Asymmetric War of Attrition: Analysis of Public Good Provision

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1 Introduction

When two native Hawaiians are standing under a coconut tree, on the one hand, knowing that both of them are thirsty and willing to climb up to pluck down a coconut which is big enough for quenching both sides' thirst, however, on the other, not knowing the exact extent of their opponents' eagerness and willingness, what will happen to these two individuals? Without a third party capable of holding full commitment, no mechanism eliciting truth from individuals about their values of the coconut is available, and therefore a war of attrition under imperfect information is inevitable. In this war, each Hawaiian chooses the length of waiting till providing according to its private information, the value. This story illustrates the natural form of public-good provision that represents the very proper and commonly seen cases in reality where commitment seems costly or even impossible.

The notion of public-good provision under imperfect information has a long history, but most relevant works only have investigated into symmetric settings in which each player's private information (e.g., value, cost, discount factor) is anticipated to be subjected to the same prior distribution. Nonetheless, I propose that asymmetric analysis is important, since it allows us to look into the most generic pattern of people's freeriding tendency; and, as what I will show you, the asymmetric variance engenders counterintuitive results of individual behavior and aggregate ex ante welfare. Besides, in the simplified version of this war of attrition where no uncertainty of value exists, multiple equilibria including pure and mixed ones are possible; however, the property of equilibrium uniqueness of my asymmetric game with uncertainty allows me to offer a novel way of equilibrium refinement of the former game, namely only one equilibrium of the certain version of war of attrition can be approximated by the unique solution of the uncertain version.

Section 2 introduces the model taking into consideration the cost of provision, the discount factor measuring impatience, and privately known values. Section 3 discusses the properties of solution, and now that the mathematical structure of my model is similar to that of Fudenberg and Tirole (1986), many conclusions characterizing the existence and uniqueness of equilibrium are directly borrowed from their established analysis, and I only supply necessary proofs and comments. Specifically, I elucidate some peculiar geometric characteristics of the solution and the insight of uniqueness property. Section 4 exhibits detailed discussion on comparative static analysis under both symmetric and asymmetric variance. I will explain the paradox that when both players are anticipated to have higher values, they tend to freeride consistently longer, whereas when one of the players is anticipated to have higher value than the other, the former tends to provide consistently sooner than the latter. The term *higher* represents the consistent dominance of one player's hazard rate over that of the other, and *consistently* means that the propensity is definite regardless of the type of players. Also, in section 4, I will provide an equilibrium refinement approach which indicates that, in the certain version of war of attrition, the only reasonable equilibrium turns out to be that the player with higher value provides immediately at the beginning while the other chooses to wait forever. Section 5 discusses the aggregate ex ante welfare of the two players, and compares the result of war of attrition with alternative settings, e.g., direct mechanism and coercive dictation. Section 6 concludes.

2 The Model

There are two individuals, $i = 1, 2$, facing a public-good provision problem. Both players participate in this game at time $t = 0$, and at every moment thereafter, each of them needs to decide whether he will provide at this time point or just continue to wait. The player, say i, knows the cost $c_i > 0$ of his provision, the rate $r_i > 0$ that he obeys to discount his gain according to an exponential function $e^{-r_i t}$, and his value $v_i > 0$ of the public good. For both sides, the costs c_i and discount rates r_i, $i = 1, 2$, are common knowledge, whereas each value v_i is private information anticipated by the opponent j as a random variable subjected to cumulative distribution function $F_i : [0, +\infty) \to [0, 1)$. I assume that F_i can yield a continuous and bounded dense function $f_i : [0, +\infty) \to \mathbb{R}^+$ which ensures the existence of v_i's variance, and also that v_1 and v_2 are extracted independently from these distributions. For convenience, I will call player i with private value v_i as player v_i.

Similar to Fudenberg and Tirole's (1986) discussion, I apply Harsanyi's

(1967) Bayesian equilibrium for analysis. Player i's strategy is a measurable function $T_i : [0, \infty) \to [0, \infty]$ which gives the time point of provision $T_i(v_i)$ as the pure strategy conditional on i's type v_i. Noticeably, only when no provision happens before $T_i(v_i)$ will player v_i provide at this point. If any provision happens, the game ends and both players gain lump-sum payoff, and if both sides choose to wait forever, they earn zero as reserve utility.

The expected gain of player v_i with a strategy choice of t_i fixing the opponent's strategy $T_j(.)$ can be written as:

$$\begin{aligned} R(t_i|v_i) =& Prob(T_j(v_j) \geq t_i)(v_i - c_i)e^{-r_i t_i} \\ &+ \int_{\{v_j|T_j(v_j)<t_i\}} v_i e^{-r_i T_j(v_j)} f_j(v_j) dv_j \end{aligned} \tag{1}$$

The first term above represents i's expected gain if j has not yielded until t_i, and the second presents the situation under which j provides before t_i. I define that strategy $(T_1(v_1), T_2(v_2))$ is a Bayesian equilibrium if for all $i = 1, 2$ and $v_i, t_i \in (0, +\infty)$, $R_i(T_i(v_i)|v_i, T_j(.)) \geq R_i(t_i|v_i, T_j(.))$. And as there is no need to consider players' behavior after a provision happens, any Bayesian equilibrium in this model is Bayesian-perfect.

3 Properties of Solution

In this section, I firstly list several geometric characteristics which offer a cursory glance of the solution's shape, the differential equations that an equilibrium has to obey, and also two critical boundary conditions ensuring the uniqueness of its solution. Then I show the existence of equilibrium and the uniqueness property based on the notion of *Relative Toughness* raised by Fudenberg and Tirole (1986); and I discuss over the intuition behind the uniqueness property that cannot be satisfied if the domain on which f_i allocates positive value does not overlap with $(-\infty, c_i)$ or include any large but finite point.

Lemma 1 *For* $\{T_1(v_1), T_2(v_2)\}$ *a Bayesian equilibrium:*

(i) $T_i(v_i)$ *are nonincreasing.*

(ii) $T_i(v_i) = +\infty$ *on* $[0, c_i)$, *and* $T_i(v_i) < +\infty$ *on* $(c_i, +\infty)$[1].

(iii) $\lim_{v_i \to +\infty} T_i(v_i) = 0$.

[1] At v_i=0, the player is indifferent between providing immediately and waiting forever, but this only corresponds to zero measure.

(iv) *There exists* $\overline{T} \geq 0$ *such that* $\lim_{v_1 \to c_1+0} T_1(v_1) = \lim_{v_2 \to c_2+0} T_2(v_2) = \overline{T}$. $\overline{T}$ *can be* $+\infty$.

(v) *There exist* $m_i \geq c_i$ *such that* $T_i(m_i) = 0$, $T_i(v_i) = 0$ *on* $[m_i, +\infty)$, *and* $T_i(v_i) > 0$ *on* $[0, m_i)$, $i = 1, 2$. *At least one* $m_i = +\infty$[2].

(vi) $T_i(v_i)$ *is continuous and strictly decreasing on* (c_i, m_i). *Thus* $T_i(v_i)$*'s inverse function* $\Phi_i(t_i)$ *exist and are continuous on* $(0, \overline{T})$

(vii) $\Phi_i(t)$ *is differentiable on* $(0, \overline{T})$ *and its first order derivative is given by:*

$$\Phi_i'(t) = -\frac{r_j}{c_j}(\Phi_j(t) - c_j)\frac{F_i(\Phi_i(t))}{f_i(\Phi_i(t))} \tag{2}$$

Proof of Lemma 1 The proof of Lemma 1's (i),(vi) and the differentiability of (vii) can be duplicated from Fudenberg and Tirole (1986) without much modification, so I omit these parts. Please see the proof of their Lemma 1's (i), (iii) and (iv) and you can easily figure out.

(ii) Obviously, the dominant strategy for player $v_i < c_i$ is $T_i(v_i) = +\infty$, since no matter what the opponent chooses, the provision of i yields negative gain while waiting forever brings zero. And for player $v_i > c_i$, $T_i(v_i)$ cannot be $+\infty$, since if so, player v_i can benefit by shifting provision time to a sufficiently large but finite point, say T_m. T_m is so large that there is a very little probability, say $q_\epsilon(T_m)$, of the rival's providing after T_m, which costs i less than $q_\epsilon e^{-r_i T_m}$; and on the other hand, since there is a considerable probability $q =$Prob$(v_j < c_j)$ of the rival's waiting forever, changing to T_m can increase the payoff by $q(v_i - c_i)e^{-r_i T_m}$. Clearly, there exists large but finite T_m such that $q(v_i - c_i)e^{-r_i T_m} > q_\epsilon e^{-r_i T_m}$, indicating T_m is more profitable than $T_i = +\infty$.

(iii) If there exists some $\epsilon > 0$ such that $T_i(v_i) \geq \epsilon$ for all $v_i \in [0, +\infty)$, there must be no such player v_j that $T_j(v_j) \in (0, \epsilon)$, because any of this kind can benefit by changing T_j to 0. Thus, no player will provide during $(0, \epsilon)$, however, this makes it possible for player $v_i > c_i/(1 - e^{-r_i \epsilon})$ to benefit by providing at $t = 0$. In this way, a contradiction emerges, and along with (i), we have $T_i(+\infty) = 0$.

(iv) Based on (i)-(iii), I only need to prove that, for finite situation $\overline{T} < +\infty$, we have $\lim_{v_1 \to c_1+0} T_1(v_1) = \lim_{v_2 \to c_2+0} T_2(v_2) = \overline{T}$. If not, namely one of the limit is greater than the other, say $\lim_{v_1 \to c_1+0} T_1(v_1) = \overline{T}_1 > \overline{T}_2 = \lim_{v_2 \to c_2+0} T_2(v_2)$, then any player v_1^* satisfying that $T_1(v_1^*) > \overline{T}_2$ can benefit by lowing its

[2] $m_i = +\infty$ implys that $T_i(v_i) > 0$ holds everywhere.

provision time to $\overline{T}_2$, for there is no possibility of provision by player 2 after this time point and waiting longer is only costly. Thus, a contradiction follows.

(v) Based on (i)-(iii) and the fact that $T_i \geq 0$, there is only necessity to prove that at least one $m_i = +\infty$. If m_i is finite, then there is strictly positive probability $q =$Prob$(v_i > m_i)$ of i's providing immediately. Given this, for all $v_j \in [0, +\infty)$, player v_j will always wait after $t = 0$, since if not, he can benefit by providing a little later at $t = \epsilon$ so that he gains by $qc_i/2$ and loses only $(v_i - c_i)(1 - e^{-r_i\epsilon}) \propto \epsilon$. This means that m_j has to be $+\infty$.

(vii) Now that (vi) holds, equation (1) is differentiable. On domain $(0, \overline{T})$, the optimality of strategy $T_i(v)$ and equivalently $\Phi_i(t)$ can be acquired by $dR(t_j|v_j)/dt_j = 0$ at $t_j = T_j(v_j)$. This gives equations (2) after algebraic calculation. ■

Here I supply the equations qualifying equilibrium in Lemma 1(vii) along with two boundary conditions that are shown in (ii), (iii), (iv) and (v) depicting the behavior of solution on the left side ($v_i = c_i$) and the right side ($v_i = +\infty$). Since the boundary conditions are not explicit and possibly locate on the infinity, the discussions that follow will be more difficult, and thus I will develop some solutary tricks and add necessary explanations and comments to proofs, even for those that can be borrowed directly from previous works.

What will be significant in all my later discussion is that the solution curves denoted by $T_i(v)$, or equivalently $\Phi_i(t)$, are *smoothly extensible*. This term is raised to allay the annoying uncertainty brought by (iv) and (v) in Lemma 1 which suggest that the boundary conditions on the four axes, $v_i = c_i$ and $t_i = 0$ for $i = 1, 2$, have much flexiblity as parameters like $\overline{T}$ and m_i can change. To avoid repeated discussions, I extend those curves, and each of them, say $T_i(v)$ or $\Phi_i(t)$, is comprised of three parts: (a) a half-line acquired by intercepting the line $v_i = c_i$ on domain $t \in [\overline{T}, +\infty)$; (b) another half-line acquired by intercepting the line $t_i = 0$ on domian $v \in [m_i, +\infty)$[3]; (c) the differentiable part linking $(c_i, \overline{T})$ and $(m_i, 0)$ which has been characterized in detail in Lemma 1. The purpose of doing this is that the extended solution is smooth and can be utilized in simplifing analysis. Specifically, it is easy to verify that the two points of junction of these three parts, $(c_i, \overline{T})$ and $(m_i, 0)$, are differentiable points of $\Phi_i(t)$ and $T_i(v)$ respectively, and this makes these two functions differentiable on $(0, +\infty)$ and $(c_i, +\infty)$ respectively. Besides, for function $\Phi_i(t)$, the equations (2)'s applicable domain is also extended to the entire $\mathbb{R}$. However, the smooth extension does not correspond to any

[3]If $\overline{T}$ or m_i is infinite, the corresponding part as a half-line should be omitted.

economic insight, since the half-line (a) has no practical meaning, and this should only be construed as a technical tool. In the subsequent discussion, if not particularly indicated, when speaking of extended form, I refer to these three parts of curves by (a), (b) and (c).

I propose the following conclusions as preparative lemmas for the proof of existence and uniqueness of equilibrium.

Lemma 2 *For equations (2) along with the boundary conditions* $(ii) - (v)$ *given by Lemma 1, we have:*

(i) If there is a solution $(\Phi_1(t), \Phi_2(t))$ *of (2) satisfying the boundary conditions, this solution corresponds to a perfect Bayesian equilibrium.*

(ii) The "Relative Toughness" holds. That is, for two distinct solutions of (2) satisfying Lemma 1, say $\{\Phi_1(t), \Phi_2(t)\}$ *and* $\{\hat{\Phi}_1(t), \hat{\Phi}_2(t)\}$ *whose overlapping domain is* $(0, T_m)$*, where* $T_m = \min(\overline{T}, \hat{\overline{T}})$*, we have* $(\Phi_1(t) - \hat{\Phi}_1(t))(\Phi_2(t) - \hat{\Phi}_2(t)) < 0$ *for all* $t \in (0, T_m)$*.*

Proof of Lemma 2 The proof of (i) is trivial. The proof of (ii) is similar to Fudenberg and Tirole's (1986) proof of their Lemma 3 only with subtle difference. We coincide in the first step of the proof: they firstly articulate that distinct solutions with the same subscript cannot intersect, that is, if for some $s \in (0, T_m)$ $\Phi_i(s) > \hat{\Phi}_i(t)$, we have $\Phi_i(t) > \hat{\Phi}_i(t)$ for all $t \in (0, T_m)$.

For the next step, I let some $\tau \in (0, T_m)$ exists such that, without loss of generality, $\Phi_1(\tau) > \hat{\Phi}_1(\tau)$ and $\Phi_2(\tau) > \hat{\Phi}_2(\tau)$. The intersection-free property gives that all Φs are consistently greater than $\hat{\Phi}$s on $(0, \tau)$. And there must be at least one pair Φ_i and $\hat{\Phi}_i$ such that $m_i, \hat{m}_i = +\infty$, because if not so, both $\hat{m}_1$ and $\hat{m}_2$ will be finite and this contradicts Lemma 1's (v). Without loss of generality, I let this pair be Φ_1 and $\hat{\Phi}_1$. Then consider some $u \in (0, \tau)$ and another curve $\tilde{\Phi}_1$ which is defined on $(0, u]$ by:

$$\begin{aligned} \tilde{\Phi}_1(u) &= \hat{\Phi}_1(u), \\ \tilde{\Phi}_1'(t) &= -\frac{r_2}{c_2}(\Phi_2(t) - c_2)\frac{F_1(\tilde{\Phi}_1(t))}{f_1(\tilde{\Phi}_1(t))}, \end{aligned}$$

which gives $\tilde{\Phi}_1$ the same equation as Φ_1 but with a lower initial value $\tilde{\Phi}_1(u) < \Phi_1(u)$. The same equations indicate:

$$\frac{\tilde{\Phi}_1'(t) f_1(\tilde{\Phi}_1(t))}{F_1(\tilde{\Phi}_1(t))} = \frac{\Phi_1'(t) f_1(\Phi_1(t))}{F_1(\Phi_1(t))}, \ for\ all\ t \in (0, u].$$

Integrate this equation with respect to t, and we get the linear relationship which is $F_1(\Phi_1(t))/F_1(\tilde{\Phi}(t)) = K = F_1(\Phi_1(u))/F_1(\tilde{\Phi}(u)) > 1$.

By replicating Fudenberg and Tirole's (1986) method, I also derive the relation that $\hat{\Phi}_1(t) < \tilde{\Phi}_1(t) < \Phi_1(t)$ on $(0, u)$. Since $m_1, \hat{m}_1 = +\infty$, $\hat{\Phi}_1(t)$ and $\Phi_1(t)$ both hike up to positive infinity with t approaching 0 from the right side, and this leads to $\lim_{t \to 0+0} \tilde{\Phi}_1(t) \to +\infty$ as well.

Here I can provide the contradiction. On the one hand, we have $F_1(\Phi_1(t))/F_1(\tilde{\Phi}(t)) = K > 1$; but on the other, there is a small $\epsilon > 0$ such that $1/(1-\epsilon) < K$, and also a sufficiently small $\delta > 0$ such that $\tilde{\Phi}(\delta)$ can be very large and $F_1(\tilde{\Phi}(\delta))$ very close to 1, namely, $1 > F_1(\tilde{\Phi}(\delta)) > 1 - \epsilon$. Then we have $F_1(\Phi_1(\delta))/F_1(\tilde{\Phi}(\delta)) < 1/(1-\epsilon) < K$, a contradiction. ■

This toughness property reveals the rivaling nature of this war of attrition. When one side of a solution goes higher, or equivalently one player tends to provide later, the other player will realize the opponent is freeriding more and he has to offset the loss by providing sooner himself. In the reverse, when one tends to provide sooner, the other will seize the opportunity to freeride. It seems that the opportunity cost of waiting and the freeriding propensity form the intrinsic countervailing forces that drive the strategy of both sides, and therefore we may anticipate the existence and uniqueness of the equilibrium.

Proposition 1 *Founded on Lemma 1 and Lemma 2, I propose:*

(i) There exists at least one such perfect Bayesian equilibrium.

(ii) The equilibrium is unique.

Proof of Proposition 1 Please see Fudenberg and Tirole's (1986) proof of the existence in their Appendix B, which is extremly technical but holds the same here. The uniqueness is proved with both the toughness property and the smooth extensibility.

If there are two distinct solutions of (2) satisfying boundary conditions in Lemma 1, say $(\Phi_1(t), \Phi_2(t))$ and $(\hat{\Phi}_1(t), \hat{\Phi}_2(t))$, according to Lemma 2's (ii), we have that, without loss of generality, $\Phi_1 > \hat{\Phi}_1$ while $\Phi_2 < \hat{\Phi}_2$ on $(0, T_m)$ where $T_m = \min(\overline{T}, \hat{\overline{T}})$. Now let Φ^e and $\hat{\Phi}^e$ denote the extended solution of Φ and $\hat{\Phi}$, and each of this extended curve contains two parts of the half-line (a) and the smooth curve (c). As explained before, the extended form also satisfies equations (2), and therefore we have:

$$\frac{dln(F_i(\Phi_i^e(t)))}{dt} = -\frac{r_j}{c_j}(\Phi_j^e(t) - c_j), \ \textit{for all } t \in (0, +\infty)$$

and it holds the same if Φ is replaced by $\hat{\Phi}$. Integrate the equation above under $i=1$ with respect to t on (s,τ) where $s \in (0, T_m)$, and we get:

$$\begin{aligned} ln(F_1(\Phi_1(s))) - ln(F_1(\Phi_1^e(\tau))) &= \int_s^\tau \frac{r_2}{c_2}(\Phi_2^e(u) - c_2)du \\ &< \int_s^\tau \frac{r_2}{c_2}(\hat{\Phi}_2^e(u) - c_2)du \\ &= ln(F_1(\hat{\Phi}_1(s))) - ln(F_1(\hat{\Phi}_1^e(\tau))), \end{aligned}$$

Now let τ move to the positive infinity, and the inequation above becomes $ln(F_1(\Phi_1(s))) < ln(F_1(\hat{\Phi}_1(s)))$ indicating that $\Phi_1(s) < \hat{\Phi}_i(s)$, a contradiction to the toughness property.

Along with Lemma 2's (i), all above gives the existence and uniqueness of a perfect Bayesian equilibrium. ■

The equilibrium uniqueness is not trivial, because when the domain on which f_i allocates positive value does not overlap with $(-\infty, c_i)$, or equivalently when there is no possibility of occurance of a player whose dominant strategy is to wait forever, this property will not hold. Fudenberg and Tirole (1986) also pointed out this critical premise that uniqueness requires that players' prior belief contains types which choose the stop time at infinity, and they compared their setting with that of Riley (1980) to illustrate this.

Specifically, in the case with dense functions $f(.)$s defined on a shrinked domain, there are at least two salient equilibria that player i always provides immediately and player j waits forever, for all $i = 1, 2$. And the reason why these two cannot exist in my model is that player j faces a positive probabilty of i's waiting forever if he has value $v_i < c_i$, and therefore j is inclined to alter his strategy to offset this loss. Actually, there will be a continuum of equilibria in this shrinked game, because the boundary condition given by Lemma 1's (iv) can be substituted with a continuum set of alternatives, that is, a maximal time of waiting $\overline{T}$ is also needed but $\overline{T}$ will not be only fixed on axis $v_i = c_i$ but on any axes $v_i = v \in [c_i, a_i]$ where a_i is the infimum of f_i's domain.

As Fudenberg and Tirole (1986) have suggested that they "picked out a unique equilibrium by a small perturbation of the information structure", which coincides with Binmore's (1981) opinion, the occurance of population that only freerides serves as the perturbation to refine the equilibrium set of my model. This spirit of refinement is similar to Selten's (1975) and Kreps and Wilson's (1982) method of considering a totally mixed alteration to check if the off-path beliefs also yield rationality; and in my model, instead of looking into totally mixed strategy as they did, the range of prior anticipation of players is set to be wide enough to cover the entire $\mathbb{R}^+$, and the off-path beliefs corresponding to the arbitrary selection of boundary conditions on $[c_i, a_i]$ are examined and only the condition in Lemma 1 is qualified.

4 Comparative Statics

In this section, I firstly present some symmetric comparative statics involving the variance of cost, discount rate and prior distribution. Namely, by holding identical on both sides' parameters and information structure, I derive the variance in the pairs of solutions $\Phi_1 = \Phi_2 = \Phi$ and $\hat{\Phi}_1 = \hat{\Phi}_2 = \hat{\Phi}$ given by two distinct equations (2). Next I consider asymmetric comparative statics, that is, I investigate into the difference between the strategies, say $\Phi_1 \neq \Phi_2$, given by the same (2) which takes but an asymmetric setting.

Finally, I propose the refinement strategy that a sequence of random value pairs $\{(\hat{v}_1, \hat{v}_2)\}$, in which $\hat{v}_i$ represents the value of player i as a random variable for j and has a real limit $v_i^0 > c_i$ such that $\hat{v}_i = v_i^0$ *a.s.*, corresponds to a sequence of equilibria $\{(\Phi_1, \Phi_2)\}$ converging to some (Φ_1^0, Φ_2^0). Specifically, (Φ_1^0, Φ_2^0) is one of the equilibria of the public-provision game where player i has value v_i^0 which is common knowledge, or equivalently this is the certain version of the model depicted in Section 2. Thus, taking advantage of the uniqueness property of my model with *total uncertainty*, I find the most reasonable pure strategy equilibrium for the certain version.

4.1 Symmetric Analysis

For two pairs of equations (2) between which one of parameters or the information structure is distinct, we respectively have two unique perfect Bayesian equilibria denoted as (Φ_1, Φ_2) and $(\hat{\Phi}_1, \hat{\Phi}_2)$. Some preparative points need to be mentioned: (i) due to the uniqueness property, the strategies of both players subjuected to the same equations are identical, because if not, we can get a second equilibrium by interchanging the distinct strategies, and therefore I omit all the subscripts of symmetric game; (ii) based on Lemma 1's (v) and point (i) here, we have $m_1 = m_2 = +\infty$ which means both Φ_i goes to infinity with t approaching 0 from the right side; (iii) in the following proposition, I employ variable-controling method, namely when I say, e.g., two equations are distinct in cost, I mean there is only difference between two costs and all other parts are controlled to be identical; (iv) I let the statement that "players facing higher parameter provide sooner" mean that when Φ and $\hat{\Phi}$ respectively correspond to equations with higher and lower parameters, then for all $t \in (0, T_m)$ we have $\Phi(t) < \hat{\Phi}(t)$, where $T_m = \min(\overline{T}, \hat{\overline{T}})$, and any statement alike should be construed in this way. And now I can present the results.

Proposition 2 *Founded on notations defined above, we have: (i) players facing higher cost provide later; (ii) players who are less impatient provide sooner; (iii) players who anticipate each other with "higher" value distribution provide later, where "higher" is defined as the consistent dominance of hazard rate*[4] *function $f(v)/F(v)$.*

Proof of Proposition 2 Fudenberg and Tirole (1986) also provided several symmetric comparative statics in similar form, but they proved it in a different manner. Instead of Φ, I consider its reverse function T and I provide the differential equation of which T is the unique solution:

$$T^{'}(v) = -\frac{f(v)}{F(v)}\frac{1}{r}\frac{1}{\frac{v}{c}-1} \tag{3}$$

This equation is directly acquired from (2), and because of Φ's good mathematical properties and that $m = +\infty$, $T(v)$ is absolutely continuous and strictly decreasing on $(c, +\infty)$. Clearly, equation (3) perfectly separates the effects of cost, discount rate and hazard rate.

For clause (i), supposing $c > \hat{c}$, we have $T^{'}(v) < \hat{T}^{'}(v)$ on $(c_m, +\infty)$ where $c_m = \max(c, \hat{c})$. Then integrate (3) with respect to v from some v^* to $\overline{v} > v^*$, and we get $T(\overline{v}) - T(v^*) < \hat{T}(\overline{v}) - \hat{T}(v^*)$. By letting $\overline{v}$ goes to positive infinity, $T(\overline{v}) = \hat{T}(\overline{v}) = 0$ gauranteed by Lemma 1's (iii) and point (b) mentioned at the beginning of this section, and then we find $T(v^*) > \hat{T}(v^*)$ for all $v^* \in (c_m, +\infty)$. And reaily we can verify that $\Phi(t) > \hat{\Phi}(t)$ consistently on $(0, T_m)$. By this means, similar proofs of (ii) and (iii) can be given.

However, I have to show that the improper integral $\int_v^{+\infty} T^{'}(u)du$ makes sense[5]. As what I have assumed, $f(v)$ warrants the existence of v's variance, and this implys that there exists a large A, a small ϵ and a bounded constant C such that for all $v > A$ we have $f(v) < C/(v-c)$ and $F(v) > 1-\epsilon$, since if not the existence of variance cannot hold. This generates a bounded control function $g(v)$ of (3):

$$|T^{'}(v)| = \frac{f(v)}{F(v)}\frac{1}{r}\frac{1}{\frac{v}{c}-1} < \frac{C}{r(1-\epsilon)}\frac{1}{(v-c)^2} = g(v).$$

Obviously, the improper interal of $g(v)$ is bounded, and therefore $T(v^*) = -\int_{v^*}^{+\infty} T^{'}(u)du$ makes sense and all analysis above is substantiated. ■

[4]In other literature, hazard rate is defined as $\frac{f(v)}{1-F(v)}$, but here I define it differently only for simplicity.

[5]Even though one can easily see this point in an economic way, I still find it necessary to provide the mathematical explanation.

The economic insight of Proposition 2's first two results is straightforward, since higher cost marks higher difficulty of providing and higher discount rate indicates higher degree of impatience, or equivalently higher opportunity cost. The propensity of evading cost is explicit here.

The third concludes that when players allocate more probability to occurance of opponents with higher value, namely for all $v \in \mathbb{R}^+$ we have $f(v)/F(v) > \hat{f}(v)/\hat{F}(v)$, both sides tend to wait longer. As suggested before, each player faces the dilemma that he can either provide sooner to reduce opportunity cost or wait longer to freeride. Nonetheless, according to analysis above, it seems astounding that in symmetric setting, we observe only a consistent inclination of freeriding if anticipated value is "higher". To intepret this, one may see the hazard rate function of player j, $h_i(v) = f_j(v)/F_j(v)$, as player i's *Right to Freeride*. This notion indicates that when $h_i(v)$ is consistently higher, player i is stimulated by the "high" reputation of player j to freeride. We will see similar function of this concept in the following asymmetric analysis.

4.2 Asymmetric Analysis

I now only consider variation within one pair of equations (2) whose unique solution is given as (Φ_1, Φ_2). Similar to the previous subsection, I also apply variable-controling method; and with a little modification, I let the statement that "player facing higher parameter provides sooner" mean that when, without loss of generality, player 1 faces a higher parameter than player 2, then for all $t \in (0, \overline{T})$ we have $\Phi_1(t) < \Phi_2(t)$. I offer the following results.

Proposition 3 *Comparing both sides' strategies under asymmetric setting, we have: (i) player facing higher cost provides later; (ii) player who is less impatient provides sooner; (iii) player who is anticipated to be likely to value higher, namely its hazard rate $f_i(v)/F_i(v)$ consistently dominates that of the other player, provides sooner.*

Proof of Proposition 3 (i) Providing $c_1 > c_2$, I denote the extended solutions of equations (2) as $(\Phi_1^e(t), \Phi_2^e(t))$ and each of them consists of two parts of the half-line (a) and the smooth curve (c). And the equations that the solutions obey are given as:

$$\begin{aligned}
\Phi_1^{e'}(t) &= -r\left(\frac{\Phi_2^e(t)}{c_2} - 1\right)\frac{F(\Phi_1^e(t))}{f(\Phi_1^e(t))} \\
\Phi_2^{e'}(t) &= -r\left(\frac{\Phi_1^e(t)}{c_1} - 1\right)\frac{F(\Phi_2^e(t))}{f(\Phi_2^e(t))}, \ \textit{for all } t \in (0, +\infty).
\end{aligned} \tag{4}$$

Since $\Phi_i^e(t)$ converges to c_i with t approaching positive infinity, there exist a large t^* such that for all $t \in [t^*, +\infty)$ we have $\Phi_1^e(t) > \Phi_2^e(t)$.

Now, let Φ_1^e and Φ_2^e intersect on $(0, t^*)$. Then denote the intersection point nearest to t^* as s, and due to continuity of Φ^es, for u slightly larger than s, $\Phi_1^e(u) > \Phi_2^e(u)$ and therefore we have $\Phi_1^{e'}(s) > \Phi_2^{e'}(s)$. Nonetheless, equations (4) indicates that, at an intersection point, $\Phi_1^{e'}(s) > \Phi_2^{e'}(s)$, which gives a contradiction. Hence, there is no intersection on $\mathbb{R}^+$ and $\Phi_1 > \Phi_2$ consistently.

(ii) Providing $r_1 > r_2$, I also utilize the extended solutions $(\Phi_1^e(t), \Phi_2^e(t))$. The equations (2) are rewritten as the following integrated form:

$$\begin{aligned}
\Phi_1^{e'}(t) &= -r_2(\frac{\Phi_2^e(t)}{c} - 1)\frac{F(\Phi_1^e(t))}{f(\Phi_1^e(t))} \\
\Phi_2^{e'}(t) &= -r_1(\frac{\Phi_1^e(t)}{c} - 1)\frac{F(\Phi_2^e(t))}{f(\Phi_2^e(t))},\ for\ all\ t \in (0, +\infty).
\end{aligned} \tag{5}$$

Noticeably, if $\overline{T}$ is finite, $\Phi_1(\overline{T}) = \Phi_2(\overline{T}) = c$, and if not, both solutions converge to c at positive infinity.

Now, let some point $s \in (0, \overline{T})$ exists such that $\Phi_1(s) > \Phi_2(s)$. Then on the right side of point s, $\Phi_1^e(t) \geq \Phi_2^e(t)$ consistently on $(s, +\infty)$, because if not, there must be some intersection points on $(s, \overline{T})$ and we have the intersection point nearest to s denoted as t'. Since Φs are continuous and $\Phi_1(s) > \Phi_2(s)$, then for any u slightly less than t', $\Phi_1(u) > \Phi_2(u)$ and therefore we have $\Phi_1'(t') < \Phi_2'(t')$. However, this contradicts to the result given by equations (5) that, at an intersection point, $\Phi_1'(t') > \Phi_2'(t')$. Consequently, $\Phi_1^e(t) \geq \Phi_2^e(t)$ on $(s, +\infty)$.

Along with the deduction above, at point s, the integrated version of equations (5) gives the following relation:

$$\begin{aligned}
lnF(\Phi_1(s)) = lnF(\Phi_1^e(s)) &= lnF(c) + \int_s^{+\infty} r_2(\Phi_2^e(\tau)/c - 1)d\tau \\
&\leq lnF(c) + \int_s^{+\infty} r_1(\Phi_1^e(\tau)/c - 1)d\tau \\
&= lnF(\Phi_2^e(s)) = lnF(\Phi_2(s)),
\end{aligned}$$

which gives that $\Phi_1(s) \leq \Phi_2(s)$ contradicting to the definition of s. In this way, there is no such s; and also there is no point $s^* \in (0, \overline{T})$ such that $\Phi_1(s^*) = \Phi_2(s^*)$, because if so, $\Phi_1'(s^*) > \Phi_2'(s^*)$ and for any u slightly larger than s^* we have $\Phi_1(u) > \Phi_2(u)$, which is not possible. So for all $t \in (0, \overline{T})$, $\Phi_1 < \Phi_2$ consistently.

(iii) Providing $f_1(v)/F_1(v) > f_2(v)/F_2(v)$ for all $v \in \mathbb{R}^+$. The extended solutions obey the following equations:

$$\begin{aligned}
\Phi_1^{e'}(t) &= -r(\frac{\Phi_2^e(t)}{c} - 1)\frac{F_1(\Phi_1^e(t))}{f_1(\Phi_1^e(t))} \\
\Phi_2^{e'}(t) &= -r(\frac{\Phi_1^e(t)}{c} - 1)\frac{F_2(\Phi_2^e(t))}{f_2(\Phi_2^e(t))}, \ for\ all\ t \in (0, +\infty).
\end{aligned} \tag{6}$$

Similar to the proof of (ii), we let some point $s \in (0, \overline{T})$ exists such that $\Phi_1(s) > \Phi_2(s)$. Since (6) suggests that at any intersection point on $(s, \overline{T})$, say $t^{'}$, $\Phi_1^{'}(t^{'}) > \Phi_2^{'}(t^{'})$, the analysis from (ii) gaurantees that $\Phi_1^e(t) \geq \Phi_2^e(t)$ on $(s, +\infty)$.

Given that $\Phi_1(s) > \Phi_2(s) > c$, integrating equations (6) yields:

$$\begin{aligned}
lnF_2(\Phi_1(s)) - lnF_2(c) > lnF_2(\Phi_2^e(s)) - lnF_2(c) &= \int_s^{+\infty} r(\frac{\Phi_1^e(u)}{c} - 1)du \\
&\geq \int_s^{+\infty} r(\frac{\Phi_2^e(u)}{c} - 1)du = lnF_1(\Phi_1^e(s)) - lnF_1(c) \\
&= lnF_1(\Phi_1(s)) - lnF_1(c),
\end{aligned}$$

or equivalently, the inequality above can be written as $L(\Phi_1(s)) < L(c)$ where L is defined as $L(x) = lnF_1(x) - lnF_2(x)$. However, $L(.)$ is a strictly increasing function, as $L^{'}(x) = f_1(x)/F_1(x) - f_2(x)/F_2(x) > 0$, and this gives that $L(\Phi_1(s)) > L(c)$, a contradiction. Therefore, there is no such s, and for similar reason in (ii), $\Phi_1 < \Phi_2$ consistently.