Robust Equilibrium

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I. Introduction

Literature of Refinement. Selten(1975 [1]) proposed the very important concept of perfect equilibrium, also known as *trembling hand* equilibrium, to refine from solution given by nash's(1950 [2]) equilibrium. He presented that the robustness of extensive game equilibria need to incorporate some perturbation examination, namely, an equilibrium is perfect if for all players i, there exist some totally mixed paths of their opponents' strategies[1 2] that keep i's strategy optimal. To illustrate this, consider the game in Figure 1 where there exist two nash equilibria (U,L) and (D,R)[3]. It is reasonable to question the second equilibrium (D,R) for lack of robustness, namely, if player 2 diverts even slightly from strategy R and thus allocates positive probability on L, strategy D is no longer the best reaction of player 1. On the contrary, (U,L) can endure any trembling of each player and therefore becomes the unique perfect equilibrium.

1, 1	0, 0
0, 0	0, 0

Fig. 1

Nonetheless, Kreps and Wilson(1982a [4]) pointed out the indirect nature of Selten's definition, for perfect equilibrium refines game solution actually by rationalizing off-path belief but not necessarily by imposing trembling perturbations on equilibria. They developed the concept of sequential equilibrium defined directly upon the spirits of robustness of off-path belief, and this definition from an absolutely distinct angle remains equivalent to Selten's under very weak conditions[4]. Enlightened by pioneer works, scholars have elaborated game refinement within similar framework and discussed many other equilibrium concepts, e.g. proper equilibrium by Myerson(1978 [5]), strategically stable equilibrium by Kohlberg and Mertens(1986 [6]) and else[5].

Reexamination of Trembling. The examination of off-path consistency seems to dominate, whereas the initial trial of *trembling* method is obscured. Actually, the trembling hand equilibrium is not necessarily about trembling. Consider the game in Figure 2 where σ_0=($\frac{1}{2}$U+$\frac{1}{2}$D,$\frac{1}{2}$L+$\frac{1}{2}$R) is the only nash equilibrium and also perfect. The perfectness lies in the existence of a totally mixed sequence of

1, -1	-1, 1
-1, 1	1, -1

Fig. 2

[1]The notations used in this proposal accord with that in *Game Theory* by Fudenberg and Tirole(1991 [3]).

[2]Such path means a sequence converging to player i's opponents' strategies, say $\{\sigma_{-i}^{(n)} \to \sigma_{-i}\}$, of which any action at any information set allocates strictly positive probability to every pure action at that information set. See Selten(1975 [1]).

[3]For simplicity, in this proposal I denote the strategy of 2×2 matrix game in the same way that the left number within each box is the utility gained by player 1 when the event of this box happens, while the right number utility of player 2; and the two pure strategies of player 1 are denoted as U(up) and D(down) corresponding to the rows, while that of player 2 L(left) and R(right) corresponding to the columns. The strategy of both players is given by a tuple.

[4]Specifically, a perfect equilibrium is sequential, but the converse is not true; however, for generic games the two concepts coincide. See Kreps and Wilson(1982a [4])

[5]For more instances, persistent equilibrium by Kalai and Samet(1984 [7]), justifiable equilibrium by McLennan(1985 [8]) and explicable equilibrium by Reny(1992 [9]). See Kohlberg [10] for a detailed review of game refinement.

strategy that converges to σ_0 and maintains the optimality of it. However, the only qualified sequence turns out to be $\{\sigma^{(n)} = \sigma_0\}$, that is, the "trembling" used to justify this equilibrium is exactly a constant sequence of itself which fails to accord with what we commonly mean by *trembling*. And consequently, the literature of refinement neglects the approach of perturbation examination in the first place.

What I intend to point out is that definition of equilibrium robustness ought to incorporate trembling method rather than omitting it. In the field of nonlinear system[6], the stability of a specific equilibrium is examined by generating arbitrary perturbation within some small neighborhood and checking whether the perturbed system has the inclination to regress. Thus, we can observe the necessity and feasibility of deriving a trembling approach in game refinement. Such an approach contains two steps: first, to implement perturbation according to certain rules; and second, to examine whether the perturbation entails inclination of regression. Besides, to make the concept robustness reasonable, namely, to guarantee the existence and the close connection with nash equilibrium of the new definition, the rules of perturbation need to be regularized and regression inclination well-defined.

Equilibrium Robustness. The first step of trembling approach mentioned above can be expressed as follows: a perturbation from player i's strategy σ_i toward some $\hat{\sigma}_i$ in some strategy set P_i with an extent of $\lambda \in (0,1)$ is denoted as $\sigma_i^{(p)} = (1-\lambda)\sigma_i + \lambda\hat{\sigma}_i$. Here the set P_i corresponds to the rules of perturbation, which I will discuss later.

On the other hand, I define the regression inclination based on player's *anticipation* of others. It is undeniable that if the anticipation of opponents' strategy is given, that of this player will be almost surely definite, and this lead to the idea that we need to check the robustness of every player's anticipation of opponents before deciding that of the equilibrium. As for player i's anticipation, say σ_{-i}, if any perturbation of it according to P_{-i} is "unreasonable", we call σ_{-i} a robust anticipation, which probably makes up part of a robust equilibrium.

[6]e.g., see Khalil(2002 [11])

Actually, this angle of anticipation is also contained in nash equilibrium. In other words, when player i anticipates others' strategy σ_{-i}, i will perform best reaction $r_i(\sigma_{-i})$ to it, and if for all players i there exist some strategy, say $\sigma_i \in r_i(\sigma_{-i})$, such that: given i's choosing σ_i, there is no space of improvement for any perturbation of σ_{-i} toward any direction $\hat{\sigma}_{-i} \in P_{-i}$ with some extent $\lambda \in (0, \epsilon)$, we call ($\sigma_i$,$\sigma_{-i}$) is a nash equilibrium.

My definition of robust equilibrium is an improved version of nash equilibrium in such interpretation of anticipation. Clearly, there is a flaw occurring in nash's definition depicted in the last paragraph that it neglects to consider update of best reaction corresponding to perturbation. Namely, when player i supposes a slight deviation from its initial belief σ_{-i}, it should take on a different best reaction, that is, the best reaction to $\sigma_{-i}^{(p)}$. Hence I provide the intuitive definition of robust anticipation: an anticipation σ_{-i} of player i is robust, if there is no profitable improvement for any perturbation direction $\hat{\sigma}_{-i} \in P_{-i}$ with any extent $\lambda \in (0,1)$ under *immediate best reaction*[7].

We can construe the insight of this definition better by revisiting the game in Figure 1. Nash equilibrium fail in this game for an unreasonable point (D,R) is included in the nash solution. Here I show that the anticipations in this strategy are both not robust: as for player 1's anticipation σ_2=R, if we allow a deviation toward L with amplitude $\lambda \in (0,1)$ and the anticipation becomes λL+(1-λ)R, the immediate best reaction of player 1 will be U, and given this reaction, the utility gained by player 2 is $u_2 = \lambda$ which is strictly increasing with respect to λ. Therefore, the perturbation toward L is justified and player 1's original anticipation is not robust. Likewise, player 2's anticipation is not robust in (D,R).

The imperative problem here is to qualify the definition by specifying the perturbation direction set P_i and the concept of regression inclination. I will present the technical definition of these issues and robust equilibrium in the next section, and also justify them in detail.

[7]I will use this term in the following of the proposal to refer to player i's updating its best reaction according to the perturbation of anticipation within its mind.

II. Robust Equilibrium

In the anticipation interpretation of nash equilibrium in the last section, for lack of the strong condition of immediate reaction, perturbation can be implemented toward arbitrary direction with any extent. However, this cannot hold for a well-defined robust equilibrium, namely, restrictions on both the direction and amplitude of perturbation. Otherwise, the equilibrium may not even exist.

First, perturbation need to be infinitesimal. Consider the game in Figure 3 where (U,L), (D,R) and ($\frac{1}{2}$U+$\frac{1}{2}$D,$\frac{1}{2}$L+$\frac{1}{2}$R) are all three nash equilibria. We

1, 1	0, 0
0, 0	1, 1

Fig. 3

examine the robustness of anticipation of player 1 in the first equilibrium with perturbation direction $\hat{\sigma}_2$=R and amplitude $\lambda \in (\frac{1}{2}, 1)$. The best reaction of player 1 is updated to D under such circumstances, and this renders player 2's payoff function strictly increasing with respect to λ, indicating that player 1's anticipation appears to be not robust. In the same way, we can readily find sufficiently large λ to undermine every one of the three equilibria and thus no outcome exists to be qualified. The insight of requiring perturbation to be infinitesimal lies in the fact that the information needed to decide whether one strategy point is robust is completely included in some small neighborhood about this point, and if the amplitude becomes redundantly large, the information on other points' robustness will come into confounding.

Second, the direction of perturbation should be restricted within *rational* ones. We shall see the game in Figure 4 in which player 1 has three pure strategy A, B and C while player 2 still gets L and R. Apparently, (C,R) is the unique nash equilibrium. Nonetheless, if we suppose a perturbation of player 2 in this equilibrium toward $\hat{\sigma}_1$=B with infinitesimal amplitude, the best reaction of player 2 will be updated to L and this engenders further deviation from initial anticipation. Actually, without limitation on direction, no robust equilibrium will exist in this game. The reason for this is that

A	2, -1	-2, 0
B	1, 1	-1, 0
C	0, 0	0, 0

Fig. 4

player 1's second pure strategy B is not included in rationalizable strategies[8], which firstly raised by Bernheim(1984 [12]) and Pearce(1984 [13]). The rationalizable strategy set $\times_i R_i$ forms a rational *core* of the game in which any strategy of any player is probable to be optimal and thus constructs an equilibrium. Based on this concern, I simply define the perturbation direction set as $P_{-i} = \times_{j\in -i} R_j$, that is, I assume that rational players only perturb their anticipation of opponents in those "probable" directions.

Now I provide the definition of robustness. But before this, I need to denote several functions for convenience. Let $u_i(\sigma_i, \sigma_{-i})$ be the payoff function of player i under strategy profile (σ_i, σ_{-i}), and $r_i(\sigma_{-i})$ be the best reaction function of player i given opponent strategy σ_{-i}. It is worth noticing that r_i is a set map. Now I give the definition of General Decision Function(GDF).

Definition 1: The general decision function of player j anticipating player i given σ_{-ij} is defined as $g_i^j(\sigma_i|\sigma_{-ij}) = \max_{\sigma_j \in r_j(\sigma_i, \sigma_{-ij})} u_i(\sigma_i, \sigma_j, \sigma_{-ij})$. This function represents the maximal payoff of i when it performs σ_i, given j's immediate reaction and conditional on other players' strategy σ_{-ij}.

The connection between GDF and the robustness I elaborated previously is overt: when player j anticipates the behavior of player i with immediate

[8] I provide the technical definition here: Let Σ_i represents the mixed strategy space of player i and u_i the payoff gained by i. Set $\widetilde{\Sigma}_i^0 = \Sigma_i$, and for each i recursively define

$$\widetilde{\Sigma}_i^n = \Big\{\sigma_i \in \widetilde{\Sigma}_i^{n-1} | \exists \sigma_{-i} \in \underset{j\neq i}{\times}\, convex\ hull(\widetilde{\Sigma}_i^{n-1})\ such\ that$$
$$u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i^{'}, \sigma_{-i})\ for\ all\ \sigma_i^{'} \in \widetilde{\Sigma}_i^{n-1}\Big\}.$$

The rationalizable strategies for player i are $R_i = \cap_{n=0}^{\infty} \widetilde{\Sigma}_i^n$.

reaction, i's payoff change with respect to its anticipated strategy σ_i corresponds to the slope of hyperplanes of GDF. Along with the requirement that perturbation need to be infinitesimal, the robustness of a specific point is determined by the geometric characteristics of GDF within a (deleted) local neighborhood of this point. Besides, the derivative value of GDF can serve as proxy of regression inclination. With all these preparations, I give the definition of robust anticipation.

Definition 2: Let $P_k = R_k$. Player j's anticipation σ_i^* on player i conditional on others' strategy σ_{-ij} is robust, if σ_i^* is a deleted maximal point of GDF on the domain $P = \times_k P_k$; equivalently, if $\sigma_i \in P_i$, $\sigma_{-ij} \in P_{-ij}$ and r_j is defined on P_j, and if for all $\sigma_i \in P_i$, there exists $\epsilon \in (0,1]$ such that function $h(\lambda) = g_i^j((1-\lambda)\sigma_i^* + \lambda\sigma_i|\sigma_{-ij})$ is decreasing on $\lambda \in (0,\epsilon)$.

The mathematical definition of robustness is rather obscure, so I present here some catchy examples. The first example is the game in Figure 3: if the probability player 1 allocates on U is denoted as p_1, the GDF(g_1) of player 2 can be depicted as in Figure 5. We can see that only

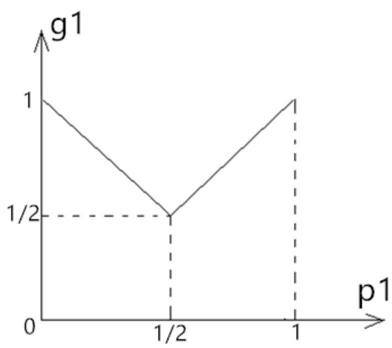

Fig. 5

$p_1 = 0$ and $p_1 = 1$ are deleted maximal points[9], and therefore are robust anticipations which respectively make up two nash equilibria (D,R) and (U,L). What is interesting is that the third nash equilibrium ($\frac{1}{2}$U+$\frac{1}{2}$D,$\frac{1}{2}$L+$\frac{1}{2}$R) is not robust, and it even appears to be the "least" robust strategy for i gains the least at this point. This example provides a novel and necessary examination on mixed strategy, since mixed equilibria are usually derived by "balancing" opponents' payoff of each of their pure strategies, which seems to have little or implicit connection with individual optimization. Specifically, when a mixed nash equilibrium lies between some pure equilibria, it tend to be not robust at all.

Another example is as well intriguing. Revisit the game in Figure 2 and also denote p_1 as the probability allocated by player 1 on U. The GDF of player 2 is shown in Figure 6. We can find that the unique

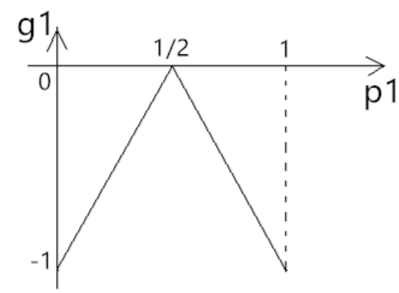

Fig. 6

robust anticipation is $p_1 = \frac{1}{2}$ corresponding to the unique equilibrium ($\frac{1}{2}$U+$\frac{1}{2}$D,$\frac{1}{2}$L+$\frac{1}{2}$R). In comparison with the mixed equilibrium excluded in the last example, this mixed strategy provides player 1 with the highest payoff. It is compelling to conclude that different mixed equilibria may take on completely distinct properties, which seems to be neglected by literature. Specifically, when players are playing the game in Figure 3, with considerable probability they tend to anticipate opponent's behavior as one of pure strategies rather than the mixed one; however, when playing game in Figure 3, they will anticipate opponent to perform ($\frac{1}{2}$,$\frac{1}{2}$) for sure.

Lastly I supply an instance to explain why I use *deleted* maximal point in the definition. Consider the game in Figure 7 where (D,R) is the unique equilibrium. We denote p_1 as the probability allo-

1, 1	0, 0
2, 0	1, 1

Fig. 7

cated to D in player 1's anticipated strategy, and the GDF of player 2 is illustrated in Figure 8. The local

[9]This kind of maximal point can only be construed through Definition 2.

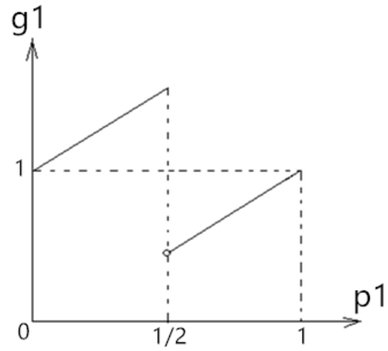

Fig. 8

maximal points in Figure 8 are $p_1 = \frac{1}{2}$ and $p_1 = 1$, but there is only one equilibrium corresponding to the latter. The reason for this is that on the right side of $p_1 = \frac{1}{2}$ the first-order derivative of GDF has the orientation toward the opposite direction of $p_1 = \frac{1}{2}$ and thus an inclination of regression does not appear in this case. Actually, the direction of derivative over the entire domain $(0, 1)$ is oriented toward $p_1 = 1$, which corresponds to the fact that pure strategy D strictly dominates U. In this way, I manifest an important intuition that the information of robustness is completely contained in the orientations of first-order derivatives of GDF within a small deleted neighborhood, and to conclude robustness with affirmative these orientations need to take on tendency of regressing.

III. Discussion

The insight of robust equilibrium has been justified in the last section, nonetheless, some important properties that make this new approach of refinement qualified need to be discussed. I start this by firstly presenting the definition of robust equilibrium. Denote I as the set of players.

Definition 3: A profile σ is a robust equilibrium, if for every $i \neq j \in I$, σ_i is a robust anticipation of j on i, conditional on σ_{-ij}.

Furthermore, I propose two propositions on the existence of robust equilibrium and its connection with nash equilibrium without providing the proofs.

Proposition 1: For every finite game, there exists at least one robust equilibrium.

Proposition 2: If profile σ is a robust equilibrium, it is also a nash equilibrium.

Remark 1: The proofs of these two propositions are not very involved, but it takes much time to formulate them. So I temporarily omit technical content in this proposal.

Through numerical experiments, robust equilibrium provides much more reasonable solution sets than nash's method. Under particular circumstances, such as in finite two-player static games, robust equilibrium even surpasses perfect equilibrium by Selten. However, it is necessary to point out that robust equilibrium has some limits, especially when it comes to multi-player or dynamic games, since the examination of anticipation robustness only includes bilateral connection, that is, it merely considers j's expectation of i, but that of others(σ_{-ij}) is seen as exogenous constant. This nature that the examination can only be implemented between two individual each time makes this method undermined in multi-player games, which goes even more severe when games are dynamic where off-path chaos tends to exacerbate the situation.

Yet robust equilibrium should only be seen as an improved version of nash equilibrium, and in this matter, it does perform outstandingly. In the same way that literature has upgraded nash's definition, the flaws of robustness appearing in multi-player and dynamic games can be alleviated by combining it with other refinement approaches, for example, we can easily define "subgame-perfect robust equilibrium" and "sequential robust equilibrium" by sticking to spirits of subgame and sequential perfectness.

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