Scientific Computing

Lecture 07: Linear Programming
Part 03: Simplex Algorithm

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Introduction

In this lecture, we will cover the simplex algorithm to solve SMPs (standard maximum problems we covered in Lectures 05 and 06 last week).

The usefulness of the simplex method rests on the fact that, when implemented in a computer program, it can be used to solve problems with very large (hundreds. sometimes thousands!) numbers of decision variables.

The fundamental theorem of LP tells us that if there is a solution, it will be at a corner point of the feasible set.

Introduction

Of course, n constraints and n variables may give us a feasible set with 2^n corner points. We can easily visualize this situation in 2D with 2 variables and 2 constraints forming a rectangular feasible set with 4 corner points (when the constraint lines intersect each other and the x and y coordinate lines).

Thus, a problem with 20 decision variables and 20 constraints may have a feasible set with $\approx 2^{20} = 1,048,576$, i.e., the number of subsets of the set with 20 elements, (that's over $1,000,000!$) corner points. Not a fun set to work with if we have to evaluate the objective function at every corner point.

Introduction

The simplex method evaluates the objective function at one corner point after another in such a way that the objective function achieves better and better values; better may be larger if we are maximizing and smaller if we are minimizing.

So what makes this algorithm fast? The algorithm **does not evaluate the objective function at all corner points**.

A great advantage of the simplex algorithm is that it is an algebraic method, which is especially useful in higher-dimensional spaces. Where graphics is of limited value.

Introduction

There are multiple versions of the simplex algorithm, each effective on problems of certain types.

The simplex algorithm can be applied to any LP problem no matter how it is formulated, but the operation of the algorithm is easier to understand and implement if the problem is formulated in a specific way. In this lecture, we'll look at one such way, which is known as the **tableau formation** method.

An SMP Problem (SMP 01)

Let us start our discussion with the tableau formulation method by considering the following SMP with 2 decision variables.

Maximize $p = 2x + y$ subject to:

1. $x \geq 0$;
2. $y \geq 0$;
3. $x \leq 40$;
4. $3x + 4y \leq 240$;
5. $2x + y \leq 100$.

Graphing the Feasible Set of SMP 01

Let us graph the feasible set for SMP 01. My source code is in `color_feasible_set.py`. It shows you how to color a feasible set.

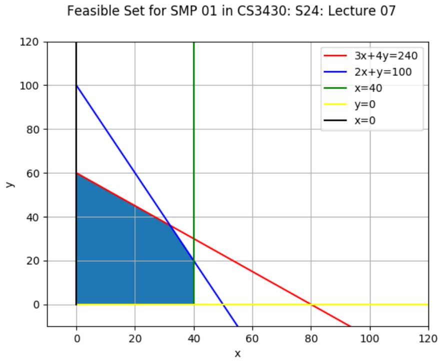

Figure: Feasible Set of SMP 01

In the subsequent slides, we will outline several basic solutions for this SMP. Keep in mind that each basic solution corresponds to a corner point of this feasible set.

Slack Injection

Recall that for each constraint (except those that make the decision variables non-negative), we can add a non-negative number (i.e., **slack variable**) to turn each constraint into an equation.

Thus, we need to introduce the slack variables to the last three constraints on the previous slide. Let's call these slack variables u, v, and w. Our slack equations are:

1. $x + u = 40$;
2. $3x + 4y + v = 240$;
3. $2x + y + w = 100$.

Basic Solution 1

We can re-write the equations with the following variable order: u, v, w, x, y. In other words, every constraint will be written so that u comes first, v comes second, w comes third, x comes fourth, y comes fifth. We will also make 0 coefficients explicit in every constraint. Also, recall that the sign $\equiv$ means *equivalent to*. Here we go!

$$
\begin{array}{llcccl}
1. & u + x & = & 40 & \equiv & u + 0v + 0w + x + 0y = 40; \\
2 & v + 3x + 4y & = & 240 & \equiv & 0u + v + 0w + 3x + 4y = 240; \\
3. & w + 2x + y & = & 100 & \equiv & 0u + 0v + w + 2x + y = 100.
\end{array}
$$

Basic Solution 1

We can turn the equations on the previous slide into another tableau. Remember that our current variable order is u, v, w, x, y, which means that the columns of the tableau will be u, v, w, x, y. The last column is always named B.S. which abbreviates "Basic Solution (BS)."

How many rows? 3! Why? Because we have 3 constrains. What are the names of the rows? The first three variables in our current variable order. They correspond to our current decision variables. Let me box it for emphasis.

The order of the variables (decision variables and slacks) matters!

Basic Solution 1

We are ready to formulate the tableau.

	u	v	w	x	y	B.S.
u	1	0	0	1	0	40
v	0	1	0	3	4	240
w	0	0	1	2	1	100

In this tableau, we have 3 row variables (u, v, w; these are the first three in our current variable order) and 5 column variables (u, v, w, x, y; that's our current order).

What do we make of this tableau? Here's how we read it. Consider the 3x3 matrix formed by the the 3 rows and the first three columns. If the column of a row variable has a pivot its value is given in the B.S. column. The values of the variables that are not in any row (they are not our current decision variables) have their values at 0 by default.

With these conventions in place, the tableau gives us the following basic solution in the B.S. column: $u = 40$

(because there's a pivot of 1), $v = 240$ (because there is a pivot of 1), $w = 100$ (because there's a pivot of 1),

$x = 0$ (not a row variable so defaults to 0), $y = 0$ (not a row variable so defaults to 0).

Basic Solution 2

We can re-write the slack equations in a different way as follows.

1. $x + u = 40 \equiv x + 0v + 0w + u + 0y = 40$;
2. $3x + v + 4y = 240 \equiv 3x + v + 0w + 0u + 4y = 240$;
3. $2x + w + y = 100 \equiv 2x + 0v + w + 0u + y = 100$.

Basic Solution 2

We can turn the equations on the previous slide into the following tableau.

	x	v	w	u	y	B.S.
x	1	0	0	1	0	40
v	3	1	0	0	4	240
w	2	0	1	0	1	100

In the above tableau, we have 3 row variables (x, v, and w) and 5 column variables (x, v, w, u, y). However, we cannot readily read the values of the row variables, except for x, because there're no 1-pivots in the v and w rows. Thus, we have to apply a few row operations to obtain 1-pivots in the v and w rows.

Basic Solution 2

Let us apply row $2 = -3\cdot$ row $1 +$ row 2 and row $3 = -2\cdot$ row $1 +$ row 3 to the tableau on the previous slide to obtain this tableau.

	x	v	w	u	y	B.S.
x	1	0	0	1	0	40
v	0	1	0	-3	4	120
w	0	0	1	-2	1	20

In the above tableau, we have 3 row variables (x, v, w) and 5 column variables (x, v, w, u, y). But now we have the 1-pivots in all rows. Thus, the second basic solution is $x = 40$, $v = 120$, $w = 20$, $u = 0$, $y = 0$.

Basic Solution 3

We can re-write the equations with the slack variables as follows:

1. $x + u = 40 \equiv x + 0y + 0v + u + 0w = 40$;
2. $3x + 4y + v = 240 \equiv 3x + 4y + v + 0u + 0w = 240$;
3. $2x + y + w = 100 \equiv 2x + y + 0v + 0u + 1w = 100$.

Basic Solution 3

We can put the equations on the previous slide into the following tableau.

	x	y	v	u	w	B.S.
x	1	0	0	1	0	40
y	3	4	1	0	0	240
v	2	1	0	0	1	100

In the above tableau, we have 3 row variables (x, y, v) and 5 column variables (x, y, v, u, w). But we don't have the 1-pivots in the y and v rows. We have to apply a few row operations to obtain the 1-pivots in these rows.

Basic Solution 3

Let us apply the following row operations to the tableau on the previous slide: 1) row 2 $= -3\cdot$ row 1 $+$ row2; 2) row 3 $= -2\cdot$ row 1 $+$ row 3; 3) interchange row 2 and row 3; 4) row 3 $= -4\cdot$ row 2 $+$ row 3. We obtain the following tableau.

	x	y	v	u	w	B.S.
x	1	0	0	1	0	40
y	0	1	0	-2	1	20
v	0	0	1	5	-4	40

In the above tableau, we have 3 row variables (x, y, v) and 5 column variables (x, y, v, u, w). Since we now have the 1-pivots in the x, y, and v rows, we obtain another basic solution: $x = 40$, $y = 20$, $v = 40$, $u = 0$, $w = 0$.

Observations

Each basic solution of the system of equations is a corner point of the feasible set.

Each corner point of the feasible set is a basic solution of the system of equations.

To solve an SMP, we proceed by finding basic solutions of the system of equations and evaluating the objective function at each basic solution.

Our tableaus for SMP01 are incompolete in that they dont' include any information about the objective function. Let us see how we can form a complete tableau with the next example SMP.

An SMP Problem (SMP 02)

Let us consider this SMP with 3 decision variables that we worked out on the board on 01/31/2024.

maximize: $p = 10x + 6y + 2z$

subject to:

1. $x \geq 0$;
2. $y \geq 0$;
3. $z \geq 0$;
4. $2x + 2y + 3z \leq 160$;
5. $5x + y + 10z \leq 100$;

Adding Slack Variables into SMP 02's Constraints

We can introduce the slack variables $u \geq 0$ and $v \geq 0$ and put them into equations 4 and 5 on the previous slide to turn them into these equations:

- $2x + 2y + 3z + u = 160$;
- $5x + y + 10z + v = 100$.

Complete Tableau Formation for SMP 02

1. We can take the equations on the previous slide and form the following tableau:

	x	y	z	u	v	B.S.
u	2	2	3	1	0	160
v	5	1	10	0	1	100

2. Let us add the third row (the so-called p-row) with the information on the objective function where the entries are the **negatives** of the variable coefficients of the objective function. Here's what we obtain.

	x	y	z	u	v	B.S.
u	2	2	3	1	0	160
v	5	1	10	0	1	100
p	-10	-6	-2	0	0	0

The basic solution of this complete tableau is $u = 160$, $v = 100$, $x = 0$, $y = 0$, $z = 0$, $p = 0$.

Choosing the Pivot

An important operation that the simplex algorithm does with the complete tableau is choosing the pivot. The choice of the pivot is described by the following rules.

Rule 1: Find the most *negative* entry in the p-row (ignore the B.S. column!). Break ties arbitrarily. The variable that labels the column is the *entering variable*. The entering variable enters one of the rows.

Rule 2: Divide each *positive* entry in the column of the entering variable into the same row entry in the B.S. column. The variable in the row that gives the *smallest* quotient is the *departing variable*. Break ties arbitrarily. The departing variable leaves one of the rows.

Rule 3: The pivot is in the row of the departing variable and the column of the entering variable.

Tableau Formation for SMP 02

Let us find the pivot in the complete tableau for SMP 02 given below.

	x	y	z	u	v	B.S.
u	2	2	3	1	0	160
v	5	1	10	0	1	100
p	-10	-6	-2	0	0	0

Rule 1: The most negative entry in the p-row is -10. Thus, x is the entering variable.
Rule 2: $\frac{160}{2} = 80$ and $\frac{100}{5} = 20$. Since $20 < 80$, v is the departing variable.
Rule 3: The pivot is at (2, 1), i.e., it's 5. Sometimes you will see the notation $p_v = 5$, i.e., thte value of the pivat is 5.

A Simplex Theorem

Theorem: The rules of choosing the pivot yield a corner point of the feasible set at the next iteration.

This gives us a mathematical assurance that the algorithm will stay at corner points.

The Pivoting Operation

The pivoting operation that the simplex algorithm does after the pivot is chosen consists of 3 steps.

1. Set the value of the pivot (p_v) to 1 by multiplying the pivot's row (i.e., the row of the departing variable) by $\frac{1}{p_v}$.

2. Add the suitable multiples of the pivot's row to the other rows in the tableau (including the p-row) to turn into 0's all other entries in the pivot's column.

3. Put the entering variable into the row of the departing variable.

Tableau Example 1

Let us apply the pivoting operationg to this tableau.

	x	y	z	u	v	B.S.
u	2	2	3	1	0	160
v	5	1	10	0	1	100
p	-10	-6	-2	0	0	0

1. We multiply row 2 (the pivot's row) by $\frac{1}{5}$ to obtain this tableau.

	x	y	z	u	v	B.S.
u	2	2	3	1	0	160
v	1	0.2	2	0	0.2	20
p	-10	-6	-2	0	0	0

Tableau Example 1

2. We do row1 = -2 $\cdot$ row2 + row1 to obtain the following tableau.

	x	y	z	u	v	B.S.
u	0	1.6	-1	1	-0.4	120
v	1	0.2	2	0	0.2	20
p	-10	-6	-2	0	0	0

3. We do row3 = 10 $\cdot$ row2 + row3 to obtain the following tableau.

	x	y	z	u	v	B.S.
u	0	1.6	-1	1	-0.4	120
v	1	0.2	2	0	0.2	20
p	0	-4	18	0	2	200

Tablelau Example 1

4. Put the entering variable x into the row of the departing variable v to obtain the following table.

	x	y	z	u	v	B.S.
u	0	1.6	-1	1	-0.4	120
x	1	0.2	2	0	0.2	20
p	0	-4	18	0	2	200

This tableau is the result tableau of applying the pivoting operation.

This tableau corresponds to the following basic solution: $x = 20$, $y = 0$, $z = 0$, $u = 120$, $v = 0$, $p = 200$; the values of y, z, and v are all 0's, because they are not the row variables.

We can check the p value (the value of the objective function in the p-row) by computing $p = 10x + 6y + z = 10 \cdot 20 + 6 \cdot 0 + 2 \cdot 0 = 200$.

Tableau Example 2

Let us find a pivot and apply the pivoting operation to this tableau.

	x	y	z	u	v	w	B.S.
u	20	4	4	1	0	0	6000
v	8	8	4	0	1	0	10000
w	8	4	2	0	0	1	4000
p	-3	-8	-6	0	0	0	0

Now we choose the pivot with the three rules.

Rule 1: The most negative entry in the p-row is -8; thus, y is the entering variable.

Rule 2: The fractions are $\frac{4000}{4} = 1000$; $\frac{10000}{8} = 1250$; $\frac{3000}{2} = 1500$. So, 1000 is the smallest quotient, which makes w the departing variable.

Rule 3: The pivot is at position (3, 2) in the tableau. In other words, $p_v = 4$.

Tableau Example 2

The pivoting operation is done on column 2 (the column of the entering variable y). Our objective is to turn into 0's all entries in column 2 except the pivot. We'll apply the following operations to the tableau on the previous slide.

1. row 3 $= \frac{1}{4} \cdot$ row 3;
2. row 1 $=$ -4 $\cdot$ row 3 $+$ row 1;
3. row 2 $=$ -8 $\cdot$ row 3 $+$ row 2;
4. row 4 $=$ 8 $\cdot$ row 3 $+$ row 4.

Tableau Example 2

We obtain the following tableau and let's not forget to swap the entering variable y with the departing variable w.

	x	y	z	u	v	w	B.S.
u	12	0	2	1	0	-1	2000
v	-8	0	0	0	1	-2	2000
y	2	1	0.5	0	0	0.25	1000
p	13	0	-2	0	0	2	8000

The basic solution given by the tableau is $x = 0$, $y = 1000$, $z = 0$, $u = 2000$, $v = 2000$, and $p = 8000$.

The value of the objective function p (the rightmost cell in the p-row) can be directly verified as $p = 3x + 8y + 6z = 3 \cdot 0 + 8 \cdot 1000 + 6 \cdot 0 = 8000$. So, we managed to raise p from 0 up to 8000, which is not bad.

All pieces are in place for us now to tackle the simplex algorithm.

Simplex Algorithm

1. Make sure that the problem you're solving is an SMP (standard maximum problem).

2. Write the slack equations and form the initial tableau.

3. Execute the following loop.

```
While True:
 1) choose a pivot and do the pivoting operation.
 2) if no pivot can be chosen, because there’s no
    entering variable (all entries in the p-row
    are non-negative with a possible exception of the
    B.S. column),return the current tableau as the solution.
 3) if no pivot can be chosen, because there’s an
    entering variable but no departing variable (there’re
    no positive entries in the column of the entering
    variable), return None, because there’s no solution.
 4) break all ties arbitrarily.
```

Simplex Algorithm

As the pseudocode on the previous slide indicates, the simplex algorithm stops under one of the two conditions.

1. There is no entering variable, because all entries in the p-row are non-negative. In this case, the algorithm has found a solution, which is encoded in the current tableau.
2. There is an entering variable, but there's no departing variable, beause there're no positive entries in the column of the entering variable. In this case, the SMP has no solution.

Simplex Example

Let us apply the simplex algorithm to the following problem.
Maximize $p = 3x + 8y + 6z$ subject to

1. $x \geq 0$;
2. $y \geq 0$;
3. $z \geq 0$;
4. $20x + 4y + 4z \leq 6000$;
5. $8x + 8y + 4z \leq 10000$;
6. $8x + 4y + 2z \leq 4000$.

Simplex Example

1. This is an SMP, because p is linear and all constraints are linear and have a positive number to the right of $\leq$.

2. We proceed to write the slack equations for all the $\leq$ constraints.

   1. $20x + 4y + 4z + u = 6000$;
   2. $8x + 8y + 4z + v = 10000$;
   3. $8x + 4y + 2z + w = 4000$.

3. We form the initial tableau from the slack equations; we've already worked with this tableau in Example 2 of this lecture that starts on slide 9.

	x	y	z	u	v	w	B.S.
u	20	4	4	1	0	0	6000
v	8	8	4	0	1	0	10000
w	8	4	2	0	0	1	4000
p	-3	-8	-6	0	0	0	0

Simplex Example

4. In Example 2, we determined that the pivot is at position (3,2).

5. When we apply the pivoting operation with the pivot at (3, 2), we obtain the following tableau (see Example 2 for details).

	x	y	z	u	v	w	B.S.
u	12	0	2	1	0	-1	2000
v	-8	0	0	0	1	-2	2000
y	2	1	0.5	0	0	0.25	1000
p	13	0	-2	0	0	2	8000

6. In the above tableau, the pivot is at (1, 3) and its value is 2. So, the entering variable is z and the departing variable is u.

Simplex Example

7. Applying the pivoting operation on the column of the entering variable z gives us the following tableau.

	x	y	z	u	v	w	B.S.
z	6	0	1	0.5	0	-0.5	1000
v	-8	0	0	0	1	-2	2000
y	-1	1	0	-0.25	0	0.5	500
p	25	0	0	1	0	1	10000

8. In this tableau, we cannot choose the entering variable, because all entries in the p-row are non-negagtive. So the simplex algorithm returns the above tableau as the solution.

9. The solution encoded in the tableau is $x = 0$, $y = 500$, $z = 1000$, $u = 0$, $v = 2000$, $w = 0$, $p = 10000$. We can verify the value of p by plugging the values of x, y, z into $p = 3x + 8y + 6z$.

Simplex Example 2

Here's an example of a tableau for which the simplex algorithm returns None (i.e., no solution).

	x	y	z	u	v	B.S.
u	1	-1	1	1	0	5
v	2	0	-1	0	1	10
p	-1	-2	-1	0	0	0

Here's why. The most negative entry in the p-row is -2. So, y is the entering variable. But, there's no *positive* entry in the column of the entering variable. Hence, we cannot find a departing variable.

References

1. D.P. Maki, M. Thompson. *Finite Mathematics*.