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Does Asymmetry Matters in Social Conflicts? A Multi-Player Asymmetric War of Attrition On Private Provision of Public Good

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Abstract

Natural-state dilemma, where public good is provided privately during a war of attrition, is ubiquitous. I model a generalized multi-player asymmetric war of attrition with incomplete information to investigate the role of asymmetry in such conflicts. In the unique equilibrium, asymmetry differentiates players into different incentive positions leading to a stratified behavior pattern. Analysis shows that the welfare level is mainly determined by the strongest player. He determines by ranking all types' incentive positions, instead of providing directly by himself. Moreover, the influence of asymmetry on welfare can be both positive and negative. I discuss and interpret the conditions for both possibilities.

KEYWORDS. War of attrition, private provision of public good, ex ante asymmetry, multiple players, incomplete information.

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1 Introduction

Natural state, in which communication and cooperation are prohibitively costly for a society and thus any public good can only be provided privately during a war of attrition, is one of the major sources of welfare loss from strategic interaction. It happens in daily basis and covers a large amount of applications. Examples are the dilemmas where multiple political groups try to shift the burden of economy stabilization onto others, and where adjacent countries or provinces pass the buck when there is a riot infesting their common border. Strategic delay is known to be a major cause of inefficiency in these situations.

One important question is: Does *asymmetry* matters in such natural-state conflicts? Here, asymmetry refers to ex ante asymmetry which means that different members' information is anticipated differently. This case corresponds to the presence of several incumbent political groups who feel more pressured to conduct fiscal stabilization, and countries or provinces which face different costs of eliminating a riot due to their asymmetric economic levels. The interesting questions that follow are: How does asymmetry change behavior? Does heterogeneous groups equally contribute to social welfare? And if not, who matters the most? Whether asymmetry alleviates or exacerbates delay? Can we make a society better off by sharpening or equalizing its asymmetry?

I answer these questions by presenting a generalized war-of-attrition model that combines ex ante asymmetry, multiple players, and incomplete information. Such a general combination is missing in the literature of war of attrition and other similar forms of contest[1], probably because previous studies have regarded it as merely a more involved extension with no novel insights. However, this paper proves otherwise.

Bliss and Nalebuff (1984) first realized the necessity to model private provision of public good as a war of attrition. They modeled a one-exit continuous-time game in which each player chooses a provision time at the beginning to optimize his expected utility, and once someone provides first, the game ends and everyone gains lump-sum payoff according to their information. My model looks similar to theirs nonetheless differing in that, instead of seeing provision cost as uncertain like them, I let valuation be the incomplete information, and that, more importantly, I allow all players' costs, discount rates, and valuation distributions to be asymmetric. Additionally, to guarantee a unique equilibrium, I assume that every player is anticipated with positive probability to value the public good less than his cost.

This paper finds that heterogeneous individuals manifest a *stratified behavioral pattern*. One

[1] I list some examples with two of the three elements. For those looking into multi-player asymmetric wars of attrition with complete information, see Ghemawat and Nalebuff (1985, 1990), Whinston (1988), and Bildeau and Slivinski (1994). Examples into multi-player symmetric wars of attrition with incomplete information are Riley (1980), Bliss and Nalebuff (1984), Alesina and Drazen (1991), Bulow and Klemperer (1999), and Sahuguet (2006). The two-player asymmetric incomplete-information case is the most widely studied, for example, Riley (1980), Nalebuff and Riley (1985), Fudenberg and Tirole (1986), Kornhauser, Robinstein, and Wilson (1988), Ponsati Sakovics (1995), Abreu and Gul (2000), Myatt (2005), and Horner and Sahuguet (2010). Also, there are special cases including all three elements. For example, a third party strategically interfering in a two-player war of attrition (e.g., Casella and Eichengreen (1996) and Powell (2017)), and two-group bargaining game which is basically a two-player game (e.g., Ponsati and Sakovics (1996)). The most related study is Kambe (2019) which investigated a model similar to mine but with two-type incomplete information.

degenerate example of this concept commonly seen in two-player wars of attrition is *instant exit*. That is, when two players are asymmetric enough, one and only one of them will have positive probability to concede immediately. A seminal work that mentioned instant exit is Nalebuff and Riley (1985), and among more recent studies characterizing this behavioral feature are Ponsati and Sakovics (1995), Riley (1999), Abreu and Gul (2000), and Myatt (2005).

I achieve further in my multi-player game. Apart from instant exit, there is also *strict waiting* which means that for some players each of their minimal waiting time among all types is strictly positive, rather than zero as is in two-player cases, and each player's strict-waiting time can be different. In other words, when parametrization is asymmetric enough, an equilibrium is endogenously determined to be such that it begins with a period during which only two players have the probability to provide and after it starts a three-player period, and so on. Eventually, only when the game has endured for a sufficiently long time will all players become active. Obviously, instant exit should be construed as a special one-player "period", the length of which is zero because there is no provision from others and any delay is pure waste.

This stratified equilibrium results from the asymmetric incentive positions of different types. Simultaneous revelation requires the types being revealed at the same time to mutually balance each other's incentive. Namely, for each being-revealed type it finds the extra gain from providing immediately equal that from waiting slightly longer. Some types exit instantly because they value the public good so much that no simultaneous revelation with other types can offest their high valuations. In contrast, some wait strictly because even their highest types still value the good too low to be mutually balanced with earlier revealed types. That asymmetry affects the outcome by changing the scale of active players has been mentioned by Bergstrom et al. (1986), Hillman and Riley (1989), and Kambe (2019).

Based on this interpretation, I call a player who either strictly waits shorter or provides instantly as a stronger one. Comparative statics tell that either lower cost, more impatience, or "consistently higher" valuation distribution reduces a player's provision time and increases his relative incentive position.

On the uniqueness of equilibrium. A huge literature acquires unique equilibrium by perturbing a war of attrition. Namely, for each player there must be a positive probability of some others' waiting forever. For example, Fudenberg and Tirole (1986) introduced a positive probability of each player's being better off in a duopoly than in a monopoly. Kornhauser, Rubinstein, and Wilson (1989) used a slight probability of irrational type who only plays a fixed strategy, the spirits of which were also borrowed by Kambe (1999, 2019) and Abreu and Gul (2000). Finally, Myatt (2005) considered three forms of perturbation: exit failure, hybrid payoff, and time limit. Yet, most of asymmetric wars modeled in literature involve only two players or essentially complete information[2].

So far as I know, Kambe (2019) is the only paper that shares the spirits above. Nonetheless, the difference between us is critical. First, he studied a war of attrition basically with complete information and used the existence of a non-compromising type as perturbation. So, the techniques are quite different from those employed in this paper. Besides, his analysis mainly

[2]By *essentially* I refer to Abreu and Gul (2000) and Kambe (2019) where they investigated wars of attrition with two-type incomplete information, but since one of the types is non-compromising, their setups are basically perturbed complete-information games.

focused on equilibrium behavior, for example, he discussed the probability of instant exit which he took for granted as a direct measure of welfare. I prove this effort to be less important by showing examples where the arbitrary variation of players' behavior does not affect the outcome or, equivalently, the welfare level. More generally, welfare analysis introduced later stresses that the outcome is mainly determined by the *ranking* of players' behavior, instead of directly by the behavior itself.

In addition to equilibrium analysis, I investigate the relationship between ex ante asymmetry and social welfare. First, I answer: Who matters the most? Following Bliss and Nalebuff's (1984) approach, I allow the population to grow large in a certain way, and find that each player's interim welfare is solely determined by, apart from his valuation, the strongest type in the society. I also find that each member's incentive position in the equilibrium is solely determined by his highest possible type.

This the-strong-determine result does not necessarily mean that the strongest type directly provides the public good, but that it decides the behavior "level" of an equilibrium. The explanation lies in the asymmetric dependence of players' behavior on parameters. Specifically, I introduce a special case, AD war, to show that while the parameter variation of a weak player sheds no influence on the behavior of stronger types, the variation of a strong player effectively changes the behavior of weaker types. This further implies that the strongest type is directly responsible for the determination of how types' incentive positions are ranked. In contrast, weaker players merely alter their strict-waiting times to suit this established ranking without affecting it much. And it is this ranking of stratified incentives determines the welfare level.

Second, I study the impact asymmetry has on welfare. That ex ante asymmetry may improve the welfare level has been mentioned in previous studies in which an efficient degenerate equilibrium was selected with perturbation. For example, Riley (1999) modeled a family of two-player complete-information contests including a war of attrition. He selected a unique equilibrium by letting a sequence of other family members approximate a pure war of attrition, and he discovered that any introduction of asymmetry in this equilibrium makes one player concede immediately, while the symmetric case is far from efficient. Kornhauser, Rubinstein, and Wilson (1988) and Myatt (2005) derived similar results in wars of attrition with incomplete information. Also, Kambe (2019) argued, without formal proof though, that higher probability of instant exit improves efficiency.

My objective clearly differs from theirs. First, I investigate the efficacy of introducing asymmetry in non-degenerate conflicts[3]. A degenerate equilibrium is of less interest, since in most true-life applications delay does occur. Second, I generalize the result to multi-member societies to make powerful policy implications.

Surprisingly, the conclusion is not definite, because asymmetry's influence highly relies on parametrization. I consider two representative cases to disclose both the condition for and the insight behind asymmetry as improvement or deterioration. The first case is redistribution among groups in a large-population society, and any unequalizing redistribution[4] increases the

[3] Actually, Riley (1999) also did this by conducting comparative statics on perturbed contests and found that even in those where delay emerges, asymmetry yields lower expected total expenditure. However, this result is based on numerical computation and limited to complete-information case.

[4] This term was used by Bergstrom, Blume, and Varian (1986).

welfare level of a symmetric society. The second case examines AD wars with the strongest type being fixed. A straightforward condition for improvement states that when the cost of symmetry, which is measured by the welfare-level discrepancy between an N-player symmetric game and the associated infinite-player symmetric game, is positive, any introduction of asymmetry alleviates the loss out of delay. In contrast, when this cost of symmetry is negative, the symmetric war of attrition becomes most efficient. I give examples for both cases which reveal the possibility for asymmetry to either improve or deteriorate the outcome.

This paper is organized as the following. Section 2 describes the model setup and the equilibrium concept. Section 3 characterizes equilibrium and further proves the existence and uniqueness. I introduce a special case, AD war, in this section to illustrate both behavioral features and welfare implications formally discussed later. Finally, this section performs comparative statics. Section 4 establishes connection between asymmetry and social welfare by answering two questions: Who matters the most? and Does and When does asymmetry improve welfare? Section 5 discusses possible applications associated with previous analysis and makes concluding remarks.

2 Model

There is an indivisible public good potentially beneficial to N different individuals. I denote each player by $i \in I_N$ where $I_N = \{1, 2, ..., N\}$. In the natural state, cooperative provision is not an option and therefore a continuous-time war of attrition becomes inevitable. This war begins at $t = 0$ and each player chooses a stop time when, if no one has provided the good yet, he will provide. Before the war of attrition ends, each player can change his decision anytime, although later analysis shows that this change is not likely (See footnote 5).

Here is the information structure: one player, say i, knows exactly the cost of his individual provision $c_i > 0$, the rate $r_i > 0$ that he obeys to exponentially discount his expected gain at time t with $e^{-r_i t}$, and his valuation of this public good v_i. All values of cost and discount rate are common knowledge, whereas each valuation v_i is private information independently extracted from a cumulative distribution function $F_i : [\underline{v}_i, \overline{v}_i] \to [0, 1]$ in which $0 \leq \underline{v}_i < c_i < \overline{v}_i < +\infty$ for all i. For convenience, I sometimes call player i with type v_i simply as player v_i. Note that this strict relationship of $\underline{v}_i$, $\overline{v}_i$, and c_i is important because it necessarily guarantees the uniqueness of equilirium. The analysis requires some harmless assumptions: each F_i yields a dense function $f_i : [\underline{v}_i, \overline{v}_i] \to \mathbb{R}^+$ which is differentiable and strictly bounded from 0.

Player i's pure strategy is a function $T_i : [\underline{v}_i, \overline{v}_i] \to \mathbb{R}^+ \cup \{0, +\infty\}$ referring to the stop time that player v_i chooses. Only when no provision happens before $T_i(v_i)$ will this player provide at this time. If some provide first, each player gains his valuation while those who provide additionally pay their share of provision cost. Namely, if $m \geq 1$ players provide at this moment, each of them, say player i, pays c_i/m. And if all players choose to wait forever, each earns zero. All payoffs are lump-sum, and the moment the public good is provided, the game ends.

I consider pure-strategy perfect Bayesian equilibrium, and in the following sections by *equilibrium* I mean this kind, unless otherwise specified.

3 Equilibrium Analysis

In this section, I firstly show a group of sufficient and necessary conditions for an equilibrium including some analytical properties of an equilibrium, and the differential-equation system with three associated boundary conditions that characterizes the strategy functions under this equilibrium. This group of conditions reveals that in an equilibrium players demonstrate a stratified behavior pattern. Then, I prove the existence and uniqueness of equilibrium. Moreover, I introduce a special but widely applicable case, AD war, to illustrate some significant behavioral and welfare insights which are formally discussed in this and the next sections. Finally, I perform comparative statics.

3.1 Characterization

I introduce some definitions and notations before presenting the sufficient and necessary conditions. First, I give the expected gain of player v_i with his choice of stop time being t_i, while fixing other opponents' strategies $\{T_j(.), j \neq i\}$. For convenience, I define v_{-i} as the vector of valuations of players other than i, $T_{-i}^{min}(v_{-i}) = \min_{j\neq i} T_j(v_j)$, and $F_{-i}^{min}(t) = \text{Prob}(T_{-i}^{min} \leq t)$. This expected gain, denoted by $R_i(t_i|v_i)$, is:

$$R_i(t_i|v_i) = v_i \int_0^{t_i} e^{-r_i s} dF_{-i}^{min}(s) + (v_i - c_i)e^{-r_i t_i}(1 - F_{-i}^{min}(t_i)) \tag{1}$$

The first term above represents the case when someone else provides before t_i, and the second corresponds to the situation where others decide not to provid before t_i. Since the characterization shown later indicates that the equilibrium strategy is strictly monotonous, I omit the simultaneous-provision situation in (1).[5]

Further, define $d_i = \min_{v_i} T_i(v_i)$ as the minimal waiting time among all types of player i, and $m_i = \min_{T_i(v_i)=d_i} v_i$ as the minimal type of player i that provides at d_i.[6]

Now, I present the sufficient and necessary conditions for an equilibrium in Lemma 1:

Lemma 1 *A profile $\{T_j, j \in I_N\}$ is a perfect Bayesian equilibrium if and only if:*

(i) $T_i(v_i) = +\infty$ *on* $[\underline{v}_i, c_i]$, *and* $T_i(v_i) < +\infty$ *on* $(c_i, \overline{v}_i]$, *for all* $i \in I_N$.

(ii) $\lim_{v_i \to c_i+0} T_i(v_i) = +\infty$, *for all* $i \in I_N$.

(iii) N-2 strict waiting(s): there exists a set $I_0 = \{j_1, j_2, ..., j_{n_0}\} \subset I_N$, *where* $n_0 \geq 2$, *being such that for all* $j_k \in I_0$, $d_{j_k} = 0$, *and for all* $j \in I_N - I_0$, $d_j > 0$.

[5]I discuss why players will not change their decision during the war of attrition. Note that $R_i(t_i|v_i)$ is linear with respect to F_{-i}^{min}, so when any time arrives before which no one provides, the first-order derivative of (1) is only a multiple of that at $t = 0$. And since the equilibrium strategy is determined by the first-order condition for maximizing (1), the generated optimal choice of stop time at this moment is equivalent to that selected at the beginning. Consequently, the notion of Bayesian equilibrium here coincides with perfect Bayesian equilibrium.

[6]The monotonicity and continuity of each $T_i(.)$, proved later in Lemma 1, guarantee the existence of both concepts and these analytical properties does not hinges on it.

(iv) One instant exit: there exists at most one $j \in I_0$ being such that $m_j < \overline{v}_j$, while for all $k \in I_N$ other than j, $m_k = \overline{v}_k$.

(v) For all $i \in I_N$, $T_i(v_i)$ is continuous and strictly decreasing on $(c_i, m_i]$. Thus, $T_i(v_i)$'s inverse function $\Phi_i(t_i)$ exists on $[d_i, +\infty)$.

(vi) For all $t \in \mathbb{R}^+$, if there are M Φs that have definition at t, denoted by $I(t) = \{j_1, j_2, ..., j_M\}$ where $M \leq N$, then they are differentiable at t and characterized by:

$$\Phi_{j_i}'(t) = \frac{F_{j_i}(\Phi_{j_i}(t))}{f_{j_i}(\Phi_{j_i}(t))} \left[\frac{r_{j_i}}{c_{j_i}}(\Phi_{j_i}(t) - c_{j_i}) - \frac{1}{M-1} \sum_{k=1}^{M} \frac{r_{j_k}}{c_{j_k}}(\Phi_{j_k}(t) - c_{j_k}) \right] \tag{2}$$

For all $j_i \in I(t)$.

Proof of Lemma 1 Omitted.

Apart from some stylized properties like clauses (i), (ii), and (v), the other three (iii), (iv), and (vi) demonstrate a peculiar geometric feature of an asymmetric equilibrium involving multiple players: while at most one player concedes at $t = 0$ with strictly positive probability, at most $N - 2$ players will strictly wait for a while before each of them has any probability to provide.

This feature implies that ex ante asymmetry differetiates players' incentives and requires their behavior to be stratified. In a metaphorical way, the shape of equilibrium curves seems like a long cascade with dams allocated in its middle: around $t = 0$ the number of curves is small, and every time it arrives at some players' strict-waiting time, at least one curve will appear, as if the river is getting stronger when passing every dam.

For example, consider three potential providers with an informational setting asymmetric enough to make both instant exit and strict waiting possible. I denote the strictly free rider by 1 and his minimal waiting time by $t_1 = d_1$. Then according to Lemma 1, before t_1 only player 2 and 3 have the probability to supply, but after that all three individuals become active. Let each player's cost of provision be the same, and I depict a possible solution of this three-player war in Figure 1.

I introduce some definitions and notations for analytical convenience. I call those strict-waiting time points as *division*s and players' strategy functions, the Φs, as *curve*s, if there is no confounding. Between different pair of adjacent divisions exist different numbers of curves obeying different versions of (2), and I call the players corresponding to these existing curves as *active player*s. Curves of active players on both sides of one division are continuous at this division. Further, I denote the time domain between two adjacent divisions where there are M active players by $\Upsilon(M)$ and call the group of differential equations corresponding to this domain as the M problem. The boundary conditions of an M problem needs to be specified. Finally, let $I(t)$ denote the set of subscripts of active players at t or, equivalently, I_M denote the set corresponding to the players in the M problem.

Importantly, the presence of instant exit and strict waiting discloses the relative *incentive position* among all types of all players. The core here is that only the types in the same position

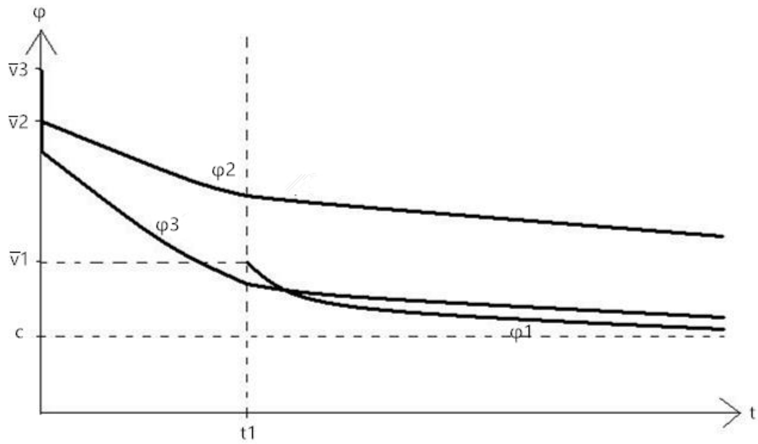

Figure 1: This figure demonstrates the geometric properties in Lemma 1. The third curve, $\Phi_3(t)$, starts at $\Phi_3(0) = m_3 < \bar{v}_3$ corresponding to an instant exit, while the first, $\Phi_1(t)$, does not exist until $t = t_1$ referring to a strict waiting. t_1 is a division, and on its left and right sides curves obey the two-player and three-player versions of (2), respectively.

are revealed at the same time. Formally, rearranging (2) generates a more intuitive equation below which gives the incentive trade-off faced by each active player in an M problem:

$$c_{j_i} \sum_{k=1,k\neq i}^{M} \frac{-dF_{j_k}(\Phi_{j_k}(t))}{F_{j_k}(\Phi_{j_k}(t))} = r_{j_i} dt(\Phi_{j_i}(t) - c_{j_i}),\ j_i \in I(t) \tag{3}$$

The interpretation of (3) is based on the facts that: *a)* the $\Phi_{j_i}(t)$ on (3)'s right side represents the type of player j_i being revealed at t, and thus $r_{j_i} dt(\Phi_{j_i}(t) - c_{j_i})$ is the extra gain of this type's providing at t instead of $t + dt$; and *b)* each $F_{j_k}(\Phi_{j_k}(t))$ is equivalent to the probability of j_k's providing after t, so the left side of (3) equals the extra gain of waiting during this infinitesimal period. So, the revelation of types is such that the two countervailing incentives, the gain of providing immediately and the gain of waiting slightly longer, faced by the type being revealed are balanced by the revelation of other players' types. So, types being revealed at the same time under equilibrium make the incentives of each other mutually balanced, and therefore they are in this sense equivalent in incentive position.

The instant-exit types are in the highest position, namely, they value the public good so much that no simultaneous revelation with others is able to offset this high incentive to provide. Between each pair of adjacent divisions, even the highest types of the inactive players generate incentives too low to balance the incentives faced by the active types. On the other hand, types of active players are revealed continuously during the period. Instant exit should be seen as a special case of strict waiting during which only one player's types are continuously revealed. Thus, the multi-player asymmetric setting requires that in an equilibrium valuations of different individuals are, in a way, ranked by both the decreasing monotonicity of curves and the presence of instant exit and strict waiting.

That asymmetry affects equilibrium by changing the number of active players has the same spirits as Bergstrom, Blume, and Varian (1986). They discussed a public-good provision problem with a static model with complete information, and they argued that considerable redistribution of wealth will cause some players incapable of providing any public good and this reduction of the number of active players alleviates underprovision. My model incorporates asymmetry in valuations[7], and at each moment all unrevealed types of all players play a similar static game in which everyone chooses either to wait to gain from the probability of others' providing first or to concede immediately to be active. These two-sided incentives are similar to the trade-off between private and public consumptions in Bergstrom et al. (1986).

What is new lies in that the introduction of incomplete information and timing game makes this scale-change process of active group endogeneous, as the requirement that the incentives faced by active players are mutually balanced automatically divides the time into different periods during which different numbers of players are active.

Above, I explain the stratified behavior pattern including both instant exit and strict waiting. Hereafter, by calling a type *stronger* than another I mean in equilibrium the former type selects a shorter stop time or the former values the public good more than the latter if they are both instant-exit types. Likewise, a player is stronger or has a higher incentive position[8] if his highest type strictly waits for a shorter period or this type exits instantly.

Additionally, the importance of instant exit and strict waiting also lies in that these two conditions complement the degrees of freedom of solution. The flexible boundary conditions including one variable instant exit and $N-2$ variable strict-waiting times help to adjust the solution of equations (2) to satisfy another boundary condition, Lemma 1(ii). Combining all three kinds of boundary conditions, it matches right $1+1+(N-2)=N$ degrees of freedom so that this condition set provided by Lemma 1 is potential in selecting a unique equilibrium.

3.2 Existence and Uniqueness

A huge literature applies perturbation to guarantee a unique equilibrium. Specifically in my model, this perturbation lies in that every player is anticipated with positive probability of being incapable of individual provision or, equivalently, with $F_i(c_i) > 0$. The straightforward intuition is that the possibility of arrival at any time imposes perfection on off-equilibrium path to eliminate unreasonable solutions.

To show how this positive probability of waiting forever guarantees a unique equilibrium in my multi-player game, the following Lemma 2 demonstrates, in a technical sense, how this perturbation affects the solution behavior of an M problem.

I introduce some notations before presenting Lemma 2. First, consider an M problem with boundary conditions all on its left-side division, and the satisfaction of Lipschitz condition guarantees a unique solution of (2) associated with each selection of boundary conditions.

Then, I extend the definition of each (F_i, f_i) to domain $(\overline{v}_i, +\infty)$ to cope with the situation where the left-side boundaries make some solution curve, say Φ_i, reach the line $v = \overline{v}_i$. And

[7] This is equivalent to setting the utility functions of different players in Bergstrom et al. (1986) to be asymmetric which yields the same reduction of active population.

[8] Here, I abuse the notions of being strong and incentive position a little, for they actually describes the relative position between types, rather than players. However, later analysis tells that under certain conditions the highest type of a player can be representative, and this abuse becomes harmless. But still, the meaning should be differentiated when describing different objects.

since in an equilibrium solution curves do not step into these domains, this extension is harmless. To conserve the differentiability and boundedness of Fs and fs, I choose $\Delta F \in (0,1)$ and $\Delta f \in (0, \min_{i\in I_N} f_i(\overline{v}_i))$, and for all $i \in I_N$ and $v > \overline{v}_i$ I define:

$$f_i(v) = f_i(\overline{v}_i) + \frac{f_i'(\overline{v}_i)}{|f_i'(\overline{v}_i)|}\Delta f[1 - e^{-\frac{|f_i'(\overline{v}_i)|}{\Delta f}(v-\overline{v}_i)}]$$
$$F_i(v) = F_i(\overline{v}_i) + \Delta F[1 - e^{-\frac{f_i(\overline{v}_i)}{\Delta F}(v-\overline{v}_i)}]$$

One can easily verify that with such extension the equation system (2) of the M problem still satisfies local Lipschitz condition everywhere.

Finally, I define a P_M problem associated with this M problem. Denote all active players by $I_M = \{1, 2, ..., M\}$ and define $m = (m_1, m_2, ..., m_M) \in B_M = \prod_{i=1}^{M}[c_i, +\infty)$ as a vector being such that $\{\Phi_i(0) = m_i, i \in I_M\}$ is the corresponding left-side boundary selection. I rewrite the solution as $\{\Phi_i(t, m), i \in I_M\}$, each element of which is a multivariate function with respect to t and m. Formally, $P_M = \{I_M, B_M, (c_i, r_i, F_i, f_i)_i, (\Phi_i(t, m))_i\}$ denotes an M problem with the M-player version of (2) as its differential equations and B_M as the set containing all possible left-side boundary selections.

In an equilibrum, after the rightmost division stands a P_N problem, and the importance of Lemma 2 lies in that it provides useful tools to prove that an equilibrium-like solution of this problem uniquely exists. Lemma 2 tells that: *a)* each solution curve is consistently monotonous in response to the variation of left-side boundaries; and *b)* when time approximates infinity, for any selection of boundaries on the left, the solution curves can be either convergent or divergent, and further, a convergent solution must satisfy Lemma 1(ii) and (v); and finally, *c)* the convergent case is asymptotically sensitive, namely, any change in boundaries on the left makes convergence collapse to divergence.

Lemma 2 *For a P_M problem defined above, its solution has the following properties:*

(i) Comparative monotonicity: for all $i \in I_M$, any $m^{(0)} = (m_i^{(0)}, m_{-i})$[9] *and $m^{(1)} = (m_i^{(1)}, m_{-i}) \in B_M$ where $m_i^{(0)} < m_i^{(1)}$ lead to that* a) *$\Phi_i(t, m^{(0)}) < \Phi_i(t, m^{(1)})$ for all $t \geq 0$, and* b) *$\Phi_j(t, m^{(0)}) > \Phi_j(t, m^{(1)})$ for all $t > 0$ and $j \neq i$.*

(ii) Two asymptotic patterns: when $t \to +\infty$, solution $\{\Phi_i, i \in I_M\}$ can only take on one of two patterns: a) convergence*: $\Phi_i(t) \to c_i + 0$ for all i meanwhile satisfying Lemma 1(ii) and (v), and* b) divergence*: there exists at least one $k \in I_M$ being such that Φ_k approximates $+\infty$ or $\underline{v}_k$.*

(iii) Asymptotic sensitivity: for all $i, j \in I_M$, $m \in B_M$, and $t > 0$, partial derivative $\partial\Phi_j(t, m)/\partial m_i$ exists; and if the solution is convergent as defined in (ii), then $\partial\Phi_i(t, m)/\partial m_i \to +\infty$ while $\partial\Phi_j(t, m)/\partial m_i \to -\infty$ as $t \to +\infty$ where $j \neq i$.

Proof of Lemma 2 Omitted.

[9] m_{-i} represents the vector $(m_1, m_2, ..., m_{i-1}, m_{i+1}, ..., m_M)$, and (a, m_{-i}) is the vector acquired by substituting a for the ith component of m.

The proof of Lemma 2 indicates that the key of uniqueness is the positive probability of each player's valuation being less than cost, because line $v_i = c_i$ divides the space into two areas in which solution manifests upper- and lower-divergence, respectively, and thus convergence only appears around $v_i = c_i$ and is sensitive to the variation of boundary conditions.

To observe the multiplicity of the opposite case, I assume that $\underline{v}_i > c_i$ for all i. Consider a P_2 problem whose $B_2 = \prod_{i=1}^{2}[\underline{v}_i, +\infty)$. First, there exist degenerate equilibria in which all types of one player provides immediately while all types of the other wait forever. However, even with slight probability of the immediate provider's valuation being less than his cost, his opponent will find waiting forever no longer optimal because doing so leaves the chance for both to earn zero. Second, multiplicity can also occur even if equilibrium is selected according to Lemma 1(ii)[10] and (v). In this case, no matter what boundaries are selected, (2) always tells that $\Phi_i'(.) < 0$ and $\Phi_i \to \underline{v}_i + 0$ naturally satisfying Lemma 1, which means that the lack of asymptotic sensitivity makes the boundary selection arbitrary[11].

Finally, I prove the existence and uniqueness of equilibrium. On the rightmost domain stands a P_N problem where all players are active, and the first step is to prove the existence and uniqueness of this problem's solution that satisfies Lemma 1. Then, from the right side to the left, I employ a backward specification strategy similar to Kambe (2019). I sequentially determine the solution between each adjacent pair of divisions according to Lemma 1(iii) and (iv) by adjusting the total number of divisions and the relative distances between them. This sequential specification process also yields a unique solution. Now I present this result:

Theorem 1 *There uniquely exists a perfect Bayesian equilibrium.*

Proof of Lemma 3 Omitted.

I introduce some notations to link a public-good-provision war to its unique equilibrium. Define Ω as the space of all proper wars of attrition. By *proper* I mean: if $w = \{N, (r_i, c_i, F_i, \underline{v}_i, \overline{v}_i)_i\} \in \Omega$, then w is parametrized such that $N \geq 2$, and that for all $i \in I_N$ $r_i > 0$, and $\overline{v}_i > c_i > \underline{v}_i \geq 0$, and $F_i(.)$ is a differentiable distribution function defined on $[\underline{v}_i, \overline{v}_i]$, and $f_i(.)$ is the corresponding density function which is differentiable and bounded from zero. On the other hand, define Ξ as the space of unique equilibria of all possible proper wars, and its element is written as $e = \{N, \overline{K}, (\Phi_i)_i, (d_K, M_K, I_K)_K\}$, in which $N \geq 2$ and $\overline{K} \geq 0$, and for all $i \in I_N$ $\Phi_i(.)$ denotes the inverse function of player i's equilibrium strategy $T_i(.)$, and for all $K = 1, 2, ..., \overline{K}$ d_K marks the location of the Kth division from the left whereas $d_0 = 0$ and $d_{\overline{K}+1} = +\infty$, and M_K and I_K represent the number of active players and the set of subscripts of these players during $t \in [d_K, d_{K+1})$, respectively. Finally, I define the mapping $E : \Omega \to \Xi$ being such that $E(w)$ is the unique equilibrium of $w \in \Omega$. In later discussion, I apply similar form to denote wars and associated equilibria, that is, I use braces to contain all parameters or information describing an

[10] Here, clause (ii) should be adjusted to be $\lim_{v_i \to \underline{v}_i + 0} T_i(v_i) = +\infty$, for all i.

[11] The multi-player case also suffers from multiplicity in this way but can be more difficult to analyze, because according to the proof of Lemma 2(iii) partly asymptotic sensitivity may appear. However, there always exists a k being such that $\frac{r_k}{c_k}(\underline{v}_k - c_k) < \frac{1}{M-1}\sum_{i=1}^{M}\frac{r_i}{c_i}(\underline{v}_i - c_i)$ so that the solution behavior is insensitive to the kth left-side boundary condition and in this way one can construct a non-singleton subset of B_M whose each element corresponds to a satisfactory solution.

object in which those with subscripts denote the parameters or information of the unit with the same subscript, while those without denote the parameters or information shared by all units. Here, by units I refer to players or divisions, and unless otherwise specified, I use subscript i to denote all players and K all divisions in an equilibrium.

3.3 An Important Case: AD War

To demonstrate some behavioral and welfare insights, I introduce an important family of proper wars, aligned-distribution war (AD war). This class of war is important for two reasons: *a)* apart from this case, there are very few mathematically tractable examples, and *b)* it corresponds to wide economic applications.

In each AD war, players have identical costs and dicount rates, but their valuation distributions are all lower-conditional distributions of one fixed distribution with different upper bounds. Formally, let F be a proper distribution whose upper bound and lower bound are denoted by $\overline{v}$ and $\underline{v}$, respectively, and $(\overline{v}_i)_{i=1}^N$ a set containing N players' upper bounds, each of which is no greater than $\overline{v}$ and greater than the provision cost. Then, the ith player's valuation distribution is the lower-conditional distribution of F corresponding to $\overline{v}_i$, that is, $F_i(v) = F(v|v \leq \overline{v}_i)$ for all i. Define $\Omega_{AD} \subset \Omega$ as the space of all such wars, and each of its element is denoted by $w_{AD} = \{N, r, c, F, \underline{v}, (\overline{v}_i)_i\}$.

This case corresponds to a large category of daily-life scenarios, because it can be seen as a partly revealed symmetric war in which players who are initially anticipated to be symmetric may have engaged in some previous games whose unequal outcomes shape the asymmetry of the current war.[12] On the other hand, an AD war is easy to analyze, for the rates on the right side of (2), $f_i(v)/F_i(v)$, of different players are identical on their overlap domain, so between each adjacent pair of divisions active players' behavior is characterized by symmetric equations, and it is easy to verify that backward specification gives symmetric solution[13].

First, I demonstrate how the equilibrium of an AD war looks like. Without loss of generality, let $\overline{v}_1 \geq \overline{v}_2 \geq ... \geq \overline{v}_N > c > \underline{v}$. Note that the active curves on each $\Upsilon(M)$ are symmetric. Formally, let $\Phi_{AD}(t|M, r, c, F, u, \underline{v})$ denote the solution of the initial-value problem derived from the symmetric M-player version of (2) with boundary condition $\Phi_{AD}(0|M, r, c, F, u, \underline{v}) = u$.[14]

[12] One should see this interpretation as merely intuitive. It needs the assumption that the decision a player made in a previous game is not strategically related to other games, for example, when the interval between games is so large that exponential discounting makes any such inter-game strategic move worthless.

[13] The proof of Theorem 1 suggests the uniqueness of the rightmost P_N problem's solution, which requires the solution characterized by symmetric equations to be symmetric as well. This rightmost symmetry further ensures symmetric boundary conditions on all domains on its left side, and therefore the symmetric equations on each domain also results in symmetric solution.

[14] Namely, $\Phi_{AD}(t|M, r, c, F, u, \underline{v})$ is the solution of $\Phi^{'}(t) = -\frac{1}{M-1}\frac{r}{c}\frac{F(\Phi)}{f(\Phi)}(\Phi - c)$ with boundary condition $\Phi(0) = u$.

Then, the equilibrium $E(w_{AD})$ is given by:

$$\begin{aligned}
&d_0 = 0,\ d_{\overline{K}+1} = +\infty \\
&d_K - d_{K-1} = \Phi_{AD}^{-1}(\overline{v}_{K+2}|K+1, r, c, \overline{v}_{K+1}, \underline{v}),\ K = 1, 2, ..., N-2 \\
&\begin{cases} T_1(v) = 0,\ v \in (\overline{v}_2, \overline{v}_1] \\ \Phi_1(t) = ... = \Phi_n(t) = \Phi_{AD}(t - d_{n-2}|n, r, c, \overline{v}_n, \underline{v}),\ t \in [d_{n-2}, d_{n-1}),\ n = 2, 3, ..., N \end{cases}
\end{aligned} \tag{4}$$

The tractability of this case depends on the behavior feature that it manifests local symmetry, that is, players with identical type provide at the same time. In this case, the ranking of incentive positions coincides with that of valuations, which gives a straightforward interpretation of the former. In Figure 2, I depict the solution of a three-player AD war.

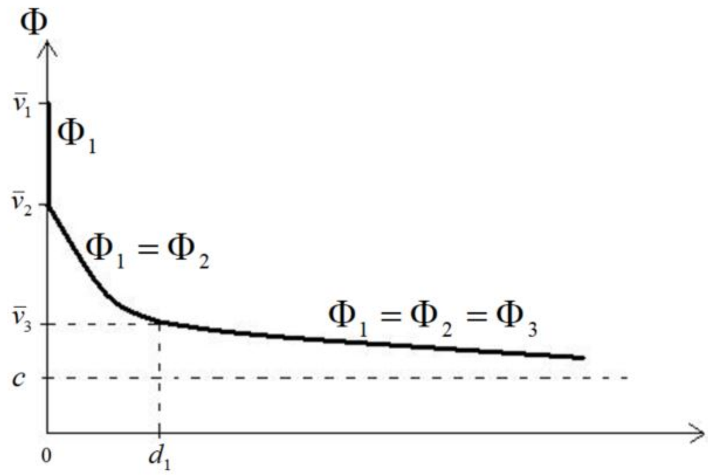

Figure 2: This figure demonstrates the equilibrium of a three-player AD war. The player with the highest upper bound provides instantly when his realized type is no less than the second-large upper bound. Before the first division two players are active, whereas after it the third becomes active as well. Active players always reveal their types symmetrically at the same time.

Another important feature is that players' behavior is asymmetrically dependent on each other. Namely, if the upper bound of a weak player varies, all stronger types' behavior stay unchanged, while if that of a strong player changes, all weaker types alternate their decisions. More formally, consider N players in an AD war and one of them, say k who is not the strongest player, has distribution upper bound $\overline{v}_k$. Now, let his and only his upper bound rise (or drop) to $\widetilde{v}_k$ to construct a new game. Then, for all players any type greater than $max\{\overline{v}_k, \widetilde{v}_k\}$ chooses the same stop time as in the old game, whereas any type less than $max\{\overline{v}_k, \widetilde{v}_k\}$ chooses a longer (or shorter) waiting time. The strongest player may seem like an exception, because variation of his upper bound only changes his probability of instant exit, but one should notice that this lowers all lower types' incentive positions by directly adding strong types at the top of the ranking.

This result demonstrates the dominant position of the strongest player in determining the incentive ranking, because any variation of a weaker player's distribution mainly changes his strict-waiting time to suit this ranking, while any variation of the strongest player changes the ranking directly.

This saliency of the strongest player further implies his dominant position in the contribution to welfare level. On the one hand, the parameter change of the strongest player sheds monotonous influence on all types' incentive positions. On the other, the influence of a weak player's change is offset[15] by the fact that the direction of the behavior alternation of those valuing the public good less than him is opposite to the change of his incentive position. Namely, when this weak player's upper bound rises (or drops), types lower than him delay longer (or shorter) and thus the welfare change out of this is moderate.

A surprising result is that it is the strongest player's parameters, but not his behavior or exact provision time, that mainly determine a society's welfare level. This is at odds with the idea mentioned in literature that asymmetry changes welfare by making some players concede sooner or later. For example, Kambe (2019) argued that higher probability of instant exit[16] necessarily improves efficiency. Others, like Myatt (2005), also stressed the importance of instant exit. However, Proposition 1 below shows a special but telling example that refutes such a statement, since in that case the welfare level becomes completely irrelevant to how players exactly behave as long as the strongest type is fixed.

Now, I use the uniform-distribution example to show the insight on welfare discussed above. I call this example the aligned-uniform-distribution war (AUD war) which has analytical solution. I let $\Phi_{AUD}(t|M, r, c, \overline{u}, \underline{v}) = \Phi_{AD}(t|M, r, c, F, \overline{u}, \underline{v})$ which is given by:

$$\Phi_{AUD}(t|M, r, c, \overline{v}, \underline{v}) = \underline{v} + \frac{c - \underline{v}}{1 - \lambda e^{-\frac{1}{N-1}\rho r t}} \tag{5}$$

Above $\lambda = 1 - (c - \underline{v})/(\overline{v} - \underline{v})$ and $\rho = 1 - \underline{v}/c$, and sometimes I simply denote this curve by $\Phi_{AUD}(t|M, \lambda)$, for other parameters are shared by all players in the same AUD war. Combining (4) and (5), one acquires the equilibrium.

I see the expectation of a decreasing exponential function with respect to stop time, $E_{t_m}[e^{-\rho r t}]$[17] where ρ is defined above, as a measure of welfare level. One important property of AUD war is that under equilibrium this measure is only determined by the maximal upper bound. Therefore, it is unaffected by the variation of population and other distribution upper bounds, both of which necessarily determine how each type will behave. I present it in the following proposition:

Proposition 1 *Any N-player asymmetric AUD war, denoted by $w = \{N, r, c, \underline{v}, (\overline{v}_i)_i\}$, gives:*

$$E_{t_m}[e^{-\rho r t}] = 1 - \frac{c - \underline{v}}{\max\limits_{i \in I_N} \overline{v}_i - \underline{v}}$$

$\rho = 1 - \underline{v}/c$, and $E_{t_m}[.]$ calculates the expectation with respect to stop time under equilibrium.

[15] As shown later in Proposition 1 and Theorem 3, the extent of this offset depends on parametrization and can be either partial, excessive, or complete.

[16] Kambe used the term *initial volunteering* instead for instant exit.

[17] A more reasonable measure of welfare level is $E_{t_m}[e^{-rt}]$ which I will use for analysis in Section 3, since if some player, say v_i, values the public good less than his cost and thus he chooses to wait forever, he earns an expected gain $v_i E_{t_m}[e^{-rt}]$. However, the economic insight here is not sensitive to this bias brought by the shrinked power, as $e^{-\rho r t}$ is still monotonous with respect to stop time t.

Proof of Lemma 4 Omitted.

Proposition 1 favors the statement that the strongest player determines the welfare level by ranking all types' incentive positions, instead of providing directly by himself. In this case, the effect of any rise or drop of a weak player's upper bound is completely offset by the longer or shorter provision time of all types lower than him. Therefore, the arbitrary change of weak players' behavior that results from different selections of their upper bounds does not affect welfare level, and it is the incentive ranking decided by the fixed strongest type that essentially decides the welfare level.

Another insight shown by Proposition 1 is that the loss out of delay always occurs, because the expectation of $e^{-\rho rt}$ is strictly less than one. The exception is the case where some players' upper bound is unbounded from infinity.

In Section 3.1, I formalize these two intuitions by considering societies with large population. Surprisingly, in the sense of welfare any such society is equivalent to its counterpart zero-lower-bound AUD war, as shown in Theorem 2.

3.4 Comparative Statics

In this subsection, I perform some comparative statics on the equilibrium. I first compare the behavior of different players in the same war conditional on some relationship between their parameters. Specifically, I investigate how the difference between costs, discount rates, and *revelation rate*s affects the relative provision time of two players. Here, revelation rate refers to $f(.)/F(.)$, the density of the type being revealed conditional on the revelation of all types above it. The interpretation of this rate will be discussed later. Now I present this result:

Proposition 2 *Consider an N-player proper war, say $w \in \Omega$, if there exists a pair of players, denoted by $\alpha, \beta \in I_N$ whose solution curves both exist on $[t_0, +\infty)$, parametrized such that:*

(i) $r_\alpha = r_\beta$, $c_\alpha = c_\beta = c$, *and* $f_\alpha(v)/F_\alpha(v) \gtreqless f_\beta(v)/F_\beta(v)$ *on their overlap domain, then* $\Phi_\alpha(t) \lesseqgtr \Phi_\beta(t)$ *for all* $t \in [t_0, +\infty)$.

(ii) $r_\alpha = r_\beta$, $f_\alpha(v)/F_\alpha(v) = f_\beta(v)/F_\beta(v)$ *on their overlap domain, and* $c_\alpha \gtrless c_\beta$, *then* $\Phi_\alpha(t) \gtrless \Phi_\beta(t)$ *for all* $t \in [t_0, +\infty)$.

(iii) $c_\alpha = c_\beta = c$, $f_\alpha(v)/F_\alpha(v) = f_\beta(v)/F_\beta(v)$ *on their overlap domain, and* $r_\alpha \gtrless r_\beta$, *then* $\Phi_\alpha(t) \lessgtr \Phi_\beta(t)$ *for all* $t \in [t_0, +\infty)$.

Proof of Lemma 5 Omitted.

The last two clauses of Proposition 2 convey straightforward intuitions: higher cost reduces the gain of provision entailing a delayed strategy, while higher discount rate corresponding to impatience increases the opportunity cost of waiting so that player tends to provide sooner.

However, the interpretation of the first clause associated with revelation rate is unclear, because under different assumptions this result generates different economic outcomes. For example, if player α's and β's upper bounds are set to be equal, Proposition 2(i) indicates that the player with consistently higher revelation rate stands in a relatively higher incentive position. In contrast, if two players' probability of valuation being lower than cost is set to be equal, the one with consistently lower revelation rate becomes stronger[18].

This discrepancy between these two cases must result from the different economic interpretations of the revelaion-rate domination under different settings. In the first case, this domination is equivalent to subjecting higher probability to higher valuations. So, the dominated player tends to anticipate the other to face higher opportunity cost of waiting and thus in average to provide sooner, and this belief leads to his free riding which in turn forces the high-revelation-rate player to actually provide sooner. In the second case, when the probability of waiting forever is controlled, the revelation-rate domination requires the dominant player's distribution to have a greater upper bound than that of the other, and these salient types value the public good so much that they stand in a higher incentive position.

Yet, Proposition 2 only exhibits horizontal comparison of players' behavior in the same war, and for a more clear look into how the change of parameters affects players' relative incentive position in different wars, I present the following result:

Proposition 3 *Consider a two-player proper war, denoted by w, where the players are parametrized by $\{r_i, c_i, F_i, \overline{v}_i\}$ for $i = 1, 2$, respectively. Denote the ratio of both players' left-side boundaries by $q(w)$, namely, $q(w) = \Phi_2(0)/\Phi_1(0)$ where Φs are w's equilibrium solution. Now, by changing one of the parameters of player 1 and fixing all others including the upper bound, it yields a new war, $\widetilde{w}$, with a new associated ratio, $q(\widetilde{w}) = \widehat{\Phi}_2(0)/\widetilde{\Phi}_1(0)$. Then, if the changed parameter is:*

(i) Cost and $\widetilde{c}_1 > c_1$, $q(\widetilde{w}) < q(w)$.

(ii) Discount rate and $\widetilde{r}_1 > r_1$, $q(\widetilde{w}) > q(w)$.

(iii) Valuation distribution and $\widetilde{f}_1(v)/\widetilde{F}_1(v) > f_1(v)/F_1(v)$ for all $v \in (c_1, \overline{v}_1)$, $q(\widetilde{w}) > q(w)$.

Proof of Lemma 6 Omitted.

The ratio $q(w)$ defined in Proposition 3 is a direct measure of the relative incentive position of player 1 over player 2, since the higher this ratio is, the higher the probaility of player 1's instant exit becomes, or the lower the probability of player 2's instant exit becomes. Therefore, Proposition 3 reveals that either lowering the cost of, increasing the discount rate of, or consistently increasing the revelation rate of one player rises his relative incentive position[19] in an asymmetric war of attrition.

[18] If α has a consistently higher revelation rate, then $\Phi_\alpha(t) < \Phi_\beta(t)$ for all $t \in [t_0, +\infty)$ which gives $lnF_\alpha(\Phi_\alpha(t)) - lnF_\beta(\Phi_\beta(t)) = \frac{r}{c}\int_t^{+\infty}(\Phi_\beta(s) - \Phi_\alpha(s))ds > 0$. This further implies that when Φ_α reaches its upper bound, Φ_β must be lower than its upper bound and therefore β is stronger.

[19] This result also holds when the number of players becomes greater than two, but it requires introduction of highly convoluted definitions and notations for a clear proof while providing no novel insight, so I omit it.

4 Asymmetry and Social Welfare

In this section, I investigate how ex ante asymmetry influences the welfare level of a war of attrition. First, I discuss the different roles played by different members in the contribution to social welfare. Next, I argue that under certain conditions symmetry is ineffcient and any introduction of asymmetry can alleviate the loss.

I introduce some definitions and notations. A society is *divided* if it consists of multiple heterogeneous groups and in each group all members are homogeneous. Formally, define $s = \{N, L, (p_\iota, r_\iota, c_\iota, F_\iota, \overline{v}_\iota)_\iota\}$ to represent an N-member divided society facing a proper war of attrition. This society is divided into L different groups, and the ιth group takes up p_ι proportion of the total population and $\sum_{\iota=1}^{L} p_\iota = 1$. Each member in the ιth group is symmetrically parametrized by $\{c_\iota, r_\iota, F_\iota, \overline{v}_\iota\}$. The rules of notation are the same as those introduced in Section 2.2 only differing in that L denotes the total number of groups and subscript ι denotes all groups.

4.1 Who Matters the Most?

To make analysis tractable, I follow the seminal work by Bliss and Nalebuff (1984) where they discussed a large-population symmetric war of attrition. They acquired a surprising result on social welfare, that is their Theorem 6 which states that when the lower bound of the distribution, from which each player's provision cost is independently extracted, reaches zero, inefficiency out of delay vanishes as the population grows large.

This subsection derives the counterpart conclusion of their limit theorem for my asymmetric war. However, my objective differs from theirs in that I intend to answer how differently do unequally positioned individuals contribute to the welfare level. For simplicity, I let all members have identical provision costs and discount rates while their valuation distributions may differ from each other. To begin with, I prove a useful lemma:

Lemma 3 *Consider a divided society parametrized by $s = \{N, L, r, c, (p_\iota, F_\iota, f_\iota, \overline{v}_\iota)_\iota\}$ which yields an aligned equilibrium: there is no instant exit or strict waiting, and thus $\Phi_\iota(0) = \overline{v}_\iota$ for all ι. Then, when N becomes sufficiently large (while maintaining the equilibrium aligned):*

(i) To maintain alignment, all upper bounds must be the same, namely, $\overline{v}_\iota = \overline{v}$ for all ι.

(ii) A player with valuation v earns expected gain $v(1 - c/\sum_{k=1}^{L} p_k \overline{v}_k) = v(1 - c/\overline{v})$.

Proof of Lemma 7 Omitted.

And I present the symmetric version of Lemma 3 which is equivalent to Bliss and Nalebuff's (1984) Theorem 6:

Corollary 1 *Consider a symmetric N-player war parametrized by $w = \{N, c, \overline{v}\}$. When N becomes sufficiently large, the expected welfare of player v approximates $R(v) = v(1 - c/\overline{v})$*[20].

[20]Some remarks on the condition for efficiency in a symmetric war. The condition that Bliss and Nalebuff (1984) found is the possible occurrence of zero-cost player. In contrast, Corollary 1 suggests that the efficiency

The condition of an aligned equilibrium in Lemma 3 simplifies the proof by only considering a society whose each group shares the same incentive position. This lemma gives three insightful results: *a)* no matter how large the population grows, inefficiency always occurs, unless some groups subject positive probability to the extreme type with infinite valuation; and *b)* when population grows large, the welfare level is solely determined by the strongest types and the population proportions associated with these types; and finally *c)* a group's highest type is representative of the incentive position of this group.

The proof of Lemma 3 provides more information. First, all possible provision is realized immediately at the beginning of the war of attrition, almost surely. But why inefficiency still occurs given this instant provision? This is because when population grows large any type below the upper-bound type tends to free ride for a sufficient long time, so that only those in an almost-zero-measure set very close to the upper bound actually contribute and the probability of everyone's not providing at $t = 0$ remains considerable. Likewise, in each subgame, the conditional distribution of stop time subjects most of the probability to two events: either someone provides instantly or nobody provides within finite time.

For better understanding of the relationship between upper bound and incentive position in the limit case which is implied by Lemma 3(i), I present a complementary lemma:

Lemma 4 *Consider a divided society with sufficiently large total population. Then, the group with higher distribution upper bound has higher incentive position in the society.*

Proof of Lemma 8 Omitted.

What is curious is that the results given by Lemma 3 and Lemma 4 are irrelevant to the shapes of distribution functions or even the density of the highest type. The explanation lies in that, when population grows large, the law of large numbers ensures that even the type with very small density will occur with probability one, and this makes all types subjected with positive probability appear with equal chances.

As a result, the welfare level of an arbitrary divided society is equivalent to that of some uniform-distribution war of attrition. Specifically, this war of attrition is the *counterpart 0-AUD war* associated with this society. A 0-AUD war is a zero-lower-bound AUD war which has been introduced in Section 2.3, and the counterpart 0-AUD war is constructed by changing all valuation distributions of a society to be uniform and their lower bounds to be zero while fixing all other parameters including the upper bounds. This equivalency is in the sense of welfare which is summarized in the theorem below:

Theorem 2 *Consider a divided society with sufficiently large total population. Then, this society is equivalent to its counterpart 0-AUD war in the sense that, at every moment in a war of attrition, the welfare level of the society is solely determined by the possibly highest valuation. Denote this*

of my model hinges on the positive probability subjected to the infinite valuation. Both conditions are equivalent, because it is the valuation-cost ratio determines the incentive one faces. The intuition is that providing immediately is the dominant strategy for the types close enough to the extreme type with either zero cost or infinite valuation. The measure of these types remains considerable even when the population grows large, and consequently, the law of large numbers guarantees instant exit with probability one.

highest valuation at t by $\overline{v}(t)$, and then the expected welfare of a member with valuation v is $v(1 - c/\overline{v}(t))$ where c is the cost of provision.

Proof of Theorem 1 This is a corollary of Proposition 1, Lemma 3, and Lemma 4.

It is surprising that the highest types are decisive in a large society. First, the ranking of upper-bound types completely determines the ranking of all groups' incentive positions. Second, the possibly highest type in a society solely determines the welfare level of this society, which also holds for every subgame. The economic logic has been mentioned in Section 2.3: the strongest type has a dominant position in determining the incentive ranking which further determines the welfare level.

4.2 Improvement or Deterioration?

Whether the introduction of asymmetry improves welfare? The insight seems straightforward: asymmetry makes some players salient so that they are anticipated and thus incentivized to provide sooner.

However, the result is acutally highly dependent on parametrization. For example, compare a pure war involving two strong players and a mixed war involving a strong and a weak. The latter may yield either higher welfare level due to the probability of instant exit, or lower welfare level due to the even worse free-riding problem in other no-instant-exit cases. The comparative strength of both countervailing forces relies on the particular geometric shape of distributions and their relationship with other parameters.

Notwithstanding this dependence, I find it possible to discuss asymmetry's influence on expected welfare in two important cases, and since both cases cover a large class of wars of attrition, the analysis is adequate in telling most of the insight.

The first case is the asymmetry in a large divided society. Borrowing the notations used in the last subsection, I present this insight in the following proposition:

Proposition 4 *Consider a family of divided societies in which each has identical cost of provision, discount rate, and average distribution upper bound, that is, $\sum_{k=1}^{L} p_k \overline{v}_k = \overline{v}$ for all societies. Then, the symmetric society, which satisfies $\overline{v}_k = \overline{v}$ for all group k, yields the lowest welfare level among all in this family.*

Proof of Lemma 9 Omitted.

The intuition is simple. Since Theorem 2 suggests that it is the possibly highest type in a large-population society that decides the welfare level, any break of symmetry that gives a greater maximal upper bound makes every member better off.

An important economic application, the redistribution of revenue, corresponds to the condition in Proposition 4 that the average upper bound of each society holds the same. Imagine that, before the war of attrition starts, a central party who can credibly make coercive redistribution promises to redistribute players' valuations in a fixed pattern the moment the public good is provided. This manner of redistribution is equivalent to move the supports of groups' valuation distributions upwards or downwards, so if there is no side payment from or profit transferred to the central party, the average upper bound of this society remains the same under redistribution. Consequently, Proposition 4 indicates that any redistribution exerted on a symmetric society improves efficiency.

The second case discusses the asymmetry in an AD war. The analysis gives a condition for asymmetry to be an improvement which perfectly illustrates how the comparative strength of the two countervailing forces triggered by asymmetry, the incentive to provide sooner and the incentive to free ride longer, is determined.

I recall and introduce some notations first. Consider an AD war parametrized by $w_{AD} = \{N, r, c, F, \underline{v}, (\overline{v}_i)_i\}$. Define function $e_n(u)$ as the expected discount factor[21] calculated under the equilibrium of a n-player symmetric AD war with upper bound u, namely, $e_n(u)$ measures the welfare level of $E(w_n(u))$ where $w_n(u) = \{n, r, c, F, \underline{v}, \overline{v}_i = u\}$. Denote the symmetric equilibrium strategy of this war by $T_n(v|u)$, and $e_n(u)$ and $T_n(v|u)$ are given by:

$$\begin{aligned} e_n(u) &= E_{t_m}^{w_n(u)}[e^{-rt}] = \frac{1}{F^n(u)} \int_c^u e^{-rT_n(s|u)} nF^{n-1}(s) f(s) ds \\ T_n(v|u) &= \frac{n-1}{r} \int_v^u \frac{f(s)}{F(s)} \frac{c}{s-c} ds \end{aligned} \tag{6}$$

Further, according to Theorem 2, $e_\infty(u)$ is defined as $1 - c/u$. Now, I present the result:

Theorem 3 *Consider a family of N-player AD wars in which each is generated from the same distribution, say F, and has the same maximal upper bound, say $\overline{v}_m$. Then, if $e_N(u) - e_\infty(u)$ is positive (or negative) for all $u \in (c, \overline{v}_m]$, the symmetric AD war, which satisfies $\overline{v}_i = \overline{v}_m$ for all player i, yields the lowest (or the highest) expected discount factor among all in this family.*

Proof of Lemma 10 Omitted.

Theorem 3 is surprising because it indicates that under certain condition an all-strong society is the least efficient, and by changing any player to be weaker, one can make the situation better off. However, this conclusion hinges on the satisfaction of the condition, $e_N(u) > e_\infty(u)$ for all $u \in (c, \overline{v}_m]$, and when the parameters satisfy the opposite condition the conclusion also reverses to be that the all-strong society becomes the most efficient.

What does this condition represent? $e_N(u) > e_\infty(u)$ requires that the welfare level of a symmetric AD war decreases when the population grows larger. That is, under symmetry the loss out of the elongated free-ride time overwhelms the benefit out of the increased opportunity for sooner provision. In short, $e_N(u) - e_\infty(u)$ measures how costly it is to introduce symmetry

[21]Here, I use expected discount factor to measure welfare level for two reasons. First, a player who values the public good less than the cost he faces will wait forever and thus his expected gain is exactly his valuation multiplying this expected discount factor. Second, the expectation of a decreasing function of stop time is a better measure than ex ante welfare, because the latter calculated in different games with different valuation distributions is not directly comparable. The spirits follow Bliss and Nalebuff (1984), Alesina and Drazen (1991), Bulow and Klemperer (1999), and Riley (1999), where expectation of different functions of stop time were considered.

in a society, and when it is consistently positive for u, any symmetry tends to reduce welfare level while asymmetry tends to enhance it.

Numerical experiments give one example satisfying this condition, the family of positive-lower-bound AUD wars. Specifically, the expected discount factor of such a war of attrition is decreasing with respect to the population[22][23]. An interpretation is: a higher lower bound means that more probability is subjected to valuations greater than provision cost, and this makes each player anticipate his opponents with higher incentive to provide, which however only incurs higher incentive to free ride but not enought incentive to actually provide sooner.

5 Discussion and Conclusion

[22] For example, consider a symmetric AUD war in which $r = 1$, $c = 2$, $\overline{v} = 3$, $\underline{v} = 1$, and then the expected discount factor calculated under the equilibria of from the 2- to the 10-player wars, respectively, is: 0.386, 0.364, 0.355, 0.350, 0.347, 0.345, 0.343, 0.342, 0.341, whereas the limit case with large population yields 0.333.

[23] Some remarks. This example gives a different insight from Bliss and Nalebuff's (1984) Theorem 2 and Grandstein's (1992) Lemma 1(ii), both of which state that increasing the population in a symmetric public-good provision game makes everyone better off. Bliss and Nalebuff's result sensitively relies on their model setup that it is the provision cost that is uncertain. Nonetheless, in my model valuation is uncertain and the relationship between expected gain and total population varies with parameters. Consider a symmetric war parametrized by $w = \{n, r, c, F\}$. I follow their approach and calculate $\partial^2 R(v)/\partial v \partial n$ where $R(v)$ is the symmetric version of (1):

$$\frac{\partial^2 R(v)}{\partial v \partial n} = -(n-1) \int_v^{\overline{v}} g^{n-1}(u) \left[\frac{1}{n-1} + lng(u)\right] \frac{f(u)}{F(u)} \frac{c}{u-c} du$$

Above, $g(u) = F(u)e^{-rT(u)}$ where $T(u)$ is the equilibrium strategy of the two-player war of attrition. Notice that $R(\overline{v}) = \overline{v} - c$ remains constant, so the second-order partial derivative $\partial^2 R(v)/\partial v \partial n$ determines how the types lower than $\overline{v}$ changes as n increases. However, the term in the integral, $\frac{1}{n-1} + lng(u)$, is positive near $u = \overline{v}$ but approximates negative infinity as $u \to c + 0$, so although $\partial^2 R(v)/\partial v \partial n$ is negative when v is large, its sign relies on parametrization when v becomes small. A consistent monotonicity is unlikely.

6 References

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