7 Green's Functions and Feynman Rules

In this section we will consider the evaluation of scattering matrix elements for a theory with the Lagrangian density

$$\mathcal{L} = \tfrac{1}{2}(\partial_\mu \phi(x))(\partial^\mu \phi(x)) - \tfrac{1}{2}m^2\phi^2(x) - \frac{\lambda}{4!}\phi^4(x). \tag{7.1}$$

This leads to a normal-ordered interaction term in the S-matrix operator of the type

$$-i\int d^4x : \mathcal{H}_{int}(x) := -i\frac{\lambda}{4!}\int d^4x : \phi^4(x) : . \tag{7.2}$$

Therefore, the Green's function we wish to calculate is of the form

$$G^{n+m} = \frac{\langle 0|T\left(\phi_I(x_1)\dots\phi_I(x_{n+m})\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\left(\frac{\lambda}{4!}\int d^4y : \phi_I(y) :\right)^k\right)|0\rangle}{\langle 0|T\left(\sum_{k=0}^{\infty}\frac{(-i)^k}{k!}\left(\frac{\lambda}{4!}\int d^4y : \phi_I(y) :\right)^k\right)|0\rangle}. \tag{7.3}$$

This is useful in the case where the coupling constant λ is small, since we can use eq. (7.3) to perform a perturbation expansion in powers of λ. (The normal-ordering of the interaction Hamiltonian is not necessary, but its absence would only result in unphysical constants in the same way as for the energy.)

In order to calculate the scattering matrix elements we have to calculate Green's functions of the type

$$\langle 0|T\big(\phi_I(x_1)\dots\phi_I(x_l)\big)|0\rangle. \tag{7.4}$$

In order to do this we need something called Wick's theorem. Consider the time-ordered product

$$T(A_1(x_1)A_2(x_2)) = \Theta(t_1-t_2)A_1(x_1)A_2(x_2) + \Theta(t_2-t_1)A_2(x_2)A_1(x_1). \tag{7.5}$$

By commuting operators so that all annihilation operators lie to the right of all creation operators we create a normal-ordered product part plus a number. The number is then equal to the vacuum expectation value of the product of the two operators since the normal-ordered part has zero vacuum expectation value by definition. For example, denoting the creation part of the operator by A^+ and the annihilation operator part by A^-,

$$\begin{aligned}
A_1(x_1)A_2(x_2) &= (A_1^+(x_1)+A_1^-(x_1))(A_2^+(x_2)+A_2^-(x_2)) \\
&= A_1^+(x_1)A_2^+(x_2) + A_1^-(x_1)A_2^-(x_2) + A_1^+(x_1)A_2^-(x_2) + A_2^+(x_2)A_1^-(x_1) \\
&+ [A_1^-(x_1), A_2^+(x_2)] \\
&= : A_1(x_1)A_2(x_2) : + \langle 0|[A_1^-(x_1), A_2^+(x_2)]|0\rangle \\
&= : A_1(x_1)A_2(x_2) : + \langle 0|A_1(x_1)A_2(x_2)|0\rangle.
\end{aligned} \tag{7.6}$$

Therefore, for the time-ordered product

$$\begin{aligned} T(A_1(x_1)A_2(x_2)) &= :A_1(x_1)A_2(x_2): (\Theta(t_1-t_2)+\Theta(t_2-t_1)) \\ &+ \langle 0|(A_1(x_1)A_2(x_2)\Theta(t_1-t_2)+A_2(x_2)A_1(x_1)\Theta(t_2-t_1)|0\rangle \\ &= :A_1(x_1)A_2(x_2): + \langle 0|T(A_1(x_1)A_2(x_2))|0\rangle, \end{aligned} \tag{7.7}$$

where the normal-ordered product $:A_1(x_1)A_2(x_2):$ is the same as $:A_2(x_2)A_1(x_1):$, and similarly for higher number of operators. This expression for time-ordered products can be extended to an arbitrary number of operators, e.g.

$$\begin{aligned} T(A_1(x_1)A_2(x_2)A_3(x_3)) =: A_1(x_1)A_2(x_2)A_3(x_3): &+ :A_1(x_1): \langle 0|T(A_2(x_2)A_3(x_3))|0\rangle \\ + :A_2(x_2): \langle 0|T(A_1(x_1)A_3(x_3))|0\rangle &+ :A_3(x_3): \langle 0|T(A_1(x_1)A_2(x_2))|0\rangle, \end{aligned} \tag{7.8}$$

and

$$\begin{aligned} T(A_1(x_1)A_2(x_2)A_3(x_3)A_4(x_4)) =: A_1(x_1)A_2(x_2)A_3(x_3)A_4(x_4): \\ + :A_1(x_1)A_2(x_2): \langle 0|T(A_3(x_3)A_4(x_4))|0\rangle + \text{perms.} \\ + \langle 0|T(A_1(x_1)A_2(x_2))|0\rangle |T(A_3(x_3)A_4(x_4))|0\rangle + \text{perms.} \end{aligned} \tag{7.9}$$

It is now easy to see the general form, and this expression is known as Wick's theorem. However, we must note that when calculating time-ordered products including terms in $\mathcal{H}_{int}(x)$ all fields within the interaction Hamiltonian at a given point in space-time are already normal ordered, and no commutations are needed to make them so. Hence, none of the $\langle 0|T(A_1(x_1)A_2(x_2))|0\rangle$ terms will contain two terms from the interaction Hamiltonian at one particular point.

When we calculate $\langle 0|T\big(\phi_I(x_1)\dots\phi_I(x_l)\big)|0\rangle$ and S-matrix elements, all terms involving any normal-ordered products will vanish. Hence, the results for S-matrix elements will be expressed in the form

$$P\langle 0|T(\phi(x_1)\phi(x_2))|0\rangle\cdots\langle 0|T(\phi(x_{l-1})\phi(x_l))|0\rangle, \tag{7.10}$$

where P contains all the integrals in d^4y and factors of $-i\lambda/4!$ coming from the terms involving the interaction Hamiltonian. Thus, as a first step we need to know what $\langle 0|T(\phi(x)\phi(y))|0\rangle$ is. Using our previous expression for $\phi(x)$

$$\begin{aligned} \langle 0|(\phi(x)\phi(y))|0\rangle &= \langle 0|\int \frac{d^3\mathbf{p}}{2E(\mathbf{p})(2\pi)^3}\int \frac{d^3\mathbf{q}}{2E(\mathbf{q})(2\pi)^3}\big(a_{\mathbf{p}}e^{-ip\cdot x}+a^\dagger_{\mathbf{p}}e^{ip\cdot x}\big)\big(a_{\mathbf{q}}e^{-iq\cdot y}+a^\dagger_{\mathbf{q}}e^{iq\cdot y}\big)|0\rangle \\ &= \langle 0|\int \frac{d^3\mathbf{p}}{2E(\mathbf{p})(2\pi)^3}\int \frac{d^3\mathbf{q}}{2E(\mathbf{q})(2\pi)^3}(a_{\mathbf{p}}e^{-ip\cdot x}a^\dagger_{\mathbf{q}}e^{iq\cdot y})|0\rangle. \end{aligned} \tag{7.11}$$

However,

$$\begin{aligned} \langle 0|a_{\mathbf{p}}a^\dagger_{\mathbf{q}}|0\rangle &= \langle 0|[a_{\mathbf{p}},a^\dagger_{\mathbf{q}}]|0\rangle \\ &= (2\pi)^3 2E(\mathbf{p})\delta^3(\mathbf{p}-\mathbf{q}). \end{aligned} \tag{7.12}$$

Inserting this into eq. (7.11) we obtain

$$\langle 0|(\phi(x)\phi(y))|0\rangle = \int \frac{d^3\mathbf{p}}{2E(\mathbf{p})(2\pi)^3} e^{ip\cdot(y-x)}. \tag{7.13}$$

Therefore our 2-point time-ordered Green's function is

$$\begin{aligned} G_F(x-y) &= \theta(t-t')\langle 0|(\phi(x)\phi(y))|0\rangle + \theta(t'-t)\langle 0|(\phi(y)\phi(x))|0\rangle \\ &= \int \frac{d^3\mathbf{p}}{2E(\mathbf{p})(2\pi)^3}\big(\theta(t-t')e^{ip\cdot(y-x)} + \theta(t'-t)e^{ip\cdot(x-y)}\big). \end{aligned} \tag{7.14}$$

Using the method of contour integration this can be shown to be equal to

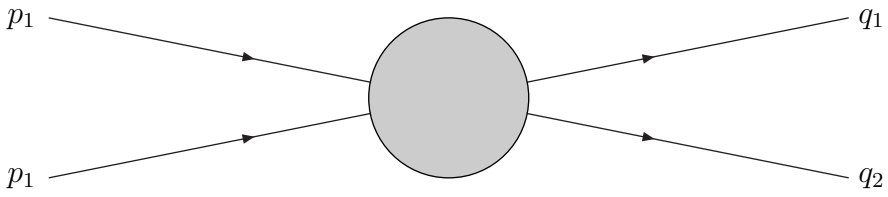
$$G_F(x-y) = \int i\frac{d^4p}{(2\pi)^4}\frac{e^{-ip\cdot(x-y)}}{p^2-m^2+i\epsilon}, \tag{7.15}$$

where ϵ is an infinitesimal constant. This is known as the Feynman propagator, and represents a free particle created at y and annihilated at x if $t > t'$ and created at x and annihilated at y if $t' > t$. It is easy to show that this Feynman Green's function is a Green's function of the Klein-Gordon operator:

$$(\partial_x^2 + m^2)G_F(x-y) = -i\delta^4(x-y). \tag{7.16}$$

Other Green's functions of this operator may be found, amounting to different prescriptions for the form of the denominator in eq. (7.15), e.g. we could have $-i\epsilon$ rather than $+i\epsilon$. The prescription in eq. (7.15) corresponds to the physical interpretation of a particle travelling forwards in time as described above.

Using Wick's theorem and our form for the propagator eq. (7.15), we are now equipped to calculate the Green's function eq. (7.3) appropriate for the scattering matrix elements. Let us begin with two particles with momentum p_1 and p_2 scattered to two particles with momentum q_1 and q_2, as presented below.



Let us calculate $G^4(x_1, x_2, x_3, x_4)$ to first order in λ. To this order the denominator in (7.3) is

$$\langle 0|T\big(1 - i\lambda/4!\int d^4y : \phi^4(y) :\big)|0\rangle = 1. \tag{7.17}$$

The numerator is

$$\langle 0|T\big(\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\big)|0\rangle - \frac{i\lambda}{4!}\langle 0|T\big(\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\int d^4y : \phi^4(y) :\big)|0\rangle. \tag{7.18}$$

Using Wick's theorem the first of these terms is

$$\langle 0|T\big(\phi_I(x_1)\phi_I(x_2)\phi_I(x_3)\phi_I(x_4)\big)|0\rangle = G_F(x_1-x_2)G_F(x_3-x_4) \quad + \quad \text{perms.} \tag{7.19}$$

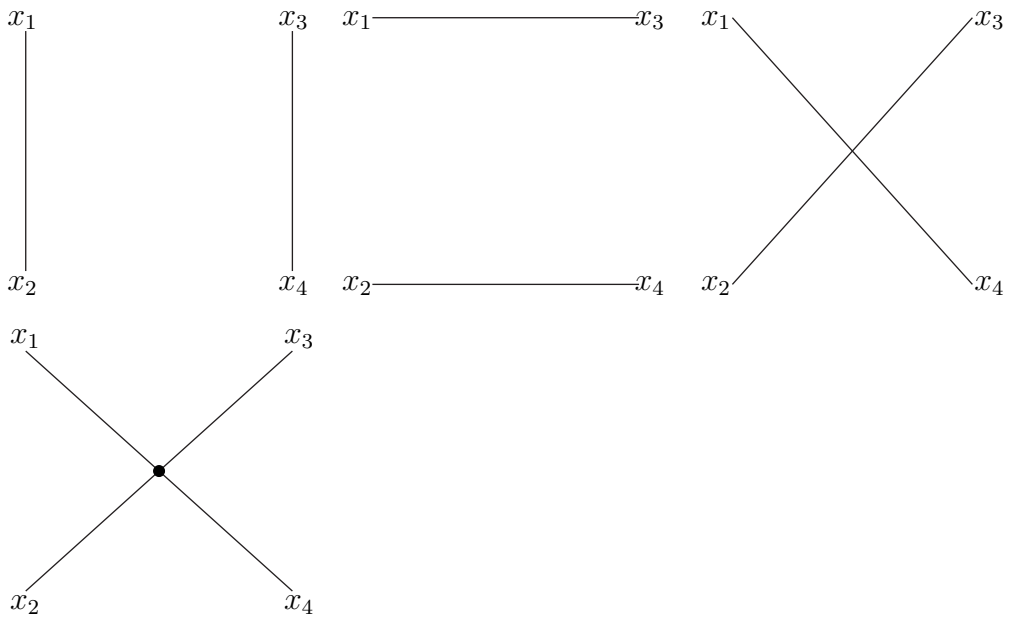
The second term is

$$-\frac{i\lambda}{4!}\int d^4y\, G_F(x_1-y)G_F(x_2-y)G_F(x_3-y)G_F(x_4-y)\times 4!, \tag{7.20}$$

where the normal ordering means there are no contractions between any of the $\phi_I(y)$ terms and the 4! factor come from the number of ways of making the contraction, i.e. $\phi_I(x_1)$ may contract with any of the 4 $\phi_I(y)$ fields, leaving 3 remaining fields for $\phi_I(x_2)$ to contract with etc. Thus to first order in λ

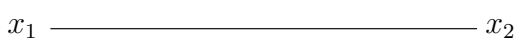
$$\begin{aligned} G^4(x_1,x_2,x_3,x_4) &= G_F(x_1-x_2)G_F(x_3-x_4) \quad + \quad \text{perms.} \\ &\quad - i\lambda\int d^4y G_F(x_1-y)G_F(x_2-y)G_F(x_3-y)G_F(x_4-y). \end{aligned} \tag{7.21}$$

This may be represented diagrammatically as

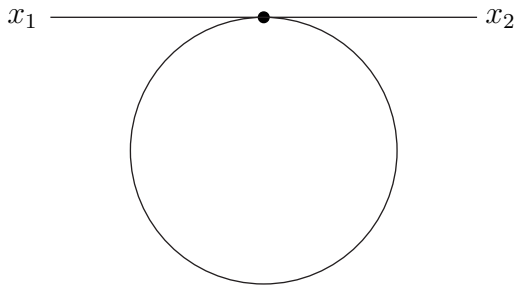


where the interaction vertex where 4 fields meet is represented by $-i\lambda/4!\int d^4y$, the lines represent the Feynman propagator $G_F(x-y)$ and the number of ways of drawing the diagram is counted. The upper set of diagrams may be thought of as the particles simply propagating without interactions, while the lower one may be thought of as two free particles created at say x_1 and x_2, propagating freely until they scatter at space-time point y, and two resulting free particles propagating to x_3 and x_4. Since the interaction may be anywhere we integrate over all y.

We can also look at the 2-point function out to second order in λ. At zeroth order we have

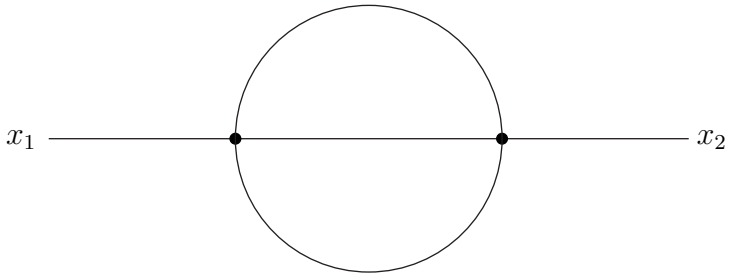


At first order we would have



but this diagram is not allowed due to the normal ordering, which allows no lines to start and end at the same vertex.

At second order we have



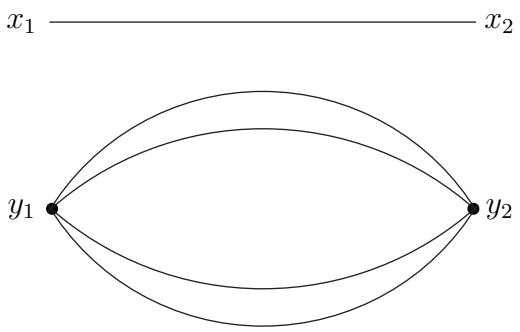
which is equal to

$$\left(\frac{-i\lambda}{4!}\right)^2 \int d^4y_1 \int d^4y_2\, G_F^3(y_1 - y_2) G_F(x_1 - y_1) G_F(y_2 - x_2) \times C, \tag{7.22}$$

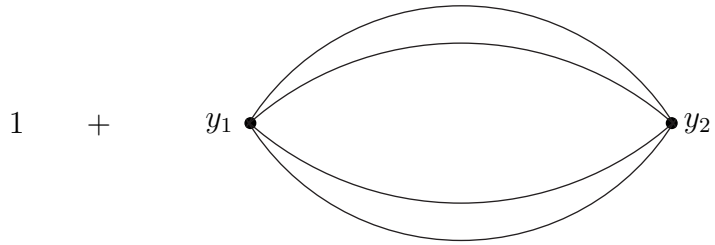
where C is the number of ways to join up the diagram, $C = 4 \times 4!$. Therefore, this diagram is equal to

$$-\frac{\lambda^2}{6} \int d^4y_1 \int d^4y_2\, G_F^3(y_1 - y_2) G_F(x_1 - y_1) G_F(y_2 - x_2). \tag{7.23}$$

Also in principle we would get in the numerator for the 2-point function



i.e. an unconnected diagram leads to the numerator being $\sim G_F(x_1 - x_2)(1 - \lambda^2 K + \cdots)$ (ignoring the connected diagram contribution at $\mathcal{O}(\lambda^2)$). However, in the denominator we also get



which is equal to $(1 - \lambda^2 K + \cdots)$. So the effect of the denominator is to cancel out the unconnected part of the numerator leaving just $G_F(x_1 - x_2)$ in this case above. It is possible to show that the total effect of the denominator is to factor out all unconnected diagrams containing vacuum sub-diagrams in the numerator just as in the example above. We also ignore all unconnected diagrams without vacuum sub-diagrams since these represent processes where not all the particles have been involved in a scattering process. Hence we only need to consider connected diagrams in the calculation of eq. (7.3).

Having seen the above few examples it should be clear that there is a pattern emerging for the calculation of the Green's functions. The method of calculation may be summarised by the Feynman rules. In order to calculate the Green's function for $n + m$ external particles one must:

1. Draw all distinct, connected diagrams with $n + m$ external lines (legs) and any number of 4-point vertices.
2. Assign a factor $-i\lambda/4! \int d^4 y_k$ to the vertex k.
3. Assign a factor $G_F(x_i - x_j)$ to the line $x_i \to x_j$.
4. Multiply by the number of equivalent ways to draw the diagram.

All diagrams may be thought of as being built up out of the propagation of free particles which have local 4-point particle scatterings at particular space-time points with strength denoted by the coupling constant λ. Thus, even though we are really working in a theory where the fields interact, by using the Interaction picture and a perturbation expansion we can work in a basis of free fields and treat the interactions as something analogous to elastic scattering of these free particles. As long as the coupling λ is small this simplistic picture will provide very accurate results.