University of Moratuwa

Department of Electronic and Telecommunication Engineering

BM4152

Bio-signal Processing

Assignment 3

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200641T

Contents

1. Continuous Wavelet Transform

1.2. Wavelet properties

The Mexican hat wavelet m(t) can be derived using the well-known gaussian function g(t) with mean zero and standard deviation one.

The gaussian function is,

$$g(t) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{t-\mu}{\sigma}\right)^2}$$

Let $\mu = 0$ and $\sigma = 1$.

Then,

$$g(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}t^2}$$

Then the Mexican hat wavelet m(t) can be derived as follows,

$$m(t) = -\frac{d^2}{dt^2} g(t)$$

$$\frac{d}{dt} g(t) = \frac{d}{dt}\left(\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\right)$$

$$\frac{d}{dt} g(t) = -\frac{t}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

$$\frac{d^2}{dt^2} g(t) = -\frac{1}{\sqrt{2\pi}}\left(t\left(-\mathrm{t}e^{-\frac{t^2}{2}}\right) + e^{-\frac{t^2}{2}}\right)$$

$$\frac{d^2}{dt^2} g(t) = -\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}(1 - t^2)$$

$$m(t) = -\frac{d^2}{dt^2} g(t) = \frac{(1 - t^2)}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$$

But the energy of a wavelet should be equal to 1. Therefore, now let's find a normalizing factor of $m(t)$ such that

$$E = \int_{-\infty}^{\infty} m^2(t)\ dt = 1$$

Let the normalizing factor of m(t) be k where k > 0.

Then let's find k such that

$$\int_{-\infty}^{\infty} \frac{m^2(t)}{k^2}\, dt = 1$$

$$\int_{-\infty}^{\infty} m^2(t)\ dt = k^2$$

$$\int_{-\infty}^{\infty} \frac{(1-t^2)^2}{2\pi} e^{-t^2}\, dt = k^2$$

$$\frac{1}{2\pi}\left[\int_{-\infty}^{\infty} e^{-t^2} dt - 2\int_{-\infty}^{\infty} t^2 e^{-t^2} dt + \int_{-\infty}^{\infty} t^4 e^{-t^2} dt\right] = k^2$$

The above integration can be evaluated using the well-known Gaussian Integral ($\int_{-\infty}^{\infty} e^{-t^2} dt = \sqrt{\pi}$) and the generic form of the gaussian integral.

Generic Form of a Gaussian Integral

$$\begin{aligned}\int_{-\infty}^{\infty} x^{2n} e^{-\alpha x^2}\, dx &= (-1)^n \int_{-\infty}^{\infty} \frac{\partial^n}{\partial \alpha^n} e^{-\alpha x^2}\, dx \\ &= (-1)^n \frac{\partial^n}{\partial \alpha^n} \int_{-\infty}^{\infty} e^{-\alpha x^2}\, dx \\ &= \sqrt{\pi}(-1)^n \frac{\partial^n}{\partial \alpha^n} \alpha^{-\frac{1}{2}} \\ &= \sqrt{\frac{\pi}{\alpha}} \frac{(2n-1)!!}{(2\alpha)^n}\end{aligned}$$

$$\frac{1}{2\pi}\left(\sqrt{\pi} - 2\cdot\frac{\sqrt{\pi}}{2} + \frac{3\sqrt{\pi}}{4}\right) = k^2$$

$$\frac{3}{8\sqrt{\pi}} = k^2$$

Normalizing factor is $\frac{\sqrt{3}}{2\sqrt{2}\pi^{1/4}}$.

Hence, the normalized Mexican hat mother wavelet $\psi(t)$ is.

$$\psi(t) = \frac{m(t)}{k}$$

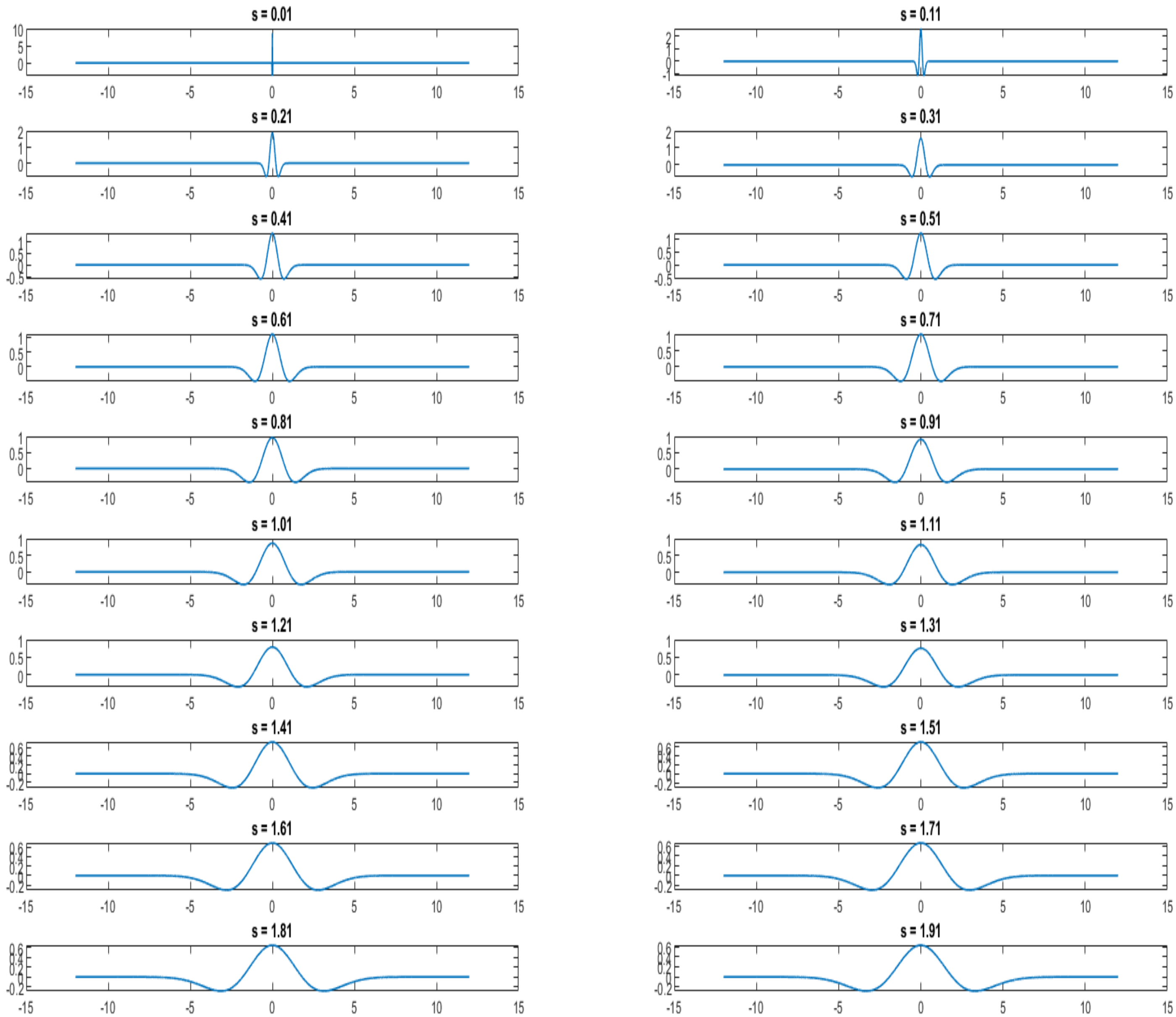
$$\psi(t) = \frac{(1-t^2)}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \cdot \frac{2\sqrt{2}\pi^{\frac{1}{4}}}{\sqrt{3}}$$

$$\psi(t) = \frac{2(1-t^2)}{\sqrt{3}\,\pi^{\frac{1}{4}}} e^{-\frac{t^2}{2}}$$

Then the generic wavelet function is

$$\psi_s(t) = \frac{1}{\sqrt{s}}\psi\left(\frac{t}{s}\right)$$

The shape of the daughter wavelets for scaling factors ranging from 0.01 to 2 (i.e. 0.01:0.1:2) has shown below,

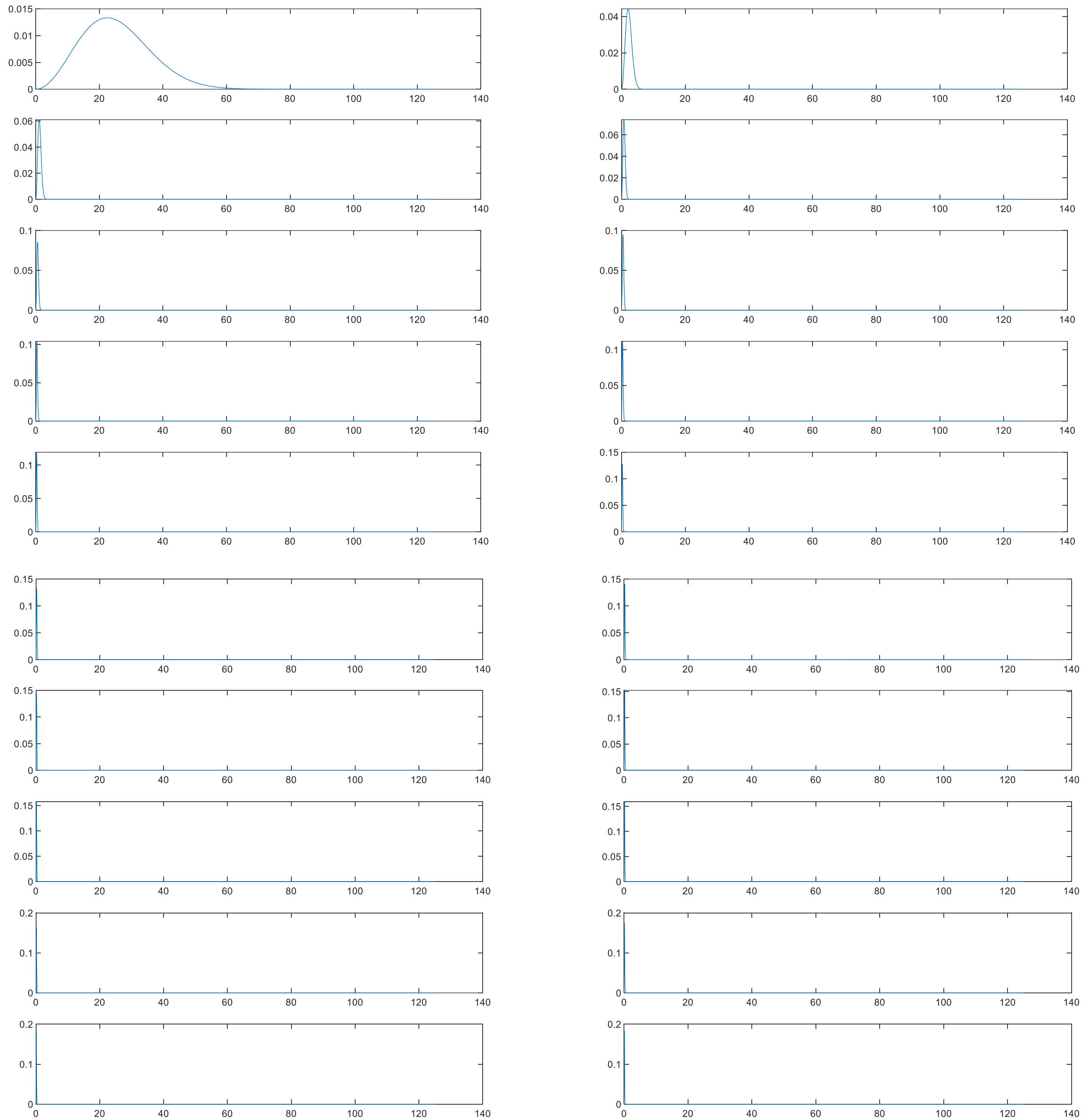


- X - Time
- Y - Amplitude

From the observations of the daughter wavelets, it can be concluded that each exhibit zero mean, unity energy, and compact support. For further verification, the mean and energy have been calculated.

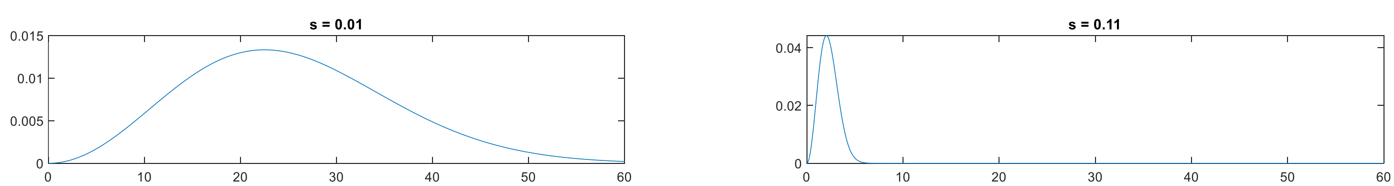
```
Scaling factor: 0.01    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 0.11    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 0.21    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 0.31    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 0.41    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 0.51    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 0.61    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 0.71    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 0.81    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 0.91    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 1.01    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 1.11    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 1.21    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 1.31    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 1.41    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 1.51    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 1.61    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 1.71    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 1.81    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
Scaling factor: 1.91    |   The mean of the wavelet is: 0.00000    |   The energy of the wavelet is: 1.00000
```

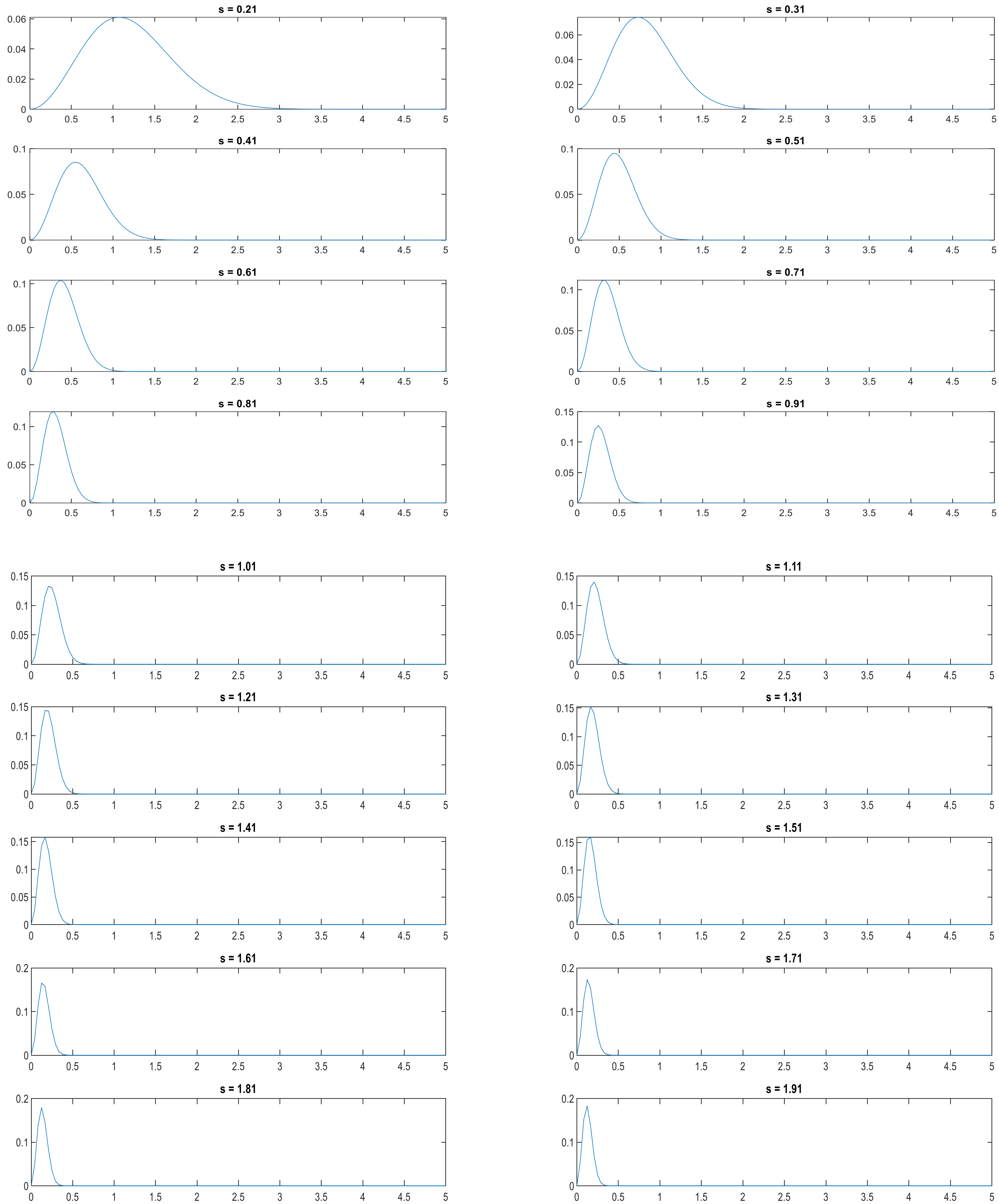
The spectra of the daughter wavelet are shown as follows:



- X – Frequency(Hz)
- Y - Amplitude

A zoomed version of the



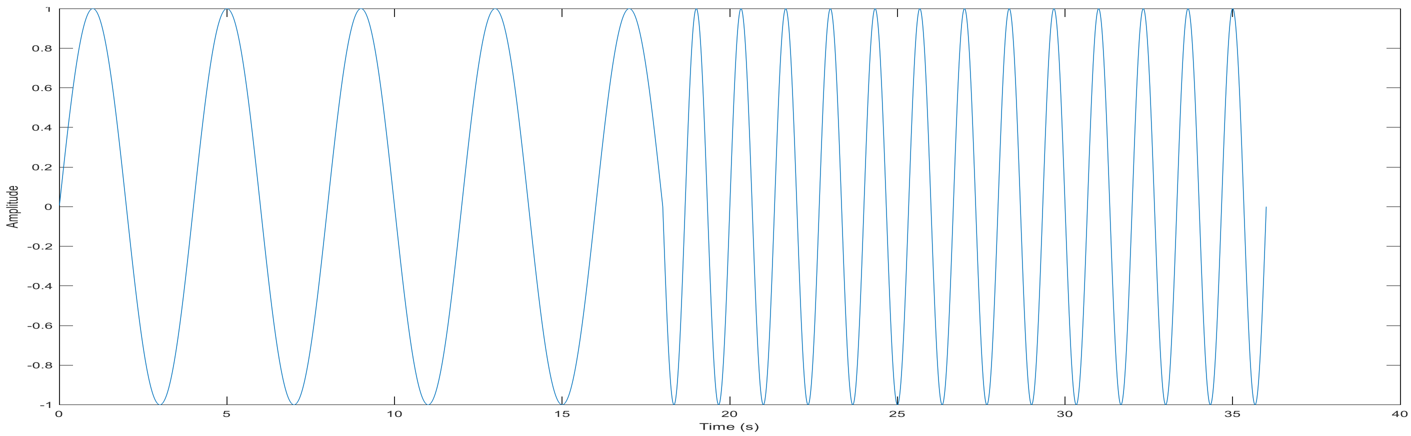


As scale increases, the wavelet broadens in the time domain, while its spectral profile narrows in the frequency domain, focusing on lower frequencies. This reflects the trade-off between time and frequency resolution, with broader wavelets capturing lower-frequency components effectively

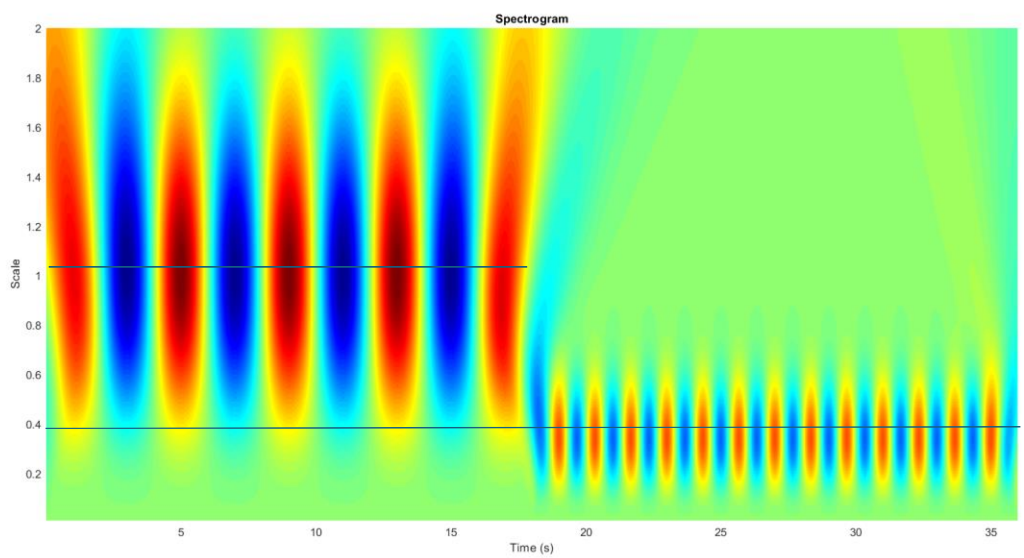
1.3. Continuous Wavelet Decomposition

Define a waveform, where N =3000 and fs = 250Hz.

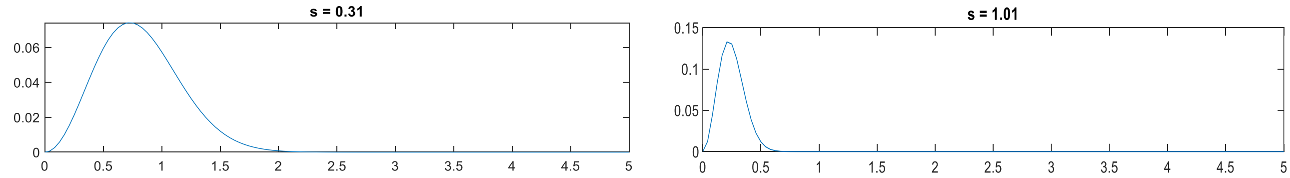
$$x[n] = \begin{cases} \sin\left(\frac{0.5\pi n}{fs}\right), & 0 \leq n < \frac{3N}{2} \quad (frequncy = 0.25Hz) \\ \sin\left(\frac{1.5\pi n}{fs}\right), & \frac{3N}{2} \leq n < 3N \quad (frequency = 0.75Hz) \end{cases}$$



Then the continuous wavelet transformation is performed with Mexican hat wavelet for scales ranging from 0,01 to 2 (i.e. 0.01:0.01:2).



Based on the spectrogram, it can be concluded that the signal contains two distinct frequency components at scales 1 and 0.3, with one component occurring before approximately 17.5 seconds and the other after. To gain further insights into these frequencies, we can examine the spectra of the daughter wavelet at each scale.



Examining the spectra at scales 0.31 and 1.01 reveals corresponding frequencies of 0.75 Hz and 0.25 Hz, respectively. These frequency values closely match the defined signal components, confirming the accuracy of the spectral analysis.

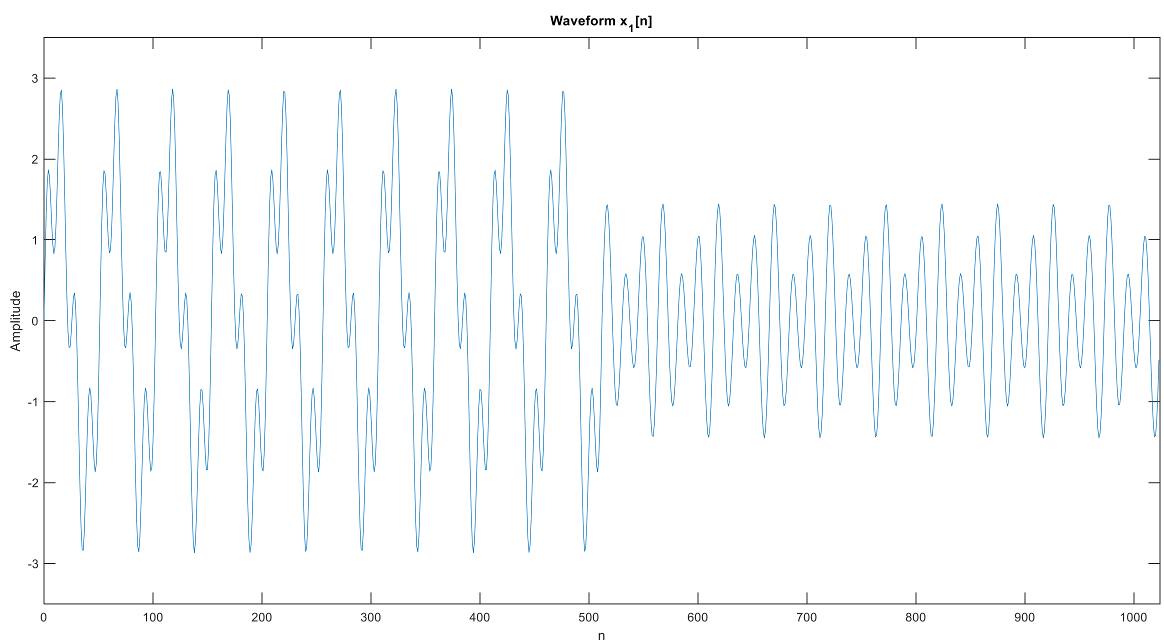
2. Discrete Wavelet Transform

2.2. Applying DWT with the Wavelet Toolbox in MATLAB

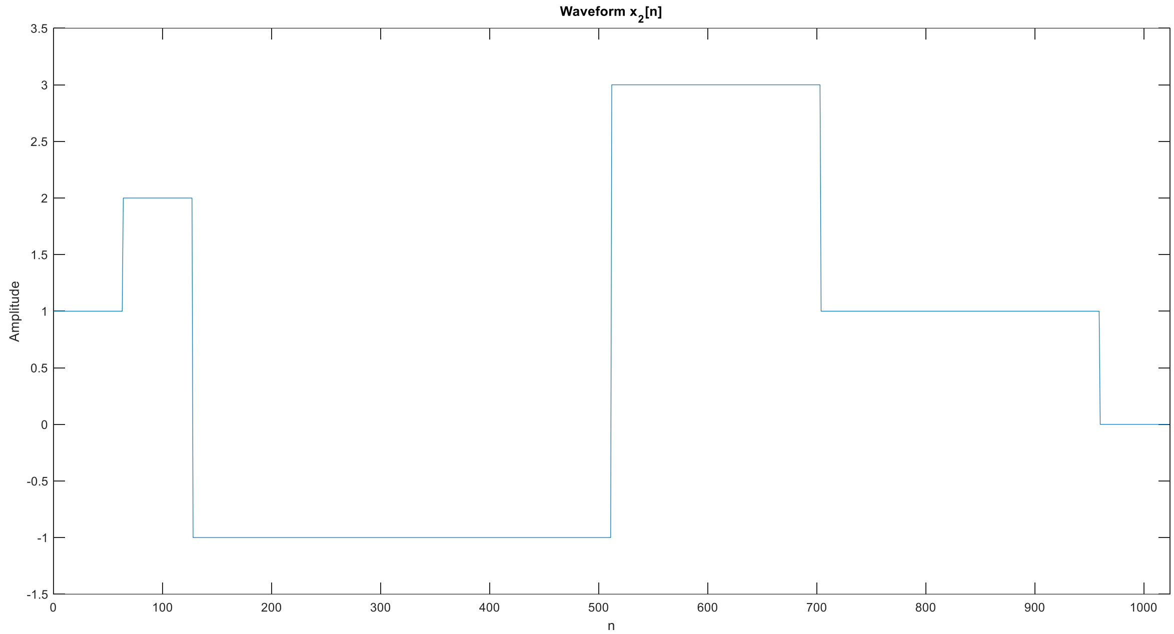
Creating waveforms ($x_1[n]$ and $x_2[n]$) for the purpose analysis.

The waveforms are specified as follows.

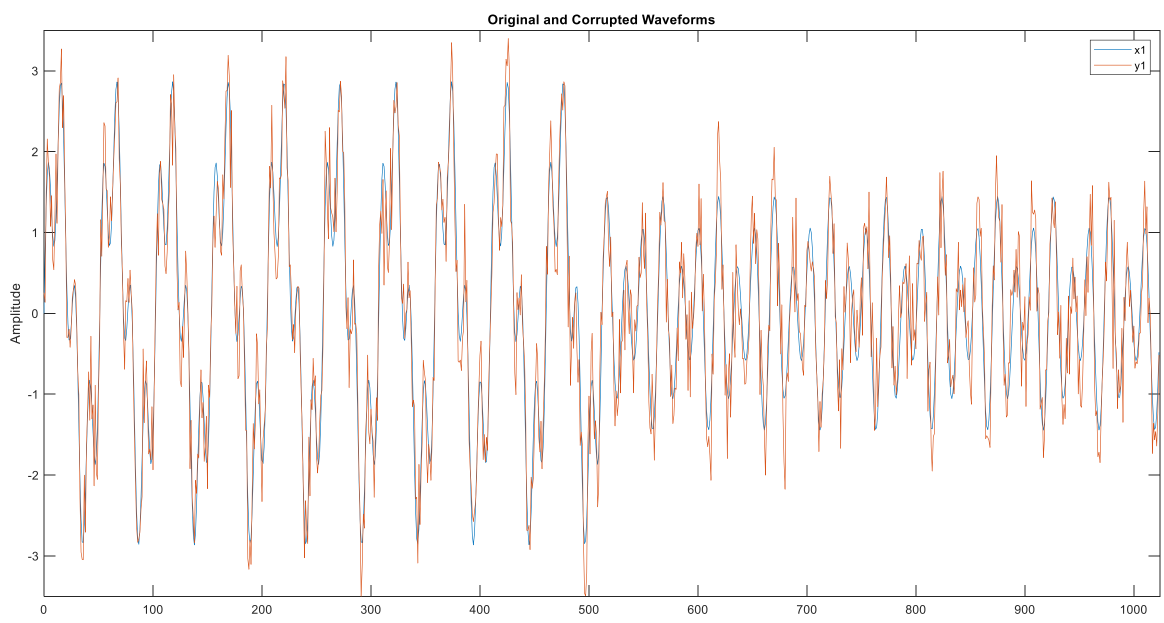
$$x_1[n] = \begin{cases} 2\sin\left(\frac{20\pi n}{fs}\right) + \sin\left(\frac{80\pi n}{fs}\right), & 0 \leq n < 512 \\ 0.5\sin\left(\frac{40\pi n}{fs}\right) + \sin\left(\frac{60\pi n}{fs}\right), & 512 \leq n < 1024 \end{cases}$$

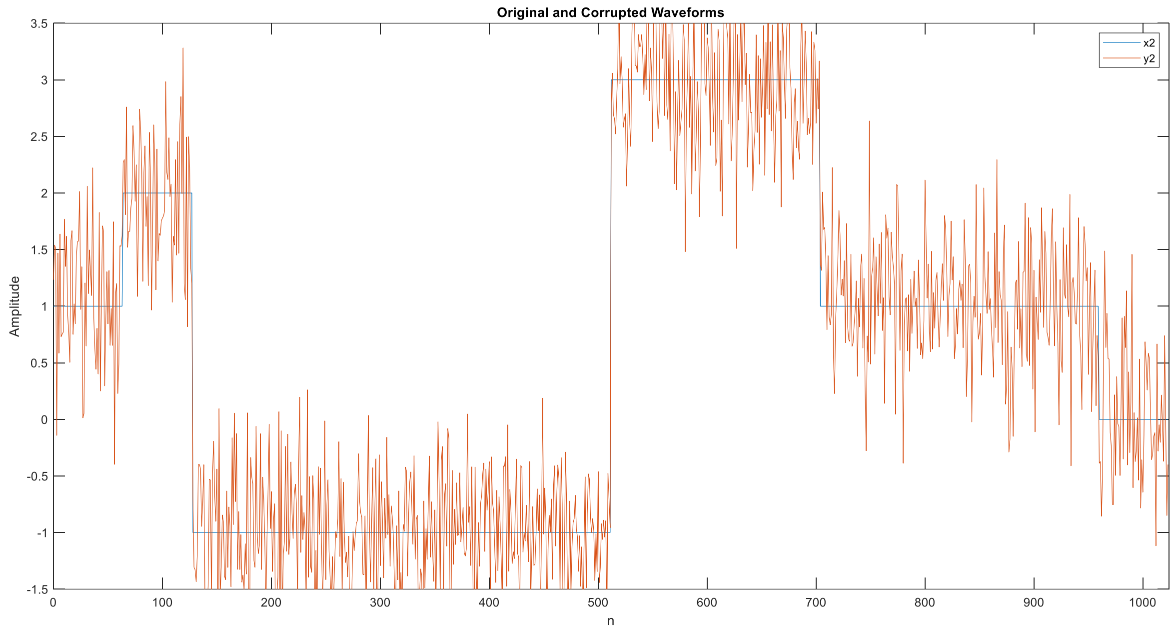


$$x_2[n] = \begin{cases} 1, & 0 \leq n < 64 \\ 2, & 64 \leq n < 128 \\ -1, & 128 \leq n < 512 \\ 3, & 512 \leq n < 704 \\ 1, & 704 \leq n < 960 \\ 0, & otherwise \end{cases}$$



Then the obove generated signals are corrupted by adding a AWGN noise so that the signal SNR is 10dB,

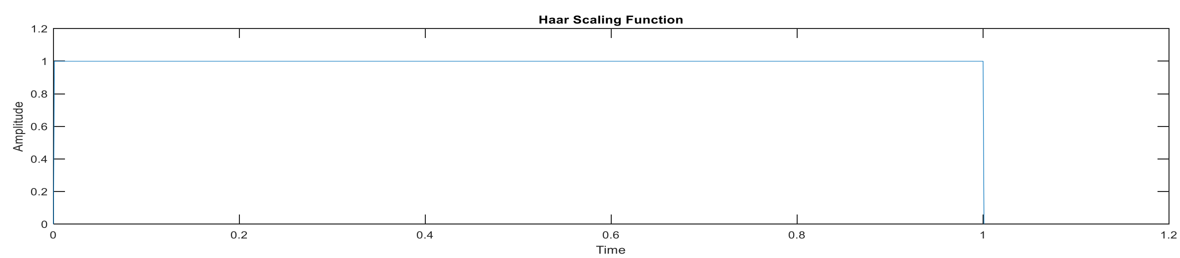


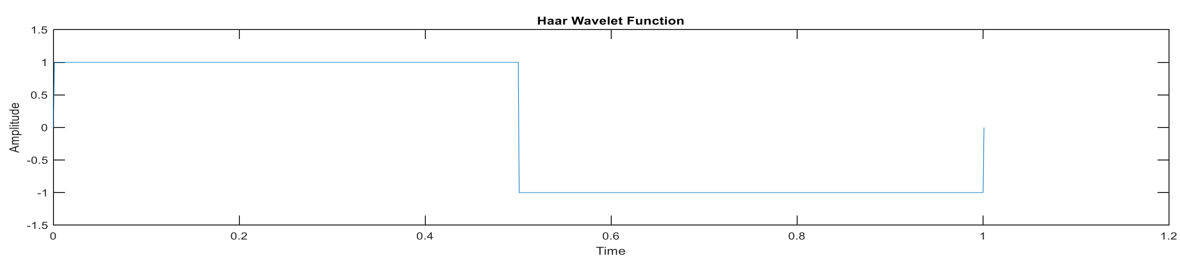


Then the morphology of the wavelet and scaling functions of Haar and Daubechies tap 9 are observed using;

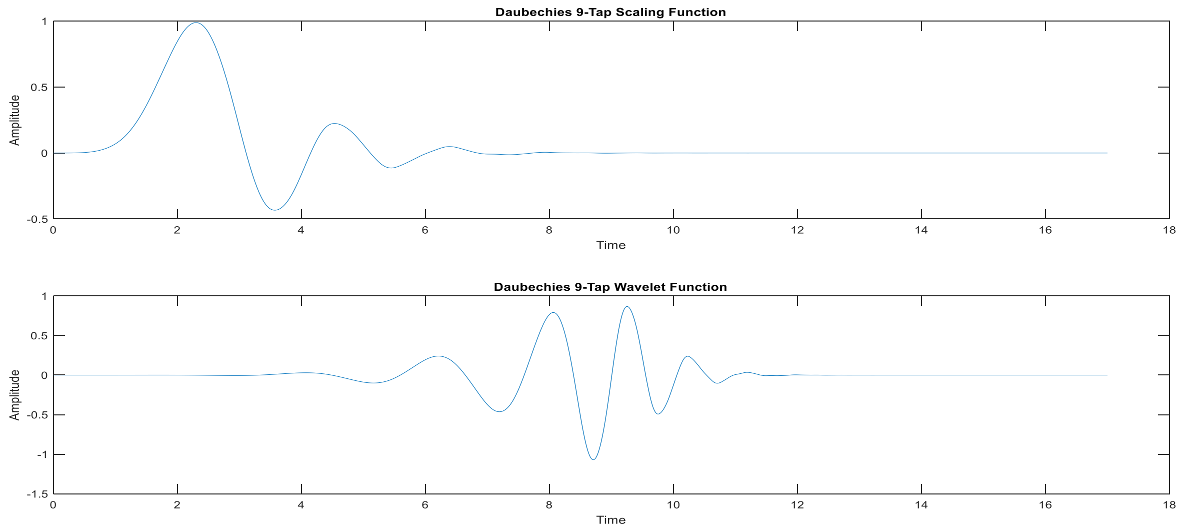
1. *wavefun()*

Haar



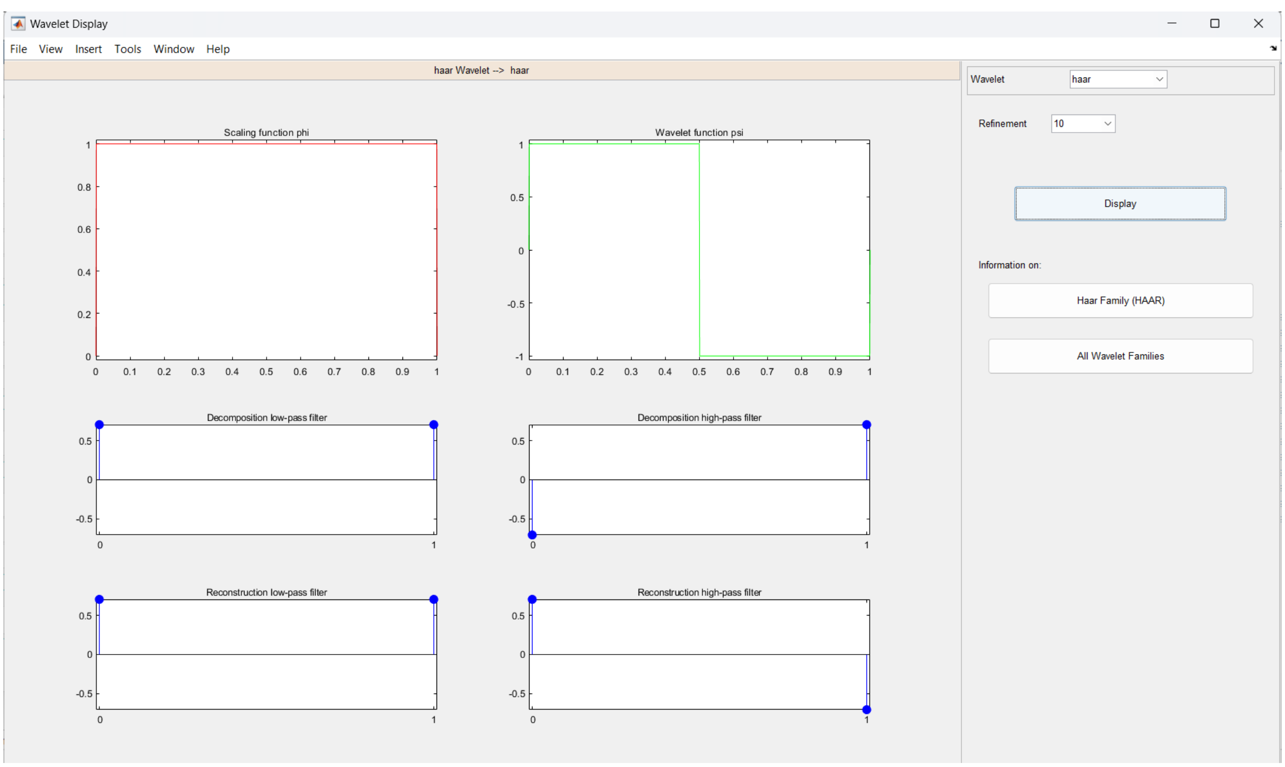


Daubechies 9-Tap

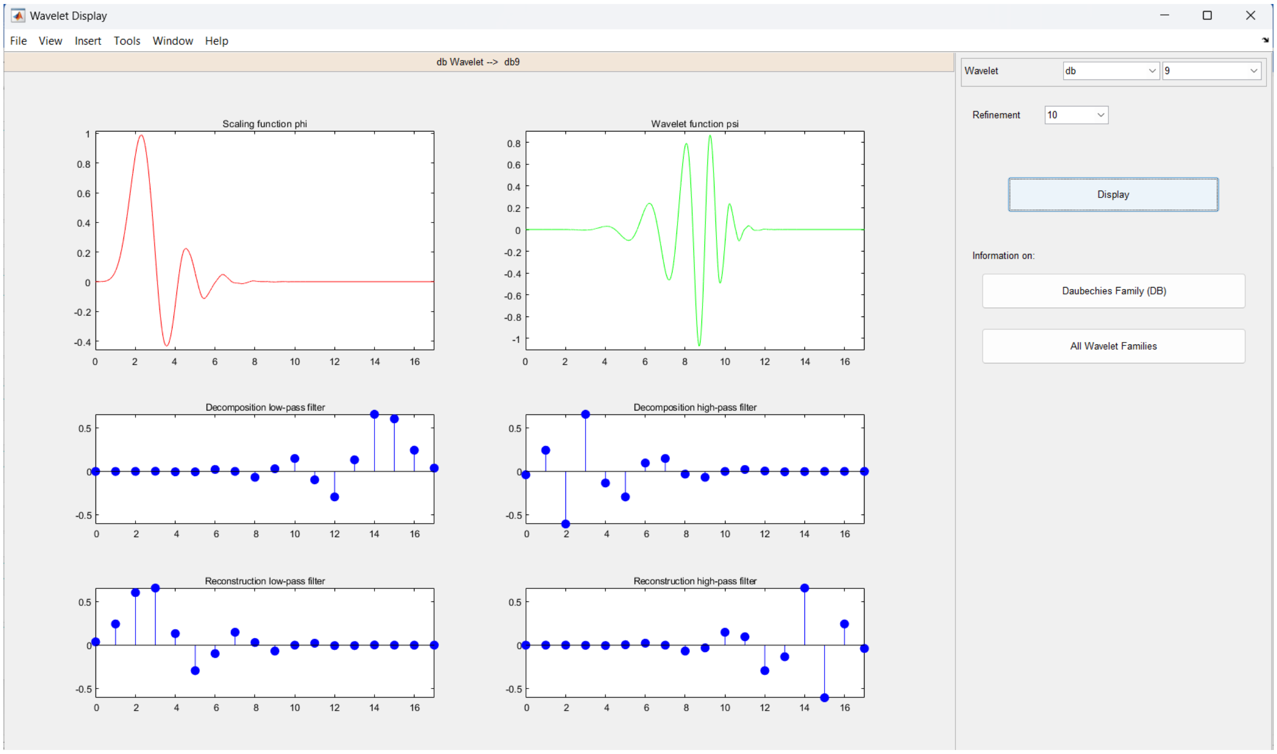


2. *waveletAnalyzer*

Haar

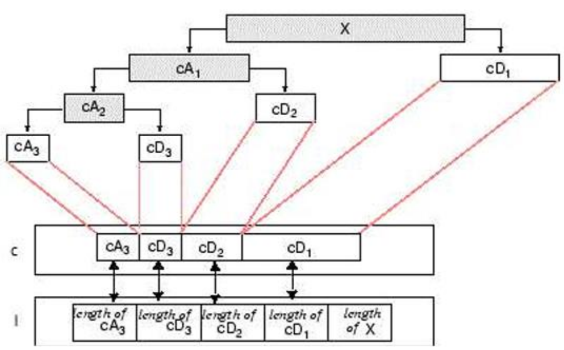


Daubechies 9-Tap



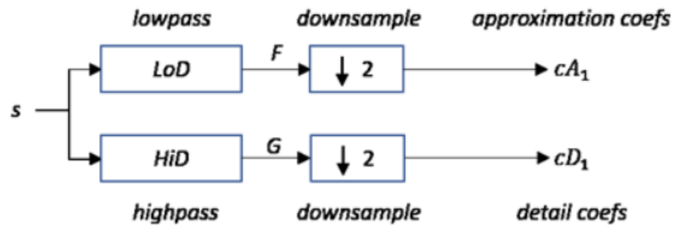
The *wavedec()* function can be used to decompose a signal into a given number of decomposition levels using a specific wavelet.

The following diagram shows how the *wavedec()* function decomposes a signal(x) into 3 decomposition levels.



Then,
Approximation coefficients can be extracted using the *appcoef()* function.
Detail coefficients for each level can be extracted using the *detcoef()* function.

However, the method that the *wavedec()* function uses to calculate the wavelet coefficients provides some redundant coefficients as well.

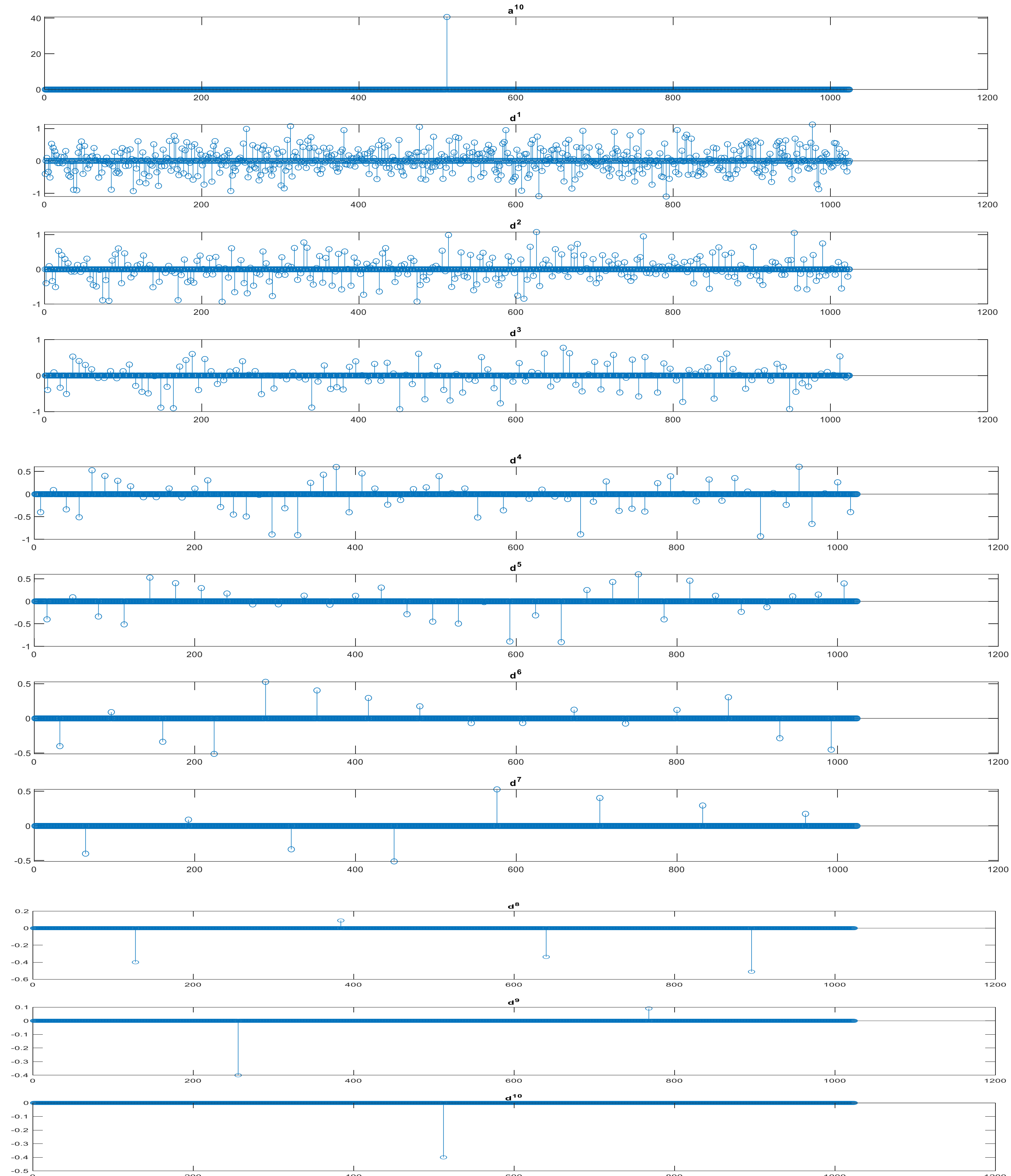


where

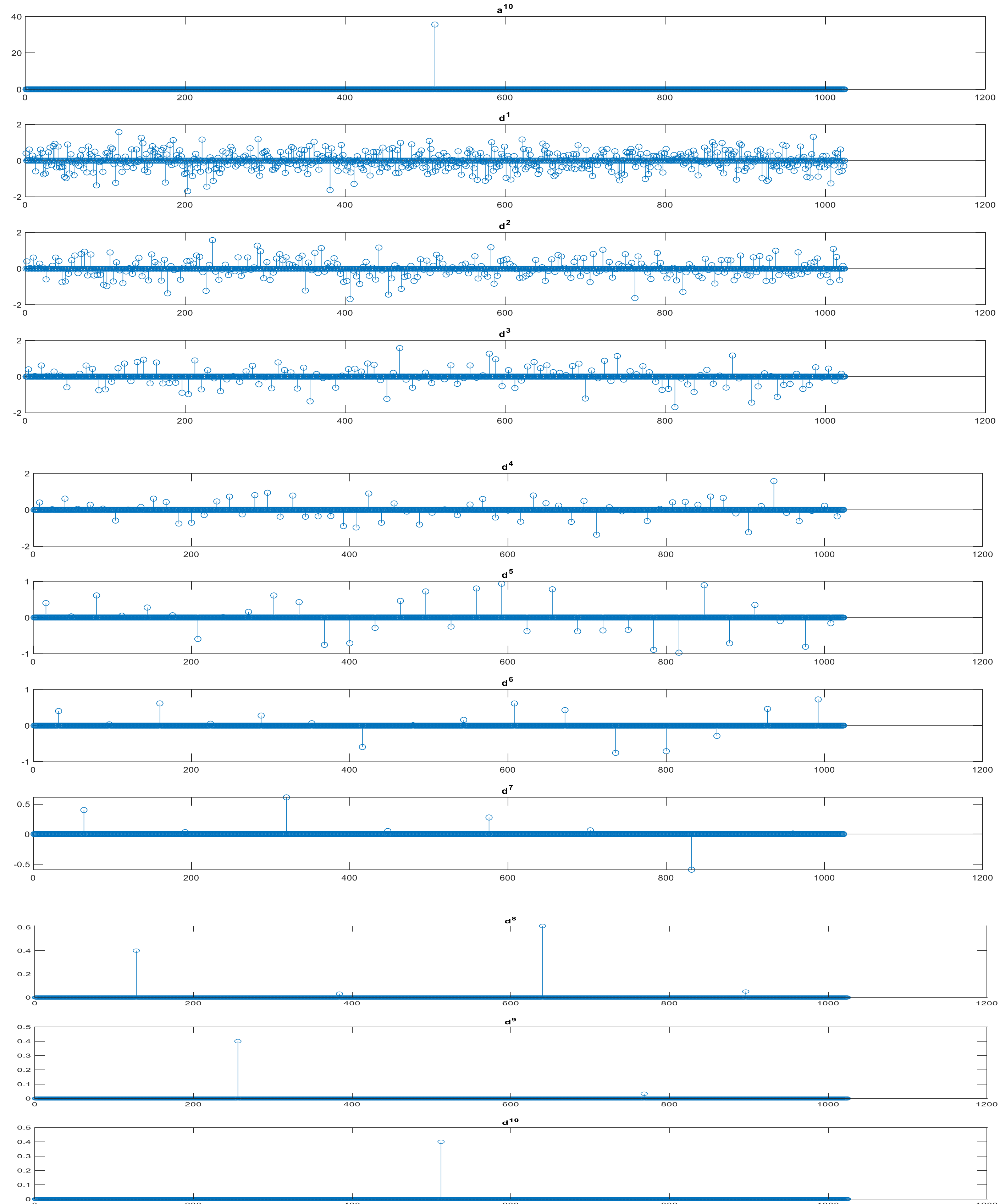
- $\boxed{X}$ — Convolve with filter *X*
- $\boxed{\downarrow 2}$ — Downsample (keep the even-indexed elements)

The length of each filter is equal to $2n$. If N = length(s), the signals F and G are of length $N + 2n - 1$ and the coefficients cA_1 and cD_1 are of length floor$\left(\frac{N-1}{2}\right) + n$.

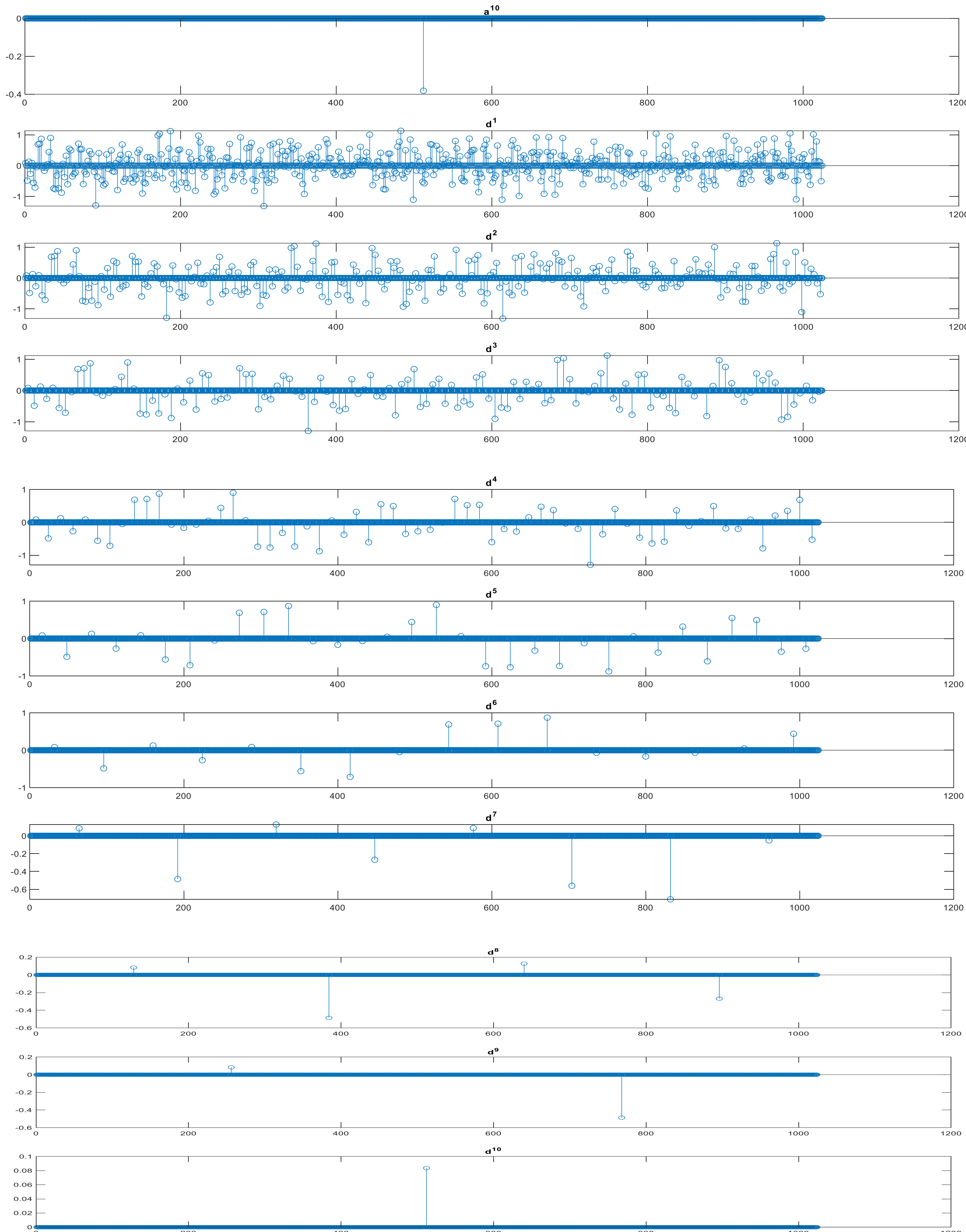
Wavelet coefficients for y1 with db9.



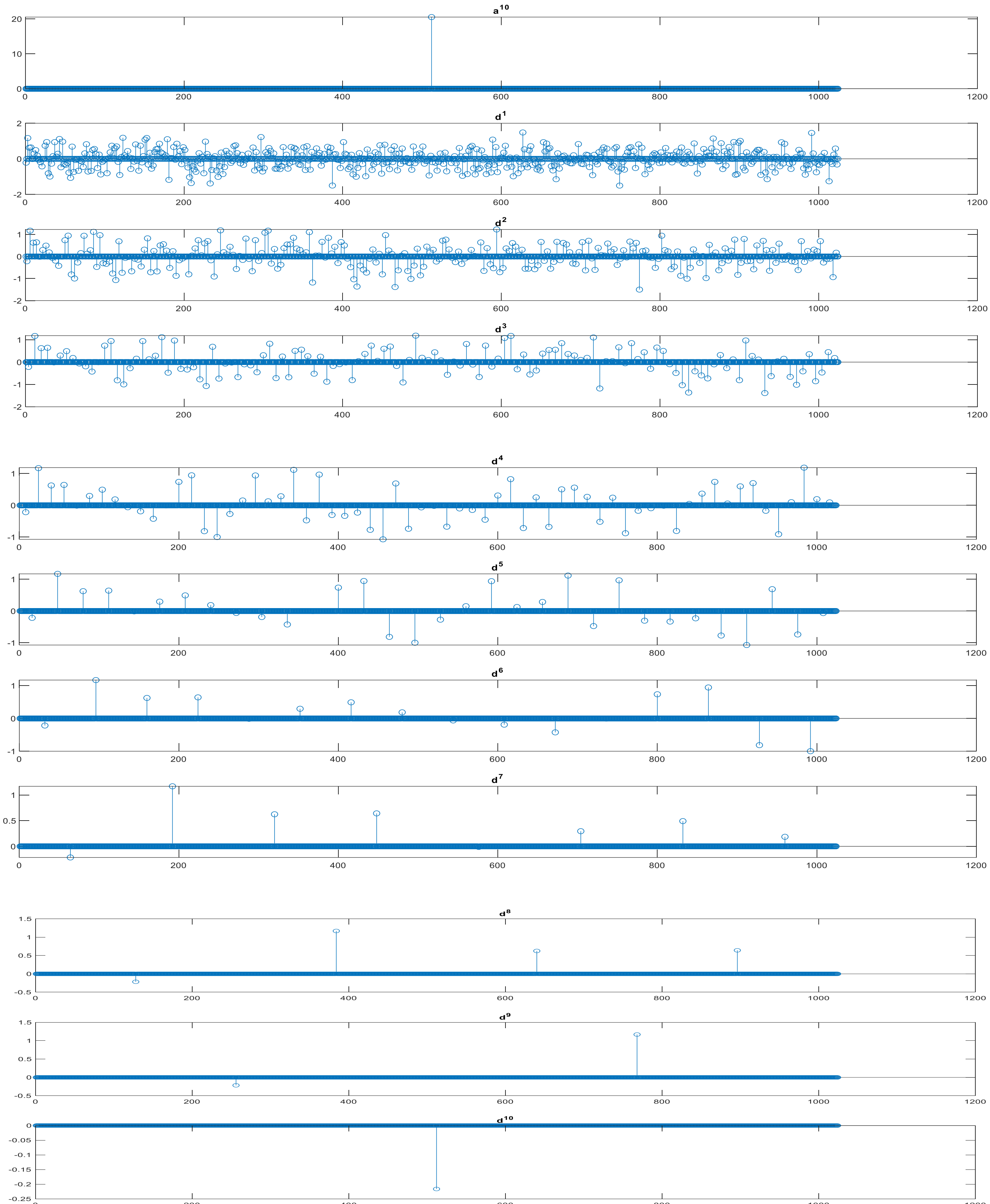
Wavelet coefficients for y2 with db9.



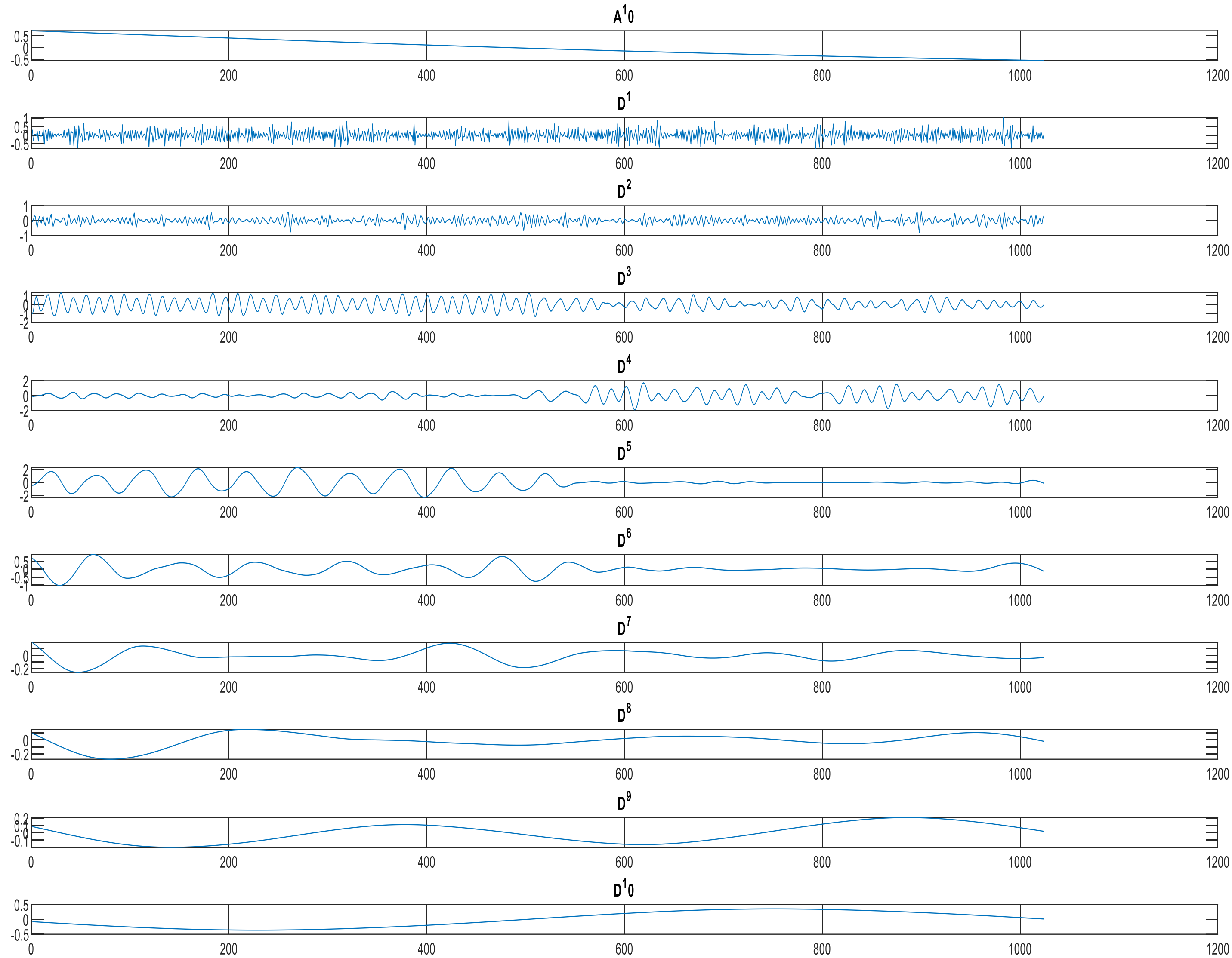
Wavelet coefficients for y1 with haar.



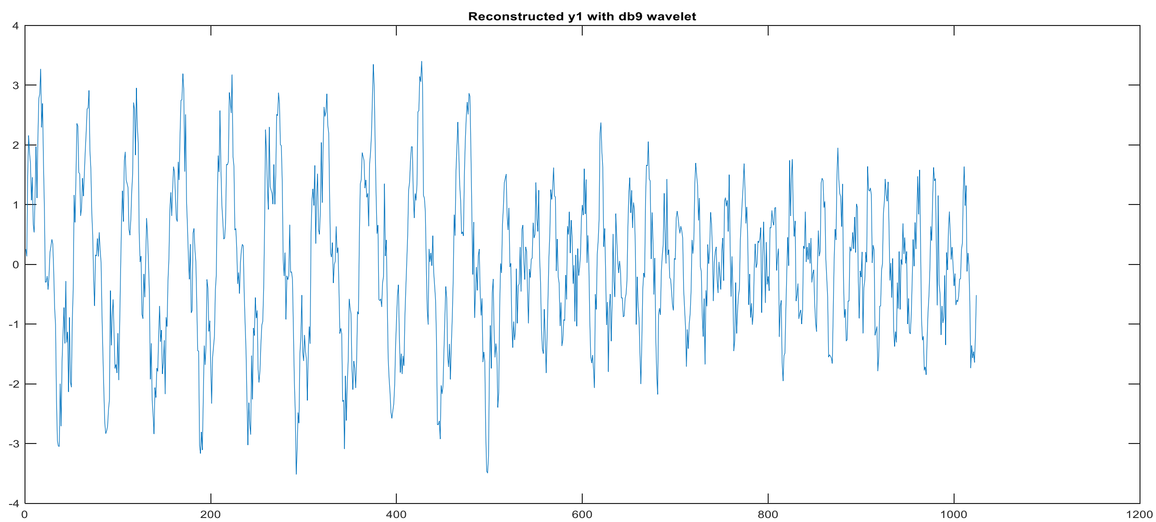
Wavelet coefficients for y2 with haar.



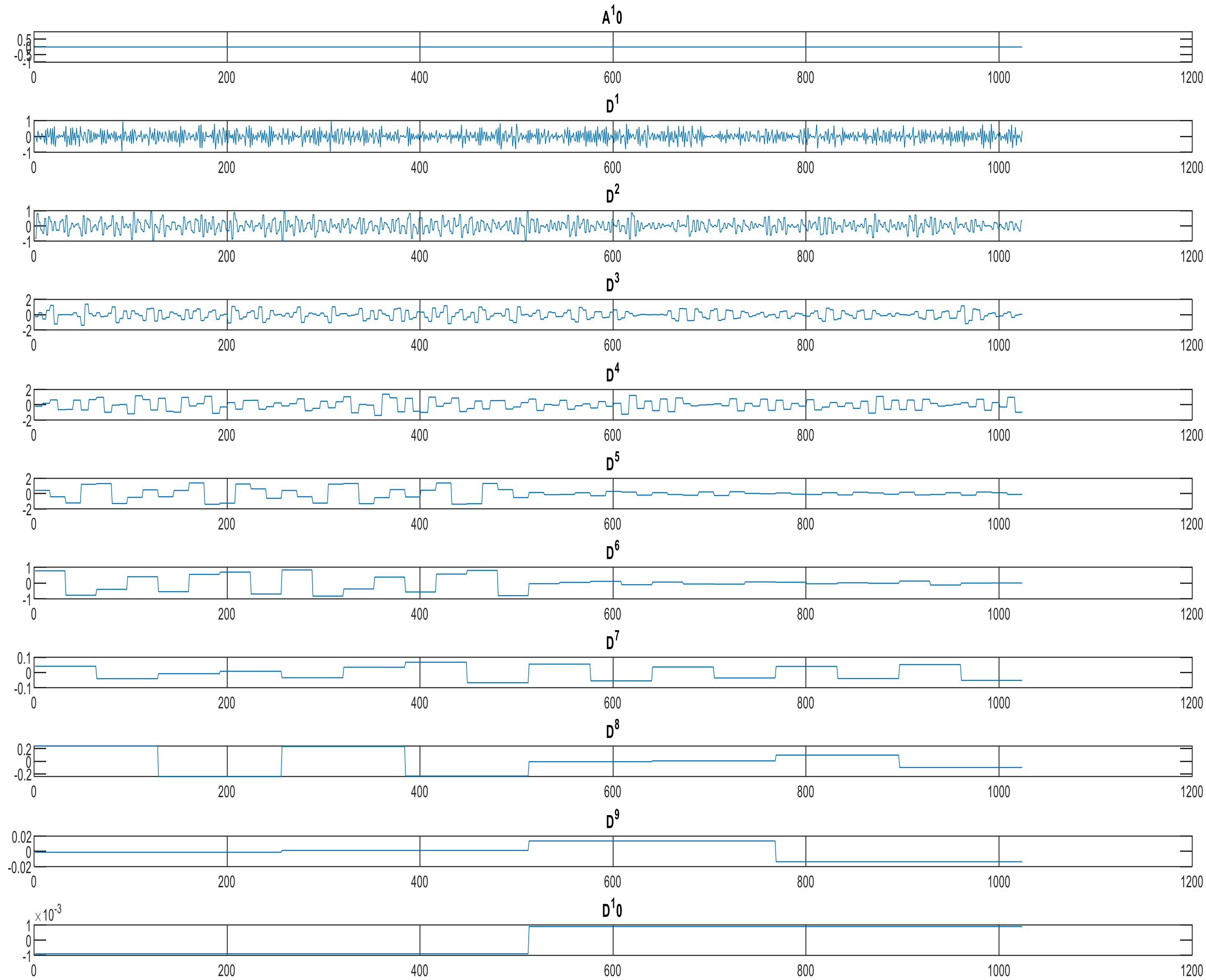
Reconstructed A10, D10, D9, …., D2, D1 for signal y1 with db9 wavelet.



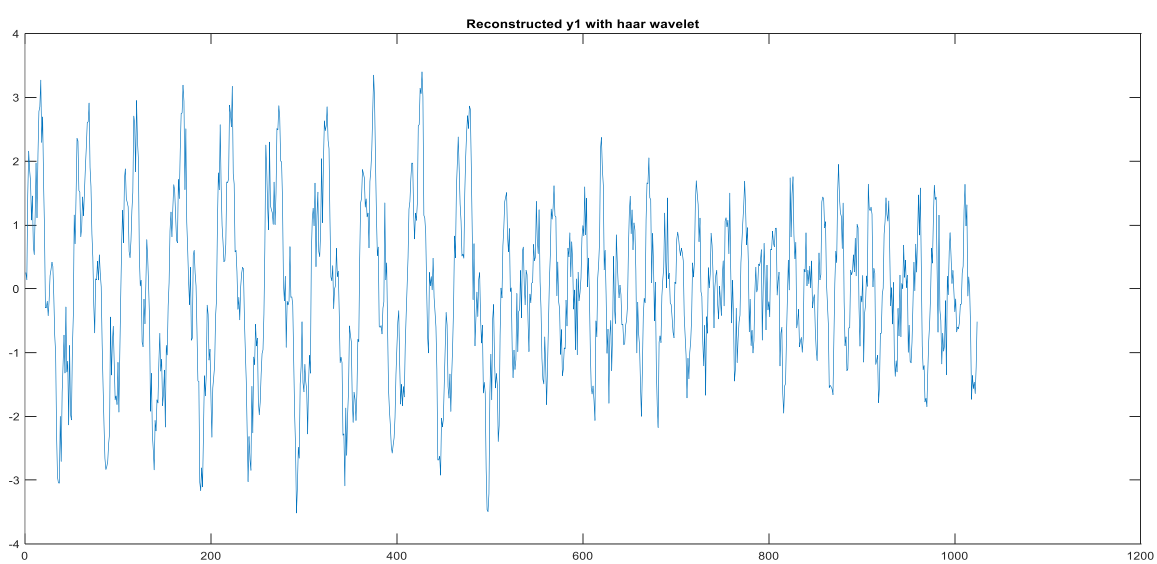
Reconstructed y1 signal using $y = \Sigma D^i + A$ with db9,



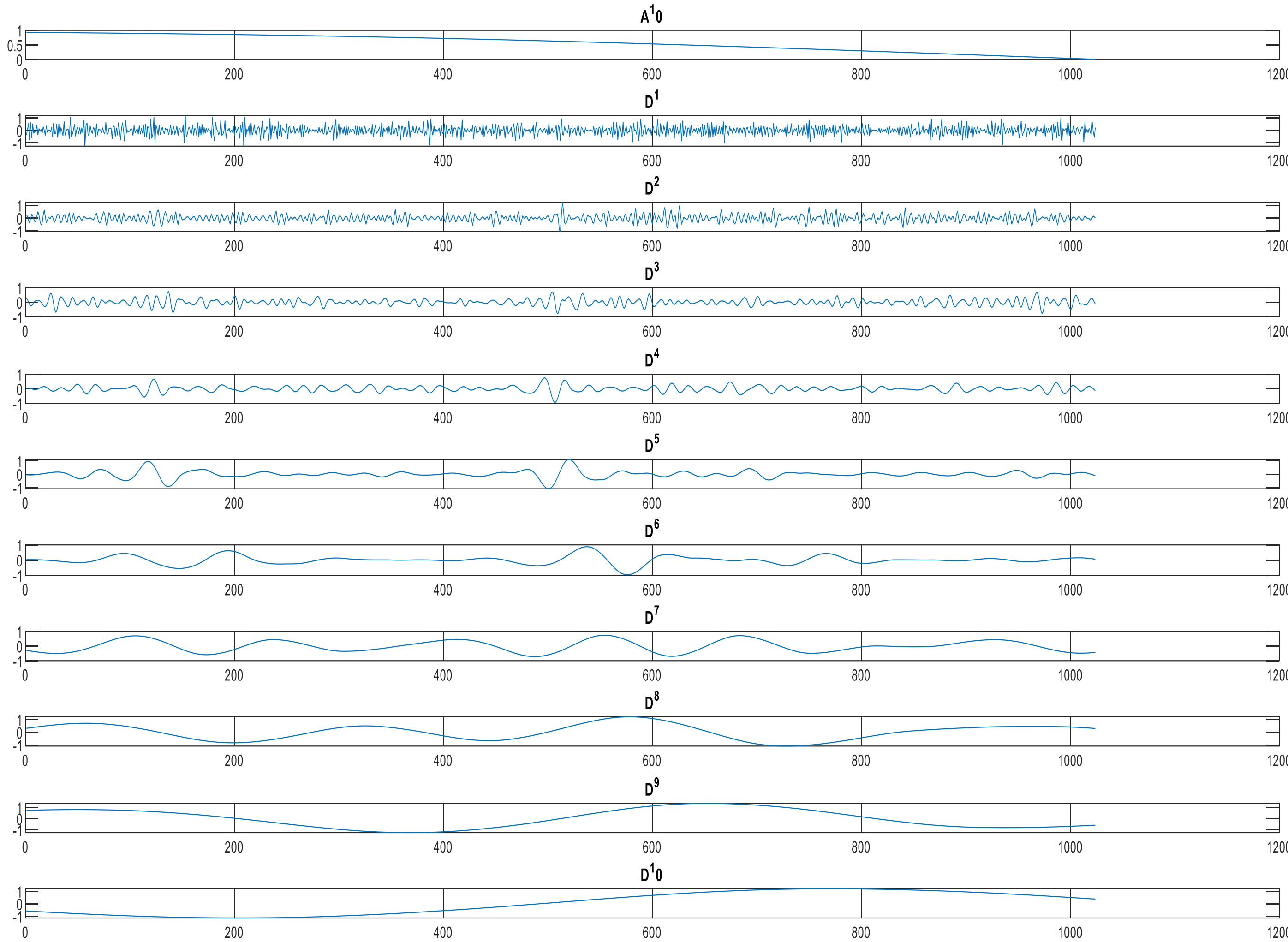
Reconstructed A10, D10, D9, …., D2, D1 for signal y1 with haar wavelet.



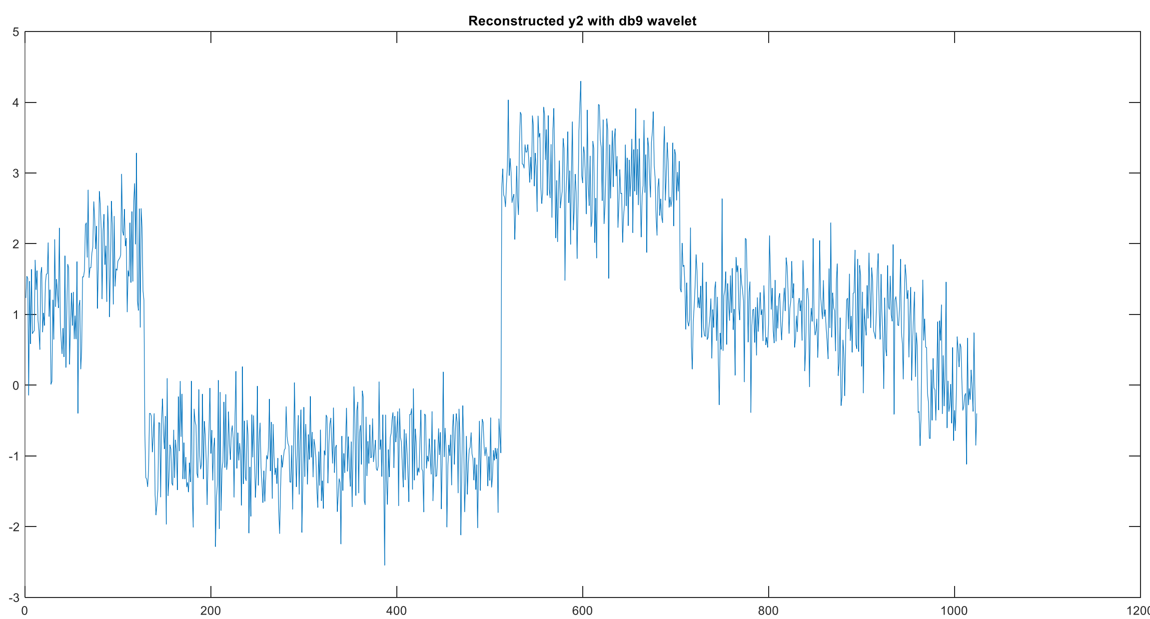
Reconstructed y1 signal using $y = \Sigma D^i + A$ with haar,



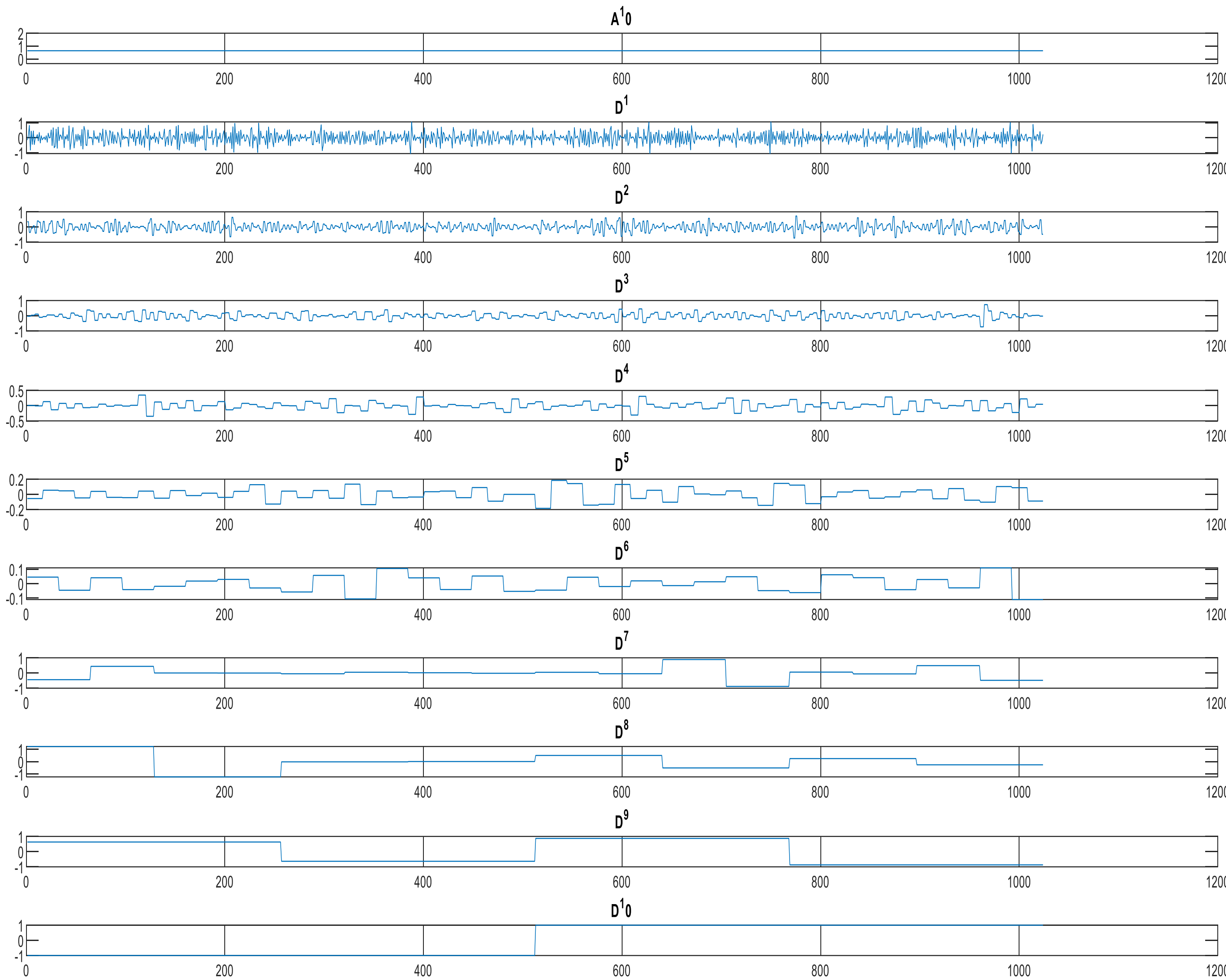
Reconstructed A10, D10, D9, …., D2, D1 for signal y2 with db9 wavelet.



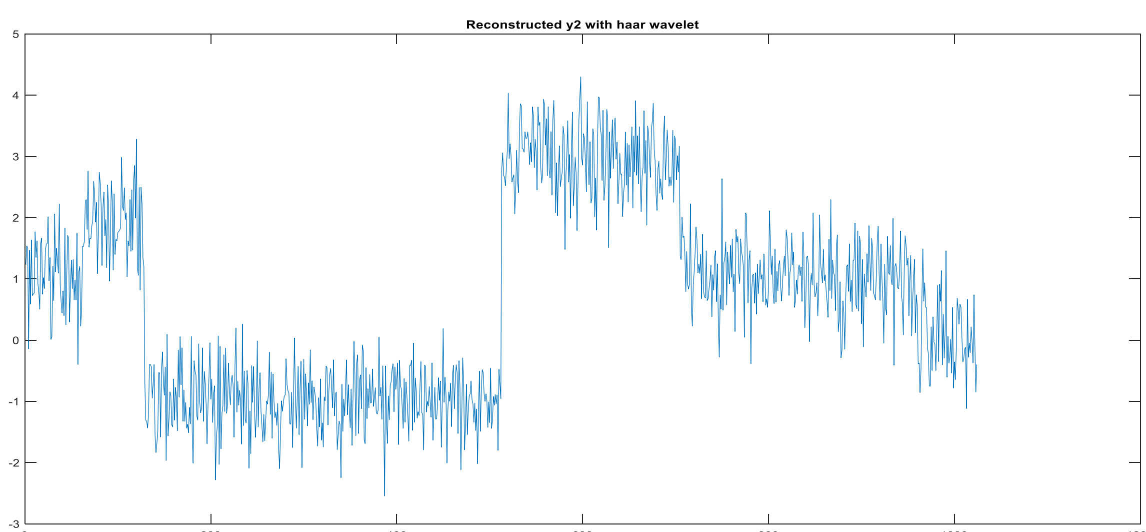
Reconstructed y2 signal using $y = \Sigma D^i + A$ with db9,



Reconstructed A10, D10, D9, …., D2, D1 for signal y2 with haar wavelet.



Reconstructed y2 signal using $y = \Sigma\, D^{i} \; + \; A$ with haar,



Calculated energy differences between original signal and the reconstructed signal.

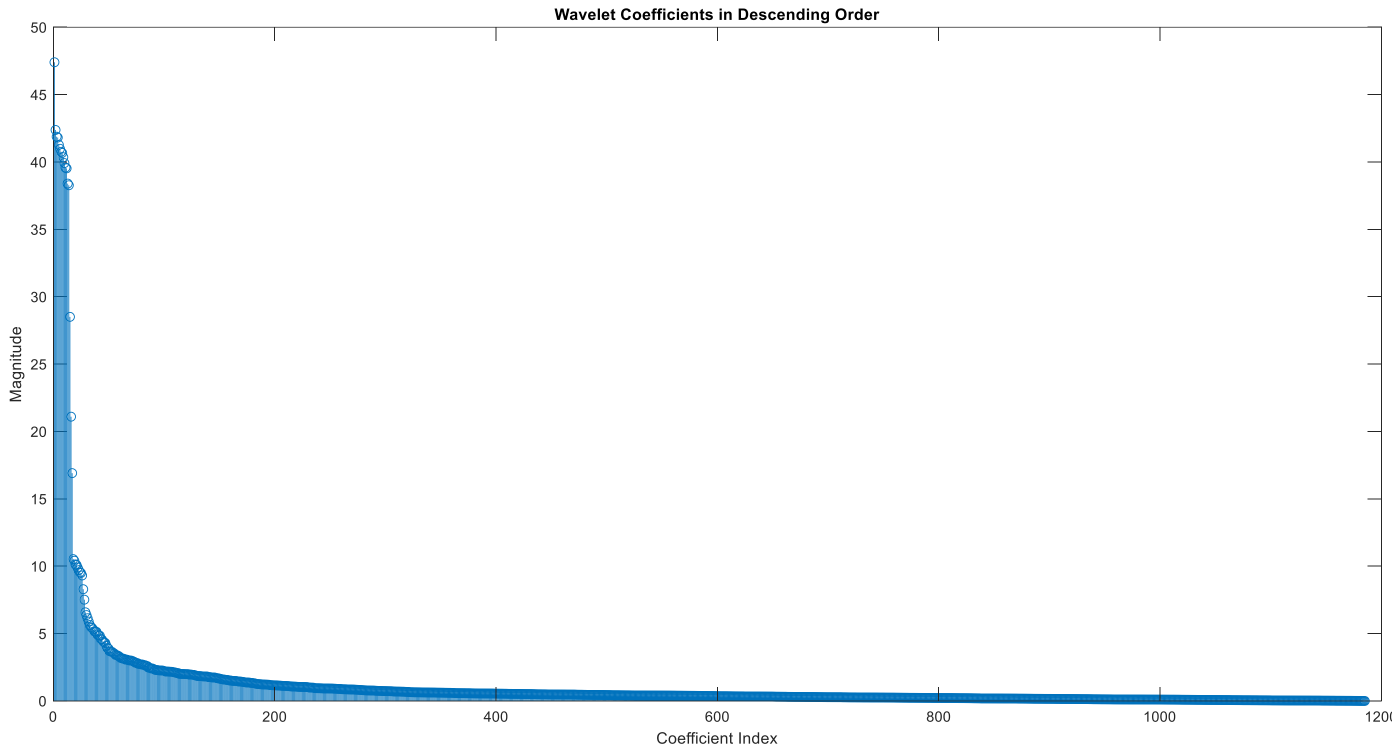
```
energy differnece between y1 and reconstructed y1 using db9: 0.00000030297406
energy differnece between y1 and reconstructed y1 using haar: 0.00000000000182
energy differnece between y2 and reconstructed y2 using db9: 0.00000050967537
energy differnece between y2 and reconstructed y2 using haar: 0.00000000000546
```

According to that we can observe that the haar wavelet has performed well when reconstructing the signal.

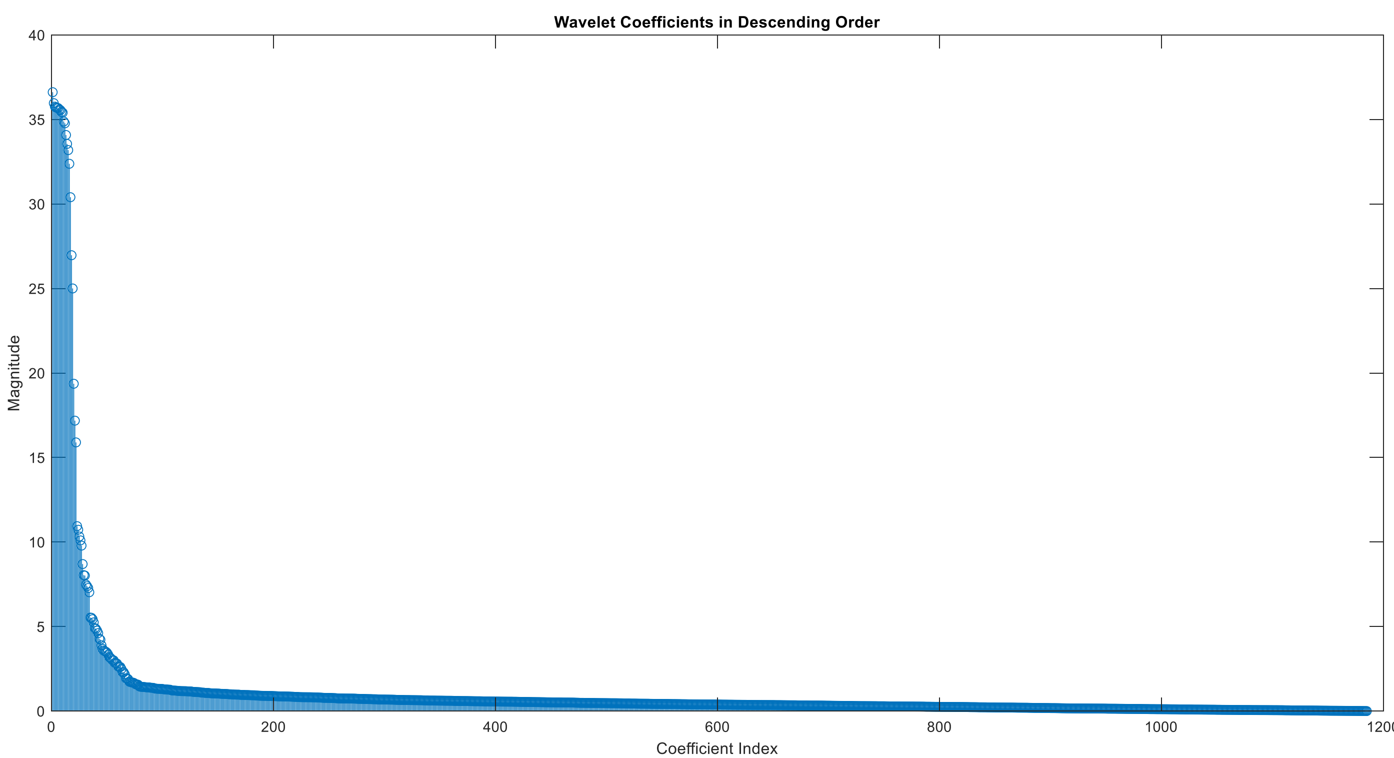
2.3. Signal Denoising with DWT

First consider the db9 wavelet.

Magnitude of wavelet coefficients (stem plot) of the y1 signal with db9 wavelet.



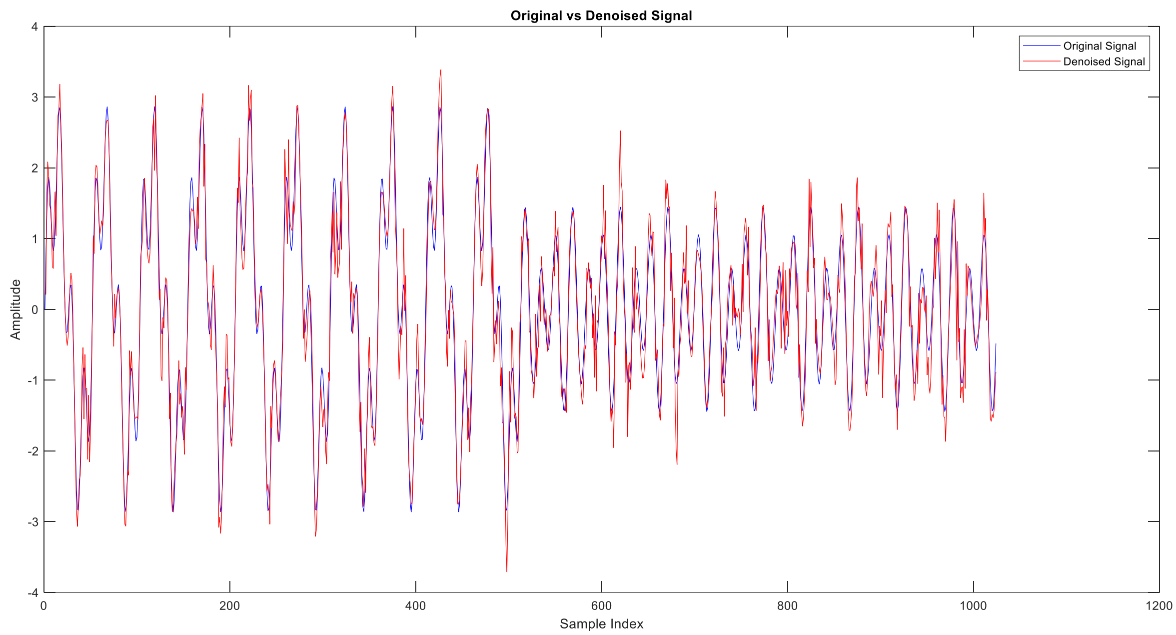
Magnitude of wavelet coefficients (stem plot) of the y2 signal with db9 wavelet.



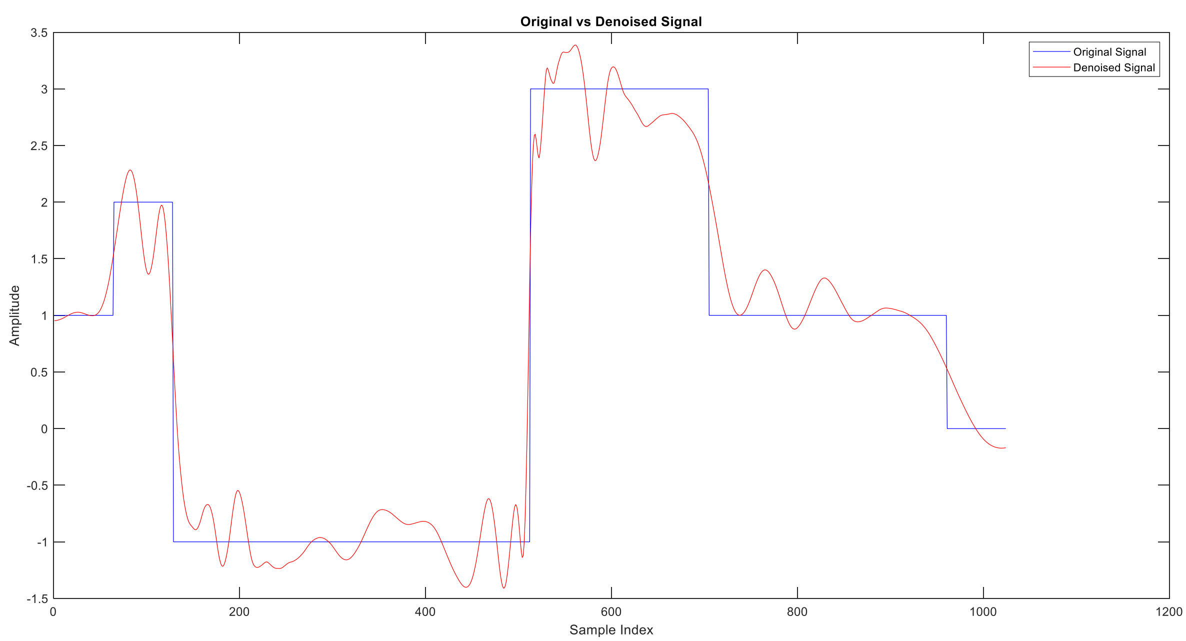
For the y1 signal the threshold is selected as **0.5** and for y2 signal the thrshold is **2.5**.

Then the coefficients which has the magnitude less than the selected threshold is suppressed. And then signal is reconstructed using *wavrec()*.

Reconstructed-Denoised y1 signal with db9.



Reconstructed-Denoised y2 signal with db9.

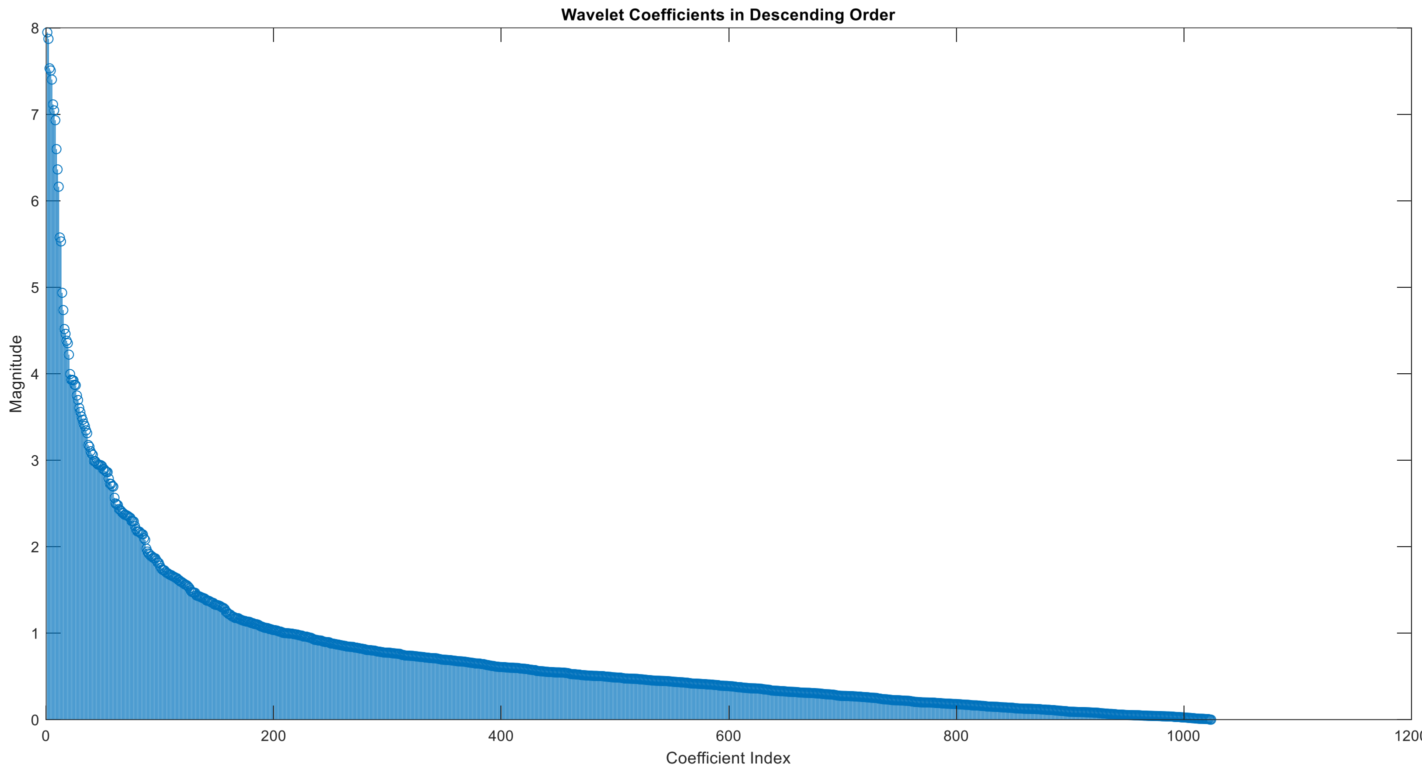


RMSE between original signal x1 and denoised y1 signal using db9: 0.34454648022603

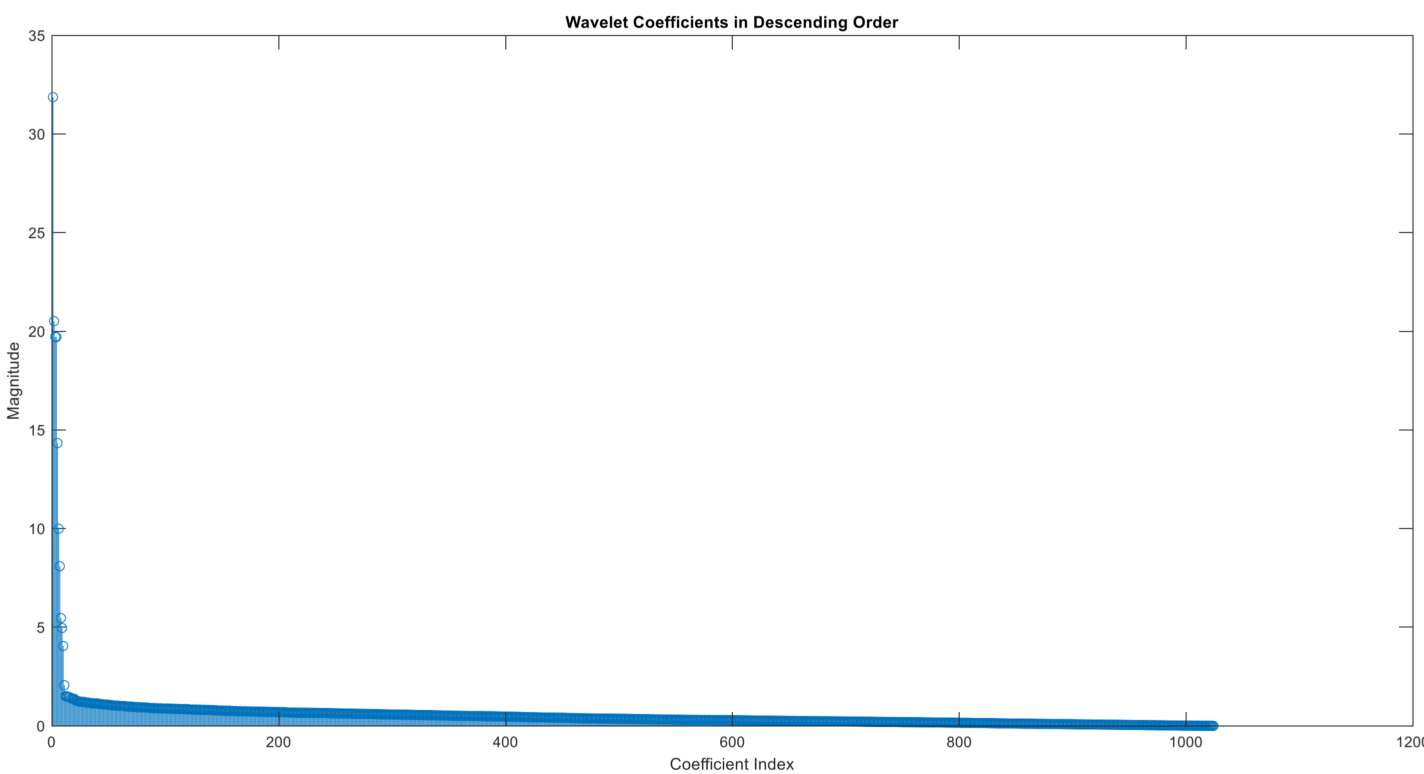
RMSE between original signal x2 and denoised y2 signal using db9: 0.30858405560891

Then consider the haar wavelet.

Magnitude of wavelet coefficients (stem plot) of the y1 signal with haar wavelet.



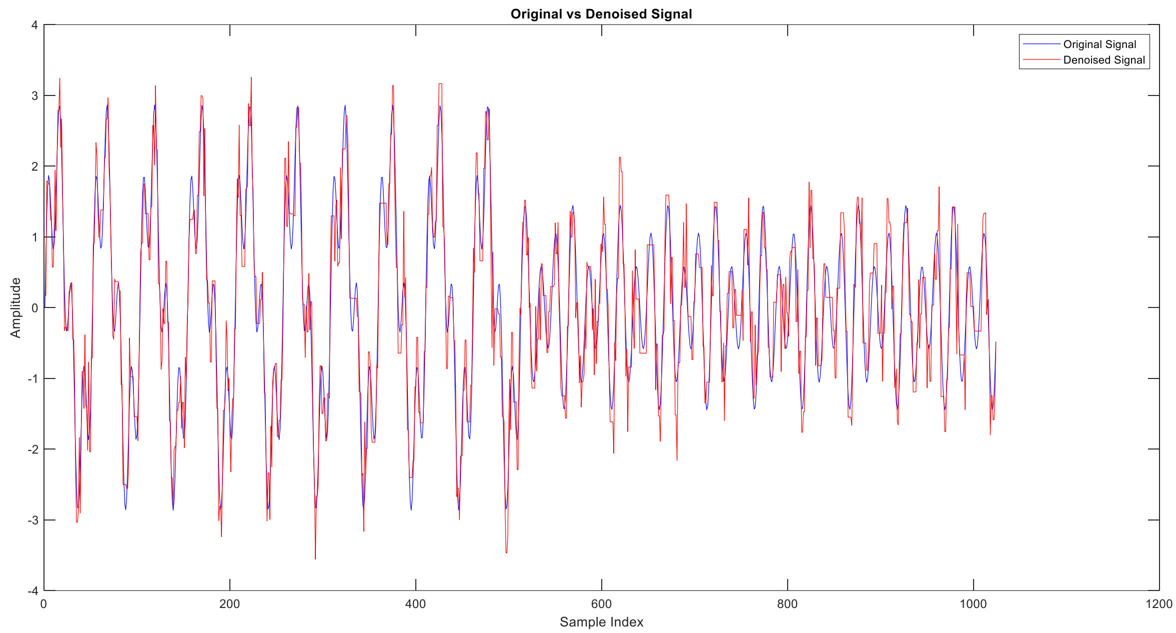
Magnitude of wavelet coefficients (stem plot) of the y2 signal with haar wavelet.



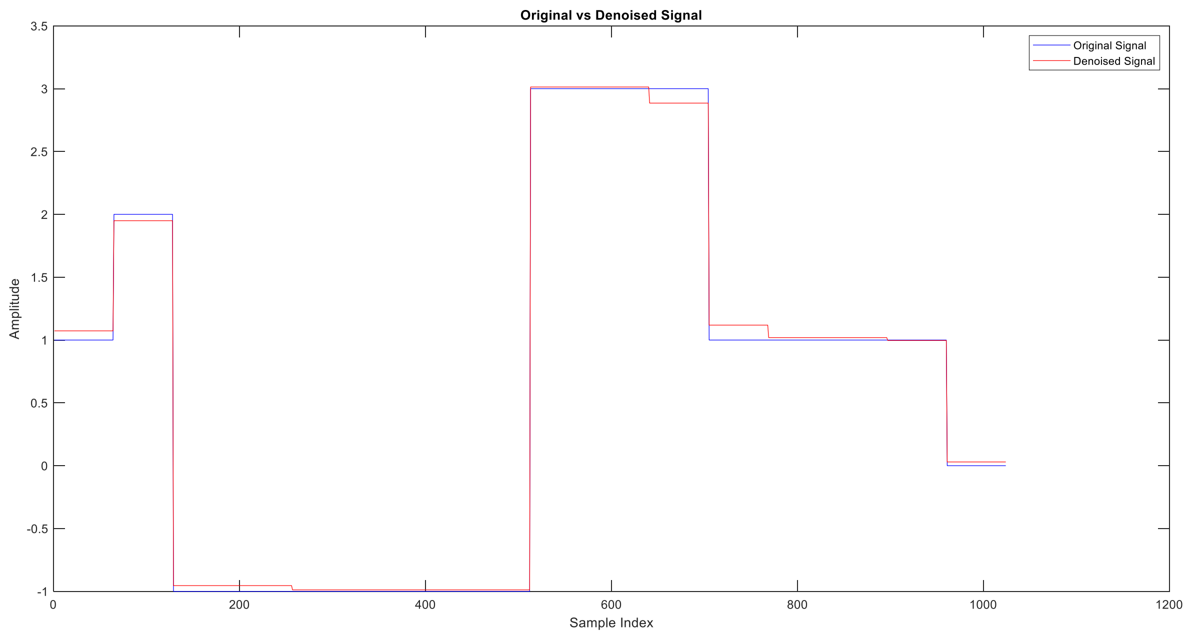
For the comparison purpose same thresholds are used with haar wavelet as well : for y1 threshold is selected as **0.5** and for y2 signal the thrshold is **2.5**.

Then,

Reconstructed-Denoised y1 signal with haar.



Reconstructed-Denoised y2 signal with haar.



RMSE between original signal x1 and denoised y1 signal using haar: 0.39923412752343

RMSE between original signal x2 and denoised y2 signal using haar: 0.05116642535226

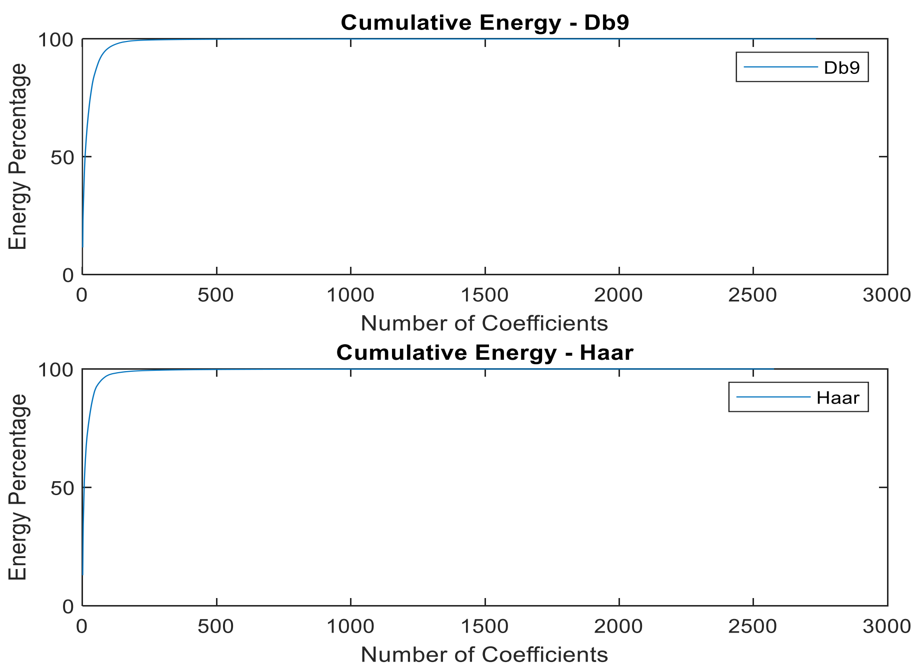
	Y1	Y2
Db9	**0.34454648022603**	0.30858405560891
Haar	0.39923412752343	**0.05116642535226**

The Haar wavelet shows optimal performance in denoising the y2 signal, achieving low RMSE and minimal distortion. For the y1 signal, the Db9 wavelet is more effective, preserving signal morphology while reducing noise, leading to better RMSE.

2.4. Signal Compression with DWT

Discrete wavelet coefficients are obtained using the *wavedec()*

Then evaluate the cumulative energy with the number of coefficients.



Then evaluated the number of wavelet coefficients need to represent 99% energy of the original signal;

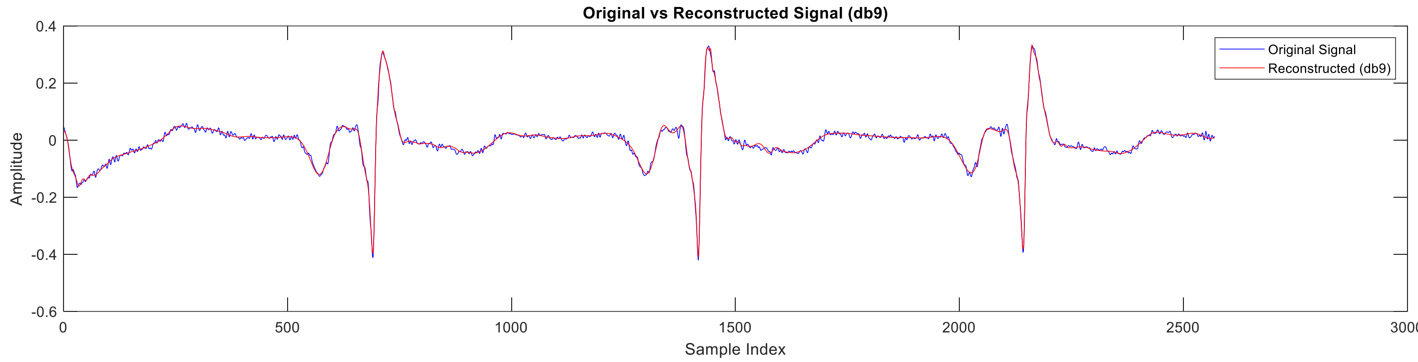
```
Number of coefficients for 99% energy (db9): 170
Number of coefficients for 99% energy (haar): 177
```

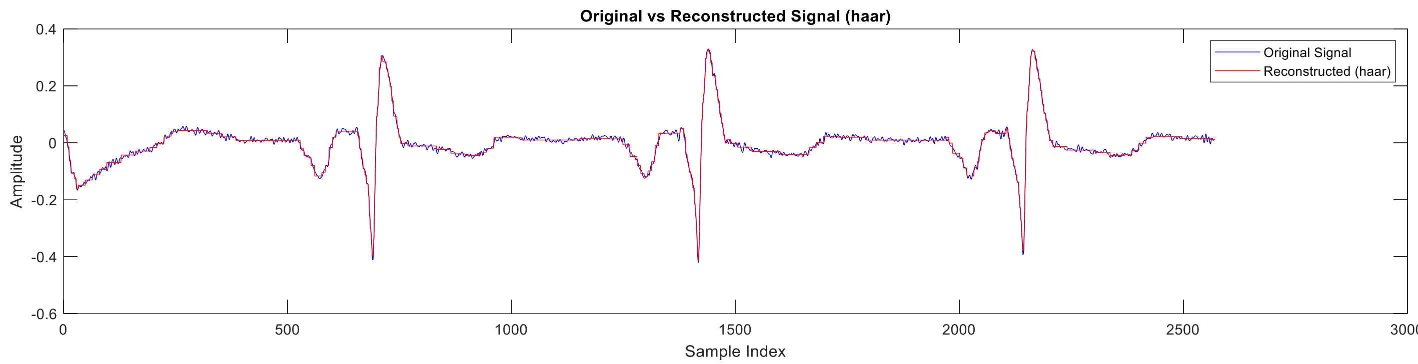
After sorting the wavelet coefficients in descending order, a threshold is set based on the amplitude values at the 170th coefficient for Db9 and the 177th coefficient for Haar. Coefficients with amplitudes below these thresholds are set to zero, effectively removing less significant components while retaining the primary features, thus optimizing signal compression.

Then the compression ratio can be evaluated using,

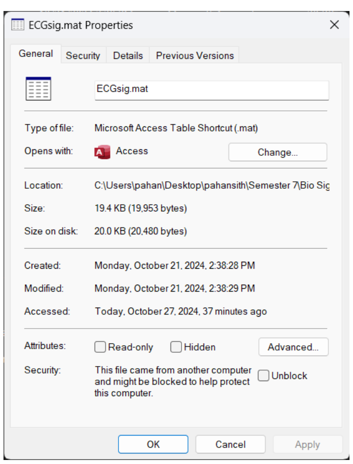
$$Compression\ Ratio\ (CR) = \frac{Original\ Number\ of\ Coefficients}{Compressed\ Number\ of\ Coefficients}$$

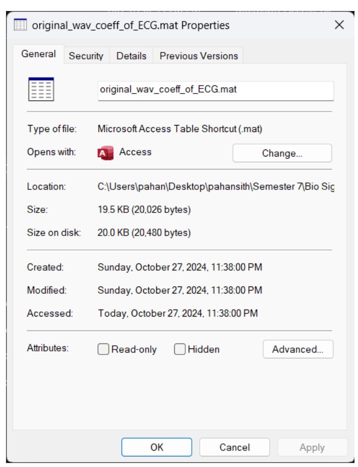
```
Compression Ratio (db9): 16.08
Compression Ratio (haar): 14.56
```

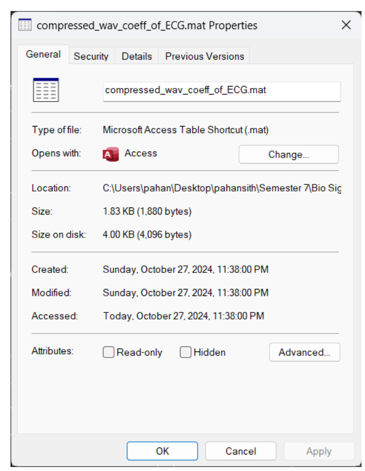




Examining the morphology of the reconstructed signal in comparison to the original ECG signal, we observe that the reconstruction retains nearly all relevant information, preserving the overall shape and key features of the ECG waveform. However, some minor details are lost in the compressed signal.







It is clear that the compressed wavelet coefficients has a very low file size.