PHASM426 / PHASG426
Advanced Quantum Theory Problem Sheet 4

Deadline: Wednesday 11th January 2017 at 13:00

Please bring the work to me in my office (B12) on Wed 11th January 2017 between 12:00 and 13:00, leave it in my pigeon hole (**at your own risk**) or scan your work and email it to me as **a single PDF file** that does not exceed 5 MB. In any case, please make sure your completed work is clearly labelled with your name and college, and **stapled** if you are handing in a paper version.

1. Consider H is the total Hamiltonian descrbing a quantum system interacting with an environment. The total state of the system and environment $\rho(t)$ is given by the unitary evolution $U(t)$ associated to the total H and at time $t = 0$ we have $\rho(t = 0) = \rho_s(0) \otimes \rho_B(0)$, where the inital state of the environment is $\rho_B(0) = |B_0\rangle\langle B_0|$. Here the basis set $\{B_k\}$ spans the enviornment states.

   (a) Show that the reduced density matrix operator for the system $\rho_s(t)$ takes the form
   $$\rho_s(t) = \sum_k S_k \rho_s(0) S_k^\dagger$$
   where $S_k = \langle B_k|U(t)|B_0\rangle$. [5]

   Model Answer: The combined state of systema and environment is given by:
   $$\rho(t) = U(t)\rho(0)U^\dagger(t) = U(t)\rho_S(0)|B_0\rangle\langle B_0|U^\dagger(t).$$
   $$\begin{aligned}\rho_s(t) &= \mathrm{tr}_B\{\rho(t)\}\\ &= \mathrm{tr}_B\{U(t)\rho_S(0)|B_0\rangle\langle B_0|U^\dagger(t)\}\\ &= \sum_k \langle B_k|U(t)\rho_S(0)|B_0\rangle\langle B_0|U^\dagger(t)|B_k\rangle\\ &= \sum_k \langle B_k|U(t)|B_0\rangle\rho_S(0)S_k^\dagger\\ &= \sum_k S_k\rho_S(0)S_k^\dagger\end{aligned}$$

   Marks:2 marks for writting $\rho(t)$, 1 marks for each of the lines. Partial marks for partial answers.

   (b) Discuss whether S_k is an operator acting on the system or on the environment, or wether it is an expected value. [2]

Model Answer: $U(t)$ is an operator defined in terms of tensor produc of operators acting on the system and on the environment. Since we project onto $\langle B_k|$, then S_k is an opertor acting on the system. **Marks:**1 mark for indicating that $U(t)$ is an operator on both. 1 mark for correct justification of why S_k is an operator on the system.

(c) Show that $\sum_k S_k^\dagger S_k = \mathbb{1}$ and discuss the physical meaning of this result.

[3]

Model Answer:

$$\begin{aligned}\sum_k S_k^\dagger S_k &= \sum_k \langle B_0|U^\dagger(t)|B_k\rangle\langle B_k|U(t)|B_0\rangle \\ &= \langle B_0|U^\dagger(t)\Big(\sum_k |B_k\rangle\langle B_k|\Big)U(t)|B_0\rangle \\ &= \langle B_0|U^\dagger(t)U(t)|B_0\rangle \\ &= \langle B_0|B_0\rangle = \mathbb{1}.\end{aligned}$$

In the above we have used $\sum_k |B_k\rangle\langle B_k| = \mathbb{1}$ and $U^\dagger(t)U(t) = \mathbb{1}$. This implies that S_k are Krauss operators and $\rho_S(t)$ satisfies all the physical properties of a physical density operator.

Marks:1 mark for using $\sum_k |B_k\rangle\langle B_k| = \mathbb{1}$, 1 mark for using $U^\dagger(t)U(t) = \mathbb{1}$ and 1 mark for stating correctly the physical implications of the result..

2. Consider a two-level atom with excited state $|e\rangle$ and ground state $|g\rangle$ such that its Hamiltonian is $H = \hbar\omega|e\rangle\langle e|$. The action of the environment interacting with the atom is described by the jump operators $L_1 = \Gamma|e\rangle\langle g|$ and $L_2 = \gamma|g\rangle\langle e|$.

(a) Assuming that at $t = 0$ the state of the atom is $\rho(0) = |g\rangle\langle g|$, show that the probability of finding the atom in the excited state at time t , $\rho_{ee}(t) = \langle e|\rho(t)|e\rangle$, is given by

$$\rho_{ee}(t) = \frac{|\Gamma|^2}{|\gamma|^2 + |\Gamma|^2}\Big(1 - e^{-(|\gamma|^2+|\Gamma|^2)t}\Big).$$

[10]

Model Answer:

$$H_{eff} = \left(\hbar\omega - i\hbar|\gamma|^2/2\right)|e\rangle\langle e| - (i\hbar/2)|\Gamma|^2|g\rangle\langle g|$$

$$\begin{aligned} L_1\rho(t)L_1^\dagger &= |\Gamma|^2\rho_{gg}(t)|e\rangle\langle e| \\ L_2\rho(t)L_2^\dagger &= |\gamma|^2\rho_{ee}(t)|g\rangle\langle g| \end{aligned}$$

where $\rho_{gg}(t) = \langle g|\rho(t)|g\rangle$ and $\rho_{ee}(t) = \langle e|\rho(t)|e\rangle$.
Now compute:

$$\begin{aligned} \frac{d}{dt}\rho_{ee}(t) =& \langle e|\frac{d\rho(t)}{dt}|e\rangle \\ =& (-i/\hbar)\left(\langle e|H_{eff}\rho(t)|e\rangle - \langle e|\rho(t)H_{eff}^\dagger|e\rangle\right) + \sum_j \langle e|L_j\rho(t)L_j|e\rangle. \end{aligned}$$

Each element in the above expression becomes:

$$\begin{aligned} \langle e|H_{eff}\rho(t)|e\rangle &= \left(\hbar\omega - i\hbar|\gamma|^2/2\right)\rho_{ee}(t), \\ \langle e|\rho(t)H_{eff}^\dagger|e\rangle &= \left(\hbar\omega + i\hbar|\gamma|^2/2\right)\rho_{ee}(t), \\ \langle e|L_1\rho(t)L_1|e\rangle &= |\Gamma|^2\rho_{gg}(t) = |\Gamma|^2(1-\rho_{ee}(t)), \\ \langle e|L_2\rho(t)L_2|e\rangle &= 0. \end{aligned}$$

The differential equation that $\rho_{ee}(t)$ satisfies is given by:

$$\frac{d\rho_{ee}(t)}{dt} + (|\Gamma|^2 + |\gamma|^2)\rho_{ee}(t) = \Gamma|^2$$

A linear differential equation of the form $\frac{dy}{dx} + ay = b$ with a and b real numbers, has the general solution

$$y(x) = e^{-ax}\left(\frac{b}{a}e^{ax} + \kappa\right),$$

where κ is to be determined by initial conditions. Using this and the initial conditon $\rho_{ee}(0) = 0$, we obtain:

$$\rho_{ee}(t) = \frac{|\Gamma|^2}{|\gamma|^2 + |\Gamma|^2}\left(1 - e^{-(|\gamma|^2+|\Gamma|^2)t}\right).$$

Marks:3 Marks for given correct expressions for H_{eff} and $L_j\rho(t)L_j$, 2 marks for projecting these operators correctly i.e. $\langle e|H_{eff}\rho(t)|e\rangle$ 2 marks for arriving to the right differential equation and 3 Marks for solving correctly the differential equation.

(b) Find a relation between Γ and γ such that in the long-time limit $\rho_{ee}(\infty)$ equals the probability of finding the atom in its excited state when it is in thermal equilibrium at temperature T. Recall that in thermal equilibrium, a system with Hamiltonian H is described by the density matrix operator $\rho_{eq} = \frac{\exp(-H/k_BT)}{\text{Tr}[\exp(-H/k_BT)]}$. Express your answer as

$$|\Gamma|^2 = C|\gamma|^2$$

and specify the value of C as a function of ω and k_BT where k_B is the Boltzman constant. [5]

Model Answer: The question asks for the conditions in which $\langle e|\rho_{eq}|e\rangle = \rho_{ee}(\infty)$. The thermal equilibrium state is defined as:

$$\rho_{eq} = \frac{\exp\left(-H/k_BT\right)}{\text{Tr}[\exp(-H/k_BT)]} = \frac{\exp\left(-\hbar\omega|e\rangle\langle e|/k_BT\right)}{\text{Tr}[\exp\left(-\hbar\omega|e\rangle\langle e|/k_BT\right)]}.$$

In the above

$$\begin{aligned}\text{Tr}[\exp\left(-\hbar\omega|e\rangle\langle e|/k_BT\right)] &= \langle g|\text{exp}(-\hbar\omega|e\rangle\langle e|/k_BT)|g\rangle + \langle e|\text{exp}(-\hbar\omega|e\rangle\langle e|/k_BT)|e\rangle \\ &= 1 + \exp\left(-\hbar\omega/k_BT\right).\end{aligned}$$

Then, the population of the excited state in thermal equilibrium becomes

$$\begin{aligned}\langle e|\rho_{eq}|e\rangle &= \frac{\exp\left(-\hbar\omega/k_BT\right)}{1+\exp\left(-\hbar\omega/k_BT\right)} \\ &= \frac{1}{\exp\left(\hbar\omega/k_BT\right)+1}.\end{aligned}$$

When $t \to \infty$ we have that

$$\rho_{ee}(\infty) = \frac{|\Gamma|^2}{|\Gamma|^2 + |\gamma|^2}.$$

We find C by equating the above expression to $\langle e|\rho_{eq}|e\rangle$, that is,

$$\frac{|\Gamma|^2}{|\Gamma|^2 + |\gamma|^2} = \frac{1}{\exp\left(\hbar\omega/k_BT\right)+1}.$$

We then find that

$$|\Gamma|^2 = \exp(-\hbar\omega/k_BT)|\gamma|^2,$$

from this we can conclude that $C = \exp(-\hbar\omega/k_BT)$.

Marks: 2 for deriving correctly the thermal equilibrium state, 1 for deriving correctly the population of the excited state in thermal equilibrium, 1 for writting the correct relationship between $|\Gamma|^2$ and $|\gamma|^2$.
