PHASM426 / PHASG426
Advanced Quantum Theory Problem Sheet 3

Deadline: 12th December 2017.

Please hand in your completed work at the **end** of the lecture on that day. Attache the coversheet. If you are unable to attend the lecture, you may scan your work, *save it as a single PDF file* and email it to me **prior** to this lecture. You may also bring the work to me in my office (B12) before the lecture. **Make sure your completed work is clearly labelled with your name and college**. Please note that UCL places severe penalties on late-submitted work.

1. **Generalisation of the Ehrenfest theorem**. The Heisenberg picture leads to equations of motion that are formally similar to those obtained in classical mechanics.

   (a) Consider the Shrödinger picture Hamiltonian of a particle of mass m under the influence of a potential $V(\hat{x})$:

   $$H_S = \frac{1}{2m}\hat{p}^2 + V(\hat{x}).$$

   Show that the Hamiltonian operator in the Heisenberg picture becomes:

   $$H_H(t) = \frac{1}{2m}\hat{p}_H^2(t) + V(\hat{x}_H(t)).$$

   [1]

   (b) Consider the Heisenberg equation

   $$\frac{\partial}{\partial t}\hat{O}_H(t) = \frac{i}{\hbar}[H_H(t), \hat{O}_H(t)].$$

   and the results proved on problem 1 to show that $\hat{x}_H(t)$ and $\hat{p}_H(t)$ satisfy the following differential equations (which are similar in form to those that give the evolution of the classical quantities x and p):

   $$\begin{aligned}\frac{\partial}{\partial t}\hat{x}_H(t) =& \frac{1}{m}\hat{p}_H(t)\\ \frac{\partial}{\partial t}\hat{p}_H(t) =& -\frac{\partial}{\partial \hat{x}_H}V(\hat{x}_H(t)).\end{aligned}$$

   [1]

   (c) Consider that the particle of mass m is an electron of charge e under the influence of an electric field of intensity E such that the potential operator is given by $V(\hat{x}) = -eE\hat{x}$. Using the results of 2(b), write an expression for the expected value of $\hat{p}_H(t)$ as a function of time. Assume the particle is initially in a state $|\psi(0)\rangle$. [2]

2. **Approximations in unitary dynamics**

   (a) Prove that for any self-inverse operator $\hat{O}$, i.e. where $\hat{O}^2 = \mathbb{1}$,

   $$\exp[i\omega t\hat{O}] = \cos(\omega t)\mathbb{1} + i\sin(\omega t)\hat{O}.$$

   [2]

   (b) Using the result of part 3(a), find the evolution operator for the following Hamiltonian:

   $$H = \hbar g\frac{\sigma_x + \sigma_z}{\sqrt{2}}.$$

   use it to derive $|\psi(t)\rangle$, for a spin-half particle, initially in state $|\psi(0)\rangle = |\uparrow\rangle$.

   [2]

   (c) A first-order Trotter approximation for evolution under this Hamiltonian is

   $$U_1 = \exp[-ig\sigma_x t/\sqrt{2}]\exp[-ig\sigma_z t/\sqrt{2}].$$

   Calculate the first order approximate solution $|\psi_1(t)\rangle = U_1(t)|\uparrow\rangle$. The error in this computation can be quantified in terms of the norm of the difference between exact $|\psi(t)\rangle$ and approximate solution $|\psi_1(t)\rangle$. By expressing $|\psi(t)\rangle$ and $|\psi_1(t)\rangle$ as a power series in t up to second order, calculate this error, $||\ |\psi(t)\rangle - |\psi_1(t)\rangle||$ to the second order in t.

   Hint: Recall that the spin states $|\uparrow\rangle$ and $|\downarrow\rangle$ form an orthonormal basis for the spin-state of a spin-half particle, and that the operators σ_x and σ_z transform these states as follows:

   $$\sigma_x|\uparrow\rangle = |\downarrow\rangle \qquad \sigma_x|\downarrow\rangle = |\uparrow\rangle \qquad \sigma_z|\uparrow\rangle = |\uparrow\rangle \qquad \sigma_z|\downarrow\rangle = -|\downarrow\rangle$$

   [3]

3. The interaction picture is employed to describe the dynamics of a system whose Hamiltonian has the form $H = H_0 + V$, where the eigenstates and eigenenergies of H_0 are known. In this representation, operators take the form $O_I(t) = U_0^\dagger(t)OU_0(t)$ with $U_0(t) = \exp[-iH_0t/\hbar]$. A Hamiltonian describing the interaction between a pair of two-state atoms takes the form

   $$H = \hbar\epsilon_1|A\rangle\langle A| + \hbar\epsilon_2|B\rangle\langle B| + \hbar J\left(|A\rangle\langle B| + |B\rangle\langle A|\right),$$

   where $|A\rangle \equiv |e_1, g_2\rangle$; $|B\rangle \equiv |g_1, e_2\rangle$ and $|e_{1(2)}\rangle$ and $|g_{1(2)}\rangle$ are the excited and ground states of atom $1(2)$, respectively.

   (a) Show that $V_I(t) = \hbar J\Big(e^{i(\epsilon_1-\epsilon_2)t}|A\rangle\langle B| + e^{-i(\epsilon_1-\epsilon_2)t}|B\rangle\langle A|\Big)$. [4]

   (b) In the interaction picture, the joint state of the atoms at time t can be expressed as $|\Psi(t)\rangle_I = \alpha(t)|A\rangle + \beta(t)|B\rangle$. This state satisfies the differential equation $\frac{d}{dt}|\Psi(t)\rangle_I = (-i/\hbar)V_I(t)|\Psi(t)\rangle_I$. Assume that $\epsilon_1 = \epsilon_2$ and $|\Psi(0)\rangle = |A\rangle$. Show that the state at time t becomes $|\Psi(t)\rangle_I = \cos(Jt)|A\rangle - i\sin(Jt)|B\rangle$ [4]

4. **Applying time-dependent perturbation theory.** A spin-1 particle is held in a strong magnetic field in the z-direction. Immediately prior to time $t = -t_0$, a measurement of its spin indicates that it is in the state $|s = 1, m_s = 1\rangle$. At $t = -t_0$ the experiment is perturbed by a weak magnetic field in the x-direction which ramps up to a maximum and then decays back down to zero at time $t = t_0$.

   (a) The resulting Hamiltonian is $H = \Omega\hat{S}_z + \lambda(t)\hat{S}_x$ where $\lambda(t) = \lambda_0(1 - |t|/t_0)$ for $|t| < t_0$ and $\lambda(t) = 0$ for $|t| \geq t_0$ and $|\lambda_0| \ll \Omega$. Using perturbation theory, show that (to first-order) the probability that a measurement on the spin at time $t = t_0$ will indicate $m_s = 0$ is:

$$P^{(1)}_{1\to 0} = 2\left|\frac{\lambda_0}{\Omega^2 t_0}\right|^2 (1 - \cos(\Omega t_0))^2.$$

   You may find the following spin-1 matrix representations of $\hat{S}_z$ and $\hat{S}_x$:

$$\hat{S}_x = \frac{\hbar}{\sqrt{2}}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad \hat{S}_z = \hbar\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

   and the following indefinite integral helpful:

$$\int e^{iat}(1 - bt)dt = \frac{-e^{iat}}{a^2}(b + ia(1 - bt)) + c.$$

   [5]

   (b) Without detailed calculation, explain why, in this example, second order perturbation theory is required to see a non-zero transition probability to the state $|s = 1, m_s = -1\rangle$. [1]