# Particle Physics
# PHASM/G442

**Prof. Ruben Saakyan**

http://moodle.ucl.ac.uk/course/view.php?id=2589

Enrolment is automatic if you are registered on the course via Portico.

Yr 2017/18

# Standard Model of
# FUNDAMENTAL PARTICLES AND INTERACTIONS

The Standard Model is a quantum theory that summarizes our current knowledge of the physics of fundamental particles and fundamental interactions (interactions are manifested by forces and by decay rates of unstable particles).

## FERMIONS matter constituents spin = 1/2, 3/2, 5/2, ...

### Leptons spin =1/2

| Flavor | Mass GeV/$c^2$ | Electric charge |
|---|---|---|
| $\nu_L$ lightest neutrino* | $(0-0.13)\times10^{-9}$ | 0 |
| e electron | 0.000511 | –1 |
| $\nu_M$ middle neutrino* | $(0.009-0.13)\times10^{-9}$ | 0 |
| $\mu$ muon | 0.106 | –1 |
| $\nu_H$ heaviest neutrino* | $(0.04-0.14)\times10^{-9}$ | 0 |
| $\tau$ tau | 1.777 | –1 |

### Quarks spin =1/2

| Flavor | Approx. Mass GeV/$c^2$ | Electric charge |
|---|---|---|
| u up | 0.002 | 2/3 |
| d down | 0.005 | –1/3 |
| c charm | 1.3 | 2/3 |
| s strange | 0.1 | –1/3 |
| t top | 173 | 2/3 |
| b bottom | 4.2 | –1/3 |

*See the neutrino paragraph below.

**Spin** is the intrinsic angular momentum of particles. Spin is given in units of $\hbar$, which is the quantum unit of angular momentum where $\hbar = h/2\pi = 6.58\times10^{-25}$ GeV s $=1.05\times10^{-34}$ J s.

**Electric charges** are given in units of the proton's charge. In SI units the electric charge of the proton is $1.60\times10^{-19}$ coulombs.

**The energy** unit of particle physics is the electronvolt (eV), the energy gained by one electron in crossing a potential difference of one volt. **Masses** are given in GeV/$c^2$ (remember $E = mc^2$) where 1 GeV = $10^9$ eV $=1.60\times10^{-10}$ joule. The mass of the proton is 0.938 GeV/$c^2$ = $1.67\times10^{-27}$ kg.

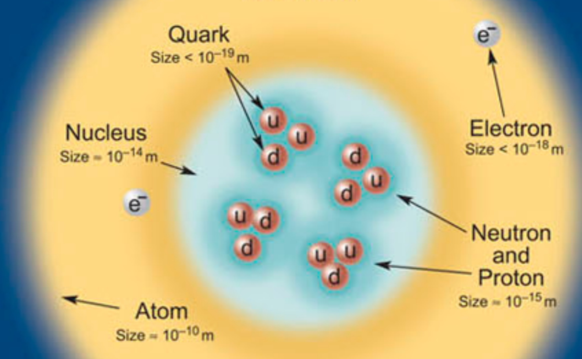
### Neutrinos

Neutrinos are produced in the sun, supernovae, reactors, accelerator collisions, and many other processes. Any produced neutrino can be described as one of three neutrino flavor states $\nu_e$, $\nu_\mu$, or $\nu_\tau$, labelled by the type of charged lepton associated with its production. Each is a defined quantum mixture of the three definite mass neutrinos $\nu_L$, $\nu_M$, and $\nu_H$ for which currently allowed mass ranges are shown in the table. Further exploration of the properties of neutrinos may yield powerful clues to puzzles about matter and antimatter and the evolution of stars and galaxy structures.

### Matter and Antimatter

For every particle type there is a corresponding antiparticle type, denoted by a bar over the particle symbol (unless + or – charge is shown). Particle and antiparticle have identical mass and spin but opposite charges. Some electrically neutral bosons (e.g., $Z^0$, $\gamma$, and $\eta_c = c\bar{c}$ but not $K^0 = d\bar{s}$) are their own antiparticles.

## Structure within the Atom




If the proton and neutrons in this picture were 10 cm across, then the quarks and electrons would be less than 0.1 mm in size and the entire atom would be about 10 km across.

## BOSONS force carriers spin = 0, 1, 2, ...

### Unified Electroweak spin = 1

| Name | Mass GeV/$c^2$ | Electric charge |
|---|---|---|
| $\gamma$ photon | 0 | 0 |
| $W^-$ | 80.39 | –1 |
| $W^+$ W bosons | 80.39 | +1 |
| $Z^0$ Z boson | 91.188 | 0 |

### Strong (color) spin =1

| Name | Mass GeV/$c^2$ | Electric charge |
|---|---|---|
| g gluon | 0 | 0 |

### Color Charge

Only quarks and gluons carry "strong charge" (also called "color charge") and can have strong interactions. Each quark carries three types of color charge. These charges have nothing to do with the colors of visible light. Just as electrically-charged particles interact by exchanging photons, in strong interactions, color-charged particles interact by exchanging gluons.

### Quarks Confined in Mesons and Baryons

Quarks and gluons cannot be isolated – they are confined in color-neutral particles called **hadrons**. This confinement (binding) results from multiple exchanges of gluons among the color-charged constituents. As color-charged particles (quarks and gluons) move apart, the energy in the color-force field between them increases. This energy eventually is converted into additional quark-antiquark pairs. The quarks and antiquarks then combine into hadrons; these are the particles seen to emerge.

Two types of hadrons have been observed in nature **mesons** $q\bar{q}$ and **baryons** qqq. Among the many types of baryons observed are the proton (uud), antiproton ($\bar{u}\bar{u}\bar{d}$), neutron (udd), lambda $\Lambda$ (uds), and omega $\Omega^-$ (sss). Quark charges add in such a way as to make the proton have charge 1 and the neutron charge 0. Among the many types of mesons are the pion $\pi^+$ ($u\bar{d}$), kaon $K^-$ ($s\bar{u}$), $B^0$ ($d\bar{b}$), and $\eta_c$ ($c\bar{c}$). Their charges are +1, –1, 0, 0 respectively.

## Properties of the Interactions

The strengths of the interactions (forces) are shown relative to the strength of the electromagnetic force for two u quarks separated by the specified distances.

| Property | | Gravitational Interaction | Weak Interaction (Electroweak) | Electromagnetic Interaction | Strong Interaction |
|---|---|---|---|---|---|
| Acts on: | | Mass – Energy | Flavor | Electric Charge | Color Charge |
| Particles experiencing: | | All | Quarks, Leptons | Electrically Charged | Quarks, Gluons |
| Particles mediating: | | Graviton (not yet observed) | $W^+$ $W^-$ $Z^0$ | $\gamma$ | Gluons |
| Strength at | $10^{-18}$ m | $10^{-41}$ | 0.8 | 1 | 25 |
| | $3\times10^{-17}$ m | $10^{-41}$ | $10^{-4}$ | 1 | 60 |

Visit the award-winning web feature *The Particle Adventure* at

**ParticleAdventure.org**


This chart has been made possible by the generous support of:

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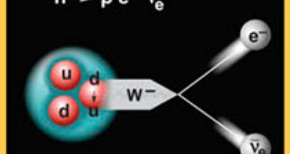
## Particle Processes

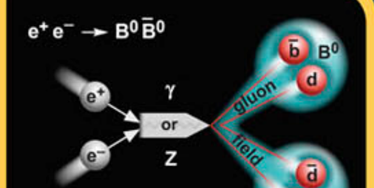
These diagrams are an artist's conception. Blue-green shaded areas represent the cloud of gluons.

$n \rightarrow p\, e^- \bar{\nu}_e$




A free neutron (udd) decays to a proton (uud), an electron, and an antineutrino via a virtual (mediating) W boson. This is neutron β (beta) decay.

$e^+ e^- \rightarrow B^0 \bar{B}^0$




An electron and positron (antielectron) colliding at high energy can annihilate to produce $\bar{B}^0$ and $B^0$ mesons via a virtual Z boson or a virtual photon.

## Unsolved Mysteries

Driven by new puzzles in our understanding of the physical world, particle physicists are following paths to new wonders and startling discoveries. Experiments may even find extra dimensions of space, mini-black holes, and/or evidence of string theory.

### Universe Accelerating?

The expansion of the universe appears to be accelerating. Is this due to Einstein's Cosmological Constant? If not, will experiments reveal a new force of nature or even extra (hidden) dimensions of space?

### Why No Antimatter?

Matter and antimatter were created in the Big Bang. Why do we now see only matter except for the tiny amounts of antimatter that we make in the lab and observe in cosmic rays?

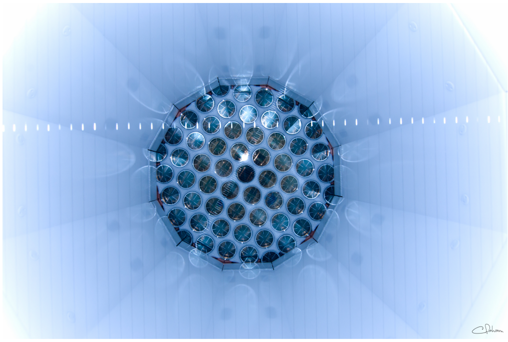
### Dark Matter?

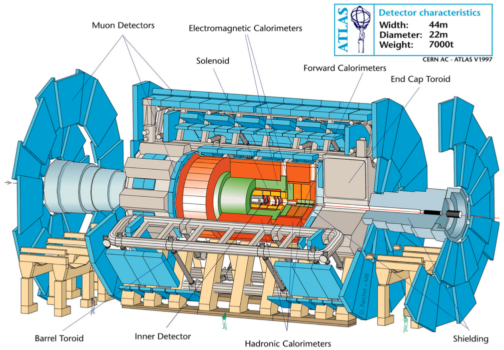
Invisible forms of matter make up much of the mass observed in galaxies and clusters of galaxies. Does this dark matter consist of new types of particles that interact very weakly with ordinary matter?

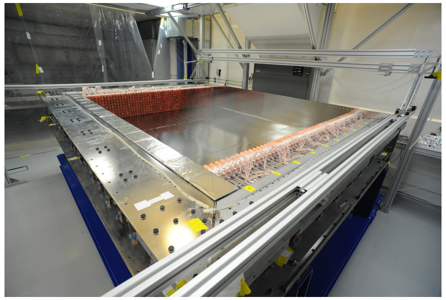
### Origin of Mass?

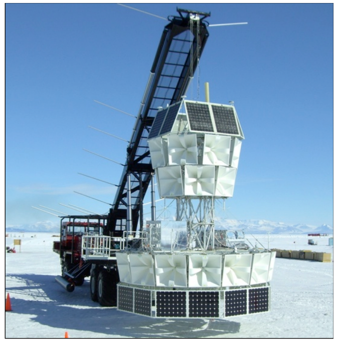
In the Standard Model, for fundamental particles to have masses, there must exist a particle called the Higgs boson. Will it be discovered soon? Is supersymmetry theory correct in predicting more than one type of Higgs?

# UCL High Energy Physics Group







**http://www.hep.ucl.ac.uk/**

- Energy Frontier with **ATLAS** at **LHC**
- Neutrino Physics with **SuperNEMO**, **CHIPS**, **NOVA** and **DUNE**
- Direct detection of dark matter with **LUX** and **LZ** experiments
- Probe physics at energies beyond LHC indirectly with UHE neutrinos at South Pole (**ANITA/ARA**), muon's magnetic moment (**FNAL g-2**), lepton flavour violation (**Mu2e, Mu3e**)
- **Theoretical studies** of strong interactions, QCD, and physics beyond the standard model
- Developing novel instrumentation for future experiments and their applications in **proton cancer therapy** and **muon tomography**
- Novel accelerator technologies, **AWAKE**

# Logistics

## Contact Details

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Web: http://moodle.ucl.ac.uk/course/view.php?id=2589

## Office Hours

Fridays 11-12:30, but life is often unpredictable — email me before coming!

# Books

- W. Cottingham, D. Greenwood : “An Introduction to the Standard Model of Particle Physics" (2nd edition)
- D. Griffiths : “Introduction to Elementary Particles”
- **M. Thomson: “Modern Particle Physics”**

also:

- F. Halzen, A. Martin : “Quarks and Leptons”
- R. Cahn, G. Goldhaber : “The Experimental Foundations of Particle Physics”

# Assessment

- 90% 2.5 hr exam (3 questions from 5) + 10% coursework (problem sheets)
- Module incomplete unless mark > 15% achieved on 4-problem sheets for MSci & MSc.

## Lecture & Course Notes

- This course uses a combination of powerpoint slides and old fashioned writing on the board.
- Lecture slides will be on Moodle in pdf format
- Working / examples and additional material at lectures
  - YOU SHOULD take notes during the lectures

## Experimental Methods (beta)

- Last ~20 minutes of most lectures devoted to discussing experimental methods
- Intention is to finish with a little less mathematical rigour and provide experimental (and historical) examples that reinforce some of the concepts covered in the course.

# Prerequisites

- 3rd year/BSc Quantum Mechanics
- Special Relativity (4-vector notation)
- 3rd year/BSc Electromagnetism
- 3rd year/BSc Particle Physics

- Without BSc Particle Physics* – you may struggle (please discuss with me) - it's possible to catch up by reading a BSc Particle Physics textbook eg.
  - "Nuclear and Particle Physics - An Introduction" : Brian R. Martin

* Hereafter referred to as NPP (3$^{rd}$ year Nuclear and Particle Physics course)

# Coursework

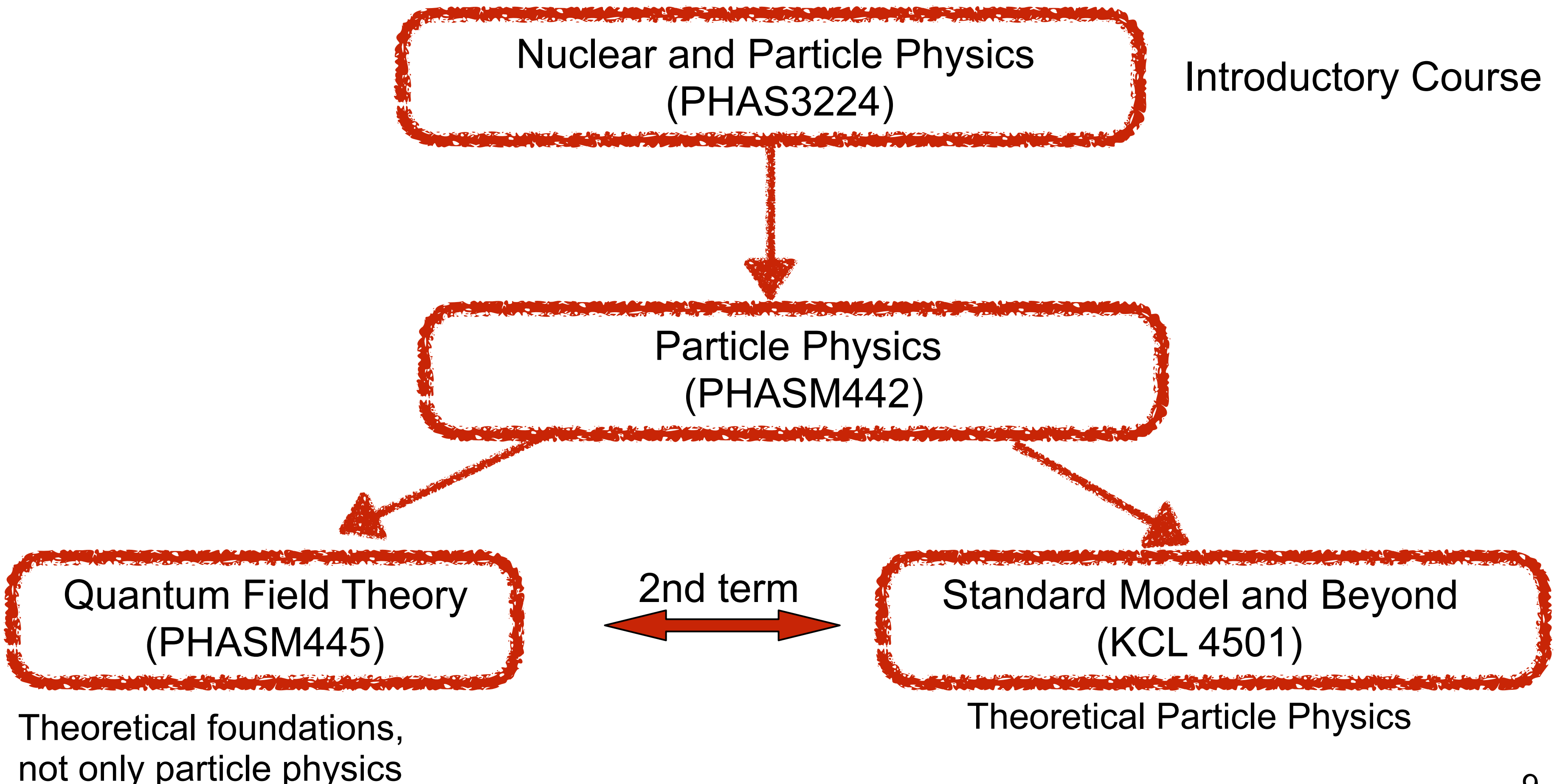
## 4 Problem Sheets

- Posted on web. It's up to you to check the course moodle web-page.

## Tentative schedule

| Coursework | hand out | hand in |
|---|---|---|
| PS1 | 12-Oct-17 | 26-Oct-17 |
| PS2 | 26-Oct-17 | 16-Nov-17 |
| PS3 | 16-Nov-17 | 7-Dec-17 |
| PS4 | 7-Dec-17 | 16-Jan-18 |

- **Particle Physics** studies properties of the fundamental constituents of matter and their interactions.
- This is a (much) more in depth, more quantitative version of UCL's 3rd year Nuclear and Particle Physics course. Relates **theoretical concepts** to **experimental measurements**.



# Course Outline

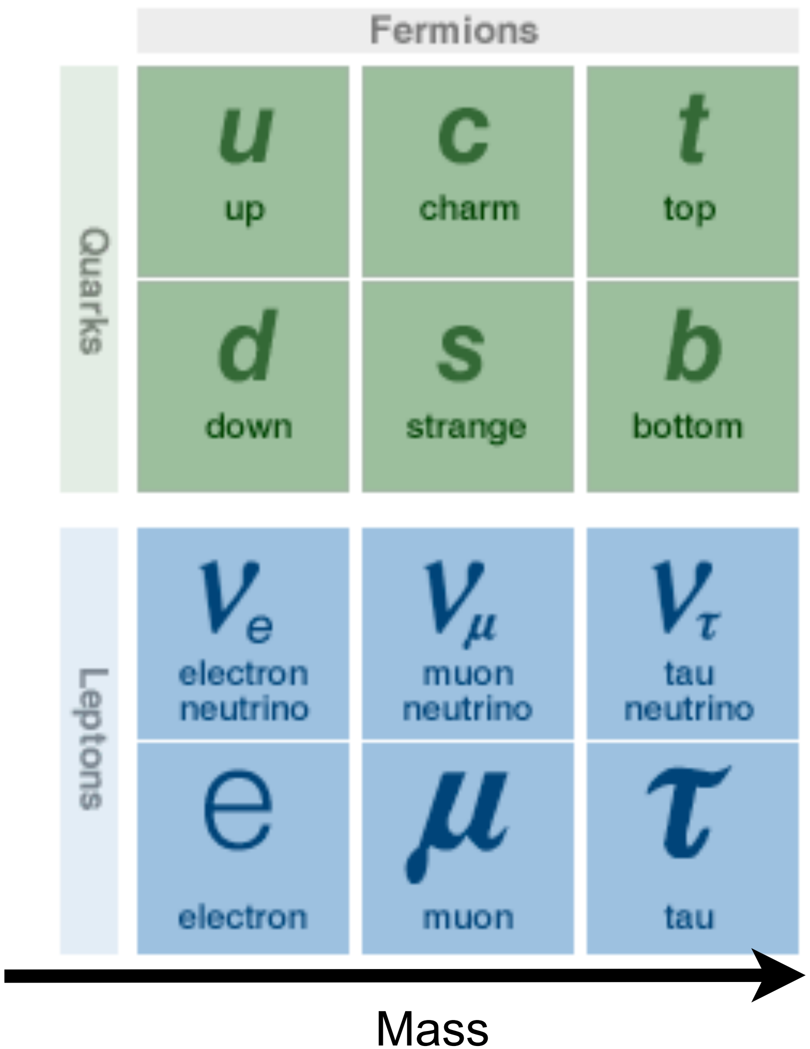
- Recap, formalism, reaction rates, Feynman Rules, coupling constants (**Module 1**)
- Symmetries and conservation laws (**Module 2**)
- The Dirac Equation (**Module 3**)
- Dirac + Maxwell ==> Interaction by particle exchange (**Module 4**)
- QED Calculations (**Module 5**)
- Quark properties, proton structure and QCD (**Module 6**)
- Weak Interactions (**Module 7**)
- Electroweak Unification and Higgs (**Module 8**)
- Neutrino Phenomenology and Beyond the Standard Model (**Module 9**)
- Revision (**Term 3**)

**NB: <u>NO</u> Reading Week for intercollegiate courses**

# Module 1: Outline

- BSc recap :  particles & forces
- Natural units
- Four Vectors and Special Relativity
- Fermi's Golden Rule : Rate of reactions
- Feynman diagrams recap
- Feynman rules
- Phase space, density of states, Matrix Element
- *Phase space calculation for simple decay and scattering*
- Renormalisation / Running Coupling constants

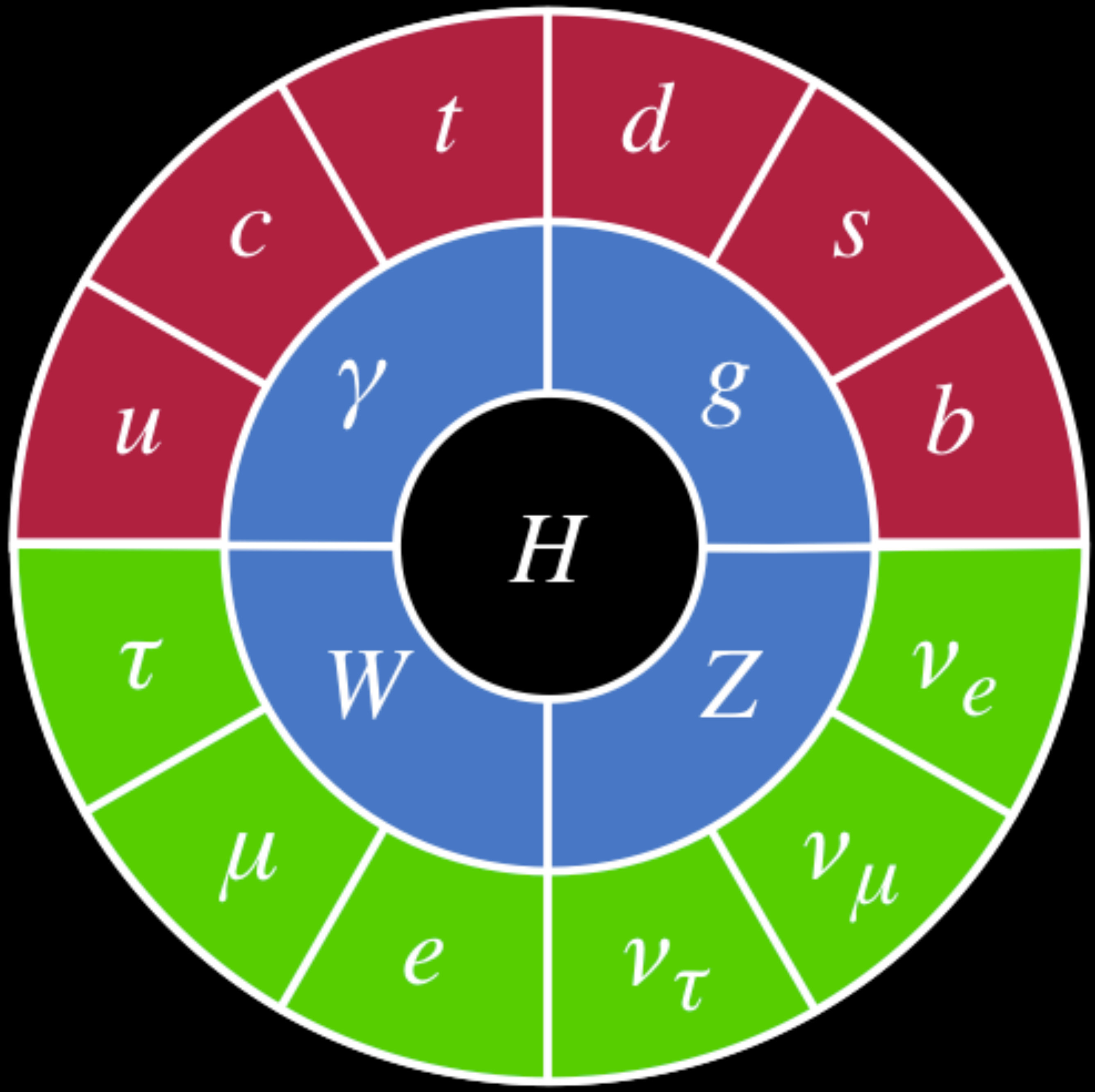
# Elementary Matter Particles (NPP recap)




| | Q | Strong | Weak | EM |
|---|---|---|---|---|
| u,c,t | +2/3 | ✓ | ✓ | ✓ |
| d,s,b | -1/3 | ✓ | ✓ | ✓ |
| nu's | 0 | ✗ | ✓ | ✗ |
| e,mu, tau | -1 | ✗ | ✓ | ✓ |

Particles of same type but different families are identical except for mass.

# Where do you put the Higgs?



# Elementary Matter Particles

- Properties:
  - Fundamental point-like particles (i.e without size)
  - Fermions -- spin ½
  - Obey Dirac Equation
  - Have anti-particles  (but do the neutrinos?)

- Questions (well some of them):
  - Why 3 families?
    - Need 2 for any matter/anti-matter asymmetry
  - Why mass hierarchy?
  - Where does mass come from?
  - Are neutrinos Majorana particles?

# Force Particles (NPP recap)

| Force | Carrier | Symbol | Number | Mass (GeV) | Coupling |
|---|---|---|---|---|---|
| EM | Photon | $\gamma$ | 1 | 0 | $\alpha_{EM}=1/137$ |
| Weak | $W^{\pm}$, $Z^0$ | $W^{\pm}$, $Z^0$ | 3 | 80.4, 91.2 | $\alpha_W \approx \alpha_{EM}$ |
| Strong | Gluon | g | 8 | 0 | $\alpha_S \sim 0.1\text{-}1$ |
| Gravity | Graviton | ? | ? | ? | $\sim 10^{-42}$ |

All bosons with spin=1 (except graviton : spin = ?  )

Photon massless & no-charge : so doesn't self-interact

Strong/Weak "mediators" carry their own "charge" and so do self-interact (they are NON-ABELIAN) - this has important ramifications.

SM provides a unified treatment of EM & Weak forces (and implies unification of electroweak with strong force), but needs the Higgs boson (which we now have!)

# Natural Units

- SI units not used in particle physics
- More practical to use a "natural" system where:

$$\hbar = c = 1$$

- Energy, Mass, Momentum all have units of energy (eV, GeV)
- Time, length have units of inverse energy ($eV^{-1}$, $GeV^{-1}$)
- *Test: Why time, length are inverse energy*
- The conversion factors are:
- 1 $GeV^{-1}$ = 0.1973 fm = $1.973 \times 10^{-16}$ m = $6.582 \times 10^{-25}$ sec
- *More tests:*
  - *Cross sections : What is 1 $GeV^{-2}$ in mb ?*
  - *What are dimensions of angular momentum (L) or spin (S) in natural units ?*

# Four Vector Notation

- Relativistic effects are critical in particle physics, we will commonly use four vector notation
- Four Vector Definition: "An object that transforms like $x^\mu$ between inertial frames"
- Invariant Definition = "A quantity that is unchanged in all inertial frames"
- Example four vectors:

$$x^\mu = \left(\text{time}, \overrightarrow{\text{position}}\right)$$

$$p^\mu = \left(\text{energy}, \overrightarrow{\text{momentum}}\right)$$

$$j^\mu = \left(\phi \text{ density}, \overrightarrow{\text{current}}\right)$$

Example Invariants:

Rest mass

- $x^\mu$ is the contra-variant four vector

# Four Vector Recap

- Special Relativity Reminder

$$E^2 = |\vec{p}|^2 + m^2$$

$$\beta = \frac{v}{c} = v$$

$$E = \gamma m \qquad \vec{p} = \gamma m \vec{\beta}$$

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1}{\sqrt{1-\beta^2}}$$

$$\gamma = \frac{E}{m} \qquad \vec{\beta} = \frac{\vec{p}}{m}$$

- Lorentz Transform:

$$(x^\mu)' = \sum_{\nu=0}^{3} \Lambda^\mu_\nu x^\nu = \Lambda^\mu_\nu x^\nu$$

$$\begin{pmatrix} t' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \\ z \end{pmatrix}$$

- Covariant metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$x_\mu = g_{\mu\nu} x^\nu$$

$$x^\mu = (t, \vec{x})$$

$$x_\mu = (t, -\vec{x})$$

# Few words on notation

**4-Vectors**

$$p^{\mu} = \{E, \vec{p}\} = \{E, p_x, p_y, p_z\}$$ Contravariant

$$p_{\mu} = g_{\mu\nu} p^{\mu} = \{E, -\vec{p}\} = \{E, -p_x, -p_y, -p_z\}$$ Covariant

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

**3-Vectors**

$\vec{p}$ or **p** or <u>p</u>

Quantities evaluated in CoM frame $\vec{p}^{\,*}$

# Four Vector Cheat-sheet

- Scalar Product

$$
\begin{aligned}
a \cdot b &= a^0 b^0 - \vec{a} \cdot \vec{b} \\
&= g_{\mu\nu} a^\mu b^\nu \\
&= a_\nu b^\nu
\end{aligned}
$$

$$
\begin{aligned}
x \cdot x &= t^2 - \vec{x} \cdot \vec{x} \\
x_\mu x^\mu &= t^2 - |\vec{x}|^2
\end{aligned}
$$

4-vector “length”
>0 “time like”
<0 “space like”
=0 “light like”

$$
\begin{aligned}
P_\mu P^\mu &= P^2 \\
P_\mu P^\mu &= E^2 - |\vec{p}|^2 \\
P_\mu P^\mu &= m^2
\end{aligned}
$$

Invariant Rest Mass

- Differential 4-vector (“four-derivative”)

$$\partial^\mu = \left( \frac{\partial}{\partial t}, -\vec{\nabla} \right)$$

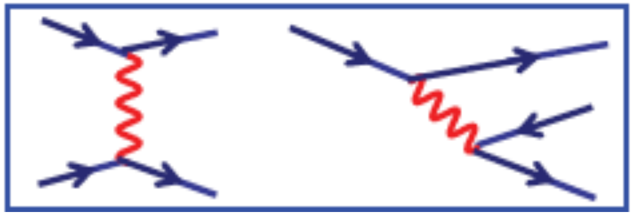
Note - sign (c.f Q.M.)

$$\partial_\mu = \left( \frac{\partial}{\partial t}, +\vec{\nabla} \right)$$

$$
\begin{aligned}
\partial_\mu a^\mu &= \frac{\partial a^0}{\partial t} + \vec{\nabla} \cdot \vec{a} \\
b_\mu \partial^\mu &= b^0 \frac{\partial}{\partial t} + \vec{b} \cdot \vec{\nabla}
\end{aligned}
$$

# Cross-Sections and Decay Rates

- In particle physics we are mainly concerned with particle **interactions** and **decays**, i.e. **transitions between states**



**Experimental Observables**

- Calculate **transition rates** from **Fermi's Golden Rule**

See Section 2.3.6 of Thomson (2013) for quantum mechanical derivation of FGR

$$\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)}$$

Fundamental Particle Physics (**Feynman Diagrams**)

$\Gamma_{fi}$ is number of transitions per unit time from initial state $|i\rangle$ to final state $\langle f|$ – not Lorentz Invariant !

$T_{fi}$ is Transition **Matrix Element**

$$T_{fi} = \langle f|\hat{H}|i\rangle + \sum_{j \neq i} \frac{\langle f|\hat{H}|j\rangle\langle j|\hat{H}|i\rangle}{E_i - E_j} + \ldots$$

$\hat{H}$ is the perturbing Hamiltonian

$\rho(E_f)$ is density of final states

"Just" kinematics (**phase space** integral)

# Observables: What can we measure?

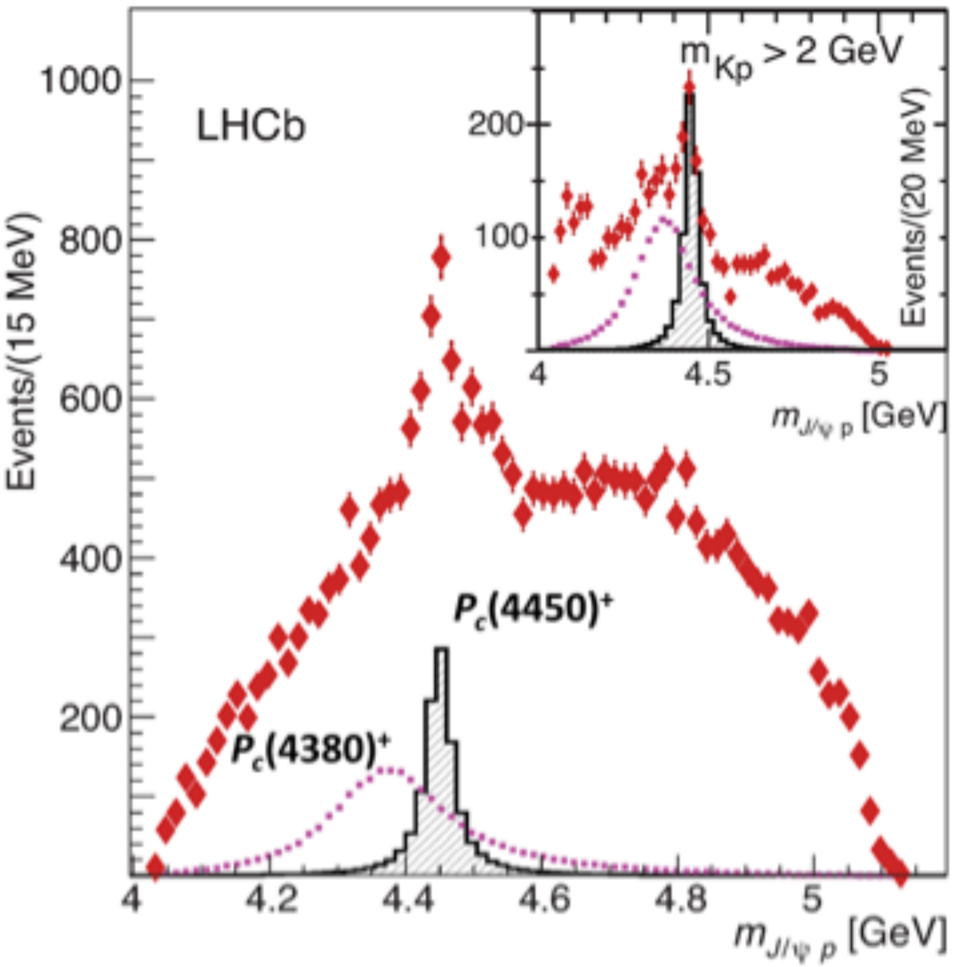
- Particle decays : A → B + C + …
  - Lifetime, $\tau$
  - Width, $\Gamma$
  - Branching Ratios
  - Angular Distributions

- Reactions/Interactions : A + B → C + D + …
  - Rate
  - Angular Distribution

- Bound states (typically strong force)
  - Peak in rate vs energy

# Aside: Bound States

Hot off the Press! LHCb, July 2015 — **Pentaquarks**




Also excited states of known hadrons:

Analogous to the energy levels in the hydrogen atom, can observe photons given off when transitions from one state to another occur.

# Particle Decay (NPP recap)

- Decay Rate, $\Gamma$ : "Probability per unit time that a particle decays"

$$\Gamma = \frac{-1}{N}\frac{dN}{dt}$$
$$N(t) = N(0)e^{-\Gamma t}$$

  - If expressed in units of energy (since it is s-1) then we call it a Decay Width
  - Lifetime, $\tau$ : "Average time it takes to decay (in particle's rest-frame)"
- $\Gamma$ and $\tau$ are simply related by: $\tau = 1/\Gamma$
- Generally a particle can have many decay modes : concept of partial widths, $\Gamma_i$

$$\Gamma_{Tot} = \sum \Gamma_i$$

- Branching Ratio (BR) defined as :

$$BR_i = \frac{\Gamma_i}{\Gamma_{tot}}$$

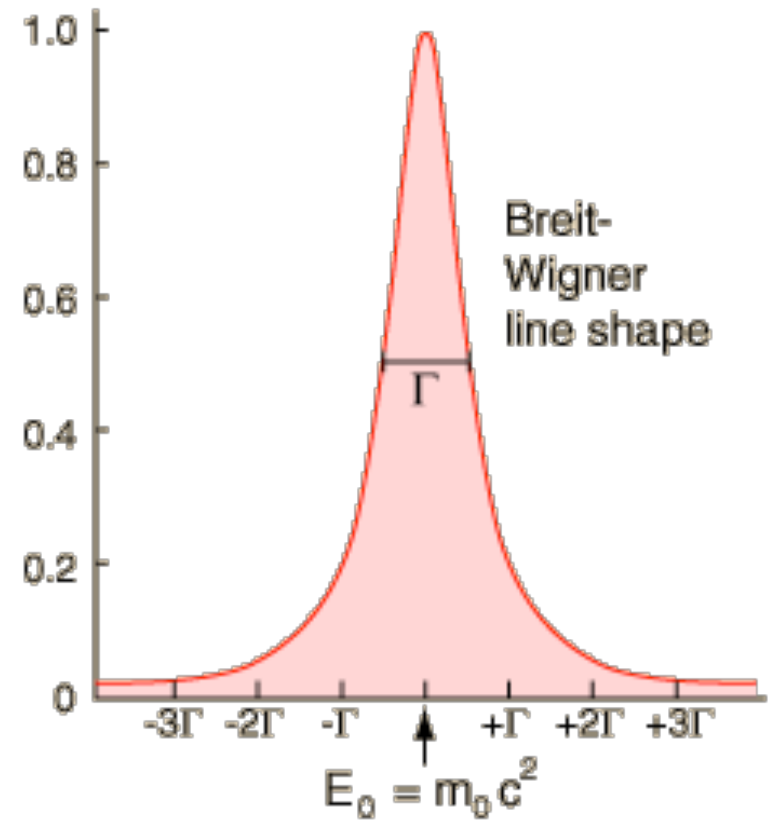
- We tend to measure : BRs and $\Gamma_{TOT}$ or $\tau$ and calculate $\Gamma_i$

# Decay Width (NPP recap)

Time of a particle's decay has uncertainty : $\Delta t = \tau$
Uncertainty Principle then predicts

$\Delta E.\tau = 1/2$ and hence $\Gamma = 2\,\Delta E$

If measure invariant mass of a state then Uncertainty principle gives it a "width" due to particle having a finite lifetime.




Distribution of mass follows Breit-Wigner form:

$$N(E) = \frac{N_{max}\left(\frac{\Gamma}{2}\right)^2}{(E - M_0)^2 + \left(\frac{\Gamma}{2}\right)^2}$$

We can only ever measure either lifetime or width due to measuring capabilities of particle detectors

E.g. $\tau_{Z,W^\pm} \approx 3\times 10^{-25}$ sec

$\Gamma_Z \approx 2.5$ GeV

# Reactions: A + B → C + D

- Rate (Probability) of a reaction often expressed in terms of cross section (σ) it is the effective cross-sectional area that A sees of B (or B of A).

- Often measure "differential" cross sections e.g. $d\sigma/d\Omega$ or $d\sigma/d(\cos\theta)$

$$\begin{aligned} d\Omega &= \sin\theta \; d\theta \; d\phi \\ &= d(\cos\theta) \; d\phi \end{aligned}$$

- Luminosity definition : $$L = \frac{n_b \times n_1 \times n_2 \times f}{A}$$

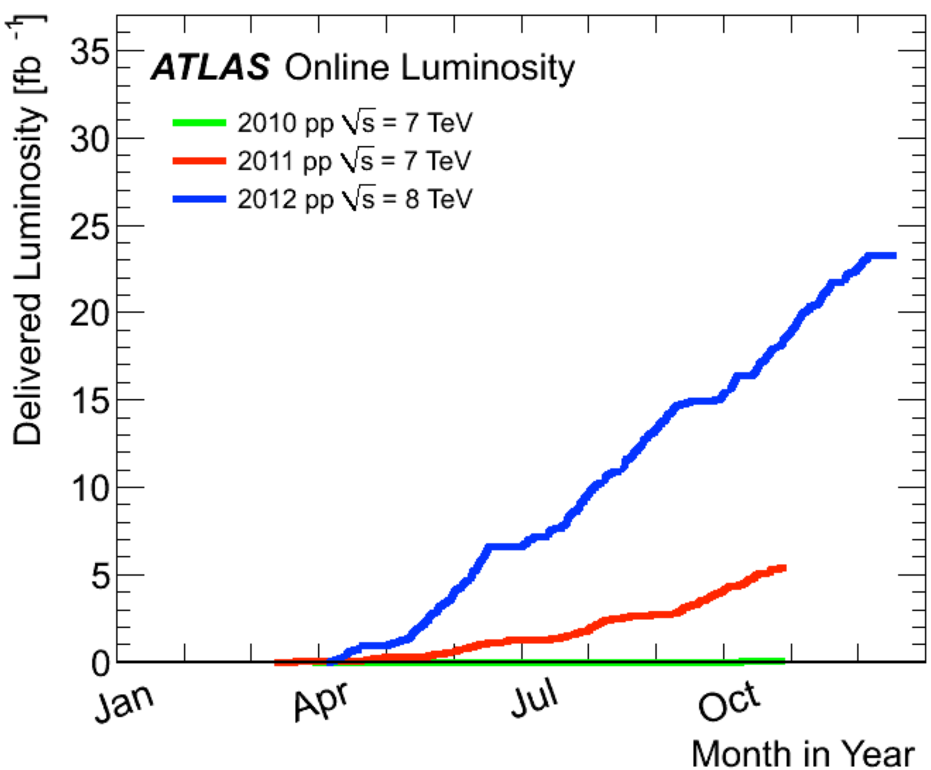
  – Typical values for accelerator : $10^{30}$-$10^{34}$ $cm^{-2}s^{-1}$

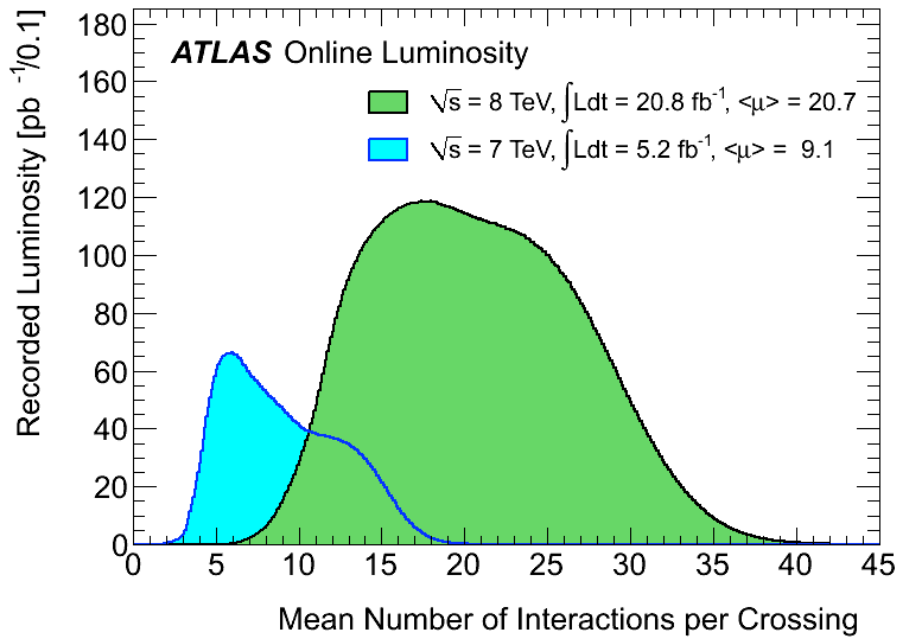
- Event rates and "integrated luminosity" :

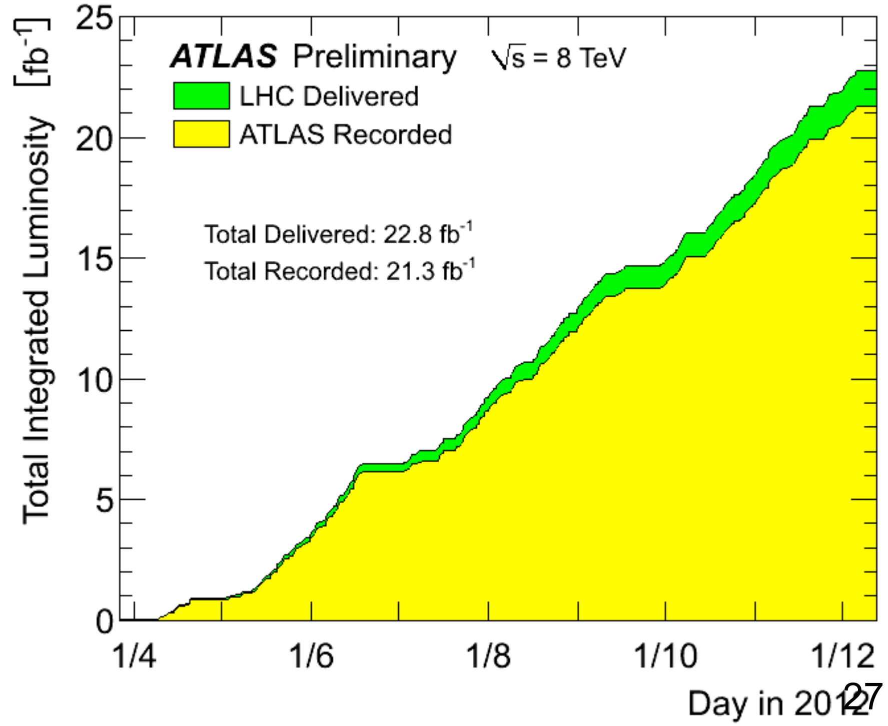
$$\frac{dN}{dt} = \sigma \times L$$

$$N = \sigma \int L dt$$

# Aside: Integrated Luminosity at LHC

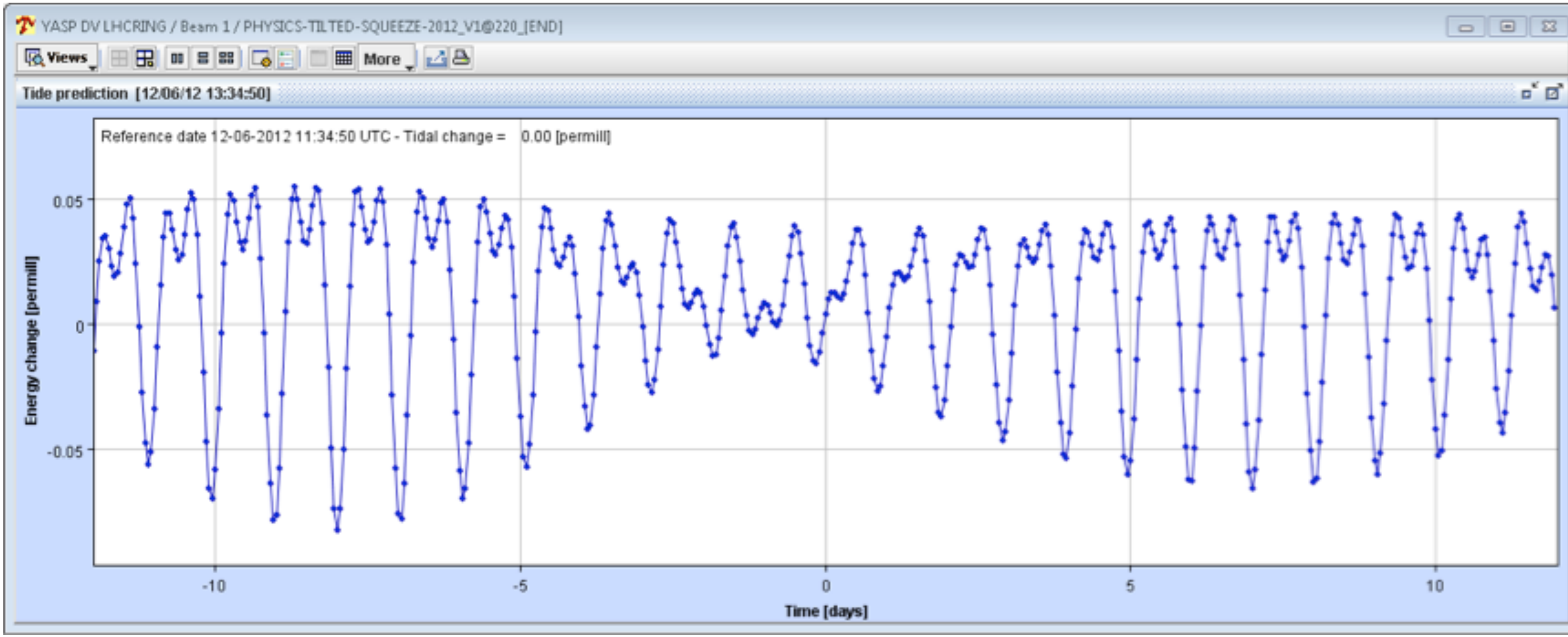


# Aside: Precision Effects

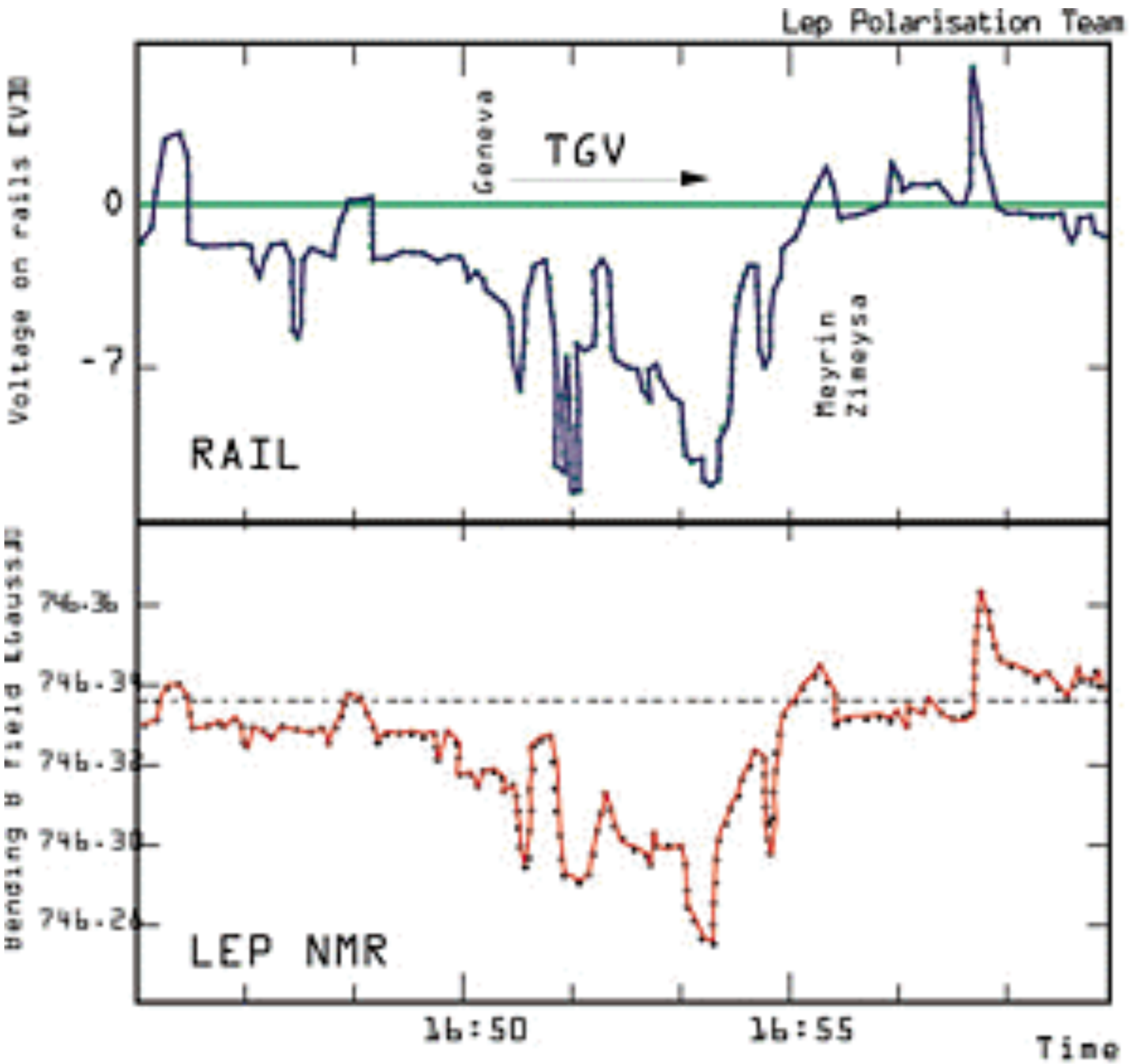
$$p[GeV] = 0.3 \times B[T] \times R[m]$$




This is the graph used by the LHC operators to compensate the accelerator displacement. Each up and down represents a day, with a high and a low tides. The external modulation comes from adding in the position of the moon with respect to the earth and sun during the month.

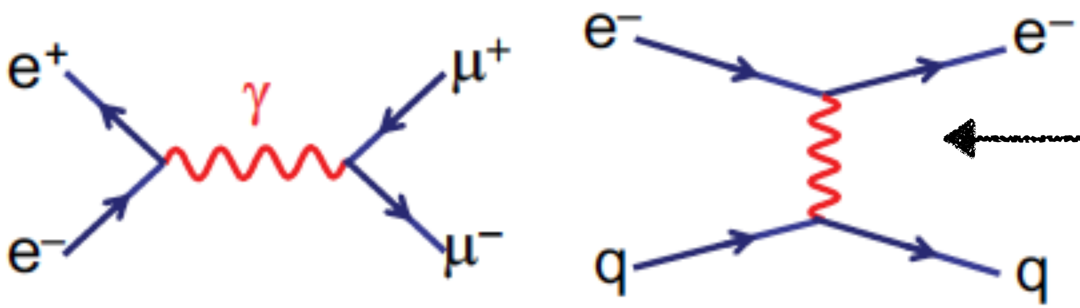
# Aside: Precision Effects

Impact on bending magnets due to current from overhead TGV trains




$$p[GeV] = 0.3 \times B[T] \times R[m]$$

- Our goal is to learn how to properly calculate particle interaction processes (albeit the simplest ones)




helps probe proton structure

- Need **relativistic** calculations of decay rates and cross-sections

$$\sigma = \frac{|M_{fi}|^2}{\text{flux}} \times (\text{phase space})$$

- Need **relativistic** treatment of spin-half particles ⇒ **Dirac Equation**
- Need **relativistic** calculation Interaction **Matrix Element** ⇒

⇒ **Interaction by particle exchange and <u>Feynman Rules</u>**

# How we calculate Reaction Rate (σ) or Decay Width (Γ)

- We follow the following recipe
  - Draw Feynman diagrams for the process
    - Need to decide to which "*order*" we want to perform the calculation
  - Invoke Feynman Rules to calculate the "*Matrix Element*"
  - Calculate the "*phase space*" and "*flux*" of the process
  - Plug these into Fermi's Golden Rule (FGR) to calculate rate or cross-section

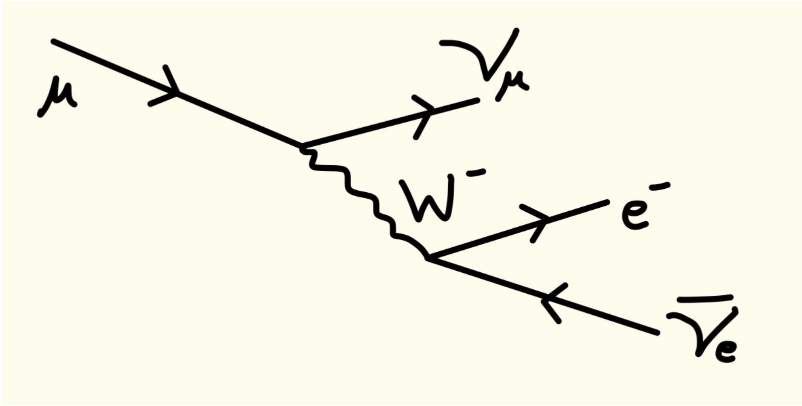
$$\mathrm{Rate} = |M|^2 \rho \prod_{in} \frac{1}{2E_{in}}$$

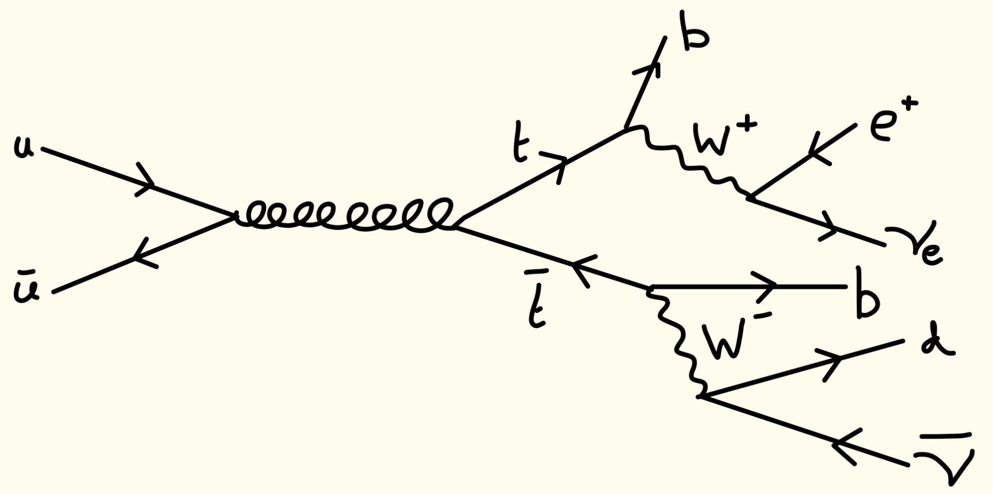
$$\sigma = \frac{\mathrm{Rate}}{\mathrm{Flux}}$$

  - See (e.g.) Section 2.3.6 of Thomson (2013) for quantum mechanical derivation of FGR
  - Those interested where Feynman Diagrams and Feynman Rules come from, see next term's Quantum Field Theory course by Prof. R. Thorne

# Feynman Diagrams — “Rules” (NPP recap)

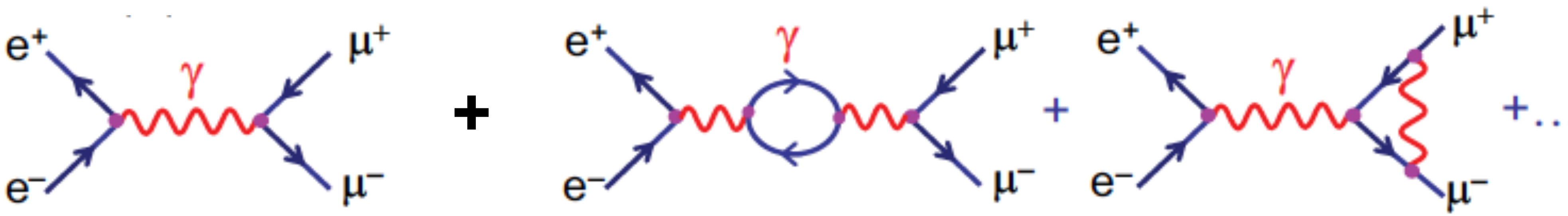
1) Time from left to right
2) Draw initial particle lines on left and final to right - there will be a propagator in the middle
3) Based on information about reaction (initial & final state, rate) determine the type of interaction : EM(γ), Weak (W,Z), Strong (g)
4) Draw interaction vertices - make sure that charge, lepton # etc are conserved
5) Draw arrow (L→R for particles) and (R→L : backward in time for anti-particles)
6) Make  sure arrows ‘flow’ through the vertex



# “Order”

- The order is determined by number of vertices / complexity of Feynman diagrams
- We talk about the lowest order process/diagram and “higher order” processes.




1$^{st}$ order

2$^{nd}$ order

- Occasionally the lowest order permissible process is quite complex e.g. $K \rightarrow \mu\mu$
- In this course we will limit ourselves to 1st order calculations

# Aside: Virtual Particles (NPP recap)

- Definition of a real particle:
  - $P^2 = (\text{Rest Mass})^2$
- Not true for  a virtual particle
- This is possible because one can violate conservation of energy (or momentum) for a short time via the uncertainty principle

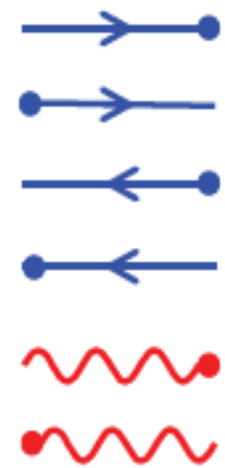
$$\Delta t \sim \frac{1}{\Delta E}$$

- The more energy that has to be "borrowed" the shorter the time interval (the particles are more virtual)
- Example:
  - Virtuality of W in pion decay

# Basic Feynman Rules for calculating |M| (QED example)

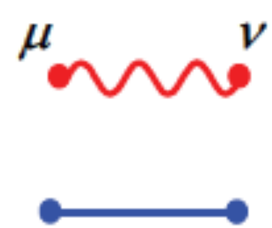
- **External Lines**

| | | | |
|---|---|---|---|
| spin 1/2 | incoming particle | $u(p)$ | |
| | outgoing particle | $\bar{u}(p)$ | |
| | incoming antiparticle | $\bar{v}(p)$ | |
| | outgoing antiparticle | $v(p)$ | |
| spin 1 | incoming photon | $\varepsilon^{\mu}(p)$ | |
| | outgoing photon | $\varepsilon^{\mu}(p)^{*}$ | |



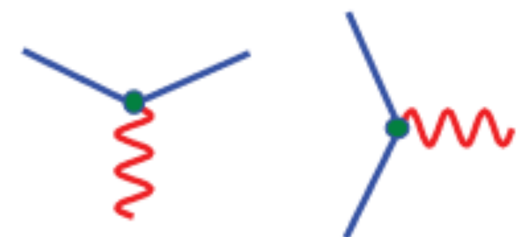
- **Internal Lines (propagators)**

| | | | |
|---|---|---|---|
| spin 1 | photon | $-\dfrac{ig_{\mu\nu}}{q^2}$ | |
| spin 1/2 | fermion | $\dfrac{i(\gamma^{\mu}q_{\mu}+m)}{q^2-m^2}$ | |




- **Vertex Factors**

| | | | |
|---|---|---|---|
| spin 1/2 | fermion (charge -\|$e$\|) | $ie\gamma^{\mu}$ | |

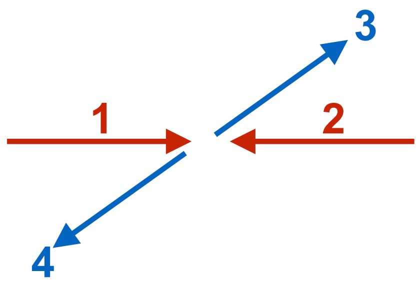


- **Matrix Element** $-iM$ = product of all factors

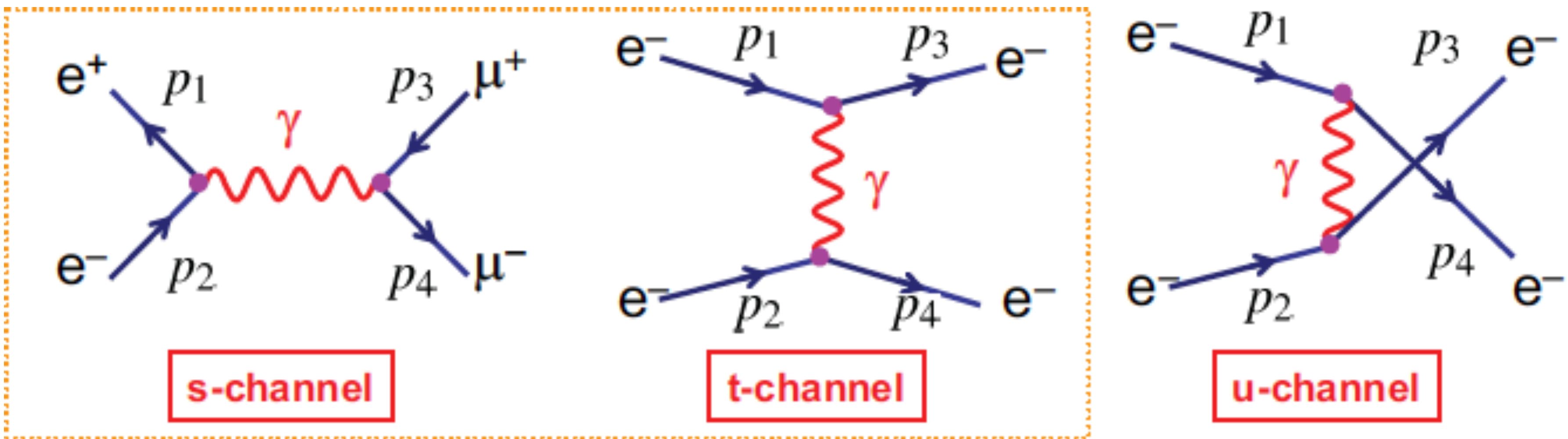
**We will come back to this to calculate real processes**

# Mandelstam variables (NPP recap)

- In particle annihilation/scattering there are 3 particularly useful **Lorentz Invariant** quantities: **s**, **t** and **u**




- (Simple) Feynman Diagrams can be categorised according to the 4-momentum of the exchanged particle




$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2$$

$$s + t + u = m_1^2 + m_2^2 + m_3^2 + m_4^2$$ — prove!

Show that $\sqrt{s}$ is the total energy of collision in the CoM frame

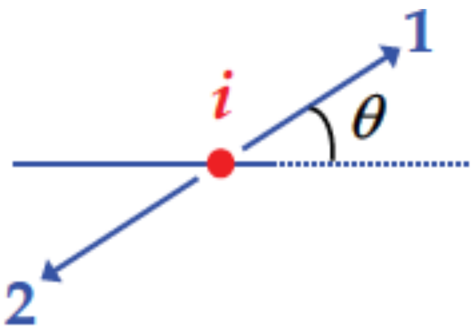
***You must be very confident at calculating particle interaction kinematics using 4-vectors (sure exam questions) — use problem sheet questions and beyond***

# Phase Space (ρ)

**FGR:** $\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)}$

We will now focus on calculating the phase space, starting from the case of **particle decay**

$$i \rightarrow 1 + 2$$




Use 1st order perturbation theory and Born approximation (particles as plane waves)

$$\psi = N e^{i(\vec{p}\cdot\vec{r} - Et)} = N e^{-p\cdot x} \quad (\vec{k}\cdot\vec{r} = \vec{p}\cdot\vec{r} \text{ as } \hbar = 1)$$

We will need:

- To recall non-relativistic phase space calculation and wave-function normalisation
- A couple of mathematical tricks — **Dirac δ-function**

Normalise to 1 particle in a cube with side *a*

$$\int \psi\psi^* \mathrm{d}V = N^2 a^3 = 1 \quad \Rightarrow \quad \boxed{N^2 = 1/a^3}$$

- Apply boundary conditions $(\vec{p} = \hbar\vec{k})$:
- Wave-function vanishing at box boundaries

  ➡ quantised particle momenta:

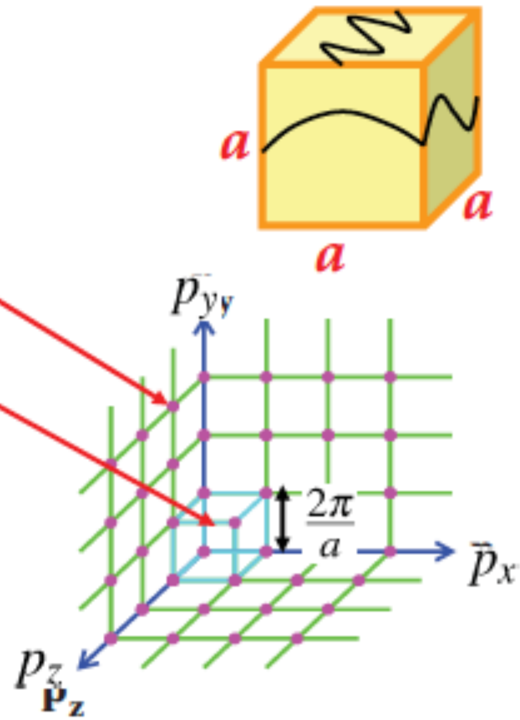
$$p_x = \frac{2\pi n_x}{a};\ p_y = \frac{2\pi n_y}{a};\ p_z = \frac{2\pi n_z}{a}$$

- Volume of single state in momentum space:

$$\left(\frac{2\pi}{a}\right)^3 = \frac{(2\pi)^3}{V}$$

- Normalising to **one particle/unit volume** gives number of states in element: $\mathrm{d}^3\vec{p} = \mathrm{d}p_x \mathrm{d}p_y \mathrm{d}p_z$

$$\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{\frac{(2\pi)^3}{V}} \times \frac{1}{V} = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}$$




- Therefore density of states in Golden rule:

$$\rho(E_f) = \left|\frac{\mathrm{d}n}{dE}\right|_{E_f} = \left|\frac{\mathrm{d}n}{\mathrm{d}|\vec{p}|}\frac{\mathrm{d}|\vec{p}|}{\mathrm{d}E}\right|_{E_f}$$

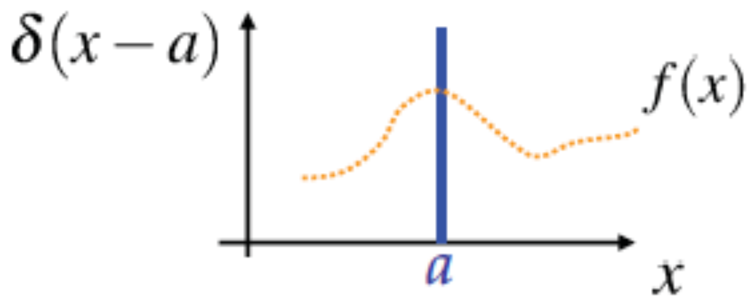
with $p = \beta E$

- Integrating over an elemental shell in momentum-space gives $(\mathrm{d}^3\vec{p} = 4\pi p^2 \mathrm{d}p)$

$$\rho(E_f) = \frac{4\pi p^2}{(2\pi)^3} \times \beta$$

# Math Appendix: Dirac δ-function

"Infinitely narrow spike of unit area"




$$\int_{-\infty}^{+\infty} \delta(x-a)\mathrm{d}x = 1$$

$$\int_{-\infty}^{+\infty} f(x)\delta(x-a)\mathrm{d}x = f(a)$$

- Any function with the above properties can represent $\delta(x)$

**e.g.** $$\delta(x) = \lim_{\sigma \to 0} \frac{1}{\sqrt{2\pi}\sigma} e^{-\left(\frac{x^2}{2\sigma^2}\right)}$$ (an infinitesimally narrow Gaussian)

- In relativistic quantum mechanics delta functions prove extremely useful for integrals over phase space, e.g. in the decay $a \to 1+2$

$$\int \ldots \delta(E_a - E_1 - E_2)\mathrm{d}E \quad \text{and} \quad \int \ldots \delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\mathrm{d}^3\vec{p}$$

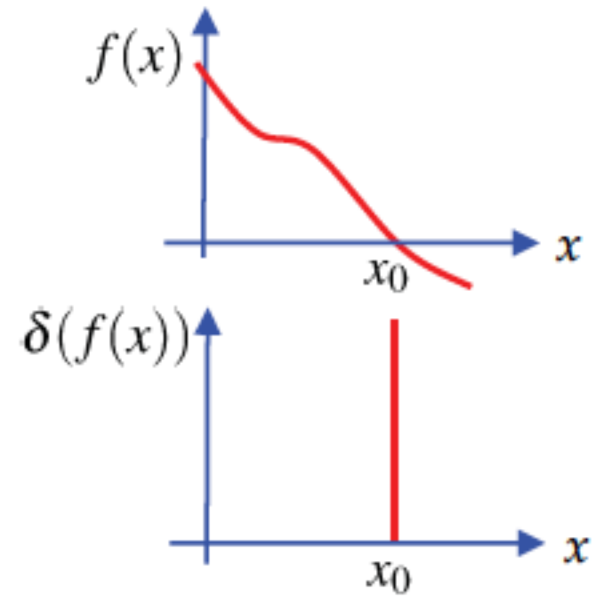
express energy and momentum conservation

# Math Appendix: Dirac δ-function

We'll also need an expression for the **delta function of a function** $\delta(f(x))$

$$\int_{y_1}^{y_2} \delta(y)\,dy = \begin{cases} 1 & \text{if } y_1 < 0 < y_2 \\ 0 & \text{otherwise} \end{cases}$$

Express $y = f(x)$ where $f(x_0) = 0$




Changing variables $$\int_{x_1}^{x_2} \delta(f(x))\frac{df}{dx}\,dx = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

Using properties of delta-function

$$\left|\frac{df}{dx}\right|_{x_0} \int_{x_1}^{x_2} \delta(f(x))\,dx = \begin{cases} 1 & \text{if } x_1 < x_0 < x_2 \\ 0 & \text{otherwise} \end{cases}$$

$$\int_{x_1}^{x_2} \delta(f(x))\,dx = \frac{1}{|df/dx|_{x_0}} \int_{x_1}^{x_2} \delta(x - x_0)\,dx \Rightarrow \boxed{\delta(f(x)) = \left|\frac{df}{dx}\right|_{x_0}^{-1} \delta(x - x_0)}$$

# Decay Rate Calculations

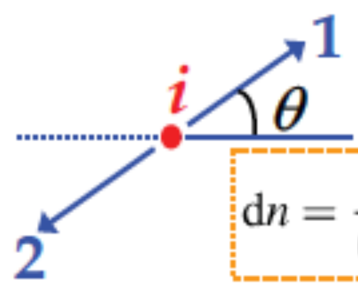
$i \rightarrow 1 + 2$

**FGR:** $\boxed{\Gamma_{fi} = 2\pi |T_{fi}|^2 \rho(E_f)}$

$$\rho(E_f) = \left|\frac{dn}{dE}\right|_{E_f} = \int \frac{dn}{dE}\delta(E - E_i)\,dE \qquad \text{since } E_f = E_i$$

Integrating over all final state energies, **energy conservation is "fixed" by delta-function**

$$\Gamma_{fi} = 2\pi \int \left|T_{fi}\right|^2 \delta(E_i - E)\,dn$$




$\boxed{\mathrm{d}n = \frac{\mathrm{d}^3\vec{p}}{(2\pi)^3}}$ "allowed" final states of any energy

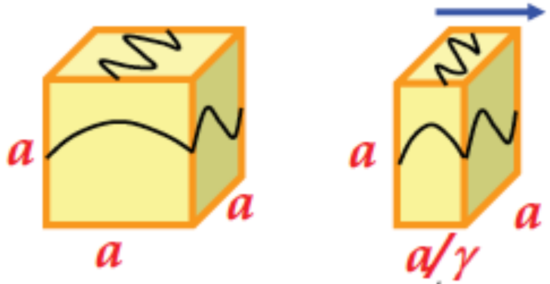
$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \underbrace{\delta(E_i - E_1 - E_2)}_{\text{Energy cons.}} \underbrace{\delta^3(\vec{p}_i - \vec{p}_1 - \vec{p}_2)}_{\text{Mom. cons.}} \underbrace{\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3}}_{\text{Density of states}}$$

---

but for 2-body decay can be integrated over 1particle momentum

# Lorentz Invariant Phase Space

- In non-relativistic QM normalise to one particle/unit volume: $\int \psi^* \psi \mathrm{d}V = 1$
- When considering relativistic effects, volume contracts by $\gamma = E/m$




- Particle density therefore increases by $\gamma = E/m$
  - ★ Conclude that a relativistic invariant wave-function normalisation needs to be **proportional to $E$ particles per unit volume**
- Usual convention: **Normalise to 2E particles/unit volume** $\int \psi'^* \psi' \mathrm{d}V = 2E$
- Previously used $\psi$ normalised to 1 particle per unit volume $\int \psi^* \psi \mathrm{d}V = 1$
- Hence $\psi' = (2E)^{1/2}\psi$ is normalised to $2E$ per unit volume
- Define Lorentz Invariant Matrix Element, $M_{fi}$ , in terms of the wave-functions normalised to $2E$ particles per unit volume

$$\boxed{M_{fi} = \langle \psi_1'.\psi_2'...|\hat{H}|...\psi_{n-1}'\psi_n'\rangle = (2E_1.2E_2.2E_3....2E_n)^{1/2}T_{fi}}$$

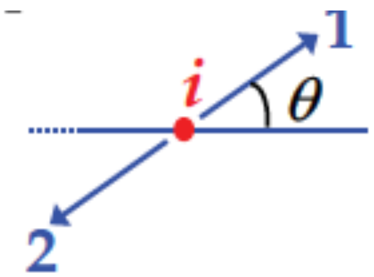
$$\boxed{\Gamma_{fi} = \frac{(2\pi)^4}{2E_i}\int |M_{fi}|^2 \delta(E_i - E_1 - E_2)\delta^3(\vec{p}_a - \vec{p}_1 - \vec{p}_2)\frac{\mathrm{d}^3\vec{p}_1}{(2\pi)^3 2E_1}\frac{\mathrm{d}^3\vec{p}_2}{(2\pi)^3 2E_2}}$$

In this form the **integral** is now **frame independent**, i.e. Lorentz Invariant

# Decay Rate Calculations

- Since the integral is L.I. we can choose the most convenient frame for our calculations
- For decay the most convenient frame is CoM:

$$E_i = m_i \quad \text{and} \quad \vec{p}_i = 0$$




$$\Gamma_{fi} = \frac{1}{8\pi^2 E_i}\int \left|M_{fi}\right|^2 \delta(m_i - E_1 - E_2)\delta^3(\vec{p}_1 + \vec{p}_2)\frac{d^3\vec{p}_1}{2E_1}\frac{d^3\vec{p}_2}{2E_2}$$

Using:

delta-function to integrate over $\vec{p}_2$ $\quad\quad \vec{p}_2 = -\vec{p}_1$

temporary notation (for simplicity) $\left|\vec{p}_1\right| = p_1$ and $E^2 = p^2 + m^2$

and $d^3\vec{p}_1 = p_1^2 dp_1 \sin\theta d\theta d\phi = p_1^2 dp_1 d\Omega$

$$\Rightarrow \quad \Gamma_{fi} = \frac{1}{32\pi^2 E_i}\int |M_{fi}|^2 \delta\left(m_i - \sqrt{m_1^2 + p_1^2} - \sqrt{m_2^2 + p_1^2}\right)\frac{p_1^2 \mathrm{d}p_1 \mathrm{d}\Omega}{E_1 E_2}$$

which can be rewritten as

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 g(p_1)\delta(f(p_1))\mathrm{d}p_1\mathrm{d}\Omega$$

where

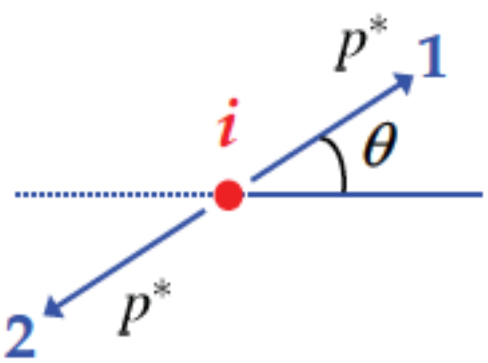
$$g(p_1) = p_1^2/(E_1E_2) = p_1^2(m_1^2+p_1^2)^{-1/2}(m_2^2+p_1^2)^{-1/2}$$

$$f(p_1) = m_i - (m_1^2+p_1^2)^{1/2} - (m_2^2+p_1^2)^{1/2}$$

**Note:**

- $\delta(f(p_1))$ imposes energy conservation.
- $f(p_1) = 0$ determines the C.o.M momenta of the two decay products

  i.e. $f(p_1) = 0$ for $p_1 = p^*$




Recall the property of the **delta function of a function**:

$$\int g(p_1)\delta(f(p_1))\mathrm{d}p_1 = \frac{1}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}} \int g(p_1)\delta(p_1 - p^*)\mathrm{d}p_1 = \frac{g(p^*)}{|\mathrm{d}f/\mathrm{d}p_1|_{p^*}}$$

where $p^*$ is the value for which $f(p^*) = 0$

Differentiating $df/dp_1$ and using the expression for $g(p_1)$ gives:

$$\Gamma_{fi} = \frac{1}{32\pi^2 E_i} \int |M_{fi}|^2 \left| \frac{p_1}{E_1 + E_2} \right|_{p_1 = p^*} d\Omega$$

But from $f(p_1) = 0$, i.e. energy conservation: $\mathrm{E}_1 + E_2 = m_i$

Moreover in the particle's **rest frame** $E_i = m_i$

Therefore:

$$\boxed{\frac{1}{\tau} = \Gamma_{fi} = \frac{|\vec{p}^*|}{32\pi^2 m_i^2} \int |M_{fi}|^2 \, d\Omega}$$

**Valid for all two-body decays!**

$p^*$ can be obtained from $f(p_1) = 0$

$$\Rightarrow \quad p^* = \frac{1}{2m_i}\sqrt{[(m_i^2 - (m_1 + m_2)^2)][(m_i^2 - (m_1 - m_2)^2)]}$$

# Cross-section once again

$$\sigma = \frac{\text{no of interactions per unit time per target}}{\text{incident flux}}$$

**Flux** = number of incident particles/ unit area/unit time

- The "cross section", σ, can be thought of as the <u>effective</u> cross-sectional area of the target particles for the interaction to occur.
- In general this has nothing to do with the physical size of the target although there are exceptions, e.g. neutron absorption

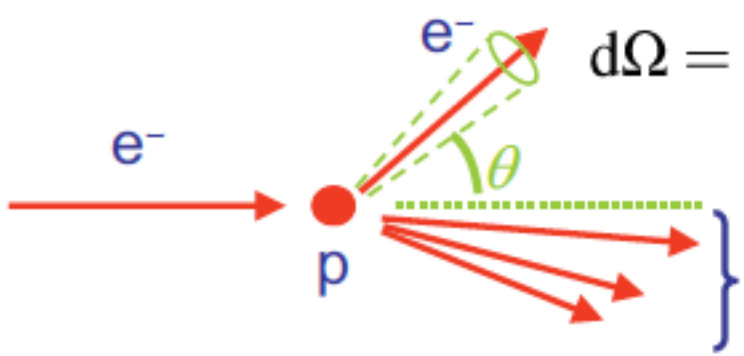
here σ is the projective area of nucleus

## Differential Cross section

$$\frac{d\sigma}{d\Omega} = \frac{\text{no of particles per sec/per target into } d\Omega}{\text{incident flux}}$$

or generally

$$\frac{d\sigma}{d\ldots}$$




$$d\Omega = d(\cos\theta)d\phi$$

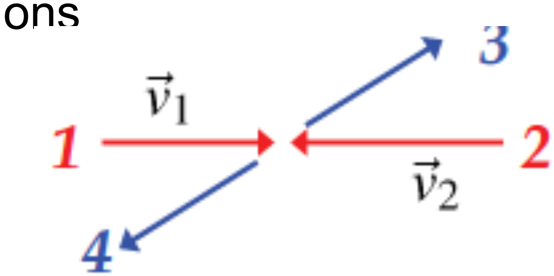
integrate over all other particles

with $\boxed{\sigma = \int \frac{d\sigma}{d\Omega} d\Omega}$

# Cross-section calculations

Now use our decay rate calculations to find analogous expressions for simple scattering process

$$1+2 \rightarrow 3+4$$




Start with **FGR**:

$$\Gamma_{fi} = (2\pi)^4 \int |T_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}$$

where $T_{fi}$ is the transition matrix for a normalisation of 1/unit volume

- Now $\text{Rate/Volume} = (\text{flux of } 1) \times (\text{number density of } 2) \times \sigma$

$$= n_1(v_1+v_2) \times n_2 \times \sigma$$

- For 1 target particle per unit volume $\text{Rate} = (v_1+v_2)\sigma$

$$\sigma = \frac{\Gamma_{fi}}{(v_1+v_2)}$$

$$\sigma = \underbrace{\frac{(2\pi)^4}{v_1+v_2}} \int \underbrace{|T_{fi}|^2} \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\underbrace{\frac{\mathrm{d}^3\vec{p}_3}{(2\pi)^3}\frac{\mathrm{d}^3\vec{p}_4}{(2\pi)^3}}$$

$\boxed{\text{the parts are not Lorentz Invariant}}$

To obtain a **L.I.** form use wave-functions normalised to $2E$ particles per unit volume

$$\psi' = (2E)^{1/2}\psi$$

Similarly to the decay case define **L.I.** matrix element

$$M_{fi} = (2E_1\,2E_2\,2E_3\,2E_4)^{1/2} T_{fi}$$

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int \underbrace{|M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{\mathrm{d}^3\vec{p}_3}{2E_3}\frac{\mathrm{d}^3\vec{p}_4}{2E_4}}_{\text{integral in L.I. form now}}$$

The quantity $F = 2E_1 2E_2(v_1+v_2)$ can be written in L.I. form $\quad F = 4[(p_1^\mu p_{2\mu})^2 - m_1^2 m_2^2]^{1/2}$

**Two important cases:**

- **Centre-of-Mass Frame**

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1E_2(|\vec{p}^*|/E_1 + |\vec{p}^*|/E_2) \\ &= 4|\vec{p}^*|(E_1+E_2) \\ &= 4|\vec{p}^*|\sqrt{s} \end{aligned}$$

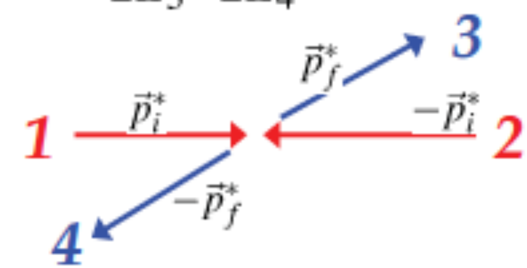
- **Target (particle 2) at rest**

$$\begin{aligned} F &= 4E_1E_2(v_1+v_2) \\ &= 4E_1 m_2 v_1 \\ &= 4E_1 m_2(|\vec{p}_1|/E_1) \\ &= 4m_2|\vec{p}_1| \end{aligned}$$

# 2→2 scattering in CoM frame

$$\sigma = \frac{(2\pi)^{-2}}{2E_1 2E_2(v_1+v_2)} \int |M_{fi}|^2 \delta(E_1+E_2-E_3-E_4)\delta^3(\vec{p}_1+\vec{p}_2-\vec{p}_3-\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

In CoM: $\vec{p}_1 + \vec{p}_2 = 0$ and $E_1 + E_2 = \sqrt{s}$




$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}} \int |M_{fi}|^2 \delta(\sqrt{s}-E_3-E_4)\delta^3(\vec{p}_3+\vec{p}_4)\frac{d^3\vec{p}_3}{2E_3}\frac{d^3\vec{p}_4}{2E_4}$$

This is the same integral as we had in the particle decay calculation (!)

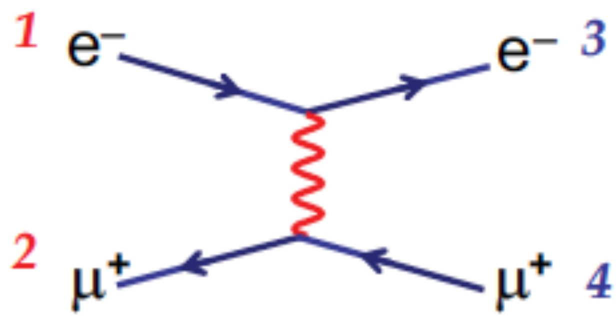
if $m_i$ is replaced with $\sqrt{s}$

$$\Rightarrow \quad \sigma = \frac{(2\pi)^{-2}}{4|\vec{p}_i^*|\sqrt{s}}\frac{|\vec{p}_f^*|}{4\sqrt{s}} \int |M_{fi}|^2 d\Omega^*$$

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{\left|\vec{p}_f^*\right|}{\left|\vec{p}_i^*\right|}\int \left|M_{fi}\right|^2 d\Omega^*}$$

In case of elastic scattering $\left|\vec{p}_i^{\,*}\right| = \left|\vec{p}_f^{\,*}\right|$

$$\sigma_{elastic} = \frac{1}{64\pi^2 s}\int \left|M_{fi}\right|^2 d\Omega^*$$

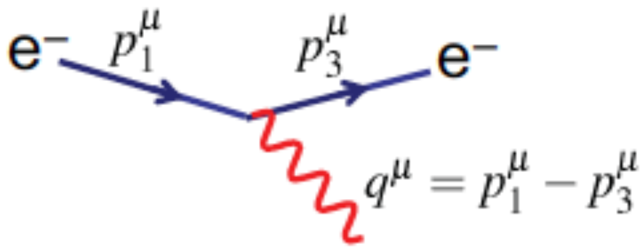



The total cross-section is **L.I.** but the differential cross-section is frame-specific due to $\mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^*$

To obtain an expression for **L.I.** $d\sigma$ use the Mandelstam variable

$$t = q^2 = (p_1 - p_3)^2$$

square of the **four-momentum transfer**



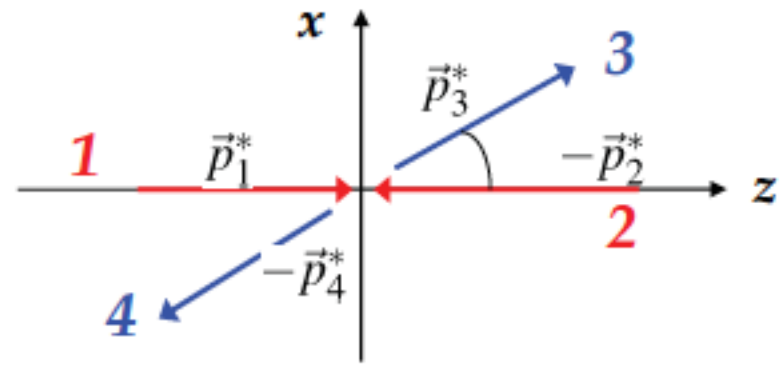

Express $d\Omega^*$ in terms of Mandelstam's $t$

# L.I. differential cross-section

$$t \equiv (p_1 - p_3)^2 = p_1^2 + p_3^2 - 2p_1.p_3 = m_1^2 + m_3^2 - 2p_1.p_3$$

- **In C.o.M. frame:**

$$\begin{aligned}
p_1^{*\mu} &= (E_1^*, 0, 0, |\vec{p}_1^*|) \\
p_3^{*\mu} &= (E_3^*, |\vec{p}_3^*|\sin\theta^*, 0, |\vec{p}_3^*|\cos\theta^*) \\
p_1^{\mu} p_{3\mu} &= E_1^* E_3^* - |\vec{p}_1^*||\vec{p}_3^*|\cos\theta^* \\
t &= m_1^2 + m_3^3 - E_1^* E_3^* + 2|\vec{p}_1^*||\vec{p}_3^*|\cos\theta^*
\end{aligned}$$




giving $\quad \mathrm{d}t = 2|\vec{p}_1^*||\vec{p}_3^*|\mathrm{d}(\cos\theta^*)$

therefore $\quad \mathrm{d}\Omega^* = \mathrm{d}(\cos\theta^*)\mathrm{d}\phi^* = \dfrac{\mathrm{d}t\,\mathrm{d}\phi^*}{2|\vec{p}_1^*||\vec{p}_3^*|}$

hence $\quad \mathrm{d}\sigma = \dfrac{1}{64\pi^2 s}\dfrac{|\vec{p}_3^*|}{|\vec{p}_1^*|}|M_{fi}|^2\mathrm{d}\Omega^* = \dfrac{1}{2\cdot 64\pi^2 s|\vec{p}_1^*|^2}|M_{fi}|^2\mathrm{d}\phi^*\mathrm{d}t$

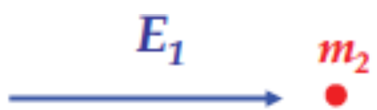
- Finally, integrating over $\mathrm{d}\phi^*$ (assuming no $\phi^*$ dependence of $|M_{fi}|^2$ ) gives:

$$\boxed{\frac{\mathrm{d}\sigma}{\mathrm{d}t} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}$$

$\left|\vec{p}_i^{\,*}\right|^2$ is a constant fixed by energy/momentum conservation

$$|\vec{p}_i^{\,*}|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

As an example of using the invariant differential cross-section $d\sigma / dt$ consider scattering of a relativistic particle with negligible mass, $E_1 >> m_1$ in a lab frame on $m_2$

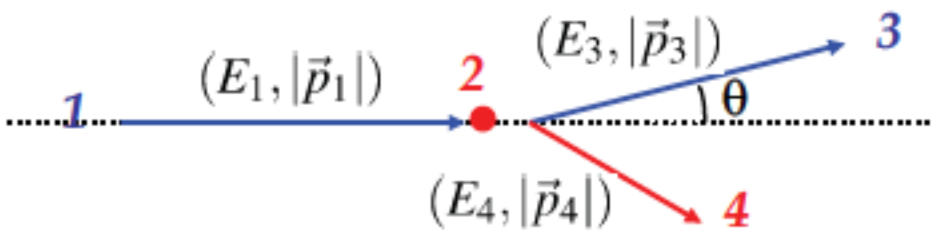



e.g. electron or neutrino scattering

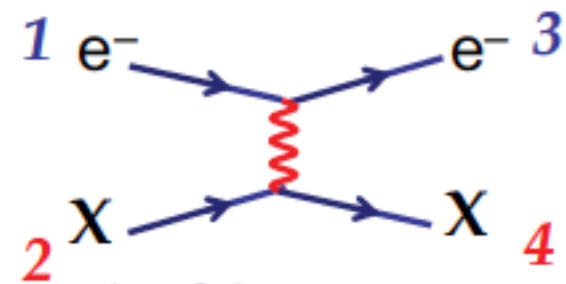
In this limit $$\left|\vec{p}_i^{\,*}\right|^2 = \frac{(s-m_2^2)^2}{4s}$$

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{16\pi(s-m_2^2)^2}\left|M_{fi}\right|^2}$$

- The other commonly occurring case is scattering from a fixed target in the Laboratory Frame (e.g. electron-proton scattering)
- First take the case of elastic scattering at high energy where the mass of the incoming particles can be neglected: $m_1 = m_3 = 0, \quad m_2 = m_4 = M$




**e.g.**




- Wish to express the cross section in terms of scattering angle of the e⁻

$$d\Omega = 2\pi d(\cos\theta)$$

therefore $$\frac{d\sigma}{d\Omega} = \frac{d\sigma}{dt}\frac{dt}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt}$$

Integrating over $d\phi$

- The rest is some rather tedious algebra.... start from four-momenta

$$p_1 = (E_1, 0, 0, E_1), \; p_2 = (M, 0, 0, 0), \; p_3 = (E_3, E_3\sin\theta, 0, E_3\cos\theta), \; p_4 = (E_4, \vec{p}_4)$$

so here $$t = (p_1 - p_3)^2 = -2p_1.p_3 = -2E_1E_3(1-\cos\theta)$$

But from (E,p) conservation $p_1 + p_2 = p_3 + p_4$

and, therefore, can also express $t$ in terms of particles 2 and 4

$$t = (p_2 - p_4)^2 = 2M^2 - 2p_2.p_4 = 2M^2 - 2ME_4$$
$$= 2M^2 - 2M(E_1 + M - E_3) = -2M(E_1 - E_3)$$

Note $E_1$ is a constant (the energy of the incoming particle) so

$$\frac{dt}{d(\cos\theta)} = 2M\frac{dE_3}{d(\cos\theta)}$$

- Equating the two expressions for $t$ gives $\boxed{E_3 = \frac{E_1 M}{M + E_1 - E_1\cos\theta}}$

so $$\frac{dE_3}{d(\cos\theta)} = \frac{E_1 M}{(M + E_1 - E_1\cos\theta)^2} = E_1^2 M\left(\frac{E_3}{E_1 M}\right)^2 = \frac{E_3^2}{M}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{2\pi}\frac{dt}{d(\cos\theta)}\frac{d\sigma}{dt} = \frac{1}{2\pi}2M\frac{E_3^2}{M}\frac{d\sigma}{dt} = \frac{E_3^2}{\pi}\frac{d\sigma}{dt} = \frac{E_3^2}{\pi}\frac{1}{16\pi(s-M^2)^2}|M_{fi}|^2$$

using $$s = (p_1 + p_2)^2 = M^2 + 2p_1.p_2 = M^2 + 2ME_1$$

Particle 1 massless ➡ $(p_1^2 = 0)$

gives $$(s - M^2) = 2ME_1$$

➡ $$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2|M_{fi}|^2}$$

In limit $m_1 \longrightarrow 0$

In this equation, $E_3$ is a function of $\theta$:

$$E_3 = \frac{E_1 M}{M + E_1 - E_1 \cos\theta}$$

giving

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2} \qquad (m_1 = 0)$$

## General form for 2→2 Body Scattering in Lab. Frame

The calculation when m1 cannot be neglected is longer but contains no additional physics. It gives:

$$\boxed{\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2} \cdot \frac{1}{p_1 m_1} \cdot \frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1 + m_2) - E_3|\vec{p}_1|\cos\theta} \cdot |M_{fi}|^2}$$

Again, θ is the only independent variable:

$$E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{\underbrace{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta}_{\vec{p}_4 = \vec{p}_1 - \vec{p}_3} + m_4^2}$$

i.e. $|\vec{p}_3|$ is a function of $\theta$

# Summary of useful results

Derived Lorentz Invariant expressions for decay rates and scattering cross-sections in terms of the **L.I. Matrix Element**

**Particle Decay:**

$$\boxed{\Gamma = \frac{|\vec{p}^*|}{32\pi^2 m_i^2}\int |M_{fi}|^2 \, d\Omega}$$

$$p^* = \frac{1}{2m_i}\sqrt{[(m_i^2-(m_1+m_2)^2)][(m_i^2-(m_1-m_2)^2)]}$$

**Scattering cross-section in CoM:**

$$\boxed{\sigma = \frac{1}{64\pi^2 s}\frac{|\vec{p}_f^*|}{|\vec{p}_i^*|}\int |M_{fi}|^2 \, d\Omega^*}$$

**Invariant differential cross-section:**

$$\boxed{\frac{d\sigma}{dt} = \frac{1}{64\pi s|\vec{p}_i^*|^2}|M_{fi}|^2}$$

$$|\vec{p}_i^*|^2 = \frac{1}{4s}[s-(m_1+m_2)^2][s-(m_1-m_2)^2]$$

# Summary of useful results (contd)

- Differential cross-section in the lab frame ($m_{projectile} = 0$)

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{E_3}{ME_1}\right)^2 |M_{fi}|^2 \Leftrightarrow \frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\left(\frac{1}{M + E_1 - E_1\cos\theta}\right)^2 |M_{fi}|^2$$

- Generic formula for $m_{projectile} \neq 0$

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\cdot\frac{1}{|\vec{p}_1|m_1}\cdot\frac{|\vec{p}_3|^2}{|\vec{p}_3|(E_1+m_2) - E_3|\vec{p}_1|\cos\theta}\cdot|M_{fi}|^2$$

with $E_1 + m_2 = \sqrt{|\vec{p}_3|^2 + m_3^2} + \sqrt{|\vec{p}_1|^2 + |\vec{p}_3|^2 - 2|\vec{p}_1||\vec{p}_3|\cos\theta + m_4^2}$

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- Having now dealt with kinematics we'll proceed to **matrix element** calculation. Fundamental particle physics is in there.
- But before doing that we will take a short detour to discuss symmetries and their role in particle physics (Module 2)