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Does Asymmetry Matters in Social Conflict? A Multi-Player Asymmetric War of Attrition on Private Provision of Public Good

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1 Model

There is an indivisible public good potentially beneficial to N different individuals. I denote each player by $i \in I_N$, where $I_N = \{1, 2, ..., N\}$. Under natural state, cooperative provision is not an option, and instead a continuous-time war of attrition becomes inevitable. This war begins at $t = 0$ and at this time point each player chooses a stop time when, if no one has provided the good yet, he will provide. At every $t > 0$, each player can change their stop time, although later analysis shows that this change is not likely.

Here is the information structure: one player, say i, knows exactly the cost of his individual provision $c_i > 0$, the rate $r_i > 0$ that he obeys to exponentially discount his expected gain at time t with $e^{-r_i t}$, and his valuation of this public good v_i. All values of cost and discount rate are common knowledge, whereas each valuation v_i is private information anticipated by other players as a random variable subjected to a cumulative distribution function $F_i : [\underline{v}_i, \overline{v}_i] \to [0, 1]$ in which $0 \leq \underline{v}_i < c_i < \overline{v}_i < +\infty$ for all i. Every v_i is extracted independently, and for simplicity I sometimes call player i with type v_i simply as player v_i. Note that this strict relationship of $\underline{v}_i$, $\overline{v}_i$, and c_i is important because it necessarily guarantees the uniqueness of equilirium. The analysis requires some harmless assumptions: each F_i yields a dense function $f_i : [\underline{v}_i, \overline{v}_i] \to \mathbb{R}^+$ which is differentiable and strictly bounded from 0.

Player i's pure strategy is a function $T_i : [\underline{v}_i, \overline{v}_i] \to \mathbb{R}^+ \cup \{0, +\infty\}$ referring to the stop time that player i chooses at $t = 0$ given his valuation. Only when no provision happens before $T_i(v_i)$ will this player provide at this time. If someone provides first, each player gains lumpsum payoff according to their decision and information; and if multiple players, say m players, provide at the same time, each of them pays c_i/m; and if every one chooses to wait forever, each earns zero. The moment the public good is provided, the game ends.

I consider pure-strategy perfect Bayesian equilibrium, and in the following sections by *equilibrium* I mean this kind, unless otherwise specified.

2 Equilibrium Analysis

In this section, I firstly show a group of necessary and sufficient conditions for equilibrium including the differential equations charactrizing an equilibrium and three associated boundary conditions which reveal two important geometric features, *instant exit* and *strict waiting.* I will discuss the economic insight behind these two features and how they determine an equilibrium. Then, I show some important behavior patterns of the differential equations' solution, and I further utilize these solution patterns to prove the existence and uniqueness of an equilibrium. Finally, I introduce a spacial case, AUD war, both useful and tractable.

2.1 Characterization

I introduce some definitions and notations before presenting the necessary and sufficient conditions. First, I give the expected gain of player v_i with the choice of stop time t_i fixing other opponents' strategy $T_j(.)$. For convenience, I define v_{-i} as a vector of valuations of players other than i, $T_{-i}^{min}(v_{-i}) = \min_{j\neq i} T_j(v_j)$, and $F_{-i}^{min}(t) = \text{Prob}(T_{-i}^{min} \leq t)$. This expected gain, denoted by $R_i(t_i|v_i)$, is:

$$R_i(t_i|v_i) = v_i \int_0^{t_i} e^{-r_i s} dF_{-i}^{min}(s) + (v_i - c_i)e^{-r_i t_i}(1 - F_{-i}^{min}(t_i)) \tag{1}$$

The first term above represents i's expected gain if others has not provided before t_i, and the second corresponds to the situation where some other player provides before t_i. Since the characterization shown later indicates that in equilibrium there is no simultaneous provision, I omit this situation in (1). Further, fixing a strategy profile $\{T_j, j \in I_N\}$, I define the dense function of player i's valuation conditional on the arrival of time point t as:

$$f_i(v_i|t) = \begin{cases} \frac{f_i(v_i)}{\text{Prob}(T_i(v_i)\geq t)}, & T_i(v_i) \geq t \\ 0, & T_i(v_i) < t. \end{cases}$$

This function is the updated information others have about player i at t, and with this each player, say j, can recalculate the conditional expected gain $R_j(t_j|v_j,t)$ and therefore update the newly optimal stop time choice. However, one can easily verify that $R_j(t_j|v_j,t) = R_j(t_j|v_j)/(1-F_{-j}^{min}(t))$ so that this proportional change of expected utility guarantees that future modification of stop time will not occur.

Now I present the necessary and sufficient conditions in Lemma 1:

Lemma 1 *If $\{T_j, j \in I_N\}$ corresponds to a perfect Bayesian equilibrium if and only if:*

(i) $T_i(v_i) = +\infty$ *on* $[\underline{v}_i, c_i)$, *and* $T_i(v_i) < +\infty$ *on* $(c_i, \overline{v}_i]$, *for all* $i \in I_N$[1].

(ii) $\lim_{v_i \to c_i+0} T_i(v_i) = +\infty$, *for all* $i \in I_N$.

(iii) N-2 strict waiting(s): for some, say every $j_k \in I_0 = \{j_1, j_2, ..., j_{n_0}\}$ *where* $n_0 \geq 2$, $T_{j_k}(\overline{v}_{j_k}) = 0$; *and for every* $\iota \in I_N - I_0$, $T_\iota(\overline{v}_\iota) > 0$.

(iv) One instant exit: for every $i \in I_N$, *there exists a* $m_i = \sup_{T_i(v_i)>0}(v_i)$; *and there can be at most one* $k \in I_N$ *whose* $m_k < \overline{v}_k$ *and therefore* $T_k(v) = 0$ *for* $v \in (m_k, \overline{v}_k]$. *Obviously,* $k \in I_0$.

(v) For all $i \in I_N$, $T_i(v_i)$ *is continuous and strictly decreasing on* (c_i, m_i) *where* m_i *is defined above. Thus* $T_i(v_i)$*'s inverse function* $\Phi_i(t_i)$ *exists.*

(vi) For all $t \in \mathbb{R}^+$, *if there are* M Φ*s that have definition at* t, *denoted by* $I(t) = \{j_1, j_2, ..., j_M\}$ *where* $M \leq N$, *then they are differentiable at* t *and characterized by:*

$$\Phi_{j_i}'(t) = \frac{F_{j_i}(\Phi_{j_i}(t))}{f_{j_i}(\Phi_{j_i}(t))}\left[\frac{r_{j_i}}{c_{j_i}}(\Phi_{j_i}(t) - c_{j_i}) - \frac{1}{M-1}\sum_{k=1}^{M}\frac{r_{j_k}}{c_{j_k}}(\Phi_{j_k}(t) - c_{j_k})\right] \tag{2}$$

For all $j_i \in I(t)$.

Apart from some stylized properties similar to those of the equilibrium of two-player wars of attrition like clauses (i), (ii), (iv), and (v), the other two conditions (iii) and (vi) demonstrate a peculiar geometric pattern of multi-player equilibrium: *cascade free ride*. In a metaphorical way, the shape of equilibrium curves appears to be a long cascade with dams allocated in its middle: around $t = 0$ the number of curves is small, which is similar to the small brook near the origin of a river; and every time it arrives at some players' strict-waiting time, at least one curve will appear, as if the river is getting stronger when passing every dam. For an example,

[1] At v_i=0, the player is indifferent between providing immediately and waiting forever, but this situation only corresponds to zero measure.

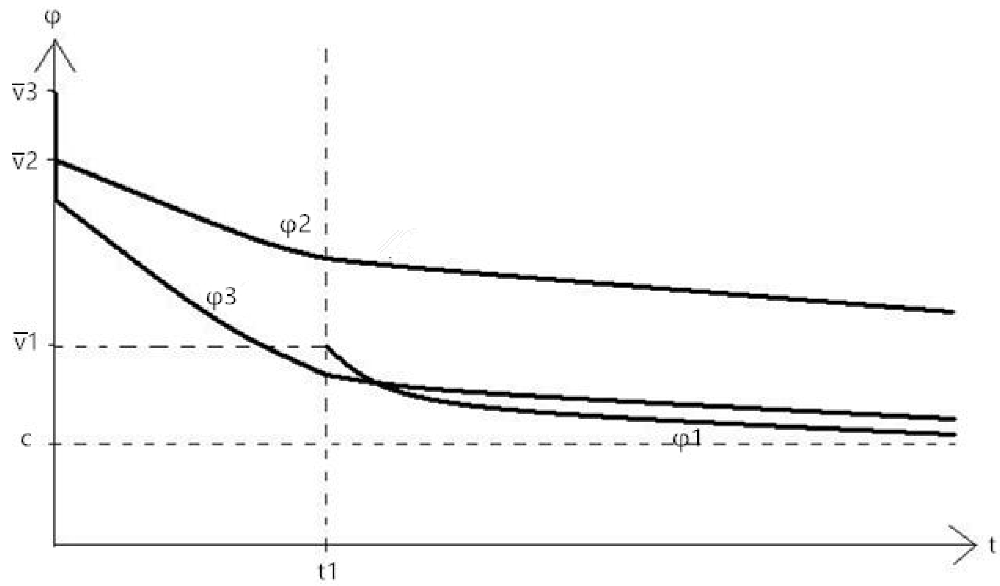

Figure 1: This figure demonstrates the geometric properties in Lemma 1. The third curve, $\Phi_3(t)$, starts at $\Phi_3(0) = m_3 < \overline{v}_3$ corresponding to an instant exit, while the first, $\Phi_1(t)$, does not exist until $t = t_1$ referring to a strict waiting. t_1 is a division, and on its left and right sides curves obey two-player and three-player versions of (2), respectively.

consider three potential providers with an information setting asymmetric enough to make both instant exit and strict waiting possible. I denote the strictly free rider by 1 and his minimal waiting time by t_1. Then according to Lemma 1(vi), before t_1 only player 2 and 3 have the probability to supply, but after that all three individuals can supply. Let each player's cost of provision be the same, and I depict a possible solution of this three-player war in Figure 1.

I introduce some definitions and notations for analytical convenience. I call those strict-waiting time points as *divisions*, if there is no confounding. Between different pair of adjacent divisions exist different number of curves obeying different version of (2), and I call the players corresponding to these curves as *active players*. Curves of players who remain active on both sides of one division are continuous at the division. Further, I denote the domain between two adjacent divisions where there are M active players by $\Upsilon(M)$ and call the group of differential equations corresponding to this domain as the M problem. The boundary conditions of a M problem needs to be specified. Finally, let $I(t)$ denote the set of subscripts of active players at t or, equivalently, I_M denote the set corresponding to M players.

Importantly, the presence of instant exit and strict waiting discloses the relative *incentive position* among all types of all players. The core here is that only the types in the same position are revealed at the same time. Formally, equation (**??**) characterizes the first-order condition for each active player in the M problem. The interpretation of (**??**) is based on the fact that: *a)* the $\Phi_{j_i}(t)$ on its right represents the type of j_i being revealed at t, and thus $r_{j_i}dt(\Phi_{j_i}(t) - c_{j_i})$

is the extra gain of providing at t instead of $t+dt$; and *b)* each $F_{j_k}(\Phi_{j_k}(t))$ on the left is equivalent to the probability of j_k's providing after t, so dt multiplying the left side of (**??**) equals the extra gain of waiting during this infinitesimal period. Intuitively, the revelation of players' types is such that the two countervailing incentives, the gain of providing immediately and keeping waiting, faced by the type being revealed are balanced by the speed of revelation of other players' types. So, types being revealed at the same time make the incentives of each other mutually balanced, and therefore they are in this sense equivalent in incentive position.

The instant-exit types are in the highest position, namely, they value the public good so much that no simultaneous revelation with others is able to offset this high incentive to provide. Between each pair of adjacent divisions, types of active players are revealed continuously. However, even the highest types of the inactive players generate incentives too low to balance the incentives faced by active types. Thus, the multi-player asymmetric setting requires an equilibrium where valuations of different individuals are, in a way, ranked by both the decreasing monotonicity of each $\Phi_i(.)$ and the presence of instant exit and strict waiting.

That asymmetry affects equilibrium by changing the number of active players has the same spirits as Bergstrom, Blume, and Varian (1986). They discussed a similar public-good provision problem with a static model with complete information, and they argued that considerable redistribution of wealth will cause some players incapable of providing any public good and this reduction of the number of active players alleviates underprovision. My model incorporates asymmetry in valuation[2], and at each moment all unrevealed types of all players play a similar static game in which everyone chooses either to wait to gain from the probability of others' providing first or to concede immediately to be active. These two-sided incentives are similar to the trade-off between private and public consumption in Bergstrom et al. (1986). What is new lies in that the introduction of incomplete information and dynamic makes this scale-change process of active group endogeneous: at the beginning, when the saliently high types have not been revealed, they tend to provide soon and this fast revelation makes others' gain of waiting so high that any of their type will not concede; and as the pass of time reveals the extremely high types, the incentives start to balance and at certain points some players with comparatively high valuations will become active.

Above, I explain how instant exit and strict waiting become into presence. Hereafter, by calling a type *stronger* than another I mean in equilibrium the former type selects a shorter stop time or the former is greater than latter if they are both instant-exit types. Likewise, a

[2]This is equivalent to setting utility function in Bergstrom et al. (1986) asymmetric which will yield the same reduction of active population.

player has a higher incentive position[3] if his highest type strictly waits for a shorter period or this type exits instantly.

Additionally, the importance of instant exit and strict waiting lies in that these two conditions complement the degrees of freedom of solution. The flexible boundary conditions including one variable instant exit and $N-2$ variable strict-waiting times help to adjust the solution of equations (2) to satisfy another boundary condition, Lemma 1(ii). Combining all three kinds of boundary conditions, it matches right $1+1+(N-2)=N$ degrees of freedom so that this condition set provided by Lemma 1 is potential in selecting a unique equilibrium.

However, N degrees of freedom are not enough. A huge literature of war and bargaining confirms that one asymmetric war of attrition cannot yield unique equilibrium unless it is perturbed, namely, for each player there must be a positive probability of some others' waiting forever. For example, Fudenberg and Tirole (1986) introduced a positive probability of each player's being benefited even without concession. Kornhauser, Rubinstein, and Wilson (1989) used a slight probability of irrational type who only plays a fixed stratgy to refine a unique equilibrium of their bargaining game, the spirits of which were also borrowed by Kambe (1999) and Abreu and Gul (2000) to acquire uniqueness. Finally, Myatt (2005) considered three forms of perturbation: exit failure, hybrid payoff, and time limit. Specifically in my model, this perturbation lies in that every player is anticipated with positive probability of being incapable of individual provision or, equivalently, with $F_i(c_i) > 0$. The straightforward intuition reveals that the possibility of arrival at any time imposes perfection on off-equilibrium path to eliminate unreasonable solutions.

Yet, most of asymmetric wars modeled in literature involves only two players. To show how this positive probability of waiting forever guarantees a unique equilibrium in my multi-player game, I present a lemma in the next subsection revealing two asymptotic properties of the solution of the initial-value problem in Lemma 1. Further, I complete the proof of existence and uniqueness.

2.2 Existence and Uniqueness

The following Lemma 2 demonstrates, in a technical sense, how this perturbation affects the behavior of the solution of a M problem. Consider a M problem with boundary conditions all on its left-side division, then *a)* the satisfaction of Lipschitz condition yields a unique solution, and

[3] Here, I abuse the notion of incentive position a little, for it actually describes the relative position between types, rather than players. However, later analysis tells that under certain conditions the highest type of a player can be representative, and this abuse becomes harmless. But still, its meaning should be differentiated when describing different objects.

each solution curve is consistently monotonous in response to variation of left-side boundaries, and *b)* when time approximates infinity, for any selection of boundaries on the left, the solution curves can be either convergent or divergent and a convergent solution must satisfy Lemma 2(ii) and (v), and finally *c)* the convergent case is asymptotically sensible, namely, any change in boundaries on the left makes convergence collapse to divergence, which is critical in later proof of uniqueness.

I introduce some notations before presenting Lemma 2. I extend the definition of each (F_i, f_i) to domain $(\overline{v}_i, +\infty)$ for analytical convenience when the left-side boundaries make curve i reach the line, $v = \overline{v}_i$. And since the final equilibrium curves do not step into these domains, this extension is harmless. To conserve the differentiability and boundedness of Fs and fs, I choose $\Delta F \in (0, 1)$ and $\Delta f \in (0, \min_{i\in I_N} f_i(\overline{v}_i))$ and for all $i \in I_N$ and $v > \overline{v}_i$ define:

$$f_i(v) = f_i(\overline{v}_i) + \frac{f_i'(\overline{v}_i)}{|f_i'(\overline{v}_i)|}\Delta f[1 - e^{-\frac{|f_i'(\overline{v}_i)|}{\Delta f}(v-\overline{v}_i)}]$$
$$F_i(v) = F_i(\overline{v}_i) + \Delta F[1 - e^{-\frac{f_i(\overline{v}_i)}{\Delta F}(v-\overline{v}_i)}]$$

One can easily verify that with such extension equations (2) still satisfy local Lipschitz condition everywhere.

Finally, for analysis of the behavior of a M problem's solution in response to the variation of left-side boundary conditions, I denote all active players by $I_M = \{1, 2, ..., M\}$ and define $m = (m_1, m_2, ..., m_M) \in B_M = \prod_{i=1}^{M}[c_i, +\infty)$ as a vector being such that $\{\Phi_i(0) = m_i, i \in I_M\}$ is the corresponding left-side boundary selection. I rewrite the solution as $\{\Phi_i(t, m), i \in I_M\}$, each element of which is a multivariate function with respect to t and m. Formally, $P_M = \{I_M, B_M, (c_i, r_i, F_i, f_i)_i, (\Phi_i(t, m))_i\}$ denotes a M problem with the M-player version of (2) as its differential equations and $m \in B_M$ as all possible left-side boundary selections.

Lemma 2 *For a P_M problem defined above, its solution has the following behavior patterns:*

(i) Comparative monotonicity: for all $i \in I_M$, any $m^{(0)} = (m_i^{(0)}, m_{-i}$[4]$)$ and $m^{(1)} = (m_i^{(1)}, m_{-i}) \in B_M$ where $m_i^{(0)} < m_i^{(1)}$ lead to that a) *$\Phi_i(t, m^{(0)}) < \Phi_i(t, m^{(1)})$ for all $t \geq 0$, and* b) *$\Phi_j(t, m^{(0)}) > \Phi_j(t, m^{(1)})$ for all $t > 0$ and $j \neq i$.*

(ii) Asymptotic sensitivity: for all $i, j \in I_M$, $m \in B_M$, and $t > 0$, partial derivative $\partial\Phi_j(t, m)/\partial m_i$ exists; and if $\Phi_r(s, m) \to c_r$ as $s \to +\infty$ for all $r \in I_M$, then $\partial\Phi_i(t, m)/\partial m_i \to +\infty$ while $\partial\Phi_j(t, m)/\partial m_i \to -\infty$ as $t \to +\infty$ where $j \neq i$.

[4] m_{-i} represents the vector $(m_1, m_2, ..., m_{i-1}, m_{i+1}, ..., m_M)$

(iii) Two asymptotic patterns: when $t \to +\infty$, solution $\{\Phi_i, i \in I_M\}$ can only take on one of two patterns: a) convergence*: $\Phi_i(t) \to c_i + 0$ for all i meanwhile satisfying Lemma 1(ii) and (v), and* b) divergence*: there exists at least one $k \in I_M$ being such that Φ_k approximates $+\infty$ or $\underline{v}_k$.*

Lemma 2 mainly demonstrate such a property of P_M problem's solution: a boundary selection yields convergent solution if and only if it satisfies Lemma 1(ii) and (v), and any change in these boundary conditions, no matter how slight, will break the convergence. The importance of this lemma lies in that after the rightmost division of an equilibrium stands a P_N problem, and Lemma 2 provides useful tools to search for an equilibrium-like solution of this problem, which means it satisfies Lemma 1, and also to prove its uniqueness.

The proof of Lemma 2 indicates that the key of equilibrium uniqueness is the positive probability of each player's valuation being less than his cost, because line $v_i = c_i$ divides the space into two areas in which solution manifests upper- and lower-divergence, respectively, and thus convergence only appears around $v_i = c_i$ and is sensitive to boundary variation. The economic insight is straightforward: any perturbation of one player's behavior at or near $t = 0$ can be expanded and transimitted to any time that follows. Clearly, this transmission from $t = 0$ to $+\infty$ cannot be realized without the perturbation that players have chance to wait forever.

To observe the multiplicity of the opposite case, I assume that for all i $\underline{v}_i > c_i$. Consider a P_M problem whose $B_M = \prod_{i=1}^{M}[\underline{v}_i, +\infty)$. First, there exist degenerate equilibria in which all types of one player provides immediately while all types of others wait forever. However, even with slight probability of the immediate provider's valuation being less than his cost, his opponents will find waiting forever no longer optimal because doing so leaves the chance for everyone to be inactive and thus to earn zero. Second, multiplicity can also occur even if equilibrium is selected according to Lemma 1. In this case, no matter what boundaries are selected, (2) always tells that: since $F_i(\underline{v}_i) = 0$ forbids each curve to go through the line $v_i = \underline{v}_i$, it yields $\Phi_i'(.) < 0$ and $\Phi_i \to \underline{v}_i + 0$ naturally satisfying Lemma 1, which means there is no asymptotic sensitivity and boundary selection can be arbitrary.

Finally, I prove the existence and uniqueness of equilibrium. To do this, I employ a *backward specification* strategy. On the rightmost domain where all players are active stands a P_N problem, and the first step of searching for an equilibrium is to prove the existence and uniqueness of the solution of this problem that satisfies Lemma 1. Then, from the right side to the leftmost $t = 0$, I sequentially determine the solution between each adjacent pair of divisions according to Lemma 1(iii) and (iv) by adjusting the total number of divisions and the relative distances between each adjacent pair. This sequential specification process also yields unique solution. Now I present this result:

Proposition 1 *There uniquely exists a perfect Bayesian equilibrium.*

Note that the proof of Proposition 1 relies highly on the asymptotic sensitivity of a convergent solution, which would not hold had I assumed that the type with the lowest valuation can also provide the good individually. This uniqueness property makes the analysis of multi-player asymmetric war become possible.

I introduce some notations to link a public-good-provision war to its unique equilibrium. Define Ω as the space of all proper wars of attrition. By *proper* I mean: if $w = \{N, (r_i, c_i, F_i, f_i, \underline{v}_i, \overline{v}_i)_i\} \in \Omega$, then w is parametrized such that $N \geq 2$, and that for all $i \in I_N$ $r_i > 0$, and $\overline{v}_i > c_i > \underline{v}_i \geq 0$, and $F_i(.)$ is a differentiable distribution function defined on $[\underline{v}_i, \overline{v}_i]$, and $f_i(.)$ is the corresponding density function which is differentiable and bounded from 0. On the other hand, define Ξ as the space of unique equilibrium of all possible proper wars, and its element is written as $e = \{N, \overline{K}, (\Phi_i)_i, (d_K, M_K, I_K)_K\}$, in which $N \geq 2$ and $\overline{K} \geq 0$, and for all $i \in I_N$ $\Phi_i(.)$ denotes the inverse function of player i's equilibrium strategy $T_i(.)$, and for all $K = 1, 2, ..., \overline{K}$ d_K marks the location of the Kth division from the left whereas $d_0 = 0$ and $d_{\overline{K}+1} = +\infty$, and M_K and I_K represent, respectively, the number of active players and the set of subscripts of these players during $t \in [d_K, d_{K+1})$. Finally, I define the mapping $E : \Omega \to \Xi$ such that $E(w)$ is the unique equilibrium of $w \in \Omega$. In later discussion, I apply similar form to denote wars and associated equilibria, that is, I use braces to contain all parameters or information describing an object in which those with subscripts denote the parameters or information of the unit with the same subscript, while th ose without denote the parameters or information shared by all units. Here, by units I refer to players or divisions. Unless otherwise specified, I use subscript i to denote all players and K all divisions of an equilibrium.

2.3 An Important Case: AD War

To demonstrate the characteristics of the equilibrium of this multi-player asymmetric war of attrition, I introduce an important family of proper wars, aligned-distribution war (AD war). This class of war is important for two reasons: *a)* apart from this case, there are very few mathematically tractable examples, and *b)* it corresponds to wide economic applications.

In each AD war, players have identical costs and dicount rates. Moreover, players' valuation distributions are all lower-conditional distributions of one fixed distribution with different upper bounds. Formally, let F be a proper distribution whose upper bound and lower bound are denoted by $\overline{v}$ and $\underline{v}$, respectively, and $(\overline{v}_i)_{i=1}^N$ a set containing N players' upper bounds, each of which is no greater than $\overline{v}$ and greater than the provision cost. Then, the ith player's distribution is the lower-conditional distribution of F corresponding to $\overline{v}_i$, that is, $F_i(v) = F(v|v \leq \overline{v}_i)$ for all i. Define $\Omega_{AD} \subset \Omega$ as the space of all such wars, and each of its element is denoted by $w_{AD} = \{N, r, c, F, \underline{v}, (\overline{v}_i)_i\}$.

This case corresponds to a large category of daily-life scenarios, because it can be seen as a partly revealed symmetric war indicating that players who are initially anticipated to be symmetric may have engaged in some previous games whose unequal outcomes shape the asymmetry of the current war. On the other hand, an AD war is also easy to analyze as well, for the rates, $f_i(v)/F_i(v)$, of different players are identical on their overlap domain implying that on each $\Upsilon(M)$ active players' behavior is characterized by symmetric differential equations.

First, I demonstrate how the equilibrium of an AD war looks like. Without loss of generality, let $\overline{v}_1 \geq \overline{v}_2 \geq ... \geq \overline{v}_N > c > \underline{v}$. Since these aligned distributions yield the same $F_i(v)/f_i(v)$, on each domain $\Upsilon(M)$ the M active curves are characterized by a group of symmetric differential equations, and it is easy to verify that backward specification gives symmetric solution[5] on each $\Upsilon(M)$. Formally, let $\Phi_{AD}(t|M, r, c, F, \overline{u}, \underline{v})$ denote the solution of the following initial-value problem derived from the symmetric M-player version of (2):

$$\begin{aligned} &\Phi'(t) = -\frac{1}{M-1}\frac{r}{c}\frac{F(\Phi)}{f(\Phi)}(\Phi - c) \\ &\Phi(0) = \overline{u} \end{aligned} \tag{3}$$

Therefore, the equilibrium $E(\{N, r, c, F, \underline{v}, (\overline{v}_i)_i\})$ is given by:

$$\begin{aligned} &\overline{K} = N-2;\ M_K = K+2,\ I_K = \{1, 2, ..., K+2\},\ K = 0, 1, ..., \overline{K} \\ &d_0 = 0,\ d_{\overline{K}+1} = +\infty;\ d_K - d_{K-1} = \Phi_{AD}^{-1}(\overline{v}_{K+2}|K+1, r, c, \overline{v}_{K+1}, \underline{v}),\ K = 1, 2, ..., \overline{K} \\ &\begin{cases} T_1(v) = 0,\ v \in (\overline{v}_2, \overline{v}_1] \\ \Phi_1(t) = ... = \Phi_n(t) = \Phi_{AD}(t - d_{n-2}|n, r, c, \overline{v}_n, \underline{v}),\ t \in [d_{n-2}, d_{n-1}),\ n = 2, 3, ..., N \end{cases} \end{aligned} \tag{4}$$

In Figure 2, I depict the solution curves of a three-player AD war whose geometric shape looks exactly like a cascade dropping sharply at the beginning and slowing down when passing

[5] The proof of Proposition 1 suggests the uniqueness of the rightmost P_N problem's solution, which requires the solution characterized by symmetric equations to be symmetric as well. This rightmost symmetry further ensures symmetric boundary conditions on all domains on its left side, and therefore the symmetric equations on each domain also results in symmetric solution.

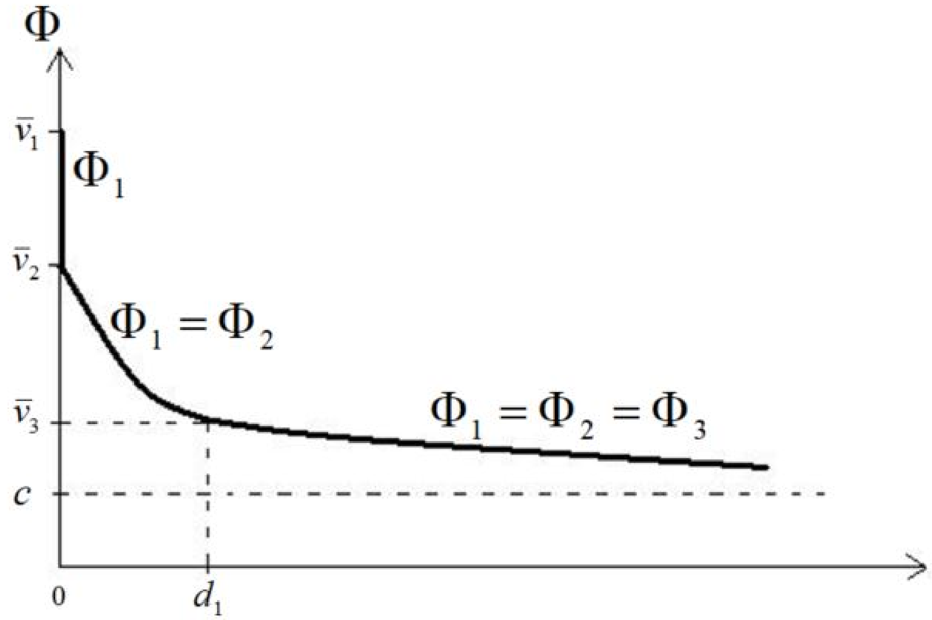

Figure 2: This figure demonstrates the equilibrium of a three-player AD war. The player with the highest upper bound provides instantly when his realized type is no less than the second-large upper bound. Before the first division two players are active, whereas after it the third becomes active as well. Active players always reveal their types symmetrically at the same time.

the division. This special case yields local symmetry at each t, and this allows valuations to be directly comparable, namely, the relationship between the incentive positions of two types is equivalent to the relationship of their numerical value. Consequently, the player with the highest upper bound has the strongest type.

Additionally, I use the uniform-distribution example to briefly show some economic insights on welfare analysis. I call this example the aligned-uniform-distribution war (AUD war). One feature of uniform distribution that simplifies the analysis is that any lower-conditional distribution of it remains uniform, and this feature guarantees that the equilibrium of such a war is analytic. I let $\Phi_{AUD}(t|M, r, c, \overline{u}, \underline{v}) = \Phi_{AD}(t|M, r, c, F, \overline{u}, \underline{v})$ which is given by:

$$\Rightarrow \Phi_{AUD}(t|M, r, c, \overline{v}, \underline{v}) = \underline{v} + \frac{c - \underline{v}}{1 - \lambda e^{-\frac{1}{N-1}\rho r t}} \tag{5}$$

Above $\lambda = 1 - (c - \underline{v})/(\overline{v} - \underline{v})$ and $\rho = 1 - \underline{v}/c$, and sometimes I simply denote this curve by $\Phi_{AUD}(t|M, \lambda)$ for other parameters are shared by all players in the same AUD war. Combining (4) and (5), one acquire the equilibrium.

One important property of the equilibrium of an AUD war is that the expectation of the exponential discount factor with shrinked power is only determined by the value of cost, lower bound, and the maximal upper bound, and therefore it remains unaffected when the scale of war changes. This property shows both the strict loss of welfare[6] out of delay and the asymmetric positions of players in determining this welfare level. I present it in the following proposition:

Proposition 2 *Any N-player asymmetric AUD war, denoted by $w = \{N, (\overline{v}_i)_i, r, c, \underline{v}\}$, gives:*

$$E_{t_m}[e^{-\rho r t}] = 1 - \frac{c - \underline{v}}{\max\limits_{i \in I_N} \overline{v}_i - \underline{v}}$$

$\rho = 1 - \underline{v}/c$, *and* $E_{t_m}[.]$ *calculates the expectation with respect to stop time under equilibrium.*

This proposition provides an example illustrating two important intuitions on welfare. First, loss from delay always occurs, because the expectation of the exponential function of stop time defined in Proposition 2 is strictly less than one. This loss appears no matter how large the scale of population becomes, which questions the Theorem 6 of Bliss and Nalebuff (1984) stating that under certain parametrization large population makes inefficiency vanishes. In the next section, I formalize this insight and discuss the condition for efficiency which can barely be satisfied in real-life scenario.

The second thing Proposition 2 tells is that different players affect the outcome very differently in an asymmetric war. If one sees the expected discount rate with shrinked power as an adequate measure of stop time and therefore a measure of welfare, then Proposition 2 indicates that the welfare level of a society is seemingly solely determined by the strongest type in it or, equivalently, by the maximal upper bound. And in the next section, I generalize this intuition to the exact welfare of each player: when population grows large, the welfare of each type only depends on his valuation and the strongest types in the society. However, this result does not mean that the highest type will undertake all cost of provision, but implies a close dependence of other players' behavior on the behavior of the strongest individual. Actually, this dependence results from the requirement that types revealed simultaneously must be in the same incentive position. Namely, when fixing the strongest player's valuation distribution, any variation of a comparatively weaker player's distribution mainly changes his strict-waiting time to adjust his strategy to suit the ranking of incentive positions which is mainly determined by the strongest player. Finally, since the welfare level depends highly on this ranking, the strongest type has the strongest determinant power on each player's expected gain.

[6] In this section and the one that follows, by *welfare* I mean *interim* welfare calculated after the realization of players' types and before the war begins.

2.4 Comparative Statics

In this subsection, I perform some comparative statics on the equilibrium. The first proposition presented below compares the behavior of different players in the same war conditional on some strict relationship between their parameters. Specifically, I investigate how the difference between cost, discount rate, and *revelation rate* affects the relative provision time of two players. Here, revelation rate refers to $f(.)/F(.)$, the density of the type being revealed conditional on the revelation of all types above it. The interpretation of this rate is discussed later, which can be different if other settings change. Now I present this result:

Proposition 3 *Consider a N-player proper war, say $w \in \Omega$, if there exists a pair of players, denoted by $\alpha, \beta \in I_N$ whose solution curves both exist on $[t_0, +\infty)$, parametrized such that:*

(i) $r_\alpha = r_\beta$, $c_\alpha = c_\beta = c$, and $f_\alpha(v)/F_\alpha(v) \gtreqless f_\beta(v)/F_\beta(v)$ on their overlap domain, then $\Phi_\alpha(t) \lesseqgtr \Phi_\beta(t)$ for all $t \in [t_0, +\infty)$.

(ii) $r_\alpha = r_\beta$, $f_\alpha(v)/F_\alpha(v) = f_\beta(v)/F_\beta(v)$ on their overlap domain, and $c_\alpha \gtrless c_\beta$, then $\Phi_\alpha(t) \gtrless \Phi_\beta(t)$ for all $t \in [t_0, +\infty)$.

(iii) $c_\alpha = c_\beta = c$, $f_\alpha(v)/F_\alpha(v) = f_\beta(v)/F_\beta(v)$ on their overlap domain, and $r_\alpha \gtrless r_\beta$, then $\Phi_\alpha(t) \lessgtr \Phi_\beta(t)$ for all $t \in [t_0, +\infty)$.

The last two clauses of Proposition 3 yield straightforward intuitions: higher cost reduces the gain of provision entailing a delayed strategy, while higher discount rate corresponding to impatience increases the opportunity cost of waiting so that player tends to provide sooner. However, the interpretation of the result associated with revelation rate is unclear, because under different assumptions this relationship between revelation rate yields different economic outcomes. For example, if the upper bounds of the two distributions are set to be equal, Proposition 3(i) indicates that the player with consistently lower revelation rate becomes active at a later time, which implies that the other player either strictly waits for a shorter while than this player or just provides instantly. In this case, the player with consistently higher revelation rate stands in a relatively higher incentive position. In contrast, if two players' probability of valuation being lower than cost is set to be equal, integrating (**??**) from some t to infinity[7] tells that the one with consistently higher revelation rate becomes active later than the other who hence stands in a higher incentive position.

[7] If α has a consistently higher revelation rate, then $\Phi_\alpha(t) < \Phi_\beta(t)$ for all $t \in [t_0, +\infty)$ which gives $lnF_\alpha(\Phi_\alpha(t)) - lnF_\beta(\Phi_\beta(t)) = \frac{r}{c}\int_t^{+\infty}(\Phi_\beta(s) - \Phi_\alpha(s))ds > 0$. This further implies that when Φ_α reaches its upper bound, Φ_β must be lower than its upper bound.

The interpretation of this discrepancy between these two cases introduced above must result from the different economic interpretations conveyed by the domination of revelation rate under different settings. In the first case, when two players' upper and lower bounds are set to be equal, respectively, the domination of revelation corresponds to subjecting higher probability to higher valuations. So, others tend to anticipate one with higher revelation rate to face higher opportunity cost of waiting so that in average he will provide sooner, and this belief leads to the free riding of other players which in turn forces each type of the high-revelation-rate player to actually provide sooner. Nonetheless, in the second case, when the probability of one's waiting forever is controlled, the domination of revelation rate implies being first-order stochasitcally dominated, which according to Myatt (2005) tends to drive this player to a weaker position without instant exit. The economic logic lies in that the low-revelation-rate player has larger upper bound of distribution and his highest valuation is salient enough to exit instantly.

Proposition 3 only exhibits horizontal comparison of players' behavior in the same war, and it seems more like a mathematical results since the interpretation hinges on the setting of parameters as discussed in the last paragraphs. For a more clear look into how the change of parameters affects players' relative incentive position, I present Proposition 4:

Proposition 4 *Consider a two-player proper war, denoted by w, where the players are parametrized by $\{r_i, c_i, F_i, \overline{v}_i\}$ for $i = 1, 2$, respectively. Denote the ratio of both players' left-side boundaries by $q(w)$, namely, $q(w) = \Phi_1(0)/\Phi_2(0)$ where Φs are w's equilibrium solution. Now, by changing one of the parameters of player 1 and fixing all others, it yields a new war, $\widetilde{w}$, with a new associated ratio, $q(\widetilde{w}) = \widehat{\Phi}_1(0)/\widetilde{\Phi}_2(0)$. Then:*

(i) If the changed parameter is cost and $\widetilde{c}_1 > c_1$, $q(\widetilde{w}) > q(w)$.

(ii) If the changed parameter is discount rate and $\widetilde{r}_1 > r_1$, $q(\widetilde{w}) < q(w)$.

(iii) If the changed parameter is valuation distribution and $\widetilde{f}_1(v)/\widetilde{F}_1(v) > f_1(v)/F_1(v)$ for all $v \in (c_1, \overline{v}_1)$, $q(\widetilde{w}) < q(w)$.

Unlike the previous proposition, Proposition 4 examines the difference between two wars with different parameters. Here, the ratio $q(w)$ is a direct measure of the relative incentive position of player 2 over player 1, since the higher this ratio, the higher the probaility of player 2's instant exit or the lower the probability of player 1's instant exit. Therefore, Proposition 4 reveals that either lowering the cost of, increasing the discount rate of, or consistently increasing the revelation rate of one player rises his relative position[8] in a war.

[8]This result also holds when the number of players becomes greater than 2, but it requires introduction of

3 Asymmetry and Social Welfare

Does asymmetry matters in social conflicts? In this section, I investigate this question by revealing the relationship between asymmetric incentives of members of society and the associated social welfare of a war of attrition. First, I discuss the different roles played by different members in the contribution to welfare level. Next, I argue that shaping asymmetry in a society may dramatically improve the outcome.

I introduce some definitions and notations. A society is *divided* if it consists of multiple heterogeneous groups and in each group all members are homogeneous. Formally, define $s = \{N, L, (r_\iota, c_\iota, p_\iota, F_\iota, f_\iota, \overline{v}_\iota)_\iota\}$ representing a N-member divided society facing a proper war of attrition. This society is divided into L different groups, and the ιth group takes p_ι proportion of the total population. Each member in the ιth group is symmetrically parametrized by $(c_\iota, r_\iota, F_\iota, f_\iota, \overline{v}_\iota)$. The rules of notation are the same as those introduced at the end of Section 2.2 only differing in that L denotes the number of groups and that ι denotes all groups.

3.1 Who Matters the Most?

The seminal work by Bliss and Nalebuff (1984) acquired a surprising result about social welfare, their Theorem 6, which states that when the lower bound of the distribution, from which each player's cost of provision is independently extracted, reaches 0, inefficiency out of delay vanishes as the population grows large.

This subsection mainly derives the counterpart conclusion of their limit theorem from my asymmetric war. However, my objective differs from theirs in two ways. First, I intend to answer how differently do unequally positioned individuals contribute to the welfare level of society. Second, I want to examine the condition that vanishes inefficiency to see whether it corresponds to daily-life application. Thus, I reconsider their approach only to look into the presentation of asymmetry in social conflicts over public-good provision.

To begin with, I demonstrate a useful lemma:

Lemma 3 *Consider a divided society parametrized by $s = \{N, c, r, L, (p_\iota, F_\iota, f_\iota, \overline{v}_\iota)_\iota\}$ which yields an aligned equilibrium: there is no instant exit or strict waiting, and thus $\Phi_\iota(0) = \overline{v}_\iota$ for all ι. Then, when N becomes sufficiently large (while maintaining the equilibrium aligned):*

(i) To maintain alignment, all upper bounds must be the same, namely, $\overline{v}_\iota = \overline{v}$ for all ι.

(ii) A player with valuation v earns expected gain $v(1 - c/\sum_{k=1}^{L} p_k \overline{v}_k) = v(1 - c/\overline{v})$.

highly convoluted definitions and notations for a clear proof while giving exactly the same insight, so I omit it.

And I present the symmetric version of Lemma 3 which is equivalent to Bliss and Nalebuff's (1984) Theorem 6:

Corollary 1 *Consider a symmetric N-player war parametrized by $w = \{N, c, \overline{v}\}$. When N becomes sufficiently large, the expected welfare of player v approximates $R(v) = v(1 - c/\overline{v})$.*

The condition of an aligned equilibrium in Lemma 3 simplifies the proof by only considering a society whose each group shares the same incentive position. This lemma gives three insightful results: *a)* no matter how large the population grows, inefficiency always occurs, unless the society subjects positive density to some group's having extreme type with infinite valuation; and *b)* when population grows large, the welfare level is solely determined by the valuations of those with the highest incentive position, namely, the upper bounds of all groups, and the population proportion of each of these highest types; and finally *c)* a group's highest valuation type somehow determines the incentive position of this group, since groups in an aligned position must have identical upper bounds.

The proof of Lemma 3 gives more information. First, provision is realized with probability one within a short time after $t = 0$. But why inefficiency still occurs given this instant provision? This is because any type below the upper-bound type tends to free ride for a sufficient long time as the population grows large, so that only the types in an almost-zero-measure set which is very close to the upper bound matter and the probability of everyone's not providing around $t = 0$ remains considerable. Consequently, at each moment, the conditional distribution of stop time subjects most of the probability to two events: either someone provides instantly or nobody provides within finite time. This implies that the distribution density of stop time decreases extremely sharply so that the probability of someone providing during a period is much smaller than that of the periods before it and much larger than that of the periods after it.

For better understanding of the relationship between upper bound and incentive position in limit case which is implied by Lemma 3(i), I present a complementary lemma:

Lemma 4 *Consider a divided society with sufficiently large total population. Then, the group with higher distribution upper bound has stronger incentive position in the society.*

It is surprising that the highest type is decisive in a large society. First, the upper-bound types completely determine the ranking of the incentive positions of all groups. Additionally, the highest unrevealed type determines the expected gain of each member of this society, even in every subgame. What is curious is that this determinant property is irrelevant to the shapes of distribution functions or even the density of the highest type. The explanation lies in that, when population grows large, the law of large numbers ensures that even the type with very small density will occur with probability one, and this makes the limit expected discount factor irrelevant to how each player's valuation is specifically anticipated. Actually, Lemma 2 tells that this limit expectation calculated under the equilibrium of an arbitrary society equals that of the *counterpart 0-AUD war*. A 0-AUD war is an AUD war with 0 as the lower bound of each player's distribution, and the counterpart 0-AUD war corresponding to a society is constructed by changing all valuation distributions of this society to be uniform and their lower bounds to be 0, while fixing all other parameters including the upper bounds. This equivalency is in the sense of welfare which is summarized in the theorem below:

Theorem 1 *Consider a divided society with sufficiently large total population. Then, this society is equivalent to its counterpart 0-AUD war in the sense that, at every moment in a war of attrition, the welfare level of the society is solely determined by the possibly highest valuation among all members. Denote this highest valuation at* t *by* $\overline{v}(t)$*, and then the expected welfare of a member with valuation* v *is* $v(1 - c/\overline{v}(t))$ *where* c *is the cost of provision.*

Finally, I discuss the condition for efficiency. Bliss and Nalebuff (1984) modeled uncertain cost in private provision of public good, and the condition they found for efficiency is the occurrence of player with zero cost. In contrast, I model uncertain valuation in war of attrition whose efficiency hinges on the positive possibility of infinite valuation. Both conditions are obviously equivalent, because it is the valuation-cost ratio determines the incentive one faces. The intuition behind this lies in that the behavior of players with types in a nonzero-measure set closely around the extreme type with either zero cost or infinite valuation is fixed, namely, providing immediately is a dominant strategy for them. Consequently, the law of large numbers guarantees instant exit with probability one.

However, this condition is hard to satisfy in a real-life scenario. For one thing, in practice it is difficult to believe that there exist some individuals who either are capable of providing a public good with no cost or attach the good with infinite valuation. And for another, since the players with zero cost or infinite valuation should not be anticipated with a high probability[9], this efficiency result makes a serious requirement on the scale of population, that is, the number of members needs to be extremely large to ensure an instant exit, which the reality for most of the time cannot satisfy. Consequently, this efficiency proposition is not so robust as the results

[9]Especially the latter case, because it is mathematically absurd to subject a considerable density to the positive infinity.

presented before on player' asymmetric contributional position in this subsection.

3.2 Asymmetry as an Improvement

In previous discussion, the main idea is that ex ante asymmetry differentiates individuals into different behavioral positions. Here, I show that under certain circumstances this asymmetric positions can improve the wefare level in comparison with symmetry.

The insight seems straightforward, as asymmetry makes some players more salient than others so that they tend to provide sooner and thus it yields an improvement of welfare level. However, the result is acutally highly dependent on parametrization. For example, a two-player war as the combination of a strong player and a weak player may yield either higher welfare level due to the probability of the strong one's providing instantly, or lower welfare level due to the even worse free-riding problem in other no-instant-exit cases. The comparative strength of both countervailing forces sensitively relies on the particular geometric shape of players' distributions and their relationship with other parameters, like cost and discount rate.

Notwithstanding this dependence of parameters, I find it possible to discuss the condition for asymmetry to be an improvement in two important cases, and since both cases cover a large class of wars of attrition, the analysis of them is adequate in telling most of the insight.

The first case is the asymmetry in a large divided society which has been discussed in the last subsection. Intuitively, since Theorem 1 suggests that it is the highest upper bound in a large-population society that decides the welfare level of this society, any break of symmetry that gives a greater maximal upper bound makes every member better off. Borrowing the notations used in the last subsection, I present this insight in the following proposition:

Proposition 5 *Consider a family of divided societies in which each has identical cost of provision, discount rate, and average distribution upper bound, that is, $\sum_{k=1}^{L} p_k \overline{v}_k = \overline{v}$ for all societies. Then, the symmetric society, which satisfies $\overline{v}_k = \overline{v}$ for all group k, yields the lowest welfare level among all in this family.*

The condition above that the average upper bound of each society holds the same corresponds to an important economic application, the redsitribution of revenue. Imagine that there exists a central party who has the coercive power to extract revenue from some groups to redistribute to others, and also that before the war starts, this party promises to redistribute the revenues in a fixed pattern the moment someone provides the public good first. This manner of redistribution is equivalent to move the support of valuation distribution of each group upwards or downwards, so if there is no side payment from or profit transferred to the central party, the average upper bound of this society remains the same under redistribution. Consequently, Proposition 5 indicates that any redistribution exerted on a symmetric society improves efficiency.

The second case lies in the asymmetry in an AD war introduced in Section 2.3. The analysis of this case gives the condition for asymmetry to be an improvement, and this condition perfectly illustrates how the comparative strength of the two countervailing forces brought by asymmetry, the incentive to provide sooner and the incentive to free ride longer, is determined.

I recall and introduce some notations first. Consider an AD war parametrized by $w_{AD} = \{N, r, c, F, \underline{v}, (\overline{v}_i)_i\}$. Define function $e_n(u)$ as the expected discount factor calculated under the equilibrium of a n-player symmetric AD war with upper bound u, namely, $e_n(u)$ measures the welfare level of $E(w_n(u))$ where $w_n(u) = \{n, r, c, F, \underline{v}, \overline{v}_i = u\}$. Denote the symmetric equilibrium strategy of this war by $T_n(v|u)$, and when $n \geq 2$ $e_n(u)$ and $T_n(v|u)$ are given by:

$$\begin{aligned} e_n(u) &= E_{t_m}^{w_n(u)}[e^{-rt}] = \frac{1}{F^n(u)} \int_c^u e^{-rT_n(s|u)} nF^{n-1}(s) f(s) ds \\ T_n(v|u) &= \frac{n-1}{r} \int_v^u \frac{f(s)}{F(s)} \frac{c}{s-c} ds \end{aligned} \tag{6}$$

Further, according to Theorem 1, $e_\infty(u)$ is defined as $1 - c/u$. Now, I present the result:

Theorem 2 *Consider a family of N-player AD wars in which each is generated from the same distribution, say F, and has the same maximal upper bound, say $\overline{v}_m$. Then, if $e_N(u) > e_\infty(u)$ for all $u \in (c, \overline{v}_m]$, the symmetric AD war, which satisfies $\overline{v}_i = \overline{v}_m$ for all player i, yields the lowest expected discount factor among all in this family.*

Theorem 2 is surprising because it indicates that under certain condition an all-strong society is the least efficient, and by changing any player in this symmetric war to be weaker, one can make the situation better off. Here, according to the geometric shape of an AD war's equilibrium, being strong or weak corresponds to having a higher or lower upper bound. However, this conclusion hinges on the satisfaction of the condition mentioned in th theorem, and when the parameters satisfy the opposite condition that $e_N(u) < e_\infty(u)$ for all $u \in (c, \overline{v}_m]$, the conclusion also reverses to be that the all-strong society becomes the most efficient.

The condition of $e_N(u)$ being greater than $e_\infty(u)$ tells a quite straightforward economic insight. That is, the condition that the welfare of a symmetric AD war decreases when the population grows larger implies that, under symmetry, the loss out of the elongated free-ride time overwhelms the benefit out of the increased population or, equivalently, the increased opportunity for sooner provision. In short, $e_N(u) - e_\infty(u)$ measures how costly it is to introduce symmetry in a society, and when it is positive, any symmetry tends to reduce welfare level while asymmetry tends to enhance it.

Numerical experiments give one example satisfying this condition, the family of the AUD wars with positive lower bounds. Recall that Proposition 2 shows that any 0-AUD war is such that $e_N(u) = e_\infty(u) = 1 - c/u$, so that the variation of player structure does not effect the welfare level. Nonetheless, when the lower bound of an AUD war is set to be positive, this independence no longer holds. Specifically, a higher lower bound means that more probability is subjected to the valuations greater than provision cost, and this change makes each player anticipate his opponents with higher incentive to provide, which however only incurs higher incentive to free ride but not enought incentive to actually provide sooner. This feature implies that in an AUD war with positive lower bound symmetry is costly.