Allan Variance

Data analysis of Time Series

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1 Introduction

Analyzing phenomena in sciences most often lead to the study of a measured quantity under certain experimental conditions, furthermore, the trace of these measurements over time give rise to new data encoding statistical information of the studied system and its evolution. As a matter of fact a set of labeled data organized in time is called a **Time Series**, a concept that will be a general envelope topic in this report. Time series analysis appear everywhere in physics, from studying the trajectories of projectiles to meteorology and seismology, and using different methods (such as spectral decomposition, regression, forecasting) to evaluate the available finite data is necessary to make supported conclusions about the observed system.

A time series in the context of a pre-existing model can be described as data composed by a true signal sampled in time that can be modeled or approximated, and random[1] noise coming from different sources that corrupts the signal. Despite the noise not being contemplated in the proposed model, it can be characterized statistically, and methods such as **Power Spectral Density** (PSD) allow to explore in detail the fluctuations in frequency and phase of these *errors*. When using this method on noise data, the spectrum in frequency space $S(f)$ follows a trend function of f based on the different properties that the noise may have.

It's worth saying that PSD's are proposed under the theoretical assumption of an infinite frequency range and measurement time, generalizing the expectancy value to a true average, but can be formulated to deal with discrete finite data and non-zero width of the frequency bins used in the PSD.

For now we are focusing our attention on a specific type of system: **oscillators**. For systems that move in regular cycles a common generalization is a sinusoidal signal $H_\nu(t)$ that follows an oscillatory behavior with angular frequency $2\pi\nu$ and, since its intended to simulate the measurement of a signal, carries some noise within.

$$H_\nu(t) = H_o sin(2\pi\nu t + \phi(t)) = H_o sin(2\pi\nu(t + x(t))) \tag{1}$$

$$x(t) = \frac{\phi(t)}{2\pi\nu}$$

where $\phi(t)$ is the noise present in the argument of the sine function or **phase noise** (adimensional), which can be seen as an error in the evaluation time of the oscillator $x(t)$, called the **time error** (seconds). Note that this measurements are instantaneous, meaning that they can be obtained for any time of the signal without knowing about other values of the signal.

As mentioned before, we are dealing with a pure signal of the form $G_\nu(t) = H_o sin(2\pi\nu t)$ corrupted by a noise signal encoded in the time error. This time error in the right side of the equality is equivalent to evaluating the pure signal, not in the corresponding time t, but in a time $t + x(t)$. So the time error can be seen as a **relative time delay** of the signal $H_\nu(t)$ respect to the ideal oscillator signal $G_\nu(t)$. In this order of ideas, any frequency measurement is made by comparing two oscillators: the study case and a stable reference oscillator calibrated previously, and therefore it is impossible to purely measure one isolated oscillator[1].

This principle is well understood by the scientists in charge of defining the standard measurements we use in our every day life. By defining reference values is that the rest of the world can be measured according to these standards, and the time is no exception, since calibrating a clock means to have a device with the same frequency and phase as a well known reference oscillator. This example will be explored in further sections.

When studying the statistics of the time error $x(t)$ of atomic clocks, David W. Allan concerned by the increasing need of quality atomic frequency standards, proposes studying the **standard deviation of the frequency fluctuations** and its relation with the type of noise in the system[2]. This statistical assessment of time error in oscillators is called the **Allan Variance** and will be the main topic of this report.

[1] Non predictable and ideally uncorrelated (Homocedasticity)

2 Allan Variance

The framework for now on is the measurement of a two-oscillator system[2], specifically of the amplitude values at selected sampling times, providing as well information on the **instantaneous relative phase** between the two oscillators.

We can observe the noise data as a consequence of errors in time such as $\phi(t)$ or may as well evaluate the changes in frequency. The **frequency noise** $\Delta\nu(t)$ is then defined as:

$$\Delta\nu(t) = \frac{1}{2\pi}\frac{d\phi(t)}{dt} \tag{2}$$

Once known the frequency noise and time error, the **frequency deviation** $y(t)$ (or *frequency error* analogous to $x(t)$) is defined as:

$$y(t) = \frac{\Delta\nu(t)}{\nu} = \frac{1}{2\pi\nu}\frac{d\phi(t)}{dt} = \frac{dx(t)}{dt} \tag{3}$$

Corresponding to the coefficient of variation of phase noise. Note how these two quantities $\Delta\nu(t)$ and $y(t)$ are no longer instantaneous since its necessary to have variation in time by definition, therefore in finite set of data these values are averaged using the values of the series in a finite interval of time of length τ, the **averaging time**.

$$y(t) \approx \frac{x(t+\tau) - x(t)}{\tau} = \bar{y}(t) \tag{4}$$

With the aim of characterizing in time the variation of the frequency deviation of an oscillator, the two-sample variance of $\bar{y}(t)$ for set of N data points sampled at a given rate is defined as:

$$\sigma_y^2(\tau) = \frac{1}{2(M-1)}\sum_{i=1}^{M-1}\left[\bar{y}(t_i+\tau) - \bar{y}(t_i)\right]^2 \tag{5}$$

With $M = N - 1$ time intervals of length τ for which $\bar{y}(t)$ is averaged. This can be obtained from the time error as:

$$\sigma_y^2(\tau) = \frac{1}{2(N-2)\tau^2}\sum_{t=1}^{N-2}\left[x(t_i+2\tau) - 2x(t_i) + x(t_i)\right]^2 \tag{6}$$

Recall that the time error is given in units of time, and is proportional to the instantaneous phase difference in time obtained from the comparison of the two clocks (oscillators) at a time t and $t+\tau$[1]. The Allan variance is estimated in a clock relative to a reference clock, therefore is a **relative measurement**.

2.1 Physical interpretation

The Allan Deviation ($\sqrt{\sigma_y^2(\tau)}$) tells how the noise drifts in time and frequency vary at different timescales. This can be understood as having an oscillator active for a time τ and after this time check how the frequency of the noise changed relative to a reference oscillator.

$$\sigma_y^2(\tau) = \frac{\Delta t}{\tau} = \frac{\Delta f}{\nu} \tag{7}$$

2.2 Definitions

2.2.1 Absolute Allan Variance: Three Cornered Hat Method

The (two-sample) Allan variance is a relative measurement since it is a relation obtained by comparing two oscillators,then is impossible to establish this statistic taking into account only one oscillator. But a possible direct method of estimating an **Absolute** Allan variance is through the crossed comparisons of a few clocks.

From the article [1] recall that taking into account an ensemble of Q clocks, the measured data of time error for a clock i respect to to a clock j will correspond to $x_{ij}(t)$. Now, given that the clocks are classically independent systems[3] we can assume that the noise in frequency and phase

[2]Allan variance is also called two-sample variance since two measured oscillators (systems) are needed
[3]in the framework of atomic clocks its worth mentioning the effects of the non locality in QM

are uncorrelated between clocks. Therefore,

$$\sigma_{ij}^2(\tau) = \sigma_i^2(\tau) + \sigma_j^2(\tau) \tag{8}$$

Therefore, there exist $\binom{N}{2}$ combinations of the form $\sigma_{jk}^2(\tau)$ for N clocks. Solving the system of equations is easier with linear algebra. The Matrix A converting the vector of absolute variances $\{\sigma_i^2(t)\}$ to the vector of relative variances $\{\sigma_{ij}^2(t)\}$ has dimension Nx$\binom{N}{2}$ therefore is not always squared. Remember that, having the relative variances, we are interested in the inverse of this matrix A so is possible to retrieve the absolute variances. For the situation in the article [1] with 3 clocks is easy since the Matrix A is a squared matrix i.e. $\binom{3}{2}= 3 =$ N. In this case, the absolute Allan variance can be obtained as the following set of equations, which coefficients correspond to the inverse of matrix A:

$$\sigma_1^2(\tau) = \frac{1}{2}\sigma_{12}^2 + \frac{1}{2}\sigma_{13}^2 - \frac{1}{2}\sigma_{23}^2$$
$$\sigma_2^2(\tau) = \frac{1}{2}\sigma_{23}^2 + \frac{1}{2}\sigma_{21}^2 - \frac{1}{2}\sigma_{13}^2$$
$$\sigma_3^2(\tau) = \frac{1}{2}\sigma_{32}^2 + \frac{1}{2}\sigma_{31}^2 - \frac{1}{2}\sigma_{12}^2$$

where $\sigma_i^2(\tau)$ is the absolute Allan Variance of clock i. The matrix A can be generalized for other cases since the matrix A is a full rank matrix and thus can have a Left inverse matrix such that $A_{left}^{-1}A = I$. The Left inverse of the matrix is:

$$A_{left}^{-1} = (A^TA)^{-1}A^T$$

so that:

$$A_{left}^{-1}\sigma_{ij}^2(t) = \sigma_i^2(t)$$

Resulting in the vector of absolute variances.

2.2.2 Power Spectral Density

For the PSD $S(f)$ of frequency deviation, time error and phase the following relations are true:

$$S_y(f) = \frac{f^2}{\nu^2}S_\phi(f) = 4\pi^2 f^2 S_x(f)[Hz^{-1}] \tag{9}$$

$$S_x(f) = \frac{1}{4\pi^2\nu^2}S_\phi(f)[Hz^{-3}]$$

2.2.3 Colors of the Noise

This term usually refers to the Power Spectrum of the noise signal, and which trend function the noise follows in the frequency space. Each type of noise can be characterized with Allan Variance since it behaves as a power law of τ depending on the type of noise[3]:

- **White phase Noise** proportional to $1/\tau$
- **White frequency Noise** proportional to $1/\sqrt{\tau}$
- **Pink Noise** constant for all τ
- **Brown Noise** proportional to $1/\sqrt{\tau}$

In order to obtain the noise parameters , one must consider the following relationship between the Allan variance and the PSD of the noise:

$$\sigma_y^2(\tau) = 4\int_0^\infty S_y(f)\frac{sin^4(\pi f\tau)}{(\pi f\tau)^2}df$$

2.3 Applications

2.3.1 The Universal Coordinated Time (UTC)

In the **Bureau International des Poids et Mesures (BIPM)** the time differences (UTC-Clock) of about 450 Atomic clocks are taken at 00:00 UTC every 5 days in order to define a weighted average of the clock readings that maximizes the long-term stability of the oscillator's frequency[4]. Each clock samples the relative phase with respect to a reference clock from which the relations in phase between clocks can be inferred, leading to Absolute Allan variances. Having the relative and absolute Allan variances to make the weighted average for the clocks allow to think of the UTC as a **virtual noiseless oscillator**[1]. Data of this nature is analyzed in the python notebook for which the phase of different clocks in different laboratories is measured for 6 time samples (5 days apart each).

2.3.2 Gyroscope characterization

Gyroscopes is other good example of a system that consists on precisely calibrated oscillators. These are used in (but not limited to) guidance of aircrafts and spaceships, since these systems are **sensible to the changes in torque**. By spinning at a constant frequency, these changes in torque can be measured as off-axis forces on the gyroscope that suggest a change in orientation or angular momentum of the aircraft. Keeping precise the frequency of rotation of the gyroscopes and characterizing the possible frequency drifts is a challenge that can be approached with the Allan Variance statistic. In the data sheet [5] a 3-axis electronic gyroscope is described and the noise density values are accurately calculated within the context of Allan Variance.[4]

2.3.3 Optical Tweezers

Optical tweezers are last century technology extremely useful when working with the microscopic world. Arthur Ashkin won the Nobel Prize in Physics (2018) for discovering that the momentum of a single light beam can be used as an highly sensitive set of "tweezers" [6] by detecting a force caused by the negative light pressure. Optical traps physically implement an Ornstein-Uhlenbeck process through the harmonic trapping force field to hold in position microscopic particles (order of microns) and even detect the force to which the particle is subjected . The article [7] explain how for trapped particles in a stationary harmonic potential the noise data is analyzed in order to characterize the noise and its stability in time. Allan Variance is used to characterize low frequency drifts that may change the noise properties.

[4]For more information you can refer to a hands-on Allan variance analysis of the gyroscope data at `https://mwrona.com/posts/gyro-noise-analysis/`

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