On the Space of Orthonormal Wavelets

B. G. Sherlock and D. M. Monro

***Abstract*—The space of orthonormal wavelets is described by a set of parameters for which a simple recurrence generates the coefficients for all orthonormal perfect-reconstruction FIR filters of arbitrary length. The space splits into two halves, each containing the time reverse of the other's filters. A MATLAB implementation is given.**

I. Introduction

The theory of continuous and discrete wavelet transforms [1], [2] has inspired much basic and applied research in signal and image processing, as well as revitalized the study of subband filtering [3]–[5]. This correspondence considers the generation of two-channel perfect reconstruction quadrature mirror filter banks corresponding to compactly supported orthonormal wavelets. In defining a parameterization, we wish to achieve three objectives:

1) to cover all possible orthonormal filters;
2) to specify bounds for the parameters which permit all wavelets to be found uniquely;
3) to express the parameterization so that the coefficients can easily be generated.

The method starts from Vaidyanathan's parametrization of perfect reconstruction finite impulse response filter banks [6] and adapts it to parametrize orthonormal wavelets of arbitrary compact support. Various parametrizations of orthonormal wavelets have been published [7]–[12]. Daubechies [1] presented a complete characterization of the filter coefficients corresponding to compactly supported orthonormal wavelets. Because a spectral factorization must be performed in order to obtain the filter coefficients from the parameters, this approach has the computational disadvantage that it is necessary to solve for the roots of a polynomial. In addition, the parametrization is not unique, since for a given set of parameters, different roots of the polynomial may be chosen. The method of Zou and Tewfik [7], [8] has the advantage that it is able to parametrize wavelets having a number of vanishing moments greater than one; however, the formulae expressing the filter coefficients in terms of the parameters are fairly complicated.

II. Theory

Let the analysis lowpass filter in a two-channel perfect-reconstruction orthonormal filter bank have $2M$ coefficients $\{h_i\}$ and z transform $H_0(z) = \Sigma_{i=0}^{2M-1} h_i z^{-i} = H_{00}(z^2) + z^{-1} H_{01}(z^2)$, where $H_{00}(z)$ and $H_{01}(z)$ are the polyphase components [6], [13] of $H_0(z)$, i.e.,

$$H_{00}(z) = \sum_{i=0}^{M-1} h_{2i} z^{-1} \tag{1a}$$

$$H_{01}(z) = \sum_{i=0}^{M-1} h_{2i+1} z^{-1}. \tag{1b}$$

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The following factorization of the polyphase matrix $H_p(z)$ was proposed by Vaidyanathan [6]

$$\begin{aligned} H_p(z) &\equiv \begin{pmatrix} H_{00}(z) & H_{01}(z) \\ H_{10}(z) & H_{11}(z) \end{pmatrix} \\ &= \begin{pmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{pmatrix} \cdot \prod_{i=1}^{M-1} \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} c_i & s_i \\ -s_i & c_i \end{pmatrix} \end{aligned} \tag{2}$$

where $c_i = \cos(\alpha_i)$ and $s_i = \sin(\alpha_i)$, and $H_{10}(z)$ and $H_{11}(z)$ are the polyphase components of the highpass analysis filter $H_1(z)$. This factorization generates all two-channel perfect-reconstruction orthonormal filter banks of impulse response length $2M$, i.e., any such filter bank can be written in terms of M parameters α_i taking values in the interval $[0, 2\pi)$. In order for the filter bank to correspond to an orthonormal wavelet basis, the first-order regularity constraint $\Sigma_i h_i = \sqrt{2}$, which is equivalent to $\Sigma_{i=0}^{M-1} \alpha_i = (\pi/4)$, must also be satisfied [5], [14].

We develop our new formulation by rewriting the factorization (2) in a general recursive form expressing the polyphase matrices corresponding to filters of length $2(k+1)$ in terms of the polyphase matrices for filters of length $2k$

$$H_p^{(k+1)}(z) = H_p^{(k)}(z) \cdot \begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \begin{pmatrix} c_k & s_k \\ -s_k & c_k \end{pmatrix} \quad \text{for } k = 1, 2, 3, \cdots \tag{3}$$

with $H_p^{(1)} = \left(\begin{smallmatrix} c_0 & s_0 \\ -s_0 & c_0 \end{smallmatrix}\right)$, where superscript $^{(k)}$ refers to filters of length $2k$. From this, we derive recursive formulae expressing the coefficients $\{h_i^{(k+1)}\}$ for a filter of length $2(k+1)$ in terms of the coefficients $\{h_i^{(k)}\}$ for a filter of length $2k$. From (3), we get

$$H_{00}^{(k+1)}(z) = H_{00}^{(k)}(z) \cdot c_k - z^{-1} H_{01}^{(k)}(z) \cdot s_k \tag{4a}$$

$$H_{01}^{(k+1)}(z) = H_{00}^{(k)}(z) \cdot s_k + z^{-1} H_{01}^{(k)}(z) \cdot c_k \tag{4b}$$

with $H_{00}^{(1)}(z) = c_0$ and $H_{01}^{(1)}(z) = s_0$.

Expanding (4a) gives

$$\begin{aligned} H_{00}^{(k+1)}(z) \equiv \sum_{i=0}^{k} h_{2i}^{(k+1)} z^{-i} &= \left[\sum_{i=0}^{k-1} h_{2i}^{(k)} z^{-i}\right] c_k \\ &\quad - z^{-1} \left[\sum_{i=0}^{k-1} h_{2i+1}^{(k)} z^{-i}\right] s_k \\ &= c_k h_0^{(k)} + \sum_{i=1}^{k-1} (c_k h_{2i}^{(k)} - s_k h_{2i-1}^{(k)}) z^{-i} \\ &\quad - s_k h_{2k-1}^{(k)} z^{-k} \end{aligned}$$

with $h_0^{(1)} = c_0$ and $h_1^{(1)} = s_0$. Identifying coefficients of the various powers of z yields recursive formulae for the even-numbered filter coefficients $\{h_{2i}\}$

$$\begin{cases} h_0^{(k+1)} = c_k h_0^{(k)} \\ h_{2i}^{(k+1)} = c_k h_{2i}^{(k)} - s_k h_{2i-1}^{(k)} \quad \text{for } i = 1, 2, \cdots, k-1 \\ h_{2k}^{(k+1)} = -s_k h_{2k-1}^{(k)} \end{cases} \tag{5}$$

with $h_0^{(1)} = c_0$ and $h_1^{(1)} = s_0$. Similarly, expanding (4b) yields formulae for the odd coefficients

$$\begin{cases} h_1^{(k+1)} = s_k h_0^{(k)} \\ h_{2i+1}^{(k+1)} = s_k h_{2i}^{(k)} + c_k h_{2i-1}^{(k)} \quad \text{for } i = 1, 2, \cdots, k-1 \\ h_{2k+1}^{(k+1)} = c_k h_{2k-1}^{(k)} \end{cases} \tag{6}$$

recalling $c_k = \cos \alpha_k$ and $s_k = \sin \alpha_k$.

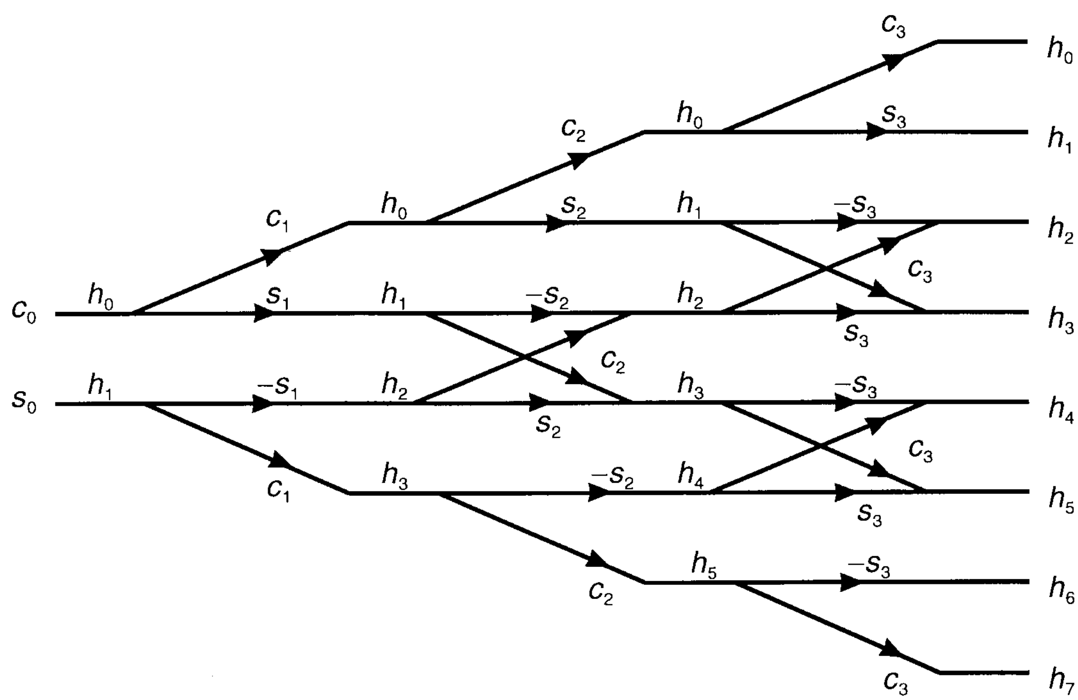

Fig. 1. Flow diagram illustrating the recurrence for generating filter coefficients. The initial coefficients s_0 and c_0 are the sine and cosine of the first free parameter α_0. At each stage of the recurrence, a new free parameter is introduced, and the filter length increases by 2.

Equations (5) and (6) express the $2M$ filter coefficients $\{h_i\}$ in terms of M angular parameters $\{\alpha_i | i = 0, \cdots, M-1\}$ taking values in the interval $[0, 2\pi)$. Any choice of $\{\alpha_i\}$ will lead to a valid orthonormal FIR filter bank system, and any such system can be expressed in terms of some choice of $\{\alpha_i\}$. The recurrence of (5) and (6) can be elegantly expressed in the form of the flow diagram of Fig. 1. In this diagram, a filter of any desired even length $2M$ can be produced by choosing M parameters α_i [with $\Sigma_i\, \alpha_i = (\pi/4)$ if regularity is desired] and passing through the flow diagram, terminating after $M-1$ stages of processing. An efficient implementation in MATLAB is presented in Fig. 2.

III. Symmetries

The space of all length-$2M$ PR orthonormal filters can be covered by allowing each of the M angular parameters α_i to vary over the full range of $[0, 2\pi)$. However, this will be a redundant representation because of inherent symmetries in the parameter space. For any i, changing α_i to $\alpha_i + \pi$ will change the sign of both c_i and s_i. This in turn changes the signs of all filter coefficients, as can be readily verified by substitution into (1) and (2). Consequently, the filter coefficients will be unchanged if any even number of the α_i are changed by π. The space can therefore be fully covered by choosing α_0 from the interval $[0, 2\pi)$ and α_1 through α_{M-1} from $[0, \pi)$.

When generating wavelets, however, the additional constraint of regularity is imposed: $\Sigma_{i=0}^{M-1}\, \alpha_i = (\pi/4)$. This reduces the number of free parameters to $M-1$ (for a length-$2M$ filter) and introduces further symmetries. For any $i \in \{0, 1, \cdots, M-2\}$, changing α_i to $\alpha_i + \pi$ must be accompanied by a change in α_{M-1} to $\alpha_{M-1} - \pi$ because of the regularity constraint. Because effectively *two* of the parameters have changed by π, the two resultant sign changes cancel, and the filter coefficients are unchanged. We therefore conclude that the space of all orthonormal wavelets of length $2M$ is covered by choosing $M-1$ parameters α_i, each chosen from the interval $[0, \pi)$.

For many applications, two wavelets that are the time reversals of each other would be regarded as effectively the same wavelet. We can find a transformation of the parameter space that transforms any wavelet into its time reversal, and this enables a further division of the parameter space that covers all orthonormal wavelets but excludes their time reversals.

Lemma: Given an orthonormal filter bank of length $2M$ determined by the M parameters α_i, under the transformation $\alpha_0' = (\pi/2) - \alpha_0$, $\alpha_i' = \pi - \alpha$, $i = 1, 2, \cdots, M-1$, the filter coefficients become $h_i' = (-1)^{M-1} h_{2M-1+i}$.

Proof: The proof is by induction. For $M = 1$, we have $h_0 = c_0$ and $h_1 = s_0$. After substituting $\alpha_0' = (\pi/2) - \alpha_0$, these become $h_0' = s_0 = (-1)^{1-1} h_1$ and $h_1' = c_0 = (-1)^{1-1} h_0$.

We now assume truth for $M = k$, i.e., application of the transformation causes the $\{h_i^{(k)}\}$ to change according to $h_i^{\prime(k)} = (-1)^{k-1} h_{2k-1-i}^{(k)}$. Additionally, since $\alpha_k' = \pi - \alpha_k$, we have $c_k' = -c_k$, and $s_k' = s_k$. Substitution into (5) and then applying (6) gives

$$\begin{cases} h_0^{\prime(k+1)} = c_k' h_0^{\prime(k)} = -c_k(-1)^{k-1} h_{2k-1}^{(k)} = (-1)^k h_{2k+1}^{(k+1)} \\ h_{2i}^{\prime(k+1)} = c_k' h_{2i}^{\prime(k)} - s_k' h_{2i-1}^{\prime(k)} \\ \qquad\quad = -c_k(-1)^{k-1} h_{2(k-i)-1}^{(k)} - s_k(-1)^{k-1} h_{2(k-i)}^{(k)} \\ \qquad\quad = (-1)^k h_{2(k-i)+1}^{(k+1)}, i = 1, 2, \cdots, k-1 \\ h_{2k}^{\prime(k+1)} = -s_k' h_{2k-1}^{\prime(k)} = -s_k(-1)^{k-1} h_0^{(k)} = (-1)^k h_1^{(k+1)}. \end{cases}$$

Similarly, substitution into (6) and then applying (6) gives

$$\begin{cases} h_1^{\prime(k+1)} = s_k' h_0^{\prime(k)} = s_k(-1)^{k-1} h_{2k-1}^{(k)} = (-1)^k h_{2k}^{(k+1)} \\ h_{2i+1}^{\prime(k+1)} = s_k' h_{2i}^{\prime(k)} + c_k' h_{2i-1}^{\prime(k)} \\ \qquad\quad = s_k(-1)^{k-1} h_{2(k-i)-1}^{(k)} - c_k(-1)^{k-1} h_{2(k-i)}^{(k)} \\ \qquad\quad = (-1)^k h_{2(k-i)}^{(k+1)}, i = 1, 2, \cdots, k-1 \\ h_{2k+1}^{\prime(k+1)} = c_k' h_{2k-1}^{\prime(k)} = -c_k(-1)^{k-1} h_0^{(k)} = (-1)^k h_0^{(k+1)}. \end{cases}$$

The above six equations simply state $h_i^{\prime(k+1)} = (-1)^k h_{2k+1-i}^{(k+1)}$, which is the required result. Q.E.D.

```
function h = orthogen(alpha);

% Constructs an array h(1 . . . N) of lowpass orthonormal
% FIR filter coefficients for any even N >= 2.

% The input array alpha(1 . . . N/2) gives N/2 free
% parameters, which are angles in radians. If the angles
% sum to pi/4 the filter corresponds to a regular wavelet.

N = 2*length(alpha);
h = zeros(1,N);
lo = N/2;
hi = lo+1;
h(lo) = cos(alpha(1));
h(hi) = sin(alpha(1));
nstages = N/2;

for stage = 1:nstages-1

  c = cos(alpha(stage+1));
  s = sin(alpha(stage+1));

  h(lo-1) = c*h(lo);
  h(lo)   = s*h(lo);
  h(hi+1) = c*h(hi);
  h(hi)   = -s*h(hi);

  nbutterflies = stage-1;
  butterflybase = lo+1;

  for butterfly = 1:nbutterflies
    hlo = h(butterflybase);
    hhi = h(butterflybase+1);
    h(butterflybase)   = c*hhi - s*hlo;
    h(butterflybase+1) = s*hhi + c*hlo;
  end;
  lo = lo-1;
  hi = hi+1;
end;
```

Fig. 2. MATLAB function for generating orthonormal FIR filter coefficients.

Theorem: For a filter of length $2M$ corresponding to an orthonormal wavelet, the transformation $\alpha_0' = (\pi/2) - \alpha_0$, $\alpha_i' = \pi - \alpha_i$, $i = 1, 2, \cdots, M-2$ transforms the filter into its time reverse, i.e., $h_i' = h_{2M-1-i}$, $i = 0, 1, \cdots, 2M-1$.

Proof: The proof will be given in terms of a filter length of $2k+2$ since this corresponds with the notation of (5) and (6). Defining $\Sigma = \sum_{i=0}^{k-1} \alpha_i$, we have, by regularity, $\alpha_k = (\pi/4) - \Sigma$. Therefore

$$\cos(\alpha_k) = \frac{1}{\sqrt{2}}(\cos\Sigma + \sin\Sigma) \quad \text{and}$$
$$\sin(\alpha_k) = \frac{1}{\sqrt{2}}(\cos\Sigma - \sin\Sigma). \tag{7}$$

The transformation specified in the theorem is

$$\begin{cases} \alpha_0' = \dfrac{\pi}{2} - \alpha_0 \\ \alpha_i' = \pi - \alpha_i, \qquad i = 1, 2, \cdots, k-1 \\ \alpha_k' = \dfrac{\pi}{4} - \displaystyle\sum_{i=0}^{k-1} \alpha_i' = \dfrac{\pi}{4} - \Sigma'. \end{cases}$$

Since $\Sigma' = -\Sigma + (k + \frac{1}{2})\pi$ it follows that

$$\cos\Sigma' = (-1)^{k-1}\sin\Sigma \tag{8a}$$
$$\sin\Sigma' = (-1)^{k-1}\cos\Sigma. \tag{8b}$$

Substituting (7) into (5b) implies

$$\begin{aligned} h_{2i}^{(k+1)} &= c_k h_{2i}^{(k)} - s_k h_{2i-1}^{(k)} \\ &= \frac{1}{\sqrt{2}}(\cos\Sigma + \sin\Sigma)h_{2i}^{(k)} - \frac{1}{\sqrt{2}}(\cos\Sigma - \sin\Sigma)h_{2i-1}^{(k)}. \end{aligned}$$

Perform the transformation by substituting α_i' in place of α_i and applying the lemma

$$\begin{aligned} h_{2u}^{\prime(k+1)} &= \frac{1}{\sqrt{2}}(\cos\Sigma' + \sin\Sigma')\cdot(-1)^{k+1}h_{2k-2i}^{(k)} \\ &\quad - \frac{1}{\sqrt{2}}(\cos\Sigma' - \sin\Sigma')\cdot(-1)^{k+1}h_{2k-2i}^{(k)} \\ &= \frac{1}{\sqrt{2}}(\sin\Sigma + \cos\Sigma)h_{2k-2i-1}^{(k)} \\ &\quad - \frac{1}{\sqrt{2}}(\sin\Sigma - \cos\Sigma)h_{2k-2i}^{(k)} \quad \text{by (8)} \\ &= c_k h_{2k-2i-1}^{(k)} + s_k h_{2k-2i}^{(k)} \quad \text{by (7)} \\ &= h_{2(k-i)+1}^{(k+1)}, \qquad i = 1, \cdots, k-1 \quad \text{by (6b)} \end{aligned}$$

In a similar fashion, substituting (7) into (5) and (6), performing the

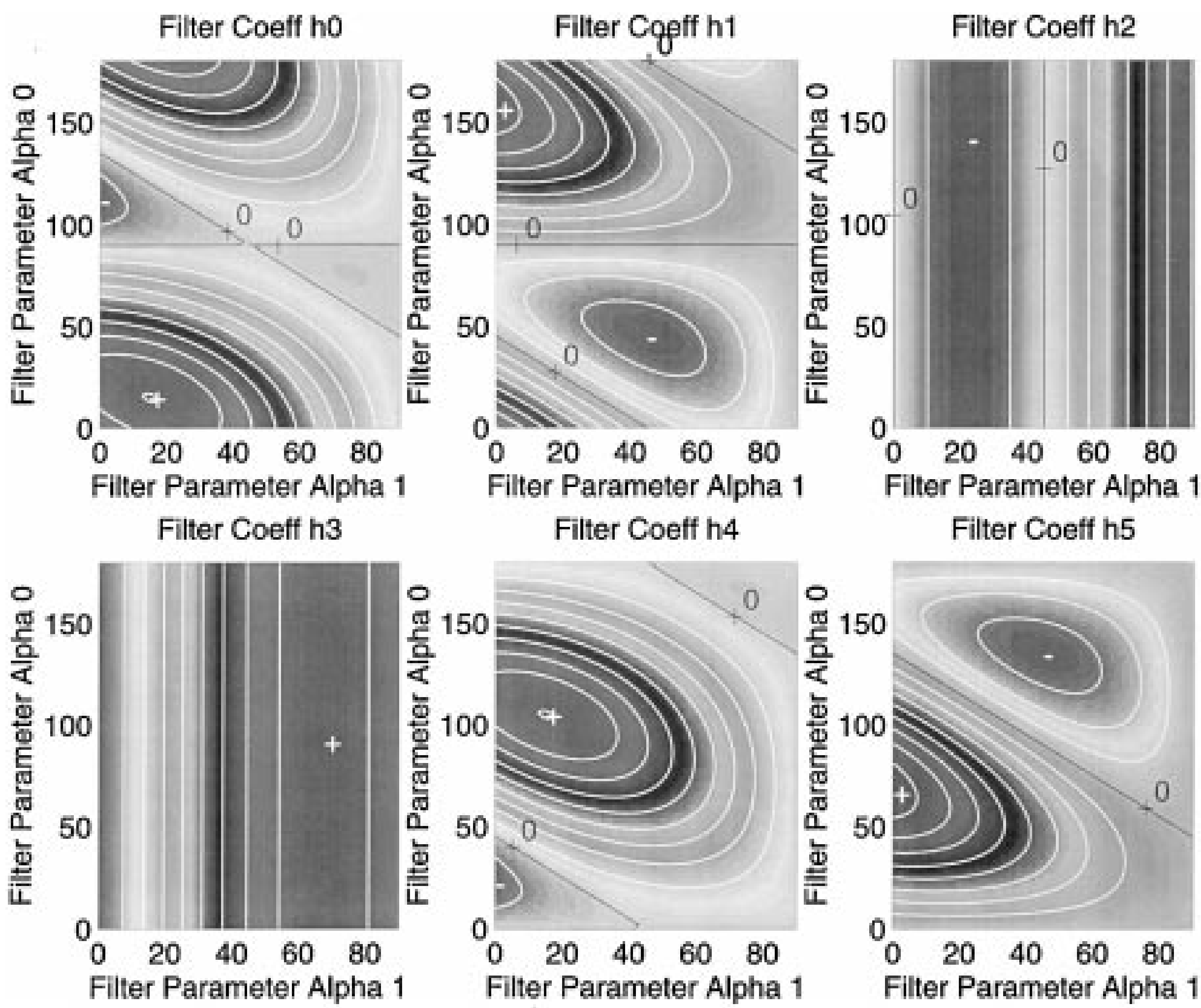

Fig. 3. Two-dimensional space of orthonormal regular FIR filter coefficients of length 6, excluding time reversals. There are two free parameters α_0 and α_1 in this case; therefore, the coefficients can be visualized as a 2-D contour plot. (The zero contour is black, the contour interval is 0.1, and maximum and minimum values are marked in white with "+" and "−," respectively.)

transformation, and applying the lemma yields

$$\begin{cases} h_0'^{(k+1)} = h_{2k+1}^{(k+1)} & \text{from (5a)} \\ h_{2k}'^{(k+1)} = h_1^{(k+1)} & \text{from (5c)} \\ h_1'^{(k+1)} = h_{2k}^{(k+1)} & \text{from (6a)} \\ h_{2i+1}'^{(k+1)} = h_{2k-1}^{(k+1)}, \quad i = 1, \cdots, k-1 & \text{from (6b)} \\ h_{2(k+1)}'^{(k+1)} = h_0^{(k+1)} & \text{from (6c)} \end{cases}$$

Thus, we have proven $h_i' = h_{2(k+1)-1-i}$, $i = 0, \cdots, 2k+1$.Q.E.D.

The theorem shows that half of the parameter space corresponds to wavelets that are time reversals of the other half. A subspace of the parameter space that covers all orthonormal wavelets but excludes their time reversals is given (for length-$2M$ wavelets) by choosing α_{M-2} from the interval $[0, \pi/2)$ and each of the other parameters (α_0 to α_{M-3}) from the interval $[0, \pi)$. Fig. 3 visualizes the two-dimensional (2-D) space of regular orthonormal filter coefficients graphically for length 6 with the time reversals excluded.

IV. Conclusion

We have presented a parameterization of the space of two-channel orthonormal FIR filters, which has enabled us to describe the generation of all possible filters using a simple recurrence. We have proven various symmetry properties of the space, which allow the exploration of all possible filters without repetition.

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A Second-Order Recursive Algorithm with Applications to Adaptive Filtering and Subspace Tracking

Åke Andersson and Holger Broman

***Abstract*— A second-order recursive algorithm for adaptive signal processing is proposed, and a similar algorithm is derived for signal subspace tracking. It is shown that the algorithm encompasses both the RLS and the LMS algorithms as special cases. The computational complexity is the same as for the RLS algorithm, but some extra memory storage is required. The associated ordinary differential equation (ODE) for the autoregressive exogenous (ARX) case algorithm is proven to be globally exponentially stable. Furthermore, it is demonstrated that the proposed algorithm has a higher ability to track time-varying signals than has the RLS algorithm. The proposed algorithm especially handles those situations well where there is a simultaneous system change and decrease of signal power.**

***Index Terms*— Adaptive algorithms, multistep algorithms, parameter estimation, recursive algorithms, subspace tracking, time-varying systems.**

I. Introduction

Recursive algorithms for spectral estimation and system identification have been established during the last few decades [1]–[3]. The recursive least squares (RLS) and least mean squares (LMS) algorithms have been thoroughly analyzed, and their properties are well known. Both algorithms make use of a first-order recursion in the estimated parameter vector, which to some extent determines and limits their behavior. In particular, when the unknown process simultaneously undergoes a system change and a loss of power, the performance of the two algorithms is poor. Attempts to boost the LMS algorithm by use of a momentum term have been described [4]–[6]. The *ad hoc* momentum term leads to a second-order recursion in the parameter vector. The so-called multistep (high-order recursive) algorithms have been studied by Benveniste [7]. With the use of hypermodels that describe the behavior of the system that generates the process, the gain matrix can be calculated beforehand. In the case of higher order hypermodels, the multistep algorithms provide significant improvements over the classical (one-step) methods for tracking the parameters of slowly time-varying systems.

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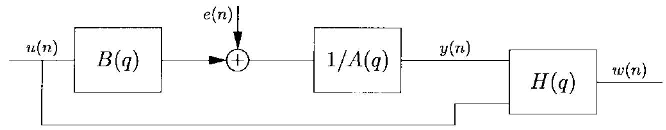

Fig. 1. ARX-process and the whitening filter used for the recursive algorithms.

In recent years, high-resolution subspace based methods, such as MUSIC and ESPRIT, have been applied to temporal and spatial domain spectral analysis [8]–[12]. However, these methods have been based on the eigendecomposition (ED) of the sample correlation matrix or the singular value decomposition (SVD) of the data matrix, which both are batch methods and are computationally expensive. In an adaptive scenario, a low-cost recursive algorithm would be advantageous. In two recent papers by Yang [13], [14], a criterion was presented from which an RLS like algorithm was derived that can be applied for the tracking of the signal subspace.

The problem addressed herein is whether a second-order recursive algorithm can do better on processes such as speech signals than can a first-order algorithm, as, for example, RLS. A second-order recursive algorithm is proposed for adaptive filtering and for subspace tracking. Simulation examples demonstrate that the proposed algorithm smoothly and rapidly can track the changes of rapidly time-varying systems.

II. The Basic Second-Order Recursive Algorithm

A. Derivation of the Algorithm

Let the linear regression

$$y(n) = \varphi^T(n)\theta(n) + e(n) \tag{1}$$

define a possibly nonstationary auto-regressive exogenous (ARX) process, where $y(\cdot)$ is the output of the system, and $e(\cdot)$ is a white, zero-mean, and possibly nonstationary stochastic process independent of the measurable input $u(\cdot)$. The regressor and the parameter vector are

$$\begin{aligned}\varphi(n) = [&-y(n-1), \cdots, -y(n-n_a), u(n), \cdots \\ &u(n-n_b+1)]^T \end{aligned} \tag{2}$$

$$\theta(n) = [a_1(n), \cdots, a_{n_a}(n), b_0(n), \cdots, b_{n_b-1}(n)]^T \tag{3}$$

where n_a and n_b denote the number of $a-$ and $b-$ parameters, respectively. The dimension of the vectors is $p = n_a + n_b$. The LMS [15], [16] and RLS [15] algorithms for estimating θ can be viewed as a filtering operation, as shown in Fig. 1. A whitening filter $H(q)$ is introduced. Both the LMS and RLS algorithms can be viewed as whitening filters, which are determined by the parameter estimates. The least squares error criterion states that the whitening filter should be chosen such that it minimizes the mean square of the output $w(\cdot)$. At minimum, the output of the whitening filter $w(\cdot)$ equals $e(\cdot)$, and the cross-covariance between $\varphi(\cdot)$ and $w(\cdot)$ equals zero.

Let the covariance matrix of $\varphi(\cdot)$ and the cross-covariance vector between $\varphi(\cdot)$ and $w(\cdot)$ be defined as

$$R(n) = E[\varphi(n)\varphi^T(n)] \tag{4}$$

$$\mathbf{r}(n) = E[\varphi(n)w(n)]. \tag{5}$$