

# Teminar 11

## Aplicatii

3.2.40

Se consideră în  $\mathbb{R}^3$  lista de vectori  $v = [v_1, v_2, v_3]^t$ . Folosind două metode (definiția bazei, respectiv lema morbității) să se verifice că  $a \in \mathbb{R}$  astfel încât  $v$  este o bază a lui  $\mathbb{R}^3$ , unde:

$$(1) \quad v_1 = (1, -2, 0) ; \quad v_2 = (2, 1, 1),$$

$$v_3 = (0, a, 1)$$

$$(2) \quad v_1 = (1, 1, -1) ; \quad v_2 = (0, 3, -1),$$

$$v_3 = (1, a, 1)$$

Iedure: (1) • Definiția :

liniar  
independenta  
este suficientă

$$d_1 v_1 + d_2 v_2 + d_3 v_3 = 0$$

$$d_1 \cdot (1, -2, 0) + d_2 (2, 1, 1) + d_3 (0, a, 1) = \\ = (0, 0, 0)$$

$$(d_1, -2d_1, 0) + (2d_2, d_2, d_2) + \\ + (0, ad_3, d_3) = (0, 0, 0)$$

$$(d_1 + 2d_2 + 0, -2d_1 + d_2 + ad_3, \\ d_2 + d_3) = (0, 0, 0)$$

$$\begin{cases} d_1 + 2d_2 + 0 \cdot d_3 = 0 \\ -2d_1 + d_2 + ad_3 = 0 \\ d_1 \cdot 0 + d_2 + d_3 = 0 \end{cases}$$

$$A = \begin{pmatrix} 1 & 2 & 0 \\ -2 & 1 & a \\ 0 & 1 & 1 \end{pmatrix}$$

Induzione nach obtemem  $d_1 = d_2 = d_3 = 0$

$\Leftarrow \det A \neq 0$ ,

$\Leftarrow \begin{vmatrix} 1 & 2 & 0 \\ -2 & 1 & a \\ 0 & 1 & 1 \end{vmatrix} \neq 0$

$$\begin{aligned}
 \Leftrightarrow 1 + (-2) \cdot 1 \cdot 0 + 0 \cdot 2 \cdot a - (0 + \\
 + a - 4) &\neq 0 \\
 1 + 4 - a &\neq 0 \\
 5 - a &\neq 0 \Rightarrow a \neq 5 \\
 \Rightarrow a \in \mathbb{R} \setminus \{5\}
 \end{aligned}$$

• Denumirea substituției  
 Fie  $b = [b_1, b_2, \dots, b_m]^t$  o  
 bază a  $K$ -spatiului vectorial  
 $V$  în  $\mathbb{R}^n$  și  $v \in V$  cu coordonatele  
 $[d_1, d_2, \dots, d_m]$  în raport cu  
 bază  $b$ . ( $v = d_1 \cdot b_1 + d_2 \cdot b_2 + \dots + d_m b_m$ )  
 Considerăm o listă de vectori  
 $b' = [b_1, \dots, v, \dots, b_m]$  care  
 rezultă din  $v$  prin înlocuirea  
 vectorului  $b_i$  cu  $v$ .  
 Atunci:  
 (a)  $b'$  este bază dacă

$d_i \neq 0$

(b) Dacă  $b'$  este bază și  $x \in V$  are coordonatele  $[x_1, x_2, \dots, x_n]$  în raport cu  $b$  și  $[x'_1, x'_2, \dots, x'_n]$  în raport cu  $b'$ , atunci:

$$\left\{ \begin{array}{l} x'_i = d_i^{-1} \cdot x_i \\ x'_{j \neq i} = d_i^{-1} (d_i x_j - d_j x_i), \end{array} \right.$$

$$v_1 = (1, -2, 0) ; v_2 = (2, 1, 1)$$

$$v_1 = \underbrace{\frac{1}{d_1} \cdot e_1}_{d_1} + \underbrace{(-2) \cdot e_2}_{d_2} + \underbrace{0 \cdot e_3}_{d_3}$$

$$\underline{d_1 = 1 \neq 0} \Rightarrow u_1 = (v_1, e_2, e_3)$$

este o bază  
în  $\mathbb{R}^3$

Coordonatele lui  $v_2$  în bază  $u_1$ :

$$x'_1 = d_1^{-1} \cdot x_1 = \frac{1}{d_1} \cdot x_1 = \frac{1}{1} \cdot 2 = 2$$

$$x_2^1 = \frac{1}{d_1} \cdot (d_1 \cdot x_2 - d_2 \cdot x_1) =$$

$$= \frac{1}{1} \cdot (1 \cdot 1 - (-2) \cdot 2) = 1 + 4 = 5$$

$$x_3^1 = \frac{1}{d_1} (d_1 \cdot x_3 - d_3 \cdot x_1) =$$

$$= \frac{1}{1} (1 \cdot 1 - 0 \cdot 2) = 1 - 0 = 1$$

$$v_2 = (2, 5, 1) = 2 \cdot v_1 + 5 \cdot l_2 + l_3$$

$$d_2 = 5 \neq 0 \Rightarrow u_2 = (v_1, v_2, l_3)$$

base in  $\mathbb{R}^3$

$$v_3 = (0, a, 1) = 0 \cdot l_1 + a \cdot l_2 + 1 \cdot l_3$$

• in base  $u_1$

$$x_1^1 = \frac{1}{d_1} \cdot x_1 = \frac{1}{1} \cdot 0 = 0$$

$$x_2^1 = \frac{1}{d_1} (d_1 x_2 - d_2 x_1) =$$

$$= \frac{1}{1} (1 \cdot a - (-2) \cdot 0) = a$$

$$x_3' = \frac{1}{d_1} (d_1 x_3 - d_3 x_1) = \\ = \frac{1}{1} (1 \cdot 1 - 0 \cdot 0) = 1$$

$$v_3 = \underbrace{0 \cdot v_1}_{x_1} + \underbrace{\alpha \cdot l_2}_{x_2} + \underbrace{1 \cdot l_3}_{x_3}$$

• im basis  $\mu_2$ :

$$x_2' = \frac{1}{d_2} \cdot x_2 = \frac{1}{5} \cdot \alpha = \frac{\alpha}{5}$$

$$x_1' = \frac{1}{d_2} (d_2 x_1 - d_1 x_2) = \\ = \frac{1}{5} (5 \cdot 0 - 2 \cdot \alpha) = -\frac{2\alpha}{5}$$

$$x_3' = \frac{1}{d_2} (d_2 x_3 - d_3 x_2) = \\ = \frac{1}{5} (5 \cdot 1 - 1 \cdot \alpha) = \frac{5-\alpha}{5}$$

$$v_3 = -\frac{2\alpha}{5} v_1 + \frac{\alpha}{5} v_2 + \frac{5-\alpha}{5} l_3$$

$$\frac{5-\alpha}{5} \neq 0 \Rightarrow 5-\alpha \neq 0$$

$$\Rightarrow \alpha \in \mathbb{R} \setminus \{5\}$$

$$(2) \quad v_1 = (2, 1, -1)$$

$$v_2 = (0, 3, -1)$$

$$v_3 = (1, \alpha, 1)$$

• Definitie :

$$d_1 v_1 + d_2 v_2 + d_3 v_3 =$$

$$= d_1 (2, 1, -1) + d_2 (0, 3, -1) + \\ + d_3 (1, \alpha, 1) =$$

$$= (2d_1, d_1, -d_1) + (0 \cdot d_2, 3d_2, -d_2)$$

$$+ (d_3, \alpha d_3, d_3)$$

$$= (2d_1 + 0 \cdot d_2 + d_3, d_1 + 3d_2 + \alpha d_3, \\ -d_1 - d_2 + d_3)$$

$$d_1 v_1 + d_2 v_2 + d_3 v_3 = 0$$

$$\Rightarrow \begin{cases} 2d_1 + 0 \cdot d_2 + d_3 = 0 \\ d_1 + 3d_2 + \alpha d_3 = 0 \\ -d_1 - d_2 + d_3 = 0 \end{cases}$$

$$A = \begin{pmatrix} 2 & 0 & 1 \\ 1 & 3 & a \\ -1 & -1 & 1 \end{pmatrix}$$

$$\det A \neq 0$$

$$\Leftrightarrow \begin{vmatrix} 2 & 0 & 1 \\ 1 & 3 & a \\ -1 & -1 & 1 \end{vmatrix} \neq 0$$

$$6 - 1 + 0 + 3 + 2a \neq 0$$

$$2a + 8 \neq 0$$

$$2a \neq -8$$

$$a \neq -4 \Rightarrow a \in \mathbb{R} \setminus \{-4\}$$

## • Darma substitutiei

$$v_1 = (2, 1, -1)' \quad v_2 = \begin{pmatrix} 0 \\ x_1 \\ 3 \\ x_2 \\ -1 \\ x_3 \end{pmatrix}$$

$$d_1 = 2 \cdot l_1 + 1 \cdot l_2 + (-1) \cdot l_3$$

$$d_2$$

$$d_3$$

$$d_1 \neq 0 \Rightarrow u_1 = [v_1, l_2, l_3]_3$$

este baza în  $\mathbb{R}^3$

Coord. lui  $v_2$  în bază  $u_1$ :

$$x_1' = \frac{1}{d_1} \cdot x_1 = \frac{1}{2} \cdot 0 = 0$$

$$x_2' = \frac{1}{d_1} (d_1 x_2 - d_2 x_1) = \\ = \frac{1}{2} (2 \cdot 3 - 1 \cdot 0) = 3$$

$$x_3' = \frac{1}{d_1} (d_1 x_3 - d_3 x_1) = \\ = \frac{1}{2} (2 \cdot (-1) - (-1) \cdot 0) = -1$$

$$v_2 = (0, 3, -1) = \frac{0 \cdot v_1}{d_1} + \frac{3 \cdot l_2}{d_2} + \frac{(-1) \cdot l_3}{d_3}$$

$$\underline{d_2 = 3 \neq 0} \Rightarrow u_2 = (v_1, v_2, l_3)_3 \text{ iste basis in } \mathbb{R}^3$$

$$v_3 = (1, u_{1,1})$$

• basis in  $u_1$ :

$$x_1' = \frac{1}{d_1} \cdot x_1 = \frac{1}{2} \cdot 1 = \frac{1}{2}$$

$$x_2' = \frac{1}{d_1} (d_1 x_2 - d_2 x_1) = \frac{1}{2} (2a - 1 \cdot 1) \\ = \frac{2a - 1}{2}$$

$$x_3' = \frac{1}{d_1} (d_1 x_3 - d_3 x_1) = \frac{1}{2} (2 \cdot 1 - (-1) \cdot 1) = \frac{3}{2}$$

$$\begin{aligned} v_3 &= \left( \frac{1}{2}, \frac{2a-1}{2}, \frac{3}{2} \right) = \\ &= \frac{1}{2} v_1 + \frac{2a-1}{2} e_2 + \frac{3}{2} e_3 \end{aligned}$$

• berechne  $u_2$ :

$$\begin{aligned} x_1' &= \frac{1}{d_2} \cdot (d_2 x_1 - d_1 x_2) = \\ &= \frac{1}{3} \cdot \left( 3 \cdot \frac{1}{2} - 0 \cdot \frac{2a-1}{2} \right) \\ &= \frac{1}{2} \end{aligned}$$

$$x_2' = \frac{1}{d_2} \cdot x_2 = \frac{1}{3} \cdot \frac{2a-1}{2} = \frac{2a-1}{6}$$

$$\begin{aligned} x_3' &= \frac{1}{d_2} (d_2 x_3 - d_3 x_2) = \\ &= \frac{1}{3} \left( 3 \cdot \frac{3}{2} - (-1) \cdot \frac{2a-1}{2} \right) = \\ &= \frac{a+4}{3} \end{aligned}$$

$$v_3 = \frac{1}{2} v_1 + \frac{2a-1}{6} v_2 + \frac{a+4}{3} l_3$$

$$\frac{a+4}{3} \neq 0 \Rightarrow a+4 \neq 0$$

$$a \neq -4$$

$$\Rightarrow a \in \mathbb{R} \setminus \{-4\}$$

3.2.41 Să se arate că

$$b = [b_1, b_2, b_3, b_4]^t \text{ unde } b_1 = [1, 2, -1, 2], b_2 = [1, 2, 1, 4]$$

$$b_3 = [2, 3, 0, -1], b_4 = [1, 3, -1, 0]$$

este o bază a lui  $\mathbb{R}^4$ .  
 să se determine coordonatele lui  $x = [2, 3, 2, 10]$  în raport cu acea bază.

Soluție:  $A = \begin{pmatrix} 1 & 1 & 2 & 1 \\ 2 & 1 & 3 & 3 \\ -1 & 1 & 0 & -1 \\ 2 & 4 & -1 & 0 \end{pmatrix}$

$$\det A = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 2 & 2 & 3 & 3 \\ -1 & 1 & 0 & -1 \\ 2 & 4 & -1 & 0 \end{vmatrix} =$$

$$\begin{array}{l} C_1 + C_2 \\ \hline C_4 + C_2 \end{array} \quad \left| \begin{array}{cccc} 2 & 1 & 2 & 2 \\ 4 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 \\ 6 & 4 & -1 & 4 \end{array} \right| =$$

$$= 1 \cdot (-1)^{2+3} \cdot \left| \begin{array}{ccc} 2 & 2 & 2 \\ 4 & 3 & 5 \\ 6 & -1 & 4 \end{array} \right| =$$

$$\begin{array}{l} C_1 - C_2 \\ = (-1) \cdot \\ C_3 - C_2 \end{array} \quad \left| \begin{array}{ccc} 0 & 0 & 0 \\ 1 & 3 & 2 \\ 7 & -1 & 5 \end{array} \right| = (-1) \cdot 2 \cdot (-1)^{1+2} \cdot \left| \begin{array}{cc} 1 & 2 \\ 7 & 5 \end{array} \right|$$

$$= 2 \cdot (5 - 14) = -18 \neq 0$$

$\Rightarrow \text{rang } A = 4$  (rang maximal)

$$\dim \mathbb{R}^4 = 4$$

$\Rightarrow$  b este bază în  $\mathbb{R}^4$

$$x = (2, 3, 2, 10)$$

$$(2, 3, 2, 10) = d_1 \textcircled{b}_1 + d_2 \textcircled{b}_2 + d_3 \textcircled{b}_3 + d_4 \textcircled{b}_4$$

$$(2,3,2,10) = \alpha_1 (1,2,-1,2) + \alpha_2 (1,2,1,4) \\ + \alpha_3 (2,3,0,-1) + \alpha_4 (1,3,-1,0)$$

$$(2,3,2,10) = (\alpha_1, 2\alpha_1, -\alpha_1, 2\alpha_1) + \\ + (\alpha_2, 2\alpha_2, \alpha_2, 4\alpha_2) + \\ + (2\alpha_3, 3\alpha_3, 0, -\alpha_3) + (\alpha_4, 3\alpha_4, -\alpha_4, 0)$$

$$(2,3,2,10) = (\alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4, \\ 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4, \\ -\alpha_1 + \alpha_2 + 0 - \alpha_4, \\ 2\alpha_1 + 4\alpha_2 - \alpha_3 + 0)$$

$$\left\{ \begin{array}{l} \alpha_1 + \alpha_2 + 2\alpha_3 + \alpha_4 = 2 \\ 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 3\alpha_4 = 3 \\ -\alpha_1 + \alpha_2 + 0 \cdot \alpha_3 - \alpha_4 = 2 \\ 2\alpha_1 + 4\alpha_2 - \alpha_3 + 0 \cdot \alpha_4 = 10 \end{array} \right.$$

$$\bar{A} = \begin{pmatrix} 1 & 1 & 2 & 1 & : & 2 \\ 2 & 2 & 3 & 3 & : & 3 \\ -1 & 1 & 0 & -1 & : & 2 \\ 2 & 4 & -1 & 0 & : & 10 \end{pmatrix}$$

$$\det A = -18 \neq 0$$

$\Rightarrow$  sistem compatibil determinat (sistem Cramer)

$$d_i = \frac{d_{di}}{\det A}$$

$$d_1 = \frac{d_{d1}}{-18}$$

$$d_{d1} = \begin{vmatrix} 2 & 1 & 2 & 1 \\ 3 & 2 & 3 & 3 \\ 2 & 1 & 0 & -1 \\ 10 & 4 & -1 & 0 \end{vmatrix} \left| \begin{array}{l} C_1 - 2C_2 \\ \\ \\ \end{array} \right. = C_4 + C_2$$

$$= \begin{vmatrix} 0 & 1 & 2 & 2 \\ -1 & 1 & 3 & 5 \\ 0 & 1 & 0 & 0 \\ 2 & 4 & -1 & 4 \end{vmatrix} =$$

$$= 1 \cdot (-1)^{2+3} \cdot \begin{vmatrix} 0 & 2 & 2 \\ -1 & 3 & 5 \\ 2 & -1 & 4 \end{vmatrix}$$

$$\underline{C_3 - C_2} = \begin{vmatrix} 0 & 2 & 0 \\ -1 & 3 & 2 \\ 2 & -1 & 5 \end{vmatrix} =$$

$$= -2 \cdot (-1)^{1+2} \begin{vmatrix} -1 & 2 \\ 2 & 5 \end{vmatrix} =$$

$$= 2 \cdot (-5 - 4) = -18$$

$$\boxed{d_1} = \frac{-18}{-18} = \boxed{1}$$

$$d_2 = \frac{d_{22}}{\det A} = \frac{d_{d_2}}{-18}$$

$$d_{d_2} = \begin{vmatrix} 1 & 2 & 2 & 1 \\ 2 & 3 & 3 & 3 \\ -1 & 2 & 0 & -1 \\ 2 & 10 & -1 & 0 \end{vmatrix} =$$

$$C_2 + 2C_1$$

$$= \begin{vmatrix} 1 & 4 & 2 & 0 \\ 2 & 7 & 3 & 1 \\ \textcircled{E}_1 & 0 & 0 & 2 \\ 4 & 14 & -1 & -2 \end{vmatrix} =$$

$$= (-1) \cdot (-1)^4 \cdot \begin{vmatrix} 4 & 2 & 0 \\ 7 & 3 & 1 \\ 14 & -1 & -2 \end{vmatrix} =$$

$$C_1 - 2C_2 = - \begin{vmatrix} 0 & \textcircled{2} & 0 \\ 1 & 3 & 1 \\ 16 & -1 & -2 \end{vmatrix} = -2 \cdot (-1)^3 \begin{vmatrix} 1 & 1 \\ 16 & -2 \end{vmatrix}$$

$$= 2 \cdot (-2 - 16) = -36$$

$$\boxed{d_2} = \frac{-36}{-18} = \boxed{2}$$

$$d_3 = \frac{d_{\lambda_3}}{-18}$$

$$d_{\lambda_3} = \begin{vmatrix} 1 & 1 & 2 & 3 \\ 2 & 2 & 3 & 3 \\ -1 & 1 & 2 & -1 \\ 2 & 4 & 10 & 0 \end{vmatrix} \begin{array}{l} C_1 - C_2 \\ = \\ C_3 - 2C_2 \\ C_4 - C_2 \end{array}$$

$$= \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -1 \\ -2 & 1 & 0 & -2 \\ -2 & 4 & 2 & -4 \end{vmatrix} = 1 \cdot (-1)^3 \cdot \begin{vmatrix} 0 & -1 & 1 \\ -2 & 0 & -2 \\ -2 & 2 & -4 \end{vmatrix}$$

$$\underline{C_2 + C_3} = \begin{vmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & -2 \\ -2 & -2 & -2 & -2 \\ -2 & -2 & -2 & -2 \end{vmatrix} = -(-1)^4 \cdot \begin{vmatrix} -2 & -2 \\ -2 & -2 \end{vmatrix}$$

$$= 0$$

$$\boxed{d_3} = \frac{0}{-18} = \boxed{0}$$

$$d_4 = \frac{d_{\lambda_4}}{-18}$$

$$d_{\lambda_4} = \begin{vmatrix} 1 & 1 & 2 & 2 \\ 2 & 2 & 3 & 3 \\ -1 & 1 & 0 & 2 \\ 2 & 4 & -1 & 10 \end{vmatrix} \begin{array}{l} C_1 + C_2 \\ = \\ C_4 - 2C_2 \end{array}$$

$$= \begin{vmatrix} 2 & 1 & 0 \\ 4 & 2 & -1 \\ 0 & 1 & 0 \end{vmatrix} = 1 \cdot (-1)^5 \cdot \begin{vmatrix} 2 & 2 & 0 \\ 4 & 3 & -1 \\ 6 & -1 & 2 \end{vmatrix}$$

$$C_1 - C_2 = - \begin{vmatrix} 0 & 0 & 0 \\ 1 & 3 & -1 \\ 7 & -1 & 2 \end{vmatrix} = -2 \cdot (-1)^3 \begin{vmatrix} 1 & -1 \\ 7 & 2 \end{vmatrix}$$

$$= 2 (2 + 7) = 18$$

$$\boxed{d_h} = \frac{18}{-18} = \boxed{-1}$$

$$X = (1, 2, 0, -1) \quad (\text{in base } 5)$$

$\downarrow$        $\downarrow$        $\downarrow$        $\downarrow$   
 $d_1$      $d_2$      $d_3$      $d_4$