TP1 : Rock paper scissors

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1 Preparation

Consider a sample $x_1, x_2, \dots, x_n$ from a distribution with an unknown density f. We want to compare different kernels and show that the associated kernel density estimator is indeed a probability density function.

A function $K : R \to R_+$ is called a statistical kernel if

$$K(x) \geq 0 \quad \text{for all } x, \quad \text{and} \quad \int_{-\infty}^{\infty} K(x)\, dx = 1.$$

1.1 Question 1

Let $\mu \in R$ be a constant. Define the translation of K by

$$\tau_\mu K(x) = K(x - \mu).$$

Show that $\tau_\mu K$ is still a statistical kernel. First, for all x, $K(x - \mu) \geq 0$ if K is nonnegative. Next,

$$\int_{-\infty}^{\infty} \tau_\mu K(x)\, dx = \int_{-\infty}^{\infty} K(x - \mu)\, dx.$$

Using the change of variable $u = x - \mu$, the integral equals

$$\int_{-\infty}^{\infty} K(u)\, du = 1.$$

Hence $\tau_\mu K$ is also a kernel.

1.2 Question 2

Let $\lambda \in R$ be a nonzero constant. Define

$$d_\lambda K(x) = \frac{1}{|\lambda|} K\Big(\frac{x}{\lambda}\Big).$$

Show that $d_\lambda K$ is still a statistical kernel. The nonnegativity follows from K being nonnegative. For the integral,

$$\int_{-\infty}^{\infty} d_\lambda K(x)\, dx = \int_{-\infty}^{\infty} \frac{1}{|\lambda|} K\Big(\frac{x}{\lambda}\Big)\, dx.$$

With $u = x/\lambda$, the integral becomes

$$\int_{-\infty}^{\infty} K(u)\, du = 1.$$

1.3 Question 3

Show that

$$K(x) = \frac{1}{2}\, 1_{[-1,1]}(x)$$

is a statistical kernel (called the uniform kernel). The function is nonnegative, and

$$\int_{-\infty}^{\infty} K(x)\, dx = \int_{-1}^{1} \frac{1}{2}\, dx = 1.$$

1.4 Question 4

Show that

$$K(x) = (1 - |x|)\, 1_{[-1,1]}(x)$$

is a statistical kernel (called the triangular kernel). The function is nonnegative for $|x| \le 1$. Its integral is

$$\int_{-1}^{1} (1 - |x|)\, dx = 2 \int_{0}^{1} (1 - x)\, dx = 1.$$

1.5 Question 5

Show that

$$K(x) = \frac{3}{4}(1 - x^2)\, 1_{[-1,1]}(x)$$

is a statistical kernel (called the Epanechnikov kernel). It is nonnegative for $|x| \le 1$. Then

$$\int_{-1}^{1} \frac{3}{4}(1 - x^2)\, dx = \frac{3}{4} \int_{-1}^{1} (1 - x^2)\, dx = 1.$$

1.6 Question 6

Show that

$$K(x) = \frac{1}{\sqrt{2\pi}} \exp\Big(-\frac{x^2}{2}\Big)$$

is a statistical kernel (the Gaussian kernel). This is the density of the standard normal distribution, hence it is positive and integrates to 1 on R.

1.7 Question 7

Define the kernel estimator with window $h > 0$ by

$$\hat{f}_h(x) = \frac{1}{n} \sum_{k=1}^{n} d_h\, \tau_{X_k} K(x) = \frac{1}{n} \sum_{k=1}^{n} \frac{1}{h} K\Big(\frac{x - X_k}{h}\Big).$$

Show that $\hat{f}_h$ is a probability density. First, it is nonnegative. Next,

$$\int_{-\infty}^{\infty} \hat{f}_h(x)\, dx = \frac{1}{n} \sum_{k=1}^{n} \int_{-\infty}^{\infty} \frac{1}{h} K\Big(\frac{x - X_k}{h}\Big)\, dx.$$

With the change of variable $u = (x - X_k)/h$, each integral is 1, so the total is $\frac{1}{n} \times n = 1$.