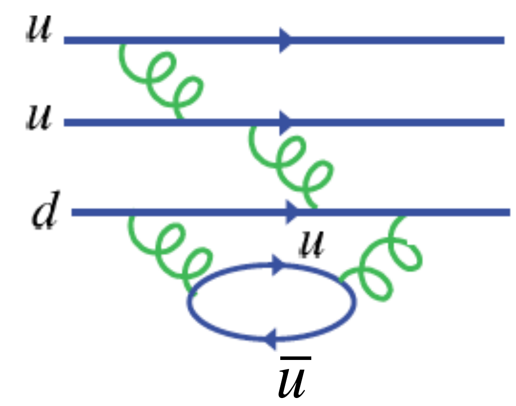
$$\Longrightarrow \left.\begin{aligned} \frac{4}{9} f_u + \frac{1}{9} f_d &= 0.18 \\ \frac{4}{9} f_d + \frac{1}{9} f_u &= 0.12 \end{aligned}\right\}$$

$$f_u \simeq 0.36 \quad ; \quad f_d \simeq 0.18$$

$$f_u + f_d \simeq 0.54 \Longrightarrow$$ QUARKS carry only ~50% of proton momentum.

Valence and Sea quarks

- PDF include contributions from **"valence"** quarks and virtual quarks produced by gluons: the **"sea"**



$$u(x) = u_V(x) + u_S(x) \qquad d(x) = d_V(x) + d_S(x)$$
$$\bar{u}(x) = \bar{u}_S(x) \qquad \bar{d}(x) = \bar{d}_S(x)$$

For proton $\int_0^1 u_V(x)\mathrm{d}x = 2 \qquad \int_0^1 d_V(x)\mathrm{d}x = 1$

- Cannot predict exactly the number of sea quarks but since $m_u = m_d$

  it's reasonable to expect $u_S(x) = d_S(x) = \bar{u}_S(x) = \bar{d}_S(x) = S(x)$

Sea

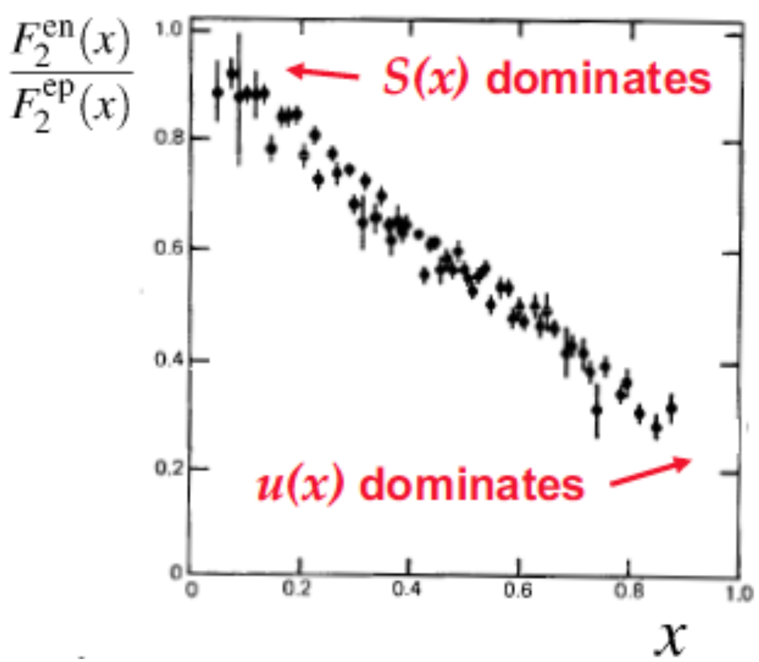
$+\frac{4}{9}\bar{u} + \frac{1}{9}\bar{d} + \frac{4}{9}u_s + \frac{1}{9}d_s = \frac{10}{9}S$

Therefore

$$F_2^{ep}(x) = x\left(\frac{4}{9}u_V(x) + \frac{1}{9}d_V(x) + \frac{10}{9}S(x)\right) \qquad F_2^{en}(x) = x\left(\frac{4}{9}d_V(x) + \frac{1}{9}u_V(x) + \frac{10}{9}S(x)\right)$$

Valence and Sea quarks

- The sea component arises from $g \to q\bar{q}$
- More likely to produce low energy gluons ($1/q^2$ in the propagator)
- Hence in the sea low momenta $q\bar{q}$ dominate
- **Expect**: $\frac{F_2^{en}(x)}{F_2^{ep}(x)} \to 1 \quad \text{as} \quad x \to 0$ $\qquad \frac{F_2^{en}(x)}{F_2^{ep}(x)} \to \frac{2}{3} \quad \text{as} \quad x \to 1$ *PS3. Q5.*
- **Experimentally**



$\frac{F_2^{en}(x)}{F_2^{ep}(x)} \to 1 \quad \text{as} \quad x \to 0$ as expected

But

$F_2^{en}(x)/F_2^{ep}(x) \to 1/4 \quad \text{as} \quad x \to 1$

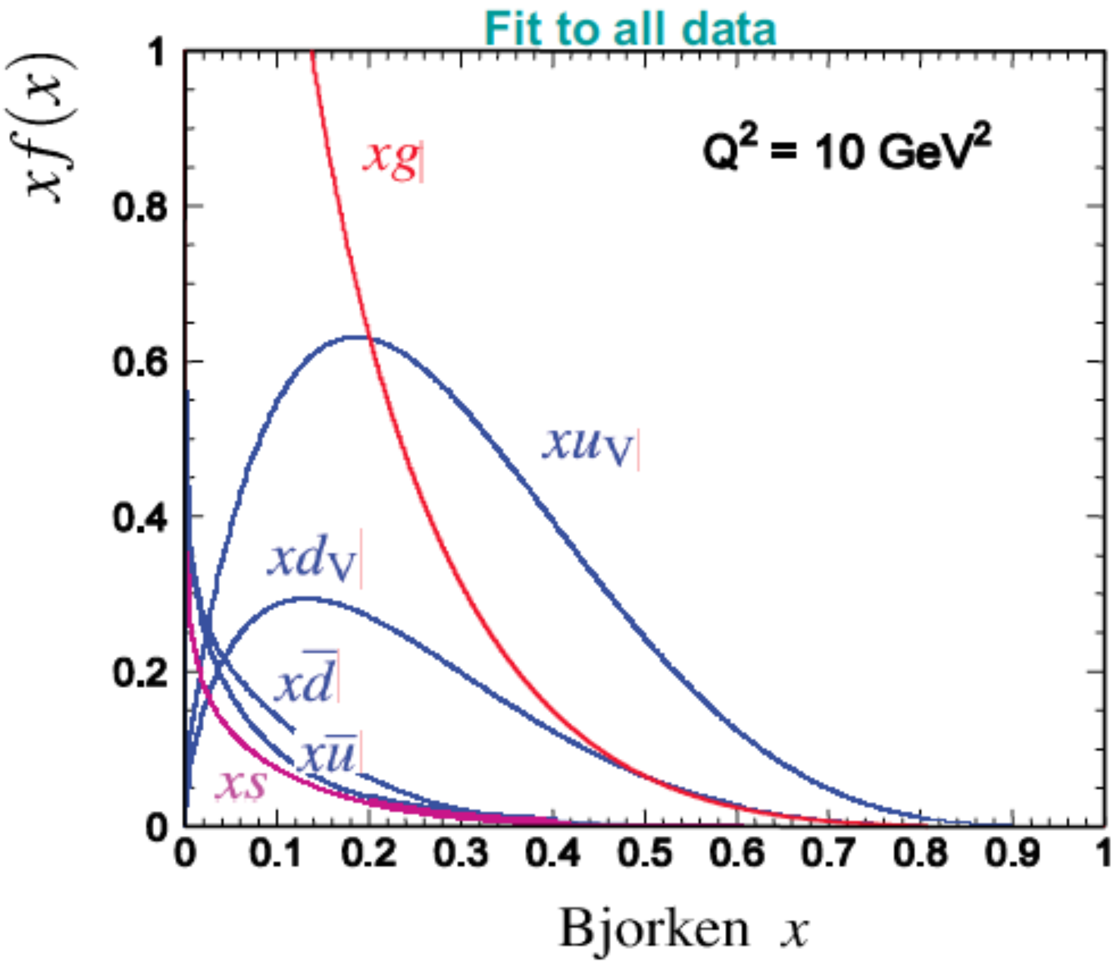
perhaps because

$d(x)/u(x) \to 0 \quad \text{as} \quad x \to 1$

Not yet understood!

Parton Distribution Functions (PDF)

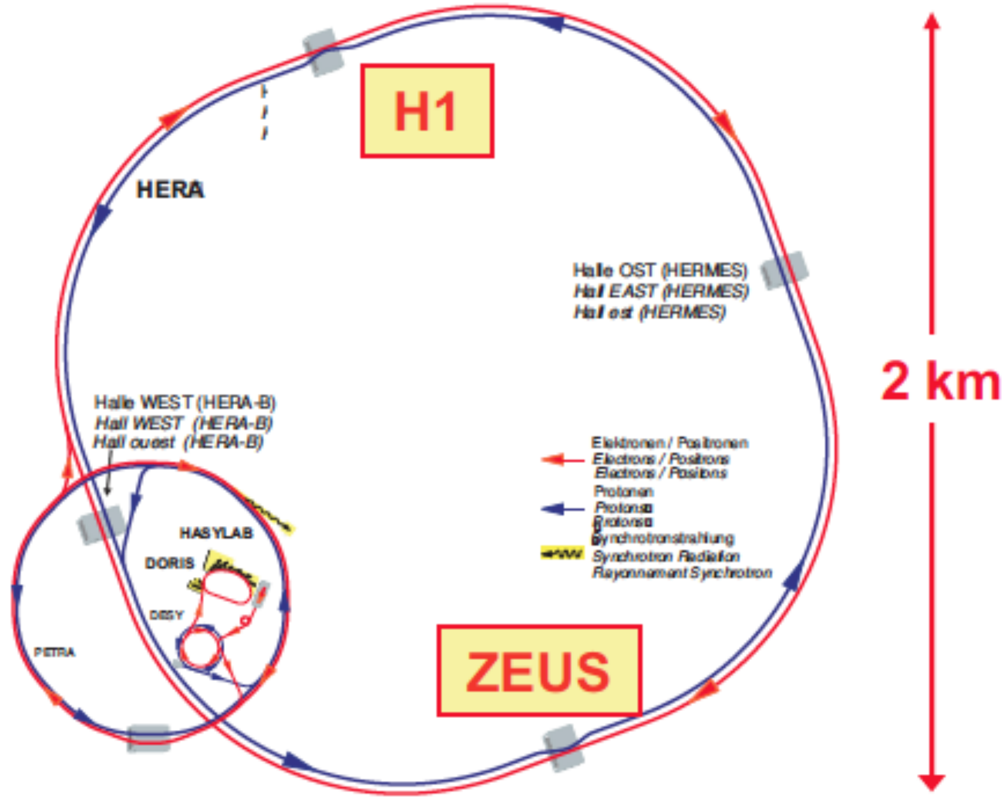
- PDFs are obtained from a global fit to all experimental data $ep, en, \nu p, \nu n, pp$



Note:

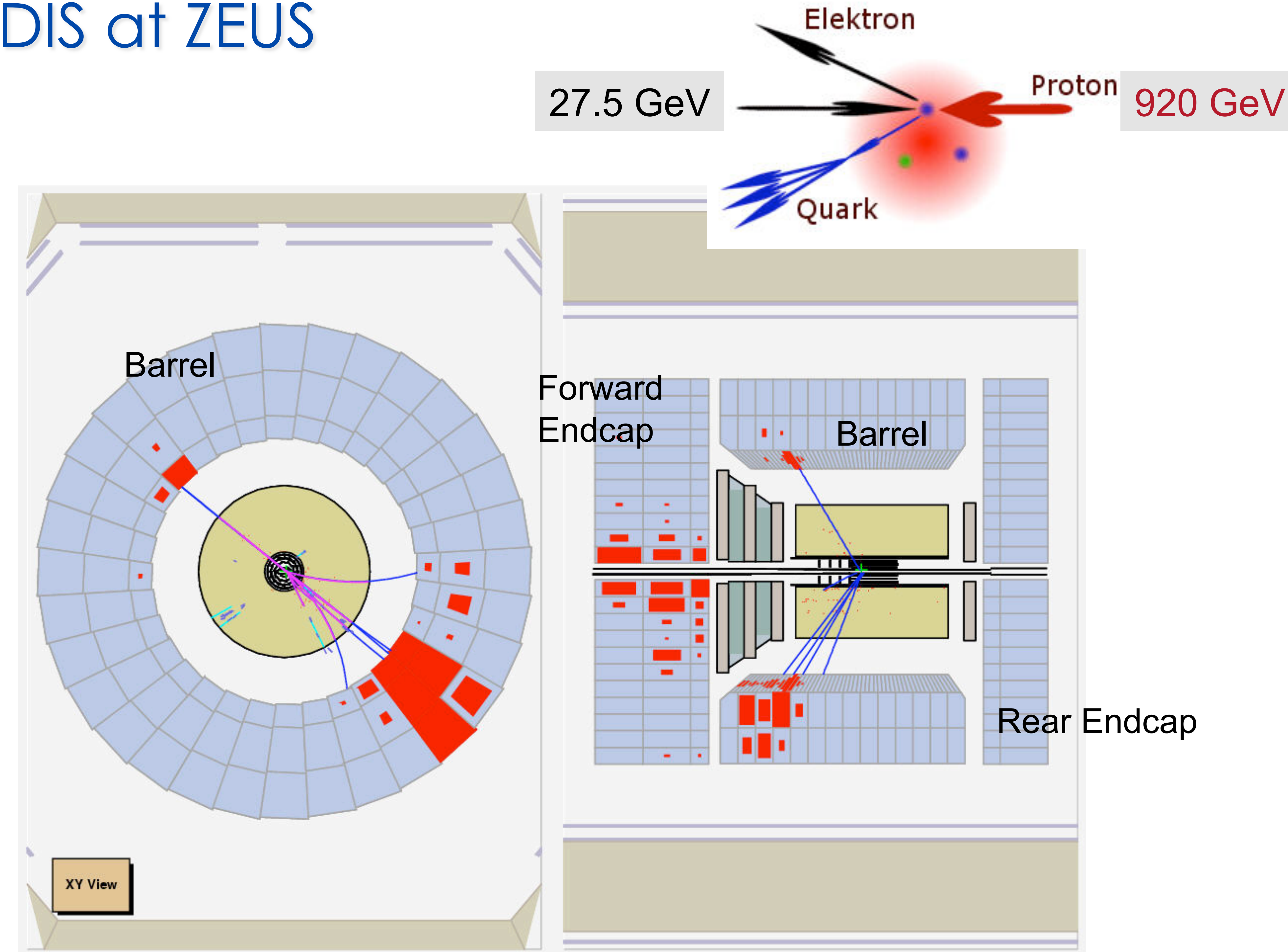
- Apart from at large x
  $u_V(x) \approx 2d_V(x)$
- For $x < 0.2$
  gluons dominate
- In fits to data assume
  $u_s(x) = \bar{u}(x)$
- $\bar{d}(x) > \bar{u}(x)$
  not understood – exclusion principle?
- Small strange quark component $s(x)$

HERA $e^{\pm}p$ Collider : 1991-2007



- ★ Two large experiments : H1 and ZEUS
- ★ Probe proton at very high Q^2 and very low x

DIS at ZEUS



Elektron
27.5 GeV
Proton
920 GeV
Quark
Barrel
Forward
Endcap
Barrel
Rear Endcap
XY View

Scaling Violations

- If quarks were not point-like, we would observe rapid decrease in x-section at high q^2. Or, equivalently, F_1, F_2 dependence on Q^2.

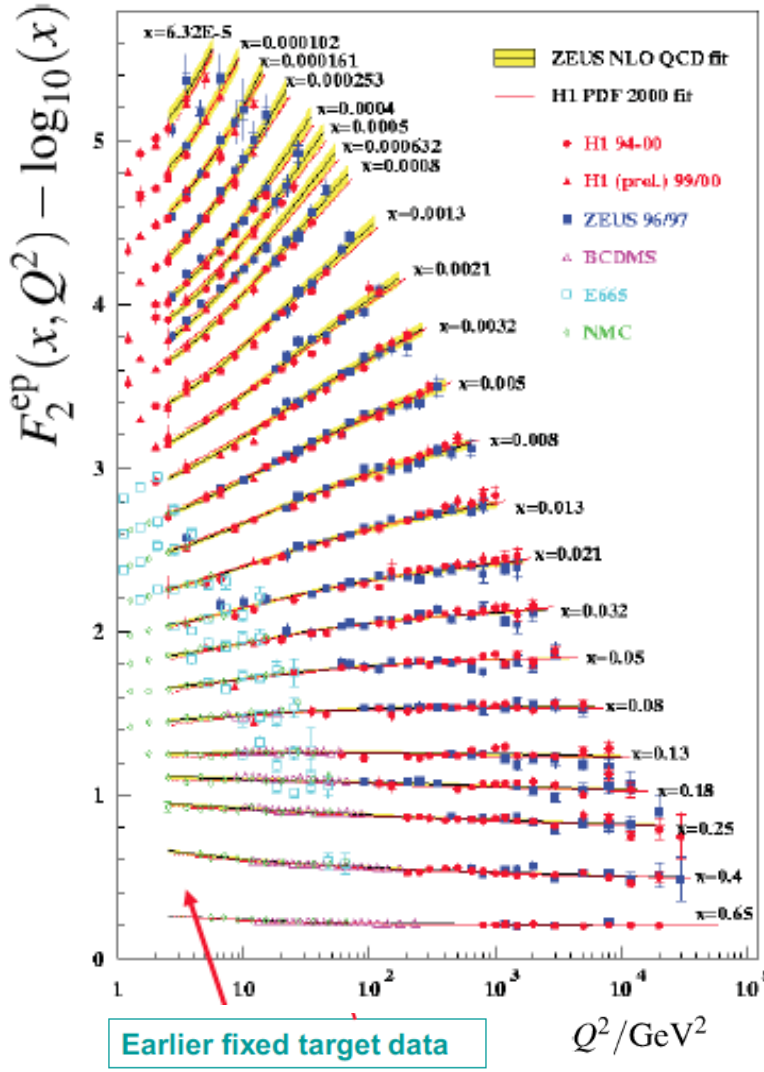
- Experimentally, no rapid decrease in x-section, F_1, F_2 are largely independent of Q^2.

$$\Rightarrow \boxed{R_{\text{quark}} < 10^{-18}\,\text{m}}$$

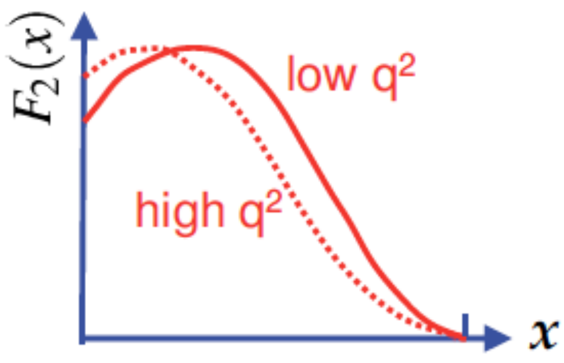
- For $x > 0.05$, only weak dependence of F_2 on Q^2.

- But clear scaling violations are observed at particular low x

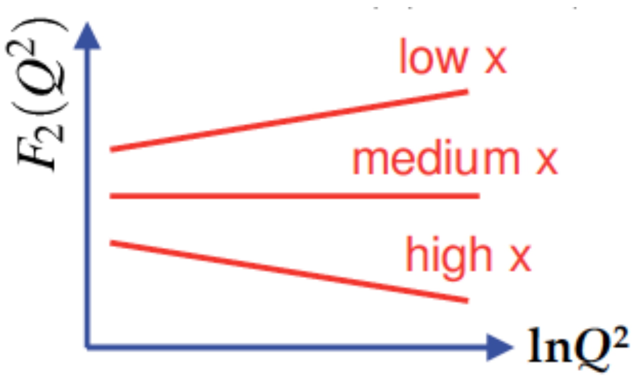
$$F_2(x, Q^2) \neq F_2(x)$$



Origin of Scaling Violations



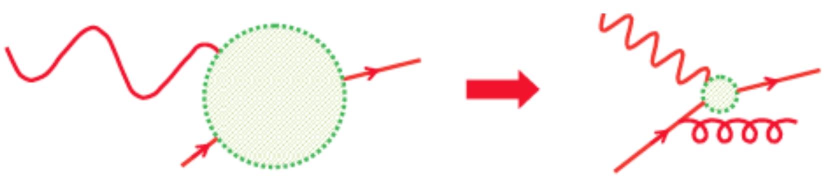
at high Q^2 observe more low-x quarks



Small deviations from exact Bjorken scaling

$$F_2(x) \rightarrow F_2(x, Q^2)$$

- "Explanation": At high Q^2 finer structures are resolved - reveal quark sharing momentum with gluon

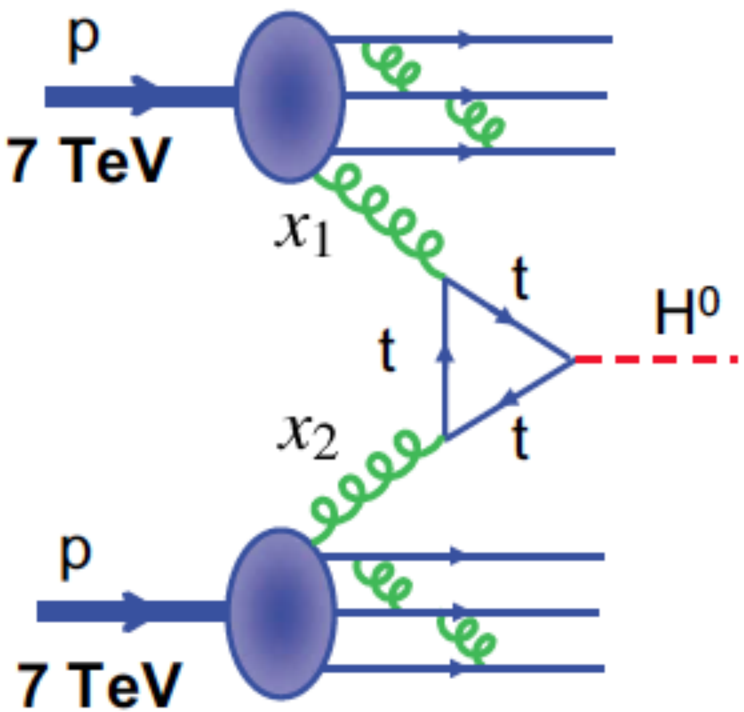


expect to see more low x quarks at high $Q^2 \Longrightarrow$ indeed observed

- QCD cannot predict *x* dependence of $F_2(x, Q^2)$
- But QCD can predict Q^2 dependence of $F_2(x, Q^2)$

- Measurements of structure functions is not only important to verify QCD. PDFs are crucial to calculate x-section at hadron colliders, $pp, p\bar{p}$

Higgs production at LHC

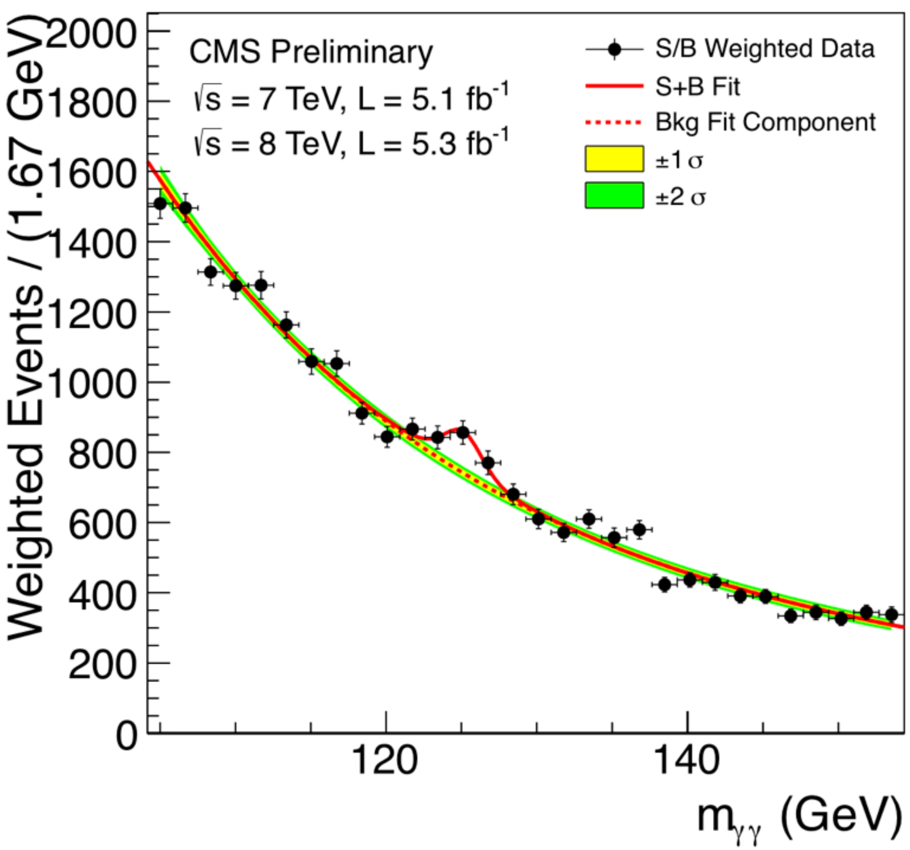


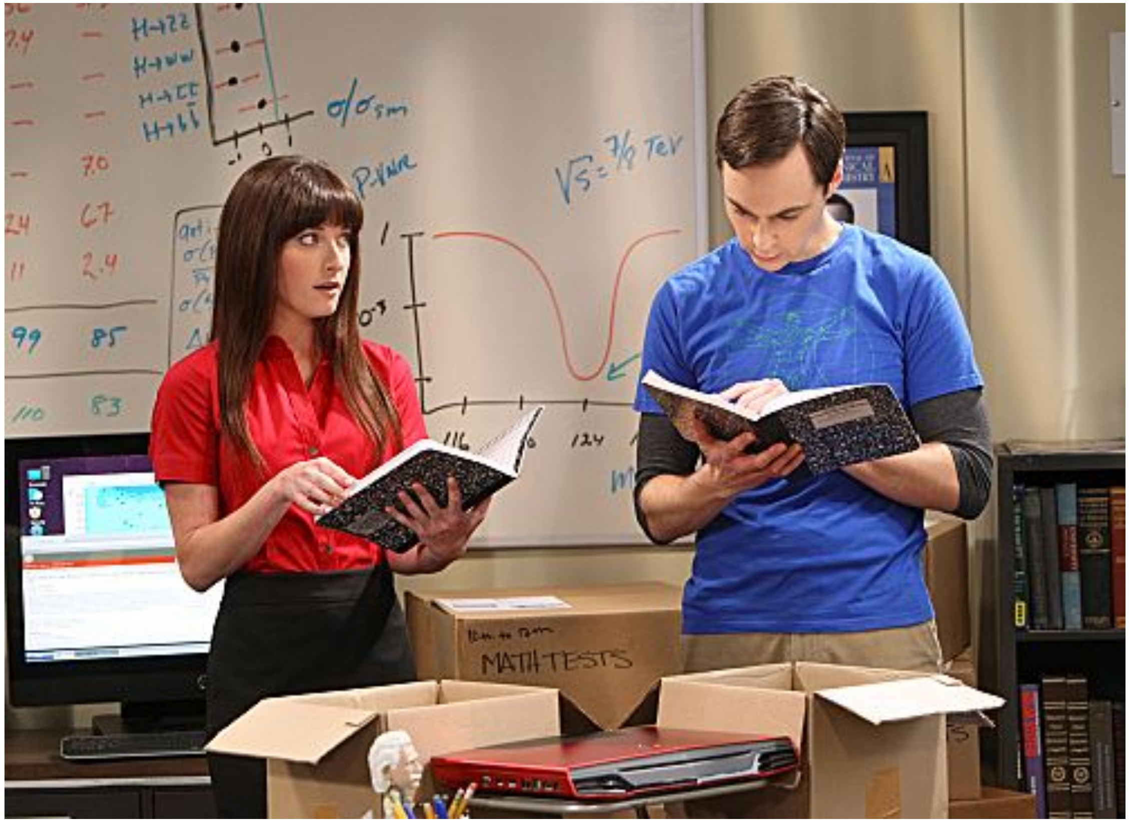
- In reality collisions are between patrons
- At LHC Higgs production is dominated by ***gluon-gluon fusion***
- X-section depends on gluon PDFs

$$\sigma(pp \to HX) \sim \int_0^1 \int_0^1 g(x_1) g(x_2) \sigma(gg \to H) \mathrm{d}x_1 \mathrm{d}x_2$$

- Prior to HERA x-section uncertainty for Higgs production was ±25% (due to gluon PDFs)
- Now it is ±5%

Why is it important?



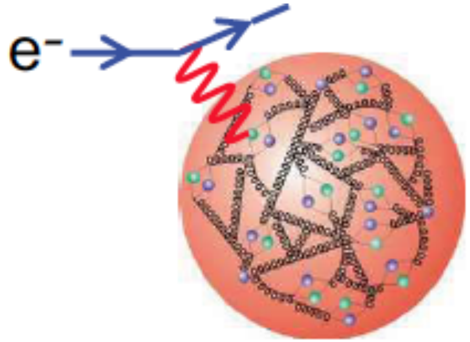


No way to discover Higgs, new physics or anything else without accurate QCD background prediction!

Summary of DIS

- At very high (projectile) energies proton is much more complex than uud, **sea of quarks and gluons**

- DIS is an elastic scattering from **quasi-free** constituent **quarks**



➡ **Bjorken Scaling** $F_1(x, Q^2) \rightarrow F_1(x)$ **point-like scattering**

➡ **Callan-Gross** $F_2(x) = 2xF_1(x)$ **Scattering from spin-1/2**

- This scattering is described in terms of Parton Distribution Functions, **PDFs** $u(x), d(x), \ldots$

- **Quarks** carry only **50%** of **proton momentum** - the rest is due to **gluons**

From QED to Theory of Strong Interactions

- Success of QED motivated applying same approach to Strong Interactions

- However specific properties of strong interactions had to be taken into account

    - nucleons have substructure — partons

    - short range of strong force, increased interaction strength with increasing distance

    - non-observation of free partons (quarks and gluons)

    - Colour charge <— > $\Omega^{-} = sss$

From QED to QCD

$\hat{U} = e^{i\vec{\alpha}\cdot\vec{\sigma}}$

Dirac equation: $i\gamma^\mu \partial_\mu \psi - m\psi = 0$

local Gauge invariance

$$\psi \to \psi' = \psi e^{iq\chi(x)}$$

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0$$

$$i\gamma^\mu (\partial_\mu - \underbrace{q A_\mu)\psi} - m\psi = 0 \;\longrightarrow\; \underbrace{i\gamma^\mu q A_\mu}$$

charge

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \chi$$

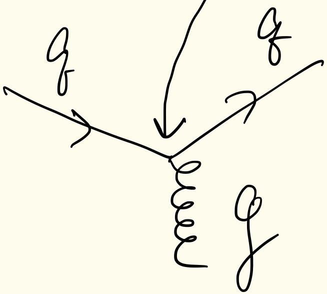
U(1) symmetry

From QED to QCD

Invariance under SU(3) local phase transformations

$$\psi \to \psi' = \psi e^{\underline{ig \vec{\lambda}\cdot\vec{\theta}(x)}}$$

$$\hat{U} = e^{\underline{i\hat{\alpha}\hat{T}}}$$



$$-\frac{1}{2} i g_s \lambda^a \gamma^\mu$$

SU(3) is a colour symmetry

SU(2) and SU(3) flavour symmetry

Isospin

- Heisenberg (1932): If you could "switch off" proton's electric charge

There would be no way to distinguish between a proton and neutron

$$V_{pp} \approx V_{np} \approx V_{nn}$$

- Proton and neutron can be considered of two states of a single entity — **nucleon**

$$p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Analogous to spin-up/spin-down states of spin-1/2 particle

ISOSPIN

p and n form an isospin doublet with **$I = 1/2$** and **$I_3 = \pm 1/2$**

- Strong interaction are **invariant** under rotation in this space

Flavour Symmetry of Strong Interactions

- Extend this to quarks $u = \binom{1}{0} \quad d = \binom{0}{1}$

- Invariance of strong interactions under $u \leftrightarrow d$

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} u \\ d \end{pmatrix}$$

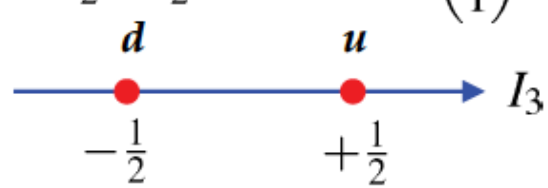
- Applying constraints: $U^\dagger U = 1$ and $\det U = 1$ obtain SU(2) group with 3 parameters (2^2 - 1, Module 2)

- The **generator** of this transformation (recall Module 2 again) can be **isospin** expressed in terms of Pauli matrices

$$\vec{T} = \frac{1}{2}\vec{\sigma} \qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$$ (unitary, with unit determinant)

- Convenient consequence of this choice: **isospin has the same algebra as spin**

$$|s, m\rangle \longrightarrow |I, I_3\rangle \qquad u = \binom{1}{0} = |\tfrac{1}{2}, +\tfrac{1}{2}\rangle \qquad d = \binom{0}{1} = |\tfrac{1}{2}, -\tfrac{1}{2}\rangle$$



$$\boxed{I = \frac{1}{2}, \quad I_3 = \pm\frac{1}{2}}$$

SU(2) symmetry. Analogies with spin algebra.

$$\begin{pmatrix} u' \\ d' \end{pmatrix} = \hat{U} \begin{pmatrix} u \\ d \end{pmatrix} = e^{i \vec{\alpha} \cdot \hat{\vec{T}}} \begin{pmatrix} u \\ d \end{pmatrix}$$

$$\vec{\alpha} \hat{\vec{T}} = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3$$

$$T_1 = \frac{1}{2} \sigma_1$$

$$T_2 = \frac{1}{2} \sigma_2$$

$$T_3 = \frac{1}{2} \sigma_3$$

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Commutation relations

$$[T_1, T_2] = i T_3 \; ; \; [T_2, T_3] = i T_1 \; \ldots\ldots$$

$$[T^2, T_3] = 0$$

Ladder operators

$$T_+ \equiv T_1 + i T_2 \quad \ldots\ldots$$

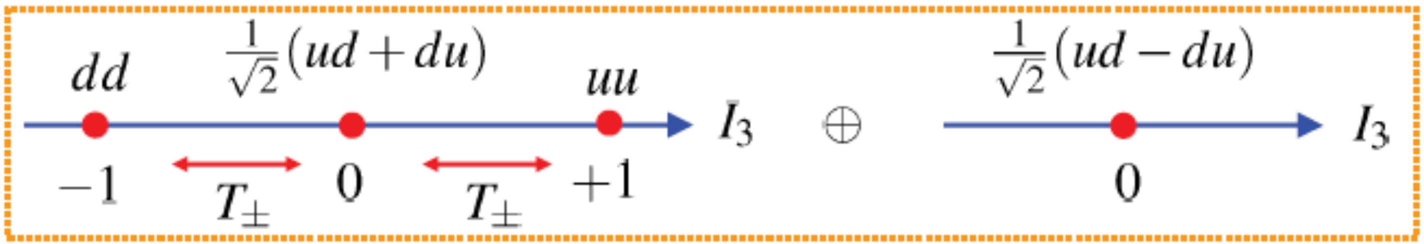
49a

Combining quarks

- The value of **isospin** is in **combining quarks** to build bound states, i.e. **hadrons**

$$\boxed{uu \equiv |\tfrac{1}{2},\tfrac{1}{2}\rangle|\tfrac{1}{2},\tfrac{1}{2}\rangle = |1,+1\rangle} \qquad \boxed{dd \equiv |\tfrac{1}{2},-\tfrac{1}{2}\rangle|\tfrac{1}{2},-\tfrac{1}{2}\rangle = |1,-1\rangle}$$

- Can define ladder operators (just like with normal spin) to move between combined quarks states



triplet of isospin-1 states

singlet of isospin-0 state

In group theory language, $2 \otimes 2 = 3 \oplus 1$

- We can proceed in the same way and build bound states of three quarks, using u and d quarks, e.g. wave function of proton (next slide)
- We can also add anti-quark to construct mesons
- The **conserved observable** quantities from this SU(2) symmetry are I_3 and I

Combining quarks:

quark a and quark b

$$I^a, I_3^a \qquad\qquad I^b, I_3^b$$

$$I_3 = I_3^a + I_3^b$$

$$|I^a - I^b| \leq I \leq |I^a + I^b|$$

proton wavefunction

$$|\tfrac{3}{2},+\tfrac{3}{2}\rangle = uuu$$
$$|\tfrac{3}{2},+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}(uud+udu+duu)$$
$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}(ddu+dud+udd)$$
$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = ddd$$

$$|\tfrac{3}{2},+\tfrac{3}{2}\rangle = \uparrow\uparrow\uparrow$$
$$|\tfrac{3}{2},+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}(\uparrow\uparrow\downarrow+\uparrow\downarrow\uparrow+\downarrow\uparrow\uparrow)$$
$$|\tfrac{3}{2},-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{3}}(\downarrow\downarrow\uparrow+\downarrow\uparrow\downarrow+\uparrow\downarrow\downarrow)$$
$$|\tfrac{3}{2},-\tfrac{3}{2}\rangle = \downarrow\downarrow\downarrow$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\frac{1}{\sqrt{6}}(2ddu-udd-dud)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{6}}(2uud-udu-duu)$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = -\frac{1}{\sqrt{6}}(2\downarrow\downarrow\uparrow-\uparrow\downarrow\downarrow-\downarrow\uparrow\downarrow)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{6}}(2\uparrow\uparrow\downarrow-\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}(udd-dud)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}(udu-duu)$$

$$|\tfrac{1}{2},-\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\downarrow-\downarrow\uparrow\downarrow)$$
$$|\tfrac{1}{2},+\tfrac{1}{2}\rangle = \frac{1}{\sqrt{2}}(\uparrow\downarrow\uparrow-\downarrow\uparrow\uparrow)$$

$$\begin{aligned}|p\uparrow\rangle = \frac{1}{\sqrt{18}}(\ & 2u\uparrow u\uparrow d\downarrow - u\uparrow u\downarrow d\uparrow - u\downarrow u\uparrow d\uparrow + \\ & 2u\uparrow d\downarrow u\uparrow - u\uparrow d\uparrow u\downarrow - u\downarrow d\uparrow u\uparrow + \\ & 2d\downarrow u\uparrow u\uparrow - d\uparrow u\downarrow u\uparrow - d\uparrow u\uparrow u\uparrow)\end{aligned}$$

SU(3) Flavour Symmetry

UCL

- Extend these ideas to add third s quark. Not an exact symmetry since $m_s > m_u, m_d$ but a good approximation

$$u = \begin{pmatrix}1\\0\\0\end{pmatrix} \quad d = \begin{pmatrix}0\\1\\0\end{pmatrix} \quad s = \begin{pmatrix}0\\0\\1\end{pmatrix} \qquad u \leftrightarrow d \leftrightarrow s \qquad \begin{pmatrix}u'\\d'\\s'\end{pmatrix} = \hat{U}\begin{pmatrix}u\\d\\s\end{pmatrix} = \begin{pmatrix}U_{11} & U_{12} & U_{13}\\ U_{21} & U_{22} & U_{23}\\ U_{31} & U_{32} & U_{33}\end{pmatrix}\begin{pmatrix}u\\d\\s\end{pmatrix}$$

- With usual constraints $U^\dagger U = 1 \quad \det U = 1$

  we arrive at **SU(3)** group with **8** parameters (3^2 - 1)

- The group generators are now 8 Gell-Mann matrices $\vec{T} = \frac{1}{2}\vec{\lambda} \qquad \hat{U} = e^{i\vec{\alpha}.\vec{T}}$

u ↔ d

$$\lambda_1 = \begin{pmatrix}0&1&0\\1&0&0\\0&0&0\end{pmatrix} \quad \lambda_2 = \begin{pmatrix}0&-i&0\\i&0&0\\0&0&0\end{pmatrix} \quad \lambda_3 = \begin{pmatrix}1&0&0\\0&-1&0\\0&0&0\end{pmatrix}$$

u ↔ s

$$\lambda_4 = \begin{pmatrix}0&0&1\\0&0&0\\1&0&0\end{pmatrix} \quad \lambda_5 = \begin{pmatrix}0&0&-i\\0&0&0\\i&0&0\end{pmatrix}$$

d ↔ s

$$\lambda_6 = \begin{pmatrix}0&0&0\\0&0&1\\0&1&0\end{pmatrix} \quad \lambda_7 = \begin{pmatrix}0&0&0\\0&0&-i\\0&i&0\end{pmatrix}$$

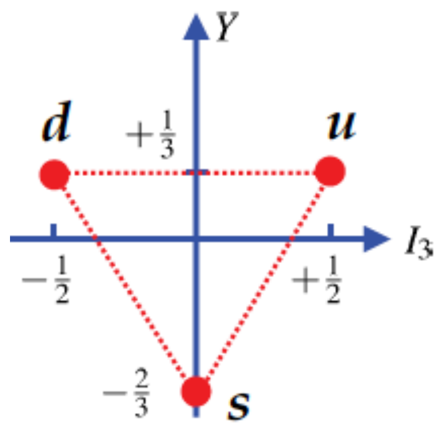
$$\lambda_8 = \frac{1}{\sqrt{3}}\begin{pmatrix}1&0&0\\0&1&0\\0&0&-2\end{pmatrix}$$

$$\begin{pmatrix}1&0&0\\0&0&0\\0&0&-1\end{pmatrix}$$

$$\begin{pmatrix}0&0&0\\0&1&0\\0&0&-1\end{pmatrix}$$

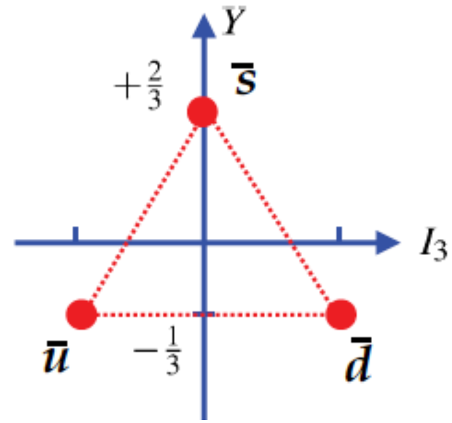
SU(3) Flavour Symmetry

- Out of 8 Gell-Mann matrices only λ_3 and λ_8 commute, i.e. the corresponding generators describe **conserved observable** quantities. I_3, and Y (**hypercharge**)

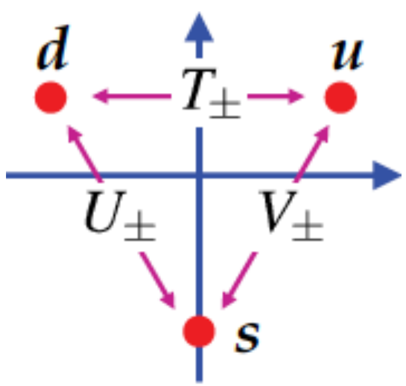


$$\boxed{Y = \frac{1}{\sqrt{3}}\lambda_8}$$

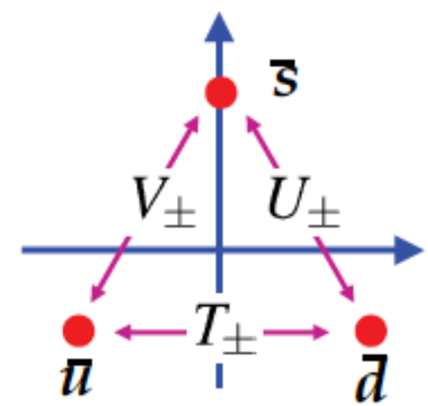
$$\boxed{I_3 = \frac{1}{2}\lambda_3}$$



- The six remaining λ-matrices are ladder operators

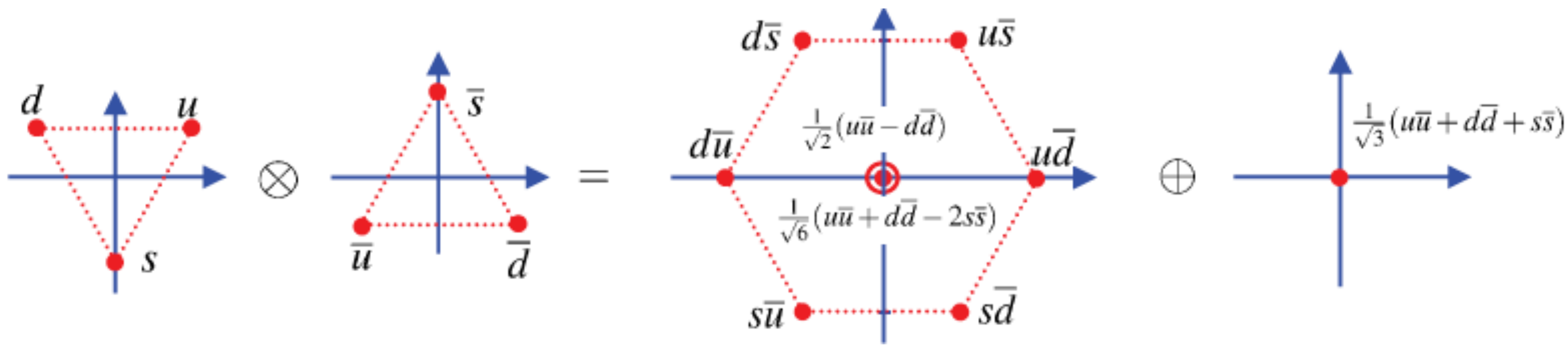


$$T_\pm = \frac{1}{2}(\lambda_1 \pm i\lambda_2)$$
$$V_\pm = \frac{1}{2}(\lambda_4 \pm i\lambda_5)$$
$$U_\pm = \frac{1}{2}(\lambda_6 \pm i\lambda_7)$$



“Building” hadrons. Mesons.

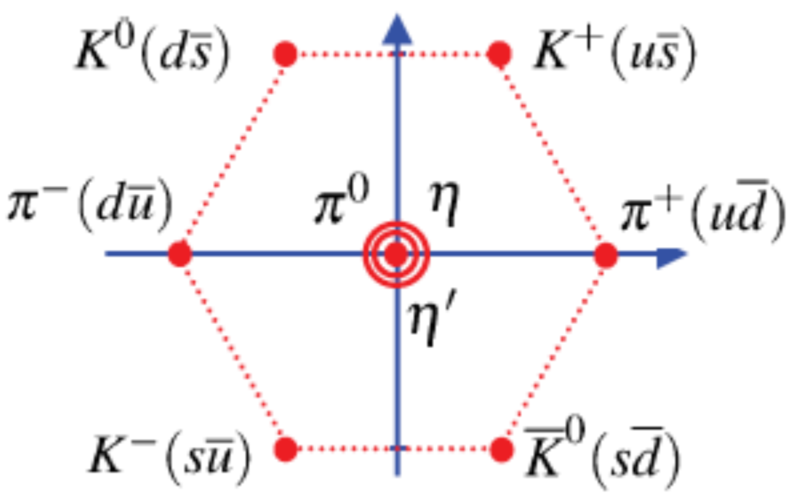
- SU(3) + ladder operators can be used to construct hadrons. For mesons:



Singlet

$$3\otimes\bar{3}=8\oplus 1$$

Octet

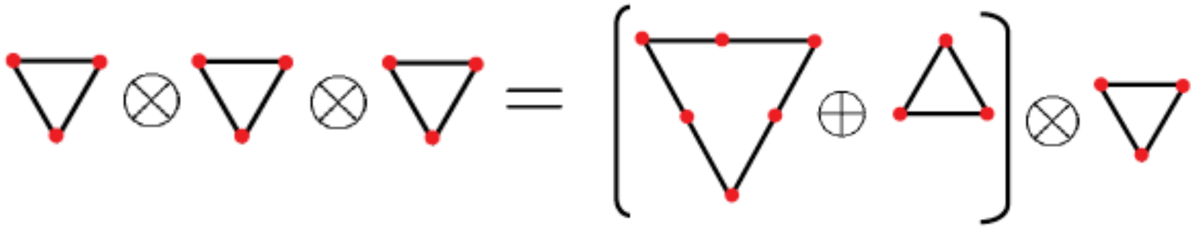


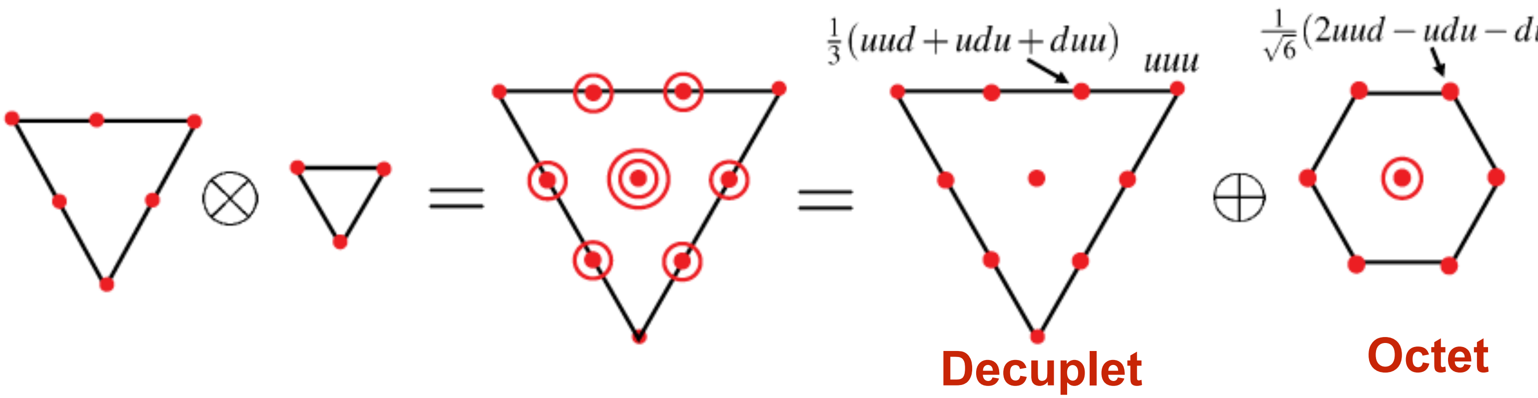
$$\pi^0=\frac{1}{\sqrt{2}}(u\bar{u}-d\bar{d})$$
$$\eta\approx\frac{1}{\sqrt{6}}(u\bar{u}+d\bar{d}-2s\bar{s})$$
$$\eta'\approx\frac{1}{\sqrt{3}}(u\bar{u}+d\bar{d}+s\bar{s})$$

singlet

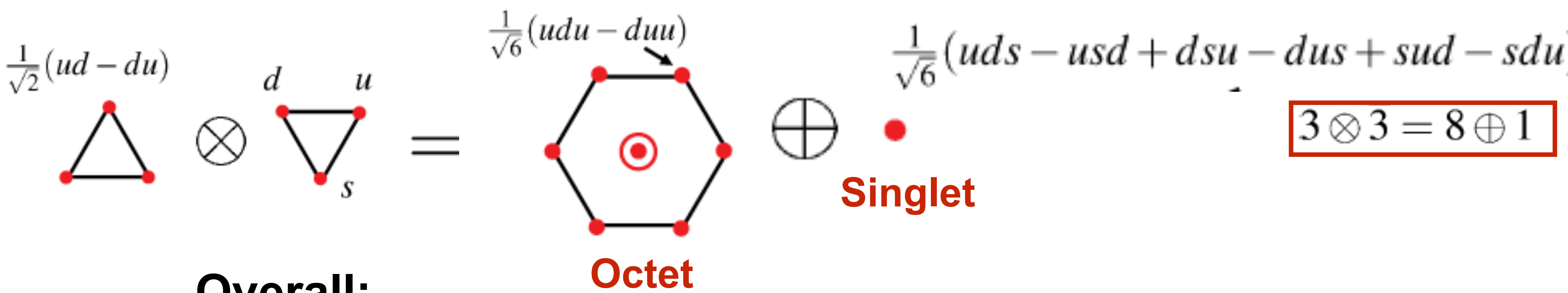
“Building” Hadrons. Baryons.

- More options when building a 3 quark-hadron





$$\boxed{3\otimes 6 = 10\oplus 8}$$



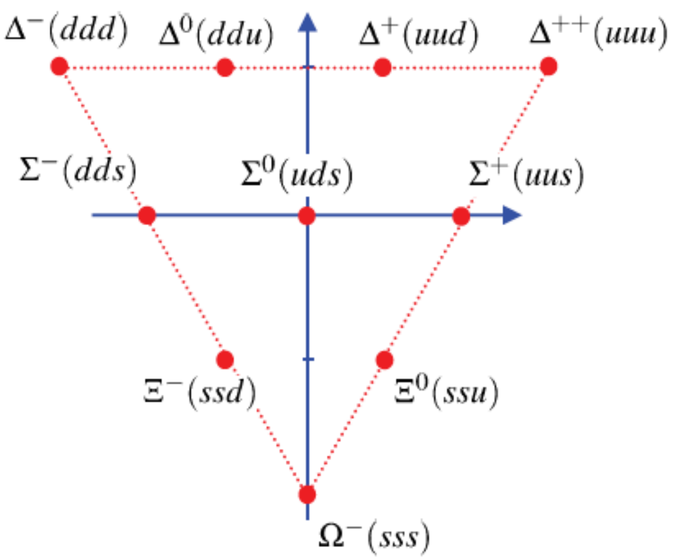
Overall:

$$\boxed{3\otimes 3\otimes 3 = 10\oplus 8\oplus 8\oplus 1}$$

Baryons

- Predicting (correctly) the observed hadron "zoo"

BARYON DECUPLET (L=0, S=3/2, J=3/2, P= +1)



Mass in MeV

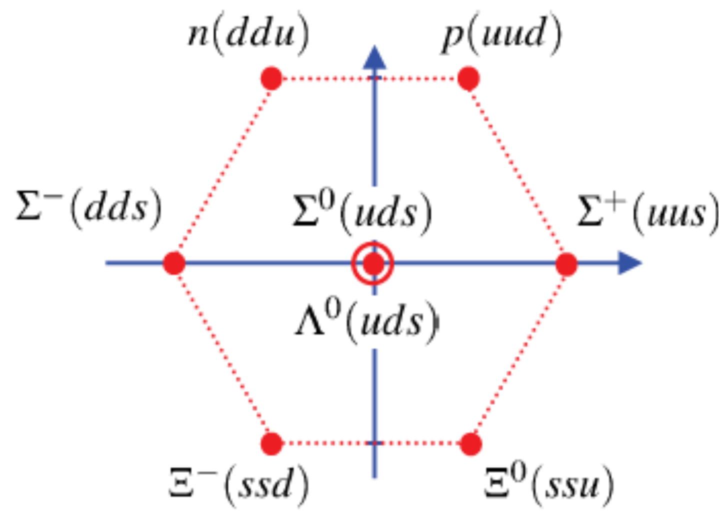
Δ(1232)

Σ(1318)

Ξ(1384)

Ω(1672)

BARYON OCTET (L=0, S=1/2, J=1/2, P= +1)



Mass in MeV

939

Σ(1193)
Λ(1116)

Ξ(1318)

- If **SU(3) Flavour** were an exact symmetry all masses would be the same.

SU(3) Flavour — "broken" symmetry

From QED to QCD

- QCD extends this idea to another fundamental symmetry

Invariance under SU(3) local phase transformations

$$\psi \to \psi' = \psi e^{ig\vec{\lambda}.\vec{\theta}(x)}$$

$\vec{\lambda}$ Eight 3x3 Gell-Mann matrices

$\vec{\theta}(x)$ 8 functions corresponding to 8 spin-1 gauge bosons (**gluons**)

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

Wave-function is now a vector in **Colour Space**

- The **invariance** under **SU(3)** local phase transformations fully specifies **QCD**

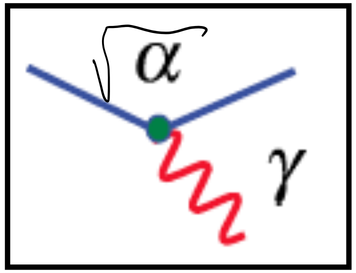
$$\boxed{-\frac{1}{2} i g_s \lambda^a \gamma^\mu}$$

QCD Interaction Vertex

QCD and Colour

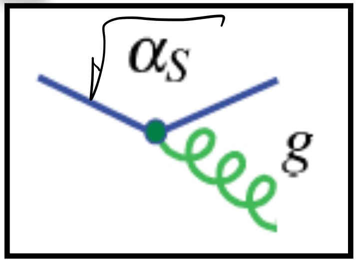
- QCD is built on the same principles as QED. However there are important differences
  - QCD predicts **8** massless gauge bosons - the **gluons**
  - QCD includes **3** conserved "**colour**" charges
  - **Gluons** carry a **colour** charge - hence **can interact with each other**

QED



- Electron carries -*e*
- Positron carries +*e*
- Force is mediated by massless gauge boson - the photon

QCD



- Quarks carry colour charge
- Anti-quarks carry anti-charge
- Force is mediated by massless gauge boson - the gluon

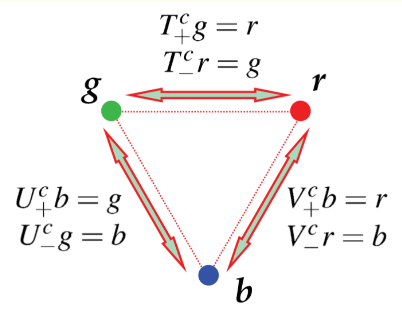
From SU(3) Flavour to SU(3) Colour

$$u = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad d = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad s = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$u \longleftrightarrow r$ (red)

$d \longleftrightarrow g$ (green)

$s \longleftrightarrow b$ (blue)



$$r = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad g = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad b = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

QCD and SU(3) Colour

- QCD: The strong interaction is invariant under rotation in colour space

$$r \leftrightarrow b;\ r \leftrightarrow g;\ b \leftrightarrow g$$

- **SU(3) colour** symmetry is an **exact symmetry** (as opposed to SU(3) flavour)

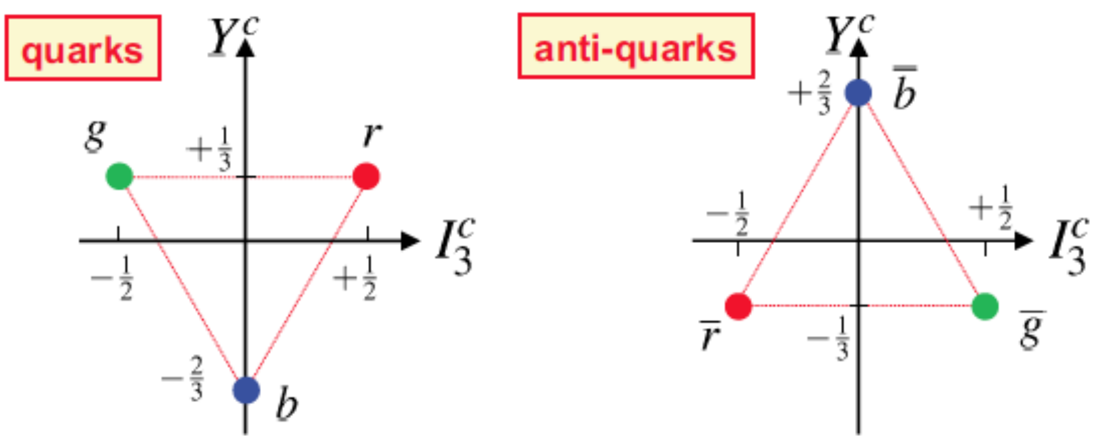
- Red, green and blue "charges" (nothing to do with actual colour!)

$$r = \begin{pmatrix}1\\0\\0\end{pmatrix};\quad g = \begin{pmatrix}0\\1\\0\end{pmatrix};\quad b = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

- Colour charges correspond to two quantum numbers

  I_3^c Colour Isospin $\qquad Y^c$ Colour Hypercharge

- For quarks



Colour Confinement

Colour Confinement Hypothesis:

only colour singlet states can exist as free particles

- All hadrons must be "colourless", i.e. **colour singlets**
- Colour singlets:
  - Have zero colour quantum numbers $I_3^c = 0,\ Y^c = 0$
  - Invariant under SU(3) colour transformation
  - Ladder operators all yield zero

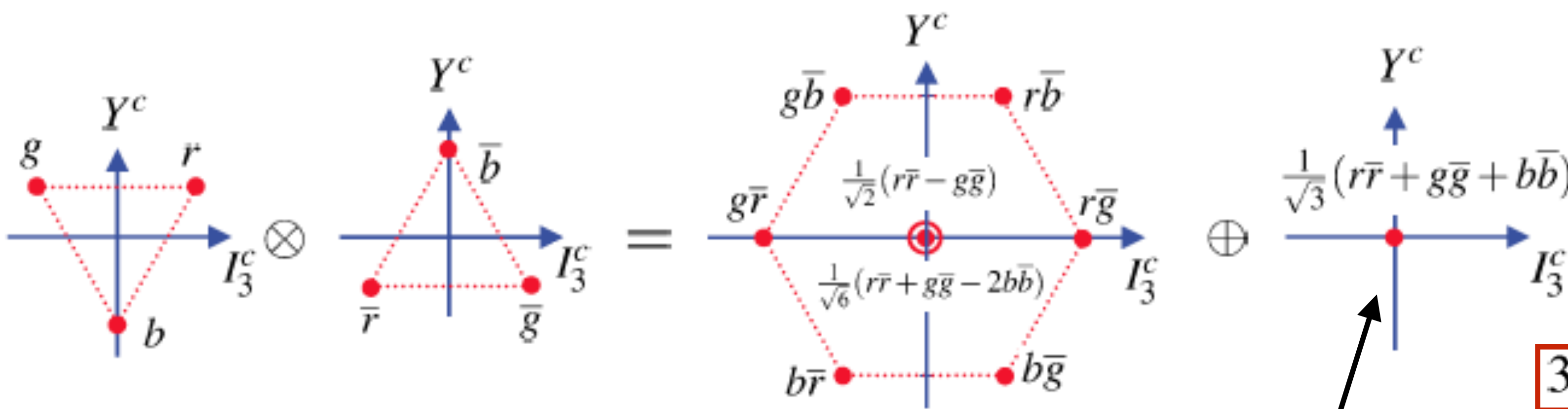
N.B. $I_3^c = 0,\ Y^c = 0$ is necessary but not sufficient condition for a singlet

- Implications: **free quarks do not exist**. In fact have never been observed (despite trying really hard). **Hadrons** exist only **colour singlets**, i.e. **colourless**.
- Hadrons observed in two forms: **Mesons** $q\bar{q}$ and **Baryons** $qqq\,(\overline{qqq})$

N.B. Some *very recent* additions to this rule (see later)

Meson Colour Wave-Function

- Borrowing heavily from SU(3) flavour we can now construct colour wave-functions from different colour combinations of $q\bar{q}$



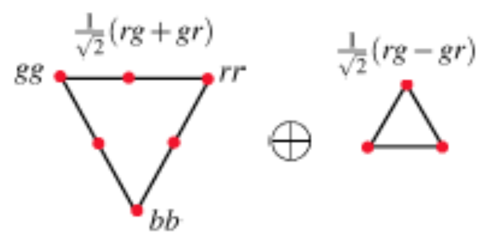
$$3 \otimes 3 = 8 \oplus 1$$

Coloured octet and a colourless singlet

- Colour confinement states that only colourless singlets exist as free observable particles. So the only combination is

$$\psi_c^{q\bar{q}} = \frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})$$

- Do qq bound states exist? **No**, as we cannot construct a colour singlet

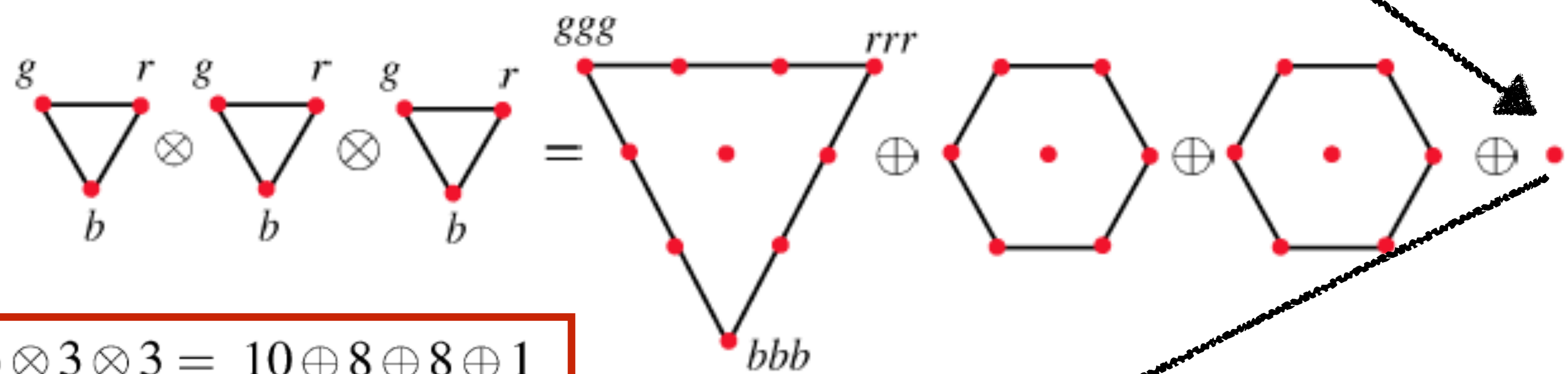


- However we can for qqq bound states…

Baryon Colour Wave-Function

- Combination of 3 quarks give a colour singlet

(recall SU(3) flavour again for uds)



$$\boxed{3 \otimes 3 \otimes 3 = 10 \oplus 8 \oplus 8 \oplus 1}$$

$$\boxed{\psi_c^{qqq} = \frac{1}{\sqrt{6}}(rgb - rbg + gbr - grb + brg - bgr)}$$

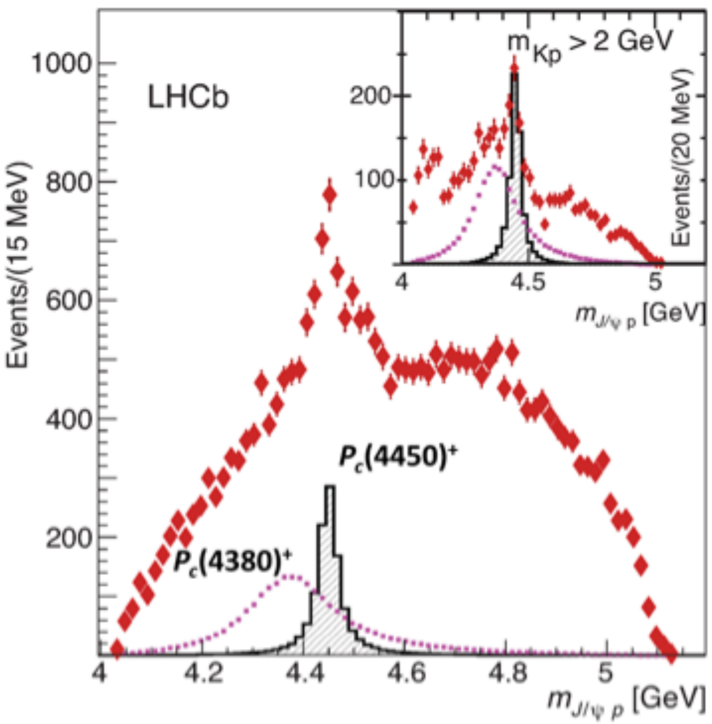
Anti-symmetric colour wave-function

- qqq bound states exist!

Therefore

- Allowed Hadrons
  - Mesons $q\bar{q}$ and Baryons $qqq(\overline{qqq})$
  - Exotic States $q\bar{q}q\bar{q},\ qqqq\bar{q}$ ⟵ pentaquarks

Until ***very*** recently the only **observed hadrons** were **mesons** and **baryons**. Last summer evidence for pentaquarks was reported.

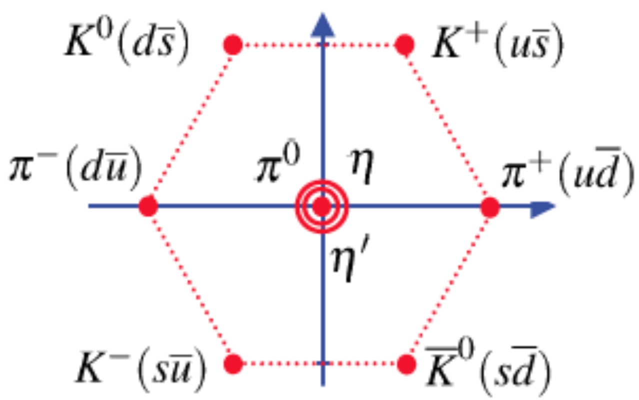


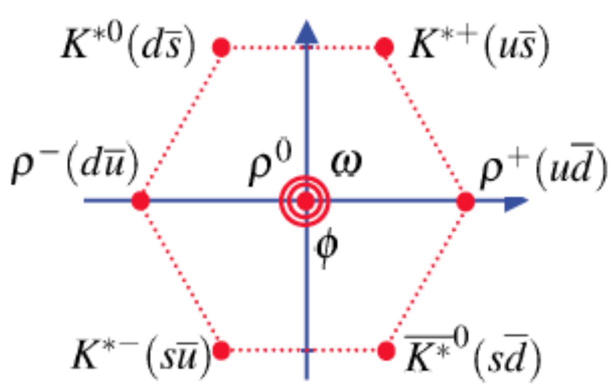
LHCb, July 2015 — **Pentaquarks**

R. Aaij et al. (LHCb Collaboration)
Phys. Rev. Lett. 115, 072001
Published 12 August 2015

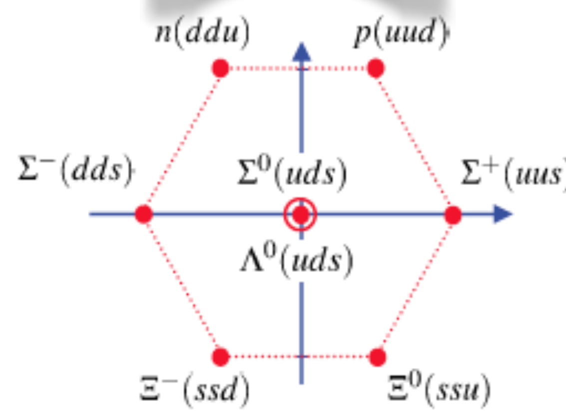
Once, again, there is a zoo of hadrons….

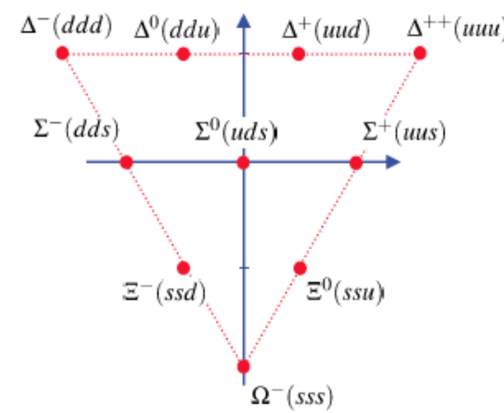
Mesons





Baryons

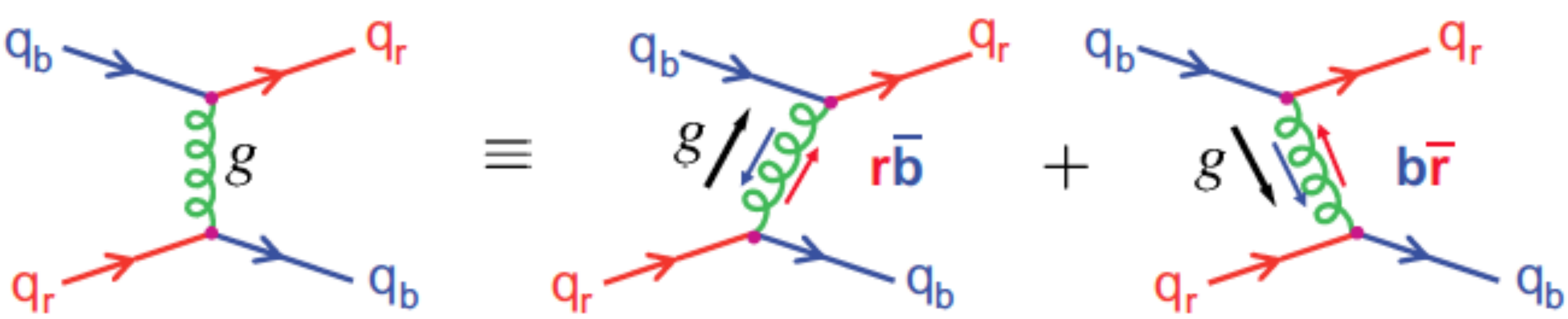




…but group theory can make sense of this diversity

Gluons in QCD

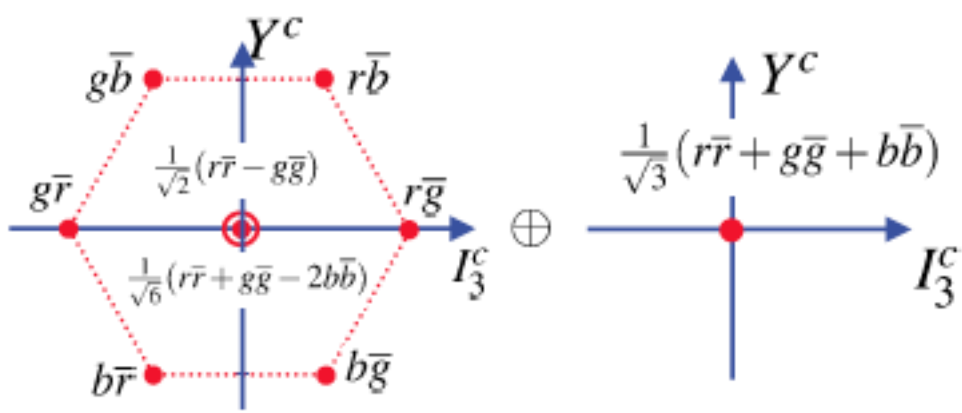
- Quarks interact by exchanging virtual massless gluons
- **Colour** charge is **conserved** at interaction vertices in FD



- Gluons carry **colour** and **anti-colour**
- Gluons colour wave-functions can be constructed in the same way as those of mesons

$3 \otimes 3 = 8 \oplus 1$

Octet + Singlet



Gluons in QCD

- 9 physical gluons are expected

Octet $r\bar{g},\ r\bar{b},\ g\bar{r},\ g\bar{b},\ b\bar{r},\ b\bar{g},\ \frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g}),\ \frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

Singlet $\frac{1}{\sqrt{3}}(r\bar{r}+g\bar{g}+b\bar{b})$

- But according to colour confinement only colour singlet states exist as free particles

- The singlet gluon would be unconfined → Infinite range Strong Force

(just like photon in QED)

- Empirically, we know that strong force is short-ranged

↓

$\boxed{\text{gluons are confined}}$

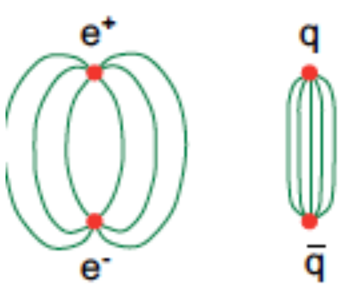
Gluon-Gluon Interaction

- Gluons having (colour) charge has most profound consequences and is the key cause of differences between QED and QCD

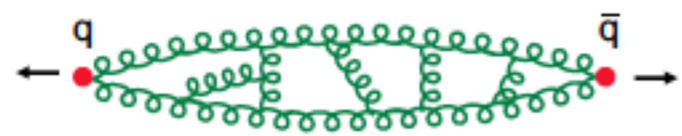
- **Gluons can interact with each other**

gluon-gluon scattering

- Gluon self-interactions are believed to give rise to colour confinement



What happens if you try to separate quark and antiquark



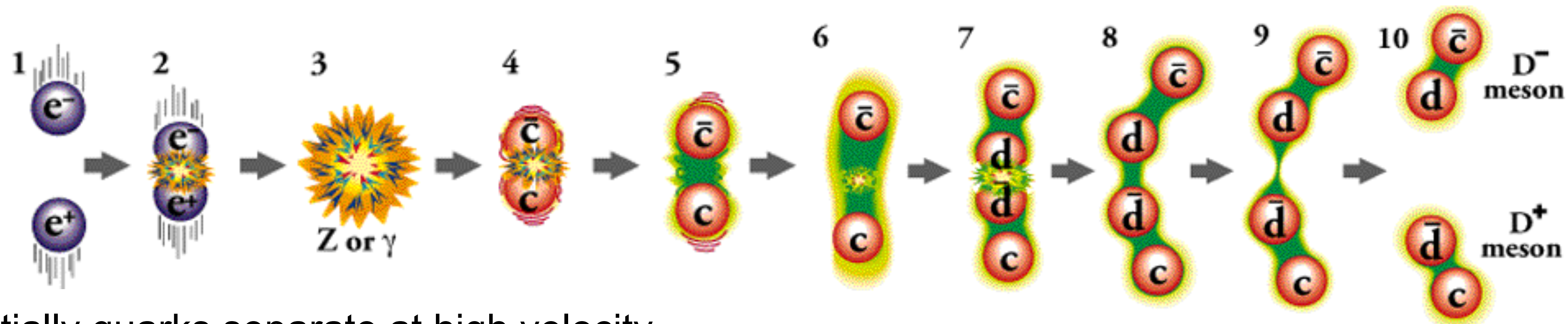
Flux tube of interacting gluons with energy density ~ 1 GeV/fm ➡ $V(r) \sim \lambda r$

⬇

Infinite energy to separate coloured objects to infinity

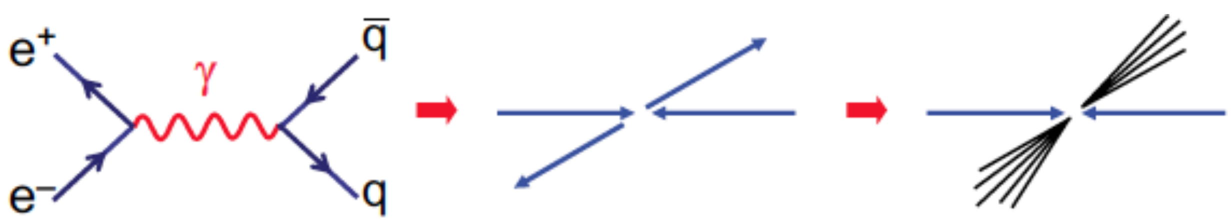
Hadronisation and Jets

Consider $e^+e^- \to q\bar{q}$



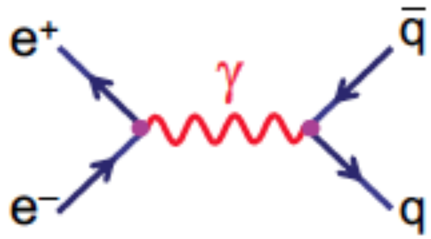
- Initially quarks separate at high velocity
- Colour flux tube forms between quarks. Energy stored increases as quarks separate $(V(r) \propto \lambda r)$
- Eventually this energy is sufficient to form $q\bar{q}$ pair (step 7 on above diagram)
- Process continues until quarks pair up into **jets** of colourless hadrons

- The above process is called **hadronisation**. It is not (yet) calculable.



Observation of jets in particle colliders is its main consequence

Evidence for quarks. QCD in e+e- collisions.



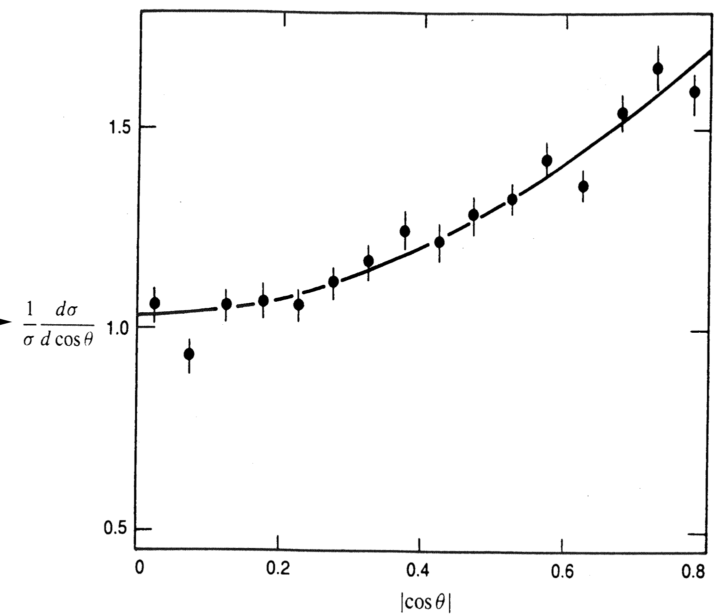
Use well understood QED to derive quark and gluon properties

- From QED $e^+e^- \to \mu^+\mu^- \Longrightarrow \sigma = \frac{4\pi\alpha^2}{3s} \quad \frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4s}(1+\cos^2\theta)$

Should be the same for $q\bar{q}$

- e^+e^- collisions produce all quark flavours with $\sqrt{s} > 2m_q$

Angular distribution of jets $\propto (1+\cos^2\theta) \Longrightarrow$



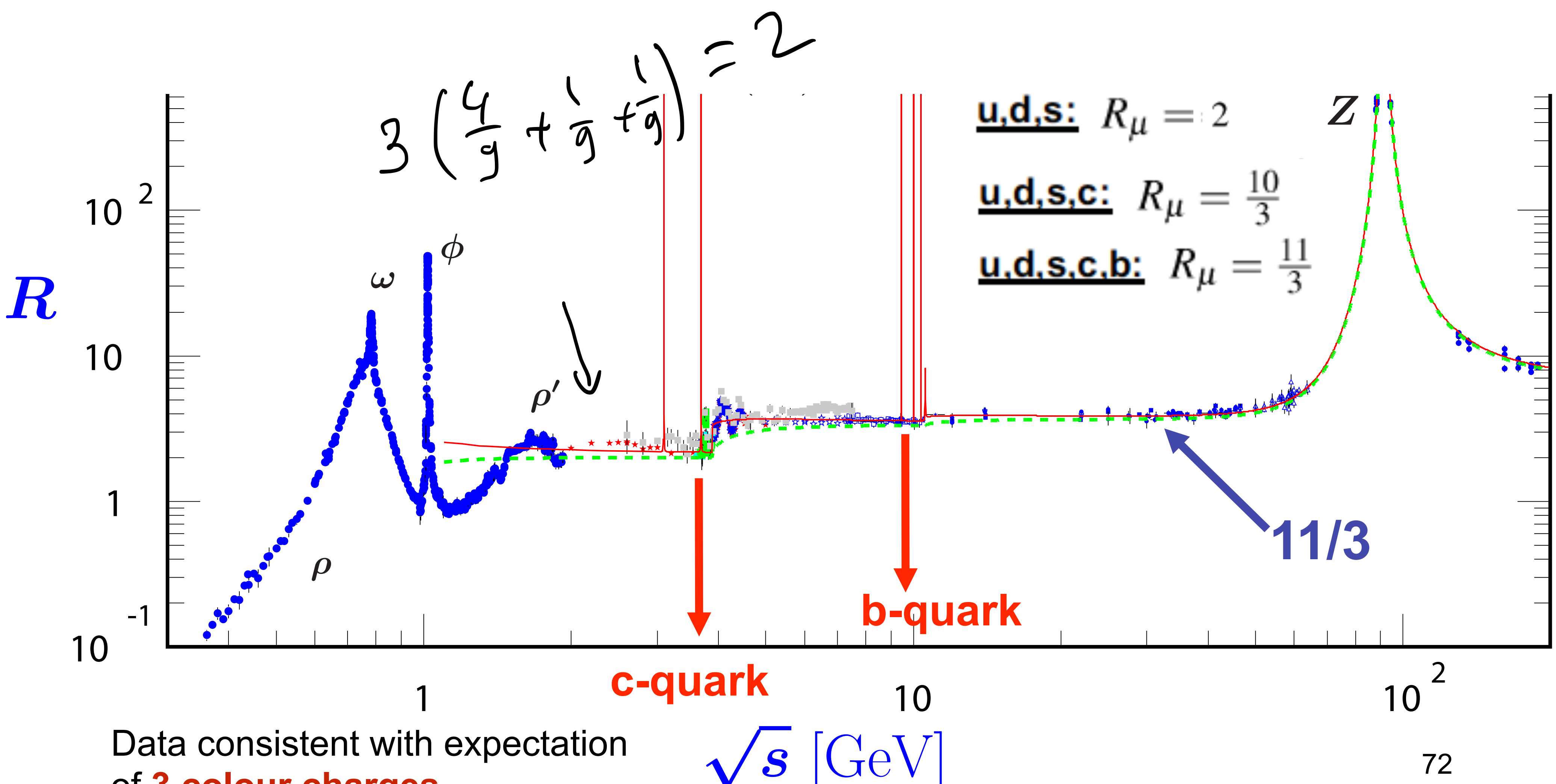
➡ **Quarks are spin ½**

Evidence for quarks and colour

- Colour is conserved, so quarks are produced as $r\bar{r},\ g\bar{g},\ b\bar{b}$
- Observe jets of hadrons, not single quarks

$$\sigma(e^+e^- \to \text{hadrons}) = 3 \sum_{u,d,s,..} \frac{4\pi\alpha^2}{3s} Q_q^2 \qquad R_\mu = \frac{\sigma(e^+e^- \to \text{hadrons})}{\sigma(e^+e^- \to \mu^+\mu^-)} = 3 \sum_{u,d,s,..} Q_q^2$$

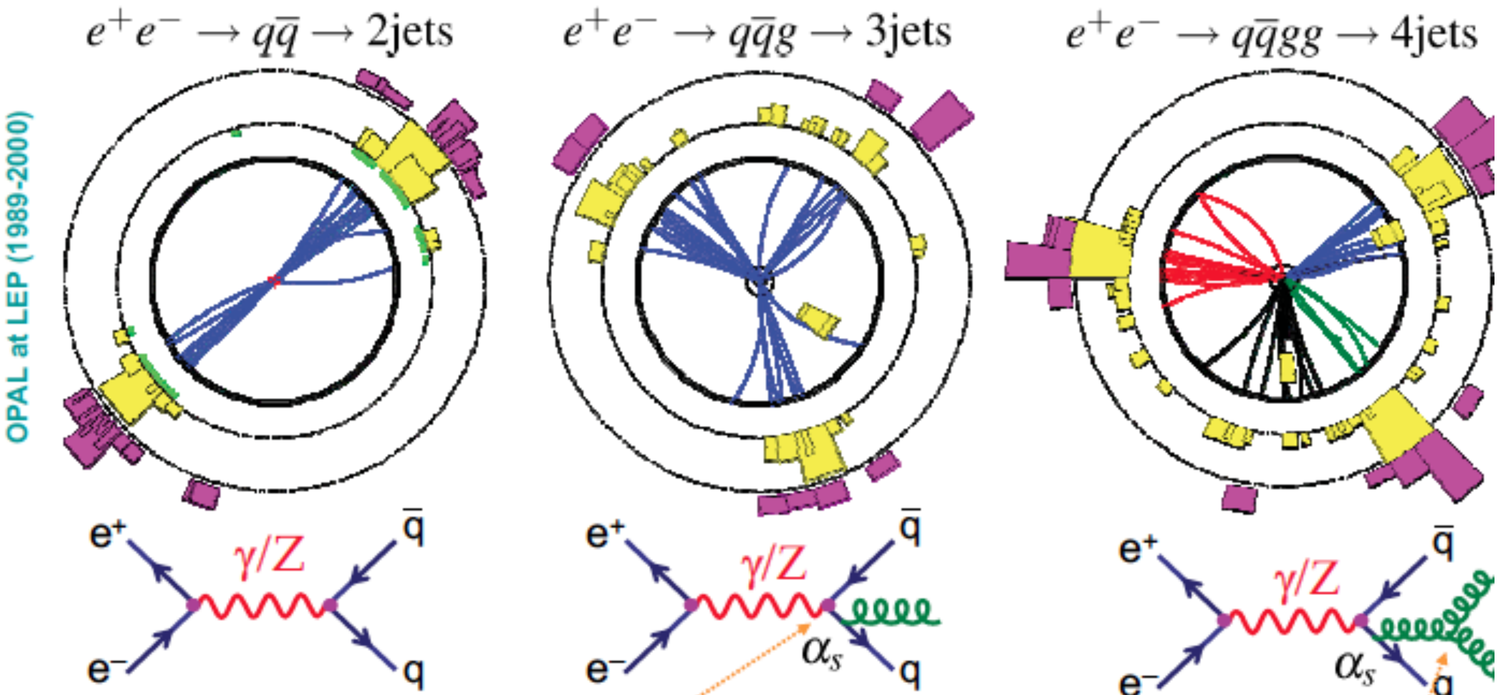
Factor 3 comes from colours

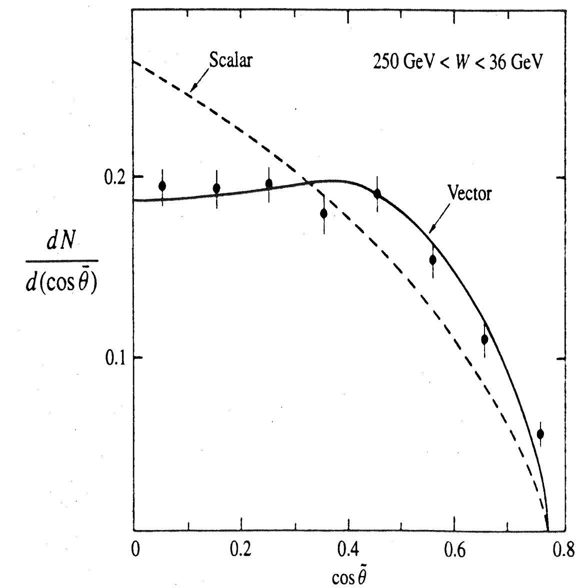


Data consistent with expectation of **3 colour charges**

Evidence for gluons. Jet production in e+e- collisions

Great place to study gluons and intrinsic strength of strong interaction





Experimentally:

- Three jet rate ➡ measurement of α_s
- Angular distributions ➡ gluons are spin-1
- Four-jet rate and distributions ➡ QCD has an underlying SU(3) symmetry

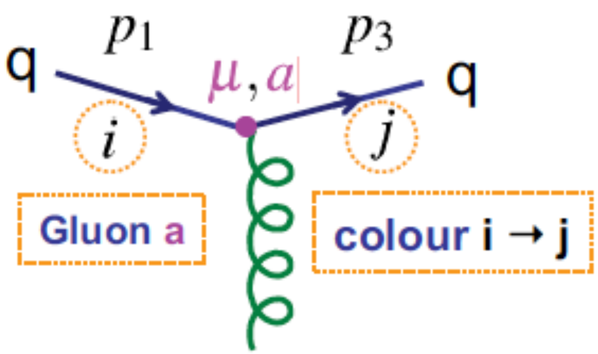
Quark-Gluon Interaction

- The colour part of the fermion wave-function can be represented by

$$r = c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad g = c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad b = c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \qquad u(p) \longrightarrow c_i u(p)$$

- The QCD vertex is

$$\overline{u}(p_3) c_j^\dagger \{ -\tfrac{1}{2} i g_s \lambda^a \gamma^\mu \} c_i u(p_1)$$

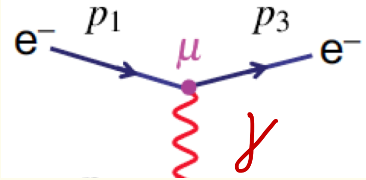


the colour part: $c_j^\dagger \lambda^a c_i = \lambda_{ji}^a$

- The quark-gluon fundamental interaction is then

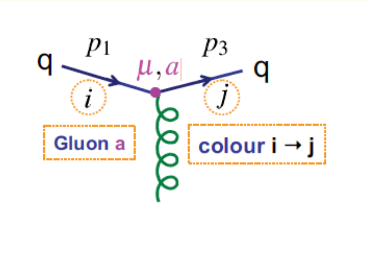
$$\overline{u}(p_3) c_j^\dagger \{ -\tfrac{1}{2} i g_s \lambda^a \gamma^\mu \} c_i u(p_1) \equiv \overline{u}(p_3) \{ -\tfrac{1}{2} i g_s \lambda_{ji}^a \gamma^\mu \} u(p_1)$$

Quark-gluon vs e-γ interaction



$$j_{ee} = \bar{u}(p_3)\left[\underbrace{ie\gamma^\mu}\right]u(p_1)$$

For QCD $\quad u \to c_i u$



$$\bar{u}(p_3)c_j^\dagger\left[-\tfrac{1}{2}\underbrace{ig_s\lambda^a\gamma^\mu}\right]c_i u(p_1)$$

$$\Downarrow$$

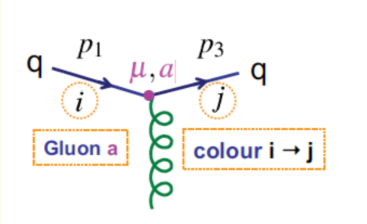
$$j_{qq}$$

$a = 1, \ldots 8 \Rightarrow$ gluons

$i, j = 1, 2, 3$
$\quad\quad\;\; r \;\; g \;\; b$

Quark-Gluon vertex and Colour Factors

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_iu(p_1)$$



$$c_j^\dagger \lambda^a c_i = c_j^\dagger \lambda^a{}_i =$$

$$= c_j^\dagger \begin{pmatrix} \lambda^a_{1i} \\ \lambda^a_{2i} \\ \lambda^a_{3i} \end{pmatrix} = \lambda^a_{ji}$$

$$c_i \Rightarrow i = 1, 2, 3$$

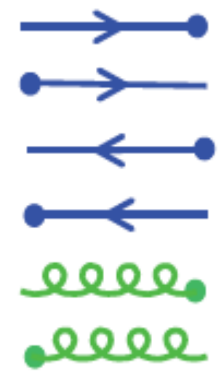
$$c_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}; \quad c_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}; \quad c_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\overline{u}(p_3)c_j^\dagger\{-\tfrac{1}{2}ig_s\lambda^a\gamma^\mu\}c_iu(p_1) \equiv \overline{u}(p_3)\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\}u(p_1)$$

Feynman Rules for QCD

- **External Lines**

spin 1/2	incoming quark	$u(p)$	
	outgoing quark	$\bar{u}(p)$	
	incoming anti-quark	$\bar{v}(p)$	
	outgoing anti-quark	$v(p)$	
spin 1	incoming gluon	$\varepsilon^{\mu}(p)$	
	outgoing gluon	$\varepsilon^{\mu}(p)^{*}$	



- **Internal Lines (propagators)**

spin 1 gluon

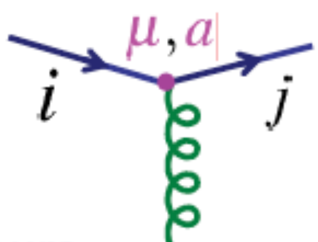
$$\boxed{\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}}$$

a, b = 1,2,…,8 are gluon colour indices

- **Vertex Factors**

spin 1/2 quark

$$\boxed{-ig_s\tfrac{1}{2}\lambda^{a}_{ji}\gamma^{\mu}}$$

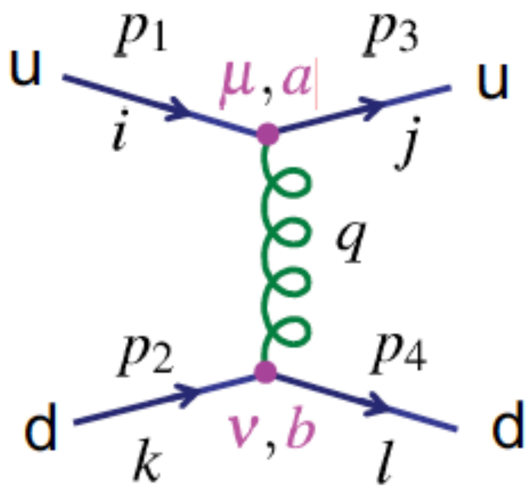


i, j = 1,2,3 are quark colours,

λ^{a} a = 1,2,..8 are the Gell-Mann SU(3) matrices

- **+ 3 gluon and 4 gluon interaction vertices**
- **Matrix Element** $-iM$ = product of all factors

Matrix Element for u-d scattering



- The incoming and out-going quark colours are labelled by $i, j, k, l = \{1,2,3\}$ (or $\{r, g, b\}$)
- In terms of colour this scattering is
$$ik \rightarrow jl$$
- The 8 different gluons are accounted for by the colour indices $a, b = 1, 2, ..., 8$
- NOTE: the δ-function in the propagator ensures ***a*** = ***b***, i.e. the gluon "emitted" at ***a*** is the same as that "absorbed" at ***b***

★ Applying the Feynman rules:

$$-iM = \left[\overline{u}_u(p_3)\left\{-\tfrac{1}{2}ig_s\lambda^a_{ji}\gamma^\mu\right\}u_u(p_1)\right]\frac{-ig_{\mu\nu}}{q^2}\delta^{ab}\left[\overline{u}_d(p_4)\left\{-\tfrac{1}{2}ig_s\lambda^b_{lk}\gamma^\nu\right\}u_d(p_2)\right]$$

where summation over ***a*** and ***b*** (and μ and ν) is implied.

★ Summing over ***a*** and ***b*** using the δ-function gives:

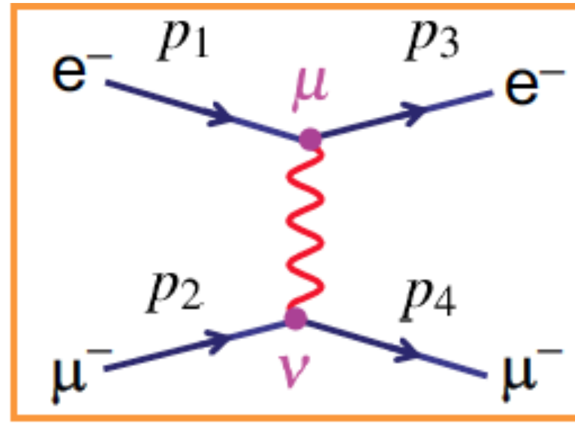
$$\boxed{M = -\frac{g_s^2}{4}\underbrace{\boxed{\lambda^a_{ji}\lambda^a_{lk}}}_{\text{Sum over all 8 gluons (repeated indices)}}\frac{1}{q^2}g_{\mu\nu}\left[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)\right]\left[\overline{u}_d(p_4)\gamma^\nu u_d(p_2)\right]}$$

Feynman Rules for QCD (vs QED)

QED

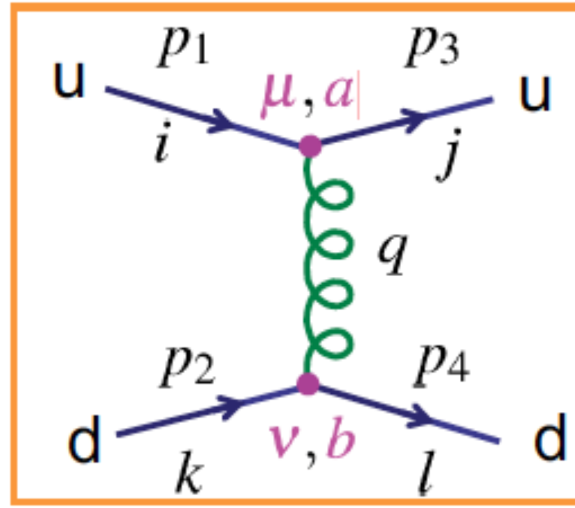
$$-iM = [\overline{u}(p_3)ie\gamma^\mu u(p_1)]\frac{-ig_{\mu\nu}}{q^2}[\overline{u}(p_4)ie\gamma^\nu u(p_2)]$$

$$\boxed{M = -e^2\frac{1}{q^2}g_{\mu\nu}[\overline{u}(p_3)\gamma^\mu u(p_1)][\overline{u}(p_4)\gamma^\nu u(p_2)]}$$



QCD

$$\boxed{M = -\frac{g_s^2}{4}\lambda^a_{ji}\lambda^a_{lk}\frac{1}{q^2}g_{\mu\nu}[\overline{u}_u(p_3)\gamma^\mu u_u(p_1)][\overline{u}_d(p_4)\gamma^\nu u_d(p_2)]}$$



★ **QCD Matrix Element = QED Matrix Element with:**

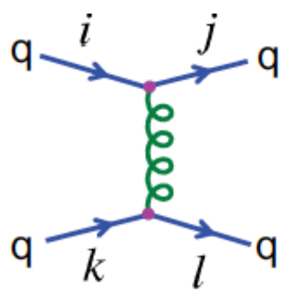
- $\boxed{e^2 \to g_s^2}$ **or equivalently** $\boxed{\alpha = \frac{e^2}{4\pi} \to \alpha_s = \frac{g_s^2}{4\pi}}$

+ QCD Matrix Element includes an additional "colour factor"

$$\boxed{C(ik \to jl) \equiv \frac{1}{4}\sum_{a=1}^{8}\lambda^a_{ji}\lambda^a_{lk}}$$

Colour Factors

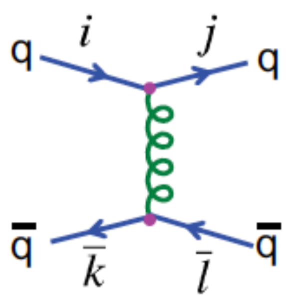
$\lambda^1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^2 = \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$ $\lambda^5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$ $\lambda^7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\lambda^8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$
Gluons: $r\bar{g}, g\bar{r}$	$r\bar{b}, b\bar{r}$	$g\bar{b}, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r} - g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r} + g\bar{g} - 2b\bar{b})$



$$\boxed{C(ik \to jl) \equiv \frac{1}{4} \sum_{a=1}^{8} \lambda^a_{ji} \lambda^a_{lk}}$$

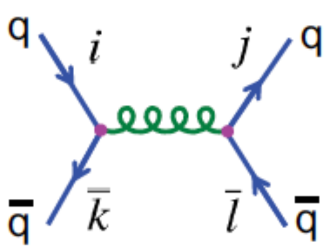
e.g.

$$C(rr \to rr) = \tfrac{1}{3}$$
$$C(rg \to rg) = -\tfrac{1}{6}$$
$$C(rg \to gr) = \tfrac{1}{2}$$



$$\boxed{C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4} \sum_{a=1}^{8} \lambda^a_{ji} \lambda^a_{kl}}$$

$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = -\tfrac{1}{6}$$
$$C(r\bar{r} \to g\bar{g}) = \tfrac{1}{2}$$



$$\boxed{C(i\bar{k} \to j\bar{l}) \equiv \frac{1}{4} \sum_{a=1}^{8} \lambda^a_{ki} \lambda^a_{jl}}$$

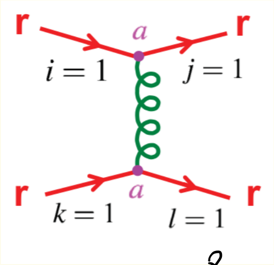
$$C(r\bar{r} \to r\bar{r}) = \tfrac{1}{3}$$
$$C(r\bar{g} \to r\bar{g}) = \tfrac{1}{2}$$
$$C(r\bar{r} \to g\bar{g}) = -\tfrac{1}{6}$$

Colour index of adjoint spinor comes first

quark-quark example

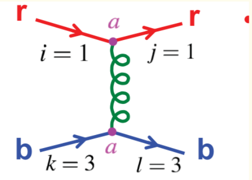
$$C = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{ji}\,\lambda^a_{lk}$$

$\lambda^1 = \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&0 \end{pmatrix}$ $\lambda^2 = \begin{pmatrix} 0&-i&0\\ i&0&0\\ 0&0&0 \end{pmatrix}$	$\lambda^4 = \begin{pmatrix} 0&0&1\\ 0&0&0\\ 1&0&0 \end{pmatrix}$ $\lambda^5 = \begin{pmatrix} 0&0&-i\\ 0&0&0\\ i&0&0 \end{pmatrix}$	$\lambda^6 = \begin{pmatrix} 0&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}$ $\lambda^7 = \begin{pmatrix} 0&0&0\\ 0&0&-i\\ 0&i&0 \end{pmatrix}$	$\lambda^3 = \begin{pmatrix} 1&0&0\\ 0&-1&0\\ 0&0&0 \end{pmatrix}$ $\lambda^8 = \frac{1}{\sqrt{3}}\begin{pmatrix} 1&0&0\\ 0&1&0\\ 0&0&-2 \end{pmatrix}$
Gluons: $r\bar{g}, g\bar{r}$	$r\bar{b}, b\bar{r}$	$g\bar{b}, b\bar{g}$	$\frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g})$ $\frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$

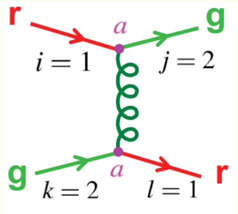


$$C = \frac{1}{4}\sum_{a=1}^{8} \lambda^a_{11}\cdot\lambda^a_{11} = \frac{1}{4}\left(\lambda^3_{11}\lambda^3_{11} + \lambda^8_{11}\lambda^8_{11}\right) = \frac{1}{4}\left(1+\frac{1}{3}\right) = \frac{1}{3}$$

$$C(rr \to rr) = C(gg \to gg) = C(bb \to bb) = \frac{1}{3}$$



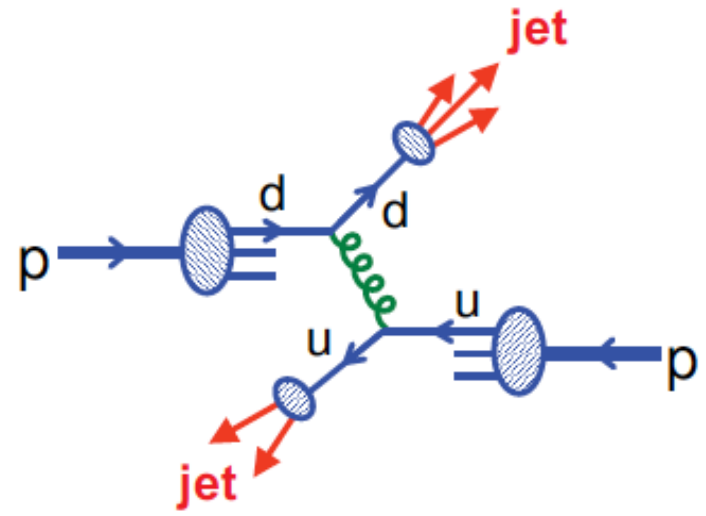
$$C(rb \to rb) = C(rg \to rg) = C(gr \to gr) = C(gb \to gb) = C(br \to br) = C(bg \to bg) = -\frac{1}{6}$$



$$C(rb \to br) = C(rg \to gr) = C(gr \to rg) = C(gb \to bg) = C(br \to rb) = C(bg \to gb) = \frac{1}{2}$$

Quark-Quark Scattering

- Consider $u + d \rightarrow u + d$
- Nine possible colour configurations of colliding quarks, which are equally likely



$$\langle |M_{fi}|^2 \rangle = \frac{1}{3} \cdot \frac{1}{3} \sum_{i,j,k,l=1}^{3} |M_{fi}(ij \rightarrow kl)|^2$$

- The average colour factor $\langle |C|^2 \rangle = \frac{1}{9} \sum_{i,j,k,l=1}^{3} |C(ij \rightarrow kl)|^2$

- For $qq \rightarrow qq$

$\boxed{rr \rightarrow rr, ..}$ $\boxed{rb \rightarrow rb, ..}$ $\boxed{rb \rightarrow br, ..}$

$$\langle |C|^2 \rangle = \frac{1}{9}\left[3 \times \left(\frac{1}{3}\right)^2 + 6 \times \left(-\frac{1}{6}\right)^2 + 6 \times \left(\frac{1}{2}\right)^2\right] = \frac{2}{9}$$

pp-Collisions

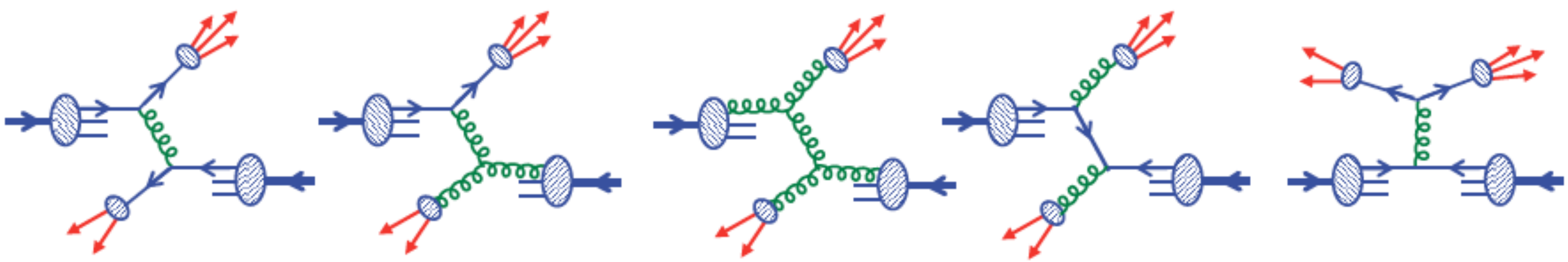
- X-sections similar to QED once colour factor is included

For $e^-\mu^- \to e^-\mu^-$
$$\frac{d\sigma}{dq^2} = \frac{2\pi\alpha^2}{q^4}\left[1+\left(1+\frac{q^2}{s}\right)^2\right]$$

For $ud \to ud$ (QCD) $\alpha \to \alpha_s$ and $\times\langle|C|^2\rangle$
$$\frac{d\sigma}{dq^2} = \frac{2}{9}\frac{2\pi\alpha_S^2}{q^4}\left[1+\left(1+\frac{q^2}{\hat{s}}\right)^2\right]$$

CoM of qq collision

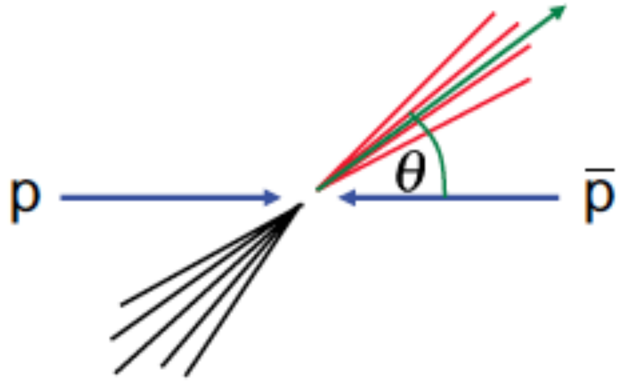
- Calculation of hadron-hadron scattering is very involved
- Examples of two-jet production (Tevatron: $p\bar{p}$ LHC: pp)

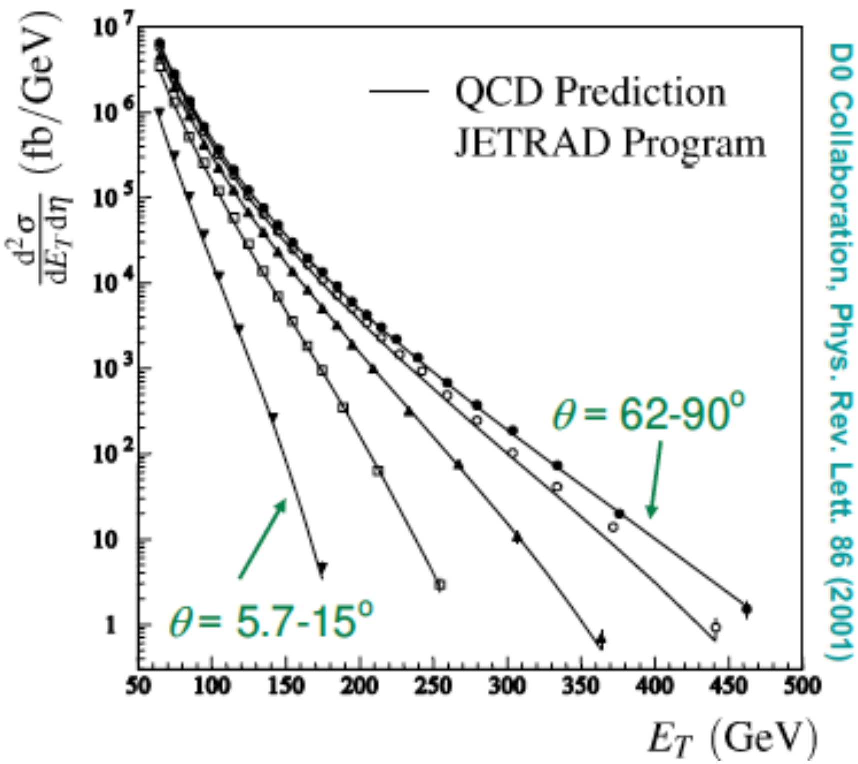


$q\bar{q} \to q\bar{q}$ $\quad qg \to qg$ $\quad gg \to gg$ $\quad q\bar{q} \to gg$ $\quad q\bar{q} \to q\bar{q}$

QCD provides good description of pp and $p\bar{p}$ data

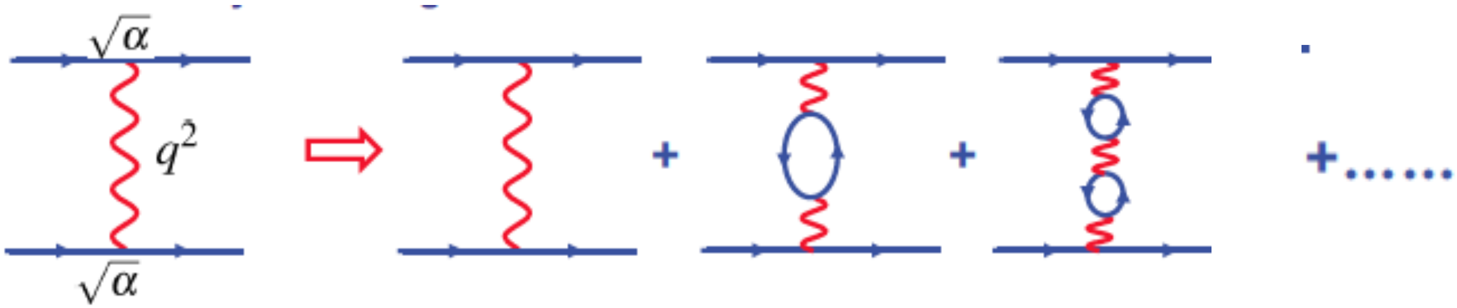
- X-section is measured in terms of
  - "transverse energy" $E_T = E_{\rm jet} \sin\theta$
  - "pseudorapidity" $\eta = \ln\left[\cot\left(\frac{\theta}{2}\right)\right]$

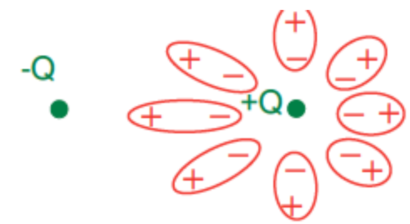




Running coupling constant. QED.

Screening effect due to virtual e^+e^- pairs





- Matrix element is obtained by **adding amplitudes** of FD of different orders

$$M = M_1 + M_2 + M_3 + \ldots$$

$$\alpha(Q^2) = \alpha(Q_0^2) \Big/ \left[1 - \frac{\alpha(Q_0^2)}{3\pi} \ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

$$Q^2 \gg Q_0^2$$

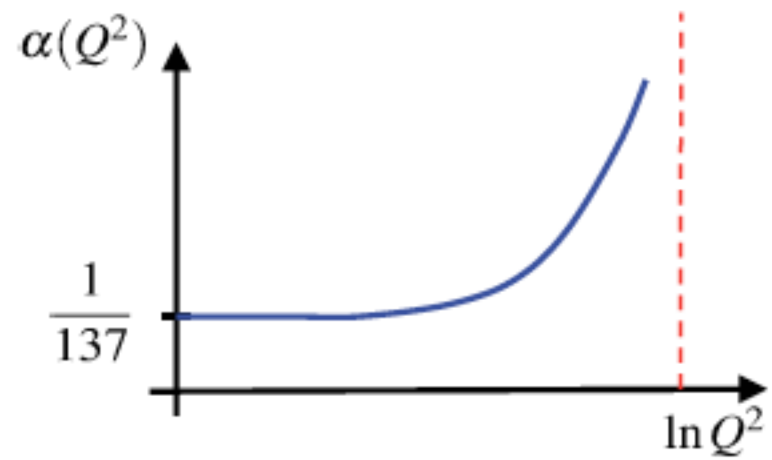
Note sign

- **Atomic physics:** $Q^2 \sim 0$

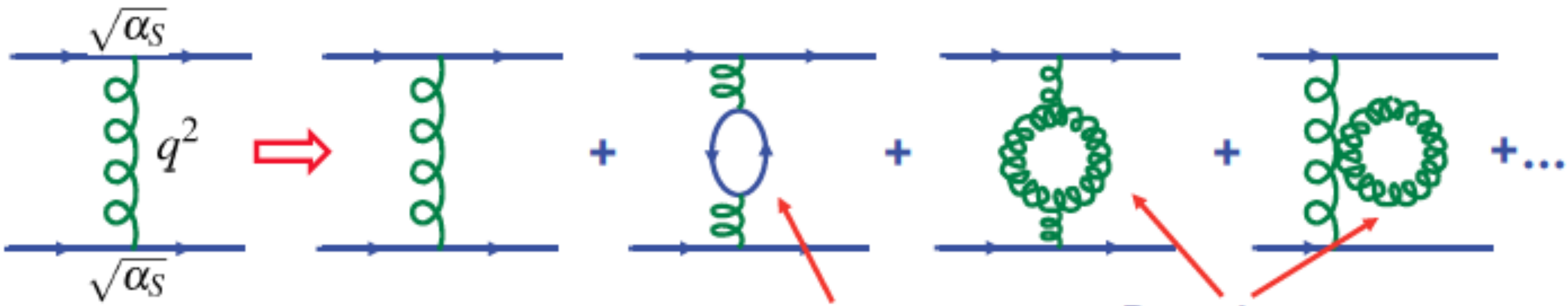
  $1/\alpha = 137.03599976(50)$

- **High energy physics:**

  $1/\alpha(193\,\text{GeV}) = 127.4 \pm 2.1$



Running coupling constant. QCD.



Boson loops interfere negatively

$$\alpha_S(Q^2) = \alpha_S(Q_0^2) \Big/ \left[1 + B\alpha_S(Q_0^2)\ln\left(\frac{Q^2}{Q_0^2}\right)\right]$$

with $B = \dfrac{11N_c - 2N_f}{12\pi}$

N_c = no. of colours

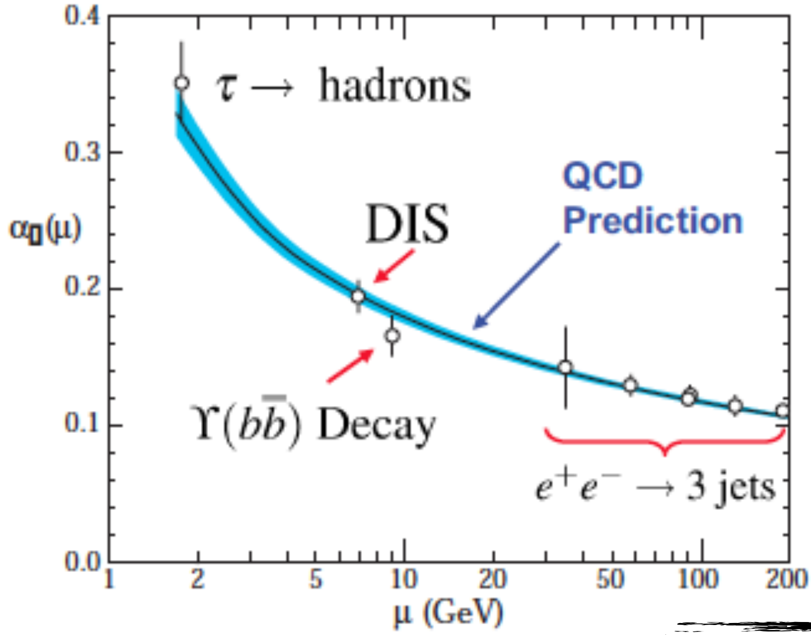
N_f = no. of quark flavours

$N_c = 3;\ N_f = 6 \Rightarrow B > 0$

$\Rightarrow$ $\boxed{\alpha_S \text{ decreases with } Q^2}$

Nobel Prize for Physics, 2004
(Gross, Politzer, Wilczek)

Strong coupling "constant" α_S



$\mu \approx Q$

- α_S is measured
  - Jets
  - τ decays to hadrons
  - Deep Inelastic Scattering
  - botommonium decays
  - etc…

α_S decreases with Q^2 as predicted by QCD

- At low Q^2 (~1 GeV^2) α_S is large, α_S~1. Cannot use perturbation theory. That's why e.g. hadronisation is not (yet) calculable.
- At high Q^2, e.g. $Q^2 \sim M_Z^2 \sim 8000\, GeV^2$ is relatively small α_S~0.12

➡ **Asymptotic Freedom**

- Perturbation theory can be used, calculations can be made. Quarks behave as **quasi-free** particles. Good agreement of QCD predictions with experiment

Summary of QCD

- Built on same principles as QED
- But gluon self-interactions result in **colour confinement**
- All hadrons are colour singles. **Mesons** and **Baryons** are observed (and probably pentaquarks, recently).
- At low energies ≤ 1 GeV) $\alpha_S \sim 1$, strong interaction processes are not (yet) calculable, e.g. hadronisation
- Strong coupling constant “runs”, smaller couplings at higher energy scales:

$\alpha_S(100\,\mathrm{GeV}) \sim 0.1$ resulting in **asymptotic freedom** $\Longrightarrow$ becomes calculable

- Where QCD calculations can be performed $\Longrightarrow$ good agreement with experimental data