Contents

1 What Will and Will Not Appear On The Test

As far as we know, function inverses, inductively defined sets, properties of binary relations, countability, partial orders, total orders, and inductive proofs will definitely appear on the test.

Other things, such as grammars, equivalence relations, recursively defined functions, equivalence classes, and partitions will supposedly not appear on the test.

The following topics appear on ”A Study Guide for Exam 2”:

- Function inverses
- Countability
- Inductive Definition of Sets
- Properties of Binary Relations
- Total Order
- Partial Order
- Mathematical Induction

Know these topics well. The other material is included as study material for the final exam, but it might be helpful for this exam as well.

2 Potential Bonus Questions

I cannot guarantee that any of these questions will be bonus questions, but I feel like they’re the type of question that he would ask as a bonus question.

I believe that these, or questions very similar to them, will appear as bonus questions on the test – particularly the sum of cubes or sum of squares. Unfortunately, I cannot check your answers for these. I will say, however, that you must use induction to prove them.

(a) through (g) are particularly likely, as they are relatively common progressions.

(a) $1 + 3 + 5 + \cdots + (2n - 1) = n^2$

(b) $2 + 4 + 6 + \cdots + (2n) = n(n + 1)$

(c) $1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}$

(d) $1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}$

(e) $1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2(n+1)^2}{4}$

(f) $1^4 + 2^4 + 3^4 + \cdots + n^4 = \frac{n(n+1)(2n+1)(3n^2+3n-1)}{30}$

(g) $1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2$

(h) $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n + 1) = \frac{n(n+1)(n+2)}{3}$

(i) $1 \cdot 3 + 2 \cdot 4 + 3 \cdot 5 + \cdots + n \cdot (n + 2) = \frac{n(n+1)(2n+7)}{6}$

(j) $1 \cdot 1! + 2 \cdot 2! + 3 \cdot 3! + \cdots + n \cdot n! = (n + 1)! - 1$

(k) $2^0 + 2^1 + 2^2 + 2^3 + \cdots + 2^n = 2^{n+1} - 1$

(l) $3^1 + 3^2 + 3^3 + \cdots + 3^n = \frac{3^{n+1}-3}{2}$

(m) $4^0 + 4^1 + 4^2 + 4^3 + \cdots + 4^n = \frac{4^{n+1}-1}{3}$

(n) $\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + \cdots + \frac{n}{(n+1)!} = 1 - \frac{1}{(n+1)!}$

(o) $\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n\cdot(n+1)} = \frac{n}{n+1}$

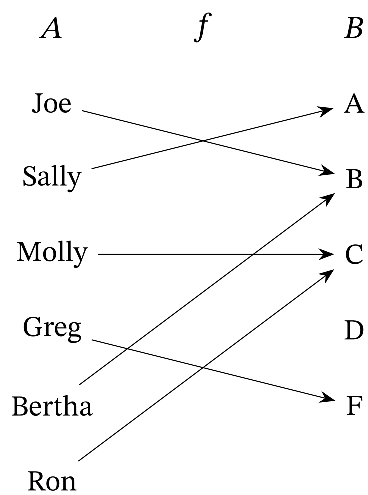
3 Chapter 2: Functions

3.1 Definition of a Function

A function f is an operator that transforms, or maps, the elements of a set A to another set B. Importantly, one input can only have one possible output, or else it is not a function. To write it formally, $f : A \mapsto B$, which is read as "f maps A to B".

3.2 Domain vs. Co-domain vs. Range

The domain of the function is the set of values that are used as inputs for the function. The co-domain is the set of values that *can* be the outputs of the function. The range of the function is the set of values that actually *are* output by the function. A good way of thinking about it can be found below:



The domain of the function is A, which is a set of students, and the co-domain B is set of possible grades a student can have. Notice that nobody has a D in the class. Obviously, that doesn't mean that it isn't a valid member of the set of grades – it just means that the inputs didn't ever yield that as a result. In this case, the *range* of the function is $\{A, B, C, F\}$, while the *co-domain* is $\{A, B, C, D, F\}$.

3.3 Floor, Ceiling, and Round Functions

The floor, ceiling, and round functions all do somewhat similar things: they convert a number with a fractional part into an integer. The floor function can be thought of as going *down*, while the ceiling function can be thought of as going *up*, as the floor function goes to the largest integer less than the value and the ceiling function goes to the smallest integer larger than the value. The floor function is written $\lfloor x \rfloor$ and the ceiling function is written $\lceil x \rceil$. If you don't have access to those symbols, you can also write floor(x) and ceil(x).

$$\lfloor 1.464 \rfloor = 1 \text{ vs. } \lceil 1.464 \rceil = 2$$

The round function behaves like an in-between of the two, as it goes to the *nearest* number, regardless of whether it's larger or smaller than the value at hand.

$$\text{round}(1.464) = 1 \text{ vs. } \text{round}(1.564) = 2$$

3.4 Onto, One-to-One, and Bijections

The more formal names for onto and one-to-one are surjective and injective, respectively. A bijection is simply a function that is both onto and one-to-one.

An onto function is a function where every element in the co-domain is mapped to by an element in the domain. In other words, the co-domain is identical to the range when a function is onto.

A one-to-one function is a function where every element in the co-domain is mapped to by only one element in the domain. In other words, there can only be one input that outputs a given value when a function is one-to-one.

Onto and one-to-one functions can exist separately, where a function is onto but not one-to-one or one-to-one but not onto, or they can co-exist, which creates a bijection. As mentioned previously, a bijection is just a function that is both one-to-one and onto.

3.5 Function Inverses

A function's inverse is an object that maps the co-domain back to the domain. If a function f is $f : X \mapsto Y$, then the inverse is $f^{-1} : Y \mapsto X$. In order for a function to have a well-defined inverse, it must be a bijection. If a function is not a bijection, it can still have an inverse, but it won't be well-defined. For example, the function x^2 is not one-to-one because $f(2) = f(-2) = 4$. It has an inverse, $\pm\sqrt{x}$, but that inverse is not well-defined because a single input yields two values, which means it is not a function. On the other hand, x^3 is a bijection, so its inverse $\sqrt[3]{x}$ is well-defined.

3.6 Countability

A set is countable if there exists a bijection from the natural numbers to that set. I.e., if there is a function that maps every individual element of the natural numbers to the elements of another set, the other set is countably infinite.

- Cardinality - The number of elements a set contains
- Equipotent - Two sets that have a bijection between one another

An example would be $f : \mathbb{N} \mapsto \mathbb{Z}$ where $f(n) = (-1)^n \left\lfloor \frac{n+1}{2} \right\rfloor$. This function takes a natural number n and outputs an integer $f(n) = z$. Examples:

$$\begin{aligned}
&f(0) = 0\\
&f(1) = (-1)^1 \left\lfloor \frac{2}{2} \right\rfloor = -1\\
&f(2) = (-1)^2 \left\lfloor \frac{3}{2} \right\rfloor = 1\\
&f(3) = (-1)^3 \left\lfloor \frac{4}{2} \right\rfloor = -2\\
&f(4) = (-1)^4 \left\lfloor \frac{5}{2} \right\rfloor = 2\\
&\qquad\vdots\\
&f(n) = (-1)^n \left\lfloor \frac{n+1}{2} \right\rfloor
\end{aligned}$$

If it isn't apparent, this function maps the odd natural numbers to the negative integers, and maps the even natural numbers to the positive integers. This also proves that the integers $\mathbb{Z}$ are countably infinite.

Finally, the size of a set can be expressed as follows: $|S| = n$. You don't need to know this because it wasn't discussed in class, but the size of all countably infinite sets can be written as $|\mathbb{N}| = \aleph_0$ (read 'Aleph null').

4 Chapter 3: Inductively and Recursively Defined Sets and Grammars

4.1 Inductively Defined Sets

A typical definition of a set is as follows: $E = \{x \mid x \equiv 0 \mod 2\}$, which is the set of even integers (All even integers are congruent to $0 \mod 2$). Alternatively, one can use induction to define this set:

$$E \coloneqq \begin{cases} 0 \in E \\ n \in E \rightarrow n + 2 \in E \end{cases}$$

$0 \in E$ is called the basis case (or base case) and $n \in E \rightarrow n + 2 \in E$ is called the induction. The induction could also be written "If $n \in E$, then $n + 2 \in E$". The induction is then used to generate every element in the set:

1. Since $0 \in E$, $2 \in E$.
2. Since $2 \in E$, $4 \in E$.
3. Since $4 \in E$, $6 \in E$.
4. *Repeats forever*

4.2 Recursively Defined Functions

Supposedly, recursively defined functions will not appear on the second test.

In order to define a recursively defined function, the function must operate on an inductively defined set S. For each basis case of the inductively defined set, you must then define a value of $f(x)$ for that set, where $x \in S$. You then give rules for any possible element of the set S that will define $f(x)$ in terms of the pre-defined values of f.

Take the function $f(n) = \sum_{i=1}^{n} i = 0 + 1 + 2 + \cdots + n$. To recursively define this function, take the base case to be $f(0) = 0$. From there, the recursive definition can be written as:

$$f(n) \coloneqq \text{If } n = 0\text{, then } 0$$
$$\text{Else } f(n-1) + n$$

4.3 Grammars

Supposedly, grammars will not appear on the test.

A grammar for a language is a set of rules that produce the strings of that language. There are four parts that must be defined to create a grammar:

1. A start symbol
2. Grammar rules, also called the productions of the grammar
3. A set of actual letters, called terminals
4. A set of additional symbols called non-terminals that have their own production rules.

The start symbol can be any letter or symbol, though it is usually written S.

The grammar rules are written with the start symbol or a non-terminal on the right side, an arrow pointing right in the middle, and the possible terminals or non-terminals on the right side. Some possible productions are:

- $S \to \Lambda$: This takes the start symbol S and produces Λ, the empty string.
- $S \to aS$: This takes the start symbol S and produces aS. The S in aS can then be used to repeat the original production infinitely.
- $S \to aB$: This takes the start symbol S and produces aB, where a is a terminal, and B is a non-terminal that has its own production rule.

If you have multiple productions that are derived from the same non-terminal or the start symbol, they can be chained as follows:

- $S \to \Lambda$, $S \to aS$, and $S \to bS$ can be chained together to form $S \to \Lambda \mid aS \mid bS$.
- $A \to \Lambda$ and $A \to a$ can be chained together to form $A \to \Lambda \mid a$.

You can also concisely defined a grammar as a tuple with four elements: $G = (N, T, S, P)$, where N is the set of non-terminals in the grammar, T is the set of terminals, S is the start symbol, and P is the set of allowed productions for the grammar, but it is more common to simply write down the start symbol as S, the non-terminals as capital letters, and write the associated productions of each of the non-terminals separately.

4.3.1 Union Rule

The union rule is a method of combining the grammars of languages into a single grammar for a new language. If the start symbols for the sets M and N are A and B, respectively, then you can combine the two sets into one by simply creating a new start symbol S for the set $L = M \cup N$ and use the chaining rule from earlier to describe the grammar of this new set:

$$S \to A \mid B$$

A and B are left untouched, so this operating is the exact same as taking the union of the original sets.

While not necessary useful for defining sets, this rule is very useful for analyzing languages. If you have a set $\{\Lambda, a, b, aa, bb, aaa, bbb, \dots\}$, then it's fairly apparent that it's really the union of two sets $\{\Lambda, a, aa, aaa, \dots\}$ and $\{\Lambda, b, bb, bbb, \dots\}$. Thanks to the union rule, you can then analyze these two sets separately. It would be somewhat tricky to define a production that would apply to the original set, so separating the sets into easier pieces simplifies the problem greatly.

4.3.2 Product Rule

Like the union rule, the product rule is a method of combining the grammars of languages into a single grammar for a new language. Using the same notation as before (M, N, A, B, and S), $L = MN$ can be written as

$$S \to AB$$

Like before, A and B are left untouched, and again, this is very useful for simplifying problems. If $L = \Lambda, a, b, ab, ba, aab, abb, bba, aabb, bbaa, \ldots$, it's not very easy to see how one would construct the grammar for L. But thanks to the product rule, we can separate out the a and b terms into separate languages such that $M = \{\Lambda, a, aa, aaa, \ldots\}$ and $N = \{Lambda, b, bb, bbb, \ldots\}$.

4.3.3 Closure Rule

Unlike the last two rules, the closure rule isn't very useful for simplifying problems. Instead, it is used to understand every possible string that can be found in a language. If you have a set of characters $A = \{a, b, c\}$ and a grammar G for those characters, in order to understand the ways A can be used, you take the closure of A. This yields every possible combination of a, b, and c.

5 Chapter 4: Binary Relations and Inductive Proofs

5.1 Properties of Binary Relations

A relation R on the set S can have several possible properties:

- **Reflexive**: $(x, x) \in R \; \forall x \in S$. In words, in order for a relation to be reflexive, every element in the set S must be related to itself.
- **Irreflexive**: $(x, x) \notin R \; \forall x \in S$. In words, in order for a relation to be irreflexive, there can be no elements in the set S that are related to themselves.

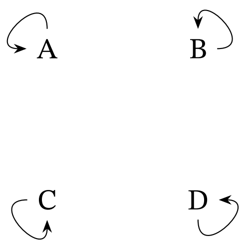
- **Neither**: There is a mix of elements where $(x, x) \in R$ and $(y, y) \notin R$. In words, in order for a relation to be neither reflexive nor irreflexive, there must be a mix of elements where the element is related to itself and where the element is not related to itself.
- **Symmetric**: $xRy \implies yRx \; \forall (x, y) \in S$. In words, in order for a relation to be symmetric, if an element x has a relation with element y, y must also have a relation with x.
- **Anti-symmetric**: xRy and $yRx \implies x = y \; \forall (x, y) \in S$. In words, in order for a relation to be anti-symmetric, the only way that two elements can have a relation that goes both ways is if the two elements are equal. Otherwise, if one element has a relation with a second element, the second element cannot have a relation with the first element.
- **Neither**: There is a mix of elements where there is symmetry and anti-symmetry.
- **Transitive**: xRy and $yRz \implies xRz \; \forall x, y, z \in S$

Importantly, symmetry and anti-symmetry are not mutually exclusive. An example of a relation that is both is the equality operator: $a = b$. Reflexive and irreflexive and transitive and not transitive, however, are mutually exclusive with one another (a relation cannot be reflexive and irreflexive, nor can it be transitive and not transitive).

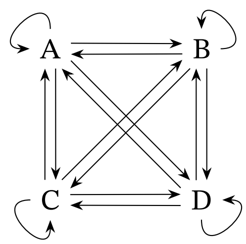
5.2 Representing Binary Relations

There are three main ways to represent binary relations: with formulas, with matrices, or with graphs. Based on matrices, you can easily determine the reflexivity and symmetry, but not so much the transitivity. With graphs, you can easily determine all three.

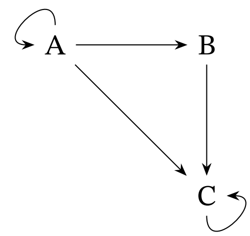
1.



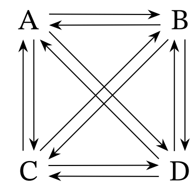
2.



3.



4.



Graph 1 is reflexive, anti-symmetric, symmetric, and transitive. It is reflexive because each element has a relation with itself. The reason it is both symmetric and anti-symmetric is because for every relation that exists, $x = x$. If you recall the definition of symmetry and anti-symmetry, anti-symmetry requires equality, while symmetry doesn't care, so the two can coexist under certain circumstances. Counter-intuitively, it is actually transitive, as $x = x = x$ is equivalent to $x = x$.

Graph 2 is reflexive, symmetric, and transitive. It is symmetric because for each arrow that goes from one relation to another, the other relation has an arrow that goes back to the original, and it is transitive because for (A, B) and (B, D), (A, D) also exists.

Graph 3 is anti-symmetric, transitive, and neither reflexive nor irreflexive. It is anti-symmetric because for each arrow that goes from one relation to another, there isn't a relation that goes back to the original. It is neither reflexive nor irreflexive because B does not have a relation with itself, while A and D do.

Graph 4 is symmetric, transitive, and irreflexive. It is irreflexive because no element has a relation with itself.

5.3 Self-Composition

The book only explains this in reference to graphs and none of the other books I have explain it mathematically, so here is what it's like with graphs.

For a self-composition of a relation R, you can express each individual composition as R^n, where n is the length of the path that you want to have from an element a to another element b. If you just have R, then the length is one. R^2 is a length of two, R^3 is three, and so on. For a relation

$R = \{(a,b),(b,c),(c,d)\}$, R, R^2, and R^3 can be expressed graphically as:

$$R \qquad\qquad R^2 \qquad\qquad R^3$$

$$\begin{tikzcd} a \arrow[r] & b \arrow[r] & c \arrow[r] & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[rr, bend left] & b \arrow[rr, bend right] & c & d \end{tikzcd} \qquad \begin{tikzcd} a \arrow[rrr, bend left] & b & c & d \end{tikzcd}$$

As sets, $R^2 = \{(a,c),(b,d)\}$ and $R^3 = \{(a,d)\}$.

5.4 Closures

The basic idea of closures of a relation is to add as few elements as possible to a set to ensure that it has the property of whatever closure is being applied. So, for example, if the transitive closure is being applied to the relation, you are adding the minimum number of elements possible to make the relation transitive.

5.4.1 Reflexive Closure

If you have a relation $R = \{(1,1),(1,2),(2,1),(3,2)\}$ on the set $A = \{1,2,3\}$, the reflexive closure of R can be obtained by adding the elements $(2,2)$ and $(3,3)$. Mathematically, the reflexive closure can be expressed as $R \cup \Delta$, where $\Delta = \{(a,a) \mid a \in A\}$

5.4.2 Symmetric Closure

With the same relation as before, the set is not symmetric. It can however be made symmetric by adding the element $(2,3)$. Mathematically, the symmetric closure can be expressed as $R \cup R^{-1}$, where $R^{-1} = \{(b,a) \mid (a,b) \in R\}$.

5.4.3 Transitive Closure

The transitive closure can be extremely tedious to calculate, as it must sometimes be calculated over and over again. Oftentimes, when new elements are added to add transitivity between the old elements, those new elements will prevent the new set from being transitive, and the process must be repeated.

Fortunately, there is a fairly simple method of denoting the transitive closure mathematically:

$$R^* = \bigcup_{n=1}^{\infty} R^n = R^1 \cup R^2 \cup R^3 \cup \cdots \cup R^\infty$$

While the general form goes on to infinity, many sets will quickly collapse to the empty set. If you recall back to the section on self-composition, the relation $R = \{(a, b), (b, c), (c, d)\}$ was given, and so were R^2 and R^3. R^4 was not given, as it is very uninteresting: $R^4 = \varnothing$. This is because there is no path of length 4 that could possibly be constructed in R. If you also recall from earlier on, $S \cup \varnothing = S$. This means that R^* of R is simply $R \cup R^2 \cup R^3$.

5.5 Equivalence Relations

Equivalence relations should likely be thought of as RST relations, as that makes it much easier to remember how to determine if a relation is an equivalence relation or not. That is because RST stands for reflexive, symmetric, and transitive. If a relation does not meet all three of those criteria, it is not an equivalence relation.

You can denote equivalence via $x \sim y$ for two elements x and y that are equivalence to one another with respect to a particular equivalence relation. $\sim$ represents an arbitrary equivalence relation.

5.6 Kernel Relations

A kernel relation of a function is the equivalence relation that is generated by the function when $x \sim y \iff f(x) = f(y)$

5.7 Equivalence Classes

An equivalence class is the set of all elements that are related to an element a of a set S under a relation R. In other words, an equivalence class is the set of all numbers where $a \sim x$ holds true for all x in the class, which is written $[a]$. For example, the odd numbers form an equivalence class such that $[1] = \{a \mid a \equiv 1 \mod 2\}$. This is also an example of a congruence class modulo m, as all of the elements are determined based on their value mod m. The other "half" of this partition would be the equivalence class of even integers $[0] = \{a \mid a \equiv 0 \mod 2\}$.

5.8 Partitions

A partition of set A is a collection of non-empty subsets of A for which each element of A is in one and only one of the subsets. Importantly, there can be many different partitions of the same set.

5.8.1 Equivalence Class vs. Partition

A partition of a set is a set of subsets of a set A where the union of all of the subsets equals the original set. Sets can be partitioned into subsets that have no "real" connections between the elements of the subsets. For example, the set $\{1, 2, 3, 4, 5\}$ can be partitioned into $\{1, 2, 5\}$, $\{3\}$, and $\{4\}$. Is there any actual meaning to those subsets? Maybe, but it certainly isn't very obvious.

If a relation is an equivalence relation, however, performing the relation on the set will automatically partition the set into equivalence classes, where the union of all of the equivalence classes forms the original set, and the intersection of any two equivalence classes equals the empty set (i.e., $[a] \cup [b] \cup \cdots = S$ and $[a] \cap [b] = \emptyset$)

5.9 Order Relations

There are two types of order relations: partial and total orders. Both require the relation in question to be anti-symmetric and transitive. Both may be reflexive or irreflexive. If the order is reflexive, it is called a reflexive order, and likewise if it is irreflexive.

Two elements a and b are *comparable* if $a \preceq b$ or $b \preceq a$, where the former denotes that $(a, b) \in R$ and the latter that $(b, a) \in R$. $a \prec b$ and $b \prec a$ also indicate comparability, but have the added stipulation that $a \neq b$ (thus, it is irreflexive).

You can write a poset mathematically as $\langle S, R \rangle$, where S is the set that contains the elements and R is the relation that is applied to the set.

5.9.1 Total Order vs. Partial Order

Partial orders are a generalization or superset of total orders. That is, any total order is a partial order (where the order applies to all elements), while a partial order is not necessarily a total order.

5.9.2 Maximal and Minimal Elements

A minimal (or least) element is an element a such that $a \preceq b$ for all elements $b \in S$. In other words, nothing comes before a. A maximal element is an element b such that $a \preceq b$ for all elements $b \in S$. Again, this means that b comes after every other element.

5.9.3 Well-ordered Set

A poset $\langle S, R \rangle$ is well-ordered if R is a total order and every non-empty subset of S has a minimal element. An example of a well-ordered set is $\langle \mathbb{N}, \leq \rangle$, as every subset of the natural numbers has a minimal element.

5.10 Inductive Proofs

5.10.1 Mathematical Induction Steps

1. Base case, usually $f(0)$ or $f(1)$ (though it can be for any $n \in \mathbb{N}$)

2. Assume $f(k)$ is true for $k \in \mathbb{N}$

3. Prove that $f(k) \to f(k+1)$

4. Conclusion: By induction, all $f(n)$ are true

5.10.2 Guidelines

1. Express the statement you are proving as "For all $n \geq b, P(n)$", where b is some integer. For positive integers, this is $b = 1$, and for non-negative integers $b = 0$. Depending on what exactly

$P(n)$ is doing, you may need to use different values of b.

2. Write "Basis:" and then show that $P(b)$ is true.

3. Write "Inductive step:" then clearly state and identify the inductive hypothesis in the form "Assume that $P(k)$ is true for some $k \geq b$."

4. Write what $P(k+1)$ says, which is also what must be proved for the inductive hypothesis.

5. Prove $P(k+1)$ without relying on $P(1)$. This is the most difficult part.

6. Clearly state, after completing the inductive step, "This completes the inductive step."

7. After completing the basis and inductive steps, state the conclusion with, "By induction, $P(n)$ is true for all integers $n \geq b$."

A final tip is to make *sure* that it is clear that the form of your final step is the $n+1^{\text{th}}$ term. It should be factored such that instead of n, you have $n+1$. As an example, if the formula that you're proving is $n(n+1)$, your final statement shouldn't be $n^2 + 3n + 2$, as that isn't clearly the $n+1$ form. It should instead be $(n+1)((n+1)+1)$, as that makes it abundantly clear that it is valid for $n+1$.

5.10.3 Strong Induction Steps

1. Prove that the base case, usually $f(0)$ or $f(1)$, is true.

2. The inductive step states that $P(1) \wedge P(2) \wedge P(3) \wedge \cdots \wedge P(k) \implies P(k+1)$ is true for all positive integers k.

5.10.4 Mathematical Induction vs. Strong (Well-Founded) Induction

The use of $P(1), P(2), \ldots, P(k)$ when using strong induction is more flexible than just relying on $P(k)$ with regular mathematical induction. Both techniques can be applied to the same problems, but mathematical induction can sometimes be more tricky to figure out.

6 Hein Problems

This is our textbook, so the questions given as exercises are not provided here. Consult your copy of the textbook for the problems. Funnily enough, if you look just a little bit farther down, yes, this textbook is piss-poor when it comes to providing examples. Because, as we all know, mathematics is a subject where you learn a definition and are magically able to do perfectly every time without doing any practice problems. Wait, you didn't know that? Wow.

1. Injections, Surjections, and Bijections are covered in 2.3 Exercises 1-6
2. Countability is covered in 2.4 Exercises 1-3
3. Inductive Sets are covered in 3.1 Exercises 1-8
4. Properties of binary relations are covered in 4.1 Exercises 1-4
5. Equivalence relations are covered in 4.2 Exercises 1-2
6. Equivalence classes are covered in 4.2 Exercises 3-6
7. Partial orders are covered in 4.3 Exercises 1-2
8. Well-foundedness is covered in 4.3 Exercises 6-9
9. Inductive proofs are covered in 4.4 Exercises 1-6

7 Rosen Problems

These problems can be found in Kenneth Rosen's *Discrete Mathematics and its Applications*. I unfortunately do not have time to typeset all of these questions, so they are left here for the reader to find. Just in the exercises for the section on functions, there are 82 questions – half of which are odds. An approximate number of questions is given. The real number will vary, depending on how many of the questions are actually relevant to the material we need to know.

1. Functions are covered in 2.3 Exercises (~40 Qs)
2. Mathematical Induction and Well-Ordering are covered in 5.1 Exercises (~40 Qs)
3. Strong (Well-founded) Induction is covered in 5.2 Exercises (~20 Qs)
4. Inductive sets are covered in 5.3 exercises. (~30 Qs)
5. Properties of binary relations are covered in 9.1 Exercises (~30 Qs)
6. Representing relations with matrices and graphs is covered in 9.3 Exercises (~15 Qs)
7. Closures are covered in 9.4 Exercises (~15 Qs)
8. Equivalence relations are covered in 9.5 Exercises (~35 Qs)
9. Partial orderings are covered in 9.6 Exercises (~35 Qs)

8 Cummings Problems

I can only provide answers for the odd-numbered problems for this textbook, so I have only listed odd-numbered problems. If you do these and would like to know if you did them correctly, send your work/answer and I will check them for you. Questions that border on irrelevance or too extreme difficulty are marked with (FFTS) at the beginning. Some questions are skipped for being irrelevant or requiring information exclusive to this book. Additionally, it is *very* important to note that this is a *proof-writing* textbook first and foremost. That means that some of the problems will be significantly more challenging than anything we would face. If you get stuck on a problem, move on. It's probably not worth doing if it's that hard. This is particularly true for the section on functions.

I *strongly* recommend that you at least attempt the relations questions.

8.1 Inductive Proofs

1. Prove that the sum of the first n odd natural numbers is equal to n^2 by induction or strong induction.

3. Use induction or strong induction to prove that the following hold for every $n \in \mathbb{N}$.

   (a) $3 \mid (4^n - 1)$

   (b) $6 \mid (n^3 - n)$

   (c) $9 \mid (3^{4n} + 9)$

   (d) $5 \mid (n^5 - n)$

   (e) $6 \mid (5^{2n} - 1)$

   (f) $5 \mid (6^n - 1)$

5. Prove that each of the following holds for every $n \in \mathbb{N}$. Also, before each proof, pick three specific n-values and verify that the result holds for those values.

   (a) $n + 2 < 4n^2$

   (b) $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \cdots + \frac{1}{\sqrt{n}} \leq 2\sqrt{n} - 1$

   (c) $1 + \frac{n}{2} \leq \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots + \frac{1}{2^n - 1} + \frac{1}{2^n}$

   (d) $2^n \leq 2^{n+1} - 2^{n-1} - 1$

   (e) $1 + 2^n \leq 3^n$

   (f) $4^{n+4} \geq (n + 4)^4$

7. Prove that $n^2 < 3^n$ for every $n \in \mathbb{N}$

9. *Omitted*

11. *Omitted*

13. Find a formula for the sum $2 + 4 + 6 + \cdots + 2n$, where $n \in \mathbb{N}$. Then, prove that your formula works in two ways: first, by the $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$, then by induction.

15. Use induction to prove that if A is a set and $|A| = n$, then $\mathcal{P}(A) = 2^{2n}$.

17. Prove that for every $n \in \mathbb{N}$, there are n distinct natural numbers $a_1, a_2, a_3, \ldots, a_n$ such that $a_1^2 + a_2^2 + a_3^2 + \cdots + a_n^2$ is a perfect square.

19. *Omitted*

21. (a) *This question contained information that we were not taught and has been omitted*

    (b) Using the fundamental theorem of arithmetic, prove that if $n \geq 2 \in \mathbb{N}$, then it has a *unique* prime factorization in the sense that if $n = p_1 p_2 \cdots p_k$ and $n = q_1 q_2 \cdots q_\ell$, where each p and q are prime, then there are the same number of primes in each list ($k = \ell$) and that the primes p and q are in fact the same, perhaps just in a different order.

23. (FFTS) In this exercise you will use strong induction to study sequences which are defined *recursively*.

    (a) Define a sequence $a_1, a_2, a_3, \ldots$ recursively where $a_1 = 1$, $a_2 = 3$, and for $n \geq 3$, $a_n = 2a_{n-1} - a_{n-2}$. Prove that $a_n = 2n - 1$ for all $n \in \mathbb{N}$.

    (b) Define a sequence $a_1, a_2, a_3, \ldots$ recursively where $a_1 = 1$, $a_2 = 4$, and for $n \geq 3$, $a_n = 2a_{n-1} - a_{n-2} + 2$. Through scratch work, conjecture a formula for a_n, then prove that your conjecture is correct.

    (c) Define a sequence $a_1, a_2, a_3, \ldots$ recursively where $a_1 = 1$, $a_2 = 2$, and for $n \geq 3$, $a_n = 2a_{n-1} - a_{n-2}$. Through scratch work, conjecture a formula for a_n, then prove that your conjecture is correct.

25. *Omitted*

27. Prove that for any natural numbers a and b, there exists a natural number m such that $mb > a$. This is a version of the so-called *Archimedean principle*.

29. *Omitted*

31. (FFTS) Prove that the following hold for every $n \in \mathbb{N}$

    (a) $F_1 + F_2 + F_3 + \cdots + F_n = F_{n+2} - 1$

    (b) $F_1 + F_3 + F_5 + \cdots + F_{2n-1} = F_{2n}$

    (c) $(F_n + 1)^2 - F_{n+1}F_n - (F_n)^2 = (-1)^n$

    (d) If $a = F_nF_{n+3}$, $b = 2F_{n+1}F_{n+2}$, and $c = (F_{n+1})^2 + (F_{n+2}^2)$, then $a^2 + b^2 = c^2$.

    (e) $F_n = \frac{1}{\sqrt{5}}(\varphi)^n - \frac{1}{\sqrt{5}}(1-\varphi)^n$, where $\varphi = \frac{1+\sqrt{5}}{2}$.

    (f) F_{3n} is even, F_{3n+1} is odd, and F_{3n+2} is odd.

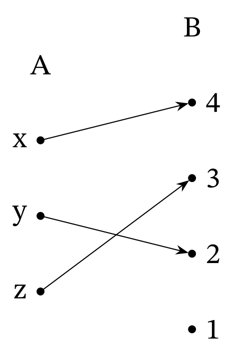
    (g) $\gcd(F_n, F_{n+1}) = 1$

    (h) $F_{n+6} = 4F_{n+3} + F_n$

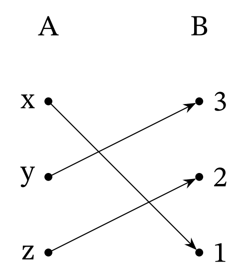
8.2 Functions

1. For each of the diagrams below, determine whether the diagram represents a function. If it does, determine whether the function is injective, surjective, bijective, or none.

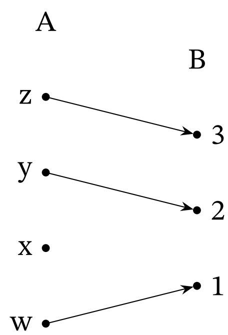
(a)



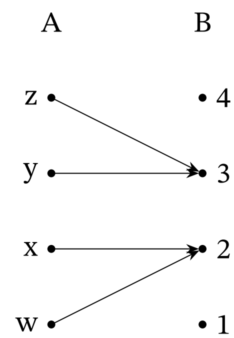
(c)



(b)



(d)



3. In precalculus, you may have written things like:

$$\frac{x^2+x-6}{x+3} = \frac{(x+3)(x-2)}{x+3} = \frac{\cancel{(x+3)}(x-2)}{\cancel{x+3}} = x-2$$

This seems to suggest that $f(x) = \frac{x^2+x-6}{x+3}$ and $g(x) = x-2$ are the same function. Explain why they are not.

5. In words, describe the range of the function $f : \mathbb{N} \times \mathbb{N} \mapsto \mathbb{N}$ where $f(m,n) = 2^m 3^n$.

7. Determine whether each of the following is an injection, surjection, bijection, or none. (FFTS Proofs) Prove your answers.

   (a) $f : \mathbb{R} \mapsto \mathbb{R}$ where $f(x) = 2x + 7$

   (b) $g : \mathbb{R} \mapsto \mathbb{Z}$ where $g(x) = \lfloor x \rfloor$

   (c) $h : \mathbb{R} \mapsto \mathbb{R}$ where $h(x) = \frac{1}{x^2+1}$

   (d) $j : \mathbb{R} \mapsto \mathbb{R}$ where $j(x) = x^2$

   (e) $q : (-\infty, -10) \mapsto (-\infty, 0)$ where $q(x) = -|x+4|$

   (f) $r : (-\infty, 0) \mapsto (\infty, 0)$ where $r(x) = -|x+4|$

   (g) $s : \mathbb{N} \mapsto \mathbb{N} \times \mathbb{N}$ where $s(x) = (x, x)$

   (h) $t : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Q}$ where $t(m,n) = \frac{m}{|n|+1}$

   (i) $u : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z} \times \mathbb{Z}$ where $u(m,n) = (m+n, 2m+n)$

(j) $v : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z}$ where $v(m, n) = 3m - 4n$

(k) $k : \mathbb{N} \mapsto \mathbb{N}$ where $k(x) = x^2$

(l) $m : \mathbb{R} \setminus \{-1\} \mapsto \mathbb{R}$ where $m(x) = \frac{2x}{x+1}$

(m) $n : \mathbb{Z} \mapsto \mathbb{N}$ where $n(x) = x^2 - 2x + 2$

(n) $p : \mathbb{N} \mapsto \mathbb{N}$ where $p(x) = |x|$

9. Let $A = \mathcal{P}(\mathcal{P}(\mathbb{N}))$ and $B = \mathbb{N}$. Define $f : A \mapsto B$ to be the function where $f(S) = \bigcup_{x \in S} x$.

11. To convert from F degrees Fahrenheit to C degrees Celsius, one can use the formula $F = \frac{9}{5}C + 32$. Determine a formula to convert from Celsius to Fahrenheit, and show that these two formulae are both inverse functions of each other (You may assume the domain and codomain are both $\mathbb{R}$).

13. Let $A = \mathbb{R} \setminus \{2\}$ and let $f(x) = \frac{3x}{x-2}$.

    (a) Determine a set B for which $f : A \mapsto B$ is a bijective function.

    (b) For the set B from part (a), find the inverse of $f : A \mapsto B$.

15. Consider the functions $f : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z} \times \mathbb{Z}$, where $f(m, n) = (5m - 3n, 2n)$, and $g : \mathbb{Z} \times \mathbb{Z} \mapsto \mathbb{Z} \times \mathbb{Z}$, where $g(m, n) = (3m + 2n, 4n - m)$. Find formulas for $f \circ g$ and $g \circ f$.

17. Give an example of functions f and g such that $f \circ g$ is injective and g is injective, but f is not injective. Write down f, g, and $f \circ g$, but you do not need to prove that your example works.

19. Consider the functions $f : \mathbb{R} \mapsto \mathbb{R}$ where $f(x) = 2x + 1$ and $g : \mathbb{R} \mapsto \mathbb{R}$ where $g(x) = 3x - 2$.

    (a) Find $(f \circ g)(x)$

    (b) Find $(f \circ g)^{-1}(x)$

    (c) Find $f^{-1}(x)$

    (d) Find $g^{-1}(x)$

(e) Find $f^{-1} \circ g^{-1}(x)$

(f) What do you notice? Prove that your observation always holds.

21. *Omitted*

23. *Omitted*

25. *Omitted*

27. *Omitted*

29. Suppose A is a set and $f : A \mapsto \emptyset$. Explain why f is a bijection.

31. (a) Prove that there are uncountably many irrational numbers.

    (b) *Omitted*

8.3 Relations

1. *Omitted*

3. Let $A = \{1, 2, 3, 4, 5\}$. Each part below is a separate definition for the relation "$a \sim b$." For each, write out all pairs that are related.

   (a) $a \sim b$ when $a < b$

   (b) $a \sim b$ when $a \mid b$

   (c) $a \sim b$ when $a \geq b$

   (d) $a \sim b$ when $a + b$ is odd.

5. Give four examples of partitions, where two are real-world examples and two are math examples.

7. Consider the relation $\sim$ on the set $\{w, x, y, z\}$ such that this is the complete list of related elements: $z \sim z$, $x \sim y$, $y \sim x$, $w \sim w$, $x \sim x$, and $y \sim y$. Is $\sim$ reflexive? Symmetric? Transitive? If a property holds, you do not need to justify it. If it doesn't, say why it fails. If all three hold, then $\sim$ is an equivalence relation; in this case, list the equivalence classes.

9. Consider the following equivalence relation $\sim$ on the set $\{1, 2, 3, 4, 5, 6\}$ such that this is the complete list of related elements: $1 \sim 1$, $2 \sim 2$, $3 \sim 3$, $4 \sim 4$, $5 \sim 5$, $6 \sim 6$, $1 \sim 2$, $2 \sim 1$, $4 \sim 5$, $5 \sim 4$, $5 \sim 6$, $6 \sim 5$, $4 \sim 6$, and $6 \sim 4$. Determine the equivalence classes of $\sim$.

11. (a) Give an example of a relation on the set $\{1, 2, 3, 4\}$ which is reflexive and symmetric, but not transitive.

    (b) Give an example of a relation on the set $\{1, 2, 3, 4\}$ which is reflexive and transitive, but not symmetric.

    (c) Give an example of a relation on the set $\{1, 2, 3, 4\}$ which is transitive and symmetric, but not reflexive

    (d) Give an example of a relation on the set $\{1, 2, 3, 4\}$ which is not reflexive, symmetric, or transitive.

13. Let $\sim$ be a relation on $\mathbb{N}$ where $a \sim b$ when $a \mid b$. Is $a \sim b$ reflexive? Symmetric? Transitive? For each property, prove that it holds or find a counterexample. Is $\sim$ an equivalence relation? If so, what are its equivalence relations?

15. Each of the following rules defines a relation on $\mathbb{Z}$. For each part, prove that $\sim$ is an equivalence relation and find its equivalence classes.

    (a) $a \sim b$ when $a \equiv b \mod 6$

    (b) $a \sim b$ when $7a - 3b$ is even

    (c) $a \sim b$ when $a^2 \equiv b^2 \mod 6$

    (d) $a \sim b$ when $a^2 + b^2$ is even

    (e) $a \sim b$ when $2a + b \equiv 0 \mod 3$

    (f) $a \sim b$ when $a + 3b \equiv 0 \mod 4$

17. Let $\sim$ be the relation on $\mathbb{R} \times \mathbb{R}$ where $(a, b) \sim (c, d)$ when $|a| + |b| = |c| + |d|$. Prove that $\sim$ is an equivalence relation.

19. Let $d \in \mathbb{N}$ and consider the set P containing an infinite arithmetic progression: $P = \{\ldots, -3d, -2d, -d, 0, d, 2d$
Let $\sim$ be the relation on $\mathbb{N}$ where $a \sim b$ if $a - b \in P$. Is $\sim$ reflexive? Symmetric? Transitive? If a property holds, you do not need to justify it. If it doesn't, say why it fails. If all three hold, then $\sim$ is an equivalence relation; in this case, list the equivalence classes.

21. *Omitted*

23. Let $A = \{(a, b) : a, b \in Z \text{ and } b \neq 0\}$. Define the relation $\sim$ to be $(a, b) \sim (c, d)$ if $ad = bc$. Prove that $\sim$ is an equivalence relation.

25. (a) Let $\sim$ be the relation on $\mathbb{Z}$ where $a \sim b$ when $a \equiv b \mod 2$ AND $a \equiv b \mod 3$. Is $\sim$ an equivalence relation?

    (b) Let $\sim$ be the relation on $\mathbb{Z}$ where $a \sim b$ when $a \equiv b \mod 2$ OR $a \equiv b \mod 3$. Is $\sim$ an equivalence relation?

27. Suppose $\sim_1$ and $\sim_2$ are equivalence relations on a set A. Let $\sim$ be the relation on A where $a \sim b$ if either $a \sim_1 b$ or $a \sim_2 b$. Is it true that $\sim$ is an equivalence relation on A? Either prove that it is an equivalence relation, or give a counterexample.

29. Determine a familiar equivalence relation whose equivalence classes are the following:

$$\{\ldots, -6, -3, 0, 3, 6\}$$

$$\{\ldots, -5, -2, 1, 4, 7\}$$

$$\{\ldots, -4, -1, 2, 5, 8\}$$

31. *Omitted*

33. (FFTS) How many relations are there from $\{1, 2, 3\}$ to $\{1, 2, 3\}$? For $n \in \mathbb{N}$, how many functions are there from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, n\}$? How many relations from $\{1, 2, \dots, n\}$ to $\{1, 2, \dots, n\}$ are not functions?

35. *Omitted*

37. *Omitted*