Does Ex-Ante Asymmetry Matter? A Modeling of Multi-Player Asymmetric War of Attrition

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Abstract

This paper models a multi-player asymmetric war of attrition game with incomplete information on the private provision of public goods to investigate how ex-ante asymmetry affects behavior and welfare. In the unique equilibrium, asymmetry leads to a stratified behavior pattern such that one player exits instantly with a positive probability, while each of others has no probability of concession until a certain moment associated with them. Efficiency measured by the cost of delay is mainly determined by the strongest type, namely the highest type of the instant-exit player. If asymmetry is introduced by strengthening the strongest type, it tends to improve efficiency, whereas if the strongest type is controlled, the effect of asymmetry coincides with the sign of an intuitive measure of the cost of symmetry.

KEYWORDS. War of attrition, private provision of public good, ex-ante asymmetry, multiple players, incomplete information.

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1 Introduction

When cooperation is prohibitively costly for a group of people, any public good can only be provided privately during a war of attrition. This happens on a daily basis and is one of the major sources of welfare loss resulting from strategic interaction. Strategic delay is known to be a major cause of inefficiency in these situations.

One intriguing question is: Does *asymmetry* matter in such conflicts? Here, asymmetry refers to the situation where individuals in different economic, political, or social positions are anticipated differently. This notion corresponds to a large number of applications. For example, when socioeconomic groups try to shift the burden of stabilization onto each other, the presence of several incumbent groups who feel more pressured to conduct fiscal stabilization creates asymmetry. For another, when several countries or provinces suffer from illegal activities on their common border, the different costs of controlling the chaos faced by different agents introduce asymmetry. Moreover, when the United Nations gathers countries to reach an agreement on how humanity should respond to climate change, the different incentives of different countries to make voluntary commitment also bring asymmetry.[1] Finally, ex-ante asymmetry also lies in the discriminative stereotypes that people have to others based on the impression of race, gender, age, and other social elements that label people.

This issue raises compelling questions: How does asymmetry change behavior? How does this asymmetric behavior pattern make each agent contribute to welfare differently? Does asymmetry alleviate or exacerbate delay? Can we be better off by sharpening or equalizing asymmetry?

This paper develops a generalized war of attrition that combines ex-ante asymmetry, multiple players, and incomplete information. Such a general combination is missing in the literature[2] of the war of attrition and other similar forms of conflict.

My model provides an asymmetric extension of Bliss and Nalebuff (1984). They discuss a continuous-time war of attrition on the private provision of an indivisible public good in which each player chooses a provision time in the beginning to optimize his expected utility, and once

[1]Detailed explanations for these three examples. The symmetric analysis of the first example has been done by Alesina and Drazen (1991). The second example corresponds to the Golden Triangle area, the common border of Thailand, Laos, and Myanmar, which is also not far from China. The rampant illegal dealing of drugs and long-lasting violent activities are the consequence of delayed and loose regulation from the neighbor countries. Asymmetry does exist in this case, as Thailand implements relatively more strict regulations than others. Besides, many other famous drug-trade areas are also the common borders of several countries, like the Golden Crescent and the Silver Triangle. For the last example, the Paris Agreement is a good manifestation. While China has shown willingness, the United States kept postponing the progress and eventually exited this agreement.

[2]I list some examples with two of the three elements. For multi-player asymmetric wars of attrition with complete information, see Ghemawat and Nalebuff (1985, 1990), Whinston (1988), and Bildeau and Slivinski (1994). Examples into multi-player symmetric wars of attrition with incomplete information are Riley (1980), Bliss and Nalebuff (1984), Alesina and Drazen (1991), Bulow and Klemperer (1999), and Sahuguet (2006). The two-player asymmetric incomplete-information case is the most widely studied, for example, Riley (1980), Nalebuff and Riley (1985), Fudenberg and Tirole (1986), Kornhauser, Robinstein, and Wilson (1988), Ponsati Sakovics (1995), Abreu and Gul (2000), Myatt (2005), and Horner and Sahuguet (2010). Also, there are special cases that consider all three elements. For example, a third party strategically interferes in a two-player war of attrition (e.g., Casella and Eichengreen (1996) and Powell (2017)), and two groups bargain over two objects, which however is basically a two-player game (e.g., Ponsati and Sakovics (1996)). The most related study is Kambe (2019) who investigates a model similar to mine but with two-type incomplete information.

someone provides first, the game ends, and everyone gains lump-sum payoff according to their information. My model differs in that I allow all players' information, like costs, discount rates, and valuation distributions, to be asymmetric. Additionally, to guarantee a unique equilibrium, I assume that every player is anticipated to have a positive probability of valuing the public good less than his cost.

This paper finds that heterogeneous individuals manifest a *stratified behavior pattern*. One degenerate example of this concept commonly seen in two-player cases[3] is *instant exit*. That is, one of the players will have a positive probability of conceding immediately.

Apart from instant exit, there is also *strict waiting* which refers to the feature that some players will have no positive probability of provision until certain time points associated with each of them. This is not possible in two-player cases, for the highest types of both players always provide instantly. For each player who waits strictly, I call the minimal waiting time among all his types the strict-waiting time.

As a result, when the environment is asymmetric enough, the equilibrium behavior begins with some probability of one player's instant exit, and what follows is a period during which only two players have the probability of provision, and after it a third player becomes active, and in this manner periods with increasing numbers of active players follow sequentially. Eventually, only when the game has endured for a sufficiently long time will all players become active. Instant exit could be construed as a "one-player period" whose length is zero because there is no provision from others and any delay is unnecessary. The idea that asymmetry affects the outcome by changing the scale of active players has been studied in earlier yet less general cases[4].

This stratified equilibrium results from the asymmetric *incentive position*s of different types. Individual optimality requires the types revealed simultaneously to balance each other's incentives mutually. Namely, the types being revealed at the same time and the manner in which they are revealed are such that each of them will find the extra gain from providing immediately equal that from waiting slightly longer at this moment. This mutual-balance requirement makes the incentives faced by different types, to some extent, comparable. For example, types that exit instantly value the public good so much that no simultaneous revelation with other types can offset their high incentives, so they are in higher incentive positions. In contrast, some players wait strictly because even their highest types still value the good too low to be mutually balanced with earlier revealed types, and thus they are in lower incentive positions.

Intuitively, I call a player *stronger* if he has a positive probability of providing instantly or if he strictly waits shorter, and the former case corresponds to the strongest player whose highest valuation is called the strongest type. Comparative statics tell that either lower cost, more impatience, or "consistently higher" valuation distribution reduces a player's provision time and thus makes him stronger.

On the uniqueness of equilibrium, a huge literature obtains uniqueness by perturbing a war of attrition. Namely, for each player, there must be a positive probability of some others' never

[3]A seminal work that mentions instant exit is Nalebuff and Riley (1985), and among more recent studies are Ponsati and Sakovics (1995), Riley (1999), Abreu and Gul (2000), and Myatt (2005).

[4]See Bergstrom et al. (1986), Hillman and Riley (1989), and Kambe (2019).

conceding.[5] Yet, most articles involve only two players or essentially complete information[6], while this paper considers multi-person interaction and continuous-type information, which generates a nonlinear differential system that has irregular boundary conditions such that each of them is either flexible or at infinity. The perturbation I employ is to allow every player to have a positive probability of valuing the public good less than his cost, in which never conceding is the dominant strategy. This assumption makes the solution of the differential system change sensitively at infinity so that the boundary conditions suffice to uniquely determine an equilibrium.

The second contribution of this paper lies in a complete discussion of the relationship between ex-ante asymmetry and social welfare.

First, I utilize the expected discount factor as a measure of welfare level to answer who matters the most and how he matters. For the former question, social welfare is mainly decided by the strongest type. Nonetheless, the answer to the second is surprising. It is commonly believed in the literature that the strongest type matters merely because it is the highest type of the instant-exit player who makes the most contribution. However, my results tell that the strongest type matters independently by determining the ranking of all types' incentives, which to some extent, is independent of the behavior of other types of the strongest player. For example, the welfare level of the special case, AD war, discussed in Section 3.3 is irrelevant to the variations in players' behavior as long as the strongest type is fixed, and this result makes the focus on the strongest player's exact behavior, like his instant-exit probability[7], less important.

The idea is that the strongest type "controls" the behavior of all types, and since the welfare is an integral with respect to all players' behavior which "closely follows" the strongest type, its level is mainly decided by the latter. The asymmetric dependence among players' behavior demonstrates one form of the "controlling". In the AD-war case, while the parameter variation of a weak player sheds no influence on stronger types' behavior, the variation of a strong player effectively changes weaker types' behavior. Additionally, the analysis of large-population societies in Section 4.1 presents a more general result that the highest type of each player completely determines his incentive position, which also implies asymmetric dependence.

Second, I investigate the impact that asymmetry has on welfare. That introducing asymmetry will enhance efficiency is a point commonly made in the literature.[8] However, the model an-

[5]For example, Fudenberg and Tirole (1986) introduce a positive probability of each player being better off in a duopoly than in a monopoly. Kornhauser, Rubinstein, and Wilson (1989) use a slight probability of irrational type who only plays a fixed strategy, the idea of which is also borrowed by Kambe (1999, 2019) and Abreu and Gul (2000). Myatt (2005) considers three forms of perturbation: exit failure, hybrid payoff, and time limit.

[6]By *essentially* I refer to Abreu and Gul (2000) and Kambe (2019) where they investigate wars of attrition with discrete-type incomplete information, but since only one type is rational while all others never concede, their setups are basically perturbed complete-information games.

[7]For example, Myatt (2005) and Kambe (2019) stress the importance of the probability of instant exit.

[8]One extreme example is to select an efficient yet degenerate equilibrium with refinement. For instance, Riley (1999) lets a sequence of members of a contest-game family approximate a war of attrition, and he finds that any introduction of asymmetry makes one player concede immediately with probability one. Kornhauser, Rubinstein, and Wilson (1988) and Myatt (2005) derive similar results in more complex cases. Besides, Kambe (2019) argues that asymmetry increases the probability of instant exit and further improves efficiency. Some others analyze welfare directly. For example, Riley's (1999) numerical calculation of a welfare measure shows a consistently positive welfare effect of asymmetry; however, the result may hinge on his complete-information setup. Static public-provision games like Bergstrom, Blume, and Varian (1986) argue similarly.

alyzed in this paper incorporates more dimensions as there are multiple players and continuous-type distributions, so the influence of ex-ante asymmetry highly depends on its definition and the parametrization. Informed by the insight that the strongest type has a decisive influence on the outcome, I manage to provide a complete discussion by considering two kinds of asymmetries, one of which allows the strongest type to change while the other controls it. Specifically, the first way of introducing asymmetry is to make the strongest type stronger under certain control, whereas the other is to fix this type and make others weaker. Such a division constitutes a complete discussion and both cases generate distinct insights. Thus, it is reasonable enough to help us to understand the effect of asymmetries under different circumstances.

The result for the first kind of asymmetry shows that by strengthening the strongest, any slight introduction of asymmetry reduces the cost of delay. I conduct numerical experiments to show that for a large number of applications, the phrase above "slight introduction of asymmetry" can be expanded to "any introduction of asymmetry". The intuition is consistent with the previous finding that the strongest type has a decisive influence on the outcome.

Nonetheless, the other case manifests dependence on parametrization. Any introduction of asymmetry improves efficiency if the cost of symmetry is positive, which is measured by the welfare-level discrepancy between an N-player symmetric game and the associated infinite-player symmetric game, while asymmetry always impairs efficiency if the thus measured cost of symmetry is negative. An explanation for this dichotomy is that the cost of symmetry defined above actually evaluates the cost brought by increasing population and since asymmetry makes the scale of active players smaller during the beginning period, the effect of asymmetry has the same sign as that of the cost of symmetry.

This paper is organized as the following. Section 2 describes the model setup and the equilibrium concept. Section 3 first characterizes the equilibrium and proves the existence and uniqueness. I introduce a special case, AD war, in this section to illustrate both behavior features and welfare implications formally discussed later. Finally, this section performs comparative statics. Section 4 shows the relationship between ex-ante asymmetry and social welfare. Section 5 discusses possible applications and differentiates from the related research.

2 Model

There is an indivisible public good potentially beneficial to N different individuals. I denote each player by $i \in I_N$ where $I_N = \{1, 2, ..., N\}$. A continuous-time war of attrition that requires one exit begins at $t = 0$ and each player chooses a stopping time when, if no one has provided the public good yet, he will provide. Since there is no dynamic interaction during the procedure, this game is strategically static.

The information structure: one player, say i, knows exactly the cost of his individual provision $c_i > 0$, the rate $r_i > 0$ at which he exponentially discounts his expected gain, and his valuation of this public good v_i. The costs and discount rates of all players are common knowledge, whereas each valuation v_i is private information independently extracted from a cumulative distribution function $F_i : [\underline{v}_i, \overline{v}_i] \to [0, 1]$ in which $\underline{v}_i < c_i < \overline{v}_i < +\infty$ and $c_i > 0$ for all i. Assume that each F_i yields a density function $f_i : [\underline{v}_i, \overline{v}_i] \to \mathbb{R}^+$ which is differentiable and strictly bounded from 0.

Note that the strict relationship of $\underline{v}_i$, $\overline{v}_i$, and c_i is important because it necessarily guarantees the uniqueness of equilirium. For convenience, I sometimes call player i with valuation v_i simply as player v_i.

Player i's pure strategy is a function $T_i : [\underline{v}_i, \overline{v}_i] \to \mathbb{R}^+ \cup \{0, +\infty\}$ referring to the stopping time that player v_i chooses. Only when no provision happens before $T_i(v_i)$ will this player provide at this moment. If some players provide first, all players gain their valuations while the providers additionally pay their share of the provision cost. Namely, if $m \geq 1$ players provide at this moment, they respectively pay $\frac{1}{m}$ of the cost associated with each of them. If all players choose to wait forever, each earns zero. All payoffs are lump-sum paid at the provision menment, at which the game ends.

I consider pure-strategy Bayesian equilibrium, and in the following sections by *equilibrium* I refer to this notion unless otherwise specified.

3 Equilibrium Analysis

In this section, I first show a set of sufficient and necessary conditions which reveal that the equilibrium behavior demonstrates a stratified behavior pattern. Then, I prove the existence and uniqueness of equilibrium. Moreover, I introduce a widely applicable case, AD war, to illustrate some significant behavior and welfare insights which are formally discussed later. Finally, I perform comparative statics.

3.1 Characterization

I introduce some definitions and notations before presenting the set of sufficient and necessary conditions. First, I formulate the expected gain of player v_i with his choice of stopping time being t_i and the strategies of other players fixed as $\{T_j(.); j \neq i\}$. For convenience, I denote the probability of at least one of the other players providing before t by $F_{-i}^{min}(t) = \text{Prob}(\min_{j \neq i} T_j(v_j) \leq t)$. This expected gain, denoted by $R_i(t_i|v_i)$, is:

$$R_i(t_i|v_i) = v_i \int_0^{t_i} e^{-r_i s} dF_{-i}^{min}(s) + (v_i - c_i)e^{-r_i t_i}(1 - F_{-i}^{min}(t_i)) \tag{1}$$

The first term above represents the case where someone else provides before t_i, and the second corresponds to the situation where no one provides before t_i and thus player i pays the cost. I omit the simultaneous-provision situation in (1) since this cannot appear in equilibrium, as proved later. Further, define $d_i = \min_{v \in [\underline{v}_i, \overline{v}_i]} T_i(v)$ as the minimal waiting time among all types of player i, and $u_i = \min_{T_i(v_i)=d_i} v_i$ as the minimal type of player i that provides at d_i.

The set of sufficient and necessary conditions presented below in Lemma 1 shows a strictly decreasing monotonicity, which means that the higher one values the public good, the shorter he will wait before concession. Besides, the asymmetric environment requires a set of flexible boundary conditions, that is for every player i:

$$(\overline{v}_i - u_i)d_i = 0,\ \overline{v}_i - u_i \geq 0,\ and\ d_i \geq 0.$$

If player i is such that $\overline{v}_i > u_i$ and $d_i = 0$, then by monotonicity, he will provide the public good instantly when his valuation is higher than u_i. I call him an *instant-exit player*. On the other hand, if his equilibrium behavior satisfies $\overline{v}_i = u_i$ and $d_i > 0$, no matter what his valuaion is, he will wait strictly until d_i when his highest type starts to contribute. In this case, he is a *strict-waiting player*.[9] Finally, player i is called an *active player* at moment t if there exists some $v_i \in [\overline{v}_i, \underline{v}_i]$ such that $T_i(v_i) = t$, which means that he is anticipated to have a positive probability of provision around this moment. Now, I present the set of conditions for an equilibrium:

Lemma 1 *A profile $\{T_j(.); j \in I_N\}$ corresponds to a Bayesian equilibrium if and only if:*

(i) For all $i \in I_N$, $T_i(v_i) = +\infty$ on $[\underline{v}_i, c_i]$, and $T_i(v_i) < +\infty$ on $(c_i, \overline{v}_i]$.

(ii) For all $i \in I_N$, $\lim_{v_i \to c_i+0} T_i(v_i) = +\infty$.

(iii) There are at most $N-2$ strict-waiting players.

(iv) There is at most one instant-exit player.

(v) For all $i \in I_N$, $T_i(v_i)$ is continuous and strictly decreasing on $(c_i, u_i]$. Thus, $T_i(v_i)$'s inverse function $\Phi_i(t_i)$ exists on $[d_i, +\infty)$ and is also continuous and strictly decreasing.

(vi) At every moment $t > 0$, if there are M active players, denoted by $I(t) = \{j_1, j_2, ..., j_M\}$, then for all $j_i \in I(t)$, $\Phi_{j_i}(t)$ exists and is differentiable at t, and its derivative satisfies:

$$\Phi'_{j_i}(t) = \frac{F_{j_i}(\Phi_{j_i}(t))}{f_{j_i}(\Phi_{j_i}(t))} \left[\frac{r_{j_i}}{c_{j_i}}(\Phi_{j_i}(t) - c_{j_i}) - \frac{1}{M-1} \sum_{k=1}^{M} \frac{r_{j_k}}{c_{j_k}}(\Phi_{j_k}(t) - c_{j_k}) \right] \tag{2}$$

Proof of Lemma 1 See appendix.

The clauses (iii), (iv), and (vi) of Lemma 1 together demonstrate an intriguing feature of the asymmetric equilibrium involving multiple players: the war of attrition starts with a positive probability of one player's instant exit, which is followed by a period during which only two players have the probability of concession, and this two-player period ends with the start of a three-player period. Likewise, players leave the inactive state sequentially as the game proceeds so that after a sufficiently long time, everyone becomes active.

To have a visual impression, consider three players with an information structure asymmetric enough to make both instant exit and strict waiting possible. I denote the strict-waiting player by 1 and his minimal waiting time by d_1. Assume that three players have identical provision costs. Then, I depict a possible solution of this three-player war in Figure 1.

I introduce some definitions and notations for convenience. I call a strict-waiting time poins as a *division*, like the d_1 in Figure 1, and the inverse function of each player's strategy as a *curve*, like the Φs in Figure 1. I denote the time interval between two adjacent divisions where there are M active players by $\Upsilon(M)$ and call the group of differential equations that characterizes the

[9]The case where $\overline{v}_i = u_i$ and $d_i = 0$ is the same as the behavior demonstrated in a symmetric game, so it is of less interest here.

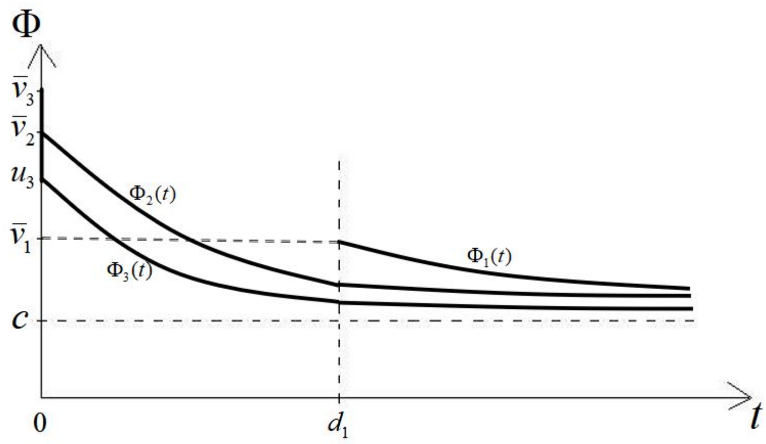

Figure 1: Player 3's behavior, $\Phi_3(t)$, starts at $\Phi_3(0) = u_3 < \overline{v}_3$ corresponding to an instant exit, while that of player 1, $\Phi_1(t)$, remains inactive until d_1 referring to a strict waiting. Before and after d_1, the active curves obey the two-player and three-player versions of (2), respectively, and they are continuous at d_1.

behavior of these players during this interval *the M problem*, whose boundary conditions need to be further specified. Let I_M denote the set of active players in the M problem.

Importantly, the presence of instant exit and strict waiting discloses the relative strength of incentives faced by different types of players. The key is that the information revelation through time makes the types revealed at the same time mutually balance each other's incentive. Formally, rearranging (2) generates (3) which shows the tradeoff faced by the active player j_i at moment t in the M problem:

$$c_{j_i} \sum_{k=1, k\neq i}^{M} \frac{-dF_{j_k}(\Phi_{j_k}(t))}{F_{j_k}(\Phi_{j_k}(t))} = r_{j_i} dt(\Phi_{j_i}(t) - c_{j_i}) \tag{3}$$

The interpretation of (3) is based on the facts that: *a)* $\Phi_{j_i}(t)$ is the type revealed at t, and thus $r_{j_i} dt(\Phi_{j_i}(t) - c_{j_i})$ represents how much this type will gain extra if he provides at t instead of at $t+dt$; and *b)* $F_{j_k}(\Phi_{j_k}(t))$ equals the probability of player j_k's providing after t, so the left side of (3) represents the extra gain from an infinitesimal delay after t. So, individual optimality requires that types revealed simultaneously make each other indifferent between providing and waiting at their revelation moment. In other words, they mutually balance each other's incentive, and intuitively, I call these types equivalent in their *incentive positions*. For example, the instant-exit types have the highest incentive position, since they value the public good so much that no simultaneous revelation with others can offset their high incentive. And, between each pair of adjacent divisions, even the highest type of an inactive player generates the incentive too low to balance those of the active types. Thus, the revelation through time offers a natural way to compare and rank the different types of different players.

That asymmetry affects the equilibrium by changing the scale of active players shares the similar idea with Bergstrom, Blume, and Varian (1986), who develop a complete-information model without timing. They find that considerable redistribution can make the incentives faced by different players change so differently that the number of contributors of the provision of public goods will decrease. Further, my model incorporates the incomplete information and timing, both of which make this scale-changing process endogenous.

Hereafter, a type is called *stronger* than another if the former type selects a shorter waiting time, or if the former values the public good more than the latter when they are both instant-exit types. Likewise, a player is *stronger* than another if the highest type of the former strictly waits shorter, or if he is an instant-exit player.

3.2 Existence and Uniqueness

I apply a perturbation strategy to guarantee a unique equilibrium. Specifically, every player has a positive probability of valuing the public good less than his cost or, formally, with $F_i(c_i) > 0$. The intuition is that the probability of never conceding makes the arrival of any moment possible, which imposes perfection on all the off-equilibrium paths to eliminate unreasonable solutions.

To determine the equilibrium, I employ a backward induction strategy. The first step is to consider the last M problem where all N players are active.

Now, I define *the P_N problem*. Recall that the N problem refers to the N-player version of (2) defined on $[\overline{d}, +\infty)$ where $\overline{d}$ denotes the largest strict-waiting time. Since (2) is time-invariant, we can substitute $t + \overline{d}$ for t so that the rightmost division $\overline{d}$ becomes the new origin of time. Then, I extend the definition of both $F_i(.)$ and $f_i(.)$ to the domain $(\overline{v}_i, +\infty)$ for every player i. The form of this extension is such that both functions remain entirely differentiable and bounded from both infinity and zero. Clearly, such an extension exists and with it, the N problem still satisfies local Lipschitz condition wherever it has definition. Further, define $B_N = \times_{i=1}^{N}[c_i, +\infty)$, and I call $m \in B_N$ a left-side boundary selection which generates $\{(0, m_i); i \in I_N\}$ as a set of boundary conditions at $t = 0$ for the N problem. The satisfaction of Lipschitz condition implies that every boundary selection yields a unique solution of the N problem, so we can denote by $\Phi_i(t, m)$ the curve of player i associated with the boundary selection m. Finally, I call $P_N = \{I_N, B_N, (c_i, r_i, F_i, f_i), \Phi_i(t, m); i \in I_N\}$ the P_N problem, which consists of the N problem with the origin reset at $\overline{d}$, the set of all possible left-side boundary selections, and the set of solution curves written as functions of both the time and the boundary selection.

According to Lemma 1(ii) and (v), a boundary selection m^* of the P_N problem that yields the equilibrium-like solution of the N problem should be such that every solution curve i, $\Phi_i(t, m^*)$, is strictly decreasing and convergent to c_i with respect to t. Besides, the clauses (iii) and (iv) also require a "just-touch" condition: $m_i^* \leq \overline{v}_i$ for all player i while $m_k^* = \overline{v}_k$ for some player k, in which such and only such player ks strictly wait till the arrival of the N problem.

To prove that such a boundary selection uniquely exists, I first show several properties of the solution curves of the P_N problem, which are summarized in Lemma 2. This lemma shows that: *a)* each solution curve is monotonous with respect to each component of the boundary selection; and *b)* when the solution is convergent such that each curve converges to a finite number, it must satisfy Lemma 1(ii) and (v); and finally, *c)* any variation of one component of the boundary selection makes the convergence collapse to divergence.

Lemma 2 *For the P_N problem defined above, its solution curves have:*

(i) Monotonicity: for all $t > 0$, $m \in B_N$, and $i, j \in I_N$ such that $j \neq i$, $\Phi_i(t, m)$ is strictly increasing with respect to m_i, while strictly decreasing with respect to m_j.

(ii) *Two patterns: when $t \to +\infty$, the solution can take on* convergence*: it satisfies Lemma 1(ii) and (v); or* divergence*: Φ_i approximates $+\infty$ or $\underline{v}_i$ for all $k \in I_N$.*

(iii) *Sensitivity: for all $m \in B_N$, and $i, j \in I_N$ such that $j \neq i$, if the solution is convergent, then as $t \to +\infty$, $\partial\Phi_i(t,m)/\partial m_i \to +\infty$, while $\partial\Phi_j(t,m)/\partial m_i \to -\infty$.*

Proof of Lemma 2 See appendix.

Lemma 2 indicates that solution uniqueness hinges on the perturbation condition that $\underline{v}_i < c_i$ for all i. This condition makes the solution that satisfies Lemma 1(ii) and (v) the only convergent solution of the P_N problem, and also makes it sensitive to the boundary seletion. To observe the multiplicity problem of the opposite case, consider a 2-player problem in which $\underline{v}_1 > c_1$ and $\underline{v}_2 > c_2$, and in the associated P_2 problem, $B_2 = [\underline{v}_1, +\infty) \times [\underline{v}_2, +\infty)$. First, there exist degenerate equilibria in which all types of one player provide instantly. Additionally, multiplicity also occurs even if the equilibrium is selected according to Lemma 1[10], since for both $i = 1, 2$ and all $m \in B_2$, (2) always gives that $\Phi_i'(t,m) \leq -(\underline{v}_i - c_i)F_i(\Phi_i)/f_i(\Phi_i) < 0$, which naturally satisfies Lemma 1(ii) and (v).

Now, I summarize the backward induction process with which I prove the existence and uniqueness of equilibrium. Lemma 2(ii) tells that to find an equilibrium-like solution of the P_N problem, it suffices to find a boundary selection that generates a convergent solution and also satisfies the "just touch" condition. First, for every constant $m_0 \geq c_1$, I show that the boundary selection that has $m_1 = m_0$ and generates a convergent solution uniquely exists. To prove so, I utilize the sensitivity property in Lemma 2(iii) to show that if the set of such boundary selections described above is nonempty and compact, then this set contains a unique element. Next, I construct a sequence of auxiliary functions such that the limit points of the sequence of their fixed points are exactly the boundary selections we want, and I prove that this set of limit points is nonempty and compact. Therefore, we can rewrite the unique boundary selection associated with $m_1 = m_0 \geq c_1$ as N well-defined functions, $\{m_i(m_0); i \in I_N\}$. Second, I show that for every $i \in I_N$ $m_i(c_1) = c_i$ and $m_i(.)$ is strictly increasing and continuous. All these properties together guarantee the unique existence of an m_0^* such that $m^* = (m_0^*, m_2(m_0^*), ..., m_N(m_0^*))$ satisfies the "just touch" condition that $c_i \leq m_i(m_0^*) \leq \overline{v}_i$ for all player i while $m_k(m_0^*) = \overline{v}_k$ for some player k. Now, consider the penultimate M problem defined on the time interval $\Upsilon(M)$. Note that any player k such that $m_k^* = \overline{v}_k$ is no longer active in this problem, and since a set of boundary conditions that generates a unique solution on $\Upsilon(M)$ is already set on the right-side division by the m^* solution of the P_N problem, the only thing to determine is the relative distance between the left-side and the right-side divisions, that is the length of $\Upsilon(M)$. I show that the strictly decreasing monotonicity of the solution curves of the P_N problem implies that this property still holds on $\Upsilon(M)$, and this yields a unique length of $\Upsilon(M)$ such that all active curves satisfy the "just touch" condition on the left-side division. Likewise, the backward sequential satisfaction of "just touch" determines the relative distance between each pair of adjacent divisions and also gradually reduces the number of active players. This process stops when the active population decreases to either zero or one, which corresponds to an equilibrium without or with one instant-exit player, respectively. Whereby, I finally obtain the main result of this section:

[10] Here, clause (ii) should be modified to be: for all i, $\lim_{v_i \to \underline{v}_i + 0} T_i(v_i) = +\infty$.

Theorem 1 *There uniquely exists a Bayesian equilibrium.*

Proof of Theorem 1 See appendix.

I introduce some notations to link a war of attrition game to its unique equilibrium. Define Ω as the set of all proper wars of attrition, where by *proper* I mean that if $\omega \in \Omega$, then it can be written as $\omega = \{N, (r_i, c_i, F_i, \underline{v}_i, \overline{v}_i); i \in I_N\}$ parametrized as in Section 2. On the other hand, define Ξ as the set of the unique equilibria of all such proper games, and for every $e = \{N, \overline{K}, \Phi_i, (d_K, M_K, I_K); i \in I_N, K \in I_{\overline{K}} = \{1, 2, ..., \overline{K}\}\} \in \Xi$, $N \geq 2$ and $\overline{K} \geq 0$, and for all $i \in I_N$ $\Phi_i(.)$ denotes the inverse of player i's equilibrium strategy, and for all $K \in I_{\overline{K}}$ d_K marks the location of the Kth division from the left whereas $d_0 = 0$ and $d_{\overline{K}+1} = +\infty$, and I_K and M_K denote the set of active players and the number of them during $t \in (d_K, d_{K+1})$, respectively. Finally, I define the mapping $E : \Omega \to \Xi$ such that $E(\omega)$ is the unique equilibrium of ω.

3.3 An Important Case: AD War

To demonstrate some behavior and welfare insights, I introduce a special yet significant family of proper wars, aligned-distribution war (AD war). This class of war is important for two reasons: *a)* apart from this case, there are very few mathematically tractable examples and *b)* it corresponds to wide economic applications.

In each AD war, players have identical costs and dicount rates, but their valuation distributions are all lower-conditional distributions of one fixed distribution with different upper bounds. Formally, let F be a proper distribution whose upper bound and lower bound are denoted by $\overline{v}$ and $\underline{v}$, respectively, and $(\overline{v}_i)_{i=1}^N$ a set containing N players' upper bounds, each of which is no greater than $\overline{v}$ and greater than the provision cost. Then, the ith player's valuation distribution is the lower-conditional distribution of F corresponding to $\overline{v}_i$, that is, $F_i(v) = F(v|v \leq \overline{v}_i)$ for all i. Define $\Omega_{AD} \subset \Omega$ as the space of all such wars, and each of its element is denoted by $\omega_{AD} = \{N, r, c, F, \underline{v}, (\overline{v}_i)_i\}$. From now on, I will sometimes call such a war of attrition an AD war generated from distribution F with upper bounds $(\overline{v}_i)_{i=1}^n$.

This case corresponds to a large category of daily-life scenarios because it can be seen as a partly revealed symmetric war in which players who are initially anticipated to be symmetric may have engaged in some previous games whose unequal outcomes shape the asymmetry of the current war.[11] On the other hand, an AD war is easy to analyze, for the rates on the right side of (2), $f_i(v)/F_i(v)$, of different players are identical on their overlap domain, so between each adjacent pair of divisions active players' behavior is characterized by symmetric equations, and it is easy to verify that backward induction gives symmetric solution[12].

First, I demonstrate how the equilibrium of an AD war looks like. Without loss of generality, let $\overline{v}_1 \geq \overline{v}_2 \geq ... \geq \overline{v}_N > c > \underline{v}$. Note that the active curves on each $\Upsilon(M)$ are symmetric.

[11] One should see this interpretation as merely intuitive. It needs the assumption that the decision a player made in a previous game is not strategically related to other games, for example, when the interval between games is so large that exponential discounting makes any such inter-game strategic move worthless.

[12] The proof of Theorem 1 suggests the uniqueness of the rightmost P_N problem's solution, which requires the solution characterized by symmetric equations to be symmetric as well. This rightmost symmetry further ensures symmetric boundary conditions on all domains on its left side, and therefore the symmetric equations on each domain also result in symmetric solution.

Formally, let $\Phi_{AD}(t|M, r, c, F, u, \underline{v})$ denote the solution of the initial-value problem derived from the symmetric M-player version of (2) with boundary condition $\Phi_{AD}(0|M, r, c, F, u, \underline{v}) = u$.[13]

Then, the equilibrium $E(w_{AD})$ is given by:

$$
\begin{aligned}
&d_0 = 0,\ d_{\overline{K}+1} = +\infty \\
&d_K - d_{K-1} = \Phi_{AD}^{-1}(\overline{v}_{K+2}|K+1, r, c, \overline{v}_{K+1}, \underline{v}),\ K = 1, 2, ..., N-2 \\
&\begin{cases} T_1(v) = 0,\ v \in (\overline{v}_2, \overline{v}_1] \\ \Phi_1(t) = ... = \Phi_n(t) = \Phi_{AD}(t - d_{n-2}|n, r, c, \overline{v}_n, \underline{v}),\ t \in [d_{n-2}, d_{n-1}),\ n = 2, 3, ..., N \end{cases}
\end{aligned}
\tag{4}
$$

The tractability of this case depends on the behavior feature of local symmetry, that is, types with identical numerical value provide at the same time. In this case, the ranking of incentive positions coincides with that of valuations, which gives a straightforward interpretation of the former. In Figure 2, I depict the solution of a three-player AD war.

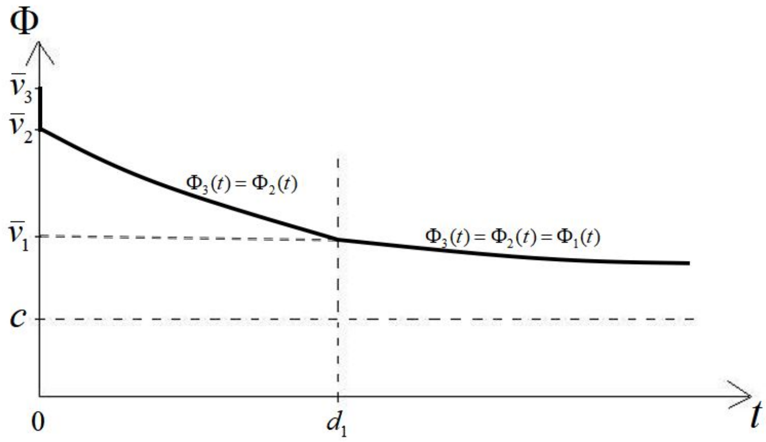

Figure 2: This figure demonstrates the equilibrium of a three-player AD war. The player with the highest upper bound provides instantly when his realized type is no less than the second-large upper bound. Before the first division two players are active, whereas after it the third becomes active as well. Active players always reveal their types symmetrically at the same time.

Another noteworthy feature is that players' behavior is asymmetrically dependent on each other. Namely, if the upper bound of a weak player varies, all stronger types' behavior stays unchanged, while if that of a strong player changes, all weaker types alternate their decisions. More formally, consider N players in an AD war and one of them, say k who is not the strongest player, has distribution upper bound $\overline{v}_k$. Now, let his and only his upper bound rise (or drop) to $\widetilde{v}_k$ to construct a new game. Then, for all players, any type greater than $max\{\overline{v}_k, \widetilde{v}_k\}$ chooses the same stopping time as in the old game, whereas any type less than $max\{\overline{v}_k, \widetilde{v}_k\}$ chooses a longer (or shorter) waiting time. The strongest player may seem like an exception because the variation of his upper bound only changes his probability of instant exit, but one should notice

[13] Namely, $\Phi_{AD}(t|M, r, c, F, u, \underline{v})$ is the solution of $\Phi'(t) = -\frac{1}{M-1}\frac{r}{c}\frac{F(\Phi)}{f(\Phi)}(\Phi - c)$ with boundary condition $\Phi(0) = u$.

that this lowers all lower types' incentive positions by directly adding strong types at the top of the ranking.

This result demonstrates the dominant position of the strongest player in determining the incentive ranking, because any variation of a weaker player's distribution mainly changes his strict-waiting time to suit this ranking, while any variation of the strongest player changes the ranking directly.

This saliency of the strongest player further implies his dominant position in the determination of welfare level. On the one hand, the parameter change of the strongest player sheds consistent influence on all types' incentive positions. On the other, the influence of a weak player's change is offset[14] due to the fact that the behavior alternation of those valuing the public good less than him is in the opposite direction of the change of his incentive position. Namely, when this weak player's upper bound rises (or drops), types lower than him delay longer (or shorter) and thus the welfare change out of this is moderate.

A surprising result is that it is the strongest player's highest type, rather than his exact behavior, that mainly determines a society's welfare level. This is at odds with the idea mentioned in the literature that asymmetry changes welfare by making some players concede sooner or later. For example, Kambe (2019) argued that a higher probability of instant exit necessarily improves efficiency. Others, like Myatt (2005), also stressed the importance of instant exit. However, Proposition 1 below shows a telling example that refutes such a statement since in this case, the welfare level becomes completely irrelevant to how players exactly behave as long as the strongest type is fixed.

Now, I use the uniform-distribution example to show the insight on welfare discussed above. I call this example the aligned-uniform-distribution war (AUD war) which has analytical solution. I let $\Phi_{AUD}(t|M, r, c, \overline{u}, \underline{v}) = \Phi_{AD}(t|M, r, c, F, \overline{u}, \underline{v})$ which is given by:

$$\Phi_{AUD}(t|M, r, c, \overline{v}, \underline{v}) = \underline{v} + \frac{c - \underline{v}}{1 - \lambda e^{-\frac{1}{N-1}\rho r t}} \tag{5}$$

Above $\lambda = 1 - (c - \underline{v})/(\overline{v} - \underline{v})$ and $\rho = 1 - \underline{v}/c$, and sometimes I simply denote this curve by $\Phi_{AUD}(t|M, \lambda)$, for other parameters are shared by all players in the same AUD war. Combining (4) and (5), one obtains the equilibrium.

I see the expectation of a decreasing exponential function with respect to stopping time, $E_{t_m}[e^{-\rho r t}]$[15] where ρ is defined above, as a measure of welfare level. One important property of AUD war is that in equilibrium this measure is only determined by the maximal upper bound. Therefore, it is unaffected by the variation of population and other players' upper bounds, both of which necessarily determine how each type will behave. I present it in the following proposition:

[14] As shown later in Proposition 1 and Theorem 4, the extent of this offset depends on parametrization and can be either partial, excessive, or complete.

[15] A more reasonable measure of welfare level is $E_{t_m}[e^{-rt}]$ which I will use for analysis in Section 3, since if some player, say v_i, values the public good less than his cost and thus he chooses to wait forever, he earns an expected gain $v_i E_{t_m}[e^{-rt}]$. However, the economic insight here is not sensitive to this bias brought by the shrunk power, as $e^{-\rho r t}$ is still monotonous with respect to stopping time t.

Proposition 1 *Any N-player asymmetric AUD war, denoted by $\omega = \{N, r, c, \underline{v}, (\overline{v}_i)_i\}$, gives:*

$$E_{t_m}[e^{-\rho r t}] = 1 - \frac{c - \underline{v}}{\max\limits_{i \in I_N} \overline{v}_i - \underline{v}}$$

$\rho = 1 - \underline{v}/c$, and $E_{t_m}[.]$ calculates the expectation with respect to stopping time under equilibrium.

Proof of Proposition 1 First, I show that this lemma holds in symmetric case. Let $\overline{v}_i = \overline{v}$ for all i. Then, (5) and (4) give that the symmetric solution $\Phi(t) = \Phi_{AUD}(t|N, \lambda)$, where $\lambda = 1 - (c - \underline{v})/(\overline{v} - \underline{v})$ and $\rho = 1 - \underline{v}/c$. From this I further derive $E_{t_m}[e^{-\rho r t}]$:

$$\begin{aligned}
E_{t_m}[e^{-\rho r t}] &= \int_0^{+\infty} e^{-\rho r t} d\,\mathrm{Prob}(t_m \le t) \\
&= \int_0^{+\infty} e^{-\rho r t} d(1 - F^N(\Phi(t))) \\
&= \rho r \frac{N}{N-1}(1-\lambda)^N \int_0^{+\infty} \frac{\lambda e^{-\frac{1}{N-1}\rho r t} e^{-\rho r t}}{(1 - \lambda e^{-\frac{1}{N-1}\rho r t})^{N+1}} dt \\
&= \lambda N (\frac{1-\lambda}{\lambda})^N \int_{1-\lambda}^{1} \frac{(1-u)^{N-1}}{u^{N+1}} du = \lambda
\end{aligned}$$

Above, the second-last step substitutes u for $1 - \lambda e^{-\frac{1}{N-1}\rho r t}$, and the last step calculates the integration by parts[16]. This concludes the first part of the proof.

Next, I prove that this lemma holds for any 2-group AUD war. A 2-group AUD war involves two kinds of individuals with upper bounds $\overline{v}_1$ and $\overline{v}_2$, respectively, whose associated uniform distribution functions are denoted by $F_1(.)$ and $F_2(.)$. Without loss of generality, let $\overline{v}_1 > \overline{v}_2$. Let the number of the first group be n, and thus the population of the second group is $N - n$. (5) and (4) give that there is one division, denoted by t^*, and that on $(0, t^*)$ exist the n identical strategy curves of the players in the first group, denoted by $\Phi_1(t) = \Phi_U(t|n, \lambda_1)$, and on $(t^*, +\infty)$ exist the N identical curves of all players, denoted by $\Phi_2(t) = \Phi_U(t - t^*|N, \lambda_2)$, where $\lambda_i = 1 - (c - \underline{v})/(\overline{v}_i - \underline{v})$ for $i = 1, 2$. Continuity of solution requires that $e^{-\rho r t^*} = (\lambda_2/\lambda_1)^{n-1}$. Similar to the proof of the symmetric case, I write $E_{t_m}[e^{-\rho r t}]$ as:

$$\begin{aligned}
E_{t_m}[e^{-\rho r t}] &= \int_0^{t^*} e^{-\rho r t} d\,\mathrm{Prob}(t_m \le t) + \int_0^{+\infty} e^{-\rho r (t + t^*)} d\,\mathrm{Prob}(t_m \le t + t^*) \\
&= \int_0^{t^*} e^{-\rho r t} d(1 - F_1^n(\Phi_1(t))) + F_1^n(\overline{v}_2) e^{-\rho r t^*} \int_0^{+\infty} e^{-\rho r t} d(1 - F_2^N(\Phi_2(t))) \\
&= \lambda_1 n (\frac{1-\lambda_1}{\lambda_1})^n \int_{1-\lambda_1}^{1-\lambda_2} \frac{(1-u)^{n-1}}{u^{n+1}} du + (\frac{\overline{v}_2 - \underline{v}}{\overline{v}_1 - \underline{v}})^n (\frac{\lambda_2}{\lambda_1})^{n-1} \lambda_2 \\
&= \lambda_1 (\frac{1-\lambda_1}{\lambda_1})^n \left[(\frac{\lambda_1}{1-\lambda_1})^n - (\frac{\lambda_2}{1-\lambda_2})^n \right] + (\frac{1-\lambda_1}{1-\lambda_2})^n (\frac{\lambda_2}{\lambda_1})^n \lambda_1 = \lambda_1
\end{aligned}$$

[16]Define $\mathrm{Int}(N) = \int_{1-\lambda}^{1} \frac{(1-u)^{N-1}}{u^{N+1}} du$, and integration by parts gives the recursion $\mathrm{Int}(N) = \frac{1}{N} \frac{\lambda^{N-1}}{(1-\lambda)^N} - \frac{N-1}{N} \mathrm{Int}(N-1)$. Boundary condition $\mathrm{Int}(1) = (1-\lambda)/\lambda$ gives $\mathrm{Int}(N) = N(\frac{\lambda}{1-\lambda})^N$.

The last two steps above borrow the results of the first part of the proof.

Finally, I use the propositions proved in the previous two steps to conduct mathematical induction to prove the lemma for any AUD war. Let $e = \{N, \overline{K}, \Phi, (d_K, M_K)_K\}$ denote the equilibrium of an AUD war, and $\lambda_K = 1 - (c - \underline{v})/(\Phi(d_K) - \underline{v})$ for all K. Obviously, by the first proposition proved above, $E_{t_m}[e^{-\rho rt}|t_m \geq d_{\overline{K}}] = \lambda_{\overline{K}}$ because the subgame on $[d_{\overline{K}}, +\infty)$ is a symmetric N-player AUD war with upper bound $\Phi(d_{\overline{K}})$. And I conduct induction from the rightmost division to $t = 0$: if for K such that $0 < K \leq \overline{K}$ we have $E_{t_m}[e^{-\rho rt}|t_m \geq d_K] = \lambda_K$, then $E_{t_m}[e^{-\rho rt}|t_m \geq d_{K-1}] = \lambda_{K-1}$, which is ensured by the second proposition proved above. Consequently, $E_{t_m}[e^{-\rho rt}|t_m \geq 0] = E_{t_m}[e^{-\rho rt}] = \lambda_1$.

Q.E.D.

Proposition 1 favors the statement that the strongest player determines the welfare level by ranking all types' incentive positions, instead of providing directly by himself. In this case, the effect of any rise (or drop) of a weak player's upper bound is completely offset by the longer (or shorter) provision time of all types lower than him. Therefore, the arbitrary change of weak players' behavior that results from different selections of their upper bounds does not affect welfare level, and it is the incentive ranking decided by the fixed strongest type that essentially decides this level. Another insight shown by Proposition 1 is that the loss out of delay always occurs because the expectation of $e^{-\rho rt}$ is strictly less than one. The exception is the case where some players' upper bound is unbounded from infinity. Section 3.1 revisits these insights.

3.4 Comparative Statics

In this subsection, I perform some comparative statics on the equilibrium. I first compare the behavior of different players in the same war conditional on some relationship between their parameters. Specifically, I investigate how the difference between costs, discount rates, and *revelation rate*s affects the relative provision time of two players. Here, revelation rate refers to $f(.)/F(.)$, the density of the type being revealed conditional on the revelation of all types above it. The interpretation of this rate will be discussed later. Now I present this result:

Proposition 2 *Consider an N-player proper war, say $\omega \in \Omega$, if there exists a pair of players, denoted by $\alpha, \beta \in I_N$ whose solution curves both exist on $[t_0, +\infty)$, parametrized such that:*

(i) $r_\alpha = r_\beta$, $c_\alpha = c_\beta = c$, and $f_\alpha(v)/F_\alpha(v) \gtreqqless f_\beta(v)/F_\beta(v)$ on their overlap domain, then $\Phi_\alpha(t) \lesseqqgtr \Phi_\beta(t)$ for all $t \in [t_0, +\infty)$.

(ii) $r_\alpha = r_\beta$, $f_\alpha(v)/F_\alpha(v) = f_\beta(v)/F_\beta(v)$ on their overlap domain, and $c_\alpha \gtrless c_\beta$, then $\Phi_\alpha(t) \gtrless \Phi_\beta(t)$ for all $t \in [t_0, +\infty)$.

(iii) $c_\alpha = c_\beta = c$, $f_\alpha(v)/F_\alpha(v) = f_\beta(v)/F_\beta(v)$ on their overlap domain, and $r_\alpha \gtrless r_\beta$, then $\Phi_\alpha(t) \lessgtr \Phi_\beta(t)$ for all $t \in [t_0, +\infty)$.

Proof of Proposition 2 See appendix.

The last two clauses of Proposition 2 convey straightforward intuitions: higher cost reduces the gain of provision and thus leads to a delayed strategy, while higher discount rate corresponding to impatience increases the opportunity cost of waiting so that player tends to provide sooner.

However, the interpretation of the first clause associated with revelation rate is unclear, because under different assumptions this result generates different economic outcomes. For example, if player α's and β's upper bounds are set to be equal, Proposition 2(i) indicates that the player with consistently higher revelation rate stands in a relatively higher incentive position. In contrast, if two players' probability of valuation being lower than cost is set to be equal, the one with consistently lower revelation rate becomes stronger[17].

This discrepancy between these two cases must result from the different economic interpretations of the revelation-rate domination under different settings. In the first case, this domination is equivalent to subjecting higher probability to higher valuations. So, the dominated player tends to anticipate the other to face higher opportunity cost of waiting and thus in average to provide sooner, and this belief leads to the free-riding of the former, which in turn forces the high-revelation-rate player actually to provide sooner. In the second case, when the probability of waiting forever is controlled, the revelation-rate domination requires the dominant player's distribution to have a greater upper bound than that of the other, and these salient types value the public good so much that they stand in a higher incentive position.

4 Asymmetry and Social Welfare

In this section, I investigate how ex-ante asymmetry influences the welfare outcome of a war of attrition. First, by considering large-population societies, I discuss the different roles played by different members in the determination of social welfare. Next, I argue that under different conditions, the introduction of asymmetry can affect efficiency either positively or negatively.

4.1 Asymmetric Contribution

To make analysis tractable, I follow the seminal work by Bliss and Nalebuff (1984) where they discussed a large-population symmetric war of attrition. This subsection derives the counterpart conclusion of their limit theorem for my asymmetric war. However, my objective differs from theirs in that I intend to answer how differently do unequally positioned individuals contribute to social welfare.

I introduce some definitions and notations.A society is *divided* if it consists of multiple heterogeneous groups, and in each group, all members are (ex ante) homogeneous. Formally, define $s = \{N, L, (p_\iota, r_\iota, c_\iota, F_\iota, \overline{v}_\iota)_\iota\}$ to represent an N-member divided society which faces a proper war of attrition. This society is divided into L different groups, and the ιth group takes up p_ι

[17] If α has a consistently higher revelation rate, then $\Phi_\alpha(t) < \Phi_\beta(t)$ for all $t \in [t_0, +\infty)$ which gives $lnF_\alpha(\Phi_\alpha(t)) - lnF_\beta(\Phi_\beta(t)) = \frac{r}{c}\int_t^{+\infty}(\Phi_\beta(s) - \Phi_\alpha(s))ds > 0$. This further implies that when Φ_α reaches its upper bound, Φ_β must be lower than its upper bound and therefore β is stronger.

proportion of the total population and $\sum_{\iota=1}^{L} p_\iota = 1$. Each member in the ιth group is symmetrically parametrized by $\{c_\iota, r_\iota, F_\iota, \overline{v}_\iota\}$. The rules of notation are the same as those introduced in Section 2.2 only differing in that L denotes the total number of groups and subscript ι denotes all groups. For simplicity, all divided societies considered in this subsection consist of members who have identical provision costs and discount rates, respectively denoted by c and r, while their valuation distributions may differ from each other.

To begin with, I prove a useful lemma:

Lemma 3 *Consider a divided society parametrized by $s = \{N, L, r, c, (p_\iota, F_\iota, f_\iota, \overline{v}_\iota)_\iota\}$ which yields an aligned equilibrium: there is no instant exit or strict waiting, and thus $\Phi_\iota(0) = \overline{v}_\iota$ for all ι. Then, when N becomes sufficiently large (while maintaining the equilibrium aligned):*

(i) To maintain alignment, all upper bounds must be the same, namely, $\overline{v}_\iota = \overline{v}$ for all ι.

(ii) A player with valuation v earns expected gain $v(1 - c/\sum_{k=1}^{L} p_k \overline{v}_k) = v(1 - c/\overline{v})$.

Proof of Lemma 3 I first prove the second clause. Let $n_\iota = p_\iota N$ be the population of the ιth group, and $\Phi_\iota(t)$ be the strategy curve of this group. Define $g(t) = \frac{1}{N-1}\sum_{k=1}^{L} n_k(\Phi_k(t) - c)$, and each curve's differential equation is:

$$\Phi_\iota'(t) = -\frac{r}{c}\frac{F_\iota(\Phi_\iota(t))}{f_\iota(\Phi_\iota(t))}[g(t) - (\Phi_\iota(t) - c)]$$

By the equation above, $\Phi_\iota'(t) \to 0$ as $N \to \infty$, which means the stopping time chosen by members increases to infinity when the population grows large, thus the second term of (1) vanishes in the limit case while the first term approximates the product of the valuation and the expected discount factor, $E_{t_m}[e^{-rt}]$, defined in Proposition 1. Therefore, any player's expected gain in the limit case is determined by this expected discount factor. To calculate the expected discount factor, I first derive the distribution of the stopping time, F^{min}, and the associated density, f^{min}:

$$F^{min}(t) = \text{Prob}(t_m \le t) = 1 - \prod_{k=1}^{L} F_k^{n_k}(\Phi_k(t))$$

$$\begin{aligned} f^{min}(t) &= -\prod_{k=1}^{L} F_k^{n_k}(\Phi_k(t)) \sum_{k=1}^{L} n_k \frac{f_k(\Phi_k(t))}{F_k(\Phi_k(t))}\Phi_k'(t) = \prod_{k=1}^{L} F_k^{n_k}(\Phi_k(t)) \sum_{k=1}^{L} n_k \frac{r}{c}[g(t) - (\Phi_k(t) - c)] \\ &= \frac{r}{c} g(t) \prod_{k=1}^{L} F_k^{n_k}(\Phi_k(t)) \end{aligned}$$

The expected discount factor is given by:

$$
\begin{aligned}
E_{t_m}[e^{-rt}] &= \int_0^{+\infty} e^{-rt} dF^{min}(t) = r\int_0^{+\infty} e^{-rt} F^{min}(t)dt \\
&= 1 - r\int_0^{+\infty} e^{-rt}\prod_{k=1}^{L} F_k^{n_k}(\Phi_k(t))dt \\
&\leq 1 - \frac{c}{\frac{N}{N-1}(\sum_{k=1}^{L} p_k\overline{v}_k - c)}\int_0^{+\infty} e^{-rt}\frac{r}{c}g(t)\prod_{k=1}^{L} F_k^{n_k}(\Phi_k(t))dt \\
&= 1 - \frac{c}{\frac{N}{N-1}(\sum_{k=1}^{L} p_k\overline{v}_k - c)} E_{t_m}[e^{-rt}] \\
\Rightarrow E_{t_m}[e^{-rt}] &\leq \frac{1}{1+\frac{c}{\frac{N}{N-1}(\sum_{k=1}^{L} p_k\overline{v}_k - c)}} \to 1 - c/\sum_{k=1}^{L} p_k\overline{v}_k
\end{aligned}
$$

The third-last step utilizes $g(t) = \frac{N}{N-1}\sum_{k=1}^{L} p_k(\Phi_k(t)-c) \leq \frac{N}{N-1}(\sum_{k=1}^{L} p_k\overline{v}_k - c)$. On the other hand, let $\epsilon > 0$ be sufficiently small, and a limit lower bound is given by:

$$
\begin{aligned}
E_{t_m}[e^{-rt}] &\geq \int_0^{\epsilon} e^{-rt} dF^{min}(t) = e^{-r\epsilon}\left[1 - \prod_{k=1}^{L} F_k^{n_k}(\Phi_k(\epsilon))\right] + r\int_0^{\epsilon} e^{-rt}F^{min}(t)dt \\
&\to e^{-r\epsilon} + 1 - e^{-r\epsilon} - r\int_0^{\epsilon} e^{-rt}\prod_{k=1}^{L} F_k^{n_k}(\Phi_k(t))dt \\
&\geq 1 - \frac{c}{\frac{N}{N-1}(\sum_{k=1}^{L} p_k\Phi_k(\epsilon) - c)}\int_0^{+\infty} e^{-rt}\frac{r}{c}g(t)\prod_{k=1}^{L} F_k^{n_k}(\Phi_k(t))dt \\
&= 1 - \frac{c}{\frac{N}{N-1}(\sum_{k=1}^{L} p_k\Phi_k(\epsilon) - c)} E_{t_m}[e^{-rt}] \\
\Rightarrow E_{t_m}[e^{-rt}] &\geq \frac{1}{1+\frac{c}{\frac{N}{N-1}(\sum_{k=1}^{L} p_k\Phi_k(\epsilon) - c)}} \to 1 - c/\sum_{k=1}^{L} p_k\Phi_k(\epsilon)
\end{aligned}
$$

Since the selection of $\epsilon > 0$ is arbitrary, the two inequations obtaind above, $1-c/\sum_{k=1}^{L} p_k\Phi_k(\epsilon) \leq \lim_{N\to\infty} E_{t_m}[e^{-rt}] \leq 1 - c\sum_{k=1}^{L} p_k\overline{v}_k$, must give the limit value $E_{t_m}[e^{-rt}] \to 1 - c\sum_{k=1}^{L} p_k\overline{v}_k$. Consequently, for a player with valuation v, his limit expected gain is $v(1 - c/\sum_{k=1}^{L} p_k\overline{v}_k)$.

Now, I prove clause (i). Summing both sides of each group's differential equations yields:

$$
\sum_{k=1}^{L}\frac{dlnF_k(\Phi_k(t))}{dt} = -\frac{r}{c}\left[\frac{N}{N-1}\sum_{k=1}^{L} n_k(\Phi_k(t)-c) - \sum_{k=1}^{L} n_k(\Phi_k(t)-c)\right] \to 0
$$

Combining this approximation and the fact that each term on the left is negative, we have $\Phi_k'(t) \to 0-0$ for all k, with which each player's differential equation further gives that $\overline{v}_\iota - c \to g(0)$ at $t = 0$ for all ι, and note that $\overline{v}_\iota = c + g(0) = \overline{v}$, which proves (i) and completes (ii).

Q.E.D.

And I present the symmetric version of Lemma 3 which is equivalent to Bliss and Nalebuff's (1984) Theorem 6:

Corollary 1 *Consider a symmetric N-player war parametrized by $\omega = \{N, c, \overline{v}\}$. When N becomes sufficiently large, the expected gain of player v approximates $R(v) = v(1 - c/\overline{v})$.*

The condition of an aligned equilibrium in Lemma 3 simplifies the proof by only considering a society whose each group shares the same incentive position. This lemma gives three insightful results: *a)* when population grows large, the welfare level is solely determined by the strongest types and the population proportions associated with these types; and *b)* a group's highest type is representative of the incentive position of this group; and finally, *c)* no matter how large the population grows, inefficiency always occurs, unless some groups subject positive probability to the extreme type with infinite valuation.

The proof of Lemma 3 provides more information. First, all possible provision is realized immediately at the beginning of the war of attrition, almost surely. But why inefficiency still occurs given this instant provision? This is because when the population grows large, any type below the upper-bound type tends to free ride for a sufficiently long time so that only those in an almost-zero-measure set very close to the upper bound actually contribute and the probability of everyone's not providing at $t = 0$ remains considerable. Consequently, the distribution of stopping time subjects all of the probability to two events: either someone provides instantly, or nobody provides within finite time.

For a better understanding of the relationship between upper bound and incentive position in this limit case which is implied by Lemma 3(i), I present a complementary lemma:

Lemma 4 *Consider a divided society parametrized by $s = \{N, L, r, c, (p_\iota, F_\iota, f_\iota, \overline{v}_\iota)_\iota\}$. Then, when N becomes sufficiently large, the strict-waiting times of different groups have the relationship that $T_{\iota_1}(\overline{v}_{\iota_1}) \lesseqqgtr T_{\iota_2}(\overline{v}_{\iota_2})$ if and only if $\overline{v}_{\iota_1} \gtreqqless \overline{v}_{\iota_2}$ for all ι_1 and ι_2.*

Proof of Lemma 4 Suppose at some moment, say t_0, there are $M > 1$ active groups whose subscripts are denoted by $\iota \in I_M = \{1, 2, ..., M\}$. Further, suppose $\Phi_\iota(t_0)$ for all ι are not identical, which indicates the existence of some $\iota^* \in I_M$ such that $\Phi_{\iota^*}(t_0) > \sum_{k=1}^{L} p_k \Phi_k(t_0)$. Summing up both sides of the differential equations of all players in these M groups, we have $\Phi'_{\iota^*}(t) \to 0 - 0$, similar to the proof of Lemma 3(i). However, this gives that $dlnF_{\iota^*}(\Phi_{\iota^*}(t_0))/dt = (r/c)(\Phi_{\iota^*}(t_0) - g(t_0)) \to (r/c)(\Phi_{\iota^*}(t_0) - \sum_{k=1}^{L} p_k \Phi_k(t_0)) > 0$, an absurdity. Thus, the values of the strategy curves of the groups active at the same moment must be identical.

For sufficiency, consider two groups with different upper bounds, respectively denoted by $\overline{v}_{\iota_1} > \overline{v}_{\iota_2}$. Then, denote the strcit-waiting times of both groups respectively by t_1 and t_2, and the time when $\Phi_{\iota_1}(t)$ equals $\overline{v}_2$ by t_{12}, the existence of which is ensured by $\overline{v}_{\iota_1} > \overline{v}_{\iota_2}$. Since $\Phi_{\iota_1}(.)$ is decreasing and continuous, $t_{12} > t_1$, and by the conclusion proved in the previous paragragh, $t_{12} = t_2$, and therefore we have $t_1 < t_2$. The proof of necessity is only an inverse process.

Q.E.D.

Lemma 4 corresponds to the asymmetric-dependence behavior feature stressed in Section 3.3. Specifically, the variation of one player's upper bound will not influence the behavior of types higher than him while those lower than this upper bound will change to wait infinitely longer than his provision time. At every moment, the highest type in this society will provide instantly with positive probability, while any distribution-wise variation of lower types will not at all change the outcome. This is another demonstration of how the strongest type determines the welfare level in addition to the special case considered in Section 3.3.

What is curious is that the results given by Lemma 3 and Lemma 4 are irrelevant to the shapes of distribution functions or even the density of the highest type. The explanation lies in that, when the population grows large, the law of large numbers ensures that the types with either large or small density will occur homogeneously with probability one, and because of this, the variation of the density subjected to valuations becomes less important.

The insights of the previous two lemmas are summarized in the following theorem which characterizes the level of expected welfare of a large-population divided society at every moment after the war of attrition begins:

Theorem 2 *Consider a divided society parametrized by $s = \{N, L, r, c, (p_\iota, F_\iota, f_\iota, \overline{v}_\iota)_\iota\}$. Then, when N becomes sufficiently large, for all $\tau \geq 0$, $E_{t_m}[e^{-rt}|t \geq \tau]$ is solely determined by the highest type anticipated at the moment τ. Specifically, denote this highest anticipation at τ by $\overline{v}(\tau)$, and the expected welfare calculated at τ of a member with valuation v is $v(1 - c/\overline{v}(\tau))$.*

Proof of Theorem 2 This is a corollary of Lemma 3 and Lemma 4.

4.2 The Welfare Effect of Asymmetry

In this section, I discuss the effect that introducing asymmetry has on efficiency in different cases. I do this by considering two ways of introducing asymmetry: *a)* to make the strongest player stronger under the control that the expectation of the sum of players' valuations is fixed; and *b)* to fix the strongest player and make others weaker. As mentioned in the introduction, such division constitutes a complete discussion.

I use AD war for discussion to simplify analysis while preserving generality.[18] Besides, I measure the welfare level with the expected discount factor[19], $E_{t_m}[e^{-rt}]$, the expectation of a decreasing function of stopping time under equilibrium.

[18]Specific explanation. First, this family of wars of attrition is simple enough, since, on the one hand, the local-symmetry property of equilibrium strategy makes the measure of welfare tractable, and on the other, asymmetry is directly represented by the difference among players' distribution upper bounds. Second, AD war is generalized enough to be representative, because the dimension of information structure remains infinite as the (proper) distribution from which each AD war is generated can be arbitrary.

[19]I apply this measure of welfare for two reasons. First, a player who values the public good less than the cost he faces will wait forever, and thus his expected gain is exactly his valuation multiplying this expected discount factor. Second, one minus this value should be seen as a justified cost of delay, for e^{-rt} is directly related to the delayed time and the exponential form measures how players undertake cost because of it. One thing to be admitted is that the infinite-dimension information structure makes it difficult to find a direct measure of welfare. However, literature always shifts the attention to the cost of delay, which makes it possible for discussion. Thus, in this section by *welfare level* I mean one minus *cost of delay* and it should be seen as an intuitive, rather than accurate, measure of welfare. The spirits follow Bliss and Nalebuff (1984), Alesina and Drazen (1991), Bulow and Klemperer (1999), and Riley (1999), where the expectation of different functions of stopping time is considered.

I recall and introduce some notations first. Consider an AD war parametrized by $w_{AD} = \{N, r, c, F, \underline{v}, (\overline{v}_i)_i\}$. Define function $e_n(u)$ as the expected discount factor calculated under the equilibrium of an n-player symmetric AD war generated from F with upper bound u, namely, $e_n(u)$ equals $E_{t_m}[e^{-rt}]$ calculated under $E(\omega_n(u))$ where $\omega_n(u) = \{n, r, c, F, \underline{v}, \overline{v}_i = u\}$. Denote the symmetric equilibrium strategy that players choose in this game by $T_n(v|u)$, and $e_n(u)$ and $T_n(v|u)$ are given by:

$$\begin{aligned} e_n(u) &= E_{t_m}[e^{-rt}] = \frac{1}{F^n(u)} \int_c^u e^{-rT_n(s|u)} nF^{n-1}(s) f(s) ds \\ T_n(v|u) &= \frac{n-1}{r} \int_v^u \frac{f(s)}{F(s)} \frac{c}{s-c} ds = (n-1) T_2(v|u) \end{aligned} \tag{6}$$

Further, $e_\infty(u)$ is defined as $1 - c/u$ according to Theorem 2, and for simplicity $T_2(.|.)$ is written as $T(.|.)$ if there is no confounding.

As stated before, I first investigate the welfare implication of the introduction of asymmetry that allows the strongest to change. To let different welfare levels be comparable, I consider the symmetric game and the asymmetric games that have the same expected sum of valuations. Similar to the prediction given by the last subsection, the theorem below shows that by making the strongest player slightly stronger, asymmetric games always yield higher welfare levels than the symmetric game.

Theorem 3 *Consider a family of two-group AD wars in which each game is generated from the same $F(.)$. The first group consists of $m \geq 1$ identical members with upper bound $\overline{v}_1$, while the second has $n \geq 1$ identical members with upper bound $\overline{v}_2 \leq \overline{v}_1$. Both upper bounds can change but under the control that the expectation of the sum of all players' valuations is fixed, namely $mE[v_1] + nE[v_2]$ is constant. Let $\overline{v}_0 > c$ be the common upper bound of the symmetric game (where $\overline{v}_1 = \overline{v}_2$). Then, there exists a $v^* \in (c, \overline{v}_0)$ such that all asymmetric games with $\overline{v}_2 \in [v^*, \overline{v}_0)$ yield higher expected discount factor than the symmetric game with $\overline{v}_2 = \overline{v}_0$.*

Proof of Theorem 3 For each $(\overline{v}_1, \overline{v}_2)$, the expected discount factor calculated under the equilibrium of the AD war parametrized with this upper-bound pair is given by:

$$\tilde{e}(\overline{v}_1, \overline{v}_2) = \int_{\overline{v}_2}^{\overline{v}_1} e^{-r(m-1)T(u|\overline{v}_1)} m \frac{F^{m-1}(u)}{F^m(\overline{v}_1)} f(u) du + e^{-r(m-1)T(\overline{v}_2|\overline{v}_1)} \frac{F^m(\overline{v}_2)}{F^m(\overline{v}_1)} e_{m+n}(\overline{v}_2)$$

Above, $T(.|.)$ and $e_{m+n}(.)$ are defined as in (6). For the control of the expectation of the sum of all players' valuations, $m \int_{\underline{v}}^{\overline{v}_1} u \frac{f(u)}{F(\overline{v}_1)} du + n \int_{\underline{v}}^{\overline{v}_2} u \frac{f(u)}{F(\overline{v}_2)} du = C$ where C is constant. By differentiating both sides of this equation with respect to $\overline{v}_1$ and rearranging the resulting equaition, we have:

$$\frac{d\overline{v}_2}{d\overline{v}_1} = -\frac{m(\overline{v}_1 - E[v_1]) \frac{f(\overline{v}_1)}{F(\overline{v}_1)}}{n(\overline{v}_2 - E[v_2]) \frac{f(\overline{v}_2)}{F(\overline{v}_2)}}$$

The equation above tells how the variations of both upper bounds are related to each other, whereby $\tilde{e}$ can be written as $\tilde{e}(\overline{v}_1)$. To investigate how the welfare level is affected by the introduction of asymmetry, I need to calculate $\frac{d\tilde{e}}{d\overline{v}_1} = \frac{\partial \tilde{e}}{\partial \overline{v}_1} + \frac{\partial \tilde{e}}{\partial \overline{v}_2}\frac{\overline{v}_2}{\overline{v}_1}$. Based on the expression of $\tilde{e}(\overline{v}_1, \overline{v}_2)$ and (6), the two partial derivatives on the right are presented below:

$$
\begin{aligned}
\frac{\partial \tilde{e}}{\partial \overline{v}_1} =& \left[\frac{\partial}{\partial \overline{v}_1} + \frac{\partial T(u|\overline{v}_1)}{\partial \overline{v}_1}\frac{\partial}{\partial T(u|\overline{v}_1)} + f(\overline{v}_1)\frac{\partial}{\partial F(\overline{v}_1)}\right]\int_{\overline{v}_2}^{\overline{v}_1} e^{-r(m-1)T(u|\overline{v}_1)} m \frac{F^{m-1}(u)}{F^m(\overline{v}_1)} f(u)du \\
&+ \left[\frac{\partial T(\overline{v}_2|\overline{v}_1)}{\partial \overline{v}_1}\frac{\partial}{\partial T(\overline{v}_2|\overline{v}_1)} + f(\overline{v}_1)\frac{\partial}{\partial \overline{v}_1}\right] e^{-r(m-1)T(\overline{v}_2|\overline{v}_1)}\frac{F^m(\overline{v}_2)}{F^m(\overline{v}_1)} e_{m+n}(\overline{v}_2) \\
=& \frac{f(\overline{v}_1)}{F(\overline{v}_1)}\left[\frac{c}{\overline{v}_1 - c}\tilde{e} + m\left(1 - \frac{\tilde{e}}{e_\infty(\overline{v}_1)}\right)\right]
\end{aligned}
\tag{7}
$$

$$
\begin{aligned}
\frac{\partial \tilde{e}}{\partial \overline{v}_2} =& - e^{-r(m-1)T(\overline{v}_2|\overline{v}_1)} m \frac{F^m(\overline{v}_2)}{F^m(\overline{v}_1)}\frac{f(\overline{v}_2)}{F(\overline{v}_2)} + \left[\frac{\partial}{\partial \overline{v}_2} + \frac{\partial T(\overline{v}_2|\overline{v}_1)}{\partial \overline{v}_2}\frac{\partial}{\partial T(\overline{v}_2|\overline{v}_1)}\right. \\
&\left. + \frac{\partial T(u|\overline{v}_2)}{\partial \overline{v}_2}\frac{\partial}{\partial T(u|\overline{v}_2)} + f(\overline{v}_2)\frac{\partial}{\partial F(\overline{v}_2)}\right] e^{-r(m-1)T(\overline{v}_2|\overline{v}_1)}\frac{F^m(\overline{v}_2)}{F^m(\overline{v}_1)} e_{m+n}(\overline{v}_2) \\
=& \eta \times n \frac{f(\overline{v}_2)}{F(\overline{v}_2)}\left(1 - \frac{e_{m+n}(\overline{v}_2)}{e_\infty(\overline{v}_2)}\right)
\end{aligned}
$$

Above, $\eta = e^{-r(m-1)T(\overline{v}_2|\overline{v}_1)}\frac{F^m(\overline{v}_2)}{F^m(\overline{v}_1)}$. Finally, combining $\frac{\partial \tilde{e}}{\partial \overline{v}_1}$, $\frac{\partial \tilde{e}}{\partial \overline{v}_2}$, and $\frac{d\overline{v}_2}{d\overline{v}_1}$ by definition yields:

$$
\frac{d\tilde{e}}{d\overline{v}_1} = \frac{f(\overline{v}_1)}{F(\overline{v}_1)}\left[\frac{c}{\overline{v}_1 - c}\tilde{e} + m\left(1 - \frac{\tilde{e}}{e_\infty(\overline{v}_1)}\right) - m\left(1 - \frac{e_{m+n}(\overline{v}_2)}{e_\infty(\overline{v}_2)}\right) \times \eta \frac{\overline{v}_1 - E[v_1]}{\overline{v}_2 - E[v_2]}\right] \tag{8}
$$

Now, consider the value of this derivative in the symmetric case where $\overline{v}_1 = \overline{v}_2 = \overline{v}_0$, which implies $\tilde{e} = e_{m+n}(\overline{v}_2)$, $e_\infty(\overline{v}_1) = e_\infty(\overline{v}_2)$, $\eta = 1$, and $\frac{\overline{v}_1 - E[v_1]}{\overline{v}_2 - E[v_2]} = 1$. All these equations give:

$$
\left.\frac{d\tilde{e}}{d\overline{v}_1}\right|_{symmetry} = \frac{f(\overline{v}_0)}{F(\overline{v}_0)}\frac{c}{\overline{v}_0 - c}\tilde{e} > 0
$$

Consequently, any slight upward variation of $\overline{v}_1$ from $\overline{v}_0$ or, equivalently, any slight downward variation of $\overline{v}_2$ from $\overline{v}_0$ will increase the value of expected discount factor. Further, since $\tilde{e}$ is continuous with respect to either $\overline{v}_1$ or $\overline{v}_2$, there must exist a $v^* < \overline{v}_0$ such that for all $\overline{v}_2 \in (v^*, \overline{v}_0)$ the associated asymmetric game yields higher expected discount factor than that by the symmetric game.

Q.E.D.

The intuition conveyed by Theorem 3 is straightforward: the strongest type "controls" the behavior of all types and thus he has a decisive influence on the outcome, so the stronger he becomes, the lower cost of delay tends to be. Particularly, when $\overline{v}_1$ moves slightly upwards from $\overline{v}_0$, the second and the third terms in (8) completely offset each other, which corresponds to that the increase of the second group's stopping time out of the rise of $\overline{v}_1$ offsets the decrease out of the drop of $\overline{v}_2$, and this leaves the positive first term which is purely caused by the rise of $\overline{v}_1$ to be the only effect.

One problem Theorem 3 must confront is how much the positive efficacy of asymmetry relies on the slightness of its introduction or, equivalently, whether v^* is closer to $\overline{v}_0$ than c or not. Although the complexity of (8) implies that a clear answer might be unlikely, numerical experiments on AUD wars suggest that $v^* \to c$ always seems to hold, that is, for uniform-distribution cases, any introduction of asymmetry under the control of the expected sum of valuations improves efficiency. In the appendix, I demonstrate several examples and comment on them.

More remarks on this theorem. First, the two-group setting can be relaxed to general cases where there is a multi-player strongest group, and all other weaker players can have arbitrary upper bounds. Theorem 3 still holds if it is restated to be: slightly splitting the strongest group into two groups with controlling the expected sum of valuations always increases the expected discount factor. Second, other forms of control such that $\frac{d\overline{v}_2}{d\overline{v}_1} = -\frac{m}{n}$ when $\overline{v}_1 = \overline{v}_2$, which is reasonable, generate the same results as fixing the expected sum of valuations, for example, controls like fixing the expected sum of upper bounds or fixing the value of $F^M(\overline{v}_1)F^N(\overline{v}_2)$. Therefore, the positive efficacy of the first kind of asymmetry is quite robust.

Now, I investigate the effect of the introduction of asymmetry that controls the strongest player. Unlike the previous case, the strong-strong symmetric game may yield either lower or higher expected discount factor than the associated strong-weak asymmetric games. In other words, the result depends on the parametrization. In the following theorem, I give the condition for asymmetry either alleviating or aggravating cost of delay, and then I interpret its insight.

Theorem 4 *Consider a family of N-player AD wars in which each game is generated from the same $F(.)$ and has identical maximal upper bound, namely $\max_{i\in I_N} \overline{v}_i = \overline{v}_m$ where $\overline{v}_m$ is constant. Then, if $e_N(u) - e_\infty(u)$ is positive (or negative) for all $u \in (c, \overline{v}_m]$, the symmetric AD war where $\overline{v}_i = \overline{v}_m$ for all i yields the lowest (or the highest) expected discount factor.*

Proof of Theorem 4 I only show the proof of the case where $e_N(u) - e_\infty(u)$ is positive for all $u \in (c, \overline{v}_m]$, and that of the other can be similarly done. Without loss of generality, let $\overline{v}_1 = \overline{v}_m \geq \overline{v}_2 \geq ... \geq \overline{v}_N > c$. I need to show that if $e_N(u) > e_\infty(u)$ for all $u \in (c, \overline{v}_m]$, the minimal expected discount factor is obtaind when all unequal signs above become equal.

To prove this, I first consider a two-group AD war (not necessaily a member of the family defined in the theorem) generated also from $F(.)$ where the first group contains M players with the identical upper bound, $\overline{v}_1$, whereas the second contains $N - M$ players with the identical upper bound, u such that $u \leq \overline{v}_1 \leq \overline{v}_m$. Denote the expected discount factor calculated under the equilibrium of this AD war by $e_{M,N}(\overline{v}_1, u)$. Then, by definition $e_{M,N}(\overline{v}_1, u)$ is equivalent to $\tilde{e}(\overline{v}_1, \overline{v}_2)$ given in the proof of Theorem 3 but with $m = M$, $n = N - M$, and $\overline{v}_2 = u$. Let $\overline{v}_1$ be controlled as constant, and by directly modifying the second partial derivative in (7), I give how $e_{M,N}$ varies with respect to u:

$$\frac{\partial e_{M,N}}{\partial u} = (N-M)e^{-r(M-1)T(u|\overline{v}_1)}\frac{F^{M-1}(u)}{F^M(\overline{v}_1)}f(u)\left(1 - \frac{e_N(u)}{e_\infty(u)}\right)$$

From above, it is easy to see that $\partial e_{M,N}(\overline{v}_1, u)/\partial \overline{v}_2 < 0$ for all $u \in (c, \overline{v}_1]$, because $e_N(u) > e_\infty(u)$ for all $u \in (c, \overline{v}_1]$ as assumed. This result suggests that to get a lower expected discount factor under this two-group AD war is to increase the value of u, and the lowest expected discount is obtained when this war degenerates to a one-group symmetric war, namely when $u \to \overline{v}_1$.

Now, suppose that the lowest expected discount factor of the AD war in Theorem 4 is obtaind when $\overline{v}_1 = \overline{v}_m \geq \overline{v}_2 \geq ... \geq \overline{v}_N$ in which at least one of the unequal signs is strict. Then, denote the upper bound on the left side of the last of such strictly unequal signs by $\overline{v}_M$, and therefore the N players do not all become active until player $M+1$ becomes active. Denote the moment when player M becomes active by t_M. However, there occurs a contradiction: the lemma proved above indicates that the expected discount factor conditional on the arrival of t_M, $E_{t_m}[e^{-rt}|t \geq t_M]$, can be further reduced by increasing the upper bounds of players $M+1, M+2, ..., N$ to $\overline{v}_M$; and according to (4), this variation will not affect the equilibrium behavior during $t \in [0, t_M]$, thus the expected discount factor under the entire equilibrium is also reduced in this way. Therefore, the game that yields the lowest expected discount factor can only be the symmetric game.

Q.E.D.

The term $e_N(u) - e_\infty(u)$ can be interpreted as a measure of the cost of symmetry since it calculates the loss or gain brought by adding people to a symmetric game. If this term is positive (or negative), then a larger population is worse (or better), and when asymmetry makes several players become active sooner than others, the expected gain generated during the beginning period with fewer players being active will rise (or drop), while the gain generated after that will not be very different. Thus, Theorem 4 argues that when the cost of symmetry is positive (or negative), any introduction of asymmetry improves (or impairs) efficiency.

The force that decides the sign of $e_N(u) - e_\infty(u)$ is two-sided: *a)* in an M-player symmetric game each type's decision is $M-1$ times the decision made by the same type in the associated 2-player game, and this redoubled waiting time presents severer free riding which impairs efficiency; and *b)* the potential probability of provision[20], defined as the provision probability calculated in the counterfactual world where every type valuing the public good higher than his cost provides immediately (that is, a world without delay), is $1 - F^M(c)$ which is an increasing function of M, and the rise of this probability is potential to improve efficiency.

Numerical experiments on AUD war give examples satisfying either of the two conditions. When the lower bound of the uniform distribution from which an AUD war is generated is positive, the introduction of asymmetry improves welfare[21], while in negative-lower-bound cases ex-ante asymmetry impairs welfare. In the appendix, I give two numerical examples that favor both cases, respectively. The critical example is 0-AUD war, which refers to the family of AUD wars with zero lower bounds, as Proposition 1 suggests that $e_N(u) = e_\infty(u)$ always holds under 0-AUD wars and asymmetry has no influence.

Notice that in the AUD-war example, when the probability of the type valuing the public good less than his cost increases, the introduction of asymmetry tends to impair efficiency. This regularity can be generalized to other AD wars. Consider a sequence of symmetric N-player AD

[20] This notion of potential probability is only an intuitive, rather than accurate, demonstration of the positive effect of increasing population on welfare.

[21] Some remarks. This example gives a different result from Bliss and Nalebuff's (1984) Theorem 2 and Grandstein's (1992) Lemma 1(ii), both of which state that increasing the population in a symmetric public-good provision game makes everyone better off. The reason lies in the difference in model setups, for example, my model regards player's valuation as uncertain and incorporates the possibility of $v < c$.

wars in which each game has identical provision cost, c, and identical upper bound, $\overline{v}$, and this sequence converges to a *dead* game where for each player the probability of his valuation being less than his cost is one. Then, when integer n becomes sufficiently large, the nth game in this sequence yields an expected discount factor smaller than that under the associated infinite-player game. The reason is that as the sequence converges to the dead game, the associated expected discount factor approaches zero, however, Theorem 2 tells that for each game in this sequence the expected discount factor calculated under its associated infinite-player game is $1 - c/\overline{v} > 0$.

5 Discussion

5.1 Insights

The main insights of my analysis are summarized as follows. First, ex-ante asymmetry differentiates individuals into distinct incentive positions, whereby they become active sequentially in equilibrium. Second, the strongest type mainly determines the cost of delay generated in a war of attrition by "controling" the behavior of all types. Lastly, introducing asymmetry that reinforces the strongest type tends to improve efficiency, while in the situation where the strongest type is controlled, the positive effect of asymmetry hinges on the positive cost of symmetry.

These features have compelling power in explaining economic phenomena. Take the infamous Mekong River massacre[22] for example. A surprising consequence of this tragedy is the establishment of security system led by China that aims to repress illegal activities in the Golden Triangle, which used to be a paradise for drug dealers and outlaws. The leadership in building this system can be seen as a privately provided public good, and the situations before and after this massacre correspond to two wars of attrition differing from each other in that the incentives faced by China are drastically different. Before the event, Golden Triangle generated more negative influence on its neighbors, Thailand, Laos, and Myanmar, while China was not directly concerned, although it has stronger executive power. As a result, countries kept passing the buck, and the riot had been dealt with in a delayed and inefficient way, as predicted by symmetric or less asymmetric equilibrium. Nevertheless, the inhuman violence against Chinese ships incentivized China to intervene not only because of the pursuit for justice but also because China realized the importance of guaranteeing security for Chinese commercial activities in Southeast Asia.[23] The asymmetry that makes China, the (potentially) strongest player in this game, salient dramatically accelerates the overdue formation of security cooperation, as predicted by Theorem 3.

[22] The introduction of this event on Wikipedia: "... on the morning of 5 October 2011, ... two Chinese cargo ships were attacked on a stretch of the Mekong River in the Golden Triangle region on the borders of Myanmar (Burma) and Thailand. All 13 crew members on the two ships were killed and dumped in the river. ... In response, China ... reached an agreement with Myanmar, Thailand and Laos to jointly patrol the river. The event was also the impetus for the Naypyidaw Declaration and other anti-drug cooperation efforts in the region."

[23] About half a year after the event, Chicago Tribune made comments on China's motives that "(t)he patrols, ostensibly conducted with Myanmar, Laos and Thailand, have been seen as an expansion of Beijing's growing role in regional security, extending its law enforcement down the highly strategic waterway."

5.2 Redistribution

The central provision of public good, referring to the instant provision by a central party who has coercive power to tax and give orders, is commonly believed to be a good way to alleviate the loss incurred by the private provision.[24] However, this solution fails to avoid the mistakes out of the central party's lack of information. Since the central party makes its decision only based on incomplete information, the provision of public good may generate negative ex-post total surplus ($\sum_{i=1}^{n} v_i < c$) with considerable probability, even though the ex-ante total surplus is positive ($\sum_{i=1}^{n} E[v_i] > c$), for which the provision decision is made.

Following the ideas displayed in Section 4, I propose that when there is private information, a central party can do at least not worse than central provision by making redistribution in a certain way to asymmetrize players' incentives in a war of attrition, and sometimes this strategy is strictly better than central provision. Redistribution absorbs the advantage of private provision (war of attrition) that private information is partially revealed. Specifically, any provision in a war of attrition implies a high valuation of the provider, which always brings non-negative ex-post total surplus. Therefore, when the central party makes redistribution, what it essentially does is to use this fact to "nudge" the high-valuation players to reveal sooner with asymmetry.

Consider the following procedure. At $t = 0$, the central party announces a redistribution plan which it promises to implement at the provision moment, defined as the moment when someone provides the public good, in the following war of attrition. Assume that players trust this party and thus incorporate this plan in their incentives. Since the plan is fixed beforehand, the war of attrition remains a strategically static game but with its information structure changed.

I use a two-player symmetric 0-AUD war parametrized by $w_0 = \{c < \overline{v}\}$ to demonstrate one typical example of the redistribution plan. For instance, the central party promises to redistribute y from player 2 to player 1 at the provision moment only when player 1 provides, or no redistribution happens. This case equivalently reduces both players' cost to $c - y$ and moves player 2's valuation domain downwards to $[-y, \overline{v} - y]$, which generates asymmetry.[25]

For a generically and properly parametrized public-good provision problem, I denote the set of all possible redistribution plans by $\mathbb{D}$. To select the optimal plan, the central party computes $\max_{d\in\mathbb{D}} W(d)$ where $W(d)$ is the party's objective integral under the war-of-attrition equilibrium conditional on the implementation of d, and by d^* I denote the selected plan.

Importantly, for all $W(.)$ the associated d^* is not worse than either private provision or central provision. Obviously, the private-provision case corresponds to $d_0 =$"doing nothing"$\in \mathbb{D}$ which thus has $W(d_0) \leq W(d^*)$. On the other hand, assume that the central party conducts central provision by ordering the lowest-cost player to provide with his cost covered by the money taxed from others, and I denote this operation by $\tilde{d}_c$. Then, the central-provision case selects d_1 such that $W(d_1) = \max\{W(d_0), W(\tilde{d}_c)\}$. Note that $\tilde{d}_c$ yields the outcome equivalent

[24]For discussions in literature, see Bergstrom et al.'s (1986) Section 5 for a classic analysis on the effect of government supply on underprovision problem, and Grandstein's (1992) Section 5 for a discussion on several solutions to delay including mechanism, redistribution (he refers to subsidization), and coercive provision.

[25]Other interesting examples are plans that employ probabilistic transfer, which refers to redistributing $\tilde{y}$ from player 2 to player 1 where $\tilde{y}$ is a random variable, to change the shape of players' valuation distributions; or plans that make time-dependent transfer $y = y(t)$ where t refers to the provision moment, and if $y(t)$ is decreasing the targeted player will be incentivized to exit sooner for there is another opportunity cost of waiting.

to that of the associated redistribution plan, denoted by $d_c \in \mathbb{D}$, in which the central party promises to tax players at the provision moment in the same way as in $\tilde{d}_c$ and meanwhile transfer the tax to the lowest-cost player only when this player provides. The reason, similar to the example given before, is that this plan reduces the cost faced by the lowest-cost player to zero, which makes instant provision with probability one a dominant strategy. Consequently, $W(d^*) \geq \max\{W(d_0), W(d_c)\} = \max\{W(d_0), W(\tilde{d}_c)\} = W(d_1)$.

Finally, I show an example where redistribution is strictly better than central provision under the objective of maximizing expected total surplus. Consider a two-player symmetric AD war parametrized by $\omega = \{F, c = \overline{v} = 1, \underline{v} > 0\}$ such that $E[v_1] + E[v_2] < c$. This example represents the situation where private provision is not possible since the highest valuation is no greater than the cost, and also central provision is not possible since the expected total surplus it generates is assumed to be negative. Consequently, the objective integral $W = 0$ in both cases. However, suppose the central party announces a redistribution plan that it promises to transfer a small ϵ from player 2 to player 1 at the provision moment only when the latter provides. As a result, player 1 will provide instantly if his type is almost 1, namely $v_1 > 1 - \epsilon$, and therefore the expected total surplus must be greater than $\text{Prob}(v_1 > 1 - \epsilon) \times (1 - \epsilon + E[v_2] - c) = \text{Prob}(v_1 > 1 - \epsilon) \times (E[v_2] - \epsilon)$ which is positive if ϵ is selected to be less than $E[v_2]$, which is positive for $\underline{v} > 0$. Thus, there exists some redistribution that yields higher objective integral than that by the central provision. This example demonstrates how the central party uses partial revelation to make fewer mistakes, which makes the partial instant provision possible in this case.

5.3 Related Paper

I differentiate this paper from Kambe (2019). His article is the most related to mine, but the results are nonetheless very different due to both the two-type information structure considered in his setup and his distinct focus on economic issues.

For one thing, Kambe (2019) assumes that each player is anticipated to have only two possible types. This paper shows that players' behavior is different when the continuous-type distribution is assumed. First, the continuity at $v_i = c_i$ for all players allows provision to occur after any sufficiently long time (Lemma 1(ii)). An explanation is that: first, the right-above-cost type $v_i = c_i + \epsilon$ where $\epsilon > 0$ is sufficiently small is always willing to wait longer if there is positive probability of others' providing after him, so that all players' maximal stopping time must be the same; and further, the rate on the right side of equation (2) at which each player reveals his information drops quickly to zero when $\epsilon \to 0$, and thus the choice made by this type is delayed rapidly so that the common maximal stopping time cannot be finite.

The second difference is a consequent proposition of the previous one. In my continuous-type equilibrium, players become active sequentially, and once one becomes active, he cannot return to the passive state. However, Kambe (2019) gives an example in which asymmetry is fairly large and hence some active player will exhaust the possibility of exit before the arrival of the maximal stopping time, and some passive player will become active at this exhaustion moment. In other words, in a two-type equilibrium, active players may become passive again after ruling out the possibility of him being a regular type. The explanation for this difference is that continuous-type distribution subjects positive density to the right-above-cost type which, as stated before, does not allow any provision probability of others to exist after his stopping time.

For the other, Kambe (2019) pays more attention to Who provides first? and How much does he provide? Specifically, Kambe's (2019) Section 4 carries out discussion on instant exit. He argues that the probability of instant exit is positively related to efficiency. However, he does not validate this argument by comparing instant-exit probability with other possible measures of welfare, for example, expected discount factor employed in this paper which also includes the non-instant-exit case. The truth is that the two-type setup actually makes the integral with respect to stopping time very complex, and thus this validation is difficult. In contrast, there are several special yet representative cases of my continuous-type setup whose expected-discount-factor functions are tractable, like AD war and large-population society.

What makes my alternative measure of welfare important is that certain examples show that the probability of instant exit may not be a good representation of efficiency. For example, Proposition 1 tells that the expected discount factor under 0-AUD-war equilibrium manifests complete irrelevance to instant-exit probability when the strongest type is fixed. Moreover, Theorem 4 tells that when the cost of symmetry is negative (for example, negative-lower-bound AUD war) asymmetry that controls the strongest type impairs efficiency, which implies that instant-exit probability may even have a negative influence.

6 Appendix

6.1 Proof of Lemma 1.

I first prove the necessity. (i) For all $i \in I_N$, the dominant strategy for player $v_i < c_i$ is $T_i(v_i) = +\infty$, since any provision of i yields negative gain while waiting forever brings zero. In contrast, for player $v_i > c_i$, $T_i(v_i)$ cannot be $+\infty$, since otherwise, player v_i can be better off by changing his provision time to a finite point, say T_m, whereby he gains at least $(v_i - c_i)e^{-r_i T_m}$.

(ii) I show that there exists a $\overline{T} \in \mathbb{R}^+ \cup \{0, +\infty\}$ such that $\lim_{v \to c_i + 0} T_i(v) = \overline{T}$ for all $i \in I_N$. Assume otherwise that $\min_{j \in I_N} \lim_{v \to c_j + 0} T_j(v) = \underline{t} \neq \bar{t} = \max_{j \in I_N} \lim_{v \to c_j + 0} T_j(v)$, and this means that the former player associated with the minimal limit, denoted by k, faces a positive probability q^* of someone providing after $\underline{t}$. Then, for a sufficiently small $\epsilon > 0$, player $c_k + \epsilon$ can be better off by changing his stopping time to $\bar{t}$, whereby he gains at least $q^* v_k e^{-r_k \bar{t}} - \epsilon(e^{-r_k \underline{t}} - e^{-r_k \underline{t}})$, which becomes positive when $\epsilon \to 0$. The property that $\overline{T} = +\infty$ can be easily proved after I show the validity of (v) and (vi), and I will demonstrate this later.

(iii) First, I show that $T_i(.)$ is non-increasing for all $i \in I_N$. In equilibrium, player v_i and player $\tilde{v}_i$ both reach optimality, which gives:

$$R_i(T_i(v_i)|v_i) \geq R_i(T_i(\tilde{v}_i)|v_i)$$
$$R_i(T_i(\tilde{v}_i)|\tilde{v}_i) \geq R_i(T_i(v_i)|\tilde{v}_i)$$

Adding these two inequations together and by the definition of $R_i(.|.)$ in (1), we have $(v_i - \tilde{v}_i)[\phi_i(T_i(v_i)) - \phi_i(T_i(\tilde{v}_i))] \geq 0$ for all $i \in I_N$ and $v_i, \tilde{v} \in [\underline{v}_i, \overline{v}_i]$ where $\phi_i(t) = \int_0^t e^{-r_i s} dF_{-i}^{min}(s) + e^{-r_i t}(1 - F_{-i}^{min}(t))$. This means that $\phi_i(T_i(.))$ is non-decreasing. Since dF_{-i}^{min}/dt has meaning in the sense of measurable function, the derivative of $\phi_i(.)$ can be written as $\phi_i^{'}(t) = -r_i e^{-r_i t}(1 - F_{-i}^{min}(t)) \leq 0$ implying that $\phi_i(.)$ is non-increasing, and thus $T_i(.)$ is non-increasing.

Now, I prove that in equilibrium, there must be at least two players whose highest types provide at $t = 0$ by contradiction. First, suppose there is no one whose highest type exits instantly, by the monotonicity proved above, there is no instant-exit type. Then, for a sufficiently small $\epsilon > 0$, there is a type v_k such that $T_k(v_k) \leq \inf_{j \in I_N, v_j \in [\underline{v}_j, \overline{v}_j]} T_j(v_j) + \epsilon$ and $T_k(v_k) > 0$. Player v_k can be better off by changing his stopping time to $t = 0$, since there is almost no probability for others to exit at $t \in [0, T_k(v_k))$. And second, if there is only one player, say j, whose highest type exits instantly, then for a sufficiently small $\epsilon > 0$, there is a type v_k such that $T_k(v_k) = \inf_{j \in I_N - \{j\}, v_j \in [\underline{v}_j, \overline{v}_j]} T_j(v_j) + \epsilon$ and $T_k(v_k) > 0$. In this case, player j should have almost no incentive to provide at $t \in (0, T_k(v_k))$, during which others have almost no probability of exit. Given this gap in j's strategy, player v_k can be better off by providing slightly earlier. Thus, there are at least two players whose highest types exit instantly or, equivalently, there can be at most $N - 2$ players whose minimal stopping times are greater than 0.

(iv) I prove by contradiction. Suppose there are $m > 1$ players who provide at $t = 0$ with positive probabilities, any instant-exit type of any one of them, say $v_k \in (m_k, \overline{v}_k]$, can be better off by slightly delaying the stopping time. Formally, define $p_k =$ Prob(anyone other than k provides at $t = 0$). v_k can be better off by waiting for some small $\epsilon > 0$ to gain at least $[p_k v_k + (1 - p_k)e^{-r_k \epsilon}(v_k - c_k)] - [p_k(v_k - \frac{1}{m}c_k) + (1 - p_k)(v_k - c_k)]$, which will be positive as $\epsilon \to 0$. Thus, there can be at most one player who exits instantly with a positive probability.

(v) First, I show the continuity. Assume otherwise that $T_i(.)$ is discontinuous at some v_i^*, and by the non-increasing monotonicity proved in (iii), we have $\lim_{v\to v_i^*-0} T_i(v) - \lim_{v\to v_i^*+0} T_i(v) = \tau > 0$. Let $t_i^* = \lim_{v\to v_i^*+0} T_i(v)$. The optimality of player v_i^* indicates that he is indifferent between providing at t^* and $t^* + \tau$, but this cannot be true unless there is no probability of others providing during $(t^*, t^* + \tau)$. However, this vacancy contradicts some players' optimality, since for the player $v_i^* - \epsilon$ where $\epsilon > 0$ is sufficiently small, he can be better off by stopping sooner during this vacancy period instead of after $T_i(v_i^*)$. Thus, no discontinuity should appear.

Further, I show the strict monotonicity. The non-increasing monotonicity and the continuity bring only one possible exception: for some $i \in I_N$, there exists some non-empty set $(\underline{u}_i, \overline{u}_i) \subset (c_i, m_i)$, on which T_i remains constant, say as t_0. However, this player's pooling exit brings discontinuity to other $T_j(.)$, because suppose they are all continuous functions, there exists a player v_k^* such that $T_k(v_k^*) = t_0 - \epsilon$ where $\epsilon > 0$ is sufficiently small. Player v_k^* can be better off by changing his stopping time to $t_0 + \epsilon$, whereby he gains at least $e^{-r_k(t_0+\epsilon)} \mathrm{Prob}(v_i \in (\underline{u}_i, \overline{u}_i))c_k - (1-e^{-2r_k\epsilon})R_k(T_k(v_k^*)|v^*)$, which becomes positive as $\epsilon \to 0$. Hence, $T_i(.)$ is continuous and strictly decreasing, and this indicates the existence of the inverse function $\Phi_i(.)$.

(vi) This part follows the spirits of Fudenberg and Tirole's (1986) proof of their Lemma 1(iv). For every $\bar{t} > \underline{t} > T_k(\overline{v}_k)$, if the war is still active at $\underline{t}$, player $\overline{v}_k$ prefers to exiting immediately to waiting any longer to exit. Consider the process in which $\overline{v}_k$ waits till $\bar{t}$ instead of $\underline{t}$. The marginal profit from winning with higher probability minus the extra waiting cost gives the negative gain of this deviation. Formally, for all $k \in I_N$:

$$
\begin{aligned}
0 &\geq v_j \frac{F_{-k}^{min}(\bar{t}) - F_{-k}^{min}(\underline{t})}{1 - F_{-k}^{min}(\underline{t})} - \int_{\underline{t}}^{\bar{t}} r_k(v_k - c_k)e^{-r_k(s-\underline{t})}ds \\
&\geq v_k \left[F_{-k}^{min}(\bar{t}) - F_{-k}^{min}(\underline{t})\right] - r_k(v_k - c_k)(\bar{t} - \underline{t}) \Rightarrow \left|F_{-k}^{min}(\bar{t}) - F_{-k}^{min}(\underline{t})\right| \leq \lambda_k |\bar{t} - \underline{t}|
\end{aligned}
$$

The inequation above shows that $F_{-k}^{min}(.)$ is Lipschitz continuous and thus also absolutely continuous. The Lipschitz constant $\lambda_k = r_k(1 - c_k/v_k)$. Further, notice that:

$$
\begin{aligned}
F_{-k}^{min}(t) &= \mathrm{Prob}(\min_{j\neq k} T_j(v_j) \leq t) = 1 - \mathrm{Prob}(\min_{j\neq k} T_j(v_j) > t) \\
&= 1 - \prod_{j=1, j\neq k}^{N} \mathrm{Prob}(T_j(v_j) > t) = 1 - \prod_{j=1, j\neq k}^{N} F_j(\Phi_j(t))
\end{aligned}
$$

For all $i \in I_N$, $F_i(\Phi_i(t))$ can be written as an expression of $F_{-k}^{min}(t)$ for all k only by simple algebraic operations that preserve the absolute continuity:

$$
F_i(\Phi_i(t)) = \frac{\prod_{j=1}^{N}(1 - F_{-k}^{min}(t))^{1/(N-1)}}{1 - F_{-i}^{min}}
$$

One thing noteworthy is that although the function above $y(x) = x^{1/(N-1)}$ is not Lipschitz continuous at $x = 0$, $1 - F_{-k}^{min}(t)$ remains strictly positive when $t < +\infty$, which guarantees the Lipschitz continuity of $y(.)$ on the domain of interest. Therefore, $F_i(\Phi_i(.))$ is also absolutely continuous. Next, because of the boundedness of $f_i(.)$, $F_i^{-1}(.)$ is Lipschitz continuous, as

$|F_i^{-1}(\overline{v}) - F_i^{-1}(\underline{v})| \leq |\overline{v} - \underline{v}| \max_{v \in [\overline{v}_i, \underline{v}_i]}(1/f_i(v))$. Thus, $\Phi_i(.) = F_i^{-1}(F_i(\Phi_i(.)))$ is preserved to be absolutely continuous, which indicates that $\Phi_i(.)$ is differentiable almost everywhere.

Now, I show equations (2). But before this, I need to clarify why there can be $M \leq N$ Φ-curves that have definition at some time point. The reason lies in the existence of strict waiting suggested in clause (iii), as once some player k chooses to wait strictly for $T_k(\overline{v}_k) = t_k > 0$, any of his opponents' stop-time choice, say $t < t_k$, will not be affected locally by $T_k(.)$, since $\text{Prob}(T_k > t) = F_k(\overline{v}_k)$ remains constant as one near this moment. Formally, when at some $t > 0$ there are M Φ-curves that have definition whose subscripts are denoted by $I(t) = \{j_1, j_2, ..., j_M\}$, the function $F_{-j_i}^{min}$ in $R_{j_i}(t|v)$ becomes $1 - \prod_{k=1,k\neq i}^{M} F_{j_k}(\Phi_{j_k}(t))$. Further, fixing the opponents' strategy profile, the first-order condition for the maximization of player j_i's expected gain gives:

$$
\begin{aligned}
\frac{d}{dt_{j_i}} R_{j_i}(t_{j_i}|v_{j_i}) = &- v_{j_i} e^{-r_{j_i} t_{j_i}} \Big(\sum_{k=1,k\neq i}^{M} \frac{d}{dt_{j_i}} lnF_{j_k}(\Phi_{j_k}(t_{j_i})) \Big) \Big(\prod_{k=1,k\neq i}^{M} F_{j_k}(\Phi_{j_k}(t_{j_i})) \Big) \\
&+ (v_{j_i} - c_{j_i}) e^{-r_{j_i} t_{j_i}} \Big(\sum_{k=1,k\neq i}^{M} \frac{d}{dt_{j_i}} lnF_{j_k}(\Phi_{j_k}(t_{j_i})) \Big) \Big(\prod_{k=1,k\neq i}^{M} F_{j_k}(\Phi_{j_k}(t_{j_i})) \Big) \\
&- r_{j_i}(v_{j_i} - c_{j_i}) e^{-r_{j_i} t_{j_i}} \Big(\prod_{k=1,k\neq i}^{M} F_{j_k}(\Phi_{j_k}(t_{j_i})) \Big) = 0
\end{aligned}
$$

$$
\Rightarrow -c_{j_i} \sum_{k=1,k\neq i}^{M} \frac{d}{dt} lnF_{j_k}(\Phi_{j_k}(t)) = r_{j_i}(\Phi_{j_i}(t) - c_{j_i}),\ j_i \in I(t)
$$

Eventually, algebraic calculation gives the expression for $\Phi'_{j_i}(t)$:

$$
\frac{d}{dt} lnF_{j_i}(\Phi_{j_i}(t)) = \frac{1}{M-1} \sum_{p=1}^{M} \sum_{k=1,k\neq p}^{N} \frac{d}{dt} lnF_{j_k}(\Phi_{j_k}(t)) - \sum_{k=1,k\neq i}^{N} \frac{d}{dt} lnF_{j_k}(\Phi_{j_k}(t))
$$

This equation above yields exactly equation (2). What is worth mentioning is that the group of equations in (2) satisfies local Lipschitz condition wherever it has definition.

Lastly, I verify $\lim_{v \to c_i + 0} T_i(v) = \overline{T} = +\infty$. Suppose not, this means that the equilibrium must coincide with the solution of the initial-value problem in which (2) serves as the group of equations and $\{\Phi_i(\overline{T}) = c_i, i \in I_N\}$ as the boundary conditions. However, $\{\Phi_i(t) \equiv c_i, i \in I_N\}$ is obviously one possible solution of this initial-value problem, and since the equations satisfy local Lipschitz condition at the boundary conditions, this is the unique solution which is however not acceptable, for $\Phi_i(0) = c_i < m_i$ for all $i \in I_N$ violates clause (iv).

Now, I prove the sufficiency. I only need to prove that $R_i(t|v_i)$ reaches global optimality at $t = T_i(v_i)$ for all $i \in I_N$. This directly results from the strictly decreasing monotonicity of each $\Phi_i(.)$. In particular, according to the derivation of (2), $dR_{j_i}(t|v_{j_i})/dt$ has the same sign as the following expression:

$$
- \sum_{k=1,k\neq i}^{M} \frac{d}{dt} lnF_{j_k}(\Phi_{j_k}(t)) - \frac{r_{j_i}}{c_{j_i}}(v_{j_i} - c_{j_i})
$$

Substituting equation (3) for the first term above, we have $(r_{j_i}/c_{j_i})(\Phi_{j_i}(t) - v_{j_i})$. Since $\Phi_{j_i}(.)$ is strictly decreasing and equals v_{j_i} when $t = T_{j_i}(v_{j_i})$, $dR_{j_i}(t|v_{j_i})/dt \gtreqless 0$ when $t \gtreqless T_{j_i}(v_{j_i})$. Thus, $T_i(.)$ gives the globally optimal choice when the rival strategy profile is fixed. *Q.E.D.*

6.2 Proof of Lemma 2.

(i) When $m = m^{(0)}$ changes its ith component higher to be $m^{(1)}$, $\Phi_i(0, m^{(1)}) = m_i^{(1)} > m_i^{(0)} = \Phi_i(0, m^{(0)})$. I show that for all $t > 0$ $\Phi_i(t, m^{(1)}) > \Phi_i(t, m^{(0)})$. Assume otherwise the existence of some $s > 0$ such that $\Phi_i(s, m^{(1)}) = \Phi_i(s, m^{(1)})$, but this is not possible, because the presence of s means that the two solutions $\{\Phi_i(., m^{(0)}), i \in I_M\}$ and $\{\Phi_i(., m^{(1)}), i \in I_M\}$ are obtaind by the same boundary conditions $\{\Phi_j(0) = m_j, j \in I_M - \{i\}\} \cup \{\Phi_i(s) = \Phi_i(s, m^{(0)})\}$ so that these two solutions must coincide violating $\Phi_i(0, m^{(1)}) \neq \Phi_i(0, m^{(0)})$.

Second, for all $j \in I_M - \{i\}$ equations (2) show that $\partial\Phi_j(0, m^{(0)})/\partial t > \partial\Phi_j(0, m^{(1)})/\partial t$, which means that there exists a small domain $t \in (0, u)$ where $\Phi_j(t, m^{(0)}) > \Phi_j(t, m^{(1)})$. It also holds that for all $t > 0$ $\Phi_j(t, m^{(0)}) > \Phi_j(t, m^{(1)})$, since otherwise there will be an s such that $\Phi_j(s, m^{(0)}) = \Phi_j(s, m^{(1)})$, which is not possible for a similar reason illustrated in the first part of proof of this clause.

(iii) For some $i \in I_M$, all $j \in I_M$, $m \in B_M$, and $t > 0$, consider the initial-value problem characterizing $\partial\Phi_j(t, m)/\partial m_i = z_j^{(i)}$. For all $r \in I_M$, denote the funtion on the right side of player r's equation (2) by $g_r(\Phi_1, \Phi_2, ..., \Phi_M)$. According to differential equation theory, the differentiability of each g_r, which is ensured by the differentiability and boundedness of all (F_r, f_r), along with the satisfaction of Lipschitz condition ensures the existence of each $z_j^{(i)}$. Specifically, the initial-value problem determining $\{z_j^{(i)}, j \in I_M\}$ is:

$$\begin{aligned}
\frac{dz_j^{(i)}}{dt} &= \sum_{k=1}^{M} \frac{\partial g_j}{\partial \Phi_k} z_k^{(i)} \\
&= \left\{ \frac{M-2}{M-1} \frac{r_j}{c_j} \frac{F_j(\Phi_j(t))}{f_j(\Phi_j(t))} + \left[\frac{r_j}{c_j}(\Phi_j(t) - c_j) \right.\right. \\
&\qquad \left.\left. - \frac{1}{M-1} \sum_{k=1}^{M} \frac{r_k}{c_k}(\Phi_k(t) - c_k) \right] \frac{d}{d\Phi_j} \left(\frac{F_j(\Phi_j(t))}{f_j(\Phi_j(t))} \right) \right\} z_j^{(i)} \\
&\qquad - \frac{1}{M-1} \frac{F_j(\Phi_j(t))}{f_j(\Phi_j(t))} \sum_{k=1, k\neq j}^{M} \frac{r_k}{c_k} z_k^{(i)}, \; j \in I_M \\
& z_1^{(i)}(0) = 0, z_2^{(i)}(0) = 0, ..., z_{i-1}^{(i)}(0) = 0, z_{i+1}^{(i)}(0) = 0, ..., z_M^{(i)}(0) = 0 \\
& z_i^{(i)}(0) = 1
\end{aligned}$$

The boundary conditions above are obvious, since m_i is the only variable parameter in differentiation. Further, now that $\Phi_\iota(s, m) \to c_\iota$ as $s \to +\infty$ for all $\iota \in I_M$, when time approximates infinity the complicated equations above are simplified to a group of linear equations with constant coefficients, which are:

$$\frac{dz_j^{(i)}}{dt} = \frac{1}{M-1}\frac{F_j(c_j)}{f_j(c_j)}\left[(M-2)\frac{r_j}{c_j}z_j^{(i)} - \sum_{k=1,k\neq j}^{M}\frac{r_k}{c_k}z_k^{(i)}\right],\ j\in I_M \tag{9}$$

Since clause (i) indicates that for all $t>0$ $z_i^{(i)}(t)>0$ and $z_j^{(i)}(t)<0$ for all $j\neq i$, according to (9), when $t\to+\infty$ $dz_i^{(i)}/dt>0$. Another fact here is in similar form to (iii): the only convergent solution satisfies $z_j^{(i)}\to 0$ for all $j\in I_M$. This is because convergence requires $dz_j^{(i)}/dt\to 0$ for all j, and this along with (9) gives that when $t\to+\infty$ $A_M y=0$, where the vector $y=(r_j z_j^i/c_j)_{j=1}^M$ and the coefficient matrix A_M is:

$$A_M = \begin{bmatrix} M-2 & -1 & \cdots & -1 \\ -1 & M-2 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & M-2 \end{bmatrix}_{M\times M}$$

A_M is invertible, because its determinant $det(A_M)=-(M-1)^{M-1}\neq 0$, so that the unique solution is all zero. Now let H_k denotes $\frac{M-2}{M-1}\frac{r_k F_k(c_k)}{c_k f_k(c_k)}$ for all k. Then, $z_i^{(i)}$'s (9) gives $dz_i^{(i)}/dt > H_i z_i^{(i)}$ suggesting that $z_i^{(i)}$ is increasing no slower than $ae^{H_i t}$, and consequently $z_i^{(i)}\to+\infty$ as $t\to+\infty$.

Next, I show that $z_j^{(i)}\to-\infty$ for $j\neq i$. Assume otherwise that for some k $z_k^{(i)}$ approximates some $0\geq -v_k^* > -\infty$[26]. This along with $z_k^{(i)}$'s (9) gives that $\sum_{r=1,r\neq k}^{M}(r_r z_r^{(i)}/c_r)\to-(M-2)(r_k v_k^*/c_k)$ because $dz_k^{(i)}/dt\to 0$. However, this is not possible, for when t becomes large $z_k^{(i)}$'s equation (9) actually becomes $dz_k^{(i)}/dt = H_k(z_k^{(i)}+v_k^*)$ giving a solution that approximates $-\infty$, a contradiction.

(ii) First consider convergent case, in which each curve i approximates a finite value, say $v_i^\infty\in(\underline{v}_i,+\infty)$. This along with (2) necessarily requires that for all $i\in I_M$:

$$(M-2)\frac{r_i}{c_i}(v_i^\infty - c_i) - \sum_{k=1,k\neq i}^{M}\frac{r_k}{c_k}(v_k^\infty - c_k) = 0$$

These are linear equations whose constant coefficient matrix is identical to the invertible A_M defined in (iii)'s proof. Consequently, the only possible solution is $\{v_i^\infty = c_i, i\in I_M\}$.

Now I show why any P_M solution converging in this way must satisfy Lemma 1(ii) and (v), namely, each curve $\Phi_i(.)$ keeps decreasing from $t=0$ and respectively approximates c_i from above. I denote the boundaries yielding this solution by m. First, each curve i must remain above c_i, since if not, the fact that $\Phi_i(0,m)>c_i$ for all i indicates that there exists a $u>0$ and $k\in I_M$ such that $\Phi_k(u,m)=c_k$ while $\Phi_j(u,m)\geq c_j$ where at least one $\geq$ is strict. However, this can only bring divergence. Notice that equations (2) are time-invariant and when I reset u as the new

[26] Since (9) is a group of linear equations with constant coefficients, when its solution curves do not diverge, they must converge to a finite point.

zero time, a new P_M problem with identical equations, $\{I_M, B_M, (c, r, F, f)_i, (\widetilde{\Phi}(t, m))_i\}$, emerges, whereas the boundary conditions become $\{\widetilde{\Phi}_k(0) = c_k\} \cup \{\widetilde{\Phi}_j(0) = \Phi_j(u, m), j \neq k\}$. Remember that these boundaries can be seen as $\widetilde{\Phi}_j(0)$ in $m_0 = \{\widetilde{\Phi}_i(0) = c_i, i \in I_M\}$ being raised from 0 to $\Phi_j(u, m)$ for all $j \neq k$, and also that m_0 yields the benchmark solution $\{\widetilde{\Phi}_i(t, m_0) \equiv c_i, i \in I_M\}$. Due to the asymptotic sensitivity introduced in (iii), any rise of each $\widetilde{\Phi}_j(0)$ will cause $\widetilde{\Phi}_k(+\infty, m_0)$ to drop sharply, namely, $\widetilde{\Phi}_k(t) \to \underline{v}_k$ in distance, a contradiction to $\widetilde{\Phi}_k(t) \to c_k$.

Additionally, I need to explain why convergent $\Phi_i(.)$s are decreasing. Since I have shown $\Phi_i > 0$ and that in distance (2) approximates linear equations with constant coefficients, the fact that $\Phi_i(t) \to c_i + 0$ implies that when $t \to +\infty$ each $\Phi_i(t)$ approximates a linear combination of several decreasing exponential terms like $e^{-\lambda t}$, $t^r e^{-\lambda t}$, and $\sin wte^{-\lambda t}$, where r, w, and $\lambda > 0$. As a result, there exists a large time T such that $\Phi_i'(t) < 0$ for all $i \in I_M$ and $t \geq T$. Then, I show that $\Phi_i' < 0$ must also hold for all i before T, because if not, there exists some s such that for some $k \in I_M$ $\Phi_k'(s) = 0$, and I denote the nearest such s as u. At u, it cannot be $\Phi_i'(u) = 0$ for all i, because if so, (2) yields:

$$(M-1)\frac{r_i}{c_i}(\Phi_i(u) - c_i) = \sum_{k=1}^{M} \frac{r_k}{c_k}(\Phi_k(u) - c_k),\ i \in I_M$$

By summing up both sides of every i's inequation above, one can easily obtain a contradiction, so there is at least one r^* such that $\Phi_{r^*}'(u) > 0$. For any one of k such that $\Phi_k'(u) = 0$, I assume Φ_k' turns to positive right before u, so that there exist two sufficiently close points, say u_1 and u_2, such that $u_1 < u < u_2$ and $\Phi_k(u_1) = \Phi_k(u_2)$. At these two points, k's (2) along with the existence of r^* gives:

$$\Phi_k'(u_1) - \Phi_k'(u_2) \approx \frac{1}{M-1}\frac{F_k(\Phi_k(u))}{f_k(\Phi_k(u))} \sum_{p=1, p\neq k}^{M} \frac{r_p}{c_p}(\Phi_k(u_1) - \Phi_k(u_2)) < 0$$

This inequation contradicts $\Phi_k'(u_1) > 0 > \Phi_k'(u_2)$, so there is no such u and therefore all $\Phi_i(.)$ are strictly decreasing. *Q.E.D.*

6.3 Proof of Theorem 1.

First, I formally introduce the backward induction strategy and some useful notations. According to Lemma 1(iii), there are $\overline{K} \leq N-2$ divisions located at $0 < D_{\overline{K}} < D_{\overline{K}-1} < ..., < D_1 < +\infty$, and I add $D_{\overline{K}+1} = 0$ and $D_0 = +\infty$ for convenience. I denote the set of subscripts of the players who are active during $t \in [D_K, D_{K-1}]$ by $I(D_K)$ and the number of these players by $M(K)$. Each division D_K starts, on its right, a $P_{M(K)}$ problem on area $\Upsilon(M(K))$. Peculiarly, the rightmost is a P_N problem with no division on its right side, so Lemma 2(ii) requires that this problem must select a set of boundary conditions yielding a convergent solution. The process of searching for equilibrium starts with finding the unique convergent solution of this P_N problem whose left-side boundaries satisfy that $\Phi_i(D_1) \leq \overline{v}_i$ for all $i \in I_N$ and $\Phi_k(D_1) = \overline{v}_k$ for at least one $k \in I_N$. Then, in a backward order, $K = 1, 2, ..., \overline{K}$, I sequentially consider the $M(K+1)$ problem on each $\Upsilon(M(K+1))$ whose boundary conditions are the values of the solution curves at $t = D_K$ determined in the previous $M(K)$ problem. For each K, adjust the distance between D_{K+1} and D_K, and fix this comparative distance in all the following problems with bigger Ks. Each adjustment of $D_K - D_{K+1}$ is determined to be such that $\Phi_i(D_{K+1}) \leq \overline{v}_i$ for all $i \in I(D_{K+1})$ and $\Phi_k(D_{K+1}) = \overline{v}_k$ for at least one $k \in I(D_{K+1})$. Also, for each K $M(K+1)$ is obtaind by eliminating any such subscript k such that $\Phi_k(D_K) = \overline{v}_k$ from $M(K)$. This sequential adjustment of D_Ks and determination of $M(.)$ end when some large K^* occurs which is such that $M(K^*)$ =1 or 0, and $\overline{K} = K^* - 1$ is thus determined. Finally, I verify that this strategy gives all possible solutions satisfying Lemma 1 and only one such solution is found. I divide the proof into 4 steps.

Step 1: For the rightmost P_N problem, define $\Delta(B_N^{(-1)})$ as the set consisting of all subsets of $B_N^{(-1)} = \prod_{k=1,k\neq 1}^{N}[c_k, +\infty)$, and define a function $\psi : [c_1, +\infty) \to \Delta(B_N^{(-1)})$. Each $m^{(-1)} \in \psi(m_1) \neq \varnothing$ is an $N-1$-dimension vector such that $(m_1, m_1^{(-1)}, m_2^{(-1)}, ..., m_{N-1}^{(-1)})$ is a boundary selection yielding a convergent solution of this P_N problem.

In this step, I show that for each $m_1 \in [c_1, +\infty)$ $\psi(m_1)$ is either empty or a singleton. Now I fix m_1 and find its corresponding boundary set, $\psi(m_1)$, with each of whose element m_1 yields a convergent solution. I begin by defining some notations. I choose a time sequence $(t_n)_{n=1}^{\infty}$ such that $t_n \to +\infty$, and a $v_0 > 0$. For each n, define $\phi^{(n)} : C_N^{(-1)} = \prod_{k=1,k\neq 1}^{N}[c_i, c_i + v_0] \to B_N^{(-1)}$ as a mapping such that, if $\phi^{(n)}(v)$ is denoted by $m^{(-1)}$, $(m_1, m_1^{(-1)}, m_2^{(-1)}, ..., m_{N-1}^{(-1)})$ is the boundary selection that yields a solution satisfying $\Phi_i(t_n) = v_{i-1}$ for all $i \in I_N - \{1\}$. Geometrically, $\phi^{(n)}$ is the link of two forms of boundary conditions, $\{\Phi_i(0) = m_{i-1}^{(-1)}, i \in I_N - \{1\}\} \cup \{\Phi_1(0) = m_1\}$ and $\{\Phi_i(t_n) = v_{i-1}, i \in I_N - \{1\}\} \cup \{\Phi_1(0) = m_1\}$ where each v_i is bounded by $[c_i, c_i + v_0]$, and because of the satisfaction of Lipschitz condition, this link is unique and invertible. Further, define $\psi^{(n)}(v_0)$ as the set $\phi^{(n)}(C_N^{(-1)}) \cap B_N^{(-1)}$.

Based on Lemma 2(ii), $\psi(m_1|v_0) \subset \psi(m_1)$, because each element of $\psi(m_1|v_0)$ with m_1 yields a solution satisfying $\Phi_i(t_n \to +\infty)$ remains finite for all $i \geq 2$, which indicates that $\Phi_1(t_n \to +\infty)$ must remains finite according to (2), and thus this is a Lemma-1(ii)-and-(v) convergent solution.

Additionally, $\psi(m_1|v_0)$ is a singleton if it exists and is non-empty. The continuity of $\phi^{(n)}(.)$, which is implied by the partial differentiability of Φs with respect to each m_i in Lemma 2(iii), tells that each $\psi^{(n)}(m_1|v_0)$ is a compact set since $C_N^{(-1)}$ and $B_N^{(-1)}$ are compact, and thus the limit $\psi(m_1|v_0)$ is compact. So, if the limit set contains more than one element, there must be two points, say $m_1^{(-1)}$ and $m_2^{(-1)}$, sufficiently close to each other, and the asymptotic sensitivity in Lemma 2(ii) requires that their difference $m_1^{(-1)} - m_2^{(-1)} = \Delta m^{(-1)} = (\Delta m_2, \Delta m_3, ..., \Delta m_N)$ must satisfy that for all i $\Phi_i(+\infty, (m_1, m_1^{(-1)})) - \Phi_i(+\infty, (m_1, m_2^{(-1)})) \approx \sum_{k=2}^{N} \Delta m_k \partial\Phi_i/\partial m_k = 0$. For convenience, define $y_j^{(i)} = r_j z_j^{(i)}/c_j = (r_j/c_j)\partial\Phi_j/\partial m_i$, and the conditions above become that $\sum_{k=2}^{N} \Delta m_k y_i^{(k)} = 0$ for all i. When $t \to +\infty$, $y_j^{(i)}$ is charactrized by (9), and according to Lemma 2(iii), $y_i^{(i)} \to +\infty$ and $y_j^{(i)} \to -\infty$ for all $i, j \neq i$. Notice that (9) is linear so that each $y_j^{(i)}$ is a linear combination of several exponential terms, like $e^{\lambda t}$, $e^{\lambda t}t^r$, and $e^{\lambda t}\sin wt$ where $r, w > 0$. An important property of this kind of combination is that, if one function has a strictly higher exponential order than another, the absolute value of the former will surpass that of the latter when $t \to +\infty$[27]. Then for all k, I select one $p \neq k$ such that $y_p^{(k)}$ has the lowest exponential order in $\{y_j^{(k)}, j \neq k\}$ and denote this p by $\underline{p}(k)$, and also select another $\overline{p}(k)$ such that $y_{\overline{p}(k)}^{(k)}$ has the highest exponential order. $\underline{p}(k)$ and $\overline{p}(k)$ may not be unique, but I just select one of the possible subscrips. Now Consider $N > 2$. Equations (9) along with the decreasing monotonicity of $y_{\underline{p}(k)}^{(k)}$ give that:

$$y_{\underline{p}(k)}^{(k)} < \frac{1}{N-1} \sum_{j=1}^{N} y_j^{(k)},\ k \in I_N - \{1\}$$

The asymptotic domination property of higher exponential order necessarily requires that the right side of the inequation above should be non-negative. This is because, if it goes negative, this must be caused by the dominant order of $y_{\overline{p}(k)}^{(k)}$, and therefore this term will eventually goes to $-\infty$ faster than $y_{\underline{p}(k)}^{(k)}$, which yields an inequation contradicting the inequation above. Hence, I finally get $\sum_{j=1}^{N} y_j^{(k)} \geq 0$, and this further implies:

$$|y_k^{(k)}| \geq \sum_{j=1, j\neq k}^{N} |y_j^{(k)}| > \sum_{j=2, j\neq k}^{N} |y_j^{(k)}|,\ k \in I_N - \{1\}$$

This strict inequation indicates that the following matrix, $\Gamma^{(-1)}$, is strictly diagonally dominant, and this further implies that $\Gamma^{(-1)}$ is invertible:

$$\Gamma^{(-1)} = \begin{bmatrix} y_2^{(2)} & y_2^{(3)} & \cdots & y_2^{(N)} \\ y_3^{(2)} & y_3^{(3)} & \cdots & y_3^{(N)} \\ \vdots & \vdots & \ddots & \vdots \\ y_N^{(2)} & y_N^{(3)} & \cdots & y_N^{(N)} \end{bmatrix}$$

Finally, notice that part of the conditions, $\sum_{k=2}^{N} \Delta m_k y_i^{(k)} = 0$ for all $i \in I_N - \{1\}$, is equivalent to $\Gamma^{(-1)}\Delta m^{(-1)} = 0$, and the invertibility of $\Gamma^{(-1)}$ concludes that all $\Delta m_k = 0$. When $N = 2$, the case becomes trivial, as $\Phi_i(+\infty, (m_1, m_1^{(-1)})) - \Phi_i(+\infty, (m_1, m_2^{(-1)})) = y_i^{(2)}\Delta m_2 = 0$ implying $\Delta m_2 = 0$. Consequently, $m_1^{(-1)} = m_2^{(-1)}$, so $\psi(m_1|v_0)$ is a singleton if it is non-empty. And obviously, v_0 here only serves as a measure of convergence and is irrelevant to $\psi(m_1|v_0)$, so $\psi(m_1) = \psi(m_1|v_0)$ contains every possible solution.

Step 2: In this step, I show that for each $m_1 \in [c_1, +\infty)$ $\psi(m_1)$ is non-empty. I begin this part by first showing that for each $v \in C_N^{(-1)}$ $\phi^{(n)}(v)$ is non-empty. Define a mapping $G^{(n)}(.|v) : B_N^{(-1)} \to B_N^{(-1)}$, and $m^G = G^{(n)}(m^{(-1)}|v)$ is an $N-1$-dimension vector satisfying $\Phi_i(t_n, (m_1, m_i^G, m_{-i}^{(-1)})$[28]$) = v_i$ for all $i \in I_N - \{1\}$. Clearly, $G(.|v)$ is non-empty, and this is because, since the benchmark boundaries $\{\Phi_i(0) = c_i, i \in I_N\}$ yield $\Phi_i(t_n) = 0$, the fact that

[27]For example, $e^{\lambda_1} > e^{\lambda_2}$ if $\lambda_1 > \lambda_2$, and $t^{r_1}e^{\lambda} > t^{r_2}e^{\lambda}$ if $r_1 > r_2$.

[28]This is an N-dimension vector with its first component being m_1, the ith component being the ith of m^G, and the rest respectively taken by $m^{(-1)}$'s components other than its ith.

$m_1 \geq c_1$ and for all k $m_k^{(-1)} \geq c_k$ gives that $\Phi_i(t_n, (m_1, 0, m_{-i}^{(-1)})) \leq c_i$ according to Lemma 2(i), and therefore $\Phi_i(t_n, (m_1, m_i^G, m_{-i}^{(-1)})) = v_i \geq c_i$ must require $m_i^G \geq c_i$, which in summary implies $m^G \in B_N^{(-1)}$.

Denote an $N-1$-dimension initial point $m^{(0)}(v) = (c_2, c_3, ..., c_N)$ and $m^{(\nu+1)}(v) = G^{(n)}(m^{(\nu)}(v)|v)$ for all $\nu \in \mathbb{N}$, and I show that the limit $m^{(\infty)}(v)$ uniquely exists. Since $\Phi_i(t_n, (m_1, m^{(0)}(v))) \leq c_i$, $\Phi_i(t_n, (m_1, m_i^{(1)}(v), m_{-i}^{(0)}(v))) = v_i \geq c_i$ along with Lemma 2(i) gives $m_i^{(1)}(v) \geq m_i^{(0)}(v)$ for all i. Further, since for all j $m_j^{(1)}(v) \geq m_j^{(0)}(v)$, $\Phi_i(t_n, (m_1, m_i^{(2)}(v)), m_{-i}^{(1)}(v)) = \Phi_i(t_n, (m_1, m_i^{(1)}(v)), m_{-i}^{(0)}(v))$ along with Lemma 2(i) gives $m_i^{(2)}(v) \leq m_i^{(1)}(v)$ for all i. And also $m_i^{(2)}(v) \geq m_i^{(0)}(v)$ for all i, because $\Phi_i(t_n, (m_1, m_i^{(0)}(v) = c_i, m_{-i}^{(1)}(v))) \leq c_i \leq v_i$. Consequently, $m_i^{(2)}(v)$ lies between $m_i^{(1)}(v)$ and $m_i^{(0)}(v)$. Then from $\nu = 3$, $m_i^{(\nu-1)}(v)$ always lies between $m_i^{(\nu-2)}(v)$ and $m_i^{(\nu-3)}(v)$ for all i, because if one assumes that this holds true for some $\nu = k \geq 3$, then since for all j $m_j^{(k-1)}(v)$ lies between $m_j^{(k-2)}(v)$ and $m_j^{(k-3)}(v)$, $\Phi_i(t_n, (m_1, m_i^{(k)}(v), m_{-i}^{(k-1)}(v))) = \Phi_i(t_n, (m_1, m_i^{(k-1)}(v), m_{-i}^{(k-2)}(v))) = \Phi_i(t_n, (m_1, m_i^{(k-2)}(v), m_{-i}^{(k-3)}(v)))$ along with Lemma 2(i) must give that $m_i^{(k)}(v)$ lies between $m_i^{(k-1)}(v)$ and $m_i^{(k-2)}(v)$ for all i, which extends this relationship to $\nu = k + 1$.

This next-in-middle property leaves only two possible results: either that $m^{(\nu)}(v) \to m^{(\infty)}(v)$ uniquely exists, or that there are two points, say $m^{(a)}(v)$ and $m^{(b)}(v)$, such that $G^{(n)}(m^{(a)}(v)|v) = m^{(b)}(v)$, $G^{(n)}(m^{(b)}(v)|v) = m^{(a)}(v)$, and, without loss of generality, $m_i^{(a)}(v) > m_i^{(b)}(v)$ for all i. I prove that the latter case never occurs. The relationship between $m^{(a)}(v)$ and $m^{(b)}(v)$ implies that for all i $\Phi_i(t_n, (m_1, m_i^{(a)}(v), m_{-i}^{(b)}(v))) = \Phi_i(t_n, (m_1, m_i^{(b)}(v), m_{-i}^{(a)}(v)))$, but the fact that for all j $m_j^{(a)}(v) > m_j^{(b)}(v)$ along with Lemma 2(i) indicates that $\Phi_i(t_n, (m_1, m_i^{(a)}(v), m_{-i}^{(b)}(v))) > \Phi_i(t_n, (m_1, m_i^{(b)}(v), m_{-i}^{(a)}(v)))$, a contradiction.

All discussion above shows one thing: $G^{(n)}(.|v)$ has a unique fixed point in $B_N^{(-1)}$. And this fixed point obviously is $\phi^{(n)}(v)$, and thus this function if non-empty. Moreover, the nth set $\psi^{(n)}(m_1|v_0) = \phi^{(n)}(C_N^{(-1)})$ is non-empty. Now rewrite $\psi^{(n)}(m_1|v_0)$ as a function of $t_n = t$, namely, $\psi(t|m_1, v_0)$. The continuity of $\phi^{(n)}(.)$, which is implied by the partial differentiability of Φs with respect to each m_i in Lemma 2(iii), plus the continuity of each $\Phi_i(.)$ tells that $\psi(.|m_1, v_0)$ is upper semi-continuous, so the limit of a sequence containing non-empty sets, $\psi(t_n \to +\infty|m_1, v_0) = \psi(m_1)$, also exists and remains non-empty.

Combining the results of the first two steps, I conclude that for each $m_1 \geq c_1$ there uniquely exists a boundary selection, $(m_1, m_2, ..., m_N)$, yielding a convergent solution in the rightmost P_N problem.

Step 3: In this step, I characterize $\psi(m_1)$. For the preceding steps have shown that ψ links each m_1 to a unique point in $B_N^{(-1)}$, I decompose this mapping into $N - 1$ new functions, $m_i^{(1)} : [c_1, +\infty) \to [c_i, +\infty)$ for all $i \in I_N - \{1\}$, which give that for all $m_1 \in [c_1, +\infty)$ $(m_1, m_2^{(1)}(m_1), m_3^{(1)}(m_1), ..., m_N^{(1)}(m_1))$ is the unique corresponding boundary selection yielding convergent solution.

First, $m_i^{(1)}(c_1) = c_i$ for all $i \in I_N - \{1\}$, which is obvious.

Additionally, I prove the strictly increasing monotonicity of each $m_i^{(1)}(.)$. Consider a change of m_1 to $m_1 + \Delta m_1$ where Δm_1 is positive but small, and denote $m_i^{(1)}(m_1 + \Delta m_1) - m_i^{(1)}(m_1)$ by Δm_i. Similar to the proof of Step 1, Lemma 2(ii) requires that $\sum_{k=1}^{N} \Delta m_k y_i^{(k)} = 0$ for all $i \in I_N$. Also, Step 1 gives that $|y_k^{(k)}| > \sum_{j=2, j\neq k}^{N} |y_j^{(k)}|$, and that $y_k^{(k)} \to +\infty$ and $y_j^{(k)} \to -\infty$ for all $k, j \neq k$. If for some k $\Delta m_k < 0$, then denote the set containing all such k by S^-, and define $S^+ = I_N - S^-$. Further select a $p \in S^-$ such that $\Delta m_p = \max_{k\in S^-} |\Delta m_k|$. Finally, we have:

$$
\begin{aligned}
0 < -\Delta m_1 y_p^{(1)} &= \Delta m_p y_p^{(p)} + \sum_{j\in S^+ - \{1\}} \Delta m_j y_k^{(j)} + \sum_{j\in S^- - \{p\}} \Delta m_j y_k^{(j)} \\
&\leq \Delta m_p y_p^{(p)} + \sum_{j\in S^- - \{p\}} \Delta m_j y_k^{(j)} \leq \Delta m_p (|y_p^{(p)}| - \sum_{j\in S^- - \{p\}} |y_k^{(j)}|) < 0
\end{aligned}
$$

The inequation above constructs a contradiction, and thus $\Delta m_i > 0$ for all i.

Finally, I show that $m_i^{(1)}(.)$ is continuous. Assume otherwise that for some $k \neq 1$ and $m_1^* \in [c_1, +\infty)$ $\lim_{m\to m_1^*-0} m_k^{(1)}(m) = \underline{m}_k < \overline{m}_k = \lim_{m\to m_1^*+0} m_k^{(1)}(m)$. Since $m_k^{(1)}(.)$ is strictly increasing, this gap suggests that there is no m_1 such that $m_k^{(1)}(m_1) \in (\underline{m}_k, \overline{m}_k)$. To form contradiction, for all $i \in I_N - \{k\}$ I define $m_i^{(k)} : [c_k, +\infty) \to [c_i, +\infty)$ satisfying that for all $m_k \in [c_k, +\infty)$ boundary selection $(m_1^{(k)}(m_k), m_2^{(k)}(m_k), ..., m_{k-1}^{(k)}(m_k), m_k, m_{k+1}^{(k)}(m_k), ..., m_N^{(k)}(m_k))$ yields a convergent solution. Based on all previous proof, these functions are non-empty and give a unique set of boundary conditions for each m_k. Thus, for all $m_k \in (\underline{m}_k, \overline{m}_k)$ $m_1^{(k)}(m_k)$ exists and this is a contradiction.

Step 4: In this step, I carry out the backward induction process and prove that it yields a unique equilibrium. Lemma 1(iii) requires that at each division D_K the solution curves existing on its right, whose subscripts are contained in $I(D_K)$, take the values such that $\Phi_i(D_K) \leq \overline{v}_i$ for all $i \in I(D_K)$ and $\Phi_k(D_K) = \overline{v}_k$ for at least one $k \in I(D_K)$. For each K the comparative distance, $D_{K-1} - D_K$, is adjusted to satisfy the condition given by Lemma 1(iii) above, and this comparative distance will be fixed when it comes to latter adjustment of divisions with bigger K. Denote the set of all such k satisfying $\Phi_k(D_K) = \overline{v}_k$ by $\Delta I(D_K)$, and on D_{K+1}'s both sides it has $I(D_{K+1}) = I(D_K) - \Delta I(D_K)$. This process starts from the determination of the rightmost division, sequentially moves to the left, and ends when some K^* occurs such that $I(K^*)$ is $\varnothing$ or a singleton, and the total number of division is determined by $\overline{K} = K^* - 1$, which is required by Lemma 1(iv) that there must be at least two curves existing at $t = 0$.

I now prove that this process yields a unique solution. First consider the rightmost division, since each $m_i^{(1)}(m_1)$ starts at c_i when $m_1 = c_1$ and continuously increases as m_1 rises, there exists a unique boundary selection $(m_1, m_2^{(1)}(m_1), m_3^{(1)}(m_1), ..., m_N^{(1)}(m_1))$ such that $\Phi_i(D_K) \leq \overline{v}_i$ for all $i \in I_N$ and $\Phi_k(D_K) = \overline{v}_k$ for at least one $k \in I_N$.

Next, consider each area between two adjacent divisions. I prove by applying mathematical induction that for each $K \geq 1$ all curves existing in $\Upsilon(M(K))$ are strictly decreasing. Obviously, Lemma 2(ii) gives that the unique convergent solution in the rightmost $\Upsilon(M(1))$ is strictly decreasing. Now, suppose that for some $K > 1$ this decreasing property holds on $\Upsilon(M(K))$. Consider the behavior of each curve, say $i \in I(D_{K-1})$, right on both sides of division D_K, and the values of this curve on both sides equals each other, while the derivative of it on the left side is less than that on the right side, namely, if $(r_k/c_k)(\Phi_k(D_K) - c_k)$ is denoted by W_k, then (2) yields:

$$\frac{f_i(\Phi_i(D_K))}{F_i(\Phi_i(D_K))}(\Phi_i^{'}(D_K - 0) - \Phi_i^{'}(D_K + 0)) = \frac{1}{M(K) - 1}\sum_{k \in I(D_K)} W_k - \frac{1}{M(K+1) - 1}\sum_{k \in I(D_{K+1})} W_k$$

$$= \frac{1}{M(K) - 1}\sum_{k \in \Delta I(D_K)} W_k - \frac{M(K) - M(K+1)}{(M(K) - 1)(M(K+1) - 1)}\sum_{k \in I(D_{K+1})} W_k$$

$$= \frac{1}{(M(K) - 1)(M(K+1) - 1)}\sum_{k \in \Delta I(D_K)} \left[(M(K+1) - 1)W_k - \sum_{j \in I(D_{K+1})} W_j\right]$$

$$= \frac{1}{(M(K) - 1)(M(K+1) - 1)}\sum_{k \in \Delta I(D_K)} \left[(M(K) - 1)W_k - \sum_{j \in I(D_K)} W_j\right]$$

$$= \frac{1}{M(K+1) - 1}\sum_{k \in \Delta I(D_K)} \frac{f_k(\Phi_k(D_K))}{F_k(\Phi_k(D_K))}\Phi_k^{'}(D_K + 0) < 0$$

I cite one of the intermediate conclusions in the proof of Lemma 2(ii) that if in a M problem at some $t \in \Upsilon(M)$ $\Phi_i^{'}(t) < 0$ for all $i \in I(t)$, then $\Phi_i^{'}(s) < 0$ for all $s \leq t$ and i. Then for $K+1$, the strictly decreasing monotonicity of curves in $\Upsilon(M(K+1))$ still holds, and this completes the proof.

Finally, I show that the comparative distance, $D_K - D_{K+1}$, can be uniquely determined for all $K \geq 1$. Since, on each $\Upsilon(M(K+1))$ for all $i \in I(D_{K+1})$, $\Phi_i(D_K) < \overline{v}_i$ and that $\Phi_i(.)$ is strictly deceasing on $\Upsilon(M(K+1))$, there is only one comparative distance satisfying $\Phi_i(D_{K+1}) \leq \overline{v}_i$ for all $i \in I(D_{K+1})$ and $\Phi_k(D_{K+1}) = \overline{v}_k$ for at least one $k \in I(D_{K+1})$. Along with the stop criterion that the total number of division $\overline{K} = K^* - 1$ where $M(K^*) = 0$ or 1, the previous conclusion confirms that there uniquely exists an object, $E^* = \{\overline{K}, (D_K - D_{K+1})_K, (m_i = \Phi_i(D_1))_i\}$, characterizing a solution found in this backward induction. Since the structure of this solution searching process is based on the requirement of Lemma 1, this process gives all qualified equilibria, which has been proved to uniquely exist. *Q.E.D.*

6.4 Proof of Proposition 2

Denote the corresponding unique equilibrium by $e = \{N, \overline{K}, \Phi_i, (d_K, M_K, I_K); i \in I_N, K \in I_{\overline{K}}\}$, and let $g_K(t) = \frac{1}{M_K - 1}\sum_{k \in I_K} \frac{r_k}{c_k}(\Phi_k(t) - c_k)$. Let K_0 denote the smallest integer in $I_{\overline{K}}$ such that $d_{K_0+1} > t_0$. For all $K \geq K_0$, the equations for α and β on $t \in [\max\{t_0, d_K\}, d_{K+1})$ are:

$$\begin{aligned} \frac{d}{dt} lnF_\alpha(\Phi_\alpha(t)) &= -[g_K(t) - r_\alpha(\frac{\Phi_\alpha(t)}{c_\alpha} - 1)] \\ \frac{d}{dt} lnF_\beta(\Phi_\beta(t)) &= -[g_K(t) - r_\beta(\frac{\Phi_\beta(t)}{c_\beta} - 1)] \end{aligned} \tag{10}$$

(i) Let $r = r_\alpha = r_\beta$. First, consider the case where $f_\alpha(v)/F_\alpha(v) = f_\beta(v)/F_\beta(v)$ for all $v \in (c, \min\{\overline{v}_\alpha, \overline{v}_\beta\})$, with which (10) gives symmetric equations for α and β on $[t_0, +\infty)$. The proof of uniqueness of the solution of the P_N problem in Section 6.4 shows that this pair of symmetric equations should yield symmetric curves in this last N problem. Therefore, the boundary conditions for both players' curves in the penultimate M problem are also symmetric, which along with the symmetric equations yields a symmetric solution. Likewise, backward induction gives that $\Phi_\alpha(t) = \Phi_\beta(t)$ for all $t \in [t_0, +\infty)$.

Next, without loss of generality, I only need to consider the case where $f_\alpha(v)/F_\alpha(v) > f_\beta(v)/F_\beta(v)$ for all $v \in (c, \min\{\overline{v}_\alpha, \overline{v}_\beta\})$. Suppose there is an intersection point, say $s \geq t_0$, such that $\Phi_\alpha(s) = \Phi_\beta(s)$, then (10) gives $\Phi'_\alpha(s) > \Phi'_\beta(s)$, so the continuity of Φs requires that there cannot be another intersection point, and thus for all $t \gtrless s$, $\Phi_\alpha(t) \gtrless \Phi_\beta(t)$. Consequently, only three possible cases can occur: *a)* one intersection point, *b)* no intersection and $\Phi_\alpha(t) > \Phi_\beta(t)$ for all $t \geq t_0$, or *c)* no intersection and $\Phi_\alpha(t) < \Phi_\beta(t)$ for all $t \geq t_0$.

Now I show that *a)* and *b)* are not possible. These two cases are similar in that there exists some $\tau \geq t_0$ such that $\Phi_\alpha(t) > \Phi_\beta(t)$ for all $t \geq \tau$. Let K_τ denote the smallest integer in $I_{\overline{K}}$ such that $d_{K_\tau+1} > \tau$, and $d_{\overline{K}+1}$ denote $+\infty$. Integrating (10) from τ to $+\infty$ yields:

$$\begin{aligned}
lnF_\alpha(\Phi_\beta(\tau)) - lnF_\alpha(c) &< lnF_\alpha(\Phi_\alpha(\tau)) - lnF_\alpha(c) \\
&= \sum_{K=K_\tau}^{\overline{K}} \int_{\max\{d_K,\tau\}}^{d_{K+1}} g_K(t)dt - \frac{r}{c}\int_\tau^{+\infty} (\Phi_\alpha(t) - c)dt \\
&< \sum_{K=K_\tau}^{\overline{K}} \int_{\max\{d_K,\tau\}}^{d_{K+1}} g_K(t)dt - \frac{r}{c}\int_\tau^{+\infty} (\Phi_\beta(t) - c)dt \\
&= lnF_\beta(\Phi_\beta(\tau)) - lnF_\beta(c)
\end{aligned}$$

Denote $h(v) = lnF_\alpha(v) - lnF_\beta(v)$ and the inequation above can be written as $h(\Phi_\beta(\tau)) < h(c)$. However, the facts that $h^{'}(v) = f_\alpha(v)/F_\alpha(v) - f_\beta(v)/F_\beta(v) > 0$ and that $\Phi_\beta(\tau) > c$ together form a contradiction to this inequation. The only possibility left is $\Phi_\alpha(t) < \Phi_\beta(t)$ for all $t \geq t_0$.

(ii) Without loss of generality, I only discuss the case where $c_\alpha > c_\beta$. This condition indicates that there exists a t^* such that for all $t > t^*$ $\Phi_\alpha(t) > \Phi_\beta(t)$. If $\Phi_\alpha(.)$ and $\Phi_\beta(.)$ intersect on the left of t^*, let u denote the largest such point and thus $\Phi'_\alpha(u) > \Phi'_\beta(u)$. However, (10) gives that $\Phi'_\alpha(u) < \Phi'_\beta(u)$, a contradiction. Consequently, there is no intersction point on $[t_0, +\infty)$ and therefore $\Phi_\alpha > \Phi_\beta$ is consistent.

(iii) Without loss of generality, I only discuss the case where $r_\alpha > r_\beta$. If $\Phi_\alpha(.)$ and $\Phi_\beta(.)$ intersect at some point, say s, (10) gives that $\Phi'_\alpha(s) > \Phi'_\alpha(s)$. This property at an intersection narrows the discussion to the three cases mentioned in the proof of (i), that is *a)*, *b)*, and *c)*. Still, *a)* and *b)* are not possible. Both cases imply the existence of a $\tau \geq t_0$ such that $\Phi_\alpha(t) > \Phi_\beta(t)$ for all $t \geq \tau$. Integrating (10) in the same way as the proof of (i) yields $h(\Phi_\beta(\tau)) < h(c)$, where $h(.)$ is also defined previously. This inequation obviously contradicts the condition that $h^{'}(v) = 0$ for all $v \in (c, \Phi_\beta(\tau))$. Therefore, $\Phi_\alpha(t) < \Phi_\beta(t)$ for all $t \geq t_0$. *Q.E.D.*

6.5 Numerical Demonstration

6.5.1 For Theorem 3

To demonstrate Theorem 3 numerically, I consider a two-group AUD war where 4 players are divided equally into the two groups. Let parameters be $r = 1$, $c = 1$, and $\overline{v}_0 = 3$. By calculating the expected discount factor (EDF) under different $\overline{v}_1 \in [3, 5)$ and different $\underline{v}$, I obtain Table 1.

EDF \ $\overline{v}_1$; $\underline{v}$	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8
0.9	0.689	0.718	0.741	0.759	0.775	0.788	0.799	0.809	0.817	0.824
0.5	0.679	0.704	0.725	0.743	0.758	0.772	0.783	0.794	0.803	0.811
0	0.667	0.688	0.706	0.722	0.737	0.750	0.762	0.773	0.783	0.792
-2	0.608	0.620	0.631	0.642	0.653	0.663	0.672	0.681	0.690	0.698
-10	0.401	0.404	0.408	0.411	0.414	0.416	0.419	0.422	0.425	0.428
-100	0.073	0.073	0.074	0.074	0.074	0.074	0.074	0.074	0.074	0.074

Table 1

Note that no matter what lower bound is selected, expected discount factor is always increasing with respect to the larger upper bound, $\overline{v}_1$. In other words, in a relatively wide range of examples any introduction of asymmetry that strengthens the strongest type reduces cost of delay. This extends Theorem 3's scope of applicability.

6.5.2 For Theorem 4

To show an example that satisfies $e_N(u) - e_\infty(u)$ for all $u \in (c, \overline{v}_m]$, I consider a symmetric AUD war in which $r = 1$, $c = 2$, $\underline{v} = 1$, and $\overline{v}_m = 4$. By calculating the expected discount factor under different $u \in (2, 4]$ and different population $N \geq 2$, I obtain Table 2.

EDF \ u ; N	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
2	0.116	0.206	0.278	0.337	0.386	0.428	0.465	0.497	0.525	0.549
4	0.102	0.183	0.251	0.307	0.355	0.397	0.433	0.466	0.494	0.520
6	0.098	0.177	0.243	0.299	0.347	0.389	0.425	0.458	0.486	0.512
8	0.096	0.174	0.240	0.296	0.343	0.385	0.422	0.454	0.483	0.509
10	0.095	0.173	0.238	0.294	0.341	0.383	0.420	0.452	0.481	0.507
∞	0.091	0.167	0.231	0.286	0.333	0.375	0.412	0.444	0.474	0.500

Table 2

Table 2 indicates several interesting numerical regularities. First, for each finite N considered the expected discount factor is always greater than the infinite-population case under all us. According to Theorem 4, this regularity implies that any introduction of asymmetry to the

N-player (N=2,4,...,10) symmetric AUD war parametrized above improves efficiency. In addition, expected discount factor calculated under greater upper bound, which increases potential provision probability, is also greater. Finally, expected discount factor is decreasing with respect to population and this directly reveals the more-is-worse nature of this parametrization.

In contrast, I change the lower bound to $\underline{v} = -1$ and preserve all other parameters used above. Similarly, I obtain the result of a negative-lower-bound case in Table 3.

EDF \ u / N	2.2	2.4	2.6	2.8	3.0	3.2	3.4	3.6	3.8	4.0
2	0.074	0.139	0.196	0.246	0.290	0.330	0.366	0.398	0.428	0.454
4	0.082	0.152	0.213	0.266	0.313	0.354	0.391	0.423	0.453	0.480
6	0.085	0.157	0.219	0.273	0.320	0.361	0.398	0.431	0.461	0.488
8	0.086	0.159	0.222	0.276	0.323	0.365	0.402	0.435	0.464	0.491
10	0.087	0.161	0.224	0.278	0.326	0.367	0.404	0.437	0.466	0.493
∞	0.091	0.167	0.231	0.286	0.333	0.375	0.412	0.444	0.474	0.500

Table 3

Obviously, the first and the third regularities found in the previous case is inverse here. With a negative lower bound, expected discount factor is always increasing with respect to population N, and this suggests that the symmetric game always yields the highest welfare level.

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