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Module - I

→ The set of Real Numbers R . $R^2, R^3, \dots R^n$ is the set of polynomials P_n.

$R^2 = \begin{bmatrix} a \\ b \end{bmatrix}$ is the column matrix where $a, b \in R$.

$R^3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is the column matrix where $a, b, c \in R$.

→ Vectors in R^3 are 3×1 column matrices with 3 entries they are represented geometrically by points in a 3 dimmensional space.

Example: Let $u, v \in R^3$

$$u = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \quad a_1, b_1, c_1 \in R$$

$$v = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix} \quad a_2, b_2, c_2 \in R$$

Suppose k is any scalar

$$k \cdot u = \begin{bmatrix} ka_1 \\ kb_1 \\ kc_1 \end{bmatrix} \in R^3$$

$$u + v = \begin{bmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{bmatrix} \in R^3$$

→ Addition and scalar multiplication of vectors in R^3 are defined.

→ If n is a positive integer R^n denotes the set of all ordered n tuples of n real numbers usually written as $n \times 1$ column matrices such as

Let $u, v \in R^n$

$$u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \qquad v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

→ Addition and scalar multiplication of vectors in R^n are defined.

* Vector space : (viva)

Let V be a non-empty set of objects called vectors on which are defined two operations called addition and multiplication by scalars (Real Numbers) subject to the 10 axioms (or) roots listed below the axioms must hold for all the vectors u, v and w in V and for all the scalars c and d.

1 to 5 → Addition
6 to 10 → Multiplication

1. $u+v \in V$
2. $u+v = v+u$
3. $(u+v)+w = u+(v+w)$
4. There is a zero vector 0 in V such that $u+0=u$.
5. For each u in v there exist a vector $-u$ in v such that $u+(-u)=0$.
6. The scalar multiple of u by c denoted by $cu \in V$.
7. $c(u+v) = cu+cv$.
8. $(c+d)u = cu+du$.
9. $c(du) = (cd)u$.
10. $1.u = u$.

Example: To prove the spaces R^n where $n \geq 1$ is a vector spaces.

$$R^n = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}, a_1, a_2, \ldots\ldots a_n \in R \right\}.$$

Sol: Let $u, v \in R^n$

consider $u = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$, $v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ $\begin{matrix} a_1, a_2, \ldots\ldots a_n \\ b_1, b_2, \ldots\ldots b_n \end{matrix} \in R.$

① $u+v = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = \begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} \in R^n$

$\therefore a_1+b_1, a_2+b_2, a_3+ \ldots\ldots a_n+b_n \in R.$

$\therefore u+v \in R^n.$

② $v+u = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} b_1+a_1 \\ b_2+a_2 \\ \vdots \\ b_n+a_n \end{bmatrix} \in R^n.$

$\therefore$ commutative property holds in R.

$\therefore \boxed{u+v = v+u}$

③ $(u+v)+w = \left[\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}\right] + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$

$= \begin{bmatrix} (a_1+b_1)+c_1 \\ (a_2+b_2)+c_2 \\ \vdots \\ (a_n+b_n)+c_n \end{bmatrix} \in R^n$

$\therefore (u+v)+w \in R^n$

$u+(v+w) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \left[\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}\right]$

$= \begin{bmatrix} a_1+(b_1+c_1) \\ a_2+(b_2+c_2) \\ \vdots \\ a_n+(b_n+c_n) \end{bmatrix} \in R^n$

$\therefore u+(v+w) \in R^n$

$\therefore$ Associative law holds in R.

$\therefore (u+v)+w = u+(v+w)$.

④ Let $0 \in V \ni u+0 = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = u$.

$u+0=u$.

$\therefore \exists\ 0 \in R^n$

$\therefore$ Identity property satisfies.

⑤ For any $u \in V \ \exists \ -u = \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix} \in R^n$ where $-a_1, -a_2, \ldots -a_n \in R$.

$\therefore u+(-u) = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} -a_1 \\ -a_2 \\ \vdots \\ -a_n \end{bmatrix}$

$= u+(-u) = 0$.

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⑥ For any $c \in R$, $u \in V$

$c \cdot u = c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} \in V$.

$\because ca_1, ca_2, \ldots\ldots ca_n \in R$.

⑦ For any $c \in R$, $u, v \in V$.

$= c(u+v) = c\left[\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}\right] = c\begin{bmatrix} a_1+b_1 \\ a_2+b_2 \\ \vdots \\ a_n+b_n \end{bmatrix} = \begin{bmatrix} ca_1+cb_1 \\ ca_2+cb_2 \\ \vdots \\ ca_n+cb_n \end{bmatrix}$

$= \begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} + \begin{bmatrix} cb_1 \\ cb_2 \\ \vdots \\ cb_n \end{bmatrix} = cu+cv$.

⑧ Let $c, d \in R$, $u \in V$

To prove $(c+d)u = cu+du$.

consider $(c+d)\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} (c+d)a_1 \\ (c+d)a_2 \\ \vdots \\ (c+d)a_n \end{bmatrix} = \begin{bmatrix} ca_1+da_1 \\ ca_2+da_2 \\ \vdots \\ ca_n+da_n \end{bmatrix} =$

$$\begin{bmatrix} ca_1 \\ ca_2 \\ \vdots \\ ca_n \end{bmatrix} + \begin{bmatrix} da_1 \\ da_2 \\ \vdots \\ da_n \end{bmatrix} = c\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} + d\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$$

$$= cu+du.$$

(9) $c, d, \in R$, $u \in V$

$$c(du) = c\begin{bmatrix} da_1 \\ da_2 \\ \vdots \\ da_n \end{bmatrix} = \begin{bmatrix} c(da_1) \\ c(da_2) \\ \vdots \\ c(da_n) \end{bmatrix} = \begin{bmatrix} cda_1 \\ cda_2 \\ \vdots \\ cda_n \end{bmatrix}$$

$$= cd\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = (cd)u.$$

(10) $1 \cdot u = u.$

$$1 \cdot u = 1\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \quad \therefore 1 \cdot u = u.$$

$\therefore (R^n, +, \cdot)$ is a Vector Space.

Problem:

→ For $n \geq 0$. Prove that P_n is a set of polynomials of degree at most n of the form $P(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$ where $a_1, a_2, \ldots, a_n$ and t are real numbers in a vector space.

Sol: Let $P(t), q(t) \in V$

$P(t) = a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n$

$q(t) = b_0 + b_1 t + b_2 t^2 + \dots + b_n t^n$, $b_0, b_1, \dots b_n, t \in R$

① $p(t) + q(t) = (a_0 + b_0) + (a_1 + b_1)t + \dots + (a_n + b_n)t^n \in V.$

② $q(t) + p(t) = (b_0 + a_0) + (b_1 + a_1)t + \dots + (b_n + a_n)t^n \in V.$

$\therefore p(t) + q(t) = q(t) + p(t).$

③ $(u+v)+w$

④ 0 polynomials with all $a_0, a_1, \dots a_n$ as 0 acts as a zero vector $\bar{0}$. Therefore, $p(t) + \bar{0} = P(t)$.

⑥ Scalar

Let $c \in R$

$c(P(t)) = c(a_0 + a_1 t + a_2 t^2 + \dots + a_n t^n)$

$= (ca_0) + (ca_1)t + \dots + (c(a_n))t^n \in V$

$= (ca_0) + (ca_1)t + \dots + (ca_n)t^n \in V.$

$\because ca_0, ca_1, \dots ca_n t \in R$

$\therefore P_n$ the set of polynomials with respective addition, multiplication.

Theorem:

Statement: If V is a vector space

(i) $\bar{0}$ zero vector is unique.

(ii) If $u \in V$. Then the $-u$ is unique in V.

Proof: Given V is a vector space.

Therefore the axioms 1 to 10 are satisfied in V.

For any $u \in V \ \exists \ \bar{0} \in V \ni u + \bar{0} = u \rightarrow ①$

Suppose w is an another vector

$u + w = u \ \forall \ u \in V \longrightarrow ②$

Suppose $\bar{0} \in V$

$\bar{0} + w = \bar{0}$ (By ②)

$\bar{0} + w = w$ (By ①)

$\therefore w = 0 + w = \bar{0}$

$\Rightarrow \boxed{w = \bar{0}}$

$\therefore \bar{0}$ is unique.

If $u \in V$, $\exists\ -u \in V \ni u + (-u) = \bar{0}$.

Let w be an another vector $\ni u + w = \bar{0}$.

$(-u) + u + w = (-u) + \bar{0}$.

$\bar{0} + w = -u$.

$\boxed{w = -u}$.

Thus $-u$ is the unique inverse of $u \in V$.

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Theorem:

Statement: Let V be a vector space. Then

(i) $0 \cdot u = \bar{0}$ For every $u \in V$

(ii) $c \cdot \bar{0} = \bar{0}$ For every scalar c.

(iii) If $u \in V$ Then $(-1)u = -u$.

Proof: (i) $0u = (0+0)u$

$= 0u + 0u$. (By distributive)

By adding negative of $0u$ on both sides.

$0u + (-0u) = (0u + 0u) + (-0u)$

$\bar{0} = 0u + \bar{0}$

$\boxed{\bar{0} = 0 \cdot u}$ (By Additive Identity)

(ii) $c \cdot \bar{0} = c(\bar{0} + \bar{0})$

$c\bar{0} = c\bar{0} + c\bar{0}$

Adding negative of $c \cdot \bar{0}$ on b.s

$$c\bar{0}+(-c\bar{0}) = c\bar{0}+c\bar{0}+(-c\bar{0})$$

$$\bar{0} = c\bar{0}+\bar{0}$$

$$\boxed{\bar{0} = c\bar{0}}$$

(iii) Let $u \in V$. Then consider

$$u+(-1)u = 1\cdot u+(-1)u.$$
$$= [1+(-1)]u.$$
$$= 0\cdot u \quad [\text{By Using property } 1].$$
$$= \bar{0}$$
$$u+(-1)u = \bar{0}$$

Adding negative of u on both sides.

$$u+(-1)u+(-u) = \bar{0}+(-u)$$
$$\bar{0}+(-1)u = \bar{0}+(-u)$$
$$\boxed{(-1)u = -u.}$$

Subspaces :

A subspace of a vector space V is a subset H of V that has the three following properties:

(i) The zero vector of V is in H.

(ii) H is closed under vector addition that is for each $u, v \in H$, $u+v \in H$.

(iii) H is closed under scalar multiplication that is for each $u \in H$ and each scalar c the vector $cu \in H$.

Theorem:

statement: A subset w of a vector space v iff and only if the following condition holds if u and v are any two vectors in w then for any two scalars c, d the vector $cu+dv$ is in w.

Proof: Given w is a subspace of v and v is a vector

space. (Necessary condition)
~~(To prove $cu+dv \in W$.)~~

$$cu+dv \in W$$

To prove W is a subspace of V.

If $c=0, d=0$

$0 \cdot u + 0 \cdot v$

$\Rightarrow \bar{0} + \bar{0}$

$\boxed{\bar{0} \in W}$

$c=1, d=1$

$u+v \in W$

$d=0$

$cu + 0 \cdot v \in W$

$cu \in W$.

$\therefore$ W is a subspace of V.

(sufficient condition)

given W is a subspace of V.

To prove $cu+dv \in W$

Let $u, v \in W$

consider two scalars c, d

$cu \in W$ & $dv \in W$ (By Additive property).

$cu+dv \in W$.

Problems:

1) Let V be the first quadrant in xy plane i.e,

Let $V = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, x \geq 0, y \geq 0 \right\}$

(i) u and v are in V is $u+v$ in V? why?

(ii) Find a specific vector ~~in~~ u in V and a specific scalar c such that cu is not in V.

Sol: $V = \left\{ \begin{bmatrix} x_1 \\ y_1 \end{bmatrix}, x_1 \geq 0, y_1 \geq 0 \right\}$

$u = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \in V$ $\quad v = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \in V$

$u+v = \begin{pmatrix} 3 \\ 7 \end{pmatrix} \in V$, because $u+v \in Q_1$ in XY-plane.

$\therefore$ $u+v$ is in V because its entries will both be non-negative.

Let $c = -1$ be a scalar

$$-1 \cdot u = \begin{bmatrix} -1 \\ -3 \end{bmatrix} \notin V.$$

2) Let w be the union of first and third quadrants in the xy-plane that is let $w = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, xy \geq 0 \right\}$.
(i) If u is in w and c is any scalar is cu in w?
(ii) Find specific vectors u and v in w such that $u+v$ is not in w?

Sol: Given $w = \left\{ \begin{bmatrix} x \\ y \end{bmatrix}, xy \geq 0 \right\}$

Let us consider

$u = \begin{bmatrix} -5 \\ -10 \end{bmatrix} \in W$ $\quad v = \begin{bmatrix} 3 \\ 11 \end{bmatrix} \in w$

$$u+v = \begin{bmatrix} -2 \\ 1 \end{bmatrix} \notin w.$$

Vector addition is not possible.

w is not a vector space.

Problem On subspace:

* Determine if the set H of all matrices of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ is a subspace of $M_{2\times2}$.

Sol: V is a Vector space of $M_{2\times2}$ matrices.

$H \subseteq V$.

$$H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} a, b, d \in R \right\}$$

To prove H is a subspace of V.

Let $0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \in H$

Let $u = \begin{bmatrix} a_1 & b_1 \\ 0 & d_1 \end{bmatrix}$ $a_1, b_1, d_1, \in R.$

$V = \begin{bmatrix} a_2 & b_2 \\ 0 & d_2 \end{bmatrix}$ $a_2, b_2, d_2 \in R.$

$u, v \in H$

$$u + v = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 + 0 & d_1 + d_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ 0 & d_1 + d_2 \end{bmatrix} \in H.$$

For any scalar

$$c \cdot u = \begin{bmatrix} ca_1 & cb_1 \\ 0 & cd_1 \end{bmatrix} \in H.$$

$\therefore$ H is a subspace of V.

Theorem:

Statement: Let H and K be subspaces of a vector space V. The intersection of H and K, written as $H \cap K$, is the set of v in V that belongs to both H and K. Show that $H \cap K$ is a subspace of V.

Proof: Given H and K are subspaces of a vector space V.

$\therefore$ H is a subspace of vectorspace in V.

$$\bar{0} \in H$$

K is a subspace of v in V.

$$\bar{0} \in K$$

$$\bar{0} \in H \cap K \rightarrow ①$$

$$u+v \in H$$
$$u+v \in K.$$

$$u+v \in H \cap K \rightarrow ②$$

For any scalar c, f $u \in H \Rightarrow c \cdot u \in H$.
For any scalar c f $u \in K \Rightarrow c \cdot u \in K$.

$$cu \in H \cap K \rightarrow ③.$$

$\therefore$ $H \cap K$ is a subspace of V.

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Note :

The set consisting of only the zero vector in a vector space v is a subspace of V called the zero subspace and written as $\{0\}$.

Problem:

1. The set $H = \left\{ \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} ; s, t \in R \right\}$ is a subset of R^3. show that H is a subspace of R^3.

Sol: Let $u = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix}$, $v = \begin{bmatrix} d \\ e \\ 0 \end{bmatrix}$ where $s, t, d, e \in R$.

$$\exists\ \bar{0} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \in H.$$

① $u+v = \begin{bmatrix} s \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} d \\ e \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} s+d \\ t+e \\ 0+0 \end{bmatrix} = \begin{bmatrix} s+d \\ t+e \\ 0 \end{bmatrix} \in H.$$

Let c is any scalar

$$cu = \begin{bmatrix} cs \\ ct \\ 0 \end{bmatrix} \in H.$$

$\therefore$ H is a subspace of R^3.

2) ~~Determine if the set H of all matrices of the form~~

2) Let F be a fixed 3×2 Matrix and let H be the set of all matrices A in $M_{2\times4}$ with the property that $FA = 0$ (The zero matrix in $M_{3\times4}$). Determine If H is a subspace $M_{2\times4}$.

Sol: $M_{2\times4} = \left\{ \begin{bmatrix} x_1 & x_2 & x_3 & x_4 \\ x_5 & x_6 & x_7 & x_8 \end{bmatrix}, \begin{array}{l} x_i \in R \\ i = 1,2,\ldots\ldots 8 \end{array} \right\}$

$$H = \{ A_{2\times4} \ni FA = 0 \}.$$

condition: (i) The zero vector $M_{2\times4}$ is $0 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore$ clearly this is in H, since

$$F_{3\times2}\, 0_{2\times4} = 0_{3\times4}.$$

(ii) Let $A, B \in H$

$\therefore A_{2\times4}$ & $F.A = 0$.

$\therefore B_{2\times4}$ & ~~B~~ $F.B = 0$.

To prove $A+B \in H$

$(A+B)_{2\times4}$

$$F(A+B) = FA + FB$$
$$= 0 + 0$$
$$= 0_{3\times4}.$$

$\therefore A+B \in H.$

For any scalar $c \Rightarrow F[CA] = c[FA]$

$$= c \cdot 0$$

$$= O_{3\times4}.$$

$\therefore CA \in H$

$\therefore H$ is a subspace of $M_{2\times4}$.

3) Give an example in R^2 to show that the union of two subspaces is not a subspace.

Sol: Let H, K be the two subspaces of R^2 and defining the H as

$$H = \left\{ \begin{bmatrix} x \\ -x \end{bmatrix} ; x \in R \right\} \rightarrow \{(x,-x), x \in R\}$$ ordered space

$$K = \left\{ \begin{bmatrix} x \\ 2x \end{bmatrix} ; x \in R \right\} \rightarrow \{(x, 2x), x \in R\}$$

$$H \cup K = \{(x,-x), (x,2x) ; x \in R\}.$$

$u = (2,-2) \in H \cup K$,

$v = (3,6) \in H \cup K$.

$u+v = (5,4) \notin H \cup K$.

$\therefore H \cup K$ is not a subspace.

Theorem:

Statement:

If H and K are two subspaces of a vector space V, then H+K is also a subspace of V.

Proof: Given H and K are subspaces of a vector space V to prove H+K is a subspace of V.

Since H is a subspace of V so $H \subseteq V$.

Since K is a ~~subset~~ subspace of V $K \subseteq V$ by additive property

$H+K \subseteq V$.

(i) $\bar{0} \in H$, $\bar{0} \in K$

$\bar{0} = \bar{0} + \bar{0} \in H+K$.

$\therefore \bar{0} \in H+K$.

① Let $u, v \in H+K$

$u = h_1 + k_1$ where $h_1 \in H$, $k_1 \in H$.
$v = h_2 + k_2$ where $h_2 \in H$, $k_2 \in H$.

$u+v = (h_1+h_2) + (k_1+k_2) \in H+K$.

$\because h_1 + h_2 \in H$
$k_1 + k_2 \in H$.

② Let c is any scalar
$u \in H+K$
$cu = c(h_1+k_1)$
$= ch_1 + ck_1$ [$\because$ H, K are subspaces]

$\Rightarrow cu \in H+K$

$\therefore$ H+K is a subspace of a vectorspace V.

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Linear Combinations And Span:

Given vectors $v_1, v_2, v_3 \ldots v_n$ and given scalars $c_1, c_2, \ldots c_n$. Then the vector y defined by $y = c_1v_1 + c_2v_2 + \ldots + c_nv_n$ is called a linear combination of $v_1, v_2, \ldots v_n$ with the coefficients $c_1, c_2, \ldots, c_n$.

Note:
The coefficients in the linear combination could be any real numbers ~~or~~ including zero.

Definition Of A Span:

Let $v_1, v_2, \ldots v_p$ be any set of p vectors. Then the set of all linear combinations of $v_1, v_2 \ldots v_p$ is denoted by span$\{v_1, v_2, \ldots, v_p\}$ and is called the set spanned by $v_1, v_2 \ldots v_p$. which means that span of $\{v_1, v_2, \ldots v_p\}$ is the collection of all vectors that can be written in the form $c_1 v_1 + c_2 v_2 + \ldots + c_p v_p$ with the scalars $c_1, c_2 \ldots c_p$.

Note:

1. span $\{v_1, v_2 \ldots v_p\}$ contains every scalar multiple of each vector. Let us consider v_1 then $c_1 v_1 =$ ~~$c_1 v_1$~~ $c_1 v_1 + 0 v_2 + \ldots + 0 v_p$.

2. In particular zero vector must be in span $\{v_1, v_2 \ldots v_p\}$ since $\bar{0} = 0v_1 + 0v_2 + \ldots + 0v_p$ is also a linear combination of $v_1, v_2 \ldots v_p$.

Theorem:

Statement: If v_1 & v_2 are in vector space V then $H = $ span$\{v_1, v_2\}$ is a subspace of V.

Proof: Given v_1, v_2 are two vectors in a Vector space V.

Given $H = \text{span}\{v_1, v_2\} = \{c_1 v_1 + c_2 v_2 : c_1, c_2 \in R\}$

(i) The zero vector $\bar{0}$ of V is in H since

$$\bar{0} = 0 \cdot v_1 + 0 \cdot v_2$$

$$\therefore \bar{0} \in H.$$

Hence zero vector $\bar{0}$ of V is in H.

(ii) Let $u, v \in H$

$u = c_1 v_1 + c_2 v_2$, $v = d_1 v_1 + d_2 v_2$ where $c_1, c_2, d_1, d_2 \in R$.

$u + v = (c_1 v_1 + c_2 v_2) + (d_1 v_1 + d_2 v_2)$

$= (c_1+d_1)v_1 + (c_2+d_2)v_2$

$\therefore\ u+v \in H.$

(iii) d is any scalar.

$du = d(c_1v_1 + c_2v_2)$

$= (dc_1)v_1 + (dc_2)v_2 \in H$

$\therefore$ H is a subspace of Vector space (V).

Thus $H = \text{span}\{v_1, v_2\}$ is a subspace of V.

1. Let W be the set of all vectors of the form given below a, b, c represents arbitrary real numbers in each case either find a set S of vectors that spans W, or give an example to show W is not a vector space.

(i) $\begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$

Sol: $y = \begin{bmatrix} -a+1 \\ a-6b \\ 2b+a \end{bmatrix}$

$= a\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} + b\begin{bmatrix} 0 \\ -6 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = av_1 + bv_2 + cv_3.$

$y = av_1 + bv_2 + cv_3.$

$= \text{span}\{v_1, v_2, v_3\}.$

$\therefore$ W is a subspace

Let us consider

$S = \text{span}\{v_1, v_2, v_3\}.$

(ii) $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix}$

Sol: given $\begin{bmatrix} a-b \\ b-c \\ c-a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \end{bmatrix} + c\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}$

$= av_1 + bv_2 + cv_3$.

$=$ span$\{v_1, v_2, v_3\}$.

$\therefore$ W is a subspace.

(iii) $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix}$

Sol: given $\begin{bmatrix} 4a+3b \\ 0 \\ a+b+c \\ c-2a \end{bmatrix} = a\begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$

$= av_1 + bv_2 + cv_3$

$=$ span $\left\{ \begin{bmatrix} 4 \\ 0 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}$.

$\therefore$ W is a subspace.

(iv) $\begin{bmatrix} s+3t \\ s-t \\ 2s-t \\ 4t \end{bmatrix} \Rightarrow s\begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix} + t\begin{bmatrix} 3 \\ -1 \\ -1 \\ 4 \end{bmatrix} \Rightarrow sv_1 + tv_2$.

The Nullspace of a Matrix:

The Nullspace of $m\times n$ matrix A is written as $\text{Nul}A$ is the set of all solutions to the homogeneous equation $A\mathbf{x}=0$.

In set notation $\text{Nul}A = \{x : x \text{ is in } R^n \ \& \ Ax=0\}$.

Problem:

1) Let A be a matrix is given as $A=\begin{bmatrix}-1 & -3 & -2\\ -5 & 9 & 1\end{bmatrix}$ and let $u=\begin{bmatrix}5\\3\\-2\end{bmatrix}$. Determine if $u\in$ null space of A.

Sol:

$$Au=\begin{bmatrix}-1 & -3 & -2\\ -5 & 9 & 1\end{bmatrix}\begin{bmatrix}5\\3\\-2\end{bmatrix}$$

$$=\begin{bmatrix}5-9+4\\ -25+27-2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}=\bar{0}.$$

$$= u\in \text{Nul}A.$$

2) Determine if $w=\begin{bmatrix}5\\-3\\2\end{bmatrix}$ is in $\text{Nul}A$ where $A=\begin{bmatrix}5 & 21 & 19\\ 13 & 23 & 2\\ 8 & 14 & 1\end{bmatrix}$

Sol:

$$WA=\begin{bmatrix}5\\-3\\2\end{bmatrix}\begin{bmatrix}5 & 21 & 19\\ 13 & 23 & 2\\ 8 & 14 & 1\end{bmatrix}$$

$$=\begin{bmatrix}25 & -63 & 38\\ 65 & -69 & 4\\ 40 & 42 & 2\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}=\bar{0}.$$

$$\therefore w\in \text{Nul}A.$$

Theorem:
Statement:
The null space of $m \times n$ matrix A is a subspace of R^n equivalently the set of all solution to a system $Ax = 0$ of m homogeneous linear equation in n annonce is a subspace of R^n.

Proof: By definition of Nul A (Nul A is a subset of R^n). (Nul $A \subseteq R^n$).

$\bar{0}$ is obviously is an element of Nul A.

(ii) Let $u, v \in$ Nul A.

To prove $u+v \in$ Nul A.

$$A \cdot u = 0, \quad A \cdot v = 0 \longrightarrow ①$$

consider

$$A(u+v) = Au + Av.$$
$$= 0 + 0$$
$$= 0.$$

$\therefore u+v \in$ Nul A.

③ Let c is any scalar.

$$A(cu) = c \cdot Au$$
$$= c \cdot 0$$
$$= 0.$$

$\therefore cu \in$ Nul A.

$\therefore$ Nul A is a subspace of R^n.

Problem:

1. Let H be the set of all vectors in R^4 whose coordinates a, b, c, and d satisfied eqn's $a - 2b + 5c = d$ and $c - a = b$. Show that H is a subgroup of R^4.

Sol: By rearranging the eqn's that described the elements of H we see that H is the set of all solutions of the following system of homogeneous linear eqn's.

$$a-2b+5c-d=0$$

$$-a \mp b+c=0.$$

$$\boxed{AX=0}$$

H is a subspace of R4.

(OR)

$$\boxed{c=b+a}$$

$$a-2b+5(b+a)=d.$$

$$a-2b+5b+5a=d.$$

$$6a+3b=d.$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ b \\ b+a \\ 6a+3b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \\ 1 \\ 6 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \text{span}\{u_1, u_2\}.$$

2) Find a spanning set for the null space of the matrix $A = \begin{bmatrix} 1 & -2 & 0 & -1 & 3 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Sol: $AX=0$.

Let $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} \Rightarrow \begin{array}{l} x_1-2x_2-x_4+3x_5=0 \\ x_3+2x_4-2x_5=0. \end{array}$

$$\begin{bmatrix} 2x_2 + x_4 + 3x_5 \\ x_2 \\ -2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$\Rightarrow x_2 u_1 + x_4 u_2 + x_5 u_3.$

$\Rightarrow \text{span}\{u_1, u_2, u_3\}.$

Every linear combination of u_1, u_2 and u_3 is an element of nul A. Therefore the set u_1, u_2, u_3 is a spanning set for Nul A.

3) Check whether the given set w is a vector space or not. If it is a vector space find the spanning set of w.

$$w = \left\{ \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} ; \quad \begin{matrix} a - 2b = 4c \\ 2a = c + 3d \end{matrix} \right\}.$$

Sol: $2a = c + 3d$

$$\boxed{a = \frac{c+3d}{2}}$$

$$\frac{c+3d}{2} - 2b = 4c.$$

$$c + 3d - 4b = 8c.$$

$$c + 3d - 8c = 4b.$$

$$\boxed{b = \frac{3d - 7c}{4}}$$

$$\boxed{a = \frac{c+3d}{2}} \quad \boxed{b = \frac{3d-7c}{4}}$$

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} \frac{c+3d}{2} \\ \frac{3d-7c}{4} \\ c \\ d \end{bmatrix} = c\begin{bmatrix} \frac{1}{2} \\ \frac{-7}{4} \\ 1 \\ 0 \end{bmatrix} + d\begin{bmatrix} \frac{3}{2} \\ \frac{3}{4} \\ 0 \\ 1 \end{bmatrix}$$

$$= cu_1 + du_2$$

$$\text{span} = \{u_1, u_2\}$$

$\therefore$ The spanning set of $w = \{u_1, u_2\}$.

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Exercise Problems:

1. Find an explicit description of Nul A by listing vectors that span the Nullspace.

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}$$

Sol:

$$A = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}_{2\times 4}$$

$$AX = 0$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

$$AX = \begin{bmatrix} 1 & 3 & 5 & 0 \\ 0 & 1 & 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \Rightarrow \begin{aligned} x_1 + 3x_2 + 5x_3 &= 0. \\ x_2 + 4x_3 - 2x_4 &= 0. \end{aligned}$$

$$X = \begin{bmatrix} -3x_2 - 5x_3 \\ x_2 \\ x_3 \\ \frac{1}{2}x_2 + 2x_3 \end{bmatrix}$$

$$\Rightarrow x_2 \begin{bmatrix} -3 \\ 1 \\ 0 \\ \frac{1}{2} \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

$$\Rightarrow x_2 u + x_3 v.$$

$\therefore$ The set $\{u, v\}$ is a spanning set for Nul$A$.

2) $W = \left\{ \begin{bmatrix} a \\ b \\ c \end{bmatrix} ; a+b+c = 2 \right\}.$

Sol: If $\overline{0} \in W$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 2-b-c \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$b = 0, \; c = 0$

$2 = 0.$

$\Rightarrow \overline{0} \notin W.$

$\Rightarrow$ W is not a vector space.

3) $W = \left\{ \begin{bmatrix} r \\ s \\ t \end{bmatrix} ; 5r - 1 = s + 2t \right\}.$

Sol: $s = 2t - 5r + 1.$

$$\Rightarrow \begin{bmatrix} r \\ s \\ t \end{bmatrix} = \begin{bmatrix} 2t - s + 1 \\ s \\ t \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow$ W is not a vector space.

4) $W = \left\{ \begin{bmatrix} -a+2b \\ a-2b \\ 3a-6b \end{bmatrix} ; a, b \text{ real} \right\}$.

Sol: If $\bar{0} \in W$.

$$\begin{bmatrix} -a+2b \\ a-2b \\ 3a-6b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-a+2b=0$$
$$\boxed{a=2b}$$

$a=0, b=0$.

$\therefore$ W is a vector space.

5) $W = \left\{ \begin{bmatrix} c-6d \\ d \\ c \end{bmatrix}, c, d \text{ real} \right\}$.

Sol: If $\bar{0} \in W$

$$\begin{bmatrix} c-6d \\ d \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c-6d=0$$
$$c=6d.$$
$$d=0, c=0.$$

$\therefore$ W is a vector space.

* Column Space Of a Matrix:

The column space of an $m\times n$ matrix A written as col A is the set of all linear combinations of the col A. If $A = [a_1, \ldots\ldots a_n]$ then $\text{col} A = \text{span}\{[a_1, a_2, \ldots \ldots a_n]\}$.

Note:

* The columnspace of $m \times n$ matrix A is a subspace of $\mathbb{R}^m$.

* The vector in col A can be written as Ax for some x that is $\text{col} A = \{b : b = Ax \text{ for some } x \text{ in } \mathbb{R}^n\}$.

Problem:

1) Find a matrix A such that $W = \text{col} A$ where

$$W = \left\{ \begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} ; a, b \in R \right\}.$$

Sol^n: $$\begin{bmatrix} 6a-b \\ a+b \\ -7a \end{bmatrix} = a\begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix} + b\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$W = \text{span}\left\{ \begin{bmatrix} 6 \\ 1 \\ -7 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

$$\therefore A = \begin{bmatrix} 6 & -1 \\ 1 & 1 \\ -7 & 0 \end{bmatrix}.$$

2) Find a matrix A such that $W = \text{col} A$ where W

$$= \left\{ \begin{bmatrix} 2r+3t \\ r+s-2t \\ 4r+s \\ 3r-s-t \end{bmatrix} ; r, s, t, \in R \right\}.$$

Sol^n: $W = \text{col} A$.

$$W = \begin{bmatrix} 2r+3t \\ r+s-2t \\ 4r-s \\ 3r-s-t \end{bmatrix}$$

$$= r\begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix} + s\begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix} + t\begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix}$$

$$W = \text{span}\left\{ \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 0 \\ -1 \end{bmatrix} \right\}$$

$$A = \begin{bmatrix} 2 & 0 & 3 \\ 1 & 1 & -2 \\ 4 & 1 & 0 \\ 3 & -1 & -1 \end{bmatrix}.$$

Matrix (Problems):

1) Find colA, NulA for the given matrix (or) Find the non-zero vector colA and non-zero vector for NulA.

$$A = \begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & -9 \end{bmatrix}$$

Sol: $\text{Col } A = \begin{bmatrix} 2 \\ -1 \\ -4 \\ 3 \end{bmatrix}$

$\text{Nul } A = Ax = 0$

$$\begin{bmatrix} 2 & -6 \\ -1 & 3 \\ -4 & 12 \\ 3 & 9 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$= 2x_1 - 6x_2 = 0.$

$= x_1 - 3x_2 = 0.$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_2 \\ x_2 \end{bmatrix}$$

$$x_2 \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$\text{Nul}A = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

2) Find k such that Nul A is a subspace of R^k and find k such that col A is a subspace of R^k.

$$A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}_{2\times 5}.$$

Soln: given $A = \begin{bmatrix} 4 & 5 & -2 & 6 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix}$

For Nul A

$K = 5$

col A $\Rightarrow K = 2.$

3) $A = \begin{bmatrix} 1 & -3 & 9 & 0 & -5 \end{bmatrix}_{1\times 5}.$

Sol: Nul A $= K = 5$

col A $\Rightarrow K = 1.$

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① Let $A = \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix}$ and $W = \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$ determine

W is in col A is w in Nul A.

Sol: $[Aw] = \begin{bmatrix} -8 & -2 & -9 & 2 \\ 6 & 4 & 8 & 1 \\ 4 & 0 & 4 & -2 \end{bmatrix}$.

$R_2 \to 4R_2 + 3R_1$

$R_3 \to 2R_3 + R_1$

$\Rightarrow \begin{bmatrix} -8 & -2 & -9 & 2 \\ 0 & 10 & 5 & 10 \\ 0 & -2 & -1 & -2 \end{bmatrix}$. $R_2 \to \frac{R_2}{10}$

$\Rightarrow \begin{bmatrix} -8 & -2 & -9 & 2 \\ 0 & 1 & 1/2 & 1 \\ 0 & -2 & -1 & -2 \end{bmatrix}$

$R_3 \to R_3 + 2R_2$

$\begin{bmatrix} -8 & -2 & -9 & 2 \\ 0 & 1 & 1/2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\therefore W \in \text{col} A.$

$\Rightarrow \begin{bmatrix} -8 & -2 & -9 \\ 6 & 4 & 8 \\ 4 & 0 & 4 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -16 & -2 & -18 \\ 12 & 4 & -16 \\ 8 & 0 & -8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

2) Let $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ $W = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Check whether w is in col A, w is in Nul A.

sol: $A = \begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix}$ $w = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$[A w]$

$= \begin{bmatrix} -6 & 12 & 2 \\ -3 & 6 & 1 \end{bmatrix} = \begin{bmatrix} -6 \\ -3 \end{bmatrix}$.

col $A = 0$.

Nul A

$\begin{bmatrix} -6 & 12 \\ -3 & 6 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} -12 & 12 \\ -6 & 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

* Definition : (Imp Viva)

Kernel And Range of A linear Transformations.

A Linear Transformation T from / form a Vector space into vector space W is a rule that assignes to each vector X in V in a unique vector T(x) in W, such that

i) $T(u+v) = T(u) + T(v)$ $\forall$ $u, v \in V$.

ii) $T(c.u) = c.T(u)$ for all scalars c.

Kernal (Definition):

kernal or Nullspace of T is the set of all u in V such that $T(u) = 0$.

Range : The Range of T is the set of all vectors in w of the form $T(x)$ for some x in V.

Note Point :
If T is Matrix transformation say $T(x) = Ax$ for some matrix A. Then the kernel and Range of T are just Nullspace A and ColA.

Theorem: Problem:

1) ~~Statement~~ : Let T from: $R^3 \to R^2$. Defined by $T(x, y, z) = (x, y)$. $T = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ or $= \begin{bmatrix} x \\ y \end{bmatrix}$. Verify whether T is linear or not.

Sol: ① Let $u, v \in R^3$

$$u = (x_1, y_1, z_1) \; ; \; v = (x_2, y_2, z_2)$$

$$u+v = (x_1+x_2, y_1+y_2, z_1+z_2)$$

$$T(u+v) = T(x_1+x_2, y_1+y_2, z_1+z_2)$$

$$= (x_1+x_2, y_1+y_2)$$

$$= (x_1+y_1) + (x_2+y_2)$$

$$= T(u) + T(v).$$

② $T(c \cdot u) = T \cdot (cx_1, cy_1, cz_1)$

$$= (cx_1, cy_1) = c(x_1, y_1)$$

$$= cT(u).$$

$\therefore$ T is linear.

H/W

2) $T : R^3 \to R^3$, defined by T of $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ y \\ x-z \end{bmatrix}$.

3. $T(x,y) = (x+1, y, x+y)$.

Sol: Let $u, v \in R^2$

$u = (x_1, y_1)$, $v = (x_2, y_2)$

$T(u) = (x_1+1, y_2, x_1+y_1)$

$T(v) = (x_2+1, y_2, x_2+y_2)$

$T(u+v) = T(x_1+x_2, y_1+y_2) \longrightarrow LHS$

$= [x_1+x_2+1, y_1+y_2+x_1+x_2+y_1+y_2]$

$= T(u)+T(v)$.

$= (x_1+x_2+2, y_1+y_2, x_1+y_1+x_2+y_2)$

$T(u+v) \neq T(u)+T(v)$

$\therefore$ T is not linear.

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1) Let $M_{2\times2}$ be the vector space of all 2×2 matrixs, and defined T from $M_{2\times2} \longrightarrow M_{2\times2}$ by $T(A) = A + A^T$. where $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(i) Show that T is linear transformation.

(ii) Let B be any element of $M_{2\times2}$ such that $V^T = V$. Find an A in $M_{2\times2}$ such that $T(A) = B$.

(iii) Describe the kernel of T.

(iv) Show that the Range of T is the set of V in $M_{2\times2}$ with the property that $V^T = V$.

Sol: (i) Let $A, B \in M_{2\times2}$

$$A = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \quad B = \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}$$

$$T(A) = A + A^T = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_1 & c_1 \\ b_1 & d_1 \end{bmatrix} = \begin{bmatrix} 2a_1 & b_1+c_1 \\ b_1+c_1 & 2d_1 \end{bmatrix}$$

$$T(B) = \begin{bmatrix} 2a_2 & b_2+c_2 \\ b_2+c_2 & 2d_2 \end{bmatrix}$$

$$A+B = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

$$(A+B)^T = \begin{bmatrix} a_1+a_2 & c_1+c_2 \\ b_1+b_2 & d_1+d_2 \end{bmatrix}$$

$$T(A+B) = (A+B) + (A+B)^T$$

$$= \begin{bmatrix} 2(a_1+a_2) & b_1+b_2+c_1+c_2 \\ b_1+b_2+c_1+c_2 & 2(d_1+d_2) \end{bmatrix}$$

$$\therefore T(A+B) = T(A) + T(B)$$

(ii) ~~TP~~ To prove $TC(A) = cT(A)$

$$CA = \begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix}$$

$$T(CA) = CA + (CA)^T$$

$$= \begin{bmatrix} ca_1 & cb_1 \\ cc_1 & cd_1 \end{bmatrix} + \begin{bmatrix} ca_1 & cc_1 \\ cb_1 & cd_1 \end{bmatrix} = \begin{bmatrix} 2ca_1 & cb_1+cc_1 \\ cb_1+cc_1 & 2cd_1 \end{bmatrix}$$

$$\therefore T(CA) = cT(A)$$

$\therefore$ T is linear.

(iii) $B = B^T$

$T(A) = B.$

$A + A^T = B.$

$A + A = B.$

$$2A = B$$

$$\boxed{A = \frac{B}{2}}$$

(iv) If $A \in M_{2\times 2}$

$$T(A) = O_2 \times 2$$

$$A + A^T = O_2 \times 2$$

$$\begin{bmatrix} 2a_1 & b_1+c_1 \\ b_1+c_1 & 2d_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\boxed{a_1 = 0} \quad \boxed{d_1 = 0}$$

$$\boxed{b_1 + c_1 = 0}$$

$$\boxed{b_1 = -c_1}$$

$$\begin{bmatrix} 0 & -c_1 \\ c_1 & 0 \end{bmatrix}$$

$\therefore$ kernel $A = \left\{ \begin{bmatrix} 0 & -c_1 \\ c_1 & 0 \end{bmatrix}, c_1 \in R \right\}$

(v) $B = B^T$

$T(A) = B$ (By condition of 3)

$$A + A^T = B$$

$$A + A = B$$

$$2A = B$$

$$\boxed{A = B/2}.$$

★ Theorem :

Statement :

If T is linear transformation then

(i) $T(\bar{0}) = 0$

(ii) $T(cu+dv) = cT(u)+dT(v)$.

(iii) $T(u-v) = T(u)-T(v)$ for all vector u, v in domain of T, c, d are scalars.

Proof :(i) Consider $T(\bar{0}) = T(0\cdot\bar{0})$

$= 0 \cdot T(\bar{0})$ [$\because T$ is linear]

$= \bar{0}$

(ii) $T(cu+dv) = T(cu)+T(dv)$ [By defn of linear Transformation]

$= cT(u)+dT(v)$.

(iii) $T(u-v) = T(u)-T(v)$

If $c=1, d=-1$.

$T(u-v) = 1\cdot T(u)+(-1)T(v)$

$= T(u)-T(v)$.

Theorem:

Statement:

Let V, W be the vector spaces and Let $T: V\to W$ be an linear transformation give a subspace $U(V)$. Let $T(U)$ denote the set of all images of the form, $T(x)$. Where x is in U such that $T(U)$ subset of W.

(OR)

Let T from $V\to W$ linear transformation. Then the range of T is subspace of W.

Proof: Given $T: V\to W$ be a linear transformation.

$T(U) = \{T(x) : x\in U\}$

U is a subspace of V.

To prove $T(U)$ is a subspace of W.

Since T is linear transformation.

(i) $T(\bar{0}) = \bar{0}$.

$\bar{0} \in$ Range of T (or) $\bar{0} \in T(U)$.

(ii) Let $T(u_1), T(u_2) \in T(U)$

Then $T(u_1) + T(u_2)$

$\Rightarrow T(u_1 + u_2) \in W$

$\Rightarrow T(u_1) + T(u_2) \in T(U)$.

(iii) $c \cdot T(u) = T(cu) \in W$.

$\Rightarrow cT(u) \in T(U)$.

$\therefore T(U)$ is a subspace of W.

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(Viva)

* Linear Independent set:

An Indexed set of vectors $\{v_1, v_2, \ldots\ldots v_p\}$ in a vector space V is said to be linearly independent if the vector equation $c_1v_1 + c_2v_2 + \ldots\ldots + c_pv_p = \bar{0}$ → ①
has only the trivial solution $c_1 = c_2 = \ldots\ldots = c_p = 0$.

The set $\{v_1, v_2 \ldots\ldots v_p\}$ is said to be linearly dependent. If equation ① has non-zero trivial solution i.e there are some scalars $c_1, c_2, \ldots\ldots c_p$ not all zero such that equation ① holds in such a case eq^n ① is called a linear dependence relation among the vector $v_1, v_2, \ldots\ldots, v_p$.

Note point:

1. A set containing only one non zero vector is always linearly independent i.e if $v \neq 0$, then $c_1v = 0$ $\Rightarrow \boxed{c_1 = 0}$.

2. The set containing only the zero vector is always linearly dependent i.e for $c_1\bar{0} = \bar{0} \Rightarrow c_1 \neq 0$.

3. Any set of vectors which include zero vector is always linearly dependent.

4. A set of two non-zero vectors is linearly dependent if and only if one vector is a scalar multiple of the other.

$H = \{v_1, v_2\}$

Problems :

$c_1v_1 + c_2v_2 = 0$

1. Determine whether the vectors $c_1v_1 = -c_2v_2$

$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$ are linearly independent or dependent. $v_1 = \left(\frac{-c_2}{c_1}\right)v_2$.

Sol:

$$c_1v_1 + c_2v_2 = 0$$

$$c_1\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} -c_1 - 2c_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$c_1 = 0, \quad c_2 = 0$$

$\therefore$ The set is linearly independent.

2. $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

Sol: There is zero vector.

Hence, the set is linearly dependent.

3. $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

Sol: $c_1\begin{bmatrix}1\\0\\0\end{bmatrix}+c_2\begin{bmatrix}1\\1\\0\end{bmatrix}+c_3\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$

$$=\begin{bmatrix}c_1+c_2+c_3\\c_2+c_3\\c_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$$

$\therefore$ $c_1=0$, $c_2=0$, $c_3=0$.

$\therefore$ The set is linearly independent.

4) Let $P_1(t)=1$, $P_2(t)=t$, $P_3(t)=4-t$. Then check the set P_1, P_2, P_3 is linearly dependent or Independent.

Sol: $P_3(t)=4P_1(t)-P_2(t)$

$\therefore$ It is linearly dependent.

* Basis :

Let H be the subspace of a vector space V and indexed set of vectors $\beta=\{b_1, b_2, \ldots\ldots, b_p\}$.
An Indexed set of vectors in V is a basis for H, if the

(i) β is linearly independent set.

(ii) The subspace spanned by β coincides with H.
i.e $H=\text{span}\{b_1, b_2, \ldots, b_p\}$.

Note :

(i) The vectors $c_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $c_2=\begin{bmatrix}0\\1\end{bmatrix}$ forms a basis for R^2.

$$c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$$\begin{bmatrix}c_1\\0\end{bmatrix}+\begin{bmatrix}0\\c_2\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$$

$\boxed{c_1 = 0}$ $\boxed{c_2 = 0}$

c_1 & c_2 are linearly independent.

(ii) Let $u = \begin{bmatrix} a \\ b \end{bmatrix} \in R^2$.

$$\begin{bmatrix} a \\ b \end{bmatrix} = a\begin{bmatrix} 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} a \\ b \end{bmatrix} = \text{span}\{c_1, c_2\}.$$

$\therefore \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}$ is a basis of R^2.

The standard basis of R^3. $c_1 = c_2 = c_3$.

(iii) In general $\{c_1, c_2, \ldots c_n\}$ is the standard basis of R^n.

Problems:

1. Let $v_1 = \begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix}$, $v_2 = \begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix}$, $v_3 = \begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix}$. Determine if $\beta = \{v_1, v_2, v_3\}$ is a basis for R^3.

Sol:

$$c_1v_1 + c_2v_2 + c_3v_3 = 0$$

$$c_1\begin{bmatrix} 3 \\ 0 \\ -6 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ 1 \\ 7 \end{bmatrix} + c_3\begin{bmatrix} -2 \\ 1 \\ 5 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3c_1 \\ 0 \\ -6c_1 \end{bmatrix} + \begin{bmatrix} -4c_2 \\ c_2 \\ 7c_2 \end{bmatrix} + \begin{bmatrix} -2c_3 \\ c_3 \\ 5c_3 \end{bmatrix} = 0$$

$$\begin{bmatrix} 3c_1 - 4c_2 - 2c_3 \\ -c_2 + c_3 \\ -6c_1 + 7c_2 + 5c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore \left\{\begin{matrix} 3c_1 = 0 \\ c_1 = 0 \end{matrix}\right\}$ $c_1 = 0,\ c_2 = 0,\ c_3 = 0.$

$\therefore$ The set is linearly independent.

Let $\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in R^3$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = xv_1 + yv_2 + zv_3 = \begin{bmatrix} 3x \\ 0 \\ -6x \end{bmatrix} + \begin{bmatrix} -4y \\ y \\ 7y \end{bmatrix} + \begin{bmatrix} -2z \\ z \\ 5z \end{bmatrix}$$

$$= \begin{bmatrix} 3x - 4y - 2z \\ y + z \\ -6x + 7y + 5z \end{bmatrix}$$

$= 3x - 4y - 2z = a \longrightarrow 1$

$y + z = b \longrightarrow 2$

$-6x + 7y + 5z = c \longrightarrow 3$

$eq^n\ 2 \times 4 + 1\ eq^n$

$\cancel{4y} + 4z + 3x - \cancel{4y} - 2z = a + b$

$3x + 2z = a + \overset{4}{b} \rightarrow 4$

$eq^n\ 2 \times (-7) + eq^n\ 3$

$-\cancel{7y} - 7z - 6x + \cancel{7y} + 5z = -7b + c$

$-6x - 2z = -7b + c \rightarrow 5$

$Eq^n\ 4 \times 2 + eq^n\ 5$

$\cancel{6x} + 4z - \cancel{6x} - 2z = 2a + 8b - 7b + c.$

$2z = 2a + b + c$

$$\boxed{z = \frac{2a + b + c}{2}}$$

Substituting z in eq^n 4

$3x + 2a + b + c = a + 4b$

$3x = -a + 3b - c$

$$\boxed{x = \frac{-a + 3b - c}{3}}$$

$$\boxed{y = \frac{-2a+b-c}{2}}$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \left[\frac{-a+3b\mp c}{3}\right] v_1 + \left[\frac{-2a+b-c}{2}\right] v_2 + \left[\frac{2a+b+c}{2}\right]$$

2. The vectors are $\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix}$. Determine the basis for R^4.

Sol: $c_1v_1 + c_2v_2 + c_3v_3 + c_4v_4 = 0$.

$$c_1\begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 3 \\ -5 \\ 4 \end{bmatrix} + c_4\begin{bmatrix} 0 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 \\ -4c_1 \\ 3c_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 3c_2 \\ -1c_2 \end{bmatrix} + \begin{bmatrix} 3c_3 \\ -5c_3 \\ 4c_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 2c_4 \\ -2c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} c_1 + 0 + 3c_3 + 0 \\ -4c_1 + 3c_2 + (-5c_3) + 2c_4 \\ 3c_1 + c_2 + 4c_3 - 2c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Note Point:

Let P_n be the set of all polynomials of Degree at most n then the set $S = \{1, t, t^2, \ldots t^n\}$ is standard basis of P_n.

Suppose let $S = \left\{ M_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, M_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, M_3 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, M_4 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ is the standard basis for $M_{2\times 2}$.

The set of all 2×2 matrix.

Spanning set Theorem:

Let $S = \{v_1, v_2, \ldots, v_p\}$ be a set in V and let $H = \text{span}\{v_1, v_2, \ldots, v_p\}$.

(i) If one of the vectors in S say v_p is the linear combination of remaining vectors in set S. Then set formed from S by removing v_p still spans H.

(ii) If $H \neq [0]$ some subset of S is a basis for H.

Proof: (i) Suppose v_p is a linear combination of remaining vectors $v_1, v_2, \ldots, v_{p-1}$ then

$$v_p = c_1v_1 + c_2v_2 + \ldots + c_{p-1}v_{p-1} \longrightarrow (1).$$

Let $\overline{x} \in H$

$$x = x_1v_1 + x_2v_2 + \ldots + x_pv_p \longrightarrow (2).$$

$$x = x_1v_1 + x_2v_2 + \ldots + x_p(c_1v_1 + c_2v_2 + \ldots + c_{p-1}v_{p-1})$$

$$\Rightarrow (x_1 + x_pc_1)v_1 + (x_2 + x_pc_2)v_2 + \ldots + x_pc_{p-1}v_{p-1}.$$

$x =$ linear combination of $\{v_1, v_2, \ldots v_{p-1}\}$.

$x = \text{span}\{v_1, v_2, \ldots, v_{p-1}\}$.

(ii) If the original spanning set is linearly independent then it is already a basis for H otherwise one of

the vectors of S depends on others and can be deleted as in part A. So long as there are two or more vectors in a spanning set we can repeat this process until the spanning set is linearly independent set and hence a basis for H. If the spanning set is reduced to one vector that vector will be non-zero because $H \neq 0$ vector.
A single non-zero vector is linearly independent and hence forms a basis for H.

Definition:

Basis for NulA:

The set of vectors which are the solution of the vector equations $Ax = \bar{0}$ is the spanning set. If the resultant set is linearly independent it forms a basis for NulA.

Problem: (Practical-3-3rd problem).

1. Find the basis for NulA where $A = \begin{bmatrix} 1 & 0 & -3 & 2 \\ 0 & 1 & -5 & 4 \\ 3 & -2 & 1 & -2 \end{bmatrix}$.

Sol: $AX = 0$

$$[A \ \bar{0}] = \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 3 & -2 & 1 & -2 & 0 \end{bmatrix}.$$

$R_3 \to R_3 - 3R_1$

$$= \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & -2 & 10 & -8 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$.

$$= \begin{bmatrix} 1 & 0 & -3 & 2 & 0 \\ 0 & 1 & -5 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$\Rightarrow x - 3z + 2w = 0$

$\Rightarrow y - 5z + 4w = 0$

$$\boxed{x = 3z - 2w}$$

$$\boxed{y = 5z - 4w}$$

$$X = \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 3z - 2w \\ 5z - 4w \\ z \\ w \end{bmatrix}$$

$$= z \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix}$$

$$X = \text{span}\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$c_1 \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$3c_1 - 2c_2 = 0$

$5c_1 - 4c_2 = 0$

$c_1 = 0$

$c_2 = 0.$

$\therefore$ set $\left\{ \begin{bmatrix} 3 \\ 5 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -4 \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent, basis for Nul A.

<u>4/7/22</u> Basis's of colA:

If A is a matrix then by the definition colA is set of all linear combinations of column of A that is colA = spans of columns.

If its linearly independent then the set of column will be the basis of column space of A. Otherwise, we can eliminate linearly dependent vectors by using spanning set theorem of find the basis.

Example : →4th prob - frac - 3

1. Find the basis of col A. Where $A = [a_1, a_2, \ldots\ldots a_5]$ which is $\begin{bmatrix} 1 & 4 & 0 & 2 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$.

Note :

The pivot col A for basis forms a col A it is basis for A.

Sol: Given, In the matrix A there are pivot col A

$$a_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad a_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}; \quad a_5 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

There are two non-pivot col A $a_2 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 0 \end{bmatrix}; a_4 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and observe that $a_2 = 4 \cdot a_1$ and $a_4 = 2a_1 - a_3$.

∴ Each non-pivot col A is linear combination of pivot col.

By spanning set theorem, we may discard a_2 and a_4 the set a_1, a_3, a_5 will span col A.

$$\therefore \text{Let } S = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Since $A_1 = 0$, and no vector in S is linear combinations of the vector that presuited it.

∴ S is linear independent.

∴ S is basis of col A.

2. Find the basis for Nul A and col A. $A = \begin{bmatrix} 1 & 0 & 6 & 5 & 0 \\ 0 & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Sol: $[A\ \bar{0}] = \begin{bmatrix} 1 & 0 & 6 & 5 & 0 \\ 0 & 2 & 5 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

$= x + 6z + 5w = 0$

$= 2y + 5z + 3w = 0.$

$x = -6z - 5w$

$y = \dfrac{-5z - 3w}{2}$

$X = \begin{bmatrix} -6z - 5w \\ \dfrac{-5z - 3w}{2} \\ z \\ w \end{bmatrix}$

$\Rightarrow z \begin{bmatrix} -6 \\ -\frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix}$

$X \Rightarrow \text{span} \left\{ \begin{bmatrix} -6 \\ -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$

$-6c_1 + 5c_2 = 0$

$-\frac{5}{2}c_1 + \frac{3}{2}c_2 = 0$

$c_1 = 0.$

$c_2 = 0.$

$\therefore$ The set $\beta = \left\{ \begin{bmatrix} -6 \\ -\frac{5}{2} \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -5 \\ -\frac{3}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$ is linearly independent. β is basis for Nul A.

(ii) $A = \begin{bmatrix} 1 & 0 & 6 & 5 \\ 0 & 2 & 5 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Sol: $\begin{bmatrix} 6 \\ 5 \\ 0 \end{bmatrix} = 6\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 5/2 \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix}$

$$a_3 = 6a_1 + \frac{5}{2}a_2$$

$$a_4 = 5a_1 + \frac{3}{2}a_2.$$

$\beta = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ 0 \end{bmatrix} \right\}$ is a basis for colA.

7/7/22 Problems :

1) Let $v_1 = \begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix}$, $v_2 = \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix}$ it can be verified that $v_1 - 3v_2 + 5v_3 = \bar{0}$. Use this information to find the basis for $H = \text{span}\{v_1, v_2, v_3\}$.

Sol: $v_1 - 3v_2 + 5v_3 = \bar{0}$

$$\begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix} - 3\begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix} + 5\begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix} = \bar{0}.$$

$$= \begin{bmatrix} 7 \\ 4 \\ -9 \\ -5 \end{bmatrix} - \begin{bmatrix} 12 \\ 21 \\ -6 \\ -15 \end{bmatrix} + \begin{bmatrix} 5 \\ -25 \\ 15 \\ 20 \end{bmatrix}$$

$$= \begin{bmatrix} 7-12+5 \\ 4+21-25 \\ -9 \mp 6+15 \\ -5-15+20 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow c_1 v_2 + c_2 v_3$

$$\Rightarrow c_1 \begin{bmatrix} 4 \\ -7 \\ 2 \\ 5 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -5 \\ 3 \\ 4 \end{bmatrix}$$

$$4c_1 + c_2 = 0$$

$$-7c_1 - 5c_2 = 0$$

$$2c_1 + 3c_2 = 0$$

$$5c_1 + 4c_2 = 0$$

$\boxed{c_2 = -4c_1} \rightarrow$ linearly independent.

$\therefore H = \text{span}\{v_2, v_3\}$.

$\therefore$ The set $\{v_2, v_3\}$ is a basis for H:

H/W

2) Let $v_1 = \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix}$, $v_3 = \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$, $H = \text{span}\{v_1, v_2, v_3\}$

it can be verified that $4v_1 + 5v_2 - 3v_3 = 0$. Use the information to find a basis for H. There is more than one answer.

Sol:

$$4 \begin{bmatrix} 4 \\ -3 \\ 7 \end{bmatrix} + 5 \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix} - 3 \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$$

$$= \begin{bmatrix} 16 + 5 - 21 \\ -12 + 45 - 33 \\ 28 - 10 - 18 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$= c_1 v_2 + c_2 v_3.$

$$= c_1 \begin{bmatrix} 1 \\ 9 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 7 \\ 11 \\ 6 \end{bmatrix}$$

$$c_1 + 7c_2 = 0$$
$$9c_1 + 11c_2 = 0$$
$$-2c_1 + 6c_2 = 0.$$

$\boxed{c_1 = -7c_2}$ $\rightarrow$ linearly independent.

$\therefore H = \text{span}\{v_2, v_3\}.$

$\therefore$ The set $\{v_2, v_3\}$ is a basis for H.

Coordinate systems:

Theorem: The Unique Representation Theorem.

statement: Let $\beta = \{b_1, b_2, \ldots\ldots, b_n\}$ be a basis for a vector space V then for each x in V $\exists$ unique set of scalars $c_1, c_2, \ldots\ldots c_n$ such that

$x = c_1 b_1 + c_2 b_2 + \ldots\ldots + c_n b_n.$

Proof: Given V is a vector space, β is a basis for V.

Since β spans V there exists scalars $c_1, c_2, \ldots\ldots, c_n$ such that $x = c_1 b_1 + c_2 b_2 + \ldots\ldots + c_n b_n \rightarrow (1)$

Suppose x have another representation, let us consider

$x = d_1 b_1 + d_2 b_2 + \ldots\ldots + d_n b_n \rightarrow (2) \quad \forall\ d_1, d_2 \ldots \in R.$

$eq^n (1) - eq^n (2)$

$0 = (c_1 - d_1) b_1 + (c_2 - d_2) b_2 + \ldots\ldots + (c_n - d_n) b_n.$

$\therefore \beta$ is linearly independent.

$c_1 - d_1 = 0$, $c_2 - d_2 = 0$, ..., $c_n - d_n = 0$.

$\therefore$ Therefore x can be expressed as a unique set of scalars $c_1, c_2, \ldots, c_n$ such that

$x = c_1 b_1 + c_2 b_2 + \ldots\ldots + c_n b_n$

$\left\{ \begin{matrix} c_1 = d_1 \\ c_2 = d_2 \\ c_n = d_n \end{matrix} \right\}$

Definition:

Co-Ordinates:

Let $\beta = \{b_1, b_2, \ldots\ldots b_n\}$ be a basis for V let $x \in V$ if $x = c_1 b_1 + c_2 b_2 + \ldots\ldots + c_n b_n$. Then the scalars $c_1, c_2, \ldots\ldots, c_n$ are called the coordinates of x with respect to the basis β.

(OR)

β coordinates of x. We denote the β coordinates of vector x [by matrix representation of x with respect to β].

$$[x]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

The mapping $x \to [x]_\beta$ is called the coordinate mapping.

Problem:

1. Consider a basis $\beta = \{b_1, b_2\}$ for R^2 where $b_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $b_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ suppose x is in R^2 has the (coordinate vector) $[x]_\beta = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Find x.

Sol: $x = c_1 b_1 + c_2 b_2$

$= -2 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

$$= \begin{bmatrix} -2 \\ 0 \end{bmatrix} + \begin{bmatrix} 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix}$$

(2) $\beta = \left\{ \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix}, \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix}, \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix} \right\}.$

Sol: $[x]_\beta = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$

$$= 3 \begin{bmatrix} 1 \\ -4 \\ 3 \end{bmatrix} + 0 \begin{bmatrix} 5 \\ 2 \\ -2 \end{bmatrix} - 1 \begin{bmatrix} 4 \\ -7 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 \\ -12 \\ 9 \end{bmatrix} + 0 - \begin{bmatrix} 4 \\ 7 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -5 \\ 9 \end{bmatrix}$$

H/W.

(3) Find the coordinate vector $[x]_\beta$ of x, relative to the given basis to the given data $\beta = \{b_1, b_2, \ldots\ldots b_n\}$. $[x]_\beta$, $b_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$, $b_2 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$, $\cancel{b_3}\ x = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$

Sol: $x = c_1 b_1 + c_2 b_2.$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ 5 \end{bmatrix}$$

$$\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{matrix} c_1 + 2c_2 \\ -3c_1 + 5c_2 \end{matrix}$$

$$c_1 + 2c_2 = -2$$
$$-3c_1 + 5c_2 = 1$$
$$\boxed{c_1 = -2 - 2c_2}.$$

Definition:

Change of Coordinate Matrix:

Let $\beta = \{b_1, b_2, \ldots\ldots, b_n\}$ be a basis for R^n let $P_\beta = [b_1, b_2, \ldots\ldots, b_n]$ be a matrix with $b_1, b_2, \ldots\ldots b_n$ as its columns then the equation $x = c_1b_1 + c_2b_2 + \ldots + c_nb_n$ is equivalent to $x = P_\beta[x]_\beta$. Where P_β is an $n \times n$ matrix and the coordinate matrix is $[x]_\beta$ is called the change of coordinate matrix from β to the standard basis in R^n, left multiplication by P_β transforms the coordinate vector into x. Since P_β is invertable then we have $P_\beta^{-1} x = [x]_\beta$.

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Definition:

The Coordinate Mapping:

Choosing a basis $\beta = \{b_1, b_2, \ldots\ldots, b_n\}$ for a vector space V introduces a coordinate system in V. The coordinate mapping from $x \to [x]_\beta$, connects the possibility unfamiliar space to the familiar space R^n. In general there is a one to one linear transformation from a vector space V onto vector space W is called isomorphism from V to W.

Theorem: (LAQ).

Statement: Let $\beta = \{b_1, b_2, \ldots\ldots, b_n\}$ be a basis for a vector space V then the coordinate mapping from $x \to [x]_\beta$ is a 1-1 (one to one) linear transformation from V onto R^n.

Proof: (i) Let u, v be the two vectors of V.

$$u = c_1b_1 + c_2b_2 + \ldots\ldots + c_nb_n.$$

$v = d_1b_1 + d_2b_2 + \dots\dots + d_nb_n.$

$u+v = (c_1+d_1)b_1 + \dots\dots\dots + (c_n+d_n)b_n.$

$$[u]_\beta = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}, \quad [v]_\beta = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$[u]_\beta + [v]_\beta = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} + \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix} = \begin{bmatrix} c_1+d_1 \\ \vdots \\ c_n+d_n \end{bmatrix}$$

$$[u+v]_\beta.$$

(ii) $$[cu]_\beta = \begin{bmatrix} cc_1 \\ \vdots \\ cc_n \end{bmatrix} = c\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

$$= c[u]_\beta.$$

Hence it is a linear transformation.

(iii) To prove the transformation is 1-1.

$$x \to [x]_\beta.$$

$$[u]_\beta = [v]_\beta.$$

$$\begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_n \end{bmatrix}$$

$$c_i = d_i \quad \forall \ i = 1, 2, \dots\dots n.$$

$u = c_1b_1 + c_2b_2 + \dots\dots + c_nb_n$

$u = d_1b_1 + d_2b_2 + \dots\dots + d_nb_n.$

$u = v.$

$\therefore$ It is 1-1.

(iv) Let $y \in R^n$ with $y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$ consider

$y_1 b_1 + y_2 b_2 + \dots\dots + y_n b_n$.

This is a linear combination of vectors of β and since β spans V such that if there β and since β spans V such that if there is an x belongs to V $[x \in V]$.

$$\therefore x = y_1 b_1 + y_2 b_2 + \dots\dots + y_n b_n.$$

Therefore the $x \to [x]_\beta$ is an onto map.

∴ The mapping $x \to [x]_\beta$ is a 1-1 linear transformation from V to R^n.

Problem:

1. Use coordinate vectors to verify that the polynomials $1+2t^2$, $4+t+5t^2$, $3+2t$ are linearly dependent in P_2.

Sol: The standard basis for P_2 is $\beta = \{1, t, t^2\}$.

Any vector in P_2 is of the form.

$$P_2(t) = a_0 + a_1 t + a_2 t^2$$

$$[P(t)]_\beta = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}.$$

The coordinate mapping from $P_2(t) \to R^3$ is defined by $P(t) \to [P(t)]_\beta$.

By using above theorem the mapping is an isomorphism from $P_2(t) \to R^3$.

$$[1+2t^2]_\beta = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$[4+t+5t^2]_\beta = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$$

$$[3+2t]_\beta = \begin{bmatrix} 3 \\ 2 \\ 0 \end{bmatrix}$$

$$A \cdot 0 = \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 2 & 5 & 0 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 - 2R_1$$

$$= \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & -3 & -6 & 0 \end{bmatrix}$$

$$R_3 \rightarrow R_3 + 3R_2$$

$$= \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & \end{array} \\ \begin{bmatrix} 1 & 4 & 3 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$$= x_1 + 4x_2 + 3x_3 = 0$$

$$x_2 + 2x_3 = 0$$

$$\boxed{x_2 = -2x_3}$$

$$x_1 + 4(-2x_3) + 3x_3 = 0$$

$$x_1 - 8x_3 + 3x_3 = 0$$

$$\boxed{x_1 = 5x_3}$$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5x_3 \\ -2x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

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$\therefore$ set of polynomials are independent.

2. The set $B = \{1+t^2, t+t^2, 1+2t+t^2\}$ is a basis P_2 find the coordinate vector $P(t) = 1+4t+7t^2$ related

to β.

Sol: $x = [P]_\beta [x]_\beta$

$$[x]_\beta = x[P^{-1}]_\beta$$

$$\begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} [x]_\beta$$

Let $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix}$ $\quad A^{-1} = ?$

$|A| = -2 \neq 0.$

co-factors of $A = \begin{bmatrix} -1 & 2 & -1 \\ 1 & 0 & -1 \\ -1 & -2 & 1 \end{bmatrix}$

$$A^{-1} = \frac{\operatorname{adj} A}{|A|}$$

Adj $A = \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$

$$A^{-1} = \frac{1}{|A|} \operatorname{adj} A$$

$$= \frac{-1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix}$$

$$[x]_\beta = \frac{-1}{2} \begin{bmatrix} -1 & 1 & -1 \\ 2 & 0 & -2 \\ -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \\ 7 \end{bmatrix}$$

$$= \frac{-1}{2} \begin{bmatrix} -1+4-7 \\ 2+0-14 \\ -1-4+7 \end{bmatrix} = \frac{-1}{2} \begin{bmatrix} -4 \\ -12 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}$$

H/W

3. The set $\beta = \{1-t^2, t-t^2, 2-2t+t^2\}$ is a basis for P_2 find the coordinate vector of $p(t) = 3+t-6t^2$ related to β.

Use coordinate vectors to test linear independence of the set of polynomials (i) $1+t^3, 3+t-2t^2, -t+3t^2-t^3$

Sol: For $AX = 0$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 1 & 0 & -1 & 0 \end{bmatrix}$$

$$R_4 \to R_4 - R_1 \to \begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 3 & 0 \\ 0 & -3 & -1 & 0 \end{bmatrix}$$

$R_3 \to R_3 + 2R_2$, $R_4 \to R_4 + 3R_2$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -4 & 0 \end{bmatrix}$$

$R_4 \to R_4 + 4R_3$

$$\begin{bmatrix} 1 & 3 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$x_1 + 3x_2 = 0$

$x_2 - x_3 = 0$

$\boxed{x_3 = 0} \Rightarrow \boxed{x_2 = 0}$ $\boxed{x_1 = 0}$

→ It is linearly independent.

(28) $1 - 2t^2 - 3t^3$, $t + t^3$, $1 + 3t - 2t^2$

For $AX = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 3 \\ -2 & 0 & -2 \\ -3 & 1 & 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ -2 & 0 & -2 & 0 \\ -3 & 1 & 0 & 0 \end{bmatrix}$$

$R_4 \to R_4 + 3R_1$ $\qquad$ $R_3 \to R_3 + 2R_2$

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ -2 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= x_1 + x_3 = 0$$

$$x_2 + 3x_3 = 0$$

$$x_3 = 0$$

$$x_4 = 0$$

It is linearly Independent.

* The Dimension Of A Vector Space:

Theorem: If a vector space V has a basis $\beta = \{b_1, b_2, \dots b_n\}$ then any set in V containing more than n vectors must be linearly dependent.

Proof: Given V is a vector space.

β is a basis of V.

$$\beta = \{b_1, b_2, \dots\dots, b_n\}$$

Let $\{u_1, u_2, \dots\dots, u_p\}$ be a set in V with more than n vectors

$$p > n$$

Consider the coordinate vectors $[u_1]_\beta, [u_2]_\beta \dots\dots$ $[u_p]_\beta$ in R^n.

Then obviously each of these coordinate vectors

is a linear combination of the vector of the basis of β.
Thus there are p linear combinations with n entries.

$$u_1 = c_1b_1 + c_2b_2 + \cdots\cdots + c_nb_n$$
$$u_2 = d_1b_1 + d_2b_2 + \cdots\cdots + d_nb_n$$
$$\vdots$$
$$u_p = d_1b_1 + \quad " \qquad "$$

$$[u_1] = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ \vdots \\ c_n \end{bmatrix}, [u_2] = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix}$$

Since $p > n$ the set $[u_1]_\beta, [u_2]_\beta \cdots\cdots [u_p]_\beta$ must be linearly dependent in R^n.
There exists scalars $c_1, c_2, \ldots\ldots c_p$ not all zeroes $\ni$

$$c_1[u_1]_\beta + c_2[u_2]_\beta + \cdots\cdots + c_p[u_p]_\beta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$\because$ The coordinate mapping is linear

$$[c_1u_1]_\beta + [c_2u_2]_\beta + \cdots\cdots + [c_pu_p]_\beta = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\because p > n$$

$$[\underbrace{c_1u_1 + c_2u_2 + \cdots\cdots + c_pu_p}_{p \text{ times}}]_\beta = \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}}_{n \text{ times}}$$

The zero vector on the right side contains the n weights needed to build the vector $c_1u_1 + c_2u_2 + \cdots + c_pu_p$ from the basis vector β.

i.e $c_1u_1 + c_2u_2 + \cdots\cdots + c_pu_p = 0 \cdot b_1 + 0 \cdot b_2 + \cdots\cdots + 0 \cdot b_n = 0.$

Since the c_1's are all not zeroes, we conclude that
$\Rightarrow \{u_1, u_2, \ldots\ldots, u_p\}$ is linearly dependent.

Note: The above theorem implies that if a Vector space V has a basis $\beta = \{b_1, b_2, \ldots, b_n\}$ then each linearly independent set in V has no more than n vectors.

Theorem: If a vector space V has a basis of n vectors then every basis of V must consists of exactly of n vectors.

Proof: Given V is a Vector space.

It has a basis of n vectors.

To prove every basis of V must consists of exactly n vectors.

Let β_1 be a basis of V with 'n' vectors.

Let β_2 be another basis of V with m vectors.

Now β_1 is a basis and β_2 is Linearly Independent.

$$\Rightarrow m \leq n \longrightarrow (1)$$

β_2 is a basis and β_1 is Linearly Independent.

$$\Rightarrow n \leq m \longrightarrow (2)$$

$$\boxed{n = m}$$

∴ Every basis of V must consists of exactly same number of vectors.

Definition:

Dimension of Vector space:

The dimension of V written as dim V is the no. of vectors in a basis for V. The dimension of the zero vector space $\{0\}$ is defined to be 0. If vector space V is spanned by finite set then V is said to be finite dimensional. If V is not spanned infinite set

then it is said to be infinite dimensional.

V is a vector space.

Note point:

→ The standard basis for R^n contains n vectors therefore dimension of $R^n = n$. $\dim R^n = n$.

→ The standard basis for P_2 is $\{1, t, t^2\}$, $\dim P_2 = 3$.

$\therefore \dim P_2 = 3$.

→ In general $\dim P_n = (n+1)$.

Problems:

1) (Find) For the subspace $H = \left\{ \begin{bmatrix} s-2t \\ s+t \\ 3t \end{bmatrix}, s,t \in R \right\}$

Find the basis and dimension.

Sol: $$H = \left\{ s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} ; s,t \in R \right\}$$

Thus the set $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$ spans H and the two vectors are linearly independent.

$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 3 \end{bmatrix} \right\}$ is a basis for H.

$\therefore \dim H = 2$.

(2) $H = \left\{ \begin{bmatrix} 4s \\ -3s \\ -t \end{bmatrix}, s,t \in R \right\}$

Sol: $$H = \left\{ s\begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} ; s,t, \in R \right\}$$

Thus the set $\left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$ spans H and the two vectors are linearly independent.

$$\therefore \left\{ \begin{bmatrix} 4 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\} \text{ is a basis for } H.$$

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(3) $H = \left\{ \begin{bmatrix} a-4b-2c \\ 2a+5b-4c \\ -a+2c \\ -3a+7b+6c \end{bmatrix} ; a, b, c, \in R \right\}$.

Find the basis and dimension.

Sol: $H = \left\{ a \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix} + b \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} + c \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} ; a, b, c, \in R \right\}$

The set H is spanned by $\left\{ u_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ -3 \end{bmatrix}, u_2 = \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix}, u_3 = \begin{bmatrix} -2 \\ -4 \\ 2 \\ 6 \end{bmatrix} \right\}$.

$u_3 = -2u_1$ discard u_3.

Hence the set $\{(u_1, u_2)\}$ also spans H and u_1 and u_2 are not multiples of each other.

$\therefore \{u_1, u_2\}$ is Linearly Independent.

$$\therefore \left\{ \begin{bmatrix} 1 \\ +2 \\ -1 \\ -3 \end{bmatrix}, \begin{bmatrix} -4 \\ 5 \\ 0 \\ 7 \end{bmatrix} \right\} \text{ is a basis for } H.$$

$\therefore \dim H = 2$.

(4) For the subspace $\{(a,b,c) : a-3b+c=0, b-2c=0, 2b-c=0\}$ Find a basis and state the dimension.

Sol: The subspace is the set of all vectors $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ which satisfy

$$a-3b+c=0$$
$$b-2c=0$$
$$2b-c=0$$

This is a homogeneous system and hence $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is a solution.

Let us check for non-zero solution

Matrix eq^n of the system is

$$\Rightarrow \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 2 & -1 & 0 \end{bmatrix}$$

$$R_3 \to R_3 - 2R_2$$

$$\Rightarrow \begin{bmatrix} 1 & -3 & 1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{bmatrix}$$

Clearly, there are 3 pivot columns and since the system has 3 variables a unique solution exists i.e zero solution

$\therefore$ Hence the subspace is a zero space.

There is no basis and the dimension is zero

(5) Find the dimension of the subspace of all vectors in R^3 whose first and third entries are equal.

Sol: Let $H = \left\{ \begin{bmatrix} a \\ b \\ a \end{bmatrix}, a, b \in R \right\}$

$H = \left\{ a\begin{bmatrix}1\\0\\1\end{bmatrix} + b\begin{bmatrix}0\\1\\0\end{bmatrix} ; a, b \in R \right\}$.

H is spanned by two vectors $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$.

These two vectors are not multiples of each other and hence they are linearly independent.

Therefore the set $\left\{ \begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix} \right\}$ is a basis for H

and dim $H = 2$.

(6) The first four laguree polynomials are $1, 1-t, 2-4t+t^2$ and $6-18t+9t^2-t^3$. show that these polynomials form a basis of P_3.

sol: P_3 is a vector space all the polynomials of degree atmost 3.

i.e $P_3 = \{ p+qt+rt^2+st^3 ; p, q, r, s, t \in R^3 \}$

To ~~state~~ show that

$H = \{ 1, 1-t, 2-4t+t^2, 6-18t+9t^2-t^3 \}$.

is a basis of P_3.

We have to check that span $H = P_3$ and H is linearly independent.

(i) Let $p+qt+rt^2+st^3$ be a general element of P_3 and suppose $p+qt+rt^2+st^3 = a(1)+b(1-t) + c(2-4t+t^2)+d(6-18t+9t^2-t^3)$

comparing the terms on both sides.

$$a+b+2c+16d = p$$
$$-b-4c-18d = q.$$

$c + 9d = r$

$s = -d$

$\Rightarrow \boxed{d = -s}$

$c - 9s = r$

$\boxed{c = r + 9s}$

$b = -q - 4c - 18d$

$b = -q - 4(r + 9s) - 18(-s)$

$b = -q - 4r - 36s + 18s$

$b = -q - 4r - 18s.$

$a = -p - (-q - 4r - 18s) - 2(r + 9s) + 16(-s).$

Thus Every vector in P_3 can be expressed as a Linear combination of Vectors of H.

$\therefore$ Span $H = P_3$.

(ii) $a(1) + b(1-t) + c(2 - 4t + t^2) + d(6 - 18t + 9t^2 - t^3) = \bar{0}$

$= 0 \cdot 1 + 0 \cdot t + 0 \cdot t^2 + 0 \cdot t^3.$

$a + b + 2c + 6d = 0$

$-b - 4c - 18d = 0$

$c + 9d = 0 \Rightarrow \boxed{c = 0}$

$-d = 0$

$\boxed{d = 0}$

$\therefore$ H is Linearly Independent.

$\therefore H = \{1, 1-t, 2 - 4t + t^2, 6 - 18t + 9t^2 - t^3\}$

is a basis of P_3.

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Subspaces of a finite dimensional space:

Statement: Let H be a subspace of a finite dimensional vector space V. Any linearly independent set in H can be expanded if necessary to a basis for H. Also, H is finite dimensional and dim H $\leq$ dim V.

Proof: Suppose if H is empty set. If $H = \{\bar{0}\}$

Then dim H = 0 = dim V

If $H \neq \{\bar{0}\}$

Let $s = \{u_1, u_2, \ldots\ldots u_n\}$ is Linearly independent set in H.

s is a basis for H.

If s spans H. Then s forms a basis for H.

Otherwise if s doesnot span H, then there exists a vector u_{K+1} in H which is not in the span H.

Now consider the $\{u_1, u_2, \ldots\ldots u_K, u_{K+1}\}$ this must be linearly independent.

Since u_{K+1} is not a linear combination of any of the preceiding vectors.

We can continue this process of expanding s to a larger linearly independent set in H. So long as the new set doesnot span H.

By using the theorem that if a vector space V has a basis $\beta = \{b_1, b_2 \ldots\ldots b_n\}$ then any set in V containing more than n vectors must be linearly independent.

Therefore the number of vectors in a linearly independent set can never exceed the dim V. So, at some point the expanded S will span H and Hence becomes a basis for H.

$\therefore \dim H \leq \dim V.$

Basis Theorem:

Statement: Let V be a p-dimensional vector space $p \geq 1$. Any set S of exactly p elements in V is a basis of V if and only if either S is linearly independent or S spans V.

Proof: Suppose S is a basis of V.
From the defn, S is linearly independent.
Let us assume that S is linearly independent.
By using the above theorem a linearly independent S can be extended to a basis for V.
But the basis must contain exactly p-elements.
Since $\dim V = p$.
So S itself must be a basis for V.
Suppose S has p-elements and spans V since V is non-zero by using spanning set theorem a subset S' of S is a basis of V. Since $\dim V = p$ S' must contain p vectors. Therefore,

$$S = S'.$$

The Dimensions Of NulA and ColA:

Dimension of NulA (Definition):

Dim of NulA = The no. of free variables in the eq^n $Ax = 0$.

Problem:

1. Find the dimension of the Nullspace where $A = [$

$$A = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 \\ 1 & -2 & 2 & 3 & -1 \\ 2 & -4 & 5 & 8 & -4 \end{bmatrix}$$

Sol: $[A\bar{0}] = \begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 1 & -2 & 2 & 3 & -1 & 0 \\ 2 & -4 & 5 & 8 & -4 & 0 \end{bmatrix}$

$\sim$ $R_2 \to 3R_2 + R_1$
$R_3 \to 2R_1 + 3R_3$ $\Rightarrow \begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 0 & 0 & 5 & 10 & -10 & 0 \\ 0 & 0 & 13 & 26 & -26 & 0 \end{bmatrix}$

$R_2/5$, $R_3/13$.

$$\to \begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$$\to \begin{array}{c} \begin{bmatrix} -3 & 6 & -1 & 1 & -7 & 0 \\ 0 & 0 & 1 & 2 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \\ \quad x_2 \qquad\quad x_4 \quad x_5 \end{array}$$

$\therefore$ dim Nul A $= 3$

There are 3 free variables.

Dimension of colA

Dim of colA = The no of pivot columns in A.

Problem:

1. Find the dimension of colA where $A = \begin{bmatrix} 1 & -6 & 9 & 0 & -2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

Sol: Dim of colA = 3

There are 3 pivot columns v_1, v_2, v_4.

2. Find the dimension of Nul A and col A for the matrices $A = \begin{bmatrix} 1 & 0 & 9 & 5 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

Sol: dim of Nul A = 2. There are 2 free variables.

dim of col A = 2. There are 2 pivot columns. v_1 & v_3.

3. If $A = \begin{bmatrix} 1 & 4 & -1 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Sol: dim of Nul A = 1.

dim of col A = 2.