Bellman's Equations

Reference : Thomas J. Sargent, *Dynamic Macroeconomic Theory*, Section 1.1-1.4

The objective of this handout is to introduce Bellman's Equations in a brief (easy and informal) way. If you are interested in this topic in dept, you can study further from any book about *Dynamic Programming.*

To understand what we had studied in class better, I will introduce the finite period version of Bellman's equation first.

Bellman's "**optimality principle**" is:

An optimal policy has the property that, whatever the initial state and decision (i.e., control) are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.

i.e., if the part of a control variable from time $t = 0$ on is an optimal program as evaluated at time $t = 0$, then at any later time $t = T$ the **same** path from $t = T$ on must be an optimal program, "in its own right", as evaluated at $t = T$.

Consider the finite period optimization. We would like to find the optimum consumption path from $t = 0$ to $t = T$ by choosing the control variable c_t (or co-state variable k_{t+1} if applicable) to maximize our objective function $\sum_{t=0}^{T} u_t(c_t)$ subject to constraints $k_{t+1} = g_t(k_t, c_t)$ for $t = 0, \ldots, T$.

Now suppose we are at time $t = T$, by Bellman's optimality principle, we can define the value function for one-period problem by,

$$
\begin{aligned}
W_1(k_T) = \quad & \max_{c_T} \quad \{u_T(k_T, c_T) + W_0(k_{T+1})\} \\
& \text{s.t.} \qquad k_{T+1} = g_T(k_T, c_T)
\end{aligned}
\tag{1}
$$

Next, suppose we are at time $t = T - 1$, we also (in similar way) can define the value function for two-period problem as,

$$
\begin{aligned}
W_2(k_{T-1}) = \quad & \max_{c_{T-1}} \quad \{u_{T-1}(k_{T-1}, c_{T-1}) + W_1(k_T)\} \\
& \text{s.t.} \qquad k_T = g_{T-1}(k_{T-1}, c_{T-1})
\end{aligned}
\tag{2}
$$

If we iterate this procedure again and again, we, finally, get the $(T + 1)$-period value function by,

$$
\begin{aligned}
W_{T+1}(k_0) = \quad & \max_{c_0} \quad \{u_0(k_0, c_0) + W_T(k_1)\} \\
& \text{s.t.} \qquad k_1 = g_0(k_0, c_0) \\
& \text{and} \qquad k_0 \qquad \text{given}
\end{aligned}
\tag{3}
$$

We can see that this problem is a recursive problem. The way to solve this kind of problem is solving backward from the last period to the initial situation.

In general, our objective function is in the form $u_t(c_t) = \beta^t u(c_t)$ and $g_t(k_t, c_t) = g(k_t, c_t)$. Therefore, our $(j+1)$-period value function becomes,

$$W_{j+1}(k_{T-j}) = \max_{c_{T-j}} \left\{\beta^{T-j} u(c_{T-j}) + W_j(k_{T-j+1})\right\}$$

Multiply both sides by β^{j-T} gives,

$$\beta^{j-T} W_{j+1}(k_{T-j}) = \max_{c_{T-j}} \left\{u(c_{T-j}) + \beta \cdot \beta^{j-1-T} W_j(k_{T-j+1})\right\}$$

Define the current value function,

$$V_{j+1}(k_{T-j}) = \beta^{j-T} W_{j+1}(k_{T-j})$$

Therefore, we can write our problem as,

$$
\begin{aligned}
V_{j+1}(k_{T-j}) = \quad & \max_{c_{T-j}} \quad \{u(c_{T-j}) + \beta V_j(k_{T-j+1})\} \\
& \text{s.t.} \qquad k_{T-j+1} = g(k_{T-j}, c_{T-j}) \\
& \text{and} \qquad k_{T-j} \qquad \text{given}
\end{aligned}
\tag{4}
$$

More compact form,

$$
\begin{aligned}
V_{j+1}(k) = \quad & \max_{c} \quad \left\{u(c) + \beta V_j(\tilde{k})\right\} \\
& \text{s.t.} \qquad \tilde{k} = g(k, c) \\
& \text{and} \qquad k \qquad \text{given}
\end{aligned}
\tag{5}
$$

where tilde denotes next-period values.

Under particular conditions, iterate (5) starting any bounded and continuous initial V_0 converges as $j \to \infty$. If the value function converges, i.e. steady state solution exists, then $V = \lim_{j \to \infty} V_j$.

Then we will have the infinite-period version of Bellman's equation:

$$V(k_t) = \max_{c_t} \{u(c_t) + \beta V(k_{t+1})\} \tag{6}$$

To solve Bellman's equations, we need to find 2 things:

1. FOC:

$$\frac{\partial V}{\partial c_t} = 0 \tag{7}$$

2. Benveniste and Scheinkman's formular (similar to envelope theorem, differentiate (6) with respect to k_t).

$$V'(k_t) = \frac{\partial u(c_t)}{\partial k_t} + \beta \frac{\partial V(k_{t+1})}{\partial k_t} \tag{8}$$

suppose the solution of FOC exists and it is $c_t = h(k_t)$ (policy function). Substitute constraint into (8), we get,

$$V'(k_t) = u'(c_t)h'(k_t) + \beta \frac{\partial g}{\partial k_t} V'(g(k_t, c_t)) \tag{9}$$

With equations (7), (9) and initial conditions, we can solve for c_t.

To make the picture clearer, let's try an example.

Optimal Growth

$$\begin{array}{ll} \max & \sum_{t=0}^{\infty} \beta^t \ln(c_t) \\ \text{s.t.} & c_t + k_{t+1} = Ak_t^{\alpha} \end{array}$$

Bellman's equation is then,

$$V(k_t) = \max_{c_t, k_{t+1}} \{\ln(c_t) + \beta V(k_{t+1})\} = \max_{k_{t+1}} \{\ln(Ak_t^{\alpha} - k_{t+1}) + \beta V(k_{t+1})\}$$

Using the recursive method, assume $V_0 = 0$. Then,

$$\begin{aligned} V_1(k) &= \max_{\tilde{k}} \left\{ \ln\left(Ak^{\alpha} - \tilde{k}\right) + \beta V_0(\tilde{k}) \right\} \\ &= \max_{\tilde{k}} \left\{ \ln\left(Ak^{\alpha} - \tilde{k}\right) \right\} \end{aligned} \tag{10}$$

The solution of (10) is corner solution, the optimal is setting $\tilde{k} = 0$. Therefore, we will have,

$$
\begin{aligned}
\tilde{k} &= 0 && (11)\\
c &= Ak^{\alpha} && (12)\\
V_1(k) &= \ln(Ak^{\alpha}) = \ln A + \alpha \ln k && (13)
\end{aligned}
$$

Next,

$$
\begin{aligned}
V_2(k) &= \max_{\tilde{k}} \left\{ \ln\left(Ak^{\alpha} - \tilde{k}\right) + \beta V_1(\tilde{k}) \right\} \\
&= \max_{\tilde{k}} \left\{ \ln\left(Ak^{\alpha} - \tilde{k}\right) + \beta\left(\ln A + \alpha \ln \tilde{k}\right) \right\} && (14)
\end{aligned}
$$

FOC:

$$
\begin{aligned}
\frac{\partial V_2}{\partial \tilde{k}} = 0 \qquad \Longrightarrow \qquad -\frac{1}{Ak^{\alpha} - \tilde{k}} + \frac{\alpha\beta}{\tilde{k}} &= 0 \\
\tilde{k} &= \frac{\alpha\beta}{1+\alpha\beta} Ak^{\alpha} && (15)
\end{aligned}
$$

Then we will get,

$$
c = \frac{1}{1+\alpha\beta} Ak^{\alpha} \qquad (16)
$$

and

$$
\begin{aligned}
V_2(k) &= \ln\left(Ak^{\alpha} - \frac{\alpha\beta}{1+\alpha\beta}Ak^{\alpha}\right) + \beta\left(\ln A + \alpha \ln\left(\frac{\alpha\beta}{1+\alpha\beta}Ak^{\alpha}\right)\right) \\
&= \ln\left(\left(1 - \frac{\alpha\beta}{1+\alpha\beta}\right)Ak^{\alpha}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{A}{1+\alpha\beta}\right) + \alpha\beta\ln(\alpha\beta) + \alpha^2\beta \ln k \\
&= \ln\left(\frac{A}{1+\alpha\beta}\right) + \alpha \ln k + \beta \ln A + \alpha\beta\ln(\alpha\beta) + \alpha\beta \ln\left(\frac{A}{1+\alpha\beta}\right) + \alpha^2\beta \ln k \\
&= \ln\left(\frac{A}{1+\alpha\beta}\right) + \beta \ln A + \alpha\beta \ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha(1+\alpha\beta)\ln k && (17)
\end{aligned}
$$

Next,

$$
\begin{aligned}
V_3(k) &= \max_{\tilde{k}} \left\{ \ln\left(Ak^{\alpha} - \tilde{k}\right) + \beta V_2(\tilde{k}) \right\} \\
&= \max_{\tilde{k}} \left\{ \ln\left(Ak^{\alpha} - \tilde{k}\right) + \beta\left(\ln(\frac{A}{1+\alpha\beta}) + \beta \ln A + \alpha\beta \ln(\frac{\alpha\beta A}{1+\alpha\beta}) + \alpha(1+\alpha\beta)\ln \tilde{k}\right) \right\} && (18)
\end{aligned}
$$

FOC:

$$
\begin{aligned}
\frac{\partial V_3}{\partial \tilde{k}} = 0 \qquad \Longrightarrow \qquad -\frac{1}{Ak^\alpha - \tilde{k}} + \frac{\alpha\beta(1+\alpha\beta)}{\tilde{k}} &= 0 \\
\tilde{k} &= \frac{\alpha\beta + \alpha^2\beta^2}{1+\alpha\beta+\alpha^2\beta^2} Ak^\alpha \qquad (19)
\end{aligned}
$$

Again, we will get,

$$
c = \frac{1}{1+\alpha\beta+\alpha^2\beta^2} Ak^\alpha \qquad (20)
$$

and

$$
\begin{aligned}
V_3(k) &= \ln\left(Ak^\alpha - \frac{\alpha\beta(1+\alpha\beta)}{1+\alpha\beta(1+\alpha\beta)} Ak^\alpha\right) \\
&\quad +\beta\left(\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta\ln A + \alpha\beta\ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \alpha\left(1+\alpha\beta\right)\ln\left(\frac{\alpha\beta(1+\alpha\beta)}{1+\alpha\beta(1+\alpha\beta)} Ak^\alpha\right)\right) \\
&= \beta\ln\left(\frac{A}{1+\alpha\beta}\right) + \beta^2\ln A + \beta^2\alpha\ln\left(\frac{\alpha\beta A}{1+\alpha\beta}\right) + \ln\left(\frac{A}{1+\alpha\beta+\alpha^2\beta^2}\right) \\
&\quad +\alpha\beta\left(1+\alpha\beta\right)\ln\left(\frac{A\alpha\beta\left(1+\alpha\beta\right)}{1+\alpha\beta+\alpha^2\beta^2}\right) + \alpha\left(1+\alpha\beta+\alpha^2\beta^2\right)\ln k \qquad (21)
\end{aligned}
$$

Repeat again for $V_4\left(k\right)$, we will get,

$$
\begin{aligned}
\tilde{k} &= \frac{\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3}{1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3} Ak^\alpha \qquad (22) \\
c &= \frac{1}{1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3} Ak^\alpha \qquad (23) \\
V_4\left(k\right) &= \ln\left(\frac{A}{1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3}\right) + \beta\ln\left(\frac{A}{1+\alpha\beta+\alpha^2\beta^2}\right) + \beta^2\ln\left(\frac{A}{1+\alpha\beta}\right) \\
&\quad +\beta^3\ln A + \alpha\beta\left(1+\alpha\beta+\alpha^2\beta^2\right)\ln\left(\frac{A\alpha\beta\left(1+\alpha\beta+\alpha^2\beta^2\right)}{1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3}\right) \\
&\quad +\beta\left[\alpha\beta\left(1+\alpha\beta\right)\ln\left(\frac{A\alpha\beta\left(1+\alpha\beta\right)}{1+\alpha\beta+\alpha^2\beta^2}\right)\right] + \beta^2\left[\alpha\beta\ln(\frac{\alpha\beta A}{1+\alpha\beta})\right] \\
&\quad +\alpha\left(1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3\right)\ln k \qquad (24)
\end{aligned}
$$

Repeat again and again. But now we can see what happen if we repeat.

Consider equation (15), (19) and (22), we observe that at steady state, it should be

$$
\tilde{k} = \frac{\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3+\ldots}{1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3+\ldots} Ak^\alpha = \frac{\alpha\beta\left(1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3+\ldots\right)}{1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3+\ldots} Ak^\alpha = \alpha\beta Ak^\alpha \qquad (25)
$$

And equation (16), (20) and (23) tell us that it should be

$$c=\frac{1}{1+\alpha\beta+\alpha^2\beta^2+\alpha^3\beta^3+\dots}Ak^\alpha=(1-\alpha\beta)Ak^\alpha \tag{26}$$

And also equation (17), (21) and (24) predict that

$$V(k)=(1-\beta)^{-1}\left\{\ln\left[A\left(1-\alpha\beta\right)\right]+\frac{\alpha\beta}{1-\alpha\beta}\ln\left(A\alpha\beta\right)\right\}+\frac{\alpha}{1-\alpha\beta}\ln k \tag{27}$$

Eventhough the recursive method is implemented directly from the definition but it is hard to calculate, take a long time and can make a mistake easily. So, to make it easier, we will **guess** the solution and substitute it into Bellman's equation.

What is the good guess? From my experience, we should guess it from the solution of one-period and two-period problem, e.g., equation (13), $V_1(k)=\ln A+\alpha\ln k$.

We should guess the value function as,

$$V(k)=E+F\ln k \tag{28}$$

where E and F are constants.

To see how it works, let's solve this problem again by guessing method.

$$V(k)=\max_{\tilde{k}}\left\{\ln\left(Ak^\alpha-\tilde{k}\right)+\beta V(\tilde{k})\right\} \tag{29}$$

FOC:

$$\frac{\partial V}{\partial \tilde{k}}=0 \qquad \Longrightarrow \qquad -\frac{1}{Ak^\alpha-\tilde{k}}+\beta V'(\tilde{k})=0 \tag{30}$$

But we assume equation (28), then

$$V'(\tilde{k})=\frac{F}{\tilde{k}} \tag{31}$$

Therefore,

$$\begin{aligned}\beta\frac{F}{\tilde{k}} &= \frac{1}{Ak^\alpha-\tilde{k}}\\ \tilde{k} &= \frac{\beta F}{1+\beta F}Ak^\alpha\end{aligned} \tag{32}$$

Differentiate (29) with respect to k,

$$
\begin{aligned}
V'(k) &= \frac{1}{Ak^{\alpha}-\tilde{k}}\left(\alpha Ak^{\alpha-1}-\frac{\partial\tilde{k}}{\partial k}\right)+\beta\frac{\partial\tilde{k}}{\partial k}V'(\tilde{k}) \\
&= \frac{1}{Ak^{\alpha}-\tilde{k}}\alpha Ak^{\alpha-1}+\left(-\frac{1}{Ak^{\alpha}-\tilde{k}}+\beta V'(\tilde{k})\right)\frac{\partial\tilde{k}}{\partial k} \\
&= \frac{1}{Ak^{\alpha}-\tilde{k}}\alpha Ak^{\alpha-1}
\end{aligned} \tag{33}
$$

In the second equality, the term in parenthesis is equal to zero by FOC.

Substitute (31) and (32) into (33), we will get,

$$
\frac{F}{k}=\frac{1}{Ak^{\alpha}-\frac{\beta F}{1+\beta F}Ak^{\alpha}}\alpha Ak^{\alpha-1} \tag{34}
$$

Solve (34) for F , then we will get,

$$
F=\frac{\alpha}{1-\alpha\beta} \tag{35}
$$

Substitute (35) back into (32), we now get the capital stock,

$$
\tilde{k}=A\alpha\beta k^{\alpha} \tag{36}
$$

Substitute (36) back into constraint,

$$
c+A\alpha\beta k^{\alpha}=Ak^{\alpha} \qquad \Longrightarrow \qquad c=(1-\alpha\beta)Ak^{\alpha} \tag{37}
$$

Substitute (28), (36) and (37) into Bellman's equation and solve for E

$$
\begin{aligned}
E+F\ln k &= \ln\left[(1-\alpha\beta)Ak^{\alpha}\right]+\beta\left[E+F\ln\left(A\alpha\beta k^{\alpha}\right)\right] \\
E+F\ln k &= \ln\left[(1-\alpha\beta)A\right]+\alpha\ln k+\beta E+\beta F\ln(A\alpha\beta)+\beta F\alpha\ln k \\
E+F\ln k &= \ln\left[(1-\alpha\beta)A\right]+\beta E+\beta F\ln(A\alpha\beta)+\alpha\left(1+\beta F\right)\ln k
\end{aligned} \tag{38}
$$

Then we will see that,

$$
E=\ln\left[(1-\alpha\beta)A\right]+\beta E+\beta F\ln(A\alpha\beta) \tag{39}
$$

Then solve for E,

$$
E=(1-\beta)^{-1}\left[\ln\left[(1-\alpha\beta)A\right]+\beta\frac{\alpha}{1-\alpha\beta}\ln(A\alpha\beta)\right] \tag{40}
$$

Finally, we get the value function,

$$
V(k)=(1-\beta)^{-1}\left[\ln\left[(1-\alpha\beta)A\right]+\beta\frac{\alpha}{1-\alpha\beta}\ln(A\alpha\beta)\right]+\frac{\alpha}{1-\alpha\beta}\ln k \tag{41}
$$

Which are exactly the same solutions as ones in the recursive method.