CS3430 S24: Scientific Computing

Lecture 02: Additional Notes on Determinants

Vladimir Kulyukin
Department of Computer Science
Utah State University

Computing Determinants without Scaling

If we don't use row scaling, the determinant of a square matrix $\mathbf{A}$ can be computed by the following procedure.

1. Reduce $\mathbf{A}$ to row echolon form using only row additions and row interchanges (no row scaling!); use a counter variable (let's call it i for the sake of discussion) to keep track of row interchanges;
2. If at any point the matrix being reduced contains a row of 0's, return 0 (i.e., $\texttt{det}(\mathbf{A}) = 0$);
3. If at no point the matrix being reduced contains a row of 0's, then $\texttt{det}(\mathbf{A}) = (-1)^i \cdot$ (product of pivots), where i is the number of the performed row interchanges.

Example 1

Let's apply the algorithm on the previous slide to compute the determinant of the following matrix.

$$\mathbf{A} = \begin{bmatrix} 2 & 3 & -1 \\ 5 & -7 & 1 \\ -3 & 2 & -1 \end{bmatrix}.$$

1. We do $-\frac{5}{2}\cdot$ row 1 + row 2 to obtain

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -14.5 & 3.5 \\ -3 & 2 & -1 \end{bmatrix}.$$

Example 1

2. We do $\frac{3}{2}\cdot$ row 1 + row 3 to obtain

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -14.5 & 3.5 \\ 0 & 6.5 & -2.5 \end{bmatrix}.$$

3. We do $\frac{6.5}{14.5}\cdot$ row 2 + row 3 to obtain

$$\begin{bmatrix} 2 & 3 & -1 \\ 0 & -14.5 & 3.5 \\ 0 & 0 & -0.9310 \end{bmatrix}.$$

Example 1

4. The matrix is in row echelon form. Since we haven't used any row interchanges, $i = 0$. Remember that i is the variable counter that keeps track of the number of row interchanges. Thus,

$$\texttt{det}(\mathbf{A}) = (-1)^0(2 \cdot -14.5 \cdot -0.9310) \approx 26.9999992.$$

5. Computing the determinant with Numpy in Python 3 on Ubuntu 18.04 LTS (Bionic Beaver) gave me the following result.

```
>>> np.linalg.det(A)
27.0
```

Example 2

Let's apply the determinant computation algorithm to another matrix and throw in a row intechange to see how it impacts the computation of the determinant.

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix}.$$

1. We interchange rows 2 and 3 to obtain

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Example 2

2. We do $-\frac{1}{3}\cdot$ row 1 + row 2 to obtain

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & 3.\overline{3} & -0.\overline{3} \\ 0 & 1 & 2 \end{bmatrix}.$$

3. We do $-\frac{1}{3.\overline{3}}\cdot$ row 2 + row 3 to obtain

$$\begin{bmatrix} 3 & 2 & 4 \\ 0 & 3.\overline{3} & -0.\overline{3} \\ 0.\overline{0} & 0.\overline{0} & 2.1\overline{0} \end{bmatrix}.$$

4. The matrix is in row echelon form. Since we've had one row interchange, $i = 1$. Thus, $\texttt{det}(\mathbf{A}) = (-1)^1(3 \cdot 3.\overline{3} \cdot 2.1\overline{0}) \approx -20.9999979$.

5. Computing the determinant with Numpy 1.16.3 in Python 3.6.7 on Ubuntu 18.04 LTS (Bionic Beaver) gives me the following result.

```
>>> np.linalg.det(A)
-21.0
```

Example 3

Let's repeat Example 2 with 2 row interchanges.

$$\mathbf{A} = \begin{bmatrix} 3 & 2 & 4 \\ 0 & 1 & 2 \\ 1 & 4 & 1 \end{bmatrix}.$$

1. We interchange rows 2 and 3 to obtain

$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Example 3

2. We interchange rows 1 and 2 to obtain

$$\begin{bmatrix} 1 & 4 & 1 \\ 3 & 2 & 4 \\ 0 & 1 & 2 \end{bmatrix}.$$

3. We do $-3\cdot$ row 1 + row 2 to obtain

$$\begin{bmatrix} 1 & 4 & 1 \\ 0 & -10 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

4. We do $\frac{1}{10}\cdot$ row 2 + row 3 to obtain

$$\begin{bmatrix} 1 & 4 & 1 \\ 0 & -10 & 1 \\ 0 & 0 & 2.1 \end{bmatrix}.$$

Example 3

5. The matrix is in row echelon form. Since we've had 2 row interchanges, $i = 2$. Thus,

$$\mathtt{det}(\mathbf{A}) = (-1)^2(1 \cdot -10 \cdot 2.1) = -21.$$

5. Computing the determinant with Numpy 1.16.3 in Python 3.6.7 on Ubuntu 18.04 LTS (Bionic Beaver) gives me the following result.

```
>>> np.linalg.det(A)
-21.0
```

Several Useful Theorems on Determinants

Theorem 1: If a single row of a square matrix $\mathbf{A}$ is multiplied by a scalar k, then the determinant of the resulting matrix is $k \cdot \mathtt{det}(\mathbf{A})$.

Theorem 2: A square matrix $\mathbf{A}$ is inevitable if and only if $\mathtt{det}(\mathbf{A}) \neq 0$.

Theorem 3: If $\mathbf{A}$ and $\mathbf{B}$ are square matrices, then $\mathtt{det}(\mathbf{AB}) = \mathtt{det}(\mathbf{A})\mathtt{det}(\mathbf{B})$.

Theorem 4: The determinant of an upper- or lower-triangular matrix is the product of its pivots.

Example

Let

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & 3 & 0 \\ 4 & 2 & 1 \end{bmatrix} \quad \mathbf{B} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}.$$

Then $\texttt{det}(\mathbf{AB}) = \texttt{det}(\mathbf{A})\texttt{det}(\mathbf{A}) = (2 \cdot 3 \cdot 1)(1 \cdot 1 \cdot 2) = 6 \cdot 2 = 12.$

References

1. J. Fraleigh, R. Beauregard. *Linear Algebra*.