PROBABILITY

Question 1

Bayes' Theorem

Formula

$$P(A \mid B) = \frac{P(B \mid A) \cdot P(A)}{P(B)}$$

A, B = events
$P(A|B)$ = probability of A given B is true
$P(B|A)$ = probability of B given A is true
$P(A), P(B)$ = the independent probabilities of A and B

Two binary random variables **R** and **F** where R is wether it rains and F is wether they forecast rain or not
Given:

P(F=1 | R=1) = 70%, when it rains they are correct 70% of the time
P(F=1 | R=0) = 30%, when it doesn't rain they say it will rain 30% of the time
P(R=1) = 73/365 or 20%, it rains 20% of the year
P(R=0) = 80%, it doesn't rain 80% of the year

Solution

I need to find P(R=1 | F=1), the probability it rains given they forecast it will
Using Bayes' Theorem I can do

$$P(R=1 \mid F=1) = \frac{P(F=1 \mid R=1) \cdot P(R=1)}{P(F=1)}$$

Since I don't know P(F = 1) I use the law of total probabilities to solve for it.

$$P(F=1) = P(F=1 \mid R=1) \cdot P(R=1) + P(F=1 \mid R=0) \cdot P(R=0)$$

$$\downarrow$$

$$(0.7 \cdot 0.2) + (0.3 \cdot 0.8)$$

$$P(F=1) = 0.38$$

Finally

$$P(R=1 \mid F=1) = \frac{P(F=1 \mid R=1) \cdot P(R=1)}{0.38}$$

$$\downarrow$$

$$\frac{0.70 \cdot 0.2}{0.38}$$

$$\downarrow$$

$$\boxed{P(R=1 \mid F=1) = 0.368 \text{ or } 36.8\%}$$

Question 2

$$\text{payout} = \begin{cases} \$1 & x = 1 \\ -\$1/4 & x \neq 1 \end{cases}$$

The probability of landing on 1 is 1/6 and not on 1 is 5/6. If I played 6 games, statistically I would win $1 and loose a $0.25 5 times or -$1.25.

Answer: So no this is not a good game to play.

Question 3

The Gaussian function can be thought of as a weighting function to the quadratic function. When you take the integral of these combined functions you find the probability of the most likely outcome.

Question 4

$$p(x) = \begin{cases} 4x & 0 \leq x \leq 1/2 \\ -4x + 4 & 1/2 \leq x \leq 1 \end{cases}$$

$$2x^2 \quad 0 \leq x \leq 1/2$$

$$-2x^2 + 4x \quad 1/2 \leq x \leq 1$$

Answer

$$\boxed{\int_{-\infty}^{1/2} 2x^2 \, dx + \int_{1/2}^{x} -2x^2 + 4x \, dx}$$

LINEAR ALGEBRA

Question 1

Transpose and Associative Property [1pt] Define matrix $B = bb^T$, where $b \in \mathbb{R}^{d\times 1}$ is a column vector that is not all-zero. Show that for any vector $x \in \mathbb{R}^{d\times 1}$, $x^T B x \geq 0$.

[*Hint*: Try to get $x^T B x$ to look like the product of two identical scalars. Note that $b^T x = (x^T b)^T$, that $a^T = a$ for scalar value a, and that matrix multiplication is associative.]

$$x^T B x = x^T b b^T x$$

$$b = \begin{array}{cc} & 1 \\ d & \begin{bmatrix} 1 \\ 2 \end{bmatrix} \end{array} \qquad b^T = \begin{array}{cc} d & \\ \begin{bmatrix} 1 & 2 \end{bmatrix} & 1 \end{array}$$

$$bb^T = \begin{array}{cc} & d \\ d & \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} \end{array}$$

$$x = \begin{array}{cc} & 1 \\ d & \begin{bmatrix} -1 \\ 1 \end{bmatrix} \end{array} \qquad bb^T x = \begin{array}{cc} 1 & \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & d \end{array}$$

$$x^T = \begin{array}{cc} d & \\ \begin{bmatrix} -1 & 1 \end{bmatrix} & 1 \end{array}$$

$$\boxed{x^T b b^T x = \text{scaler}, \ 0}$$

Question 2

a)

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 4 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix} \qquad \begin{pmatrix} 2 & 1 & 1 \\ 4 & 0 & 2 \\ 2 & 2 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 10 \\ -2 \end{pmatrix}$$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad b = \begin{pmatrix} 3 \\ 10 \\ -2 \end{pmatrix}$$

b)

$$A^{-1} = \begin{pmatrix} -1 & 1/2 & 1/2 \\ 1 & -1/2 & 0 \\ 2 & -1/2 & -1 \end{pmatrix}$$

$$\cancel{A^{-1}A}x = A^{-1}b$$

$$\downarrow$$

$$\boxed{x = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}}$$

PROVING THINGS

Question 1

$$f(x) = \ln(x) - (x-1)$$

$$f(x)' = \frac{1}{x} - 1 \qquad 0 = \frac{1}{x} - 1 \rightarrow 1 = \frac{1}{x} \rightarrow x = 1$$

$$\boxed{f(x=1)' = 0}$$

Question 2

$$\sum_{i=1}^{k} p_i = \sum_{i=1}^{k} q_i = 1$$

$$p_i > 0, q_i > 0, \quad \forall i \in \{1, \ldots, k\}$$

between p and q is given by:

$$KL(p||q) = \sum_{i=1}^{k} p_i \ln\left(\frac{p_i}{q_i}\right)$$

$$\sum_{i=1}^{K} p_i \ln\left(\frac{p_i}{q_i}\right) \leq \sum_{i=1}^{K} p_i - 1$$

$$f(x) = \sum_{i=1}^{K} p_i \ln\left(\frac{p_i}{q_i}\right) - \sum_{i=1}^{K} p_i - 1$$

$$f(x)' = p_i \frac{q_i}{p_i} - 0 \rightarrow q_i > 0 \text{ so True}$$

4 Debriefing (required in your report)

1. Approximately how many hours did you spend on this assignment?
2. Would you rate it as easy, moderate, or difficult?
3. Did you work on it mostly alone or did you discuss the problems with others?
4. How deeply do you feel you understand the material it covers (0%–100%)?
5. Any other comments?

1. 4-5 hours
2. Difficult, many topics I have learned before and understood, but with the passage of time I forgot how to do it. I had to relearn most of these topics for the assignment.
3. Worked alone.
4. 50% understanding of the material.
5. Good wake up call to the reality of doing good in a class i.e getting a good grade versus actually mastering and remembering how to solve problems.