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DSP Midterm 1

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1 Task 1

Problem: Let us define the following binary operation $(\_|\_)$ in C^2: for any two vectors $v = (v_1, v_2)$ and $w = (w_1, w_2)$ let $(v|w) = (-1)^{day} month \cdot v_1 w_1^* + year \cdot v_2^* w_2$. Is this operation a dot-product? If yes then prove, otherwise explain what dot-product axioms do hold and what do fail.

Solution: First of all, let us substitute $day = 30, month = 4, year = 2000$:

$$(v|w) = (-1)^{30} 4 \cdot v_1 w_1^* + 2000 \cdot v_2^* w_2$$

Reducing $(-1)^{30} = 1$ we get:

$$(v|w) = 4 \cdot v_1 w_1^* + 2000 \cdot v_2^* w_2$$

First of all, we need to define the field of scalars F over which vector space $V - C^2$ is given a geometric structure. By substituting arbitrary vectors $v = ((a_1 + b_1 i), (a_2 + b_2 i))$ and $w = ((c_1 + d_1 i), (c_2 + d_2 i))$:

$$(v|w) = 4 \cdot v_1 w_1^* + 2000 \cdot v_2^* w_2 = 4 \cdot (a_1 + b_1 i)(c_1 + d_1 i)^* + 2000 \cdot (a_2 + b_2 i)^* (c_2 + d_2 i) =$$

$$= 4 \cdot (a_1 + b_1 i)(c_1 - d_1 i) + 2000 \cdot (a_2 - b_2 i)(c_2 + d_2 i) =$$

$$= 4 \cdot (a_1 c_1 + b_1 d_1 + b_1 i - d_1 i) + 2000(a_2 c_2 + b_2 d_2 - b_2 i + d_2 i)$$

Since we have taken arbitrary vectors, it may be that $b_1 \neq d_2$ and $b_2 \neq d_2$, then as a result we get a complex number. Therefore, it is sufficient to take the field of scalars: $F - C$

To prove or disprove whether this binary operation is a dot product, we are required to check if axioms for dot-product from the slides hold for any $v, w, u \in C^2, a, b \in C$:

1. **Conjugate symmetry:** $(v|w) = (w|v)^*$, where * is a conjugate operation

   $$(w|v)^* = (4 \cdot w_1 v_1^* + 2000 \cdot w_2^* v_2)^* = 4^* \cdot w_1^* (v_1^*)^* + 2000^* \cdot (w_2^*)^* v_2^* =$$
   $$= 4 \cdot w_1^* v_1 + 2000 \cdot w_2 v_2^* =$$

   by commutativity of complex number multiplications

   $$= 4 \cdot v_1 w_1^* + 2000 \cdot v_2^* w_2 = (v|w)$$

   Therefore, this axiom holds

2. **Linearity in the first argument:** $(av + bu|w) = a(v|w) + b(u|w)$
   The first part of equality:

   $$(av + bu|w) = 4 \cdot (av_1 + bu_1) w_1^* + 2000 \cdot (av_2 + bu_2)^* w_2 =$$

   $$= 4 \cdot (av_1 + bu_1) w_1^* + 2000 \cdot (a^* v_2^* + b^* u_2^*) w_2$$

   The second part of equality:

   $$a(v|w) + b(u|w) = a(4 \cdot v_1 w_1^* + 2000 \cdot v_2^* w_2) +$$

   $$+ b(4 \cdot u_1 w_1^* + 2000 \cdot u_2^* w_2) = 4 \cdot a v_1 w_1^* + 4 \cdot b u_1 w_1^* +$$

   $$+ 2000 \cdot a v_2^* w_2 + 2000 \cdot b u_2^* w_2 = 4(av_1 + bu_1) w_1^* + 2000(a v_2^* + b u_2^*) w_2$$

   Since $a, b \in C$, $a \neq a^*, b \neq b^*$ and, consequently:

   $$4 \cdot (av_1 + bu_1) w_1^* + 2000 \cdot (a^* v_2^* + b^* u_2^*) w_2 \neq 4(av_1 + bu_1) w_1^* + 2000(a v_2^* + b u_2^*) w_2$$

   $$(av + bu|w) \neq a(v|w) + b(u|w)$$

   This axiom does not hold.

3. **Positive Definiteness:** $(v|v) > 0$

$$(v|v) = 4 \cdot v_1 v_1^* + 2000 \cdot v_2^* v_2$$

Let us take an arbitrary imaginary number $n = a + bi, a, b \in R$:

$$nn^* = (a + bi)(a - bi) = a^2 - (bi)^2 = a^2 - (-1)b^2 = a^2 + b^2$$

$a^2 + b^2 \geq 0$ for all $a \in R$ for all $b \in R$ and, more importantly, it is equal to 0 only if $a = 0, b = 0$.
Therefore:

$$v_1 v_1^* + 2000 \cdot v_2^* v_2 \geq 0$$

and is equal to zero only if $v_1 = 0, v_2 = 0$. This axiom holds

We have 2 axioms (**Conjugate symmetry**, **Positive definiteness**) that hold, 1 axiom (**Linearity in the first argument**) that does not hold, therefore, this operation is not dot-product.
Answer: This operation is not dot-product because 2 axioms (**Conjugate symmetry**, **Positive definiteness**) hold and 1 axiom (**Linearity in the first argument**) does not hold.

2 Task 2

Problem: Let us consider (year)-dimensional (complex) impulse-domain. Firstly, starting with/from the definition of the DFT, compute and give an explicit (in terms of trigonometric functions) representation for DFT the following two signals:

- $(\delta_{(n-month) mod\, year})$
- (day^n)

Then starting with/from the definition of the IDFT validate that IDFT applied to the results returns exactly these signals.
Solution: First of all, let us start from the definition of DFT (it is going to be taken from the slides but complex one is to be denoted as i):

- Let $m > 0$ be a fixed integer and let $W_M = e^{-2\pi i/m}$
- The discrete Fourier transform (DFT) maps each finite sequence (signal) $x = (x_0, x_1, ..., x_{m-1}) \in C^m$ to the vector of the eigen values $X = (X_0, X_1, ..., X_{m-1}) \in C^m$ of the linear system $(x^* \_)$:

$$X_k = \sum_{n=0}^{n=m-1} x_n W_m^{kn}, \;\; 0 \leq k \leq m-1$$

Now, let us proceed with the DFT computations with substituted values:
1) $(\delta_{(n-4) mod 2000})$:
By definition of Kronecker delta sequence:

$$\delta_n = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0 \end{cases}$$

For our function, we can notice, $n = 4 \implies \delta_{(4-4) mod\, 2000} = \delta_0 = 1$, therefore:

$$\delta_{(n-4) mod\, 2000} = (0, 0, 0, 0, 1, 0, ..., 0)$$

For DFT:

$$X_k = \sum_{n=0}^{n=1999} x_n W_{2000}^{kn} = \sum_{n=0}^{n=3} x_n W_{2000}^{kn} + x_4 W_{2000}^{4k} + \sum_{n=1999}^{n=5} x_n W_m^{kn} =$$

By definition of Kronecker delta sequence:

$$= \sum_{n=0}^{n=3} 0 \cdot W_{2000}^{kn} + W_{2000}^{4k} + \sum_{n=m-1}^{n=5} 0 \cdot W_{2000}^{kn} = W_{2000}^{4k} = (e^{-2\pi i/2000})^{4k} = e^{-8\pi ik/2000} =$$

By transformation: $e^{-ix} = cos(x) - isin(x),\ \ x = 8\pi k/2000$

$$= cos(\frac{8\pi k}{2000}) - isin(\frac{8\pi k}{2000}) = cos(\frac{\pi k}{250}) - isin(\frac{\pi k}{250})$$

Now, the definition of IDFT is going to be stated from the slides:

- $x_n = \frac{1}{m}\sum_{k=0}^{k=m-1} X_k W_m^{-kn},\ \ 0 \le k \le m-1$
  Converts any vector $X = (X_0, X_1, ..., X_{m-1}) \in C^m$ into a filter $x = (x_0, x_1, ..., x_{m-1} \in C^m$ such that a linear system $(x^{(*)})$ has spectrum $X = (X_0, X_1, ..., X_{m-1})$

Calculating IDFT:

$$x_n = \frac{1}{2000}\sum_{k=0}^{k=1999} X^k W_{2000}^{-nk} = \frac{1}{2000}\sum_{k=0}^{k=1999} W_{2000}^{4k} W_{2000}^{-nk} = \frac{1}{2000}\sum_{k=0}^{k=1999} W_{2000}^{(4-n)k}$$

For $n = 4$:

$$x_4 = \frac{1}{2000}\sum_{k=0}^{k=1999} W_{2000}^{0\cdot k} = \frac{1}{2000}\sum_{k=0}^{k=1999} 1 = \frac{2000}{2000} = 1$$

For $n \neq 4$, we get a sum of a geometric progression with $b_1 = 1$, $q = W_{2000}^{4-n}$, $k = 2000$:

$$x_n = \frac{1\cdot(1 - W_{2000}^{2000(4-n)})}{2000(1 - W_{2000}^{(4-n)})} = (W_{2000}^{2000(4-n)} = e^{-i2\pi(4-n)2000/2000} = e^{-i2\pi(4-n)} = cos(2\pi(4-n)) -$$

$$-isin(2\pi(4-n)) = 1) = \frac{1-1}{2000(1 - W_{2000}^{(4-n)})} = 0$$

Therefore, using IDFT we reconstructed the original signal. And we got:

$$\delta_{(n-4)mod2000} \overset{\text{DFT}}{\longleftrightarrow} (cos(\frac{\pi k}{250}) - isin(\frac{\pi k}{250}))$$

2) (30^n). DFT calculation:

$$X_k = \sum_{n=0}^{n=1999} a^n W_{2000}^{kn} = \sum_{n=0}^{n=1999} (aW_{2000}^k)^n$$

(sum of geom sequence with $b_1 = 1, q = aW_{2000}^k$)

$$= \frac{1\cdot(1 - (aW_{2000}^k)^{2000})}{1 - aW_{2000}^k} = \frac{1 - a^{2000}W_{2000}^{2000k}}{1 - aW_{2000}^k} = \frac{1 - a^{2000}e^{\frac{-2\pi i 2000k}{2000}}}{1 - ae^{\frac{-2\pi ik}{2000}}} =$$

$$= \frac{1 - a^{2000}(cos(2\pi k) - isin(2\pi k))}{1 - a(cos(\frac{2\pi k}{2000}) - isin(\frac{2\pi k}{2000}))} = \frac{1 - a^{2000}}{1 - a(cos(\frac{\pi k}{1000}) - isin(\frac{\pi k}{1000}))}$$

And, the IDFT calculation:

$$x_n = \sum_{k=0}^{k=m-1} X_k W_{2000}^{-nk} = \sum_{k=0}^{k=m-1}\sum_{p=0}^{p=m-1} x_p W_{2000}^{pk} W_{2000}^{-nk} = \sum_{k=0}^{k=m-1}\sum_{p=0}^{p=m-1} x_p W_{2000}^{(p-n)k}$$

Now, for the different elements of the inner sum:
1) $p = n$:

$$\sum_{k=0}^{k=m-1} a^n W_{2000}^{(n-n)k} = \sum_{k=0}^{k=m-1} a^n = ma^n$$

2) $p \neq n$:

$$\sum_{k=0}^{k=m-1} a^p W_{2000}^{(p-n)k} = \frac{a^p(1 - W_{2000}^{(p-n)2000})}{1 - W_{2000}^{(p-n)}} = \frac{a^p(1-1)}{1 - W_{2000}^{(p-n)}} = 0$$

Therefore, we get:

$$\sum_{k=0}^{k=m-1}\sum_{p=0}^{p=m-1} x_p W_{2000}^{(p-n)k} = \frac{ma^n}{m} = a^n$$

As a result we have proven:

$$a^n \overset{\text{DFT}}{\longleftrightarrow} \frac{1-a^{2000}}{1-a(cos(\frac{\pi k}{1000})-isin(\frac{\pi k}{1000}))}$$

Result:

- $\delta_{(n-4)mod2000} \overset{\text{DFT}}{\longleftrightarrow} (cos(\frac{\pi k}{250})-isin(\frac{\pi k}{250}))$
- $a^n \overset{\text{DFT}}{\longleftrightarrow} \frac{1-a^{2000}}{1-a(cos(\frac{\pi k}{1000})-isin(\frac{\pi k}{1000}))}$