Basic Plasma Physics

Theory and Applications

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Symbols used in the text

Please notice that some symbols have more than one definition. Index e denotes electrons and index i denotes ions throughout the text. Greek symbols are found on next page.

$\mathbf{B}$	Magnetic field
c	Speed of light
D	Diffusion constant
D_a	Ambipolar diffusion constant
e	Absolute value of the electron charge
$\mathbf{E}$	Electric field
$\mathbf{j}$	Current density
k_B	Boltzmann's constant
m_e	Electron mass
m_i	Ion mass
p	Pressure
n	Particle density
n_0	Ion density, constant background
n_i	Ion density
n_n	Atom density
r_L	Larmor radius
R_m	Mirror ratio
T_i	Ion temperature
T_e	Electron temperature
v_A	Alfvén velocity
v_g	Group velocity
v_{th}	Thermal velocity
v_{ph}	Phase velocity
Z_i	Ion charge number

$\mathbf{\Gamma}$	Particle flux
ε	Inverse aspect ratio
ϵ_0	Permittivity of free space
η	Resistivity
Θ	Pitch angle
λ_D	Debye length
λ_m	Mean free path between collisions
μ	Magnetic moment
μ	Mobility
μ_0	Permeability of free space
ν	Collision frequency
ν_i	Ionisation frequency
ρ	Charge density
ρ_m	Mass density
σ	Cross section
τ	Mean time between collisions
Φ	Magnetic flux
ω_c	Cyclotron frequency
ω_L	Larmor frequency
ω_p	Plasma frequency
ω_B	Bounce frequency

Contents

Chapter 1

Introduction

Of all the fields of physics, why should one go about to study plasma physics? Well, several more or less convincing arguments can be set up in order to answer such a question. For example, plasma physics can be said to be a very dynamical topic with applications important to mankind, such as fusion energy. Or one may say that plasma physics contains an extreme variety of interesting phenomena, and therefore enriches our understanding of the world in which we live. The list of arguments can go on and on, but the most important argument is probably the fact that there are so many arguments! This implies that plasma physics contains something for almost everyone, from those who study or work within theoretical physics, astronomy etc., to applied areas, such as electrical engineering or industrial welding.

A concise way to explain what plasma physics is all about would be to say that it is the physics of hot, ionised gases. Because of this, the plasma state is often referred to as the *fourth state of matter*. This classification is based on the following observation: Matter at very low temperatures is in *solid form*. By increasing the temperature, matter will make a transition, first to *liquid form* and then to *gaseous form*. If the temperature is increased even further, the gas will be ionised because of the high energy of the individual gas particles, and the gas goes over to the *plasma form*. Thus, a requirement for a plasma to form is high temperature. This can be expressed more precise through the Saha equation

$$\left(\frac{n_i}{n_n}\right)^2 = 2.4 \times 10^{15} \frac{T^{3/2}}{n_n} \exp\left(-\frac{U_i}{k_B T}\right), \tag{1.1}$$

giving a relation between degree of ionisation of a gas and its temperature, T. Here, n_i is the number of ionised atoms per m^3, n_n is the number of neutral atoms per m^3, T the temperature in K, U_i the ionisation energy for the gas, and k_B Boltzmann's constant. From Eq. (1.1) we see that the strongest dependence in the degree of ionisation, n_i/n_n, is contained in the

Physical example	T (K)	n_0 (m^{-3})	n_i/n_n
Good vacuum	3000	10^{16}	10^{-13}
Surface of the Sun	6000	10^{22}	10^{-4}
Solar atmosphere	10^4	10^{20}	10
Hot, confined laboratory plasmas	10^5	10^{20}	10^7
Solar Corona	10^6	10^{16}	10^{12}
Hot nebulae	10^4	10^9	10^6
Cold nebulae	10^2	10^8	10^{-334}

Table 1.1. Degree of ionisation for hydrogen gas ($U_i = 13.6$ eV).

exponential. As long as $k_B T \ll U_i$, n_i/n_n is small, but as we approach $k_B T \sim U_i$, the ionisation grows rapidly. Since the ionisation energy for most gases lies within the range 1–10 eV, this means that the temperatures required to reach the plasma state lies in the range $T \sim 10^4$–10^5 K (where we have used 1 eV $\Leftrightarrow 10^4$ K)[1]. On the other hand, we see that there is a weak dependence on the density n_n, which makes it possible to partially compensate a low temperature by keeping the density low, see Table 1.1. According to this table it is clear that plasmas can exist within a wide range of temperatures and densities.

Due to the demand for high temperatures, plasmas are rare on Earth. On the other hand, from a cosmological perspective it seems as if 99% of all the matter in the universe is in the plasma state; a remarkable figure indeed, which high-lights the importance of plasma physics in order to fully appreciate and understand many of the processes we observe in the universe, both earthbound and cosmological.

The classical fields of application for plasma physics have been astrophysics and radio astronomy, and lately space physics (i.e., the physics of the surroundings of Earth, reachable by satellites). The solar wind (the flux of charged particles ejected from the sun), Earth's radiation belts (the Van Allen belts, which is a plasma zone outside the atmosphere), and the northern lights are all examples of phenomena where plasma physics plays a crucial role. Some of the more earthly applications are discharges in gases (e.g., lightning), magnetohydrodynamical energy conversion, lasers and modern semiconductor technology.

One of the most important applications of plasma physics is *fusion*, which is the energy source of the sun. The goal of fusion research is to create controlled fusion energy by combining light atomic nuclei into heavier elements. In the case of the sun, the plasma confinement is due to gravitation, which, of course, is no viable alternative on Earth. The fusion plasma in the sun

[1] Boltzmann's constant relates energy and temperature.

is also set up by the gravitational force, since it provides the necessary high pressure in the central parts of the sun, and thereby the high temperatures needed for "igniting" the fusion plasma. Apart from the problem of confinement on Earth, the cross sections for the fusion reactions are very small until a temperature of approximately 10^8 K is reached. Thus, the problem for fusion researchers is to create and confine a plasma with such an extreme temperature with densities of the order of 10^{20} m^{-3}.

The typical physical conditions in a plasma are extreme. Even so, the theory of plasma physics accurately describes the observed characteristic phenomena, even though it can be seen as an application of classical electrodynamics to a new field—hot, ionised gases. No new fundamental concepts, conflicting with "common sense physical intuition", need to be introduced, as in the case of quantum mechanics or general relativity. On the other hand, this does not mean that plasma physics is just a straightforward application of a well known theory. Quite the contrary, as we will see!

1.1 The definition of a plasma

As we have seen, a plasma is in many respects a more or less ionised gas. However, in order for the typical plasma properties to dominate the physics of the ionised gas, certain conditions have to be fulfilled. An often used definition of a plasma is the following:

> A plasma is a quasi-neutral gas consisting of charged and neutral particles exhibiting collective behaviour.

Quasi-neutrality means that the plasma is neutral on a macroscopic level, but deviations from neutrality may occur on a microscopic level. The reason for this is the occurrence of strong electric fields even for slight deviations from neutrality. The relation between the electric field, $\mathbf{E}$, and the charge density, ρ, is given by Poisson's equation

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{e}{\epsilon_0}(n_i - n_e), \tag{1.2}$$

where $-e = -|e|$ is the electron charge, and n_i and n_e are the number densities of ions (which are supposed to be singly ionised) and electrons, respectively. This means that if we let $\nabla \cdot \mathbf{E} \sim E/L$, where L is the characteristic length over which charge neutrality does *not* hold, we find that

$$E \sim \left(\frac{en_0}{\epsilon_0}\right)\left(\frac{n_i - n_e}{n_0}\right) L. \tag{1.3}$$

Suppose we have a charge imbalance of the order of 1% over a distance of 1 mm in a plasma with density $n_0 = 10^{20}$ m^{-3}. With these numbers, Eq. (1.3) gives the electric field $E \approx 2 \times 10^7$ V/m! We do not normally observe these field strengths, and must therefore conclude that only very small imbalances on the charge neutrality occur, and when they occur it is over small distances.

The reason for including the phrase "collective behaviour" in the definition of a plasma is the following. As compared to the collisional forces acting between two neutral particles in a gas, the electrical forces between two charged particles are long range. Thus, in contrast to a neutral gas where the interactions between particles usually take place in pairs, a particle in a plasma interacts with a large number of other plasma particles at the same time.

1.2 Debye screening

One of the fundamental properties of a plasma is the ability to quickly screen out changes in the electric potential (including externally applied potentials). In order to study this, we consider the potential from an extra charge, q, placed in the origo of an otherwise neutral plasma. From Eq. (1.2) we obtain

$$\nabla \cdot \mathbf{E} = \frac{e}{\epsilon_0}(n_i - n_e) + \frac{q}{\epsilon_0}\delta(\mathbf{r}). \tag{1.4}$$

The potential, ϕ, is defined from $\mathbf{E} = -\nabla\phi$, which implies $\nabla \cdot \mathbf{E} = -\nabla^2\phi$, and in spherical coordinates, Eq. (1.4) (by assuming spherical symmetry) becomes

$$-\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{e}{\epsilon_0}(n_i - n_e) + \frac{q}{\epsilon_0}\delta(\mathbf{r}). \tag{1.5}$$

Since the electrons have less inertia than the ions in a plasma, the former will respond more quickly to changes in the electric potential. We can therefore assume that the ions are stationary, i.e., $n_i \approx n_0$, where n_0 is the constant background density. Furthermore, $n_e \to n_i$ as $r \to \infty$, since the perturbation far away is small, and that the electron density follows a Boltzmann distribution, given the potential, ϕ, in the plasma,

$$n_e \approx n_0 \exp\left(\frac{e\phi}{k_B T_e}\right), \tag{1.6}$$

where T_e is the electron temperature.

If the potential, ϕ, is assumed small compared to the thermal energy of the electrons, i.e., $|e\phi| \ll k_B T_e$, we may Taylor expand (1.6) according to

$$n_e \approx n_0\left(1 + \frac{e\phi}{k_B T_e}\right) + \mathcal{O}\left((e\phi/k_B T_e)^2\right). \tag{1.7}$$

Inserting Eq. (1.7) in Eq. (1.5), we obtain (after multiplying both sides with $-r^2$, and observing that $r^2\delta(r) \equiv 0$)

$$\frac{d}{dr}\left(r^2\frac{d\phi}{dr}\right) = \frac{e}{\epsilon_0}n_0\frac{e\phi}{k_B T_e}r^2 \equiv \frac{r^2}{\lambda_D^2}\phi, \tag{1.8}$$

where we have defined the characteristic *Debye length* according to

$$\lambda_D^{-2} \equiv \frac{n_0 e^2}{\epsilon_0 k_B T_e}. \tag{1.9}$$

Eq. (1.8) can be written as

$$\frac{d^2}{dr^2}(r\phi) - \frac{1}{\lambda_D^2} r\phi = 0, \tag{1.10}$$

which can be checked by performing the differentiations, and the solution is

$$\phi(r) = \frac{C}{r} e^{-r/\lambda_D}. \tag{1.11}$$

In order to find the value of the constant C, we argue as follows. For small r, the potential, ϕ, should have the form of the ordinary vacuum potential from the point charge q, i.e., $\phi_{\text{vacuum}} = q/(4\pi\epsilon_0 r)$. Comparing with Eq. (1.11), we see that $C = q/(4\pi\epsilon_0)$. Thus, the screening *Debye potential* reads

$$\phi(r) = \frac{q}{4\pi\epsilon_0 r} e^{-r/\lambda_D}. \tag{1.12}$$

From Eq. (1.12) we see that the screening properties of the plasma will effectively block the vacuum potential at $r = \lambda_D$, and the influence from the extra point charge, q, becomes unimportant when $r > \lambda_D$ (for examples of numerical values, see Table 1.2).

As we saw previously, the definition of a plasma contained the demands for quasi-neutrality and collective behaviour. We can now give a more quantitative description of these specific properties. It is obvious that in order for the screening abilities of the plasma to be realised, the macroscopic dimension, L, of the plasma must be much greater than the Debye length, i.e.,

$$\lambda_D \ll L. \tag{1.13}$$

In order for the assumption of a Boltzmann distribution of the electrons to hold, the statistical laws behind such a distribution must be valid. This means that the number of particles in a Debye sphere, with volume $4\pi\lambda_D^3/3$, must be large, i.e.,

$$\frac{4\pi}{3}\lambda_D^3 n_0 \gg 1. \tag{1.14}$$

Finally, if the collisions between the plasma particles and the neutral particles are too frequent, the dynamics of the ionised gas will be determined by hydrodynamical laws instead of electromagnetic theory. Thus, in order for the electromagnetic properties of the ionised gas to be important, we have to require that the collisional frequency, ν, between the neutrals and the plasma particles is much less than the frequency of a typical oscillation.

Plasma type	n (m^{-3})	T (K)	B (T)	$\omega_p/6$ (s^{-1})	λ_D (m)
Interstellar gas	10^6	10^4	10^{-10}	10^4	7
Solar corona	10^{12}	10^6	10^{-8}	10^7	7×10^{-2}
Ionosphere	10^{12}	10^3	10^{-5}	10^7	2×10^{-3}
Gas discharge	10^{18}	10^4	10^{-1}	10^{11}	7×10^{-7}
Fusion plasma	10^{20}	10^8	10	10^{12}	7×10^{-6}

Plasma type	$n\lambda_D^3$	ν (s^{-1})	$r_{L,e}$ (m)	$r_{L,i}$ (m)
Interstellar gas	3×10^8	6×10^{-4}	2×10^4	10^6
Solar corona	3×10^8	0.6	2×10^3	10^5
Ionosphere	10^4	10^4	7×10^{-2}	3
Gas discharge	30	2×10^{10}	2×10^{-5}	10^{-3}
Fusion plasma	3×10^6	5×10^6	2×10^{-5}	10^{-3}

Table 1.2. Some typical plasma parameters.

The plasma frequency (defined in section 4.3.1), ω_p, is the characteristic frequency of oscillation of the electrons defined by

$$\omega_p^2 = \frac{n_0 e^2}{m_e \epsilon_0}, \tag{1.15}$$

where m_e is the electron mass. Using this frequency, the condition becomes

$$\nu \ll \omega_p. \tag{1.16}$$

As a consequence, we will in what follows require

$$\begin{aligned} \lambda_D &\ll L, \\ \frac{4\pi}{3}\lambda_D^3 n_0 &\gg 1, \\ \nu &\ll \omega_p. \end{aligned}$$

In Table 1.2, order-of-magnitude estimates of some characteristic plasma parameters have been given. Also included in the table is the Larmor radius (defined in section 2.1) for electrons ($r_{L,e}$) and ions ($r_{L,i}$) in the presence of a magnetic field, B.

Chapter 2

Particle motion in electric and magnetic fields

A plasma consists of charged particles moving in both external and internally generated electromagnetic fields. This means that an exact description of any plasma phenomenon requires us to solve the equations of motion for each individual plasma particle, together with a self-consistent set of Maxwell's equations. Finding a solution to these equations is not possible, not only because of the complexity of the system of equations, but also because we do not have all the necessary boundary conditions. Thus, in order to obtain an understanding of the dynamics of plasmas, statistical methods have to be employed. However, before studying these methods, an understanding of single particle dynamics is valuable, and we will therefore start investigating the plasma properties by considering the motion of single charged particles in electric and magnetic fields. For simplicity, we neglect particle collisions and collective effects, and we furthermore assume that the electric and magnetic fields are known quantities.

The motion of a point particle can be described using the position vector $\mathbf{r}(t)$. The velocity vector is defined as

$$\mathbf{v}(t) = \frac{d\mathbf{r}(t)}{dt} \equiv \dot{\mathbf{r}}(t). \tag{2.1}$$

The non-relativistic equation of motion for a charged particle, known as the *Lorentz force equation*, is

$$m\frac{d\mathbf{v}}{dt} = \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{2.2}$$

where $\mathbf{E}$ is the electric field, $\mathbf{B}$ the magnetic field, and m and q the mass and charge of the particle, respectively. It is seen that the force, $\mathbf{F}$, has a component parallel to the velocity due to the electric field, and a component orthogonal to the velocity due to the magnetic field. In this chapter, the particle motion in different, chosen field configurations will be studied.

2.1 Static and homogeneous fields

The assumption that the fields are static and homogeneous implies a huge simplification for the analysis of the equations of motion for the particles, since the fields do not change over time, and look the same all over space. We will now look at the different special cases for static and homogeneous fields.

- $\mathbf{E} = \mathbf{B} = 0$

This is the simplest case, implying that $\mathbf{v}(t) = \mathbf{v}(t=0) \equiv \mathbf{v}_0 = \text{const}$, and

$$\mathbf{r} = \mathbf{v}_0 t + \mathbf{r}_0. \tag{2.3}$$

In this case, both the momentum, $m\mathbf{v}_0$, and the energy, $mv_0^2/2$, are constants of motion.

- $\mathbf{B} = 0$, $\mathbf{E} \neq 0$

Eq. (2.2) reduces to

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{E}, \tag{2.4}$$

which can be integrated to give

$$\mathbf{v}(t) = \mathbf{v}_0 + \frac{q}{m} t\mathbf{E}, \tag{2.5}$$

i.e., the electric field gives rise to a particle acceleration along the vector $\mathbf{E}$. The particle energy, with the initial condition $\mathbf{v}_0 = 0$, is

$$W = \frac{(qEt)^2}{2m}. \tag{2.6}$$

- $\mathbf{E} = 0$, $\mathbf{B} \neq 0$

The equation of motion (2.2) takes the form

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}, \tag{2.7}$$

which, when multiplied with $\mathbf{v}$, gives

$$\frac{d}{dt}\left(\frac{mv^2}{2}\right) = q\mathbf{v} \cdot (\mathbf{v} \times \mathbf{B}) = 0, \tag{2.8}$$

i.e., the kinetic energy $mv^2/2$ is a constant of motion. This can also be understood from the fact that the magnetic force is always perpendicular to the velocity.

From Eq. (2.7) we see that the component of the velocity parallel to the magnetic field, denoted by $\mathbf{v}_\parallel$, satisfies $d\mathbf{v}_\parallel/dt = 0$. Thus, both the velocity $\mathbf{v}_\parallel$ and the kinetic energy $mv_\parallel^2/2$ are constants of motion. This implies that Eq. (2.7) can be written

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}\mathbf{v}_\perp \times \mathbf{B}, \tag{2.9}$$

where $\mathbf{v}_\perp$ is orthogonal to $\mathbf{B}$. We define our coordinates such that $\mathbf{B} = B\hat{z}$ and $\mathbf{v}_\perp = (v_x, v_y, 0)$, so that Eq. (2.9) has the components

$$\dot{v}_x = \frac{q}{m}Bv_y, \tag{2.10a}$$

$$\dot{v}_y = -\frac{q}{m}Bv_x, \tag{2.10b}$$

or

$$\ddot{v}_x + \left(\frac{qB}{m}\right)^2 v_x = 0, \tag{2.11a}$$

$$\ddot{v}_y + \left(\frac{qB}{m}\right)^2 v_y = 0. \tag{2.11b}$$

Eqs. (2.11) describe a harmonic oscillator, with oscillation frequency

$$\omega_c = \frac{|q|B}{m}, \tag{2.12}$$

called the *cyclotron frequency*. The solution to Eqs. (2.11) can be written

$$v_{x,y} = v_\perp \cos(\pm\omega_c t + \delta_{x,y}), \tag{2.13}$$

where $\pm$ depends on the sign of q, and $v_\perp$ and $\delta_{x,y}$ are constants determined by the initial conditions.

Eqs. (2.11) can be integrated one step further to yield

$$x - x_0 = \pm\frac{v_\perp}{\omega_c}\sin(\pm\omega_c t + \delta_x), \tag{2.14a}$$

$$y - y_0 = \pm\frac{v_\perp}{\omega_c}\sin(\pm\omega_c t + \delta_y). \tag{2.14b}$$

We define the *Larmor radius* as

$$r_L = \frac{v_\perp}{\omega_c}, \tag{2.15}$$

and choose $\delta_x = 0$, $\delta_y = \pi/2$. The solution (2.14) then becomes

$$x - x_0 = r_L \sin(\omega_c t), \tag{2.16a}$$

$$y - y_0 = \pm r_L \cos(\omega_c t). \tag{2.16b}$$

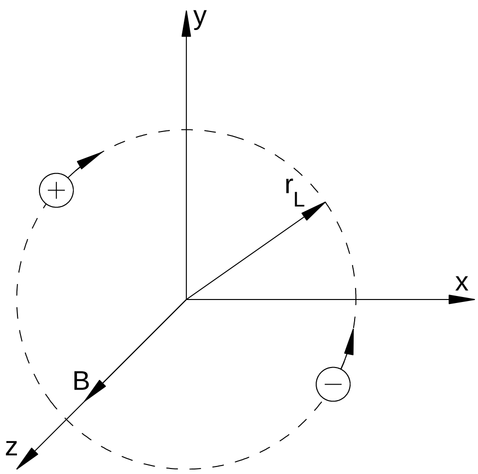


Figure 2.1. The direction of rotation during the cyclotron motion depends on the sign of the charge of the particle.

Eq. (2.16) describes a circular orbit about a *guiding centre*, positioned at (x_0, y_0), see Fig. 2.1.

A charged particle in circular motion around the magnetic field lines (*Larmor rotation*) will generate its own magnetic field. The rotational direction of the charged particle will always be such that the generated magnetic field will decrease the total magnetic field, i.e., the plasma has *diamagnetic properties*. Taking the time average of the magnetic field generated by the charged particle, we obtain the current within a circular orbit,

$$I = \frac{|q|\omega_c}{2\pi} = \frac{1}{2\pi}\frac{q^2 B}{m}. \tag{2.17}$$

The corresponding magnetic moment, μ, of the charged particle can be defined according to

$$\mu = I\pi r_L^2 = \frac{mv_\perp^2}{2B}, \tag{2.18}$$

or

$$\mu = \left|\frac{qr_L v_\perp}{2}\right|, \tag{2.19a}$$

and

$$\boldsymbol{\mu} = \frac{q\mathbf{r}_L \times \mathbf{v}_\perp}{2}, \tag{2.19b}$$

i.e., the magnetic moment is a vector in the opposite direction of the magnetic field.

• $\mathbf{B} \neq 0$ and a constant force $\mathbf{F}$

The equation of motion in this case reads

$$\frac{d\mathbf{v}}{dt} = \frac{q}{m}\mathbf{v} \times \mathbf{B} + \frac{\mathbf{F}}{m}. \tag{2.20}$$

In order to make the analysis of Eq. (2.20) transparent, we will, as in the previous case with only an external magnetic field, split the equation in one part parallel to, and one part orthogonal to the magnetic field,

$$\frac{d\mathbf{v}_\parallel}{dt} = \frac{\mathbf{F}_\parallel}{m}, \tag{2.21a}$$

and

$$\frac{d\mathbf{v}_\perp}{dt} = \frac{q}{m}\mathbf{v}_\perp \times \mathbf{B} + \frac{\mathbf{F}_\perp}{m}, \tag{2.21b}$$

respectively. Eq. (2.21a) describes a constant acceleration along the magnetic field lines, while $\mathbf{F}_\perp$ in Eq. (2.21b) gives rise to a constant drift velocity orthogonal to both $\mathbf{F}_\perp$ and $\mathbf{B}$, see Fig. 2.2. To see this, Eq. (2.21b) is solved by introducing a constant drift velocity, $\mathbf{v}_D$, such that

$$\mathbf{v}_\perp = \mathbf{v}_D + \mathbf{u}. \tag{2.22}$$

By inserting this in Eq. (2.21b) we obtain

$$\frac{d\mathbf{u}}{dt} = \frac{q}{m}\mathbf{v}_D \times \mathbf{B} + \frac{q}{m}\mathbf{u} \times \mathbf{B} + \frac{\mathbf{F}_\perp}{m}. \tag{2.23}$$

Since we have already studied the equation

$$\frac{d\mathbf{u}}{dt} = \frac{q}{m}\mathbf{u} \times \mathbf{B}, \tag{2.24}$$

we know that $\mathbf{u}$ describes the Larmor motion, and to solve Eq. (2.23) we choose

$$\mathbf{v}_D \times \mathbf{B} = -\frac{\mathbf{F}_\perp}{q} \quad \Leftrightarrow \quad \mathbf{v}_D = \frac{1}{q}\frac{\mathbf{F}_\perp \times \mathbf{B}}{B^2}. \tag{2.25}$$

Thus, Eqs. (2.24) and (2.25) show that the motion consists of a rotation around the guiding centre and a drift of the guiding centre, orthogonal to both the magnetic field and the applied force.

As a specific example of a constant force, we look at

$$\mathbf{F} = q\mathbf{E}, \tag{2.26}$$

which, using Eq. (2.25), gives the drift velocity

$$\mathbf{v}_{\mathbf{E}\times\mathbf{B}} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}, \tag{2.27}$$

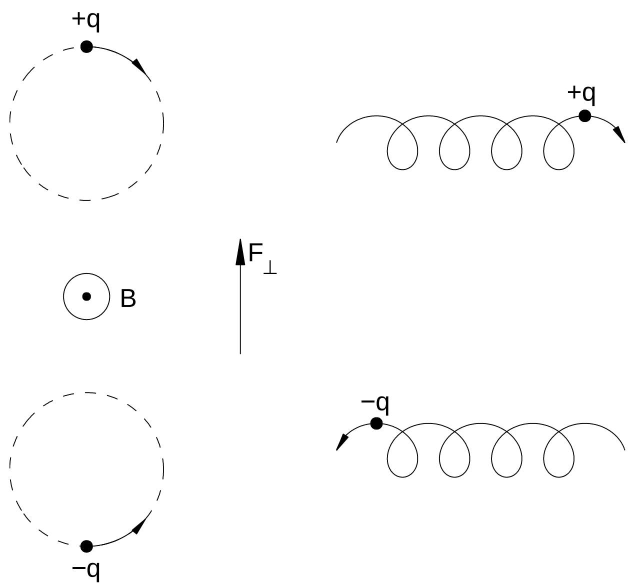


Figure 2.2. Motion in the case of a magnetic field and a constant force field. Note that the particles drift in the same direction if the force is $\mathbf{F} = q\mathbf{E}$.

representing an electric drift. Since $\mathbf{v}_{\mathbf{E}\times\mathbf{B}}$ is independent of the mass and the charge, the so-called $\mathbf{E} \times \mathbf{B}$ drift is the same for all charged particles.

For a gravitational force contribution

$$\mathbf{F} = m\boldsymbol{g}, \tag{2.28}$$

we obtain

$$\mathbf{v}_G = \frac{m}{q}\frac{\boldsymbol{g} \times \mathbf{B}}{B^2}, \tag{2.29}$$

which shows that the gravitational drift is different for electrons and ions, since it depends on m/q.

2.2 Inhomogeneous magnetic fields—the grad B drift

Whenever the magnetic field is homogeneous, we can give an exact description of the corresponding particle motion. As we shall see, the problem becomes more difficult when $\mathbf{B}$ has a spatial dependence. The drift introduced by the inhomogeneous magnetic field is termed grad B drift.

2.2.1 Derivation of the grad B drift

Let us investigate the case when the magnetic field is weakly varying in space. “Weakly” means that the characteristic length scale, L, for the inhomogeneity is much greater than the Larmor radius for the particle. Thus

$$r_L \ll L. \tag{2.30}$$

We may now Taylor expand the magnetic field around the guiding centre point r_0 (keeping only the first two terms), i.e.,

$$\mathbf{B}(\mathbf{r}) \approx \mathbf{B}(\mathbf{r}_0) + [(\mathbf{r}-\mathbf{r}_0)\cdot\nabla]\mathbf{B}(\mathbf{r})\big|_{\mathbf{r}=\mathbf{r}_0} \equiv \mathbf{B}_0 + \mathbf{B}_1, \tag{2.31}$$

From this we see that

$$\frac{|\mathbf{B}_1|}{|\mathbf{B}_0|} = \frac{|[(\mathbf{r}-\mathbf{r}_0)\cdot\nabla]\mathbf{B}(\mathbf{r}_0)|}{|\mathbf{B}(\mathbf{r}_0)|} \sim \frac{r_L}{L} \ll 1, \tag{2.32}$$

where $\nabla \sim \hat{r}/L$ and $|\mathbf{r}-\mathbf{r}_0| \sim r_L$.

We will try to solve the equations of motion using perturbative methods, by expanding the velocity according to

$$\mathbf{v} = \mathbf{v}_0 + \mathbf{v}_1 + \cdots, \tag{2.33}$$

where $|\mathbf{v}_1|/|\mathbf{v}_0| \sim r_L/L \ll 1$. Expanding the equations of motion (2.7) to first order (in r_L/L), we obtain

$$m\frac{d}{dt}(\mathbf{v}_0+\mathbf{v}_1) = q(\mathbf{v}_0\times\mathbf{B}_0 + \mathbf{v}_1\times\mathbf{B}_0 + \mathbf{v}_0\times\mathbf{B}_1). \tag{2.34}$$

To zeroth order, the equation of motion is

$$m\frac{d\mathbf{v}_0}{dt} = q\mathbf{v}_0\times\mathbf{B}_0, \tag{2.35}$$

i.e., $\mathbf{v}_0$ represents the Larmor motion, while $\mathbf{v}_1$ represents the deviation caused by the inhomogeneity. Assuming Eq. (2.35) holds, the first order equation becomes

$$m\frac{d\mathbf{v}_1}{dt} = q\mathbf{v}_1\times\mathbf{B}_0 + \mathbf{F}, \tag{2.36}$$

where

$$\mathbf{F} \equiv q\mathbf{v}_0\times\mathbf{B}_1 = q\mathbf{v}_0\times[(\mathbf{r}-\mathbf{r}_0)\cdot\nabla]\mathbf{B}(\mathbf{r}_0). \tag{2.37}$$

The reason for this notation is that the force, $\mathbf{F}$, is independent of $\mathbf{v}_1$, and in this sense represents an external force.

We will now study the force in more detail. Since $\mathbf{r}-\mathbf{r}_0 = R\hat{r}$ and $\mathbf{v}_0 = -v_\perp\hat{\theta}$ (see Fig. 2.3), we obtain from Eq. (2.37)

$$\mathbf{F} = -qv_\perp R\hat{\theta}\times\frac{\partial\mathbf{B}}{\partial r} = -qv_\perp R\left(\frac{\partial B_z}{\partial r}\hat{r} - \frac{\partial B_r}{\partial r}\hat{z}\right) \equiv \mathbf{F}_\parallel + \mathbf{F}_\perp, \tag{2.38}$$

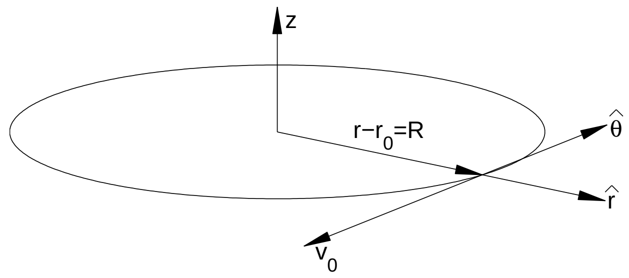


Figure 2.3. Geometry of the position and velocity vector.

where

$$\mathbf{F}_{\parallel} = q v_{\perp} R \frac{\partial B_r}{\partial r} \hat{z}, \tag{2.39a}$$

and

$$\mathbf{F}_{\perp} = -q v_{\perp} R \frac{\partial B_z}{\partial r} \hat{r}. \tag{2.39b}$$

Here, $R = |\mathbf{r} - \mathbf{r}_0| \approx r_L = v_{\perp}/\omega_c$ by definition (2.15), and using Eqs. (2.12) and (2.18) we obtain

$$|q| v_{\perp} R = |q| \frac{v_{\perp}^2}{\omega_c} = \frac{m v_{\perp}^2}{B} = 2\mu. \tag{2.40}$$

Since $\mathbf{F}$ is a time-varying force field, we will take the time average of the force over one period of rotation, denoting it by $\langle \mathbf{F} \rangle$. Expressing $\nabla \cdot \mathbf{B} = 0$ in cylindrical coordinates, we have

$$\frac{\partial B_r}{\partial r} + \frac{B_r}{r} + \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} + \frac{\partial B_z}{\partial z} = 0. \tag{2.41}$$

But

$$\left\langle \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} \right\rangle = \frac{1}{2\pi r} \oint \frac{1}{r}\frac{\partial B_\theta}{\partial \theta} r \, d\theta = 0, \tag{2.42}$$

and

$$\lim_{r \to 0} \frac{B_r}{r} = \frac{\partial B_r}{\partial r}, \tag{2.43}$$

since $B_r(r = 0) = 0$. Thus, the time average of Eq. (2.41) gives

$$2 \left\langle \frac{\partial B_r}{\partial r} \right\rangle = -\left\langle \frac{\partial B_z}{\partial z} \right\rangle = -\frac{\partial B_z}{\partial z}, \tag{2.44}$$

so that, using Eqs. (2.39a) and (2.40),

$$\langle \mathbf{F}_{\parallel} \rangle = -\mu \frac{\partial B_z}{\partial z} \hat{z} = -\frac{\mu}{B} [(\mathbf{B} \cdot \nabla)\mathbf{B}]_{\parallel}. \tag{2.45}$$

Using the vector relation

$$(\nabla \times \mathbf{B}) \times \mathbf{B} = (\mathbf{B} \cdot \nabla)\mathbf{B} - \nabla \left(\frac{B^2}{2} \right), \tag{2.46}$$

and observing that $[(\nabla \times \mathbf{B}) \times \mathbf{B}]_{\|} = 0$ (since the expression in the square bracket is orthogonal to $\mathbf{B}$), we obtain

$$[(\mathbf{B} \cdot \nabla)\mathbf{B}]_{\|} = \nabla_{\|} \left(\frac{B^2}{2} \right) = B\nabla_{\|} B, \tag{2.47}$$

and Eq. (2.45) becomes

$$\langle \mathbf{F}_{\|} \rangle = -\mu \nabla_{\|} B. \tag{2.48}$$

For the orthogonal part of the force, $\mathbf{F}_{\perp}$, we have

$$\mathbf{F}_{\perp} = -2\mu \frac{\partial B_z}{\partial r} \hat{r}. \tag{2.49}$$

Since

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y}, \tag{2.50a}$$

and

$$x = r\cos\theta, \quad y = r\sin\theta, \tag{2.50b}$$

the partial derivative can be written

$$\frac{\partial}{\partial r} = \frac{\partial x}{\partial r}\frac{\partial}{\partial x} + \frac{\partial y}{\partial r}\frac{\partial}{\partial y} = \cos\theta \frac{\partial}{\partial x} + \sin\theta \frac{\partial}{\partial y}, \tag{2.51}$$

and the average of the perpendicular force becomes

$$\begin{aligned} \langle \mathbf{F}_{\perp} \rangle &= -2\mu \left\langle \left(\cos\theta \frac{\partial B_z}{\partial x} + \sin\theta \frac{\partial B_z}{\partial y} \right) (\cos\theta\,\hat{x} + \sin\theta\,\hat{y}) \right\rangle \\ &= -\mu \left(\frac{\partial B_z}{\partial x}\hat{x} + \frac{\partial B_z}{\partial y}\hat{y} \right), \end{aligned} \tag{2.52}$$

where we have made use of the relations

$$\langle \sin^2\theta \rangle = \langle \cos^2\theta \rangle = \frac{1}{2}, \quad \text{and} \quad \langle \sin\theta\cos\theta \rangle = 0. \tag{2.53}$$

Eq. (2.52) is of the form

$$\langle \mathbf{F}_{\perp} \rangle = -\mu \nabla_{\perp} B, \tag{2.54}$$

and by combining Eqs. (2.48) and (2.54), the total average force is obtained as

$$\langle \mathbf{F} \rangle = -\mu \nabla B. \tag{2.55}$$

Thus, we see that motion parallel to the magnetic field is determined by the equation of motion

$$m\frac{d\mathbf{v}_{\|}}{dt} = -\mu \nabla_{\|} B, \tag{2.56}$$

and for motion perpendicular to the magnetic field, we obtain the grad B drift velocity

$$\mathbf{v}_{\nabla B} = \frac{\langle \mathbf{F} \times \mathbf{B} \rangle}{qB^2} = \frac{\mu}{q} \frac{\mathbf{B} \times \nabla B}{B^2}. \tag{2.57}$$

The velocity (2.57) is perpendicular to both $\mathbf{B}$ and ∇B, see Fig. 2.4.

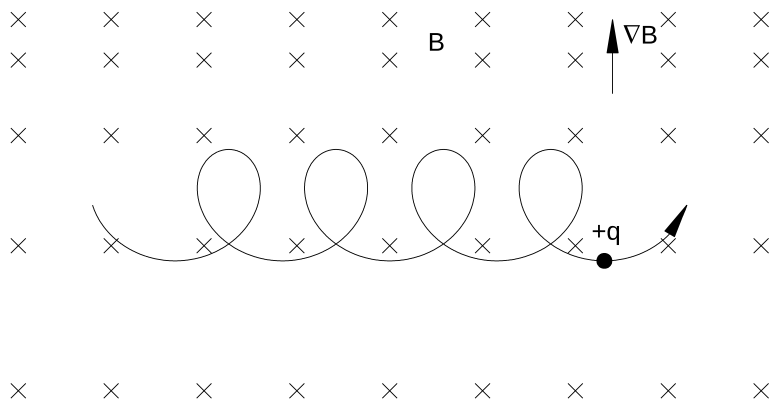


Figure 2.4. The grad B drift.

Note that contrary to the $\mathbf{E} \times \mathbf{B}$ drift, the grad B drift is charge dependent. Positively and negatively charged particles will move in opposite directions, thus causing a current to flow.

2.2.2 Heuristic derivation of the $\nabla_\perp B$ drift

In order to get a deeper physical understanding, we will make a qualitative discussion of the grad B drift. If $B = B_0 =$ const, then $\nabla_\perp B = 0$, and a negatively charged particle will rotate according to Fig. 2.5. We assume the variation of the $\mathbf{B}$-field, as function of the particle position in its orbit, to be described by Fig. 2.6, and we approximate the variations in the $\mathbf{B}$-field by a step function. Particles with negative charge, $q < 0$, that start their Larmor motion in position 1 will experience the magnetic field $B = B_+$, until it reaches position 3. The field then changes instantaneously to $B = B_-$, and the Larmor radius at position 3 satisfies

$$r_{L3} = \frac{mv_\perp}{|q|B_-} > r_{L1} = r_{L2} = \frac{mv_\perp}{|q|B_+}. \tag{2.58}$$

From position 3 to position 5 the Larmor radius is r_{L3}, and again changes instantaneously to r_{L1} as position 5 is reached. During that period of time, the particle has drifted in a direction perpendicular to $\mathbf{B}$ and $\nabla_\perp B$, see Fig. 2.7.

The direction of the drift velocity is such that $\mathbf{v}_{\nabla B} \parallel \nabla B \times \mathbf{B}$ if $q < 0$, and $\mathbf{v}_{\nabla B} \parallel \mathbf{B} \times \nabla B$ if $q > 0$, which is in agreement with Eq. (2.57). By

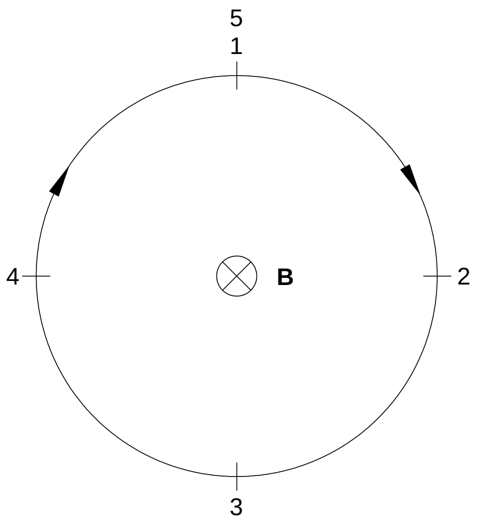


Figure 2.5. Larmor rotation of a negatively charged particle.

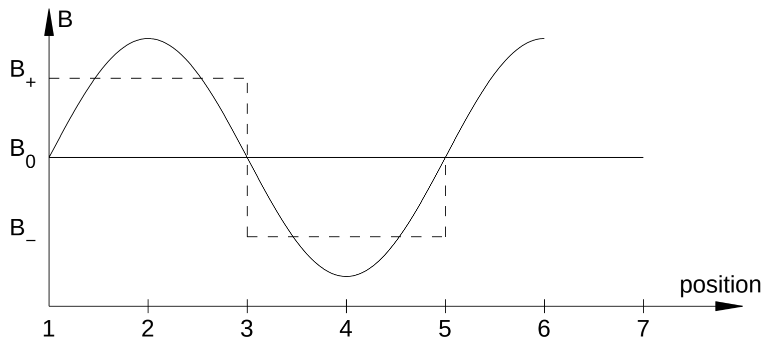


Figure 2.6. Assumed variation of the **B**-field (solid line), and a step function approximation (dashed line), as a function of the particle position (see Fig. 2.5).

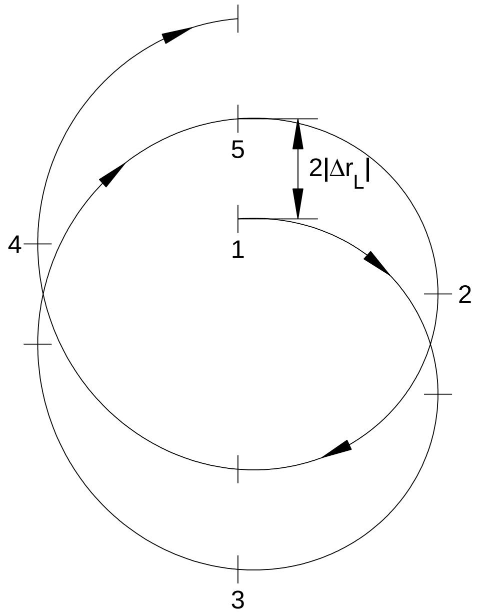


Figure 2.7. Drift of the particle between positions 1 and 5.

introducing the period of the Larmor motion, $T_c \equiv 2\pi/\omega_c$, and the drift distance per Larmor rotation, $2|\Delta r_L|$, see Fig. 2.7, the magnitude of the drift velocity can be estimated according to

$$v_{\nabla B} \approx \frac{2|\Delta r_L|}{T_c}. \tag{2.59}$$

The drift distance due to the variations in the **B**-field is estimated as

$$2|\Delta r_L| \approx 2\frac{mv_\perp}{qB^2}\Delta B \approx 2\frac{mv_\perp}{qB^2}B' \times 2r_L. \tag{2.60}$$

Thus, from Eq. (2.59) we obtain

$$v_{\nabla B} \approx \frac{2v_\perp r_L}{\pi}\frac{B'}{B} = \frac{4\mu}{\pi q}\frac{B'}{B}, \tag{2.61}$$

where also Eq. (2.19a) has been used. This is in excellent agreement with Eq. (2.57), considering the degree of approximation.

2.3 Curved magnetic fields—the curvature drift

In this section we will study the motion of charged particles in curved magnetic fields. The curvature of the magnetic field is characterised by the constant radius of curvature, R_c. A curved magnetic field will not satisfy

Maxwell's equation if $|B|$ is constant. This implies that a curved magnetic field *has* to be inhomogeneous, and a charged particle will thus always experience, apart from the curvature effect, a grad B drift (see section 2.2).

The centrifugal force acting on a particle moving along a curved magnetic field line is

$$\mathbf{F}_c = \frac{mv_\parallel^2}{R_c}\hat{r} = mv_\parallel^2 \frac{\mathbf{R}_c}{R_c^2}. \tag{2.62}$$

This force will give rise to a drift velocity

$$\mathbf{v}_c = \frac{\mathbf{F}_c \times \mathbf{B}}{qB^2} = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{R}_c \times \mathbf{B}}{R_c^2}, \tag{2.63}$$

called the *curvature drift*. In vacuum, $\nabla \times \mathbf{B} = 0$ holds, which in cylindrical coordinates (see Fig. 2.8) becomes

$$\frac{1}{r}\frac{d}{dr}(rB) = 0, \tag{2.64}$$

i.e.,

$$|B| \propto \frac{1}{r}, \quad \text{or} \quad |B| \propto \frac{1}{R_c}, \tag{2.65}$$

(which follows from the assumption that motion is along a field line with radius R_c) and therefore

$$\frac{\nabla|B|}{B} = -\frac{\mathbf{R}_c}{R_c^2}. \tag{2.66}$$

Eq. (2.66) shows that the curvature of the magnetic field always induces an inhomogeneity, i.e., a grad B drift. The curvature drift velocity can now be written, using Eqs. (2.63) and (2.66), as

$$\mathbf{v}_\parallel = \frac{mv_\parallel^2}{qB^2}\frac{\mathbf{B} \times \nabla B}{B}, \tag{2.67}$$

and the total drift velocity becomes

$$\mathbf{v}_{\nabla B} + \mathbf{v}_c = \frac{m}{qB^3}\left(\frac{1}{2}v_\perp^2 + v_\parallel^2\right)(\mathbf{B} \times \nabla B), \tag{2.68a}$$

or

$$\mathbf{v}_{\nabla B} + \mathbf{v}_c = \frac{m}{qB^2R_c^2}\left(\frac{1}{2}v_\perp^2 + v_\parallel^2\right)(\mathbf{R}_c \times \mathbf{B}). \tag{2.68b}$$

A natural illustration of a curved magnetic field is the Earth's magnetic field, as described in the following example.

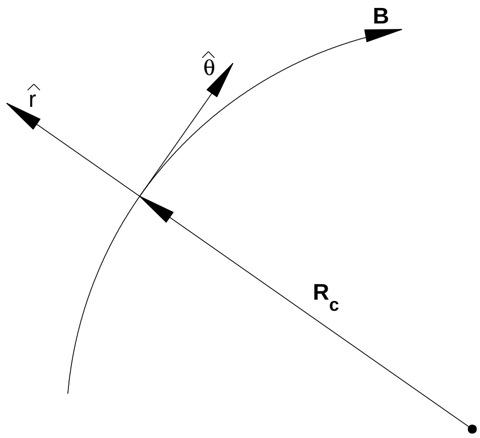


Figure 2.8. Cylindrical coordinates.

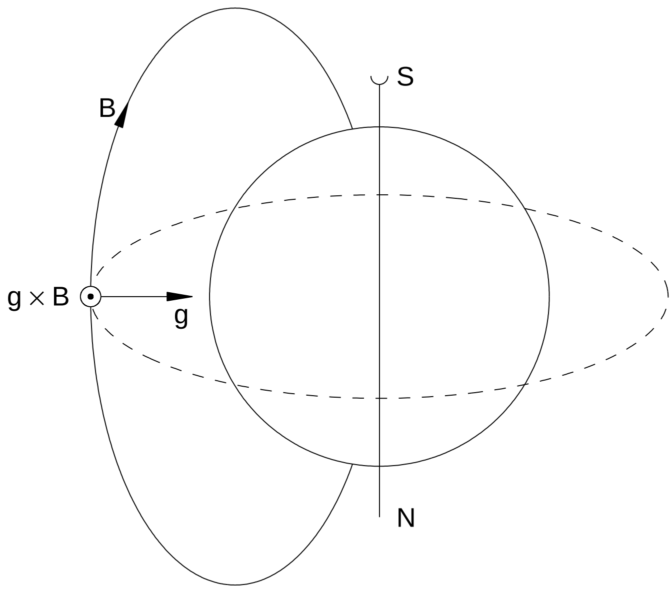


Figure 2.9. The Earth's magnetic field.

The electron drift in the Earth's magnetic field

Electrons in the outer Van Allen belt (altitude $\sim 3 \times 10^4$ km) are observed to experience an azimuthal drift, with an orbital time of approximately 1 h.

(i) Could the gravity drift $\mathbf{v}_g = m\mathbf{g} \times \mathbf{B}/(qB^2)$ be responsible for this drift (see Fig. 2.9)?

In order to make the calculations simpler, we look at what happens in the equatorial plane, and approximate the magnetic field with the field of a dipole

$$B \approx B_0 \left(\frac{R_e}{r}\right)^3, \tag{2.69}$$

where $R_e \approx 6000$ km is the Earth radius and $B_0 \approx 5 \times 10^{-5}$ T is the surface magnetic field. Furthermore,

$$g = g_0 \left(\frac{R_e}{r}\right)^2, \tag{2.70}$$

where $g_0 = 9.81$ m/s^2. The orbital period is given by

$$T \sim \frac{2\pi r}{v_g} = \frac{2\pi r q B}{mg}, \tag{2.71a}$$

i.e.,

$$T \sim \frac{2\pi q B_0 R_e}{m g_0} \approx 10^{14} \text{ s} \approx 10^7 \text{ yr}. \tag{2.71b}$$

With this extreme orbital period, the gravitational drift can not be responsible for the observed electron drift.

(ii) Could the azimuthal drift be due to the grad B drift?

According to Fig. 2.9, the magnetic field vector can be written

$$\mathbf{B} = B(r)\hat{\theta} = B_0 \left(\frac{R_e}{r}\right)^3 \hat{\theta}, \tag{2.72}$$

which implies

$$\nabla B = -\frac{3B}{r}\hat{r}, \tag{2.73}$$

and

$$\frac{\mathbf{B} \times \nabla B}{B^2} = \frac{3}{r}\hat{\varphi}, \tag{2.74}$$

where $\hat{\varphi}$ is the unit vector along the angular direction in the equatorial plane. According to Eq. (2.57), the grad B drift becomes

$$\mathbf{v}_{\nabla B} = \pm \frac{v_\perp r_L}{2} \frac{\mathbf{B} \times \nabla B}{B^2} = \pm \frac{3}{2} \frac{m v_\perp^2}{|q| B r} \hat{\varphi} = \frac{3}{q} \frac{W_\perp r^2}{B_0 R_e^3} \hat{\varphi}, \tag{2.75}$$

implying eastern drift if $q > 0$, and western drift if $q < 0$. Here $W_\perp = mv_\perp^2/2$. If we assume $r = 5R_e$ and

$$W_\perp = \begin{cases} 1 \text{ eV}, & \text{for a proton,} \\ 60 \text{ keV}, & \text{for an electron,} \end{cases} \tag{2.76}$$

we obtain, using Eq. (2.75),

$$v_{\nabla B,e} \sim 1.5 \times 10^4 \text{ m/s}, \tag{2.77}$$

$$v_{\nabla B,p} \sim 0.5 \text{ m/s}. \tag{2.78}$$

The orbital period thus becomes

$$T \sim \frac{2\pi r}{v_{\nabla B}} = \frac{2\pi r}{3W_\perp r^2}|q|B_0R_e^3, \tag{2.79}$$

which gives

$$T_e \approx 10^4 \text{ s} \approx 3 \text{ h}, \tag{2.80}$$

i.e., an orbital period comparable with the observed period.

If we include the curvature drift, i.e., according to Eq. (2.63)

$$v_c = \frac{mv_\parallel^2}{qBR_c} = \frac{2W_\parallel r^2}{qB_0R_e^3}\frac{r}{R_c}, \tag{2.81}$$

where $W_\parallel \equiv mv_\parallel^2/2$. The last factor (r/R_c) may be set to 1, since we move along a magnetic field line with radius of curvature R_c. We then obtain as the total drift velocity

$$v_{\text{tot}} = v_{\nabla B} + v_c = \frac{(3W_\perp + 2W_\parallel)}{qB_0R_e^3}r^2 = v_{\nabla B}\left(1 + \frac{2W_\parallel}{3W_\perp}\right). \tag{2.82}$$

Assuming thermodynamic equilibrium, we know that $W_\parallel = W_{\text{tot}}/3$ and $W_\perp = 2W_{\text{tot}}/3$, i.e.,

$$v_{\text{tot}} = \frac{4v_{\nabla B}}{3}, \tag{2.83}$$

implying that our earlier result should be divided by 4/3 in order to include the curvature drift.

2.4 Magnetic mirrors—the effect of $\nabla_\parallel B$

In the preceding section, it was shown that inhomogeneities in the magnetic fields causes drifts in the motion of charged particles. This effect can be used as a means for trapping charged particles in a device known as *magnetic mirror*. We will investigate the properties of such devices in the following section. However, we will first prove a fundamental property of the magnetic moment.

2.4.1 The invariance of μ

We have seen that $\nabla_\parallel B$ gives rise to a particle acceleration along the magnetic field lines according to (see Eq. (2.56))

$$m\frac{d\mathbf{v}_\parallel}{dt} = -\mu\nabla_\parallel B. \tag{2.84}$$

Consider a magnetic field characterised by $B_\theta = 0$, $B_r \neq 0$ and $\partial\mathbf{B}/\partial\theta = 0$ (see Fig. 2.10). Let $d\mathbf{s}$ represent a small line element along the magnetic field $\mathbf{B}$. If we multiply Eq. (2.84) by $\mathbf{v}_\parallel$, we obtain

$$m\mathbf{v}_\parallel \cdot \frac{d\mathbf{v}_\parallel}{dt} = \frac{d}{dt}\left(\frac{mv_\parallel^2}{2}\right) = -\mu\frac{\partial B}{\partial \mathbf{s}} \cdot \frac{d\mathbf{s}}{dt}. \tag{2.85}$$

The conservation of energy for the particle is given by

$$\frac{d}{dt}\left(\frac{mv^2}{2}\right) = \frac{d}{dt}\left(\frac{mv_\parallel^2}{2} + \frac{mv_\perp^2}{2}\right) = \frac{d}{dt}\left(\frac{mv_\parallel^2}{2} + \mu B\right) = 0, \tag{2.86}$$

where we have used the definition (2.18) of the magnetic moment. Using Eqs. (2.85) and (2.86), we find that

$$-\mu\frac{dB}{dt} + \frac{d}{dt}(\mu B) = 0, \tag{2.87}$$

which implies that $d\mu/dt = 0$, i.e., μ is constant.

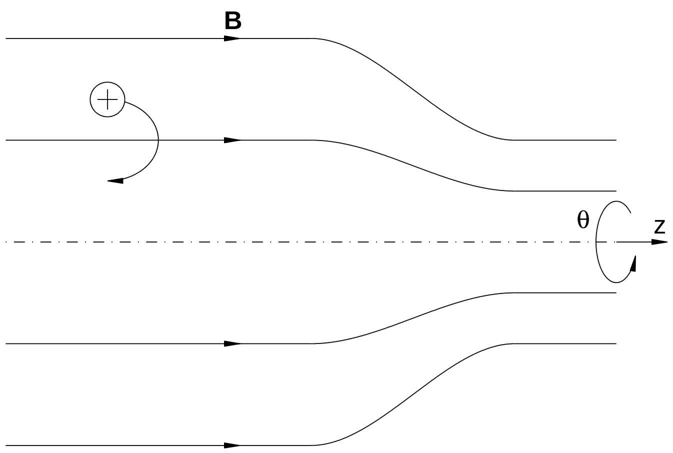


Figure 2.10. A particle in a magnetic field with cylindrical symmetry.

2.4.2 The conditions for plasma confinement in a magnetic mirror

When a particle moves from a region with a weak magnetic field to a region with stronger magnetic field, $W_\perp$ and therefore $v_\perp$ must increase, since $\mu = W_\perp/B$ is constant along the motion of the particle. Furthermore, the total kinetic energy $W = W_\| + W_\perp$ is conserved, and the parallel velocity component $v_\|$ will thus decrease. For large enough B, $v_\|$ may become zero and the particle will be reflected from the magnetic field, i.e., it reverses its motion along the magnetic field lines. Thus, the region with a strong field acts as a "magnetic mirror" for the particle.

The mirror effect is used as a means to confine plasmas in so-called mirror machines, see Fig. 2.11. When constructing the mirror field, it is of importance to know the turning point of the particles. There will always be particles with $\mu = 0$ ($v_\perp = 0$), which do not experience any force along the magnetic field lines, and thus will not be confined. The following question can therefore be posed: Which particles will be confined for the magnetic field configuration given in Fig. 2.11, where $B_m > B_0$?

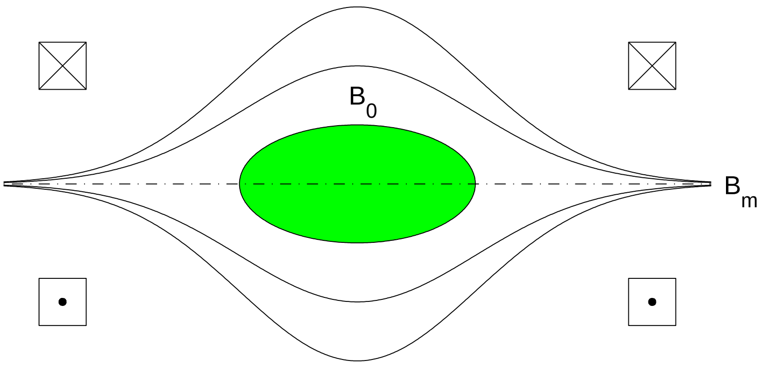


Figure 2.11. A plasma confined between magnetic mirrors.

Suppose that the particle has velocity components $v_\perp = v_{\perp 0}$ and $v_\| = v_{\|0}$ for $B = B_0$, and that the particle is reflected at $B = B_1$, the point where $v_\perp = v_{\perp 1}$ and $v_\| = 0$. The invariance of μ gives

$$\frac{mv_{\perp 0}^2}{2B_0} = \frac{mv_{\perp 1}^2}{2B_1}, \tag{2.88}$$

and energy conservation implies

$$v_{\perp 1}^2 = v_{\perp 0}^2 + v_{\|0}^2 = v_0^2. \tag{2.89}$$

Together with Eq. (2.88), this gives

$$\frac{B_0}{B_1} = \frac{v_{\perp 0}^2}{v_{\perp 1}^2} = \frac{v_{\perp 0}^2}{v_0^2} \equiv \sin^2 \Theta_0, \tag{2.90}$$

where $\Theta_0 = \arcsin(v_{\perp 0}/v_0)$ is the *pitch angle* for the particle orbit corresponding to B_0. It is seen that a small pitch angle corresponds to a large field at the reflection point, and if Θ_0 is too small, $B_1 > B_m$ and the particle is not reflected. Thus, the confinement criterion for a plasma particle is $\Theta_0 > \Theta_m$, where Θ_m is defined from

$$\sin^2 \Theta_m = \frac{B_0}{B_m} \equiv \frac{1}{R_m}, \tag{2.91}$$

and R_m is the *mirror ratio* for a magnetic mirror. This implies that plasma particles trapped between the mirrors have an anisotropic velocity distribution, since particles with pitch angles $\Theta_0 < \Theta_m$ are within the loss cone, see Fig. 2.12.

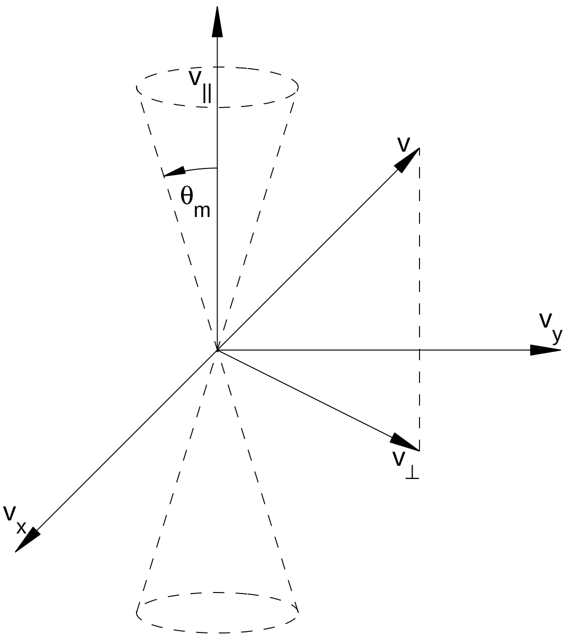


Figure 2.12. Particles within the loss cone are not confined by the magnetic mirror.

2.4.3 The number of particles in the loss cone for a magnetic mirror

In order to determine the particle losses in a magnetic mirror, we will assume that the distribution function for the plasma particles, $f(t, \mathbf{r}, \mathbf{v})$, is spatially homogeneous, time-independent, and isotropic in velocity space, i.e., depends on $v = |\mathbf{v}|$ only. We can therefore write $f(t, \mathbf{r}, \mathbf{v}) = f(v)$.

The total number of plasma particles is given by

$$N_{\text{tot}} = \int_{\mathbf{v}\text{-space}} f\, d^3v = 2\int_0^{2\pi} d\varphi \int_0^{\pi/2} d\theta \int_0^{\infty} f(v)v^2 \sin\theta\, dv, \tag{2.92}$$

where the factor 2 in (2.92) comes from integrating over the two halves of velocity space. In the same manner, the number of lost particles, N_f, can thus be written

$$N_f = \int_{\text{Loss cone}} f\, d^3v = 2\int_0^{2\pi} d\varphi \int_0^{\Theta_m} d\theta \int_0^{\infty} f(v)v^2 \sin\theta\, dv. \tag{2.93}$$

As for the full $\mathbf{v}$-space integration, the factor 2 in (2.93) is due to the integration over the two halves of the loss cone. We then obtain

$$\frac{N_f}{N_{\text{tot}}} = \frac{\int_0^{\Theta_m} \sin\theta\, d\theta}{\int_0^{\pi/2} \sin\theta\, d\theta} = 1 - \cos\Theta_m = 1 - \sqrt{1 - \frac{1}{R_m}}, \tag{2.94}$$

for the relative number of particles in the loss cone. When $R_m \gg 1$, Eq. (2.94) gives

$$\frac{N_f}{N_{\text{tot}}} \approx 1 - \left(1 - \frac{1}{2R_m}\right) = \frac{1}{2R_m} \ll 1. \tag{2.95}$$

Thus, in order to achieve small losses, the mirror ratio must be high. On the other hand, as will be shown below, the number of lost particles can be reduced even further by introducing an electric field.

2.4.4 Magnetic bottle with an electrostatic potential

Consider a magnetic bottle where we, apart from the varying magnetic field, have an electrostatic potential, $\phi(x)$, as in Fig. 2.13. The potential well set up by the electric field should improve the plasma confinement properties of the magnetic mirror, but how will the containment criterion be affected?

Using Eq. (2.56) in one dimension, and adding the force due to the electric field, the equation of motion reads

$$m\frac{dv_\parallel}{dt} = -\mu\frac{dB}{dx} + qE = -\mu\frac{dB}{dx} - q\frac{d\phi}{dx}. \tag{2.96}$$

When multiplying this equation by $v_\parallel = dx/dt$, we obtain

$$\frac{d}{dt}\left(\frac{mv_\parallel^2}{2} + \mu B + q\phi\right) = 0, \tag{2.97}$$

which we can write as

$$W_\parallel + W_\perp + q\phi = \text{const} \equiv W. \tag{2.98}$$

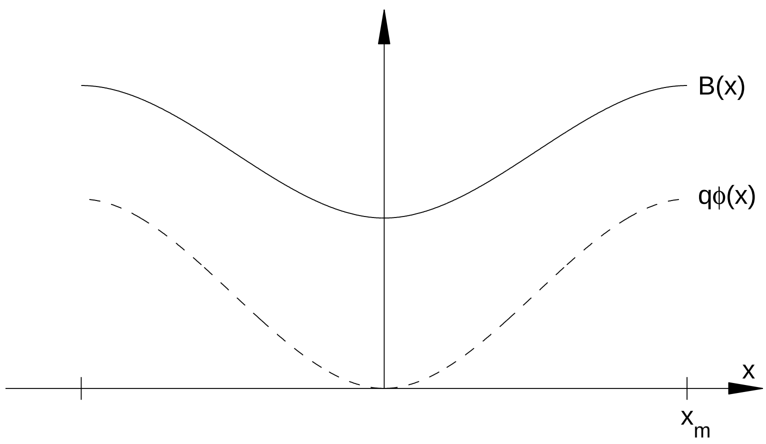


Figure 2.13. Magnetic field variation with an additional electrostatic potential.

Since W is constant, we have

$$W = W_{\|0} + W_{\perp 0} = W_{\perp m} + q\phi_m, \tag{2.99}$$

where index 0 denotes the centre of the bottle, and index m denotes the position where the particle reverses its motion. The magnetic moment, $\mu = W_\perp / B$, is still conserved, so that

$$\frac{W_{\perp 0}}{B_0} = \frac{W_{\perp m}}{B_m}. \tag{2.100}$$

Using this together with Eq. (2.99) gives

$$1 = \frac{W_{\perp m}}{W} + q\frac{\phi_m}{W} = \frac{B_m}{B_0}\frac{W_{\perp 0}}{W} + q\frac{\phi_m}{W} = R_m \sin^2\Theta_0 + q\frac{\phi_m}{W}, \tag{2.101}$$

where we have defined Θ_0 to be the pitch angle for the particle at $x = 0$. Thus, the modified containment criterion reads

$$\sin^2\Theta_0 > \frac{1}{R_m}\left(1 - q\frac{\phi_m}{W}\right). \tag{2.102}$$

For $q\phi_m > 0$ the loss cone is transformed into a loss hyperboloid, as in Fig. 2.14. For small parallel velocities, the particle does not have enough kinetic energy to escape the potential well, and the critical parallel energy, $W_{\|\mathrm{crit}}$, where the loss hyperboloid vanishes is given by $W_{\|\mathrm{crit}} = q\phi_m$.

How does the electrostatic potential modify the number of particles in the loss hyperboloid? Let us assume that the velocity distribution is isotropic and Maxwellian, i.e.,

$$f(v) = A\exp\left(-\frac{mv^2}{2k_BT}\right) \equiv A\exp\left(-\frac{v^2}{v_{th}^2}\right), \tag{2.103}$$

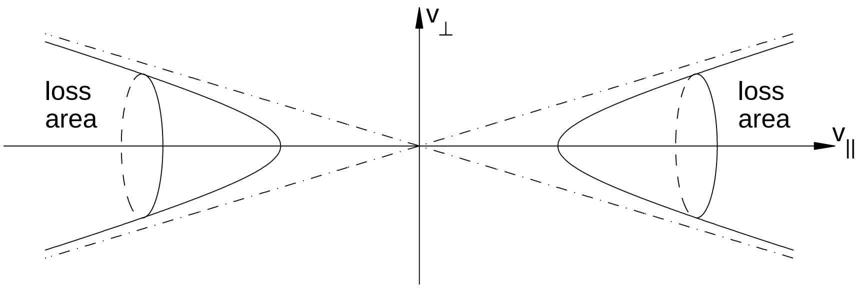


Figure 2.14. The loss hyperboloid for $q\phi_m > 0$.

where we have defined the *thermal velocity*, $v_{th} = \sqrt{2k_BT/m}$. The relative number of lost particles is then given by

$$\frac{N_f}{N_{\text{tot}}} = \frac{\int_{\text{Loss domain}} f(\mathbf{v})\, d^3v}{\int_{\mathbf{v}\text{-space}} f(\mathbf{v})\, d^3v} = \frac{\int_{v_{\text{crit}}}^{\infty} v^2 e^{-v^2/v_{th}^2} \int_0^{\Theta_m(v)} \sin\theta\, d\theta\, dv}{\int_0^{\infty} v^2 e^{-v^2/v_{th}^2}\, dv}, \tag{2.104}$$

where

$$\sin^2\Theta_m(v) = \frac{1}{R_m}\left(1 - \frac{q\phi_m}{mv^2/2}\right), \tag{2.105}$$

and

$$\frac{mv_{\text{crit}}^2}{2} = q\phi_m. \tag{2.106}$$

The angular integration is straightforward, and gives the result

$$\int_0^{\Theta_m(v)} \sin\theta\, d\theta = 1 - \cos\Theta_m(v) = 1 - \sqrt{1 - \frac{1}{R_m}\left(1 - \frac{q\phi_m}{mv^2/2}\right)}. \tag{2.107}$$

In order to proceed by analytical means, we assume that $R_m \gg 1$, which gives

$$\int_0^{\Theta_m(v)} \sin\theta\, d\theta \approx \frac{1}{2R_m}\left(1 - \frac{q\phi_m}{mv^2/2}\right). \tag{2.108}$$

By defining

$$\xi \equiv v/v_{th}, \tag{2.109}$$

$$\gamma^2 \equiv \frac{q\phi_m}{mv_{th}^2/2} = \frac{v_{\text{crit}}^2}{v_{th}^2}, \tag{2.110}$$

we obtain

$$\begin{aligned}\frac{N_f}{N_{\text{tot}}} &\approx \frac{1}{2R_m}\frac{\int_\gamma^\infty \xi^2 e^{-\xi^2}\left(1 - \frac{\gamma^2}{\xi^2}\right)\, d\xi}{\int_0^\infty \xi^2 e^{-\xi^2}\, d\xi} \\ &= \frac{1}{2R_m}\left[\frac{2\gamma}{\sqrt{\pi}}e^{-\gamma^2} + (1 - 2\gamma^2)\operatorname{erfc}\gamma\right] \equiv \frac{1}{2R_m}H(\gamma). \end{aligned} \tag{2.111}$$

Asymptotically,

$$H(\gamma) = \frac{2\gamma}{\sqrt{\pi}} e^{-\gamma^2} + (1 - 2\gamma^2)\,\mathrm{erfc}\,\gamma = \begin{cases} 1 - 2\gamma^2 & \text{when } \gamma \to 0, \\ \dfrac{2}{\gamma\sqrt{\pi}} e^{-\gamma^2} & \text{when } \gamma \to \infty. \end{cases} \tag{2.112}$$

The behaviour of $H(\gamma)$ can be seen in Fig. 2.15. From this we can conclude that the electrostatic potential gives great improvement to the confinement when $\gamma \gtrsim 1$, i.e., for

$$q\phi_m \gtrsim \frac{mv_{th}^2}{2} = k_B T. \tag{2.113}$$

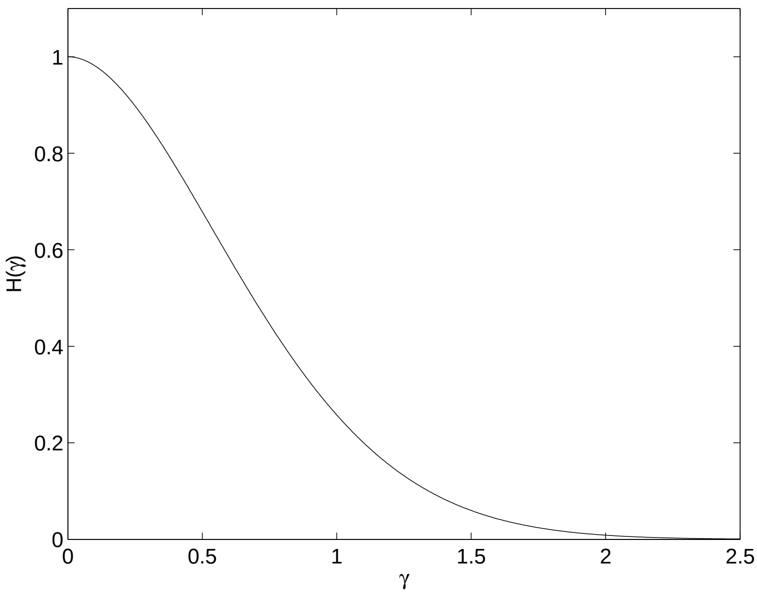


Figure 2.15. The function $H(\gamma) = 2\gamma e^{-\gamma^2}/\sqrt{\pi} + (1 - 2\gamma^2)\,\mathrm{erfc}\,\gamma$.

2.4.5 The mirror effect in the Earth's magnetic field

At large distances, the Earth's magnetic field is heavily distorted, as shown in Fig. 2.16. At distances not too far away from Earth on the other hand, the magnetic field is well described by the dipole field

$$\mathbf{B}(r, \theta) = -B_0(2\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) \left(\frac{R_e}{r}\right)^3, \tag{2.114}$$

where R_e is the Earth's radius and $B_0 \approx 3 \times 10^{-5}$ T is the strength of the magnetic field at the surface of the Earth at the equator (see Fig. 2.17).

Because charged particles are forced to move along the magnetic field lines, they will penetrate the regions around the poles where the magnetic field is stronger. They may therefore be reflected, if the right conditions are met.

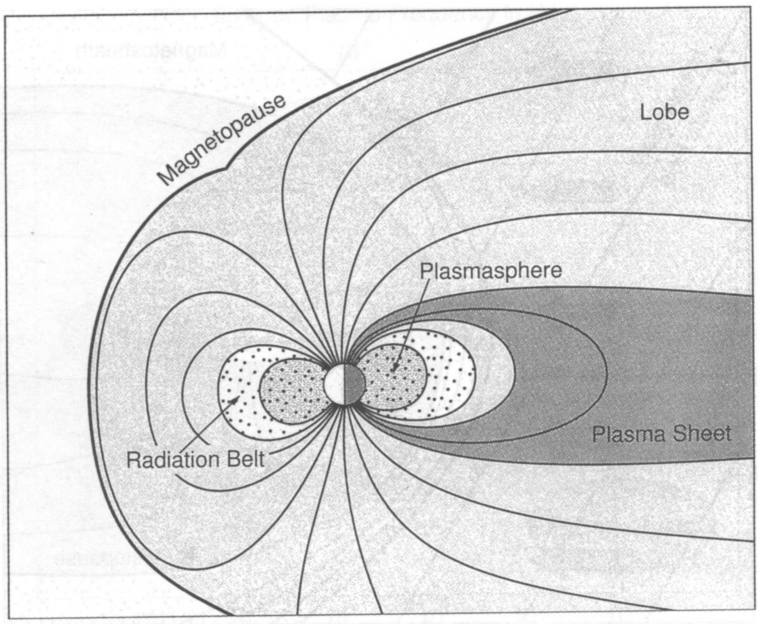


Figure 2.16. The Earth's magnetic field, distorted by the solar wind. [Picture taken from *Basic Space Plasma Physics*, W. Baumjohan and R. A. Treumann (Imperial College Press, London, 1996).]

Suppose that a particle at a height r_0 in the equatorial plane has a pitch angle Θ_e. At what height will the particle be reflected? As the particles move along a given field line, there will be a relation between r and θ along the particle orbit. This relation can be obtained as follows. From Fig. 2.18 we find that

$$\frac{r\,d\theta}{dr} = \frac{B_\theta}{B_r} = \frac{\sin\theta}{2\cos\theta}. \tag{2.115}$$

Integrating Eq. (2.115), we obtain

$$r = r_0 \sin^2\theta, \tag{2.116}$$

where, as before, r_0 is the particle position above the equator. Thus, the particle moves along a magnetic field line in such a way that its height, r, and polar angle, θ, are related by Eq. (2.116). Using Eq. (2.114), the

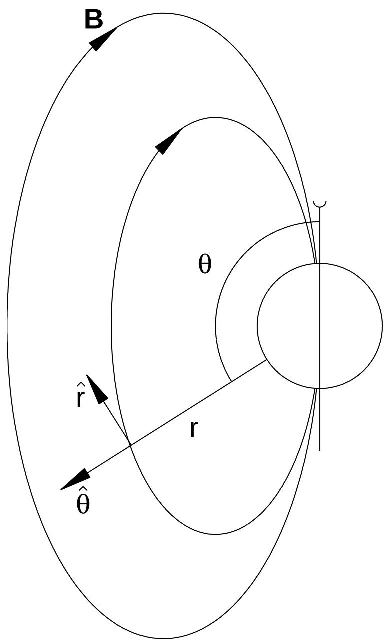


Figure 2.17. The Earth's magnetic field.

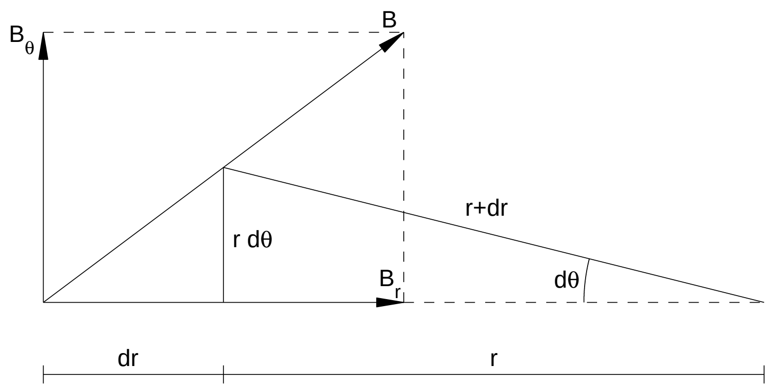


Figure 2.18. The geometry of the magnetic field outside Earth.

magnetic field strength along the particle orbit is given by

$$B(r) = B_0 \left.\sqrt{4\cos^2\theta + \sin^2\theta}\right|_{r=r_0\sin^2\theta} \left(\frac{R_e}{r}\right)^3$$
$$= B_0\sqrt{4 - 3\frac{r}{r_0}}\left(\frac{R_e}{r}\right)^3 . \qquad (2.117)$$

The invariance of μ gives

$$\mu = \frac{W_{\perp\mathrm{e}}}{B_\mathrm{e}} = \frac{W_{\perp m}}{B_m} = \frac{W_\mathrm{tot}}{B_m}, \qquad (2.118)$$

where index e denotes the equatorial plane, and index m the point of reflection. Thus

$$\frac{B_\mathrm{e}}{B_m} = \frac{W_{\perp\mathrm{e}}}{W_\mathrm{tot}} = \sin^2\Theta_\mathrm{e}. \qquad (2.119)$$

The height r_m at which the particle is reflected is therefore given by

$$\sin^2\Theta_\mathrm{e} = \frac{B(r=r_0)}{B(r=r_m)} = \frac{(R_e/r_0)^3}{(R_e/r_m)^3}\left(4 - 3\frac{r_m}{r_0}\right)^{-1/2}. \qquad (2.120)$$

Of special interest are the particles reaching the surface of the Earth. They must have a pitch angle $\Theta \leq \Theta_{cr}$, where the critical angle is determined from the condition $r_m = R_e$, i.e.,

$$\sin^2\Theta_{cr} = \left(\frac{R_e}{r_0}\right)^3\left(4 - 3\frac{R_e}{r_0}\right)^{-1/2}. \qquad (2.121)$$

For particles emanating from a region far from Earth, i.e., for which $r_0 \gg R_e$, Eq. (2.121) yields

$$\Theta_{cr} \approx \arcsin\left[\frac{1}{\sqrt{2}}\left(\frac{R_e}{r_0}\right)^{3/2}\right] \approx \frac{1}{\sqrt{2}}\left(\frac{R_e}{r_0}\right)^{3/2}. \qquad (2.122)$$

2.5 Particle motion in Tokamaks

We saw earlier in this chapter that a plasma could in principle be confined between two magnetic mirrors. In practice, however, there are problems with using a "magnetic bottle" of this type for plasma confinement. A much more efficient way to confine charged particles can be achieved by using a toroidal geometry instead of a cylindrical. This can be done by closing up the ends of the bottle, so that the particles move on circular trajectories instead of being reflected. It also follows from magnetohydrodynamic (MHD) theory, that the toroidal confinement is an equilibrium configuration for an MHD plasma. This type of doughnut-shaped confinement device is usually called a *Tokamak*, a word stemming from the contraction of the Russian designation тороидальная камера с магнитными катушками.[1]

[1] toroidal'naya kamera s magnitnymi katushkami—toroidal chamber with magnetic coils.

2.5.1 General properties of periodic magnetic mirrors

Before proceeding to the problem of particle motion in Tokamaks, we will analyse a simpler problem which still contains many of the features found in a Tokamak; motion of charged particles in a periodically varying magnetic field $\mathbf{B} = B(x)\hat{x}$, where $B(x) = B_0(1 - \alpha\cos(kx))$. The motion parallel to the magnetic field is given by

$$m\frac{dv_\parallel}{dt} = -\mu\frac{\partial B}{\partial x}. \tag{2.123}$$

If we multiply Eq. (2.123) by $v_\parallel = dx/dt$ and integrate the resulting equation, we obtain

$$\frac{mv_\parallel^2}{2} + \mu B = \text{const} = W, \tag{2.124}$$

i.e., energy is conserved. Solving Eq. (2.124) for $v_\parallel$ gives

$$v_\parallel = \pm\sqrt{\frac{2}{m}\left[W - \mu B_0(1 - \alpha\cos(kx))\right]}. \tag{2.125}$$

We can express the velocity in terms of the average energies, using $W - \mu B_0 = \overline{W}_\parallel$ and $\mu B_0 = \overline{W}_\perp$, where the bar denotes the average of the corresponding quantity. Eq. (2.125) becomes

$$v_\parallel = \pm\sqrt{\frac{2}{m}\left[\overline{W}_\parallel + \alpha\overline{W}_\perp\cos(kx)\right]} \equiv \pm\bar{v}_\parallel\sqrt{1 + \beta\cos(kx)}, \tag{2.126}$$

where

$$\bar{v}_\parallel^2 \equiv 2\overline{W}_\parallel/m, \tag{2.127}$$

$$\beta \equiv \alpha\overline{W}_\perp/\overline{W}_\parallel = \alpha\tan^2\Theta, \tag{2.128}$$

and Θ is the pitch angle when $B(x) = B_0 = \bar{B}$.

We will now draw a *phase portrait* for the particle motion, i.e., we plot the curve $v_\parallel = v_\parallel(x)$. The character of the curve depends on the size of β.

(i) $|\beta| < 1$. From Fig. 2.19, we see that particle motion is possible for all values of x. Particles with this property are called passing.

(ii) $|\beta| = 1$. In this case the velocity becomes

$$v_\parallel = \pm\sqrt{2}\bar{v}_\parallel\left|\cos(kx/2)\right|. \tag{2.129}$$

The particle motion is now modified by the magnetic field, and we have $v_\parallel = 0$ for $x = n\pi/k$, $n = \pm1, \pm3, \ldots$, see Fig. 2.20.

(iii) $|\beta| > 1$. According to Fig. 2.21, particle motion is possible only in certain intervals of x, i.e., the magnetic field modifies the orbits of the

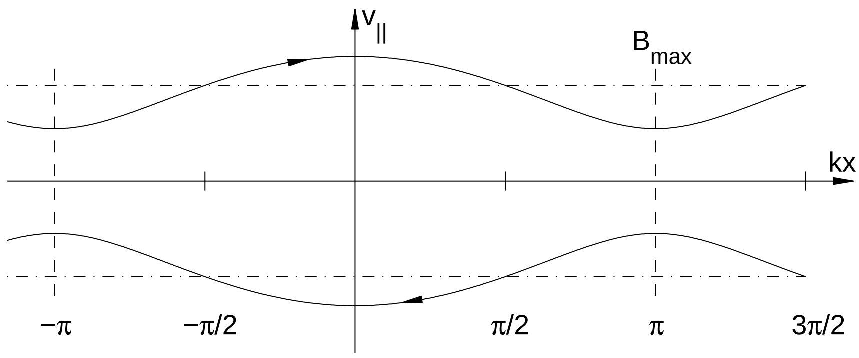


Figure 2.19. Phase portrait for $|\beta| < 1$.

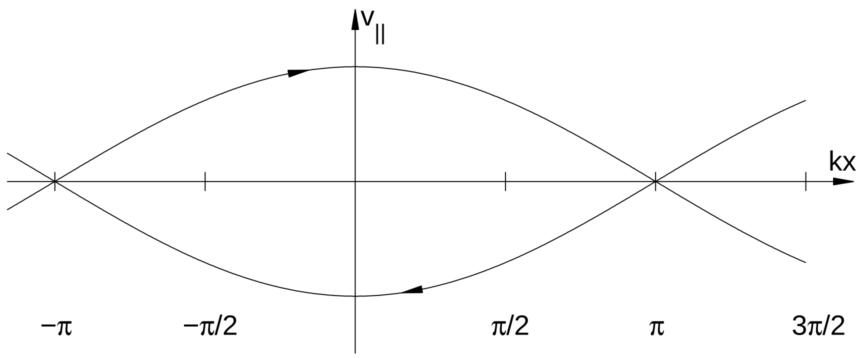


Figure 2.20. Phase portrait for $|\beta| = 1$.

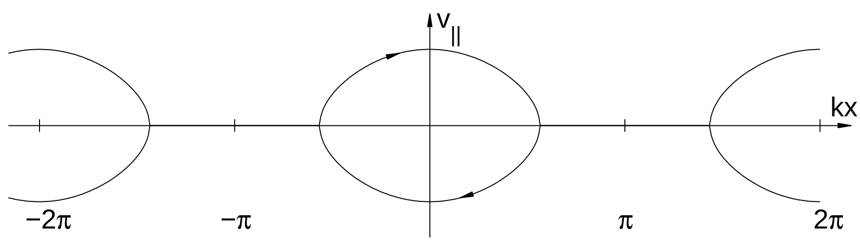


Figure 2.21. Phase portrait for $|\beta| > 1$.

particles in such a way as to give bound particles. Thus the particles are constrained to move in intervals determined by

$$1 + \beta \cos(kx_m) = 0, \tag{2.130}$$

i.e., the intervals have length $2x_m$ around the points $kx = 2n\pi$, where

$$kx_m = \pi - \arccos(1/\beta). \tag{2.131}$$

The condition for particle trapping is

$$\beta = \alpha \tan^2 \Theta > 1 \quad \Leftrightarrow \quad \tan^2 \Theta > \frac{1}{\alpha}. \tag{2.132}$$

In terms of the pitch angle, Θ_m, defined where $B = B_{\min} = B_0(1-\alpha)$, we get

$$\mu = \frac{W_\perp}{B} = \frac{\overline{W}_\perp}{\bar{B}} = \frac{W_{\perp m}}{B_{\min}}, \tag{2.133}$$

which gives

$$\sin^2 \Theta = \frac{\overline{W}_\perp}{W} = \frac{W_{\perp m}}{W}\frac{\bar{B}}{B_{\min}} = \sin^2 \Theta_m \frac{1}{1-\alpha}. \tag{2.134}$$

Since the condition for trapping (2.132) can be written

$$\sin^2 \Theta > \frac{1}{1+\alpha}, \tag{2.135}$$

we obtain, using Eqs. (2.134) and (2.135),

$$\sin^2 \Theta_m > \frac{1-\alpha}{1+\alpha} = \frac{B_{\min}}{B_{\max}} = \frac{1}{R_m}, \tag{2.136}$$

i.e., the usual mirror condition.

Finally, we may solve Eq. (2.123) for the explicit time dependence in $x(t)$ for the case of small oscillations around a minimum at $x = 0$ (or, in general, $x = n2\pi$). The right hand side is

$$-\mu \frac{\partial B}{\partial x} = -\mu \frac{\partial}{\partial x}\left[B_0(1 - \alpha \cos(kx))\right] = -\mu\alpha k B_0 \sin(kx) \approx -\mu\alpha k^2 B_0 x. \tag{2.137}$$

Using $v_\parallel = dx/dt$, Eq. (2.123) becomes

$$\frac{d^2x}{dt^2} + \frac{\mu\alpha k^2 B_0}{m} x = 0, \tag{2.138}$$

i.e., the particles perform an oscillatory motion at the minima of the magnetic potential wells. The characteristic frequency, called the *bounce frequency*, is

$$\omega_B = \sqrt{\frac{\mu\alpha k^2 B_0}{m}}. \tag{2.139}$$

2.5.2 Particle drifts in Tokamaks

The magnetic field in a Tokamak is essentially a vacuum field determined by $\nabla \times \mathbf{B} = 0$. The form $\mathbf{B} \approx B(R)\hat{\varphi}$ implies $d(RB)/dR = 0$, i.e., $RB = \text{const}$ from which we deduce

$$B = \frac{B_T}{1 + (\varepsilon/a) r \cos\theta}, \tag{2.140}$$

where $\varepsilon \equiv a/R_0$ (see Fig. 2.22) is a characteristic parameter of the Tokamak called the *inverse aspect ratio*, and B_T is the magnetic field at the centre, $r = 0$. In general $\varepsilon \ll 1$.

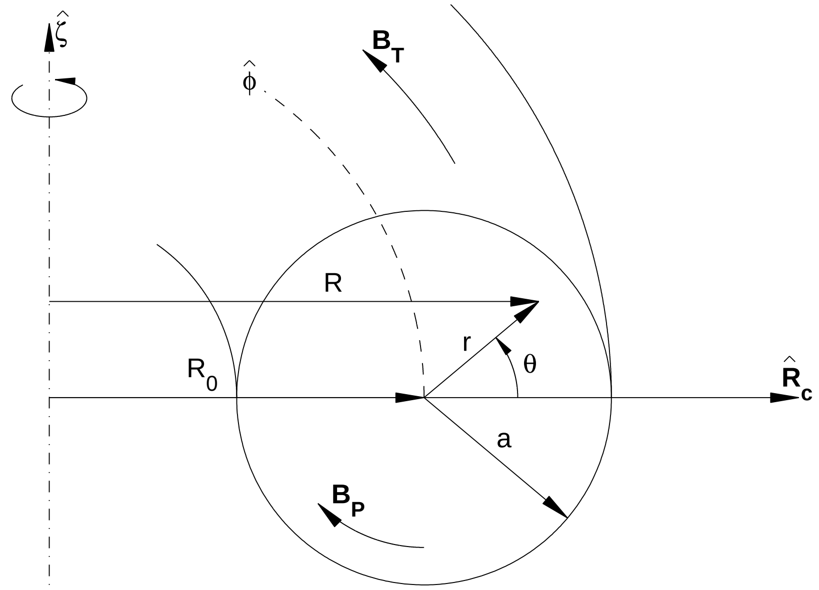


Figure 2.22. Coordinates in a Tokamak.

In a Tokamak, a current flows through the plasma in the toroidal direction. Under the influence of this current a much weaker poloidal (i.e., in the $\hat{\theta}$ direction) field is set up, giving a twisting of the magnetic field lines. The twist is determined from Fig. 2.23. We thus have

$$\frac{B_T}{B_P} = \frac{dz}{a\, d\theta}, \tag{2.141}$$

i.e.,

$$z = a \frac{B_T}{B_P} \theta. \tag{2.142}$$

Moreover, since $B_T \gg B_P$,

$$ds \approx \sqrt{dz^2 + a^2\, d\theta^2} \approx dz \approx a \frac{B_T}{B_P} d\theta. \tag{2.143}$$

If we use $\varepsilon \ll 1$ together with the relation between θ and z, we obtain

$$B \approx B_T \left[1 - \varepsilon \cos\left(\frac{B_P}{aB_T} z\right)\right]. \tag{2.144}$$

The particles moving along the magnetic field lines thus experience a periodically varying magnetic field strength, stronger at the inner region ($R < R_0$), weaker at the outer region of the torus ($R > R_0$). The particles can therefore be divided into two separate classes:

Free particles Particles with parallel velocity modulated by the periodic mirrors, but not strongly enough to be captured.

Trapped particles Particles captured in the magnetic field minima of the periodic mirrors, i.e., on the outer region at the torus ($R > R_0$).

We will now investigate some of the properties of these two types of particles.

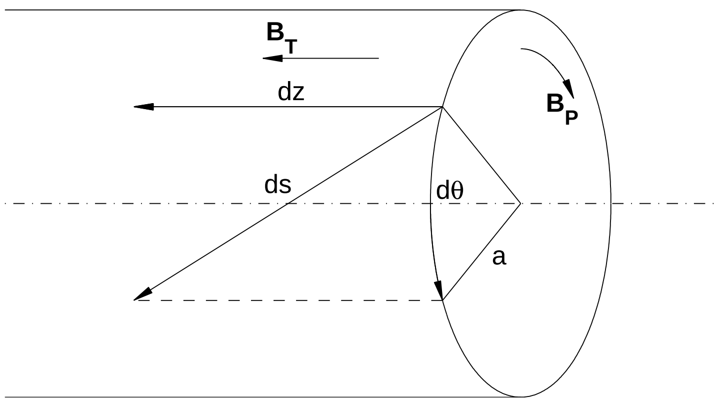


Figure 2.23. Cross section of a Tokamak with the toroidal and the poloidal magnetic fields.

Two properties of the field give rise to drift motion of the particles: The *inhomogeneity* and the *curvature* of the magnetic field.

Gradient drift

We know that the drift velocity, $\mathbf{v}_D$, due to a force, $\mathbf{F}$, is given by (see Eq. (2.25))

$$\mathbf{v}_D = \frac{1}{q}\frac{\mathbf{F} \times \mathbf{B}}{B^2}. \tag{2.145}$$

For gradient drifts, we have

$$\mathbf{F} = -\mu \nabla B = \mu \frac{B}{r} \hat{R}_c \approx \mu \frac{B}{R_0} \hat{R}_c, \tag{2.146}$$

and the drift velocity caused by the inhomogeneous magnetic field becomes

$$\mathbf{v}_{\nabla B} = \frac{\mu}{qR_0}\frac{1}{B}\hat{R}_c \times \mathbf{B} = \frac{\mu}{qR_0}\hat{\zeta}, \tag{2.147}$$

i.e., directed vertically. Here we have defined

$$\hat{\zeta} \equiv \frac{1}{B}\hat{R}_c \times \mathbf{B}. \tag{2.148}$$

Curvature drift

In this case, the force $\mathbf{F}$ is the centrifugal force, i.e.,

$$\mathbf{F} = \frac{mv_\parallel^2}{r}\hat{R}_c. \tag{2.149}$$

Thus, the drift velocity due to the curvature of the magnetic field can be written

$$\mathbf{v}_C = \frac{mv_\parallel^2}{qR_0B^2}\hat{R}_c \times \mathbf{B} = \frac{mv_\parallel^2}{qR_0B}\hat{\zeta}. \tag{2.150}$$

Total drift

Adding up the contributions (2.147) and (2.150), we obtain the total drift velocity as

$$\mathbf{v}_D = \frac{m}{qR_0B}\left(\frac{1}{2}v_\perp^2 + v_\parallel^2\right)\hat{\zeta}, \tag{2.151}$$

where we have used the definition (2.18) for the magnetic moment μ.

In order to discuss the implication of the gradient and curvature drifts on the particle orbits, we will make use of the concept *magnetic surface*, which is the surface on which the strength of the magnetic field remains constant as one moves around the Tokamak. For instance, we may choose a point $\varphi = 0$, $r = r_0$ and $\theta = \theta_0$ as our starting point for tracking a specific magnetic field line (the equation for a magnetic field line is $d\theta/B_P = (R + r\cos\theta)d\varphi/B_T$, see Eq. (2.141)). When the field line reaches $\varphi = 2\pi$, it ends up at a point $r = r_1$ and $\theta = \theta_1$. After n orbits (i.e., $\varphi = 2n\pi$), the field line is located at $r = r_n$ and $\theta = \theta_n$. Since $B_r = 0$, the radius of the field lines cannot change. Thus, in the (r, θ)-plane, the field line will have the coordinates (r_0, θ_n) after n turns around the Tokamak. All points of the field line therefore lie on a circle with radius r_0. Since the field lines always lie on circles in the (r, θ)-plane, the Tokamak is said to have circular magnetic surfaces.

Since the drift motion of the particles acts in the same direction, $\hat{\zeta}$, at all times, the average drift velocity will be zero, $\langle \mathbf{v}_D \rangle = 0$. Thus, the particle will, in the upper half of its orbit in the (r, θ)-plane, drift away from the magnetic surface, but, on the lower half of its orbit in the (r, θ)-plane, drift back to the magnetic surface again. This effect will lead to a shift in the orbit for free particles, and a widening of the orbit for trapped particles, see Figs. 2.24 and 2.25. The orbit for trapped particles is called the *banana orbit*.

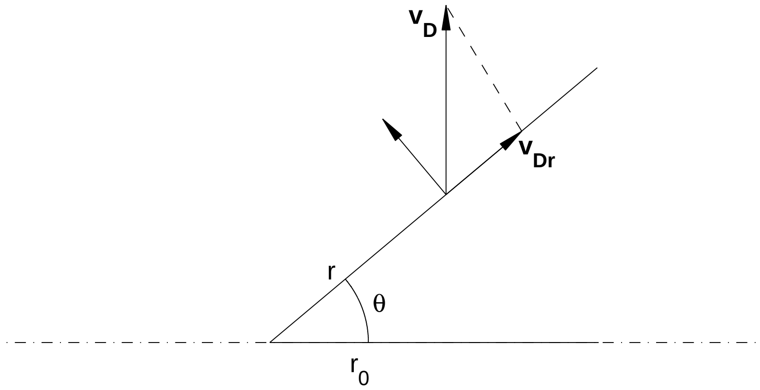


Figure 2.24. The components of the particle drift velocity in the poloidal and radial directions.

Free particles

For free particles $v_\parallel^2 \gg v_\perp^2$, and we may approximate the deviation, δ, from the magnetic surface by

$$\delta \sim \frac{T}{4} v_D \sim \frac{T}{4} \frac{m v_\parallel^2}{q R_0 B_T}, \tag{2.152}$$

where T is the time of one rotation in the toroidal direction, i.e.,

$$T = \oint \frac{ds}{v_\parallel} = \int_0^{2\pi} \frac{a B_T}{B_P v_\parallel}\, d\theta = 2\pi \frac{a B_T}{B_P v_\parallel}. \tag{2.153}$$

Using Eq. (2.153) in Eq. (2.152), we obtain

$$\delta \sim \frac{a B_T}{B_P v_\parallel} \frac{m v_\parallel^2}{q R_0 B_T} = q_s \frac{v_\parallel}{\omega_{c,T}}, \tag{2.154}$$

where $q_s \equiv a B_T/(R_0 B_P)$ is the safety factor, and $\omega_{c,T} = q B_T/m$ is the cyclotron frequency in the toroidal field.

This derivation can be done in more detail. Consider Fig. 2.24. The deviation from the magnetic surface is determined by the projected part of $\mathbf{v}_D$ along the r-direction:

$$v_{Dr} = v_D \sin\theta = \frac{dr}{dt} \approx -\frac{m v_\parallel^2}{q R_0 B} \sin\theta = -\frac{m v_\parallel^2}{q R_0 B} \sin\left(\frac{B_P}{a B_T} v_\parallel t\right). \tag{2.155}$$

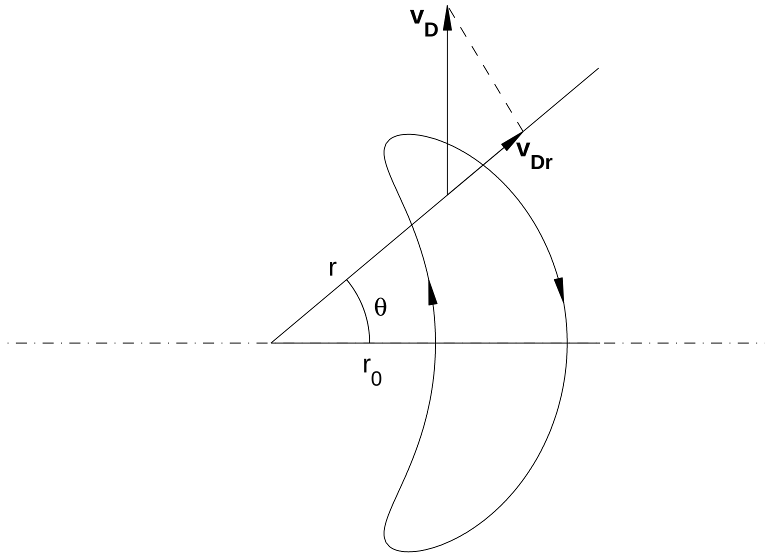


Figure 2.25. The components of the particle drift velocity in the poloidal and radial directions.

Integrating Eq. (2.155), we obtain

$$r - r_0 = \frac{mv_\parallel^2}{qR_0B}\frac{aB_T}{B_Pv_\parallel}\cos\theta, \tag{2.156}$$

i.e.,

$$\delta = q_s\frac{v_\parallel}{\omega_{c,T}}. \tag{2.157}$$

This is the same result as the one we obtained through Eq. (2.152). This shift is in general small, e.g., for 1 keV protons, using $B_T \approx 4$ T and $q_s = 2$, the shift is $\delta \sim 4$ mm.

Trapped particles

Particles with small $v_\parallel$ can be trapped in the magnetic field minima. For motion along the magnetic field lines

$$m\frac{dv_\parallel}{dt} = -\mu\frac{\partial B}{\partial s} \approx -\varepsilon\mu B_T\frac{B_P}{aB_T}\sin\left(\frac{B_P}{aB_T}z\right), \tag{2.158}$$

where the last equality follows from Eqs. (2.143) and (2.144). For well trapped particles, i.e., when the argument of the sin-function is small, we can use the approximation

$$m\frac{d^2z}{dt^2} \approx -\varepsilon\mu B_T\left(\frac{B_P}{aB_T}\right)^2 z, \tag{2.159}$$

i.e., the motion is oscillatory and given by

$$z = A\sin(\omega_B t), \tag{2.160}$$

where

$$\omega_B = \sqrt{\frac{\varepsilon\mu B_P^2}{ma^2 B_T}} \tag{2.161}$$

is the bounce frequency.

Since trapped particles satisfy $v_\parallel^2 \ll v_\perp^2$, the motion in the r-direction is determined by

$$v_{Dr} = \frac{dr}{dt} = v_D \sin\theta \approx \frac{\mu}{qR_0}\sin\theta = -\frac{\mu}{qR_0}\frac{maB_T}{\mu B_T \varepsilon B_P}\frac{dv_\parallel}{dt}, \tag{2.162}$$

where we have used Eq. (2.151). Integrating Eq. (2.162), we obtain

$$r - r_0 = -\frac{q_s}{\varepsilon\omega_{c,T}}v_\parallel, \tag{2.163}$$

i.e., the width 2δ of the banana orbit is determined by

$$\delta = \frac{q_s}{\varepsilon\omega_{c,T}}v_{\parallel\max} = \frac{v_{\parallel\max}}{\omega_{c,P}}, \tag{2.164}$$

where $\omega_{c,P}$ is the cyclotron frequency in the poloidal field.

Let us now try to estimate the maximum value $v_{\parallel\max}$ of the trapped particle velocities. We know that the critical pitch angle is determined by

$$\sin^2\Theta = \frac{1}{R_m} = \frac{B_{\min}}{B_{\max}} = \frac{1-\varepsilon}{1+\varepsilon} \approx 1 - 2\varepsilon, \tag{2.165}$$

where $\Theta \equiv \arctan(v_\perp/v_\parallel)$. Since $v_\parallel \ll v_\perp$, $\Theta \approx \pi/2$, and this gives $\sin^2\Theta = \cos^2(\pi/2 - \Theta) \approx 1 - (\pi/2 - \Theta)^2 \approx 1 - 2\varepsilon$, i.e.,

$$\frac{\pi}{2} - \Theta \approx \sqrt{2\varepsilon}, \tag{2.166}$$

and we obtain

$$v_\parallel = v\cos\Theta = v\sin(\pi/2 - \Theta) \approx v\sqrt{2\varepsilon}. \tag{2.167}$$

The widest banana orbit then has the width

$$\delta_{\max} = \sqrt{2\varepsilon}\frac{v}{\omega_{c,P}}, \tag{2.168}$$

which can be large for particles with high energy (e.g., 3.5 MeV α-particles).

Finally, we calculate the number of particles trapped in the Tokamak. If the total number of particles is N, the number of free particles, N_p, is given by

$$N_p = N\left(1 - \sqrt{1 - \frac{1}{R_m}}\right), \tag{2.169}$$

where

$$\frac{1}{R_m} = \frac{B_{\min}}{B_{\max}} = \frac{1-\varepsilon}{1+\varepsilon}. \tag{2.170}$$

Thus, the number of trapped particles is

$$N_t = N - N_p = N\sqrt{1 - \frac{1-\varepsilon}{1+\varepsilon}} \approx N\sqrt{2\varepsilon}. \tag{2.171}$$

2.6 Particle drifts in time-varying electric fields—the polarisation drift

A static electric field gives rise to particle acceleration along the magnetic field lines, and a drift (the $\mathbf{E} \times \mathbf{B}$ drift) perpendicular to the magnetic and electric fields. We will in this section study what happens if the electric field is allowed to have a slow time dependence, i.e.,

$$\left| \frac{1}{E}\frac{dE}{dt} \right| \ll \omega_c. \tag{2.172}$$

Taking the vector product of the equation of motion,

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{2.173}$$

with the magnetic field $\mathbf{B}$, we obtain

$$m\frac{d\mathbf{v}_\perp}{dt} \times \mathbf{B} = q\mathbf{E} \times \mathbf{B} - qB^2\mathbf{v}_\perp. \tag{2.174}$$

A formal solution to Eq. (2.174) is

$$\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{m}{q}\frac{(d\mathbf{v}_\perp/dt) \times \mathbf{B}}{B^2}. \tag{2.175}$$

The first term in (2.175) is the $\mathbf{E} \times \mathbf{B}$ drift velocity (see Eq. (2.27)), while the second term is the correction due to the time dependence of the electric field. We can see this more clearly as follows. Since this last term is a small correction, we can to first order write $\mathbf{v}_\perp \approx \mathbf{E} \times \mathbf{B}/B^2$, and use this approximation in the second term of the right hand side of Eq. (2.175). We then have

$$\mathbf{v}_\perp \approx \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \frac{m}{qB^2}\left[\frac{d}{dt}\left(\frac{\mathbf{E} \times \mathbf{B}}{B^2}\right)\right] \times \mathbf{B} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} + \frac{m}{qB^2}\frac{d\mathbf{E}}{dt}, \tag{2.176}$$

i.e., we obtain a combined drift motion, consisting of the usual $\mathbf{E} \times \mathbf{B}$ drift and a new drift; the *polarisation drift*, $\mathbf{v}_p$, directed along the time derivative of $\mathbf{E}$ as shown in Fig. 2.26, and given by

$$\mathbf{v}_p \equiv \frac{m}{qB^2}\frac{d\mathbf{E}}{dt}. \tag{2.177}$$

The relative magnitude between the two different drift velocities is, according to Eqs. (2.27) and (2.177),

$$\frac{|\mathbf{v}_{\mathbf{E}\times\mathbf{B}}|}{|\mathbf{v}_p|} = \frac{E}{B}\frac{|q|B^2}{m\omega E} = \frac{\omega_c}{\omega} \gg 1, \tag{2.178}$$

where we have assumed a harmonic time dependence of the electric field, $\mathbf{E} = \mathbf{E}_0 \exp(-i\omega t)$.

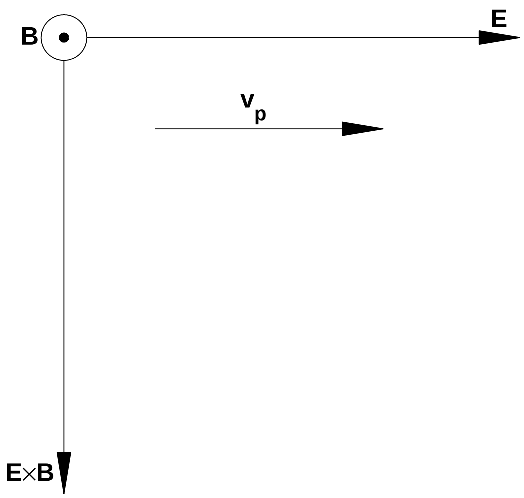


Figure 2.26. For a positively charged particle, the polarisation drift is parallel to the time derivative of the electric field.

Under the influence of these drift velocities, the guiding centre of the particle will perform an elliptic orbit according to Fig. 2.27. The elliptic orbit is distorted along $\mathbf{E} \times \mathbf{B}$. Increasing the frequency ω yields a less elongated orbit.

2.7 Particle motion in slowly time-varying magnetic fields

Often, the time-scales related to the change in the electromagnetic fields are long as compared to other time-scales within the plasma. It therefore makes sense to study the approximation of slowly time-varying fields, and usually this is also simplifies the analysis significantly.

2.7.1 Adiabatic invariants

When the fields vary slowly, some parameters related to these fields will be approximately constant over short-enough time-scales. In what follows we will discuss two of these *adiabatic invariants*.

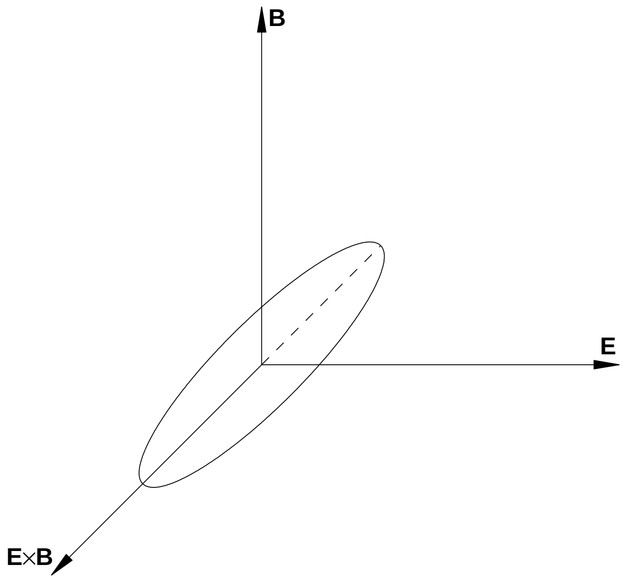


Figure 2.27. Motion of the guiding centre.

The first adiabatic invariant

In this section, we will allow the magnetic field to have a slow time dependence, defined in the same way as for the electric field (see Eq. (2.172))

$$\left|\frac{1}{B}\frac{dB}{dt}\right| \ll \omega_c. \tag{2.179}$$

Using the slow time dependence approximation, one can show that the magnetic moment, $\mu = W_\perp/B$, is still a constant. We proceed to do this as follows. First of all, since

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \tag{2.180}$$

according to Maxwell's equations, a time varying magnetic field will give rise to a corresponding electric field. Secondly, if we multiply Eq. (2.173) by $\mathbf{v}_\perp$, we obtain

$$\frac{dW_\perp}{dt} = \frac{d}{dt}\left(\frac{mv_\perp^2}{2}\right) = q\mathbf{E}\cdot\mathbf{v}_\perp. \tag{2.181}$$

We are only interested in processes which occur on time scales much longer than a Larmor period, and we can therefore replace the right hand side of Eq. (2.181) by its average, i.e.,

$$\begin{aligned}\mathbf{E}\cdot\mathbf{v}_\perp \approx \langle\mathbf{E}\cdot\mathbf{v}_\perp\rangle &\equiv \frac{1}{T}\int_0^T \mathbf{E}\cdot\mathbf{v}_\perp\, dt \\ &= \frac{1}{T}\int_0^T \mathbf{E}\cdot\frac{d\mathbf{r}_\perp}{dt}\, dt \approx \frac{1}{T}\oint_{\text{Larmor orbit}} \mathbf{E}\cdot d\mathbf{r}.\end{aligned} \tag{2.182}$$

Using Stoke's theorem, we can rewrite the closed integral as a surface integral according to (see Fig. 2.28)

$$\mathbf{E} \cdot \mathbf{v}_\perp \approx \frac{1}{T} \int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{1}{T} \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}. \tag{2.183}$$

The direction of the particle's motion along the orbit is determined by the diamagnetic property, i.e., $\mathbf{B}$ and the normal $\hat{n}_S$ to the surface S are anti-parallel. Thus, performing the integration in (2.183) and using Eq. (2.181), we obtain

$$\frac{dW_\perp}{dt} = \frac{q}{T} \pi r_L^2 \frac{dB}{dt} = \frac{mv_\perp^2}{2B} \frac{dB}{dt} = \mu \frac{dB}{dt}. \tag{2.184}$$

Combining this with the relation

$$\frac{dW_\perp}{dt} = \frac{d}{dt}(\mu B) = \frac{d\mu}{dt} B + \mu \frac{dB}{dt}, \tag{2.185}$$

we obtain $d\mu/dt = 0$, and thus

$$\mu = \frac{W_\perp}{B} = \text{const}, \tag{2.186}$$

for slowly varying magnetic fields. Since μ is constant, it is called the *first adiabatic invariant*.

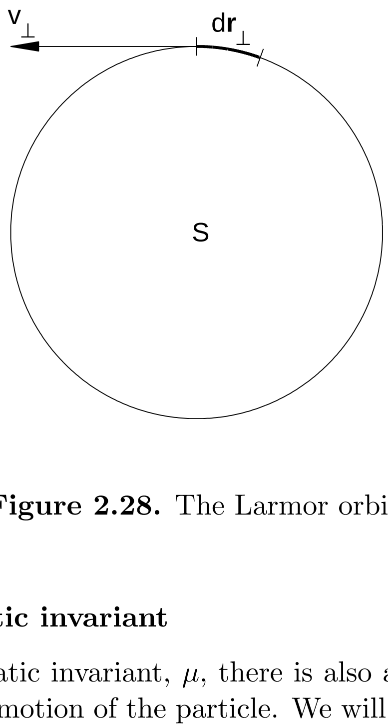


Figure 2.28. The Larmor orbit.

The second adiabatic invariant

Beside the first adiabatic invariant, μ, there is also an invariant associated with the longitudinal motion of the particle. We will show that the integral

$$J = \oint v_\parallel \, ds \tag{2.187}$$

is constant, where $v_\parallel$ is the velocity component parallel to the field lines, and the integration is over a complete cycle. The meaning of the invariant J is demonstrated in Fig. 2.29 for a particle reflected between two mirrors slowly moving towards each other. The invariance condition tells us that the area contained by $v_\parallel(x)$ is constant.

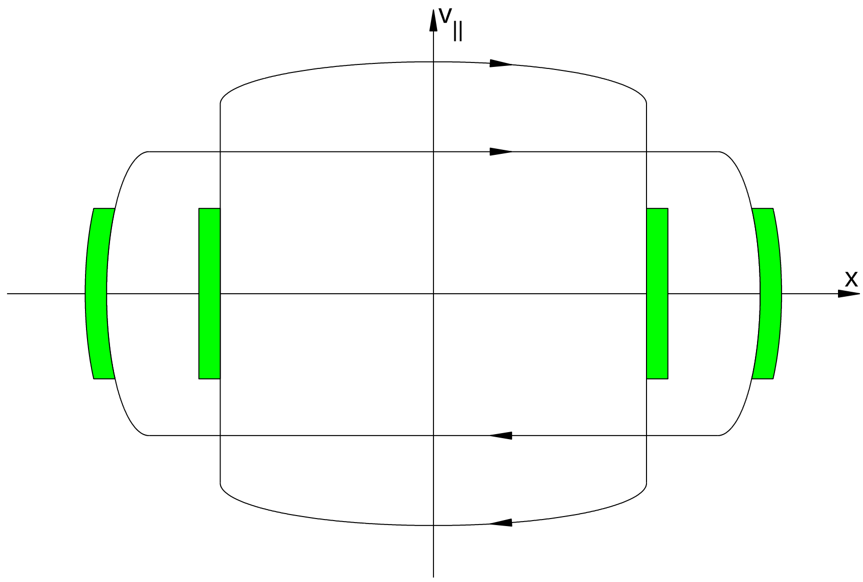


Figure 2.29. The motion of the magnetic mirrors used in the interpretation of the second adiabatic invariant.

The motion along the magnetic field lines is determined by

$$m\frac{dv_\parallel}{dt} = -\mu\frac{\partial B}{\partial s}, \tag{2.188}$$

where $v_\parallel \equiv ds/dt$, and s denotes the arc length along the orbit. If B is constant in time, we may multiply Eq. (2.188) by $v_\parallel$, integrate and obtain

$$\frac{mv_\parallel^2}{2} + \mu B = W = \text{const.} \tag{2.189}$$

If B is *not* constant in time, we may take the derivative of Eq. (2.189):

$$\frac{\partial W}{\partial t} = mv_\parallel\frac{dv_\parallel}{dt} + \mu\left(\frac{\partial B}{\partial t} + \frac{\partial B}{\partial s}\frac{ds}{dt}\right) = \mu\frac{\partial B}{\partial t}, \tag{2.190}$$

where Eq. (2.188) was used in the last step. If the time variation in the magnetic field is slow compared to the period in the longitudinal orbit, we can approximate according to

$$\frac{\partial W}{\partial t} \approx \left\langle \mu\frac{\partial B}{\partial t}\right\rangle . \tag{2.191}$$

Note that the requirement on the time variation of the magnetic field is much stricter here if compared to the first adiabatic invariant, where we used the Larmor period as our typical time scale.

Consider the functional J at an arbitrary point, s, of the orbit, i.e.,

$$J = \int_0^s v_\parallel \, ds'. \tag{2.192}$$

We solve Eq. (2.189) for $v_\parallel$, and insert into the integral for J

$$J(s, W, t) = \int_0^s \sqrt{\frac{2}{m}(W - \mu B(t))} \, ds', \tag{2.193}$$

where we consider J as a function of s, W, and t. Taking the total time derivative of the expression for J yields

$$\frac{dJ}{dt} = \frac{\partial J}{\partial s}\frac{ds}{dt} + \frac{\partial J}{\partial W}\frac{\partial W}{\partial t} + \frac{\partial J}{\partial t}, \tag{2.194}$$

and the different terms can be expressed as

$$\frac{\partial J}{\partial W} = \int_0^s \frac{ds'}{m\sqrt{2(W - \mu B(t))/m}} = \frac{1}{m}\int_0^s \frac{ds'}{v_\parallel}, \tag{2.195a}$$

$$\frac{\partial J}{\partial t} = -\int_0^s \frac{\mu}{m}\frac{\partial B}{\partial t}\frac{ds'}{\sqrt{2(W - \mu B(t))/m}} = -\frac{1}{m}\int_0^s \frac{\mu}{v_\parallel}\frac{\partial B}{\partial t} \, ds'. \tag{2.195b}$$

Furthermore, from Eq. (2.191) we have

$$\frac{\partial W}{\partial t} \approx \frac{1}{T}\int_0^T \mu \frac{\partial B}{\partial t} \, dt = \oint \mu \frac{\partial B}{\partial t}\frac{ds'}{v_\parallel} \Bigg/ \oint \frac{ds'}{v_\parallel}. \tag{2.196}$$

Collecting terms, we obtain

$$\frac{dJ}{dt} = v_\parallel^2 + \frac{1}{m}\left(\int_0^s \frac{ds'}{v_\parallel}\right)\left(\oint \mu \frac{\partial B}{\partial t}\frac{ds'}{v_\parallel}\right)\left(\oint \frac{ds'}{v_\parallel}\right)^{-1} - \frac{1}{m}\int_0^s \frac{\mu}{v_\parallel}\frac{\partial B}{\partial t} \, ds'. \tag{2.197}$$

We now choose a particular s, namely $s = s^* =$ the value after one period, i.e., the turning point of the particle where $v_\parallel^* = 0$ and all integrals over s' are closed. Thus

$$\frac{dJ^*}{dt} = 0, \tag{2.198}$$

or, equivalently,

$$J(s^*, W, t) = \oint v_\parallel \, ds' = \text{const}, \tag{2.199}$$

which is called the *second adiabatic invariant*.

2.7.2 Acceleration of cosmic particles according to Fermi

In the 1940's, E. Fermi suggested the following mechanism for accelerating cosmic particles to high energies. Two magnetic mirrors move towards each other with velocity v_m, as seen in Fig. 2.30. Particles trapped in the magnetic bottle will, at each reflection, increase their speed (and thus their energy) until they enter the loss cone and are ejected with very high energy. We consider this mechanism more closely using the second adiabatic invariant

$$J = \int_0^L v_{\|}\, ds = \text{const} \approx L(t) v_{\|}. \tag{2.200}$$

When L decreases, the speed $v_{\|}$ must increase in order for J to stay constant. The maximum attainable parallel energy is determined by the mirror ratio R_m (see Fig. 2.31)

$$\sin^2\Theta = \frac{1}{R_m} = \frac{W_{\perp 0}}{W_{\perp 0} + W_{\| \max}}, \tag{2.201}$$

from which we get

$$W_{\|\max} = (R_m - 1) W_{\perp 0}. \tag{2.202}$$

If $W_{\|0} \sim W_{\perp 0}$, we thus get an energy increase $(R_m - 1)$.

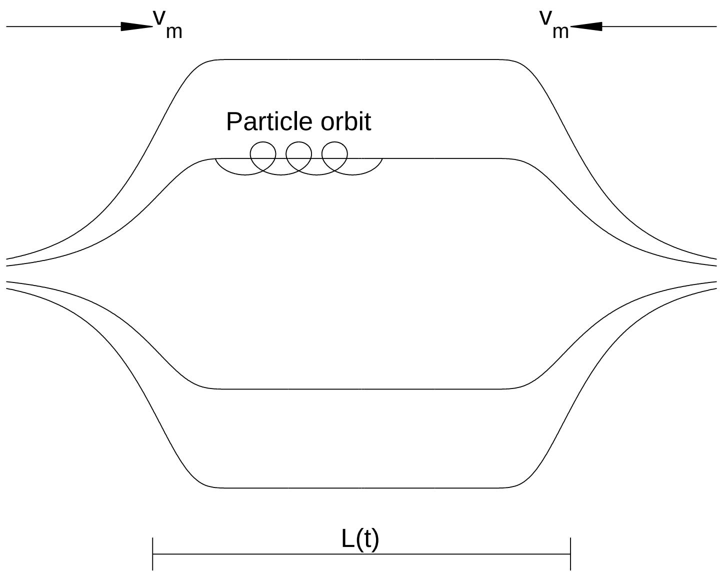


Figure 2.30. The acceleration mechanism of Fermi.

We can also calculate how $W_{\|}(t)$ grows, and determine when the particle enters the loss cone. From Eq. (2.200) we have $v_{\|} \sim 1/L(t)$ which implies

$$W_{\|}(t) = W_{\|0} \left(\frac{L_0}{L(t)} \right)^2. \tag{2.203}$$

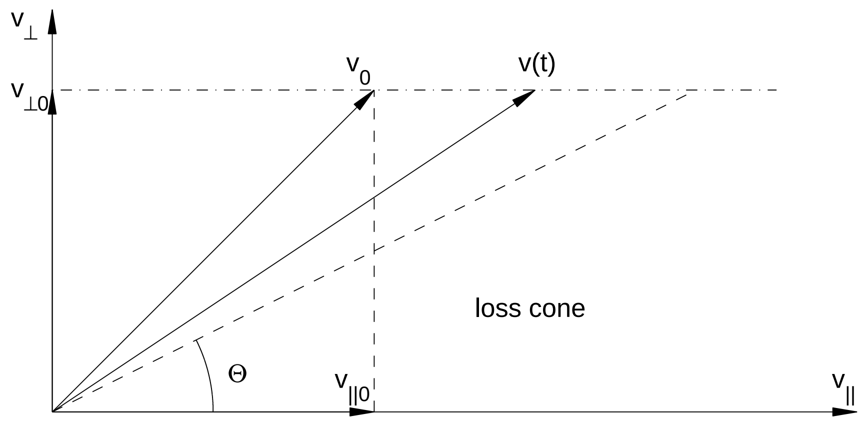


Figure 2.31. As the parallel velocity component increases, the particle eventually enters the loss cone.

Since $L(t) = L_0 - 2v_m t$, we obtain

$$W_{\|}(t) = \frac{W_{\|0}}{(1 - 2v_m t/L_0)^2}. \tag{2.204}$$

However, the particle reaches the loss cone after a time $t_{\rm loss}$ determined by $W_{\|}(t_{\rm loss}) = W_{\|{\rm max}}$, i.e.,

$$t_{\rm loss} = \frac{L_0}{2v_m}\left(1 - \sqrt{\frac{W_{\|0}}{W_{\|{\rm max}}}}\right) = \frac{L_0}{2v_m}\left(1 - \sqrt{\frac{1}{R_m - 1}\frac{W_{\perp 0}}{W_{\|0}}}\right), \tag{2.205}$$

or expressed in terms of the pitch angle Θ_0 at time $t = 0$

$$t_{\rm loss} = \frac{L_0}{2v_m}\left(1 - \frac{\tan\Theta_0}{\sqrt{R_m - 1}}\right). \tag{2.206}$$

2.7.3 The gyro relaxation effect

In this section we will give an example of a situation where the invariance of μ is broken due to particle collisions.

Consider a plasma with low density and with an external magnetic field B. When in thermodynamical equilibrium with its surroundings, the total kinetic energy of a particle is distributed on transversal and longitudinal motion in proportion to the corresponding number of degrees of freedom, i.e.,

$$W_{\rm tot} = W_{\perp} + W_{\|}, \tag{2.207}$$

where

$$W_{\perp} = \frac{2}{3}W_{\rm tot}, \quad W_{\|} = \frac{1}{3}W_{\rm tot}, \tag{2.208}$$

which implies that

$$W_\perp - 2W_\parallel = 0. \tag{2.209}$$

For deviations from equilibrium we have $W_\perp - 2W_\parallel \neq 0$, but collisions between particles tend to restore the balance. Let us assume that perturbations from equilibrium return to zero according to

$$\frac{d}{dt}(W_\perp - 2W_\parallel) = -\gamma(W_\perp - 2W_\parallel), \tag{2.210}$$

where $\gamma > 0$ (otherwise the perturbations would grow). Furthermore, the total energy is constant, i.e.,

$$\frac{d}{dt}(W_\perp + W_\parallel) = 0, \tag{2.211}$$

giving us the system of equations

$$\frac{dW_\perp}{dt} = -\frac{\gamma}{3}(W_\perp - 2W_\parallel), \tag{2.212a}$$

$$\frac{dW_\parallel}{dt} = \frac{\gamma}{3}(W_\perp - 2W_\parallel). \tag{2.212b}$$

Let us analyse what happens to the system if we let the magnetic field change slowly in time ($|B^{-1}dB(t)/dt| \ll \omega_L$). The adiabatic invariant μ relates changes in the magnetic field to changes in the transversal kinetic energy according to

$$\frac{d\mu}{dt} = \frac{d}{dt}\left(\frac{W_\perp}{B}\right) = 0 \tag{2.213}$$

or

$$\frac{dW_\perp}{dt} = \frac{W_\perp}{B}\frac{dB}{dt}. \tag{2.214}$$

The longitudinal energy is not affected, i.e.,

$$\frac{dW_\parallel}{dt} = 0. \tag{2.215}$$

By adding the relaxation and gyro effects we obtain

$$\begin{aligned}\frac{dW_\perp}{dt} &= \left(\frac{dW_\perp}{dt}\right)_r + \left(\frac{dW_\perp}{dt}\right)_g \\ &= -\frac{\gamma}{3}(W_\perp - 2W_\parallel) + \frac{W_\perp}{B}\frac{dB}{dt},\end{aligned} \tag{2.216a}$$

$$\frac{dW_\parallel}{dt} = \left(\frac{dW_\parallel}{dt}\right)_r + \left(\frac{dW_\parallel}{dt}\right)_g = \frac{\gamma}{3}(W_\perp - 2W_\parallel). \tag{2.216b}$$

We focus on the case $\gamma \ll |B^{-1}dB(t)/dt|$. The solutions to Eqs. (2.216a) and (2.216b) are then essentially determined by the gyro effects. If $\gamma = 0$, the solutions are

$$W_\perp(t) = \mu B(t), \quad W_\parallel(t) = C, \tag{2.217}$$

where μ and C are constants. In the case $\gamma \neq 0$, we look for solutions of the form

$$W_\perp(t) = \mu(t)B(t), \quad W_\parallel(t) = C(t), \tag{2.218}$$

where the slow effect of relaxation is captured by the slow time dependence of $\mu(t)$ and $C(t)$. Inserting this ansatz into Eqs. (2.216a) and (2.216b), we can simplify and thereby obtain

$$\frac{d\mu}{dt} = -\frac{\gamma}{3}\mu + \frac{2\gamma}{3}\frac{C}{B}, \tag{2.219a}$$

$$\frac{dC}{dt} = \frac{\gamma}{3}\mu B - \frac{2\gamma}{3}C. \tag{2.219b}$$

Since the variations in B are much faster than the variations in μ and C, we may replace B and $1/B$ by their respective time averages, $\langle B\rangle$ and $\langle 1/B\rangle$, respectively. Furthermore, if we are only interested in the stability of the system (2.219a)–(2.219b), we can look for solutions with a time dependence of the form $\exp(\lambda t)$. We then obtain the following linear system of equations for μ and C

$$\left(\lambda + \frac{\gamma}{3}\right)\mu - \frac{2\gamma}{3}\left\langle \frac{1}{B}\right\rangle C = 0, \tag{2.220a}$$

$$\frac{\gamma}{3}\mu\langle B\rangle - \left(\frac{2\gamma}{3} + \lambda\right)C = 0. \tag{2.220b}$$

The non-trivial solutions to this system are determined by

$$-\left(\lambda + \frac{\gamma}{3}\right)\left(\lambda + \frac{2\gamma}{3}\right) + \frac{2\gamma^2}{9}\langle B\rangle\left\langle\frac{1}{B}\right\rangle = 0. \tag{2.221}$$

By neglecting terms quadratic in λ, we obtain

$$\lambda \approx \frac{2\gamma}{9}\left(\langle B\rangle\left\langle\frac{1}{B}\right\rangle - 1\right). \tag{2.222}$$

As an example, let us assume that B varies periodically according to

$$B(t) = B_0(1 + \beta\cos\omega t), \tag{2.223}$$

where $|\beta| \ll 1$. Then

$$\langle B\rangle = B_0, \tag{2.224}$$

and

$$\begin{aligned}\left\langle\frac{1}{B}\right\rangle &= \frac{1}{B_0}\langle(1+\beta\cos\omega t)^{-1}\rangle \\ &\approx \frac{1}{B_0}\langle 1 - \beta\cos\omega t + \beta^2\cos^2\omega t\rangle = \frac{1}{B_0}\left(1 + \frac{\beta^2}{2}\right).\end{aligned} \tag{2.225}$$

Thus, in this case, the growth rate λ is given by

$$\lambda \approx \frac{\gamma\beta^2}{9}. \tag{2.226}$$

The total kinetic energy of the particle grows like

$$\begin{aligned} W_{\mathrm{tot}}(t) &= W_\perp(t) + W_\parallel(t) = [\mu(0)B(t) + W_\parallel(0)]\exp(\lambda t) \\ &= [\mu(0)(1+\beta\cos\omega t) + W_\parallel(0)]\exp(\lambda t), \end{aligned} \tag{2.227}$$

which implies that the periodic magnetic field "pumps" the plasma to higher and higher temperatures, see Fig. 2.32.

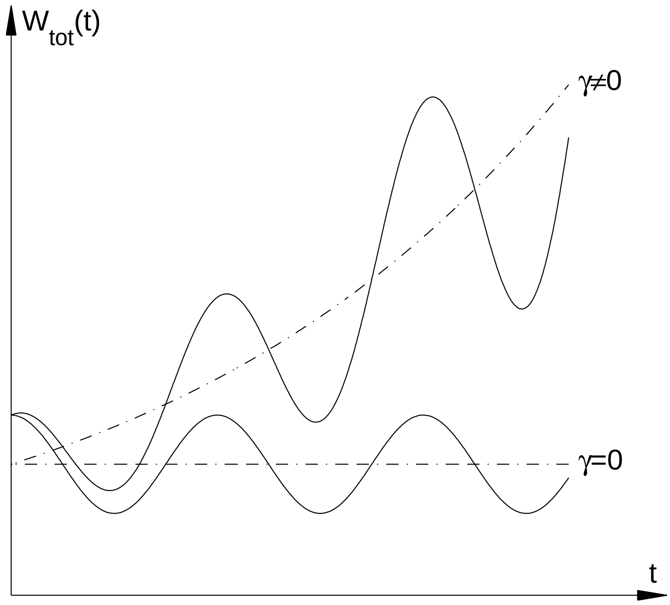


Figure 2.32. The variation of the total energy in time for $\gamma = 0$ and $\gamma \neq 0$.

2.7.4 Magnetic pumping

We have seen that the equation of motion for a particle in a magnetic bottle with periodically varying magnetic field strength $B(x) = B_0(1 - \alpha\cos(kx))$ is (see Eq. (2.138))

$$\ddot{x} + \frac{\alpha\mu k^2 B_0}{m} x = 0, \tag{2.228}$$

for small oscillations around the minima at $x = 0$ (or $x = 2\pi n$). The characteristic frequency is determined by $\omega_B^2 \equiv \alpha\mu k^2 B_0/m$. Let us now assume that the magnetic field strength B_0 varies in *time* according to

$$B_0 \to B_0(1 + \beta\cos(\omega_T t)), \tag{2.229}$$

where $|\beta| \ll 1$. This implies that the magnetic mirrors oscillate in space, i.e., the point where a particle with a given energy is reflected varies periodically. If ω_T is small enough, it follows from the second adiabatic invariant that the parallel energy will vary in resonance with the varying magnetic field strength. On the other hand, it is clear that by choosing a suitable value of the modulation frequency ($\omega_T = 2\omega_B$, which we can assume is much smaller than ω_c), the particles will be able to collide with a counter-moving mirror in every collision. Since $\omega_T \sim \omega_B$, the second adiabatic invariant is *not* conserved and the parallel energy will increase, i.e., the plasma temperature will increase.

We will now study this effect more closely. If B_0 is constant,

$$v_{\|} = \dot{x} = v_0 \cos(\omega_B t + \phi), \tag{2.230}$$

and the time average of the parallel energy is given by

$$\langle W_{\|} \rangle = \frac{m\langle v_{\|}^2 \rangle}{2} = \frac{mv_0^2}{4}. \tag{2.231}$$

With a time varying B_0, according to (2.229), Eq. (2.228) becomes

$$\ddot{x} + \omega_B^2(1 + \beta\cos(\omega_T t))x = 0, \tag{2.232}$$

i.e., the frequency of oscillation varies periodically. This type of equation is known as a *Mathieu equation*, and occurs within all fields of physics. A number of methods for solving equations of this type have been devised. We will use an approach based on Fourier transform techniques. The Fourier transform of Eq. (2.232) is

$$-\omega^2 X(\omega) + \omega_B^2 X(\omega) + \beta\omega_B^2\, \mathcal{F}[x(t)\cos(\omega_T t)] = 0, \tag{2.233}$$

where $\mathcal{F}$ denotes the Fourier transform. Since $\cos(\omega_T t) = [\exp(i\omega_T t) + \exp(-i\omega_T t)]/2$ and $\mathcal{F}[x(t)\exp(i\omega_T t)] = X(\omega - \omega_T)$, Eq. (2.233) becomes

$$-(\omega^2 - \omega_B^2)X(\omega) + \frac{\beta\omega_B^2}{2}[X(\omega+\omega_T) + X(\omega-\omega_T)] = 0. \tag{2.234}$$

As is clear from Eq. (2.234), the case $\beta = 0$ yields $\omega = \pm\omega_B$ as it should. If $\beta \neq 0$, we expect a small change in ω. Since $X(\omega)$ is only important if $\omega \approx \pm\omega_B$, $X(\omega \pm \omega_T)$ is negligible except when $\omega_B \pm \omega_T \approx \pm\omega_B$, i.e., $\omega_T \approx 0$ or $\omega_T \approx 2\omega_B$. Let us assume that the latter is the case. Then

$$\omega + \omega_T \approx 3\omega_B, \quad \omega - \omega_T \approx -\omega_B, \tag{2.235}$$

and we can approximate Eq. (2.234) as

$$(\omega^2 - \omega_B^2)X(\omega) - \frac{\beta\omega_B^2}{2}X(\omega - \omega_T) \approx 0. \tag{2.236}$$

In order to analyse $X(\omega - \omega_T)$ we let $\omega \to \omega - \omega_T$, so that Eq. (2.234) becomes

$$-[(\omega - \omega_T)^2 - \omega_B^2]X(\omega - \omega_T) + \frac{\beta\omega_B^2}{2}[X(\omega) + X(\omega - 2\omega_T)] = 0. \quad (2.237)$$

Since

$$\omega \sim \omega_B, \quad \omega - \omega_T \approx -\omega_B, \quad \omega - 2\omega_T \approx -3\omega_B, \quad (2.238)$$

we may approximate Eq. (2.237) according to

$$\frac{\beta\omega_B^2}{2}X(\omega) - [(\omega - \omega_T)^2 - \omega_B^2]X(\omega - \omega_T) \approx 0. \quad (2.239)$$

Eqs. (2.236) and (2.239) determine $X(\omega)$ and $X(\omega - \omega_T)$. In order to have non-trivial solutions to this system, we require the determinant of the system to vanish, i.e.,

$$\begin{vmatrix} \omega^2 - \omega_B^2 & -\beta\omega_B^2/2 \\ \beta\omega_B^2/2 & -[(\omega - \omega_T)^2 - \omega_B^2] \end{vmatrix} = 0 \quad (2.240)$$

Now we let $\omega_T = 2\omega_B$ and $\omega = \omega_B + i\gamma$, with $\gamma \ll \omega_B$. Then we have the approximate relations

$$\omega^2 - \omega_B^2 = (\omega - \omega_B)(\omega + \omega_B) \approx 2i\gamma\omega_B, \quad (2.241a)$$

and

$$(\omega - \omega_T)^2 - \omega_B^2 = (\omega - \omega_T - \omega_B)(\omega - \omega_T + \omega_B) \approx -2i\gamma\omega_B, \quad (2.241b)$$

and Eq. (2.240) simplifies to

$$\gamma = \frac{\beta\omega_B}{4}. \quad (2.242)$$

Since

$$\langle W_\parallel \rangle = \frac{mv_0^2}{2}\exp(2\gamma t), \quad (2.243)$$

the total energy will grow exponentially. Thus, collisions, here represented by γ, makes it possible to transfer gyration energy from the particles into $v_\parallel$.

Chapter 3

Basic models for theoretical description of plasmas

Up to now, we have considered only the electrodynamics of a single charged particle where the particle motion in the presence of specified electric and magnetic fields is governed by the equation of motion. We have, however, emphasised the collective nature of plasma behaviour. The dynamics of a plasma is governed by the interactions of the plasma particles with the internal fields produced by the particles themselves and with externally applied fields. As the charged particles in a plasma move around, they can generate local concentrations of positive or negative charge, which give rise to electric fields. Their motion can also generate electric currents and therefore magnetic fields. The dynamics of the particles in a plasma is adequately described by the laws of classical (non-quantum) mechanics.

3.1 General considerations on a self-consistent formulation

The interaction of charged particles with electromagnetic fields is governed by the Lorenz force. For a particle of charge q and mass m, moving with velocity $\mathbf{v}$ in the presence of electric, $\mathbf{E}$, and magnetic, $\mathbf{B}$, fields, the equation of motion is given by Eq. (2.2),

$$m\frac{d\mathbf{v}}{dt} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{3.1}$$

In principle, it is conceivable to describe the dynamics of a plasma by solving the equations of motion for each particle in the plasma under the combined influence of externally applied force fields and internal fields generated by all the other plasma particles. If the total number of particles is N, we will have N nonlinear coupled differential equations of motion to solve simultaneously. A *self-consistent* formulation must be used since the fields and the trajectories of the particles are coupled, i.e., the internal fields influence the particle motion which, in turn, modifies the internal fields.

The electromagnetic fields obey Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{3.2a}$$

$$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} + \frac{1}{c^2\epsilon_0}\mathbf{j}, \tag{3.2b}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{3.2c}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{3.2d}$$

where ρ, $\mathbf{j}$, ϵ_0 and c denote, respectively, the *total* charge density, the *total* current density, the electric permittivity and the speed of light. The plasma charge density and the plasma current density can be expressed, respectively, as

$$\rho_p = \frac{1}{\Delta V}\sum_i q_i, \tag{3.3a}$$

$$\mathbf{j}_p = \frac{1}{\Delta V}\sum_i q_i \mathbf{v}_i, \tag{3.3b}$$

where the summation is over all charged particles contained inside a suitably chosen small volume element ΔV. Note that since we are dealing with a discrete distribution of charges, ρ_p and $\mathbf{j}_p$ should actually be expressed in terms of Dirac delta functions.

Although this self-consistent approach is conceivable in principle, it cannot be carried out in practice. According to the laws of classical mechanics, in order to determine the position and velocity of each particle in the plasma as a function of time under action of known forces, it is necessary to know the initial position and velocity of each particle. For a system consisting of a very large number of interacting particles these initial conditions are obviously unknown. Furthermore, in order to explain and predict the macroscopic phenomena observed in nature and in the laboratory, it is not of interest to know the detailed individual motion of each particle, since the observable macroscopic properties of a plasma are due to the average collective behaviour of a large number of particles. To try to interpret the complex individual particle motions in terms of a few macroscopic observables would be a very inefficient way of making predictions.

3.2 Theoretical approaches

For the theoretical description of plasma phenomena there are basically four principal approaches with several different choices of approximations, which apply to different circumstances.

Particle orbit theory Using this approach, the motion of each charged particle in studied in the presence of specified electric and magnetic fields. It is not really plasma theory, but rather the dynamics of a charged particle in given fields. Nevertheless, it is important, since it provides physical insight, giving better understanding of the dynamical processes in plasmas.

Kinetic theory Since a plasma consists of a very large number of interacting particles, in order to provide a macroscopic description of plasma phenomena it is appropriate to adopt a *statistical approach.* This implies a great reduction in the amount of information to be handled. In the statistical description it is necessary to know only the distribution function for the system of particles. The problem consists of solving the appropriate kinetic equations, which govern the evolution of the distribution function in phase space.

Two-fluid (or many-fluid) theory In this approach each plasma species is treated as a fluid described by a local density, local macroscopic velocity and local temperature. The plasma is treated as a mixture of two or more interpenetrating fluids.

One-fluid (MHD) theory In the one-fluid theory, the whole plasma is treated as a single conducting fluid which is described by lumped macroscopic variables and their corresponding hydrodynamic conservation equations. An appropriately simplified form of this theory, applicable to the study of very low frequency phenomena in highly conducting fluids immersed in magnetic fields, is usually referred to as the magnetohydrodynamic (MHD) theory.

Particle orbit theory has been discussed in chapter 2, and we will now discuss the other approaches, starting with kinetic theory.

3.3 Kinetic description of plasma

A plasma consists of many ions and electrons, but the individual behaviour of each particle is hard to observe. What can be observed instead are statistical averages. In order to describe the properties of a plasma it is necessary to define a velocity distribution function that indicates particle density in the real space and the velocity space. In the following subsections, we introduce the macroscopic quantities defined as the velocity moments of the velocity distribution function and derive the basic equations governing the distribution function. Then, the basic two-fluid equations can be derived for a plasma composed of ions and electrons, and it is also shown that these equations may be reduced to the so-called magnetohydrodynamic plasma model if a plasma is viewed as a single-component fluid.

3.3.1 Distribution function

We introduce the concept of a kinetic equation as a description of the statistical behaviour of a system of identical interacting particles as it evolves in time. We define a *distribution function*, $f(\mathbf{r},\mathbf{v},t)$, by letting $f(\mathbf{r},\mathbf{v},t)\,d^3r\,d^3v$ be the probability of finding particles within the six-dimensional volume element $d^3r\,d^3v$, centred at the point $(\mathbf{r},\mathbf{v})$ in coordinate and velocity space at time t. Observable properties of the plasma can then be obtained from the distribution function by taking various *velocity moments* of f. By integrating the distribution function over the whole velocity space, we obtain the *particle density*,

$$n(\mathbf{r},t) = \int d^3v\, f(\mathbf{r},\mathbf{v},t). \tag{3.4a}$$

Similarly, other macroscopic quantities can be defined as

$$\text{Mean velocity} = \mathbf{u}(\mathbf{r},t) = \frac{1}{n}\int d^3v\,\mathbf{v}f(\mathbf{r},\mathbf{v},t), \tag{3.4b}$$

$$\text{Pressure tensor} = \mathsf{p}(\mathbf{r},t) = m\int d^3v\,(\mathbf{v}-\mathbf{u})(\mathbf{v}-\mathbf{u})f(\mathbf{r},\mathbf{v},t), \tag{3.4c}$$

$$\text{Energy density} = W(\mathbf{r},t) = \frac{m}{2}\frac{1}{n}\int d^3v\,|\mathbf{v}-\mathbf{u}|^2 f(\mathbf{r},\mathbf{v},t), \tag{3.4d}$$

$$\text{Heat flux vector} = \mathbf{Q} = \frac{m}{2}\int d^3v\,|\mathbf{v}-\mathbf{u}|^2(\mathbf{v}-\mathbf{u})f(\mathbf{r},\mathbf{v},t). \tag{3.4e}$$

The particle density corresponds to a zeroth-order moment (f is multiplied by $\mathbf{v}^0$) while the mean velocity is a first-order moment (f is multiplied by $\mathbf{v}^1$); pressure and energy density (or temperature) are second-order moment and the heat flux vector is an example of a third-order moment. It follows directly from Eq. (3.4a) that the total number of particles in the plasma at time t is

$$N(t) = \int d^3r\, n(\mathbf{r},t) = \iint d^3r\, d^3v\, f(\mathbf{r},\mathbf{v},t), \tag{3.5}$$

which is used to normalise the distribution function.

3.3.2 The Vlasov equation

The equation determining the distribution function is called the kinetic equation. The usual heuristic argument to obtain this equation treats the macroscopic forces (such as the applied fields) separately from the microscopic, collisional forces. Suppose first of all that particle interactions may be neglected. Then considering some given volume in $(\mathbf{r},\mathbf{v})$ space one argues that the number of particles in this volume will change in time and that the rate of change will be given by the flux of particles through the surface of the given volume. In coordinate $(\mathbf{r})$ space this flux is

$$\int f(\mathbf{r},\mathbf{v},t)\mathbf{v}\cdot\hat{n}_r\,dS_r, \tag{3.6}$$

where $\hat{n}_r$ is the unit vector perpendicular to the element of surface, dS_r, and the integral is taken over the whole surface in **r**-space; if $\hat{n}_r$ is directed outwards from the volume the integral represents a net outflow of particles. Similarly, the net flux of particles out of the volume in velocity (**v**) space is

$$\int f(\mathbf{r}, \mathbf{v}, t)\mathbf{a} \cdot \hat{n}_v \, dS_v, \tag{3.7}$$

where **a** is the acceleration experienced by particles in the velocity surface element, dS_v, with unit normal vector $\hat{n}_v$, the integral being over the whole surface in **v**-space. Hence

$$\frac{\partial}{\partial t} \int f(\mathbf{r}, \mathbf{v}, t) \, d^3r \, d^3v = -\int f(\mathbf{r}, \mathbf{v}, t)\mathbf{v} \cdot \hat{n}_r \, dS_r - \int f(\mathbf{r}, \mathbf{v}, t)\mathbf{a} \cdot \hat{n}_v \, dS_v. \tag{3.8}$$

Moving the time derivative inside the integral and using Gauss' theorem to convert the surface integrals into volume integrals, Eq. (3.8) may be written

$$\int d^3r \, d^3v \left[\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v} f) + \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a} f) \right] = 0. \tag{3.9}$$

Since the volume in $(\mathbf{r}, \mathbf{v})$ space is arbitrary, (3.9) implies that

$$\frac{\partial f}{\partial t} + \frac{\partial}{\partial \mathbf{r}} \cdot (\mathbf{v} f) + \frac{\partial}{\partial \mathbf{v}} \cdot (\mathbf{a} f) = 0. \tag{3.10}$$

Observing now that $\nabla \cdot \mathbf{v}$ vanishes since **v** and **r** are independent variables and using the explicit expression for **a** by assuming the Lorenz force

$$\mathbf{a} = \frac{\mathbf{F}}{m} = \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}), \tag{3.11}$$

we obtain

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{q}{m}(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \cdot \frac{\partial f}{\partial \mathbf{v}} = 0. \tag{3.12}$$

Eq. (3.12) is called the *Vlasov equation* or the *collisionless Boltzmann equation.* We note that it applies separately to each species of particles (electrons and ions) in the plasma.

Observe that the left hand side of (3.12) measures the change of f along a particle trajectory:

$$f[\mathbf{r}(t), \mathbf{v}(t), t] - f[\mathbf{r}(t - \Delta t), \mathbf{v}(t - \Delta t), t - \Delta t]. \tag{3.13}$$

In an exact description, our function $f(\mathbf{r}, \mathbf{v}, t)$ would become a collection of delta functions, one for each particle, and the acceleration, **a**, at any particle would (after deducting the self-field of the particle) be that due to all the other particles together. In this sense Eq. (3.12) is merely a restatement of the problem of calculating the orbits of particles. The statistical aspects of the problem are introduced when we replace f by a smooth (differentiable) function $\bar{f}$, where $\bar{f}$ may be regarded as meaningful only when applied to a volume of phase space big enough to contain a large number of particles. The question then arises whether Eq. (3.12) is satisfied by $\bar{f}$, and what value of $\mathbf{a}$ would be used. Clearly the simplest procedure is to replace $\mathbf{a}$ by an ensemble averaged value, $\bar{\mathbf{a}}$, and suppose that Eq. (3.12) then applies. We must, however, recognise that in applying the Vlasov equation to the "smoothed-out" distribution function and components of electromagnetic field, an approximation is being made. The force on any particle is assumed to be a continuous and slowly varying function of space, representing the effect of all other particles. This is true for most of the particles but there will be a few particles whose fields are varying rapidly in the region close to the particle under observation. Events involving such very close particles are of course collisions. Thus, the Vlasov equation neglects the particle collisions, and we may use it with good approximation only if the collective effect of the distant particles is much greater than any effect from collisions.

3.3.3 The Boltzmann equation

The rapidly fluctuating microscopic interactions which a particle suffers cannot be simply represented and this is why "collisions" are treated separately from the action of macroscopic forces. One formally writes

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \mathbf{a} \cdot \frac{\partial f}{\partial \mathbf{v}} = \left(\frac{\partial f}{\partial t} \right)_c , \tag{3.14}$$

where the overbars are suppressed, and $(\partial f / \partial t)_c$ is a so-called *collision operator* which represents the rate of change of f due to collisions. This is the collisional kinetic equation called the *Boltzmann equation*. There is no exact version of the collision operator, only a sequence of more accurate (and less tractable) forms. In general, the collision term is a complicated nonlinear function of f and it is the source of most of the mathematical difficulty in kinetic theory.

Fortunately, one may often neglect collisions completely. The mean time it takes for a particle to suffer an appreciable deflection due to collisions is called the *collision time*, τ_c, and its inverse is the *collision frequency*, ν_c. Thus a necessary condition for the neglect of collisions is $\nu_c \ll \omega$, where ω is a frequency which characterises the time rate of change of the macroscopic forces. This condition could also be written in terms of the *mean free path* (the average distance a particle travels between deflections), λ_c. If $\lambda_c \gg L$, where the length L characterises the distance over which the macroscopic fields vary, the collisions may be neglected and Eq. (3.14) reduces to the Vlasov equation. For the present we leave the collision operator unspecified.

3.3.4 The Maxwell distribution function

Let us consider a plasma in equilibrium (steady-state). For simplicity we concentrate on a gas consisting of only one particle species. In the absence of external forces, the equilibrium distribution function is the time-independent solution of the Boltzmann equation. Furthermore, in the equilibrium state the particle interactions do not cause any change in the distribution function with time, and there are no spatial gradients in the particle density, i.e., the distribution function is homogeneous ($\nabla f = 0$). Under these conditions there are no changes in the distribution function as a result of collisions between the particles. Hence

$$\left(\frac{\partial f}{\partial t}\right)_c = 0. \tag{3.15}$$

Consequently, we may conclude that the equilibrium distribution function should, in a certain way, provide a universal description of the equilibrium state, and it should be possible to derive it without explicitly considering the particle interactions. A derivation in these terms is in fact presented in statistical mechanics where it is shown that the so-called *Maxwell distribution function* represents the most probable distribution function satisfying the macroscopic conditions imposed on the system. The meaning of this is that if we choose, at random, any particular macroscopic state for the system, amongst all the possible states consistent with the macroscopic parameters (such as n, $\mathbf{u}$, T), the probability of choosing a Maxwellian distribution is much larger that any other distribution. Since the entropy of the system is proportional to the probability of having a given distribution, the state having maximum entropy is the most probable state consistent with the macroscopic constraints imposed on the system.

In view of the above discussion, we can base the derivation of the Maxwell distribution function on the entropy of a system which, according to thermodynamics, describes the order of the system, and can be expressed in terms of the distribution function as

$$S = -\int d^3v\, f \ln f. \tag{3.16}$$

Taking into account the following macroscopic constraints (see Eqs. (3.4))

$$\text{particle density:} \quad n = \int d^3v\, f = \text{const}, \tag{3.17a}$$

$$\text{mean particle velocity:} \quad \mathbf{u} = \frac{1}{n}\int d^3v\, \mathbf{v} f = 0, \tag{3.17b}$$

$$\text{mean particle energy:} \quad W = \frac{m}{2}\frac{1}{n}\int d^3v\, v^2 f = \text{const}, \tag{3.17c}$$

we look for a minimum of the functional consisting of a linear combination of the macroscopic quantities Eq. (3.16)–(3.17)

$$\mathscr{L} = \int d^3v\,(f \ln f + Af + Bv^2 f), \tag{3.18}$$

where A and B are constants. The conditions for having a minimum are that the first variation of $\mathscr{L}$ with respect to f is zero, and that the second variation of $\mathscr{L}$ is positive. Thus, we obtain from Eq. (3.18)

$$\delta\mathscr{L} = \int d^3v\,(1 + \ln f + A + Bv^2)\delta f = 0, \tag{3.19a}$$

and

$$\delta^2\mathscr{L} = \int d^3v\,\frac{1}{f}(\delta f)^2 > 0. \tag{3.19b}$$

Since Eq. (3.19a) should be satisfied for arbitrary δf, it follows that

$$\ln f = -(1 + A + Bv^2), \tag{3.20a}$$

i.e.,

$$f = f_M = k\exp(-Bv^2), \tag{3.20b}$$

where $k \equiv \exp(-1 - A)$. The condition (3.19b) is automatically satisfied because $f > 0$, which shows that the distribution function (3.20b) represents a minimum. The constants k and B are related to the particle density, n, and the plasma temperature, T, according to

$$n = \int d^3v\, f_M = k\left(\frac{2\pi}{B}\right)^{3/2}, \tag{3.21a}$$

$$\frac{3k_BT}{2} = \frac{1}{n}\int d^3v\,\frac{mv^2}{2} f_M = \frac{3m}{4B}, \tag{3.21b}$$

where k_B is the Boltzmann constant. Then the Maxwell distribution function becomes

$$f_M = n\left(\frac{m}{2\pi k_BT}\right)^{3/2}\exp\left(-\frac{mv^2}{2k_BT}\right). \tag{3.22}$$

3.3.5 The solution of the Vlasov equation in given fields—Jeans' theorem

The Vlasov equation (3.12) and the Maxwell equations (3.2) form a closed set of nonlinear integro-differential equations which with suitable boundary conditions in principle would describe any process of collision-free plasma dynamics. In practice though, the difficulties are formidable, and it is useful to consider first the more simplified problem of solving the Vlasov equation for one species when the electromagnetic field is given. Then we have to solve

$$\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + a_i \frac{\partial f}{\partial v_i} = 0, \tag{3.23}$$

with $\mathbf{a}(\mathbf{x}, \mathbf{v}, t)$ given since $\mathbf{E}(\mathbf{x}, t)$ and $\mathbf{B}(\mathbf{x}, t)$ are given. Here the suffixes denote Cartesian components and summation over repeated indices $i = 1, 2, 3$, is implied. Eq. (3.23) is a linear first-order homogeneous equation, which can be solved using the following method: We first solve the associated ordinary differential equations

$$\frac{dt}{1} = \frac{dx_1}{v_1} = \frac{dx_2}{v_2} = \frac{dx_3}{v_3} = \frac{dv_1}{a_1} = \frac{dv_2}{a_2} = \frac{dv_3}{a_3}, \tag{3.24}$$

to construct the characteristic curves $(\mathbf{x}, \mathbf{v}, t)$. But these equations are $d\mathbf{x}/dt = \mathbf{v}$, $d\mathbf{v}/dt = \mathbf{a}$, i.e., the equations for the particle orbit in given fields. Suppose that the general solution is

$$\mathbf{x} = \mathbf{x}(\alpha_j, t), \quad \mathbf{v} = \mathbf{v}(\alpha_j, t), \quad (j = 1, 2, \ldots, 6). \tag{3.25}$$

Then it must be possible to solve Eq. (3.25) in the form

$$\alpha_j = \alpha_j(\mathbf{x}, \mathbf{v}, t), \quad (j = 1, 2, \ldots, 6). \tag{3.26}$$

According to Lagrange's method, the general solution of the Vlasov equation (3.23) is

$$f(\mathbf{x}, \mathbf{v}, t) = H(\alpha_1, \alpha_2, \ldots, \alpha_6), \tag{3.27}$$

where H is an arbitrary function and the α_j's are to be eliminated by (3.26). (H must of course be positive everywhere and satisfy the necessary conditions at $\mathbf{v} \to \pm\infty$.)

To summarise, we may say that the orbits in phase space, expressed in terms of time as parameter, give the characteristic curves of the Vlasov equation for given electromagnetic fields, and the solutions of the equation are obtained by writing f as an arbitrary function of the invariants of the orbits. These results are called *Jeans' Theorem*.

Example: Look for solutions of the Vlasov equation in which the fields are

$$\mathbf{E}(\mathbf{x}) = -\nabla\phi(\mathbf{x}), \tag{3.28a}$$

$$\mathbf{B}(\mathbf{x}) = 0. \tag{3.28b}$$

The orbits have as an invariant the total energy

$$W = \frac{mv^2}{2} + e\phi. \tag{3.29}$$

The Vlasov equation is satisfied by taking f to be any positive function of W. Taking for example the solutions representing thermal equilibrium at temperature T, we obtain

$$\begin{aligned} f(\mathbf{x}, \mathbf{v}) &= \text{const} \times \exp\left(-\frac{W}{k_B T}\right) \\ &= \text{const} \times \exp\left(-\frac{mv^2}{2k_B T}\right) \exp\left(-\frac{e\phi}{k_B T}\right), \end{aligned} \tag{3.30}$$

which represents a Maxwellian distribution.

3.4 Two-fluid equations

It has been mentioned above that the kinetic description of a plasma is difficult to use, in particular for an inhomogeneous plasma where the full six-dimensional effects become important. There have been various attempts to derive a simplified set of equations to resolve the difficulty. One such treats the plasma as a composite of fluids (rather than a group of discrete charged particles) where the description is provided in terms of macroscopic variables such as particle density, mean velocity, kinetic pressure (temperature), and so on. These macroscopic variables are related to the moments of the distribution function. Consequently we can derive the fluid equations (so-called *moment equations*) for the macroscopic variables from the kinetic Boltzmann equation through multiplying by various functions of velocity, $\psi(\mathbf{v})$, and integrating over velocity space.

The moment $\langle\psi\rangle$ is defined as the velocity average of the function ψ,

$$\langle\psi\rangle \equiv \frac{\int d^3v\, \psi(\mathbf{v}) f}{\int d^3v\, f} = \frac{1}{n} \int d^3v\, \psi(\mathbf{v}) f. \tag{3.31}$$

The zeroth-order moment is obtained by setting $\psi = 1$. First-order and second-order moments correspond to $\psi = \mathbf{v}$ (or v_i) and $\psi = \mathbf{vv}$ (or $v_i v_j$), respectively. The general moment equation is obtained from the Boltzmann equation as

$$\int d^3v\, \psi(\mathbf{v}) \left[\frac{\partial f}{\partial t} + v_i \frac{\partial f}{\partial x_i} + a_i \frac{\partial f}{\partial v_i}\right] = \int d^3v\, \psi(\mathbf{v}) \left(\frac{\partial f}{\partial t}\right)_c. \tag{3.32}$$

We now want to perform this integration. The first term becomes

$$\int d^3v\, \psi(\mathbf{v}) \frac{\partial f}{\partial t} = \frac{\partial}{\partial t} \int d^3v\, \psi f = \frac{\partial}{\partial t}(n\langle\psi\rangle), \tag{3.33}$$

since ψ is independent of time. Similarly, since ψ is also independent of x_i, the second term equals

$$\int d^3v\, \psi(\mathbf{v}) v_i \frac{\partial f}{\partial x_i} = \frac{\partial}{\partial x_i} \int d^3v\, \psi(\mathbf{v}) v_i f = \frac{\partial}{\partial x_i}(n\langle\psi v_i\rangle) = \nabla \cdot (n\langle\psi \mathbf{v}\rangle). \tag{3.34}$$

For the third term (assuming a Lorentz force) we obtain, by partial integration,

$$\begin{aligned}\int d^3v\,\frac{q}{m}\psi(\mathbf{v})(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial f}{\partial\mathbf{v}} &= \frac{q}{m}\oint\psi f(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot d\mathbf{S}_v\\ &-\int d^3v\,\frac{q}{m}f\psi\frac{\partial}{\partial v_i}(\mathbf{E}+\mathbf{v}\times\mathbf{B})_i-\int d^3v\,\frac{q}{m}(\mathbf{E}+\mathbf{v}\times\mathbf{B})_i f\frac{\partial\psi}{\partial v_i}\\ &= -\frac{nq}{m}\left\langle(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial\psi}{\partial\mathbf{v}}\right\rangle.\end{aligned} \tag{3.35}$$

Here, the first term on the right-hand side becomes zero because we assume that $f\to 0$ sufficiently rapidly as $|\mathbf{v}|\to\infty$ (since, by Gauss' theorem, the integral is over the boundary surface in velocity space). The second term is zero because $(\mathbf{v}\times\mathbf{B})_i$ does not contain v_i.

Consequently, the general moment equation has the form

$$\frac{\partial}{\partial t}(n\langle\psi\rangle)+\nabla\cdot(n\langle\mathbf{v}\psi\rangle)-\frac{nq}{m}\left\langle(\mathbf{E}+\mathbf{v}\times\mathbf{B})\cdot\frac{\partial\psi}{\partial\mathbf{v}}\right\rangle=\frac{\partial}{\partial t}(n\langle\psi\rangle)_c. \tag{3.36}$$

The term on the right-hand side of Eq. (3.36) represents the change of the moment $\langle\psi\rangle$ due to collisions. Thus the kinetic equation for the distribution function f is replaced by a set of equations containing averaged quantities. It can be seen from (3.36) that each moment equation introduces the next higher-order velocity moment and some method of breaking the chain of equations and restricting the variables to a manageable number must be devised. We now want to write down equations for the most simple moments.

Zeroth order moment, $\psi(\mathbf{v})=1$

The third term of Eq. (3.36) vanishes in this case, as does the collision term, since in a collision the quantity 1 is not changed. Then we obtain the *continuity equation*

$$\frac{\partial n_\alpha}{\partial t}+\nabla\cdot(n_\alpha\mathbf{u}_\alpha)=0, \tag{3.37}$$

where $\mathbf{u}=\langle\mathbf{v}\rangle$, and we have added an index α to signify the particle species. In the simplest case, when the plasma consists of electrons and one kind of ions, there are two independent equations, one for the electron fluid and one for the ion fluid. Note that ionisation and recombination events have been neglected, and should be included as source terms on the right-hand side of Eq. (3.37) if they are important.

First order moment, $\psi(\mathbf{v})=\mathbf{v}$

In component form, Eq. (3.36) becomes

$$\frac{\partial}{\partial t}(nu_i)+\frac{\partial}{\partial x_j}(n\langle v_iv_j\rangle)-\frac{nq}{m}\langle[\mathbf{E}+\mathbf{v}\times\mathbf{B}]_i\rangle=\frac{\partial}{\partial t}(nu_i)_c. \tag{3.38}$$

The vector $\mathbf{v}$ is now split into the mean velocity, $\langle \mathbf{v} \rangle = \mathbf{u}$, and a random velocity, $\mathbf{w}$, i.e.,

$$\mathbf{v} = \mathbf{u} + \mathbf{w}, \quad \text{with } \langle \mathbf{w} \rangle = 0. \tag{3.39}$$

The second term of Eq. (3.38) then becomes

$$\frac{\partial}{\partial x_j}(n\langle v_i v_j \rangle) = \frac{\partial}{\partial x_j}\left[n(u_i u_j + \langle w_i w_j \rangle)\right]. \tag{3.40}$$

The expression $n\langle w_i w_j \rangle$ represents a component of the *pressure tensor*,

$$\mathbf{\Pi} \equiv nm\langle \mathbf{ww} \rangle. \tag{3.41}$$

Now, inserting Eq. (3.40) into Eq. (3.38), and using the continuity equation (3.37), we obtain

$$n_\alpha m_\alpha \left[\frac{\partial \mathbf{u}_\alpha}{\partial t} + (\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha\right] = n_\alpha q_\alpha (\mathbf{E} + \mathbf{u}_\alpha \times \mathbf{B}) - \nabla \cdot \mathbf{\Pi}_\alpha + \sum_{\beta \neq \alpha} \mathbf{P}_{\alpha\beta}, \tag{3.42}$$

where we have again added an index α to indicate the different species. The last term represents the effects on particle species α due to collisions with particle species $\beta \neq \alpha$. This is the *equation of motion* for a plasma fluid. It is noteworthy that using Eq. (3.37), the inertia term on the left-hand side of (3.42) can be written as

$$n_\alpha m_\alpha \left[\frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla\right] \mathbf{u}_\alpha = \frac{\partial}{\partial t}(n_\alpha m_\alpha \mathbf{u}_\alpha) + \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha), \tag{3.43}$$

where the first term represents the change in momentum density, while the second term represents the divergence of the momentum flow.

Another noteworthy feature of Eq. (3.42) is the treatment of the collision term. In a single collision the total momentum is conserved. As a consequence, the net momentum of the fluid is not changed in collisions between particles of the same species. However, in a collision with a particle of a different kind, momentum is transferred from one fluid to the next although the total momentum of all fluids is also conserved. Therefore, the collision term $mn\,(\partial(n\mathbf{u})/\partial t)_c$ was relabelled $\sum_{\alpha\neq\beta} \mathbf{P}_{\alpha\beta}$, indicating that there is no term $\mathbf{P}_{\alpha\alpha}$. From the total momentum conservation it follows that $\mathbf{P}_{\alpha\beta} = -\mathbf{P}_{\beta\alpha}$. For a plasma consisting of only electron- and one ion-fluid, the collision term $\sum_{\alpha\neq\beta} \mathbf{P}_{\alpha\beta}$ becomes proportional to the *resistivity* η

$$\sum_{\beta \neq \alpha} \mathbf{P}_{\alpha\beta} = -q_\alpha n_\alpha \eta \mathbf{j}, \tag{3.44}$$

where $\mathbf{j} = q_i n_i \mathbf{u}_i - e n_e \mathbf{u}_e$ is the current density.

The pressure tensor $\mathbf{\Pi}$ can be greatly simplified when the distribution function f is isotropic. We then obtain

$$\langle w_x w_x \rangle = \langle w_y w_y \rangle = \langle w_z w_z \rangle = \frac{\langle w^2 \rangle}{3}, \tag{3.45a}$$

$$\langle w_x w_y \rangle = \langle w_x w_z \rangle = \langle w_y w_z \rangle = 0. \tag{3.45b}$$

By definition the temperature is

$$\frac{3k_B T}{2} = \frac{m\langle w^2 \rangle}{2}, \tag{3.46}$$

where $\langle w^2 \rangle$ is related to the thermal velocity according to $\langle w^2 \rangle = 3v_{th}^2/2$. Therefore, the pressure tensor becomes

$$\Pi_{ij} = \frac{nm\langle w^2 \rangle \delta_{ij}}{3} = k_B T n \delta_{ij} \equiv p\delta_{ij}, \tag{3.47}$$

and the divergence of the pressure tensor in Eq. (3.41) becomes the gradient of the scalar pressure p,

$$(\nabla \cdot \mathbf{\Pi})_j = \frac{\partial}{\partial x_i}(p\delta_{ij}) = \frac{\partial p}{\partial x_j} = (\nabla p)_j. \tag{3.48}$$

Second order moment, $\psi(\mathbf{v}) = mv^2/2$

For simplicity, we assume an isotropic pressure tensor. Inserting $mv^2/2$ for ψ in Eq. (3.36), we find

$$\frac{\partial}{\partial t}\left(\frac{nm\langle v^2 \rangle}{2}\right) + \nabla \cdot \left(\frac{nm\langle \mathbf{v}v^2 \rangle}{2}\right) = nq\mathbf{E} \cdot \langle \mathbf{v} \rangle + \frac{\partial}{\partial t}\left(\frac{nm\langle v^2 \rangle}{2}\right)_c, \tag{3.49}$$

where the term containing the Lorentz force vanishes because $\mathbf{v} \times \mathbf{B}$ is perpendicular to $\mathbf{v}$. We again split $\mathbf{v}$ up into the mean velocity $\mathbf{u}$ and a random component $\mathbf{w}$. The first term in (3.49) can be written as

$$\frac{\partial}{\partial t}\left(\frac{nm\langle v^2 \rangle}{2}\right) = \frac{\partial}{\partial t}\left[\frac{nm\left(u^2 + \langle w^2 \rangle\right)}{2}\right] = \frac{\partial}{\partial t}\left(\frac{nmu^2}{2} + \frac{3nk_B T}{2}\right), \tag{3.50}$$

where we have used Eqs. (3.45) and (3.46). In a similar way the vector $\langle \mathbf{v}v^2 \rangle$, which appears in the second term in Eq. (3.49), can be written as

$$\langle \mathbf{v}v^2 \rangle = \mathbf{u}(u^2 + \langle w^2 \rangle) + \langle \mathbf{w}(2\mathbf{u} \cdot \mathbf{w} + w^2) \rangle. \tag{3.51}$$

Here we have

$$\langle 2w_i u_j w_j \rangle = \frac{2}{nm}\Pi_{ij} u_j, \tag{3.52}$$

and the term $\langle \mathbf{w}w^2 \rangle$ represents the *heat conduction vector*, $\mathbf{Q}$, (see Eqs. (3.4)

$$\mathbf{Q} = \frac{nm\langle \mathbf{w}w^2 \rangle}{2}. \tag{3.53}$$

With these definitions we obtain from Eq. (3.49)

$$\begin{aligned}\frac{\partial}{\partial t}\left(\frac{n_\alpha m_\alpha u_\alpha^2}{2} + \frac{3n_\alpha k_B T_\alpha}{2}\right) + \nabla \cdot \left(\frac{n_\alpha m_\alpha u_\alpha^2 \mathbf{u}_\alpha}{2} + \frac{3n_\alpha k_B T_\alpha \mathbf{u}_\alpha}{2}\right)\\ +\nabla \cdot (\mathbf{\Pi}_\alpha \cdot \mathbf{u}_\alpha) = - \nabla \cdot \mathbf{Q}_\alpha + n_\alpha q_\alpha \mathbf{E} \cdot \mathbf{u}_\alpha\\ + \sum_{\beta \neq \alpha} \mathbf{P}_{\alpha\beta} \cdot \mathbf{u}_\alpha + \sum_{\beta \neq \alpha} \Omega_{\alpha\beta},\end{aligned} \tag{3.54}$$

which represents the *energy conservation equation*. The first term represents the temporal evolution of the kinetic and thermal energy of the fluid. The second term is the divergence of the flow of kinetic and thermal energy. The third term represents the work done by expansion of a volume that moves with the fluid. These three terms are balanced on the right-hand side by the divergence of the heat conduction vector, the energy gained from the electric field and the energy gain or loss from the interaction with other fluids. Note that collisions of the same kind do not change the net energy of the fluid and, therefore, do not give a contribution. The interaction with other fluids has been split up into two parts. The term $\sum_{\alpha \neq \beta} \mathbf{P}_{\alpha\beta} \cdot \mathbf{u}_\alpha$ takes account of the fact that there are frictional forces acting between the fluids, while the term $\Omega_{\alpha\beta}$ takes care of heat exchange between the fluids. Using now the continuity and the momentum equations, Eq. (3.54) can be rewritten as

$$\frac{\partial}{\partial t}\left(\frac{3n_\alpha k_B T_\alpha}{2}\right) + \nabla \cdot \left(\frac{5n_\alpha k_B T_\alpha \mathbf{u}_\alpha}{2}\right) - \mathbf{u}_\alpha \cdot \nabla p_\alpha = -\nabla \cdot \mathbf{Q}_\alpha + \sum_{\beta \neq \alpha} \Omega_{\alpha\beta}, \tag{3.55a}$$

or, after a slight rearrangement.

$$\frac{3}{2}\left(\frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla\right) p_\alpha + \frac{5p_\alpha}{2} \nabla \cdot \mathbf{u}_\alpha = -\nabla \cdot \mathbf{Q}_\alpha + \sum_{\beta \neq \alpha} \Omega_{\alpha\beta}. \tag{3.55b}$$

This form of the energy equation expresses the fact that an adiabatic law holds as one moves with the fluid, provided that the right-hand side of Eq. (3.55b) can be neglected. Taking $\mathbf{Q}_\alpha = 0$, $\Omega_{\alpha\beta} = 0$, and using the continuity equation, we obtain from (3.55b)

$$\frac{d}{dt}\left(\frac{p_\alpha}{n_\alpha^{5/3}}\right) \equiv \left(\frac{\partial}{\partial t} + \mathbf{u}_\alpha \cdot \nabla\right)\left(\frac{p_\alpha}{n_\alpha^{5/3}}\right) = 0, \tag{3.56}$$

i.e.,

$$\frac{p_\alpha}{n_\alpha^{5/3}} = \text{const}. \tag{3.57}$$

In general, one obtains

$$\frac{p_\alpha}{n_\alpha^\gamma} = \text{const}, \tag{3.58}$$

which holds for the adiabatic compression. The quantity γ, i.e., the ratio of the specific heat at constant pressure to that at constant temperature, is given by

$$\gamma = \frac{\alpha + 2}{\alpha}, \tag{3.59}$$

where α is the number of degrees of freedom. Since we assumed that the pressure remains isotropic, it follows that $\alpha = 3$ and $\gamma = 5/3$.

3.5 One-fluid equations

The fact that a plasma is composed of ions and electrons has led us to describe the plasma as two fluids; an electron and an ion fluid. Another model, which we shall describe in this section, is the so-called one-fluid model, also referred to as the magnetohydrodynamic (MHD) model. In this model the plasma is described as one conducting medium which is characterised by the parameters like the mass density ρ_m, the charge density ρ, the centre-of-mass velocity $\mathbf{V}$, and the electric current density $\mathbf{j}$. These parameters are related to the two-fluid parameters by the relations

$$\rho_m \equiv \sum_\alpha n_\alpha m_\alpha = m_i n_i + m_e n_e, \tag{3.60a}$$

$$\rho \equiv \sum_\alpha q_\alpha n_\alpha = e(Z_i n_i - n_e), \tag{3.60b}$$

$$\mathbf{V} \equiv \frac{1}{\rho_m} \sum_\alpha m_\alpha n_\alpha \mathbf{u}_\alpha = \frac{m_i n_i \mathbf{u}_i + m_e n_e \mathbf{u}_e}{m_i n_i + m_e n_e}, \tag{3.60c}$$

$$\mathbf{j} \equiv \sum_\alpha q_\alpha n_\alpha \mathbf{u}_\alpha = e(Z_i n_i \mathbf{u}_i - n_e \mathbf{u}_e), \tag{3.60d}$$

where Z_i is the ion charge number, m_i and m_e are the ion and electron masses, respectively, and we have restricted ourselves to only one species of ions. The charge density, ρ, is often small and can be neglected in situations where quasi-neutrality holds, i.e., $n_e \approx n_i Z_i$.

The first one-fluid equation is derived by multiplying the continuity equation for electrons and ions by the electron and ion masses, respectively. Adding the resulting equations gives

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{V}) = 0, \tag{3.61}$$

which reflects the conservation of mass. Similarly, we can multiply the continuity equations by the charge of the respective particles, and adding the resulting equations, we obtain

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{j} = 0, \tag{3.62}$$

where the first term can be neglected. Eq. (3.62) ensures the conservation of charge.

Let us now go on to the momentum equations, Eq. (3.42), for electrons and ions, where we consider anisotropic pressure tensors. Thus, we consider

$$\begin{aligned}
&\frac{\partial}{\partial t}(n_e m_e \mathbf{u}_e) + \nabla \cdot (n_e m_e \mathbf{u}_e \mathbf{u}_e) = \\
&\qquad - n_e e(\mathbf{E} + \mathbf{u}_e \times \mathbf{B}) - \nabla \cdot \mathbf{\Pi}_e + \mathbf{P}_{ei}, && \text{(3.63a)}\\
&\frac{\partial}{\partial t}(n_i m_i \mathbf{u}_i) + \nabla \cdot (n_i m_i \mathbf{u}_i \mathbf{u}_i) = \\
&\qquad n_i Z_i e(\mathbf{E} + \mathbf{u}_i \times \mathbf{B}) - \nabla \cdot \mathbf{\Pi}_i + \mathbf{P}_{ie}. && \text{(3.63b)}
\end{aligned}$$

Adding Eqs. (3.63a) and (3.63b) and assuming quasi-neutrality (i.e., $n_e \approx Z_i n_i$), we obtain

$$\frac{\partial}{\partial t}(\rho_m \mathbf{V}) + \nabla \cdot (n_e m_e \mathbf{u}_e \mathbf{u}_e + n_i m_i \mathbf{u}_i \mathbf{u}_i) = \mathbf{j} \times \mathbf{B} - \nabla \cdot (\mathbf{\Pi}_e + \mathbf{\Pi}_i), \tag{3.64}$$

where we have also used the fact that $\mathbf{P}_{ei} + \mathbf{P}_{ie} = 0$. A difficulty arises here how one should treat the pressure tensors. Previously, the electron and ion pressure tensors were defined in a coordinate system which moved with the electron and the ion fluid, respectively, and these coordinate systems loose their meaning in the one-fluid model. Thus, we have to redefine the pressure tensors in the centre-of-mass coordinate system. For this purpose the velocity $\mathbf{v}_\alpha$ is split up into the centre-of-mass velocity $\mathbf{V}$ and the particle velocity $\mathbf{w}_\alpha^*$ in the centre-of-mass frame

$$\mathbf{v}_\alpha = \mathbf{V} + \mathbf{w}_\alpha^* = \mathbf{u}_\alpha + \mathbf{w}_\alpha. \tag{3.65}$$

In contrast to $\mathbf{w}_\alpha$, the velocity $\mathbf{w}_\alpha^*$ is not random, i.e., $\langle \mathbf{w}_\alpha^* \rangle \neq 0$. However, because of (3.65) we can write

$$\sum_\alpha \langle n_\alpha m_\alpha \mathbf{v}_\alpha \rangle = \sum_\alpha \langle n_\alpha m_\alpha \mathbf{V} \rangle + \sum_\alpha \langle n_\alpha m_\alpha \mathbf{w}_\alpha^* \rangle = \rho_m \mathbf{V} + \sum_\alpha \langle n_\alpha m_\alpha \mathbf{w}_\alpha^* \rangle \tag{3.66a}$$

and

$$\sum_\alpha \langle n_\alpha m_\alpha \mathbf{v}_\alpha \rangle = \sum_\alpha n_\alpha m_\alpha \mathbf{u}_\alpha = \rho_m \mathbf{V}, \tag{3.66b}$$

i.e.,

$$\sum_\alpha \langle n_\alpha m_\alpha \mathbf{w}_\alpha^* \rangle = 0. \tag{3.67}$$

We now define the pressure tensors in the centre-of-mass system as

$$\mathbf{\Pi}^*_\alpha \equiv n_\alpha m_\alpha \langle \mathbf{w}^*_\alpha \mathbf{w}^*_\alpha \rangle, \tag{3.68}$$

and the total pressure tensor

$$\mathbf{\Pi}^* = \sum_\alpha \mathbf{\Pi}^*_\alpha = \sum_\alpha n_\alpha m_\alpha \langle \mathbf{w}^*_\alpha \mathbf{w}^*_\alpha \rangle. \tag{3.69}$$

Using $\mathbf{w}^*_\alpha = \mathbf{u}_\alpha + \mathbf{w}_\alpha - \mathbf{V}$ we obtain

$$\mathbf{\Pi}^* = \sum_\alpha n_\alpha m_\alpha (\mathbf{u}_\alpha \mathbf{u}_\alpha - \mathbf{u}_\alpha \mathbf{V} - \mathbf{V}\mathbf{u}_\alpha + \mathbf{V}\mathbf{V} + \langle \mathbf{w}_\alpha \mathbf{w}_\alpha \rangle). \tag{3.70}$$

Since $\sum_\alpha n_\alpha m_\alpha \mathbf{u}_\alpha = \rho_m \mathbf{V}$, we find

$$\mathbf{\Pi}^* = \sum_\alpha \mathbf{\Pi}_\alpha + \sum_\alpha n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha - \rho_m \mathbf{V}\mathbf{V}. \tag{3.71}$$

Substituting (3.71) into Eq. (3.64) yields

$$\begin{aligned} &\frac{\partial}{\partial t}(\rho_m \mathbf{V}) + \nabla \cdot \left(\sum_\alpha n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha \right) \\ &\qquad = \mathbf{j} \times \mathbf{B} - \nabla \cdot \mathbf{\Pi}^* + \sum_\alpha \nabla \cdot (n_\alpha m_\alpha \mathbf{u}_\alpha \mathbf{u}_\alpha) - \nabla \cdot (\rho_m \mathbf{V}\mathbf{V}), \end{aligned} \tag{3.72}$$

i.e.,

$$\rho_m \left(\frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \right) \mathbf{V} = \mathbf{j} \times \mathbf{B} - \nabla \cdot \mathbf{\Pi}^*, \tag{3.73}$$

where we have used the continuity equation (3.61). This is the well known equation of motion. There must exist two independent equations. The other equation is obtained by dividing the momentum equation, Eq. (3.63a), for the electrons by en_e and rearranging it slightly

$$\mathbf{E} + \mathbf{u}_e \times \mathbf{B} = \eta \mathbf{j} - \frac{1}{en_e} \nabla \cdot \mathbf{\Pi}_e - \frac{m_e}{e} \frac{d\mathbf{u}_e}{dt}, \tag{3.74}$$

where we have set

$$\frac{\mathbf{P}_{ei}}{en_e} = \eta \mathbf{j}. \tag{3.75}$$

Now we systematically replace $\mathbf{u}_e$ by $\mathbf{j}$, and $\mathbf{V}$ and $\mathbf{\Pi}_e$ by $\mathbf{\Pi}^*_e$. From Eqs. (3.60c) and (3.60d) we obtain

$$\mathbf{u}_i = \mathbf{V} + \frac{m_e}{e\rho_m} \mathbf{j}, \tag{3.76a}$$

$$\mathbf{u}_e = \mathbf{V} - \frac{m_i}{Z_i e \rho_m} \mathbf{j}, \tag{3.76b}$$

so we get

$$\mathbf{u}_e \times \mathbf{B} = \mathbf{V} \times \mathbf{B} - \frac{m_i}{Z_i e \rho_m} \mathbf{j} \times \mathbf{B}. \tag{3.77}$$

Furthermore

$$\nabla \cdot \mathbf{\Pi}_e + m_e n_e \frac{d\mathbf{u}_e}{dt} = \nabla \cdot \mathbf{\Pi}_e^* - \nabla \cdot [m_e n_e(\mathbf{u}_e\mathbf{u}_e - \mathbf{u}_e\mathbf{V} - \mathbf{V}\mathbf{u}_e + \mathbf{V}\mathbf{V})] + \frac{\partial}{\partial t}(n_e m_e \mathbf{u}_e) + \nabla \cdot (n_e m_e \mathbf{u}_e \mathbf{u}_e). \tag{3.78}$$

The second term on the right hand side of Eq. (3.78) and the last term cancel. Replacing $\mathbf{u}_e$ with the help of Eq. (3.76b), we get

$$\nabla \cdot \mathbf{\Pi}_e + m_e n_e \frac{d\mathbf{u}_e}{dt} = \nabla \cdot \mathbf{\Pi}_e^* + \frac{m_e}{m_e + m_i/Z_i} \left[\frac{\partial}{\partial t}(\rho_m \mathbf{V}) + \nabla \cdot (\rho_m \mathbf{V}\mathbf{V}) \right] - \frac{1}{e} \frac{m_e m_i}{m_i + Z_i m_e} \left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{j}\mathbf{V} + \mathbf{V}\mathbf{j}) \right]. \tag{3.79}$$

Using the continuity equation (3.61) and the equation of motion (3.73) we obtain

$$\nabla \cdot \mathbf{\Pi}_e + m_e n_e \frac{d\mathbf{u}_e}{dt} = \nabla \cdot \mathbf{\Pi}_e^* - \frac{m_e}{m_e + m_i/Z_i} \nabla \cdot \mathbf{\Pi}^* + \frac{m_e}{m_e + m_i/Z_i} \mathbf{j} \times \mathbf{B} - \frac{1}{e} \frac{m_e m_i}{m_i + Z_i m_e} \left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{j}\mathbf{V} + \mathbf{V}\mathbf{j}) \right]. \tag{3.80}$$

which inserted into (3.74) gives

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{j} \frac{1}{e Z_i \rho_m} \left[m_i \nabla \cdot \mathbf{\Pi}_e^* - m_e Z_i \nabla \cdot \mathbf{\Pi}_i^* - (m_i - Z_i m) \mathbf{j} \times \mathbf{B} \right] + \frac{m_e m_i}{e^2 Z_i \rho_m} \left[\frac{\partial \mathbf{j}}{\partial t} + \nabla \cdot (\mathbf{j}\mathbf{V} + \mathbf{V}\mathbf{j}) \right]. \tag{3.81}$$

This is called the *generalised Ohm's law*. The terms in the first square brackets are called finite-Larmor radius terms, or FLR terms. They consist of electron pressure term (the ion pressure term is usually much smaller and can be neglected) and of the Hall term which is usually approximated by $(\mathbf{j} \times \mathbf{B})/(enZ_i)$. To consider the FLR terms in (3.81) we assume that all terms in the equation of motion are of the same order, i.e.,

$$\mathcal{O}(|\rho_m(\mathbf{V} \cdot \nabla)\mathbf{V}|) = \mathcal{O}(|\nabla p|) = \mathcal{O}(|\mathbf{j} \times \mathbf{B}|). \tag{3.82}$$

This means that

$$\mathcal{O}(V^2) \sim V_s^2 = \frac{k_B T_e}{m_i}. \tag{3.83}$$

Then we can compare for instance the size of $\mathbf{V} \times \mathbf{B}$ with the size of one of the FLR terms (the electron pressure term):

$$\mathcal{O}\left(\frac{|\nabla p/en|}{|\nabla \times \mathbf{B}|}\right) = \mathcal{O}\left(\frac{k_B T_e N/neL}{V_s B}\right) \mathcal{O}\left(\frac{m_i V_s}{eBL}\right) = \mathcal{O}\left(\frac{r_{L,i}}{L}\right). \tag{3.84}$$

Here L is the characteristic length of the problem and r_L^+ is the ion-Larmor radius (see Eq. (2.15)). Consequently, the FLR term can be neglected if $r_{L,i} \ll L$.

The last two terms in (3.81) are called inertia terms. Because of the smallness of the electron mass, they are typically much smaller than the FLR terms. If τ is the characteristic time for a phenomenon, then a comparison of the inertia term with an FLR term yields

$$\mathcal{O}\left(\frac{|m_e \mathbf{j}/e^2 n\tau|}{|\nabla p/en|}\right) = \mathcal{O}\left(\frac{1}{\omega_{c,e}\tau}\right), \tag{3.85}$$

where $\omega_{c,e}$ is the electron cyclotron frequency (see Eq. (2.12)). Consequently, for low frequency phenomena, $\omega \ll \omega_{c,e}$, the inertia terms can be neglected.

The most simple form of the Ohm's law is

$$\mathbf{E} + \mathbf{V} \times \mathbf{B} = \eta \mathbf{j}. \tag{3.86}$$

In very hot plasmas, when collisions can be neglected, even the resistive term in (3.86) is sometimes dropped.

Chapter 4

Waves in plasmas

It is well known, that in vacuum as well as in gases and in other media (even solids), electromagnetic fields can propagate in the form of waves. It is obvious that electromagnetic wave phenomena may also appear in plasmas. The motion of charged particles will give rise to electromagnetic fields and the field will give rise to particle motion. In fact due to the presence of many different characteristic oscillation frequencies in a plasma, there is a plethora of possible wave phenomena. In fact, the subject of waves in plasmas is so rich that it would easily motivate a course of its own. The purpose of the present chapter is just to give some illustrative examples of waves that occur in plasmas. We will start the analysis by quickly reviewing two basic wave concepts, the phase and group velocities, and then enter a short study of some fundamental wave types.

4.1 Phase and group velocity

Let us consider a wave packet with complex amplitude $E(x,t)$, which constitutes a superposition of many monochromatic waves as expressed by the Fourier representation

$$E(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k) \exp\{i[kx - \omega(k)t]\}\, dk, \tag{4.1}$$

where $\omega = \omega(k)$ is the dispersion relation for the medium where the wave propagates. The spectrum, $a(k)$, of the wave is determined by the initial condition at $t = 0$, since

$$E(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k) \exp(ikx)\, dk. \tag{4.2}$$

If we choose as initial condition a purely monochromatic wave with amplitude A, and only one wave number k_0, i.e., $E(x,0) = A\exp(ik_0x)$, the

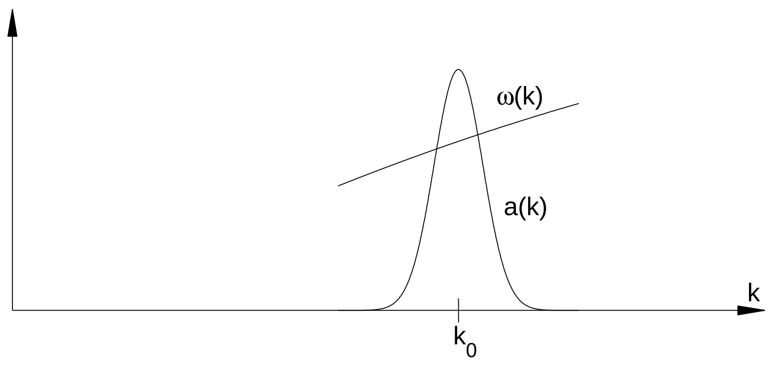


Figure 4.1. Plot showing the spectrum of the initial wave form, centred at the wave number k_0, and the variation of the dispersion relation around k_0.

spectrum will be a delta function of the form $a(k) = \sqrt{2\pi}A\delta(k - k_0)$ and the corresponding propagating wave is given by

$$E(x,t) = A\exp\{i[k_0x - \omega(k_0)t]\}. \tag{4.3}$$

The phase, $\varphi(x,t) = k_0x - \omega(k_0)t$, of the monochromatic wave is constant provided

$$\frac{d\varphi}{dt} = 0 \quad \Rightarrow \quad \frac{dx}{dt} = \frac{\omega(k_0)}{k_0} \equiv v_{ph}. \tag{4.4}$$

This characteristic velocity, along which the phase remains constant, is the *phase velocity*.

In order to illustrate the concept of group velocity, we consider a wave that is not purely monochromatic, i.e., a wave pulse with a slow amplitude modulation on a rapid oscillation. In this case, the spectrum, $a(k)$, is no longer a delta function. However, since we have assumed the superimposed amplitude variation to be slow as compared with the rapid oscillations of the carrier wave, the spectrum must be strongly localised around the carrier wave number k_0, see Fig. 4.1. It is then appropriate to expand the dispersion relation in the Fourier integral around k_0 as follows ($d\omega/dk_0$ denotes the derivative evaluated for the wave number k_0)

$$\begin{aligned}
E(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k) \exp\{i[kx - \omega(k)t]\}\, dk \approx \\
\approx \frac{1}{\sqrt{2\pi}} \exp\{i[k_0 x - \omega(k_0)t]\} \\
\times \int_{-\infty}^{\infty} a(k) \exp\left\{ i \left[(k-k_0)x - \frac{d\omega}{dk_0}(k-k_0)t \right] \right\} dk \\
= \frac{1}{\sqrt{2\pi}} \exp\{i[k_0 x - \omega(k_0)t]\} \\
\times \int_{-\infty}^{\infty} a(k) \exp\left\{ i \left[\left(x - \frac{d\omega}{dk_0} t \right) (k-k_0) \right] \right\} dk.
\end{aligned} \tag{4.5}$$

The original ($t = 0$) pulse amplitude is given by

$$\begin{aligned}
E(x,0) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k) \exp(ikx)\, dk \\
&= \frac{1}{\sqrt{2\pi}} \exp(ik_0 x) \int_{-\infty}^{\infty} a(k) \exp[i(k-k_0)x]\, dk \\
&\equiv A(x) \exp(ik_0 x).
\end{aligned} \tag{4.6}$$

where $A(x)$ denotes the initial *slowly varying envelope* of the pulse, which can be written as

$$A(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k) \exp[i(k-k_0)x]\, dk. \tag{4.7}$$

However, this also implies that

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} a(k) \exp\left[i(k-k_0) \left(x - \frac{d\omega}{dk_0} t \right) \right] dk = A\left(x - \frac{d\omega}{dk_0} t \right), \tag{4.8}$$

and we obtain for the total wave field

$$E(x,t) \approx A\left(x - \frac{d\omega}{dk_0} t \right) \exp[i(k_0 x - \omega_0 t)]. \tag{4.9}$$

From this result, it is clear that the envelope is constant when

$$\frac{dx}{dt} = \frac{d\omega}{dk_0} \equiv v_g, \tag{4.10}$$

where v_g is the so-called *group velocity*. The relation given by Eq. (4.10) expresses the fact that the envelope of the wave does not change form, it is just translated in time towards larger times. Thus the velocity of translation, or propagation velocity, of the pulse is the group velocity, which determines the propagation velocity of the "group" of wave numbers around the carrier wave number k_0.

In the above derivation of the group velocity we found that no distortion of the pulse envelope occurred. This is due to the fact that the Taylor expansion was truncated after the term linear in k. By including more terms, additional phenomena will be described. We emphasise that the present expansion of the dispersion relation, used in deriving the above result, is a legitimate procedure in two cases (see Fig. 4.1):

(i) the spectrum $a(k)$ is very narrow, and/or

(ii) the dispersion relation $\omega = \omega(k)$ is almost flat in the region around the carrier wave number k_0.

4.2 Basic equations for plasma waves in the hydrodynamic description

The basic equations on which our study will be based are the hydromagnetic plasma equations which consist of the equations of continuity, motion and state for the different plasma species according to ($\alpha = e, i$, denoting electrons and ions, respectively)

1. Equations of continuity:

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \mathbf{v}_\alpha) = 0. \tag{4.11}$$

2. Equations of motion:

$$n_\alpha m_\alpha \left[\frac{\partial \mathbf{v}_\alpha}{\partial t} + (\mathbf{v}_\alpha \cdot \nabla)\mathbf{v}_\alpha \right] = q_\alpha n_\alpha (\mathbf{E} + \mathbf{v}_\alpha \times \mathbf{B}) - \nabla p_\alpha. \tag{4.12}$$

3. Equations of state:

$$p_\alpha n_\alpha^{-\gamma_\alpha} = \text{const.} \tag{4.13}$$

This set of plasma equation is coupled to the electric, $\mathbf{E}$, and magnetic, $\mathbf{B}$, fields through Maxwell's equations

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4.14a}$$

$$\nabla \times \mathbf{B} = \mu_0 \left(\mathbf{j} + \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \right), \tag{4.14b}$$

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0}, \tag{4.14c}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{4.14d}$$

where the current, $\mathbf{j}$, and the charge density, ρ, are determined by

$$\mathbf{j} = q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i, \quad \rho = q_e n_e + q_i n_i. \tag{4.15}$$

In the present analysis of wave propagation as determined by Eqs. (4.11)–(4.15), we consider the waves as small disturbances of a stationary and homogeneous equilibrium background plasma. This implies that we can use linearisation of the system of equations in order to simplify the analysis.

Some general properties and concepts, which are often used in plasma wave theory are the following; a wave is said to be "longitudinal" if the propagation vector $\mathbf{k}$ is parallel to the electric field, i.e., $\mathbf{k} \parallel \mathbf{E}$ and "transverse" if these vectors are orthogonal, i.e., $\mathbf{k} \perp \mathbf{E}$. Note that a wave does not necessarily belong to one or the other; wave types are possible which are neither longitudinal nor transverse. It is also common to classify a wave as "electrostatic" or "electromagnetic", depending on whether the wave has an associated oscillating magnetic wave field. A plasma, in particular in the presence of a magnetic field supports the propagation of a large number of different waves. In the following presentation we will give only some illustrative examples.

4.3 Electrostatic plasma waves

We start our survey of plasma waves by considering two examples of waves where the wave vector and the wave electric field are parallel, i.e., $\mathbf{k} \parallel \mathbf{E}$. We directly note that the property of being longitudinal also implies that the wave must be electrostatic, i.e., the wave does not contain an oscillating $\mathbf{B}$-field (since $\mathbf{k} \times \mathbf{E} = 0$). Furthermore, we will consider two limits, where the waves can be assumed to have either a "high" or a "low" frequency, in which cases the plasma particle dynamics can be greatly simplified.

4.3.1 Longitudinal electron plasma waves

Since the electrons and the ions in a plasma have very different masses, they also have very different natural oscillation frequencies. Consider a situation where the wave frequency is so high that only the electrons are able to respond to the oscillating electric field. In this case the ions only experience the averaged electric field and no perturbations of the ion parameters will occur; the ions only form a stationary neutralising background. The dynamics of the plasma wave is then determined by the electron fluid equations together with Maxwell's equations. The electron equations are

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e \mathbf{v}_e) = 0, \tag{4.16a}$$

$$m_e \left[\frac{\partial \mathbf{v}_e}{\partial t} + (\mathbf{v}_e \cdot \nabla) \mathbf{v}_e \right] = -e\mathbf{E} - \frac{\nabla p_e}{n_e}, \tag{4.16b}$$

$$p_e n_e^{-\gamma_e} = \text{const.} \tag{4.16c}$$

For rapid time variations, the proper value of γ_e is $\gamma_e = 3$, and we can write the gradient of the electron pressure as $\nabla p_e = 3 p_e \nabla n_e$. Without loss of generality we can assume that the wave propagates in the x-direction and that the electric field and the electron fluid velocity consequently can be written $\mathbf{E} = E\hat{x}$ and $\mathbf{v}_e = v_e\hat{x}$, where $\hat{x}$ denotes the unit vector in the x-direction. Clearly the system then contains three equations for four unknowns and we need one of Maxwell's equations to close the system. Poisson's equation provides a direct relation between the electron density and the electric field, i.e., as our fourth equation we use

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\epsilon_0} = \frac{e(n_0 - n_e)}{\epsilon_0}, \tag{4.17}$$

where n_0 denotes the background density of the ions.

The four equations, Eqs. (4.16) and Eq. (4.17), now become scalar ones and can be written as

$$\frac{\partial n_e}{\partial t} + \frac{\partial (n_e v_e)}{\partial x} = 0, \tag{4.18a}$$

$$m_e \left(\frac{\partial v_e}{\partial t} + v_e \frac{\partial v_e}{\partial x} \right) = -eE - \frac{1}{n_e} \frac{\partial p_e}{\partial x}, \tag{4.18b}$$

$$\frac{\partial E}{\partial x} = \frac{e}{\epsilon_0}(n_0 - n_e). \tag{4.18c}$$

The propagating wave is considered as a small perturbation on the background solution $n_e = n_0$, $\mathbf{v}_e = 0$, $\mathbf{E} = 0$, $p_e = p_0 = n_0 k_B T_e$ and we write

$$\mathbf{v}_e = \mathbf{v}_1, \quad n_e = n_0 + n_1, \quad \mathbf{E} = \mathbf{E}_1, \tag{4.19}$$

where index 1 denotes the small disturbances associated with the wave field.

An important step in the wave analysis is when the system given by Eq. (4.18) is linearised around the background solution. Inserting Eq. (4.19) in Eq. (4.18), we can simplify the resulting equation by discarding all terms which are quadratic in the small wave perturbations. This yields

$$\frac{\partial n_1}{\partial t} + n_0 \frac{\partial v_1}{\partial x} = 0, \tag{4.20a}$$

$$m_e \frac{\partial v_1}{\partial t} = -eE_1 + \frac{3k_B T_e}{n_0} \frac{\partial n_1}{\partial x}, \tag{4.20b}$$

$$\frac{\partial E_1}{\partial x} = \frac{e}{\epsilon_0} n_1. \tag{4.20c}$$

We now look for monochromatic wave solutions, i.e., we assume that all wave quantities vary as $X \sim \exp[i(kx - \omega t)]$, where X stands for n_1, v_1 and E_1. The system of differential equations (4.20) is then transformed into an algebraic system of equations

$$-i\omega n_1 + ikn_0 v_1 = 0, \tag{4.21a}$$

$$-im_e \omega v_1 = -eE_1 + \frac{3k_B T_e}{n_0} ikn_1, \tag{4.21b}$$

$$ikE_1 = -\frac{e}{\epsilon_0} n_1. \tag{4.21c}$$

Since this system is homogeneous and has constant coefficients, nontrivial solutions only exist provided the determinant of the system vanishes, i.e., there is a definite relation between the wave frequency, ω, and the wave number, k. Instead of forming the system determinant we can express v_1 in terms of n_1 using the first equation, $v_1 = (\omega/kn_0)n_1$, and E_1 in terms of n_1 using the third equation, $E_1 = i(e/k\epsilon_0)n_1$. These relations are then used in the second equation, which can be rearranged as

$$\left(\omega^2 - \frac{n_0 e^2}{m_e \epsilon_0} - \frac{3k_B T_e}{m_e} k^2\right) n_1 = 0. \tag{4.22}$$

This implies that nontrivial solutions only exist provided the parenthesis vanishes, i.e.,

$$\omega^2 - \frac{n_0 e^2}{m_e \epsilon_0} - \frac{3k_B T_e}{m_e} k^2 = 0. \tag{4.23}$$

It is convenient to introduce the (electron) *plasma frequency*, ω_p, and the (electron) *thermal velocity*, v_{th}, as

$$\omega_p^2 \equiv \frac{n_0 e^2}{m_e \epsilon_0}, \qquad \frac{m_e v_{th}^2}{2} \equiv k_B T_e. \tag{4.24}$$

The relation between ω and k, i.e., the dispersion relation for the longitudinal electrostatic electron plasma wave, can then be written

$$\omega^2 = \omega_p^2 + \frac{3 v_{th}^2 k^2}{2}. \tag{4.25}$$

It is physically instructive to write the dispersion relation in the form

$$\omega^2 = \omega_p^2 \left(1 + \frac{3 v_{th}^2}{2\omega_p^2} k^2\right) = \omega_p^2 (1 + 3\lambda_D^2 k^2), \tag{4.26}$$

where we have used the relation $\lambda_D^2 = v_{th}^2/(2\omega_p^2)$. The corresponding dispersion relation is shown in Fig. 4.2. In fact, the dispersion relation, Eq. (4.26), is valid only for wavelengths much longer than the Debye length, λ_D, i.e., we have $k^2\lambda_D^2 \ll 1$. When k becomes of the order of the inverse Debye length, a new effect arises, a strong interaction between the wave and those plasma electrons, which have velocities of the order of the phase velocity of the wave. This effect, which gives rise to a damping of the wave, is called Landau damping and cannot be obtained within the plasma fluid description; a kinetic description is necessary. Thus, we conclude that within the fluid description, the natural oscillation frequency of the electron plasma wave is $\omega \geq \omega_p$. When ω (or rather k) increases, the phase velocity, v_{ph}, of the wave approaches the thermal velocity of the electrons, v_{th}, typically when $k \approx \lambda_D^{-1}$. This leads to a strong interaction between the wave and the particles, known as Landau damping, discussed in the next section.

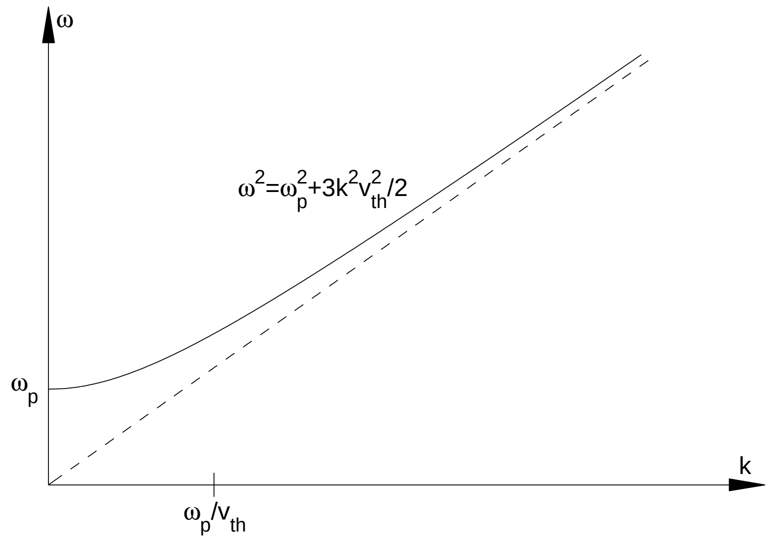


Figure 4.2. The dispersion relation for the longitudinal electron plasma wave.

4.3.2 Kinetic description of longitudinal electron plasma waves—Landau damping

As an example of the use of the kinetic approach, we will reconsider the propagation of electrostatic electron plasma waves. We still assume the ions to form a stationary and neutralising background. The relevant equations are Poisson's equation,

$$\nabla \cdot \mathbf{E} = \frac{e}{\epsilon_0}(n_i - n_e), \tag{4.27}$$

and the Vlasov kinetic equation,

$$\frac{\partial f}{\partial t} + \mathbf{v} \cdot \frac{\partial f}{\partial \mathbf{r}} + \frac{\mathbf{F}}{m_e} \cdot \frac{\partial f}{\partial \mathbf{v}} = 0, \tag{4.28}$$

which replaces the fluid equations. The electron distribution function, $f(\mathbf{v})$, is related to the electron density according to

$$n_e = \int f(\mathbf{v})\, d\mathbf{v}. \tag{4.29}$$

The stationary solution of these equations in the one-dimensional geometry studied previously is $\mathbf{E} = 0$, $n_e = n_0$, $f = f_0(v)$, where

$$n_0 = \int_{-\infty}^{\infty} f_0(v)\, dv. \tag{4.30}$$

The solution of the wave-perturbed equations can be written as

$$f(x,t,v) = f_0(v) + f_1(x,t,v), \tag{4.31a}$$
$$E(x,t) = E_1(x,t), \tag{4.31b}$$
$$n_e(x,t) = n_0 + n_1(x,t), \tag{4.31c}$$

where the perturbations are assumed to be small. Inserting the expressions for E, n and f into Eqs. (4.27)–(4.29) and linearising, we obtain two linear coupled equations for E_1 and f_1,

$$\frac{\partial E_1}{\partial x} = -\frac{e}{\epsilon_0} n_1 = -\frac{e}{\epsilon_0} \int_{-\infty}^{\infty} f_1(x,t,v)\,dv, \tag{4.32a}$$
$$\frac{\partial f_1}{\partial t} + v\frac{df_1}{dx} - \frac{eE_1}{m_e}\frac{\partial f_0}{\partial v} = 0. \tag{4.32b}$$

As before, we look for wave solutions proportional to $\exp[i(kx-\omega t)]$, which transforms the last two equations into an algebraic system for ω and k,

$$ikE_1 = -\frac{e}{\epsilon_0}\int_{-\infty}^{\infty} f_1(v)\,dv, \tag{4.33a}$$
$$-i\omega f_1 + ikvf_1 - \frac{eE_1}{m_e}\frac{df_0}{dv} = 0. \tag{4.33b}$$

Solving for f_1 from the second equation we obtain

$$f_1 = i\frac{eE_1}{m_e}\frac{df_0/dv}{\omega - kv}. \tag{4.34}$$

When this is inserted into Poisson's equation, we obtain the dispersion relation

$$k + \frac{e^2}{m_e\epsilon_0}\int_{-\infty}^{\infty}\frac{df_0/dv}{\omega - kv}\,dv = 0. \tag{4.35}$$

Note that the integral contains a singular term proportional to $1/(\omega - kv)$. Obviously, something special happens when $v = \omega/k$, i.e., for those particles in the velocity distribution for which the phase velocity of the wave, ω/k, equals the particle velocity. This effect was not observed in our previous analysis based on the fluid equations.

The evaluation of a singular integral of the form given by Eq. (4.35) is nontrivial, but for simplicity we just use the so-called Landau prescription, which implies that

$$\int_{-\infty}^{\infty}\frac{df_0/dv}{v-\omega/k}\,dv = \mathrm{P}\int_{-\infty}^{\infty}\frac{df_0/dv}{v-\omega/k}\,dv + i\pi\left.\frac{df_0}{dv}\right|_{v=\omega/k}, \tag{4.36}$$

where the first term, prefixed with P, denotes the Cauchy principal value of the integral, and the last term is half the residue at the point $v = \omega/k$.

For the evaluation of the principal part, where the singularity offers no problems, we can use the assumption that the phase velocity is much larger than the velocities in the main part of the distribution, i.e., $\omega/k \gg v \approx v_{th}$, where v_{th} is the thermal velocity of the electrons, which determines the width of the distribution. This implies that we can expand the denominator of the integral to obtain

$$\begin{aligned} \mathrm{P}\int_{-\infty}^{\infty} \frac{df_0/dv}{\omega - kv}\, dv &\approx \frac{1}{\omega}\int_{-\infty}^{\infty} \frac{df_0}{dv}\left[1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + \left(\frac{kv}{\omega}\right)^3\right] dv \\ &= -\frac{1}{\omega}\left[\frac{k}{\omega}\int_{-\infty}^{\infty} f_0(v)\, dv + \frac{3k^3}{\omega^3}\int_{-\infty}^{\infty} v^2 f_0(v)\, dv\right], \end{aligned} \tag{4.37}$$

where the last step was obtained by partial integration, and we have assumed that $f_0(v)$ is even. The velocity integral has a direct physical meaning, namely

$$\int_{-\infty}^{\infty} f_0(v)\, dv = n_0, \tag{4.38a}$$

$$\int_{-\infty}^{\infty} v^2 f_0(v)\, dv = \frac{k_B T_e n_0}{2m_e}. \tag{4.38b}$$

By rearranging the terms in Eq. (4.35) suitably, and using the definitions of electron density and temperature, where also $k_B T_e = m_e v_{th}^2/2$, the dispersion relation can be written

$$\omega^2 = \omega_p^2 + \frac{3}{2}\frac{\omega_p^2}{\omega^2} v_{th}^2 k^2 + i\pi \frac{\omega_p^2 \omega}{k} \left.\frac{d\hat{f}_0}{dv}\right|_{v=\omega/k}, \tag{4.39}$$

where $\hat{f}_0(v) = f_0(v)/n_0$ denotes the normalised electron distribution. We now consider the limit of small k, where the dominating term on the RHS of Eq. (4.39) is ω_p^2. Clearly this dispersion relation will give rise to a small imaginary part in the frequency. This implies that if we write $\omega = \omega_{\mathrm{re}} + i\omega_{\mathrm{im}}$ and insert this into the dispersion relation we obtain to lowest order

$$\omega_{\mathrm{re}}^2 \approx \omega_p^2 + \frac{3}{2} v_{th}^2 k^2, \tag{4.40a}$$

$$\omega_{\mathrm{im}} \approx \frac{i\pi}{2}\frac{\omega_p^2}{k} \left.\frac{d\hat{f}_0}{dv}\right|_{v=\omega_p/k}. \tag{4.40b}$$

The presence of an imaginary part in the frequency implies a time dependence proportional to $\exp(-i\omega_{\mathrm{re}} t + \omega_{\mathrm{im}} t)$. Consequently, for distributions which are monotonously decreasing functions of velocity, e.g., the thermal Maxwellian distribution, see Fig. 4.3, the imaginary part implies a damping of the wave. This is called *Landau damping*.

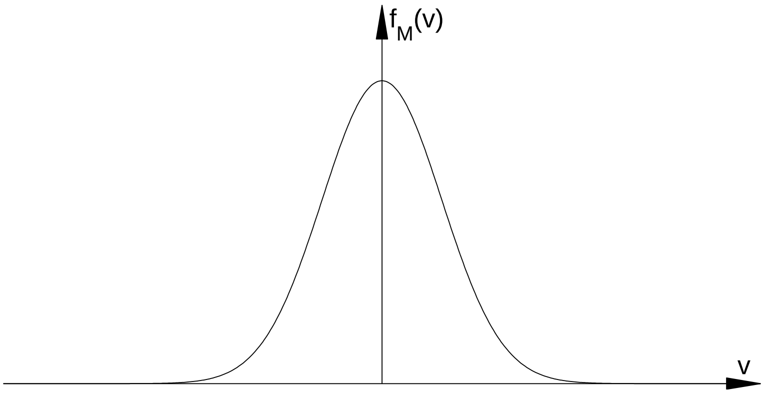


Figure 4.3. The electron velocity distribution function in thermal plasma.

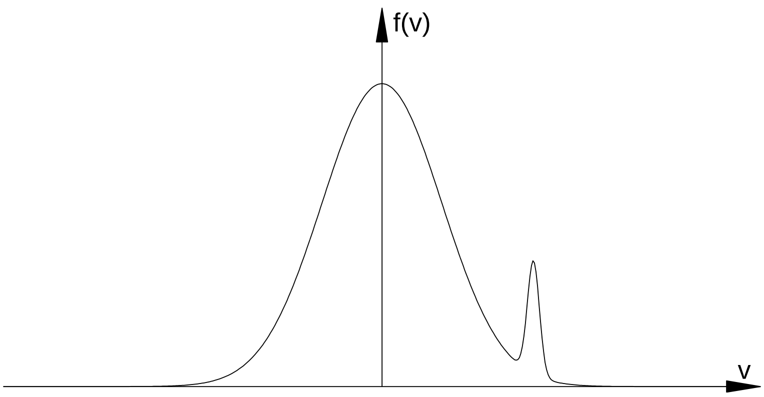


Figure 4.4. The electron velocity distribution function in a thermal plasma with a a high energy electron beam.

However, situations can be created where the distribution function is not monotonous, e.g., consider the case when the electron distribution consists of a Maxwellian background plus a high energy electron beam, see Fig. 4.4. In a certain velocity region slightly lower than the beam velocity, the distribution function has a positive slope and the wave will grow. In fact this situation is a common feature in many wave amplifiers, which are based on the interaction between the wave to be amplified and a high energy electron beam.

A qualitative explanation of the phenomenon of Landau damping is that particles with velocities approximately equal to the phase velocity of the wave tend to interact strongly. Particles with velocities slightly less than the phase velocity tend to be accelerated by the wave, i.e., the wave looses energy, whereas particles with velocities slightly larger than the phase velocity tend to be retarded by the wave, i.e., the wave gains energy. The net result, from the point of view of the wave is a damping if there are more particles with velocities $v < \omega/k$ than with $v > \omega/k$, i.e., $df_0/dv < 0$ and a gain if there are more particles with velocities $v > \omega/k$ than with $v < \omega/k$, i.e., $df_0/dv > 0$.

Finally we note that for a Maxwellian distribution of the form

$$\hat{f}_0(v) = \frac{1}{\sqrt{\pi} v_{th}} \exp\left(-\frac{v^2}{v_{th}^2}\right), \tag{4.41}$$

the damping rate becomes

$$\omega_{\text{im}} \approx -\sqrt{\frac{\pi}{8}} \frac{\omega_p}{k^3 \lambda_D^3} \exp\left(-\frac{3}{2} - \frac{1}{2k^2 \lambda_D^2}\right). \tag{4.42}$$

From this expression we infer that as long as $k\lambda_D \ll 1$, the damping rate will be negligibly small and the Landau damping in a thermal plasma will only become important when $k\lambda_D > 1$, i.e., when the phase velocity of the wave becomes comparable with the thermal velocity of the electrons.

4.3.3 Ion acoustic waves

We now turn to study the complementary case of longitudinal and electrostatic waves of very low frequencies, in fact so low that even the ions become mobile and can follow the variation of the electric field. However, in this limit we can instead simplify the electron dynamics. The small mass of the electrons makes it possible to neglect their inertia and to consider them as massless during the slow variation of the field. In other words they instantaneously follow the electrostatic potential, which implies that the electron density distribution follows the Boltzmann law, i.e.,

$$n_e \approx n_0 \exp\left(\frac{e\phi_1}{k_B T_e}\right) \approx n_0 \left(1 + \frac{e\phi_1}{k_B T_e}\right) \equiv n_0 + n_{e1}, \tag{4.43}$$

where, as before, index 1 indicates the (small) wave induced variation. Thus the whole set of dynamic equations for the electrons has boiled down to a simple relation between the electron density variation, n_{e1}, and the wave electric potential, ϕ_1.

The ion dynamics must be determined from the equations of continuity and motion. For simplicity we assume the ions to be cold so that $p_i = 0$. Linearising the ion equations and assuming wave propagation as before we obtain the following relations between the ion velocity, the ion density and the electric field, i.e., the potential ($\mathbf{E} = -\nabla\phi \Rightarrow E_1 = -ik\phi_1$)

$$v_{i1} = i\frac{e}{m_i\omega}E_1 = \frac{ek}{m_i\omega}\phi_1, \tag{4.44a}$$

$$n_{i1} = \frac{kn_0}{\omega}v_{i1}. \tag{4.44b}$$

This directly implies a relation between the perturbations in ion and electron densities,

$$n_{i1} = \frac{k^2c_s^2}{\omega^2}n_{e1}, \tag{4.45}$$

where we have introduced the characteristic *ion acoustic speed*, c_s, defined by $c_s^2 \equiv k_BT/m_i$.

In this derivation we have used the assumption of long time scales (i.e., the wave is slowly varying in time) to simplify the electron dynamics. If we make the further assumption that the ion acoustic wave also has a wavelength, which is much longer than the Debye length, we can use the additional simplification of the plasma approximation, i.e., assume that $n_e \approx n_i \to n_{e1} \approx n_{i1}$. From Eq. (4.45) we then directly obtain the small k approximation of the dispersion relation for the ion acoustic waves

$$\omega = kc_s. \tag{4.46}$$

On the other hand, if we do not restrict the analysis to long wavelengths, we cannot resort to the simplifying plasma approximation, but must instead use the full Poisson equation which implies that

$$k^2\phi_1 = \frac{e}{\epsilon_0}(n_{i1} - n_{e1}). \tag{4.47}$$

The electron density can still be considered to be Boltzmann distributed, i.e., $n_{e1} \approx n_0e\phi_1/(k_BT_e)$ and Eq. (4.47) can be written as

$$(1 + k^2\lambda_D^2)n_{e1} = n_{i1}. \tag{4.48}$$

Together with Eq. (4.45) this implies the following more general dispersion relation for the ion acoustic wave:

$$\omega^2 = \frac{k^2c_s^2}{1 + k^2\lambda_D^2}. \tag{4.49}$$

In the limit $k\lambda_D \ll 1$ this reduces to the simplified dispersion relation $\omega = kc_S$, obtained by using the plasma approximation.

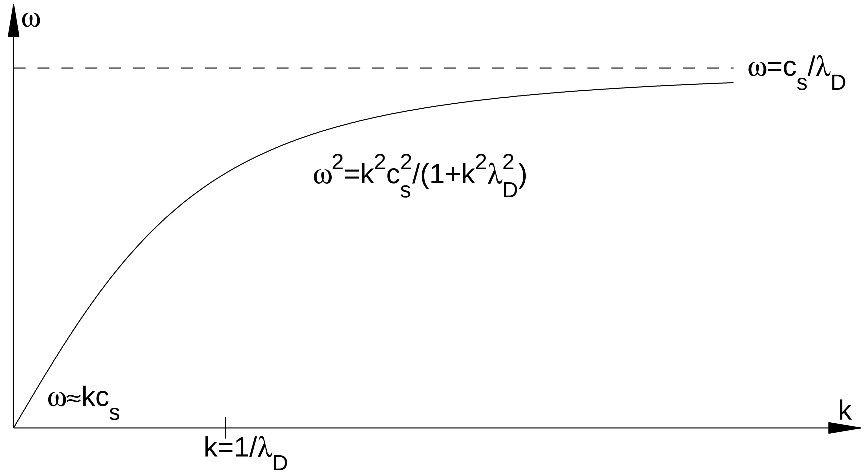


Figure 4.5. The dispersion relation of the ion acoustic wave.

4.4 Electromagnetic waves

We will now consider examples where the wave vector and the electric field are not parallel, i.e., $\mathbf{k} \times \mathbf{E} \neq 0$, which implies that the wave has both oscillating electric and magnetic fields. Such waves are called electromagnetic.

4.4.1 Transverse electromagnetic waves

We start by considering transverse electromagnetic waves in a cold and non-magnetised plasma, i.e., the plasma contains no background magnetic field. Maxwell's equations involving $\nabla \times \mathbf{E}$ and $\nabla \times \mathbf{B}$ read

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{4.50a}$$

$$c^2 \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \frac{1}{\epsilon_0}\mathbf{j}, \tag{4.50b}$$

where $\mathbf{j} = q_e n_e \mathbf{v}_e + q_i n_i \mathbf{v}_i$ is the current density. Again considering high frequency waves, where the heavy ions form a neutralising immobile background, the only equation that is needed to close the system is the equation of motion for the electrons, i.e.,

$$m_e \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v}\right] = -e(\mathbf{E} + \mathbf{v} \times \mathbf{B}). \tag{4.51}$$

Eqs. (4.50) are already linear and the linearised version of Eq. (4.51) reads simply

$$m_e \frac{\partial \mathbf{v}_1}{\partial t} = -e\mathbf{E}_1. \tag{4.52}$$

Looking for waves proportional to $\exp[i(\mathbf{k}\cdot\mathbf{r}-\omega t)]$, the three equations become

$$\mathbf{k}\times\mathbf{E}_1 = \omega\mathbf{B}_1, \tag{4.53a}$$

$$ic^2\mathbf{k}\times\mathbf{B}_1 = -i\omega\mathbf{E}_1 - \frac{n_0 e}{\epsilon_0}\mathbf{v}_1, \tag{4.53b}$$

$$\mathbf{v}_1 = \frac{e}{im_e\omega}\mathbf{E}_1, \tag{4.53c}$$

where n_0 is the constant background plasma density. By eliminating $\mathbf{B}_1$ and $\mathbf{v}_1$ in favour of $\mathbf{E}_1$, a single equation for $\mathbf{E}_1$ is obtained as follows

$$c^2\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_1) + (\omega^2-\omega_p^2)\mathbf{E}_1 = 0. \tag{4.54}$$

This expression can be simplified using the vector rule

$$\mathbf{k}\times(\mathbf{k}\times\mathbf{E}_1) = (\mathbf{k}\cdot\mathbf{E}_1)\mathbf{k} - k^2\mathbf{E}_1. \tag{4.55}$$

and using the fact that for a transverse wave $\mathbf{k}\cdot\mathbf{E}_1 = 0$, Eq. (4.54) can be written

$$(\omega^2-\omega_p^2-k^2c^2)\mathbf{E}_1 = 0. \tag{4.56}$$

Consequently, the dispersion relation for the transverse electromagnetic wave propagating in a plasma is given by

$$\omega^2 = \omega_p^2 + k^2c^2, \tag{4.57}$$

and is shown in Fig. 4.6. Some comments on this dispersion relation should be made as follows: We realise that the wave can only propagate provided $\omega > \omega_p$, since the wave number vanishes when $\omega = \omega_p$. This property of a medium is called *cut-off* and the frequency at which this happens is the *cut-off frequency*. When $\omega < \omega_p$, the wave number becomes purely imaginary,

$$k = i\frac{\omega}{c}\sqrt{1-\frac{\omega_p^2}{\omega^2}} \equiv i/\delta, \tag{4.58}$$

and becomes a measure of the penetration length, δ, of the wave since the space variation now is $\exp(ikx) = \exp(-x/\delta)$. Consider an electromagnetic wave, which propagates into a medium with slowly increasing electron density. When the wave reaches the cut-off point where $\omega = \omega_p$, it cannot propagate any further and is reflected.

4.4.2 Examples of cut-off and reflection of electromagnetic waves

In the previous section we have demonstrated that electromagnetic waves can only propagate in a plasma provided the plasma frequency, ω_p, is smaller

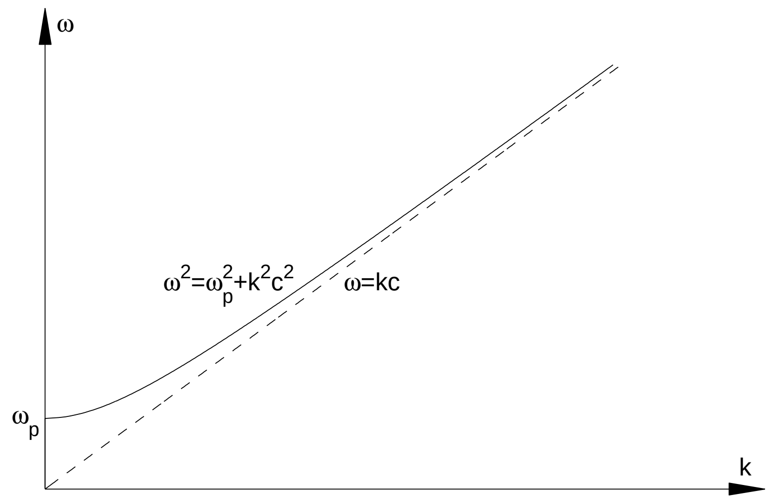


Figure 4.6. The dispersion relation of the electromagnetic "light" waves.

than the wave frequency, ω. This implies that if an electromagnetic wave propagates in an inhomogeneous plasma where the plasma density (and the plasma frequency) increases in the direction of propagation, the wave will become reflected when it reaches the point where $\omega_p = \omega$. This result has a number of important applications.

Radio wave propagation over the horizon

The communication distance of radio wave communication can be greatly enhanced and in fact be extended to reach places below the horizon by using reflection against the plasma layers in the Earth's ionosphere, see Fig. 4.7. In the ionosphere, the electron density may reach values of the order of $n_0 \approx 10^{12}$ m^{-3} at heights of the order of 300 km. This implies a plasma frequency, $f_p = \omega_p/2\pi \approx 10$ MHz. Consequently, when radio waves are emitted at an angle to the Earth's surface, they will be reflected when they reach the high electron density parts of the ionosphere and can be received at points on Earth, which are not possible to reach directly from the transmitter.

Communication silence during reentry of space vehicles

During the first manned space flights, it was found that during the reentry into the atmosphere, the microwave communication contact between the ground and the space craft was interrupted during the first part of the descent through the atmosphere. The reason for this "communication silence" was the following: During reentry into the atmosphere, the air surrounding

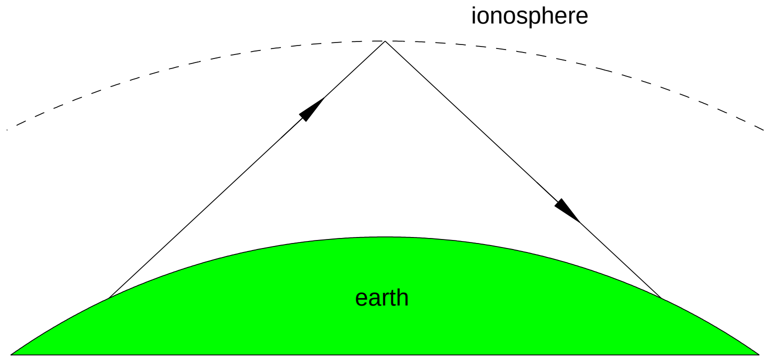


Figure 4.7. Radio wave propagation over the horizon by reflection against the ionosphere.

the space craft was strongly heated due to the friction with the fast moving space craft. The air temperature reached so high values that thermal ionisation created a significant electron density plasma around the space craft. In fact, the plasma density became temporarily so high that the incoming (and outgoing) microwave signals were reflected and the communication was broken. When the space craft reached the denser parts of the atmosphere, the velocity decreased, the temperature of the surrounding air also decreased and the plasma density fell back to its "normal" small value; and communication was restored.

Plasma limiters and TR-switches

The cut-off effect inherent in high density plasmas is used in a number of technical applications called plasma limiters. A typical example of this is the "transmit-receive" switch (TR-switch for short), which is used as a protective device in high power microwave radar installations. The TR-switch is essentially a short carefully designed waveguide, which has been manipulated in several ways so as to facilitate microwave induced *breakdown* of the gas filling the switch. The physical background of the operation of the switch is the following: A microwave propagating through a weakly ionised gas makes the free electrons oscillate in the field. If the amplitude of the microwave is sufficiently high, the energy of the oscillating electrons will be high enough to cause ionisation when they collide with the neutral particles (atoms and molecules) in the gas. This implies that one free electron, during collision, becomes two free electrons, which then go on to create four new electrons etc. This process causes a rapid avalanche-like increase of the electron density. When this density reaches the value where $\omega_p = \omega$, the breakdown plasma blocks the transmission of the microwave by reflecting the radiation. Thus, the transmission characteristics of the TR-switch are

$$P_{\text{transmitted}} \begin{cases} = P_{\text{incident}}, & P_{\text{incident}} < P_c, \\ \ll P_{\text{incident}}, & P_{\text{incident}} > P_c, \end{cases} \tag{4.59}$$

where P_c denotes the critical microwave power at which the microwave field is strong enough to cause rapid ionisation.

Consider a radar system where the antenna receives a strong microwave signal, either accidentally or due to “hostile” actions. This signal may seriously damage the internal electronics of the radar system. However, if a TR-switch is mounted between the antenna and the sensitive parts of the system, it will quickly switch to the off-mode and reflect the dangerous radiation back and out from the antenna.

The evolution of the electron density in a TR-switch can be modelled in the following simple form

$$\frac{dn_e}{dt} = \nu_i n_e, \tag{4.60}$$

where ν_i is the *ionisation frequency*, which depends on the microwave field. An often used model for ν_i is

$$\nu_i \approx \nu_{i0} \left[\left(\frac{E}{E_b} \right)^{\beta} - 1 \right], \tag{4.61}$$

where E_b is the breakdown microwave field and β is a parameter that depends on the gas. TR-switches are typically filled with Argon gas for which $\beta \approx 2$.

Consider now a high power microwave pulse incident on a TR-switch. Let us for simplicity assume that the pulse form is rectangular in time. When $E \gg E_b$, the electron density will rapidly increase exponentially as $n_e = n_e(0)\exp(\nu_i t)$. When the electron density becomes so high that the plasma frequency equals the wave frequency, transmission is stopped. This occurs at the switching time, t_s, determined by

$$n_p = n_e(0)\exp(\nu_i t_s), \tag{4.62}$$

where n_p is the electron density needed to make the plasma frequency equal to the wave frequency. Thus, the switching time is $t_s = \nu_i^{-1}\ln(n_p/n_0)$. The initial electron density in TR-switches varies somewhat and so does n_p (being determined by the microwave frequency). However, due to the logarithmic dependence on n_p/n_0, typically one has $\ln(n_p/n_0) \approx 20$ and the switching time can be approximated as $t_s \approx 20/\nu_i(E)$. This implies that the higher the incident field, the more rapid is the cut-off and for high power microwave pulses (HPM pulses), the switch closes rapidly and efficiently prevents damage on sensitive parts located after the switch in the transmission the system. An example of a TR-switch transmission curve is

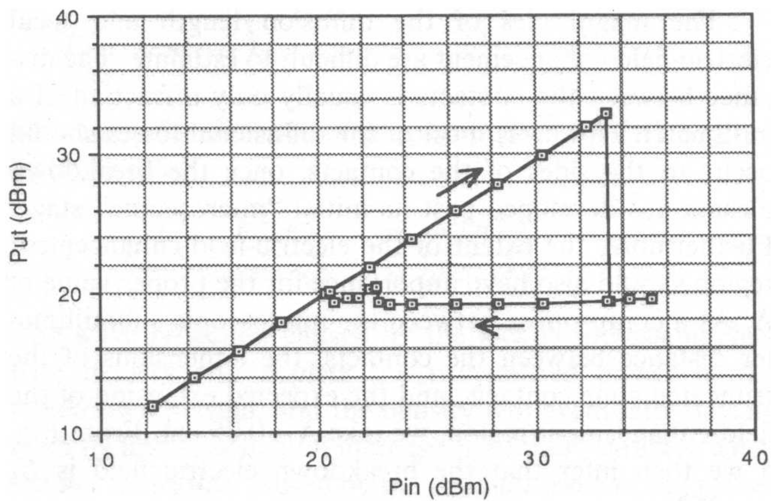


Figure 4.8. The transmitted power as a function of the incident power in a TR-switch. Also seen is the hysteresis effect discussed in section 5.2.3.

shown in Fig. 4.8, where the measured transmitted power through a TR-switch as a function of the incident power has been plotted. Note that the transmitted power equals the incident power until the breakdown threshold is reached. Above this power, only a small (constant) power leaks through, irrespective of the incident power.

4.4.3 Pulsar dispersion

Another striking and fascinating example of the importance of plasma physical effects on the propagation of electromagnetic waves, i.e., the importance of the term ω_p^2 in the dispersion relation $\omega^2 = \omega_p^2 + k^2c^2$, is provided by the observed time delay in the arrival time for different frequencies in the radiation from pulsars. Pulsars are rotating neutron stars with a strong magnetic field and located in the spiral arms of our galaxy. The pulsars emit continuous electromagnetic radiation, which sweeps around in a lighthouse beam fashion, emitting a narrow radiation cone as the star rotates. If a radio telescope on Earth happens to be in the radiation disc, the waves, which are in the radio wave frequency range, will be registered as a periodic sequence of pulses with a time between pulses equal to the rotation time of the pulsar. The interesting thing is that, although all frequencies in the radiation are generated at the same time, the different frequencies do not arrive at Earth at the same time. In fact, the high frequencies come first, lower frequencies arriving with successively longer time delays. The corresponding experimental registration of frequency with arrival time has a characteristic descending curve form, see Fig. 4.9. By measuring the time delay, Δt between two frequencies, f_1 and f_2, the radio astronomers are able to determine the distance to the pulsar! Pure magic? No, not quite, it is a simple consequence of the presence of the plasma frequency term in the dispersion relation for the electromagnetic waves emitted by the pulsar.

In order to understand the time delay between different frequencies we can consider the emitted spectrum of the pulsar radiation as composed of narrow frequency bands, centred around a central carrier frequency, i.e., as the superposition of a number of wave pulses, all emitted at the same time. Due to the interstellar dispersion, i.e., the ω_p^2-term in the dispersion relation, the group velocities of the different wave pulses will be different. Thus, in spite of the fact that all wave pulses are emitted at the same time, they will arrive to the radio telescope on Earth with a relative time delay. As we will demonstrate, this time delay is directly proportional to the distance travelled by the pulses, i.e., the distance to the pulsar.

Since the density of the interstellar plasma is very low, the plasma frequency, ω_p, is much smaller than the frequency of the emitted radiation, ω. This implies that the dispersion relation can be approximated as

$$\omega = kc\sqrt{1 + \frac{\omega_p^2}{k^2c^2}} \approx kc + \frac{\omega_p^2}{2kc}. \tag{4.63}$$

The propagation velocity of each wave packet is the group velocity, which can be expressed as

$$v_g = \frac{d\omega}{dk} \approx c - \frac{\omega_p^2}{2k^2c} \approx c - \frac{\omega_p^2 c}{2\omega^2}. \tag{4.64}$$

The time, t, it takes a pulse to propagate a certain distance, L, is then

$$t = \frac{L}{v_g} \approx \frac{L}{c(1 - \omega_p^2/2\omega^2)} \approx \frac{L}{c} + \frac{L\omega_p^2}{2c\omega^2} \equiv t_0 + \frac{L\omega_p^2}{2c\omega^2}, \tag{4.65}$$

where $t_0 = L/c$ has been introduced. The general relation between the arrival time and the corresponding frequency, $\omega = \omega(t)$ is then

$$\omega(t) = \sqrt{\frac{L\omega_p^2}{2c}} \frac{1}{\sqrt{t - t_0}}. \tag{4.66}$$

The qualitative behaviour of this relation is shown in Fig. 4.9 and can be compared with the Nobel Prize awarded observational results obtained by Hewish and Bell in 1969, see Fig. 4.10. The time delay between different frequencies and the characteristic form of the frequency registration are clearly in good agreement with the simple result given by Eq. (4.66). Starting from

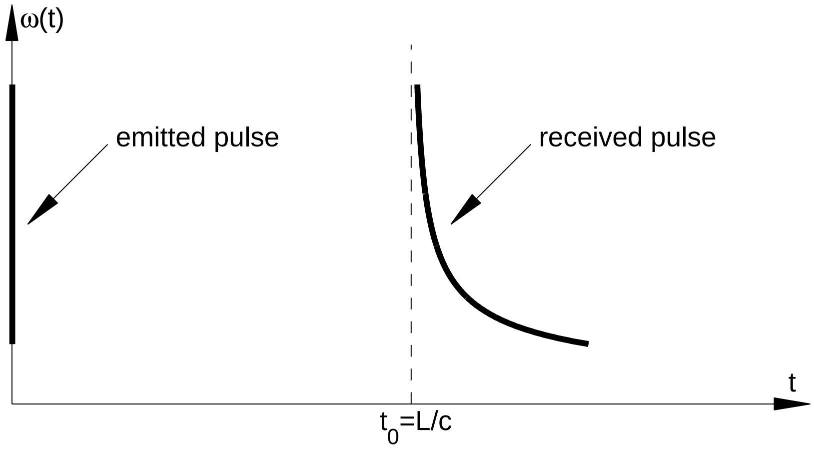


Figure 4.9. The emitted pulse contains a range of frequencies, which are detected at different times due to the frequency dependence of the group velocity.

this equation, or even better Eq. (4.65), it is easy to determine the delay in arrival times between two frequencies, f_1 and f_2 as follows

$$\Delta t = \frac{L\omega_p^2}{8c\pi^2}\left(\frac{1}{f_1^2} - \frac{1}{f_2^2}\right). \tag{4.67}$$

From the observations in Fig. 4.10 we infer that there is a time delay of approximately 3 seconds between the frequencies 408 MHz and 151 MHz. A typical value of the interstellar plasma density in the spiral arms of our galaxy is $n_0 \approx 2 \times 10^5$ m^{-3}. Inserting these values into Eq. (4.67) we obtain $L \approx 150$ pc, where the astronomically convenient length parsec is defined by 1 pc $\approx 3 \times 10^{16}$ m.

4.5 Electromagnetic waves in magnetised plasmas

The presence of a static magnetic field in a plasma gives rise to a host of new waves. The physical background of these waves is that the charged particle motion becomes more complicated since the Lorentz force, $q\mathbf{v} \times \mathbf{B}$, now survives the linearisation process and tends to give rise to motion in directions not co-linear with the direction of an applied electric field. As an illustrative and important example we consider high frequency electromagnetic waves propagating along a static magnetic field in a plasma. Without loss of generality we take the constant magnetic field to lie in the z-direction, i.e., $\mathbf{B}_0 = B_0\hat{z}$. High frequency again means that only electron dynamics is involved in the wave propagation; the ions form a stationary neutralising

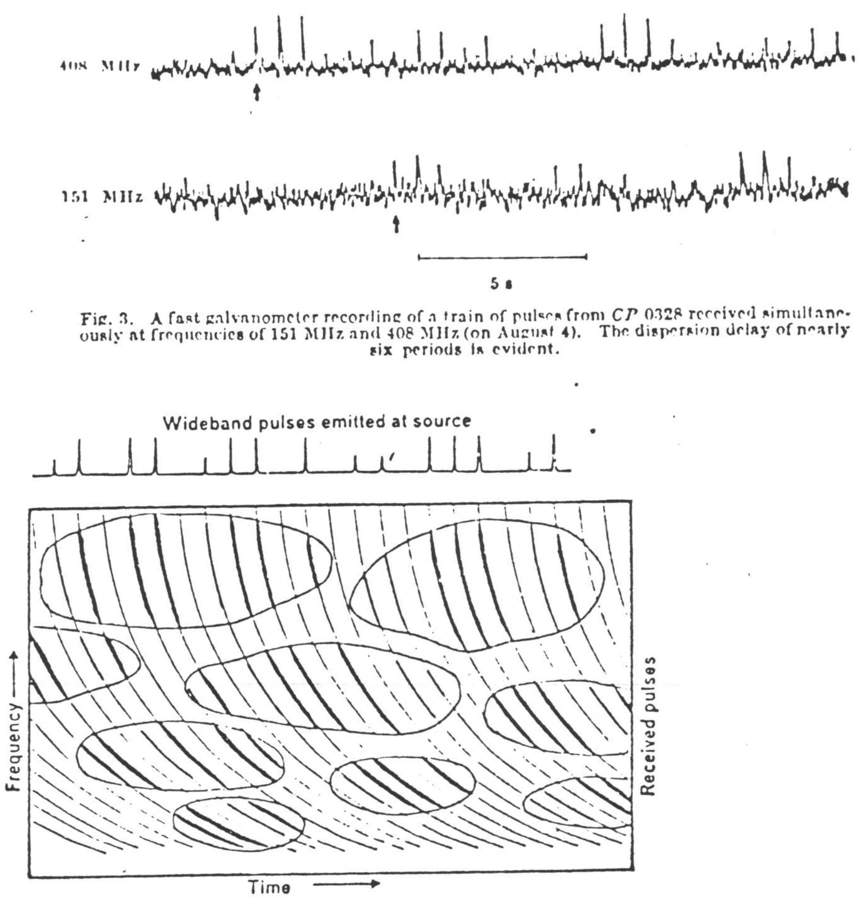


Figure 4.10. Dispersion of pulses in the interstellar medium (after Hewish et al., 1969). The heavy lines show those pulses which are observable above the noise level. The blotches are due to the fluctuations produced by the interstellar medium.

background, being too heavy to respond to the electric field on the short time scale of the wave oscillations. The electromagnetic wave is propagating along the static magnetic field, i.e., $\mathbf{k} \parallel \mathbf{B}_0$ and it is a transverse wave so that both $\mathbf{E}_1$ and $\mathbf{B}_1$ are orthogonal to $\mathbf{k}$ (and to $\mathbf{B}_0$). It is clear that if we choose the electric field to lie in the x-direction, it will give rise to a velocity, which is also in the x-direction. However, the Lorentz force in the equation of motion for the electrons, which is directed in the direction of $\mathbf{v}_1 \times \mathbf{B}_0$, will create a velocity component in the y-direction. This implies that there is a coupling between the dynamics in the x- and y-directions and that the electric field cannot be linearly polarised. In other words the electric field must necessarily contain components in both the x- and y-directions. Assuming the plasma electrons to be cold, the linearised equation of motion for the electrons becomes (we here omit the index 1 for simplicity)

$$\frac{\partial \mathbf{v}}{\partial t} = -\frac{e}{m_e}(\mathbf{E} + \mathbf{v} \times \mathbf{B}_0), \tag{4.68}$$

which upon transformation to propagating waves yields

$$\mathbf{v} = -\frac{ie}{m_e\omega}(\mathbf{E} + \mathbf{v} \times \mathbf{B}_0). \tag{4.69}$$

It is convenient to operate on this equation by $\times\mathbf{B}_0$, which gives

$$\mathbf{v} \times \mathbf{B}_0 = -\frac{ie}{m_e\omega}[\mathbf{E} \times \mathbf{B}_0 + (\mathbf{v} \times \mathbf{B}_0) \times \mathbf{B}_0] = -\frac{ie}{m_e\omega}\mathbf{E} \times \mathbf{B}_0 + \frac{ieB_0^2}{m_e\omega}\mathbf{v}. \tag{4.70}$$

We can now insert this expression back into Eq. (4.69) to obtain

$$\mathbf{v} = -\frac{ie}{m_e\omega}\mathbf{E} + \frac{\omega_c^2}{\omega^2}\mathbf{v} - \frac{e^2}{m_e^2\omega^2}\mathbf{E} \times \mathbf{B}_0, \tag{4.71}$$

which implies that the relation between $\mathbf{v}$ and $\mathbf{E}$ becomes

$$\mathbf{v} = -\frac{ie}{m_e\omega}\frac{1}{1 - \omega_c^2/\omega^2}\left(\mathbf{E} - i\frac{\omega_c}{\omega}\frac{\mathbf{E} \times \mathbf{B}_0}{B_0}\right). \tag{4.72}$$

Eliminating the magnetic wave field from Maxwell's equations in the same way as was done in the previous section and inserting the obtained expression for $\mathbf{v}$ in Eq. (4.72), we arrive at the vector equation

$$\left(\omega^2 - k^2c^2 - \frac{\omega_p^2}{1 - \omega_c^2/\omega^2}\right)\mathbf{E} + i\frac{\omega_p^2\omega_c}{\omega(1 - \omega_c^2/\omega^2)}\mathbf{E} \times \hat{z} = 0. \tag{4.73}$$

We note again the coupling between the x- and y-components of the electric field caused by the cross product $\mathbf{E} \times \hat{z} = (E_x\hat{x} + E_y\hat{y}) \times \hat{z} = E_y\hat{x} - E_x\hat{y}$.

The equation for $\mathbf{E}$, given by Eq. (4.73) can be written in matrix form as follows

$$\begin{pmatrix} \omega^2 - k^2c^2 - \dfrac{\omega_p^2}{1-\omega_c^2/\omega^2} & i\dfrac{\omega_p^2\omega_c}{\omega(1-\omega_c^2/\omega^2)} \\ -i\dfrac{\omega_p^2\omega_c}{\omega(1-\omega_c^2/\omega^2)} & \omega^2 - k^2c^2 - \dfrac{\omega_p^2}{1-\omega_c^2/\omega^2} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix} = 0. \tag{4.74}$$

Since Eq. (4.74) is a linear homogeneous system of equations, nontrivial solutions, i.e., nonzero E_x and E_y, only exist provided the determinant of the system vanishes. This yields

$$\begin{vmatrix} \omega^2 - k^2c^2 - \dfrac{\omega_p^2}{1-\omega_c^2/\omega^2} & i\dfrac{\omega_p^2\omega_c}{\omega(1-\omega_c^2/\omega^2)} \\ -i\dfrac{\omega_p^2\omega_c}{\omega(1-\omega_c^2/\omega^2)} & \omega^2 - k^2c^2 - \dfrac{\omega_p^2}{1-\omega_c^2/\omega^2} \end{vmatrix} = 0, \tag{4.75}$$

which implies

$$\omega^2 - k^2c^2 - \frac{\omega_p^2}{1-\omega_c^2/\omega^2} = \pm\frac{\omega_p^2\omega_c}{\omega(1-\omega_c^2/\omega^2)}. \tag{4.76}$$

This may be written using the refractive index, N, as

$$N^2 \equiv \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_c)}, \tag{4.77}$$

i.e., two propagation modes are possible corresponding to the plus sign (the L-wave) and the minus sign (the R-wave) respectively. The corresponding refractive indices are shown in Fig. 4.11.

These waves have cut-off when $N = 0$, which happens when

$$\omega^2 \pm \omega\omega_c - \omega_p^2 = 0 \tag{4.78}$$

i.e., for

$$\omega = \mp\frac{\omega_c}{2} + \sqrt{\frac{\omega_c^2}{4} + \omega_p^2} \equiv \begin{cases} \omega_L, & (-), \\ \omega_R, & (+). \end{cases} \tag{4.79}$$

In addition to the cut-off phenomenon, the R-wave exhibits resonance. When $\omega \to \omega_c$ for the R-wave, the refractive index (and k) approaches infinity. The physical meaning of this can be understood as follows: From Eq. (4.74) we infer that

$$\begin{aligned} \frac{E_x}{E_y} &= \pm i\frac{\omega_c\omega_p^2}{\omega(1-\omega_c^2/\omega^2)}\frac{1}{\omega^2 - k^2c^2 - \omega_p^2/(1-\omega_c^2/\omega^2)} \\ &= \begin{cases} +i, & \text{the R-wave}, \\ -i, & \text{the L-wave}, \end{cases} \end{aligned} \tag{4.80}$$

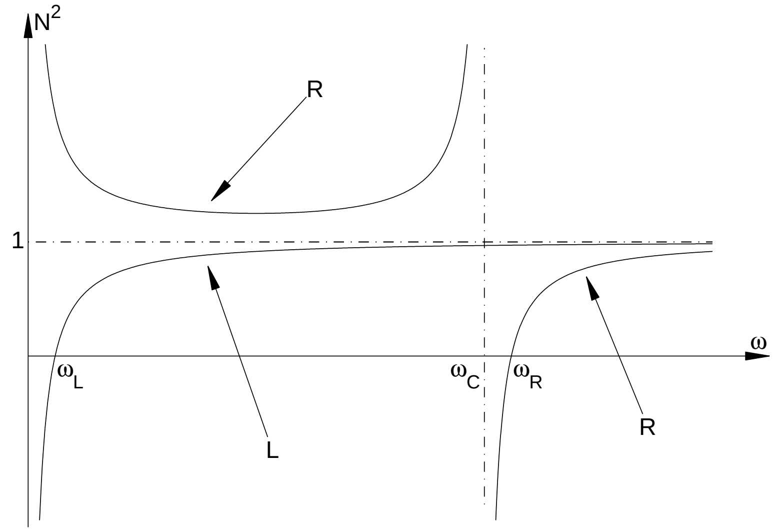


Figure 4.11. The refractive indices of the R and L waves.

where we have also used Eq. (4.76). Consequently both the R- and L-waves are circularly polarised, with the electric field vector rotating in the right (R) and left (L) directions respectively. The strong interaction between the electrons and the R-wave is now easy to understand. The electric field of the R-wave has the same direction of rotation as the electrons in their "natural" motion in terms of circular Larmor orbits. Consequently if the wave frequency is equal to the Larmor frequency, ω_c, the electrons will be in resonance with the field and will experience a constant electric field and a strong interaction. For the L-wave, on the other hand, even if the wave frequency matches the Larmor frequency, the direction of rotation is incorrect and no significant interaction occurs. There are many interesting and important applications of the R- and L-waves in many different contexts. Two examples will be given in the next sections.

4.6 Whistler waves

The low frequency branch of the R-wave, i.e., waves in the frequency range $0 < \omega < \omega_c$, are called whistler waves. Already in the early observations of radio emission from the ionosphere and magnetosphere, signals were picked up in the kHz range of frequencies, which, when transformed to acoustic waves in the detector, gave rise to a characteristic whistling signal with a rapidly falling frequency. The origin of these *whistler waves* was found to be lightning discharges, which tend to generate radio emission in a wide frequency range. The lightning could very well be on the other side of the Earth, but the waves propagated long distances through the ionosphere and magnetosphere along the Earth's magnetic field lines. Thus radiation generated at the southern hemisphere could be picked up on the northern hemisphere. However, the dispersive properties of the waves implied that different frequency components travel with different velocities, thus giving rise to a frequency drift in the received radiation. A typical frequency registration of a whistler signal is shown in Fig. 4.12. The signals labelled A, B, and C all come from the same lighting discharge, but have travelled along different paths.

The form of the observed frequency registration can easily be analysed starting from the dispersion relation, as expressed by the refractive index

$$N^2 = \frac{k^2c^2}{\omega^2} = 1 - \frac{\omega_p^2}{\omega(\omega - \omega_c)}. \tag{4.81}$$

If we assume that $\omega \ll \omega_c$ and $\omega_p^2/(\omega\omega_c) \gg 1$, we obtain the simpler dispersion relation

$$\frac{k^2c^2}{\omega^2} \approx \frac{\omega_p^2}{\omega\omega_c}, \tag{4.82}$$

i.e.,

$$\omega \approx \frac{\omega_c}{\omega_p^2}k^2c^2. \tag{4.83}$$

The group velocity then becomes

$$v_g = \frac{d\omega}{dk} = \frac{2c}{\omega_p}\sqrt{\omega\omega_c}. \tag{4.84}$$

As before we estimate the arrival time, t, of a frequency ω (emitted at time $t = 0$) as

$$t = \frac{L}{v_g} = \frac{L\omega_p}{2c}\frac{1}{\sqrt{\omega\omega_c}}, \tag{4.85}$$

where L is the propagation length. The difference in arrival times, Δt between a given frequency, ω_0, and an arbitrary frequency, ω, is then

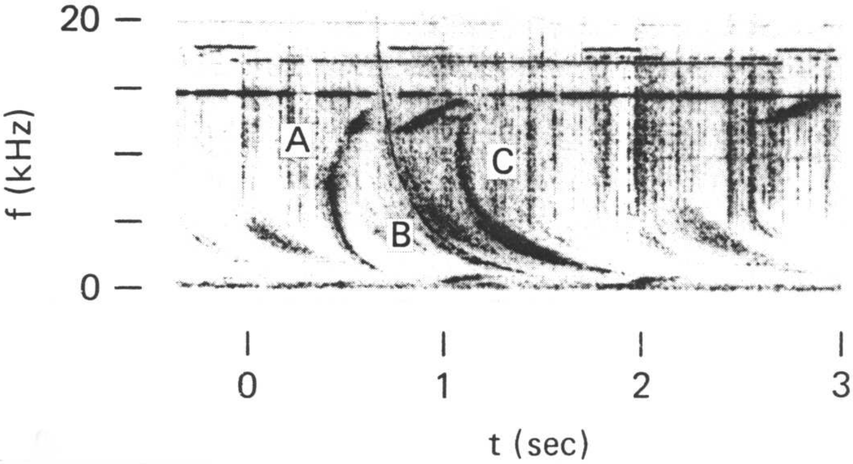
$$\Delta t = \frac{L\omega_p}{2c\sqrt{\omega_c}}\left(\frac{1}{\sqrt{\omega}} - \frac{1}{\sqrt{\omega_0}}\right), \tag{4.86}$$

or, equivalently,

$$\omega = \omega(\Delta t) = \omega_0\left(1 + \frac{2c\sqrt{\omega_c\omega_0}}{L\omega_p}\Delta t\right)^{-2}. \tag{4.87}$$

The corresponding curve is given in Fig. 4.13 and shows good qualitative agreement with the observed frequency registration, see Fig. 4.12.

However, a striking feature in the observed frequency registration is the fact that the curves seem to have an "overhang" for large frequencies, i.e.,



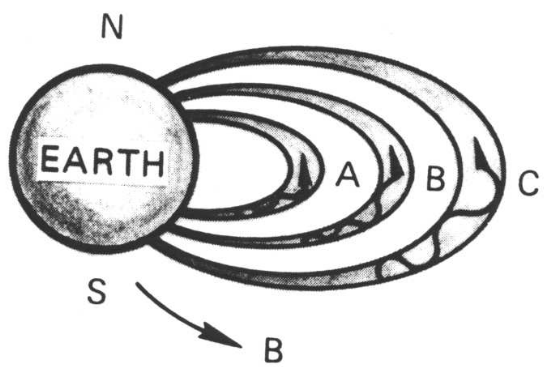


Figure 4.12. The upper figure shows spectrograms of registered whistler waves, showing the curvature of the low frequency branch of the R-wave dispersion relation (from Carpenter, J. Geophysical Research (1969)). The bottom figure shows qualitatively how the different whistler traces, labelled A, B, and C are created.

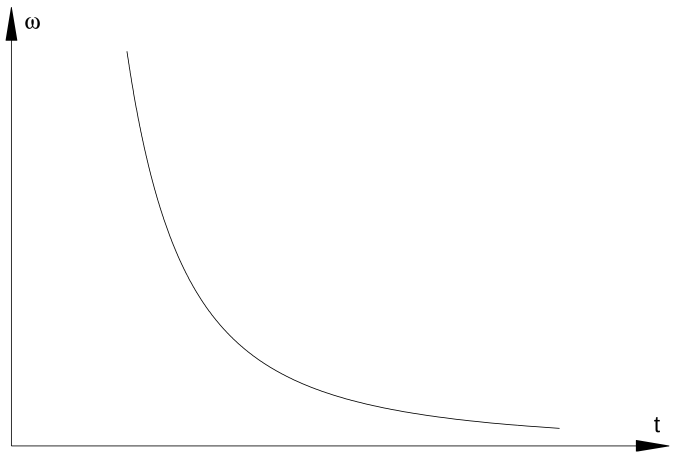


Figure 4.13. The variation of frequency with arrival time for the whistler waves as predicted by Eq. (4.87).

there is a frequency that gives a minimum propagation time. A natural explanation of this would be that the group velocity is not a monotonous function of frequency, but has a maximum at a certain frequency. When we approximated the dispersion relation we assumed that $\omega \ll \omega_c$ and that $\omega_p^2 \gg \omega\omega_c$. The first approximation is the most severe one and if we use only the last approximation, the dispersion relation becomes

$$\frac{k^2c^2}{\omega^2} \approx \frac{\omega_p^2}{\omega(\omega_c - \omega)}, \tag{4.88}$$

i.e., $\omega = \omega(k)$ can be expressed as

$$\omega = \omega_c \left(1 - \frac{\omega_p^2}{\omega_p^2 + k^2c^2}\right), \tag{4.89}$$

which implies that the group velocity becomes

$$v_g = \frac{2(\omega_c - \omega)kc^2}{\omega_p^2 + k^2c^2}. \tag{4.90}$$

This more general expression for v_g is however not a monotonous function of k. It is straightforward to show that v_g has a maximum for $\omega = \omega_{\max} = \omega_c/4$ and the corresponding k is given by $k_{\max}^2 c^2 = \omega_p^2/3$. This implies that the frequency registration should be qualitatively of the form shown in Fig. 4.14, which is in good agreement with the observed variation.

4.6.1 Faraday rotation of pulsar radiation

The fact that right- and left-polarised electron waves, propagating in a plasma, have different refractive indices leads to several fascinating and im-

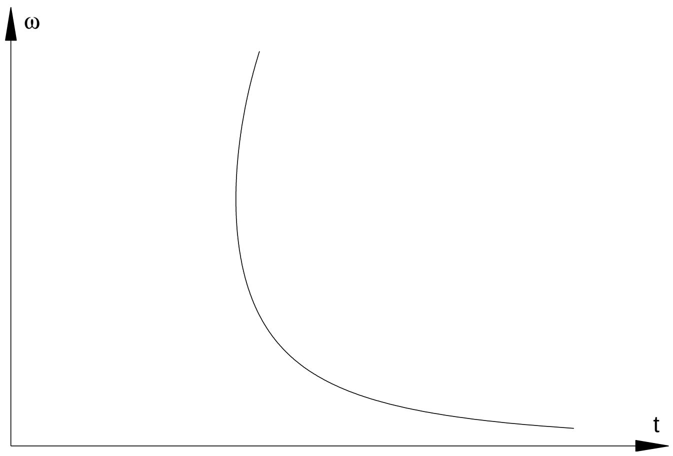


Figure 4.14. The expected qualitative form of the frequency registration of whistler waves when the low frequency approximation is improved.

portant phenomena. One beautiful example is again provided by pulsar radiation. This radiation is linearly polarised at emission, but a linearly polarised wave cannot propagate as a natural mode through the interstellar magnetic field. However, a linearly polarised electric field can be seen as the superposition of two right and left circularly polarised fields, and since the corresponding wave propagation constants for the R- and L-waves are different, the plane of polarisation will rotate during propagation, see Fig. 4.15. By measuring the difference in polarisation angle between two frequencies at arrival, an estimate can be obtained of the magnitude of the interstellar magnetic field. Let us analyse this effect in some more detail to understand why this is so.

The refractive indices for the two polarisations are

$$N_\pm^2 = 1 - \frac{\omega_p^2}{\omega(\omega \pm \omega_c)}. \tag{4.91}$$

The dispersive term in the refractive index gives rise to two effects:
(i) Since $\omega_p, \omega_c \ll \omega$, the leading order term in this relation is

$$N_\pm^2 \approx 1 - \frac{\omega_p^2}{\omega^2}. \tag{4.92}$$

The effects caused by this correction term were studied previously and we showed (see Eq. (4.65)) that the arrival time of the frequency ω was determined by

$$t = t_0 + \frac{L\omega_p^2}{2c\omega^2}, \tag{4.93}$$

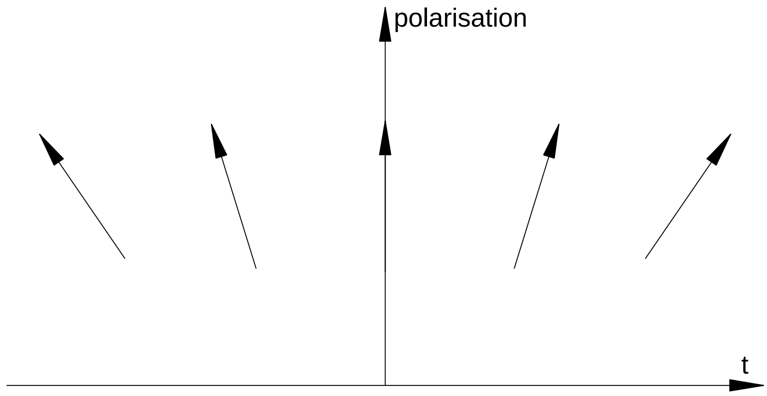


Figure 4.15. Qualitative plot of the observed rotation of the plane of polarisation (at fixed frequency) of the pulsar radiation.

i.e., the change in arrival time, Δt, is related to a change in frequency, $\Delta\omega$, as follows

$$\Delta t = -\frac{L\omega_p^2}{c\omega^3}\Delta\omega. \tag{4.94}$$

(ii) Going to next order in small quantities by introducing also terms proportional to ω_c/ω, we obtain

$$\frac{k_\pm c}{\omega} \approx 1 - \frac{\omega_p^2}{2\omega^2} \pm \frac{\omega_p^2}{2\omega^2}\frac{\omega_c}{\omega}. \tag{4.95}$$

The last term gives rise to a frequency dependent shift of the plane of polarisation. This can be understood the following way: Consider a linearly polarised wave field, $\mathbf{E}_0 = 2E\hat{x}\exp(-i\omega t)$. This field can be decomposed into two circularly polarised fields with opposite direction of rotation, since we can write

$$\begin{aligned}\mathbf{E}_0 = E[\hat{x}\exp(-i\omega t) + i\hat{y}\exp(-i\omega t)]\\ +E[\hat{x}\exp(-i\omega t) - i\hat{y}\exp(-i\omega t)] \equiv \mathbf{E}_{0L}(t) + \mathbf{E}_{0R}(t).\end{aligned} \tag{4.96}$$

The two polarisation modes propagate with the wave numbers k_L and k_R, respectively, which implies that after propagating a distance L, the resulting field can be written

$$\mathbf{E}(L,t) = \mathbf{E}_{0L}(t)\exp(ik_L L) + \mathbf{E}_{0R}(t)\exp(ik_R L), \tag{4.97}$$

which can be made symmetric by writing

$$\begin{aligned}\mathbf{E}(L,t) = 2E\exp\left[i\left(\frac{k_L+k_R}{2}L - \omega t\right)\right]\\ \times\left[\cos\left(\frac{k_L-k_R}{2}L\right)\hat{x} - \sin\left(\frac{k_L-k_R}{2}L\right)\hat{y}\right].\end{aligned} \tag{4.98}$$

This result implies that the wave is still linearly polarised, but the plane of polarisation has rotated an angle θ relative to the original orientation (along the $\hat{x}$-axis). The rotation angle is given by $\theta = -(k_R - k_L)L/2$, where

$$k_R - k_L = -\frac{\omega_p^2 \omega_c}{c\omega^2}, \tag{4.99}$$

implying that

$$\theta = -\frac{\omega_p^2 \omega_c}{2c\omega^2} L. \tag{4.100}$$

At the observation point the change of polarisation angle, $\Delta\theta$, with frequency change $\Delta\omega$ is given by

$$\Delta\theta = -\frac{\omega_p^2 \omega_c}{c\omega^3} L \Delta\omega \tag{4.101}$$

This equation can be combined with the previous result for the change in arrival time with frequency, Eq. (4.94), to give the beautifully simple result

$$\frac{\Delta\theta}{\Delta t} = \omega_c = \frac{eB}{m_e}. \tag{4.102}$$

Thus the change in polarisation angle during the arrival of the pulse is determined only by ω_c, i.e., by the magnitude of the interstellar magnetic field. The rotation of the polarisation can be measured and, e.g., for the pulsar CP 1929+10 one has observed a rotation of approximately 50° between the arrivals of the frequencies 410 MHz and 1665 MHz, which are separated by approximately 0.25 s. This directly gives $B \approx 3 \times 10^{-11}$ T. This result is in qualitative agreement with other estimates obtained by independent methods.

4.7 Low frequency electromagnetic waves in magnetised plasmas—the Alfvén[1] waves.

We will now consider low frequency ($\omega \ll \omega_{c,i}, \omega_{p,i}$) transverse electromagnetic waves in a magnetised plasma. The low frequency implies that both ion and electron motion is involved in the wave dynamics. However, since the considered wave frequency is much smaller than the cyclotron frequency of both ions and electrons, the particle motion can be approximated as consisting of the $\mathbf{E} \times \mathbf{B}$ drift, Eq. (2.27), and the polarisation drifts, Eq. (2.177), i.e., the plasma particles can be considered to be strongly magnetised. The perturbations of the particle velocities are then given by

$$\mathbf{v}_1 \approx \frac{\mathbf{E}_1 \times \mathbf{B}_0}{B_0^2} + \frac{m}{qB_0^2} \frac{\partial \mathbf{E}_1}{\partial t}. \tag{4.103}$$

[1] The Swedish name "Alfvén" is pronounced like the alternatively spelled form "Alvén". For an english-speaking person we recommend the pronunciation *Al-veen* or *Al-vein*, stressing the last syllable.

The plasma current can then be directly related to the electric field

$$\mathbf{j}_1 = n_0 e(\mathbf{v}_{i1} - \mathbf{v}_{e1}) \approx -i\omega \frac{n_0 e}{B_0}\left(\frac{1}{\omega_{c,i}} - \frac{1}{\omega_{c,e}}\right)\mathbf{E}_1 \approx -\frac{i n_0 e\omega}{B_0\omega_{c,i}}\mathbf{E}_1, \tag{4.104}$$

where we have used that the large drift velocity due to the $\mathbf{E}\times\mathbf{B}$ drift is the same for ions and electrons and consequently does not contribute to the current. From Maxwell's two $\nabla\times$-equations we have

$$\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}, \tag{4.105a}$$

$$c^2\nabla\times\mathbf{B} = \frac{\partial\mathbf{E}}{\partial t} + \frac{\mathbf{j}}{\epsilon_0}, \tag{4.105b}$$

where again we eliminate $\mathbf{B}$ to obtain

$$c^2\mathbf{k}\times(\mathbf{k}\times\mathbf{E}) = -\omega^2\mathbf{E} + \frac{i\omega}{\epsilon_0}\frac{i n_0 e\omega}{B_0\omega_{c,i}}\mathbf{E}, \tag{4.106}$$

or

$$\mathbf{k}(\mathbf{k}\cdot\mathbf{E}) - k^2\mathbf{E} = -\frac{\omega^2}{c^2}\mathbf{E} - \frac{\omega^2}{v_A^2}\mathbf{E}, \tag{4.107}$$

where we have introduced the characteristic *Alfvén velocity*, v_A, defined by

$$v_A^2 \equiv \frac{\epsilon_0 B_0^2 c^2}{n_0 m_i}. \tag{4.108}$$

The transverse character of the wave implies that $\mathbf{k}\cdot\mathbf{E} = 0$ and the dispersion relation for the low frequency Alfvén wave is

$$\omega^2 = \frac{k^2 v_A^2}{1 + v_A^2/c^2} \approx k^2 v_A^2 \quad \text{if} \quad v_A^2 \ll c^2. \tag{4.109}$$

We emphasise that we have not yet specified the direction of the constant magnetic field with respect to the direction of wave propagation. In fact, the magnetic field may be either parallel or perpendicular to the wave vector. Both waves have the same dispersion relation, but the wave dynamics are very different in the two cases:

(i) $\mathbf{B}_0 \parallel \mathbf{k}$: In this case the oscillating magnetic field associated with the wave is perpendicular to the constant magnetic field, $\mathbf{B}_0$. As the wave propagates, the resulting total magnetic field lines become rippled like waves on a string, see Fig. 4.16.

A very important result is obtained if we compare the swinging motion of the magnetic field with the oscillating motion of the particles as the wave propagates. Assume the constant magnetic field, $\mathbf{B}_0$, to be directed along $\hat{z}$, whereas the electric and magnetic fields of the wave, $\mathbf{E}_1$, and $\mathbf{B}_1$ are directed along $\hat{x}$ and $\hat{y}$ respectively. The dominating vector component of both the

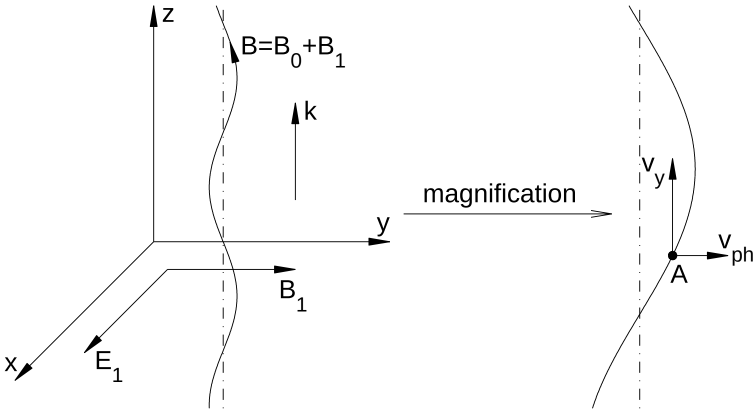


Figure 4.16. The propagation of an Alfvén wave along a magnetic field, showing the resulting "rippling" of the total magnetic field.

electron and ion velocity is the $\mathbf{E} \times \mathbf{B}$ drift, which is directed in the $-\hat{y}$ direction

$$\mathbf{v}_{e1} \approx \mathbf{v}_{i1} = \frac{\hat{\mathbf{E}}_1 \times \hat{\mathbf{B}}_0}{B_0^2} = -\frac{E_1}{B_0}\hat{y}. \tag{4.110}$$

But $E_1 = \omega B_1/k = v_{ph} B_1$, and consequently

$$\frac{v_y}{v_f} = -\frac{B_1}{B_0}. \tag{4.111}$$

As the wave propagates past a point A, the field line moves in the direction opposite to v_y. This, and Eq. (4.111) implies that the plasma particles oscillate in the same way as the magnetic field oscillates. In fact, they oscillate together as if the particles were tied to or "frozen" into the magnetic field.

(ii) $\mathbf{B}_0 \perp \mathbf{k}$: In this case we assume as before that $\mathbf{E} \parallel \hat{x}$ and $\mathbf{B}_0 \parallel \hat{z}$, but now we take $\mathbf{k} \parallel \hat{y}$. Thus the oscillating magnetic field is now parallel to the static magnetic field. Consequently, as the wave propagates, the direction of the total magnetic field does not change, but increases/decreases periodically depending on whether the oscillating field adds to or subtracts from the static field, see Fig. 4.17. Since the particles still can be considered to be tied to the field lines, the modulation of the magnetic field strength is accompanied by a corresponding plasma density variation in the direction of propagation. The wave propagates in a way similar to that of an acoustic wave and is appropriately called the *magneto-acoustic Alfvén wave*.

Thus, Alfvén waves with the same dispersion relation can propagate both perpendicular to and parallel to a constant magnetic field in a plasma. Examples of Alfvén waves play an important role in many physical situations

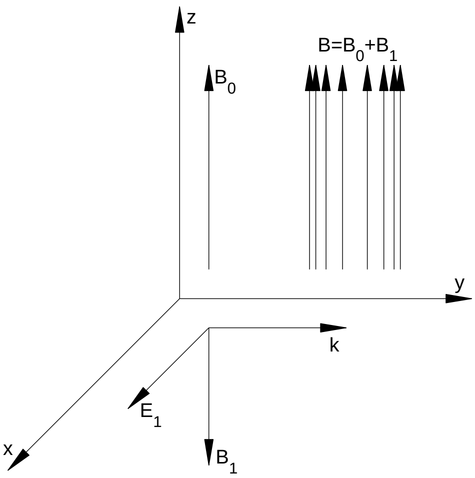


Figure 4.17. The propagation of an Alfvén wave across a magnetic field, showing the resulting "compression and decompression" of the total magnetic field.

ranging from wave heating in Tokamak fusion plasmas to astrophysical applications. An example of an Alfvén wave on a truly grand scale is given in the next section.

4.7.1 Galactic magnetic fields and Alfvén waves

In the previous section we showed how the Alfvén wave propagating along a constant magnetic field, gave rise to a "wavy" structure of the magnetic field, analogous to the "plucking of a string". This structure could be seen as if the plasma particles were tied to the magnetic field lines, oscillating in the rhythm of the transverse wave. This concept of a plasma "frozen into the magnetic field" has been very fruitful in understanding many phenomena in plasma physics in general and astrophysics in particular. We will here show how this can be used to determine the interstellar magnetic field in the spiral arms of galaxies, which should be compared with the application of Faraday rotation of the R- and L-waves presented in section 4.6.1. The analysis here is based on a short, but very elegant paper by the Nobel Prize winners S. Chandrasekhar and E. Fermi published almost 50 years ago.

It has been observed that light from distant stars tends to be linearly polarised. The observed polarisation property indicates that some mechanism is aligning the interstellar material, a possible candidate for this alignment being an interstellar magnetic field. The absorption properties of the light would then depend on whether the light is polarised perpendicular to or parallel to the magnetic field. From the polarisation maps of the galaxies, it

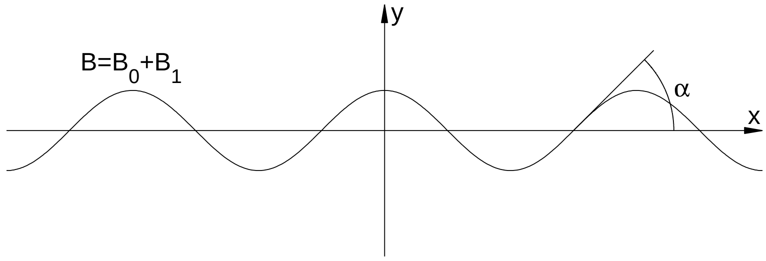


Figure 4.18. Qualitative plot showing the "waviness" of the galactic magnetic field caused by the propagating Alfvén wave.

is then inferred that there must be a galactic field oriented along the spiral arms. However, one also observes that the polarisation is not perfect, implying that the magnetic field lines are not completely straight, but rather have a wavy structure with a small mean deviation from the main direction. This deviation, α, is typically of the order of $\alpha \approx 0.2$ radians, see Fig. 4.18.

Chandrasekhar's and Fermi's interpretation of the this "waviness" was that turbulent plasma motion in the spiral arms tends to disturb the magnetic field lines, making them oscillate like a "plucked string" (remember the"frozen-in" property) when they excite Alfvén waves that propagate along the magnetic field lines, which has been qualitatively plotted in Fig. 4.18. If this picture is correct, the magnetic field oscillations can be described as a propagating Alfvén wave according to

$$y = A\cos(kx - \omega t) = A\cos[k(x - v_A t)]. \tag{4.112}$$

This implies that

$$\alpha^2 \approx \left\langle \left(\frac{\partial y}{\partial x}\right)^2 \right\rangle = \frac{A^2k^2}{2}. \tag{4.113}$$

On the other hand, since the particles can be viewed as frozen into the magnetic field, one can estimate the "plucking strength" as

$$\frac{1}{3}\langle v_{\rm turb}^2\rangle \approx \left\langle \left(\frac{\partial y}{\partial t}\right)^2 \right\rangle = \frac{A^2k^2v_A^2}{2}. \tag{4.114}$$

Eliminating the factor A^2k^2 between Eqs. (4.113) and (4.114), a relation is obtained between the Alfvén velocity, v_A, and the observables, turbulent velocity and deviation of the polarisation, as

$$v_A^2 = \frac{\langle v_{\rm turb}^2\rangle}{3\alpha^2}. \tag{4.115}$$

Using Eq. (4.108), this relation directly determines the magnetic field according to

$$B_0 = \sqrt{\frac{n_0 m_i \langle v_{\rm turb}^2\rangle}{3\epsilon_0\alpha^2c^2}}. \tag{4.116}$$

The plasma density and the averaged turbulent velocity are known from radio emission measurements, $n_0 m_i \approx 2 \times 10^{-24}$ kg/m^3 and $\langle v_{\rm turb}^2 \rangle^{1/2} \approx 5 \times 10^3$ m/s, respectively. Inserting this together with $\alpha \approx 0.2$ radians, one obtains as estimate for the galactic magnetic field strength, $B_0 \approx 7 \times 10^{-10}$ T, in rather good agreement with other independent results.

Chapter 5

Diffusion

In the present chapter we will study the influence of collisions on the dynamics of plasma particles and in particular on the confinement properties of plasmas. As a first step we will consider collisions in weakly ionised plasmas, where most of the collisions, to which the plasma particles are subjected, occur with neutral gas particles. We start by recapitulating some basic classical collision concepts.

5.1 Basic collision concepts

Consider a gas where the density of neutral particles is n_n (unit: m^{-3}), and their collision *cross section* is σ (unit: m^2). This implies that the *mean free path between collisions*, λ_m is given by

$$\lambda_m = \frac{1}{n_n \sigma}. \tag{5.1}$$

The physical background of this definition is that if a particle flux, $\Gamma(x)$ (number of particles per unit cross section area per unit time), is incident on a thin slab of width Δx, and area A, the relative area "shadowed" by the presence of neutral particles is

$$\begin{aligned} \frac{\text{blocked area}}{\text{total area}} &= \frac{\text{number of particles in slab} \times \text{cross section}}{\text{total area}} \\ &= \frac{n_n \Delta x A \sigma}{A} = \frac{\Delta x}{\lambda_m}. \end{aligned} \tag{5.2}$$

Consequently the outgoing flux from the slab, $\Gamma(x + \Delta x)$, is related to the incident flux as follows: $\Delta\Gamma(x) = \Gamma(x + \Delta x) - \Gamma(x) \approx -\Gamma(x)\Delta x/\lambda_m$, or expressed as a differential equation

$$\frac{d\Gamma(x)}{dx} = -\frac{\Gamma(x)}{\lambda_m}, \tag{5.3}$$

with the solution (assuming λ_m to be a constant) $\Gamma(x) = \Gamma_0 \exp(-x/\lambda_m)$. This implies that λ_m has the physical significance of exponential decay length for the particle flux.

We can now proceed to introduce the *mean time between collisions*, τ, through the natural definition

$$\tau \equiv \frac{\lambda_m}{v} = \frac{1}{n_n \sigma v}, \tag{5.4}$$

where v denotes the velocity of the plasma particles, and the neutrals are assumed stationary. Since the plasma particles have many different velocities, one usually averages over the corresponding velocity distribution function to obtain the collision frequency, ν, as $\nu = n_n \langle \sigma v \rangle$, where $\langle \rangle$ denotes averaging over the distribution. In particular, if the cross section, σ, is independent of energy, the collision frequency becomes simply $\nu = n_n \sigma v_{th}$, where v_{th} is the thermal velocity of the plasma particles.

5.2 Plasma diffusion in an non-magnetised weakly ionised gas

The simplest way of introducing the effects of collisions is by modifying the equation of motion. Assuming that the plasma particles loose their total momentum, $m\mathbf{v}$, at each collision, the plasma fluid will loose the momentum $nm\mathbf{v}\nu$ per second and the equation of motion, Eq. (4.12), can be augmented to read

$$nm \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{v} \right] = qn\mathbf{E} - \nabla p - mn\nu\mathbf{v}, \tag{5.5}$$

where we have used the assumption that the plasma is non-magnetised. In a stationary situation ($\partial/\partial t = 0$) and neglecting nonlinear effects, the particles will acquire a constant drift velocity, $\mathbf{v}_D$, given by

$$\mathbf{v}_D = \frac{q}{m\nu}\mathbf{E} - \frac{k_B T}{m\nu}\frac{\nabla n}{n}, \tag{5.6}$$

where we have assumed an isothermal plasma. The coefficient relating $\mathbf{v}_D$ and $\mathbf{E}$ is called the *mobility*, μ, and is defined by

$$\mu \equiv \frac{q}{m\nu}. \tag{5.7}$$

Correspondingly, the coefficient relating $\mathbf{v}_D$ and $\nabla n/n$ is called the *diffusion constant*, D, and is defined by

$$D \equiv \frac{k_B T}{m\nu}. \tag{5.8}$$

These transport coefficients are related by the so-called *Einstein relation*

$$D = \frac{k_B T}{q}\mu. \tag{5.9}$$

The particle flux, $\mathbf{\Gamma}$, corresponding to the velocity, $\mathbf{v}_D$, is given by

$$\mathbf{\Gamma} \equiv n\mathbf{v}_D = \mu n\mathbf{E} - D\nabla n. \tag{5.10}$$

In particular, for $\mathbf{E} = 0$, one obtains Fick's law

$$\mathbf{\Gamma} = -D\nabla n, \tag{5.11}$$

i.e., in an inhomogeneous plasma, a particle flux is set up by the collisions. This flux is proportional to the gradient of the density, ∇n, the constant of proportionality is the diffusion constant, D, and the flux is directed towards regions of lower density.

5.2.1 Ambipolar diffusion

The particle flux given by Eq. (5.10) implies that particles tend to leave regions of high density. Consequently the particle density must change in time. This change is governed by the continuity equation, Eq. (4.11),

$$\frac{\partial n}{\partial t} + \nabla \cdot (n\mathbf{v}) = 0, \tag{5.12}$$

or, equivalently

$$\frac{\partial n}{\partial t} = -\nabla \cdot \mathbf{\Gamma}. \tag{5.13}$$

However, the rate with which particles diffuse depends on their charge and (more importantly) on their mass. Assuming that the particles have the same temperature and have equal (but opposite) charges, we infer that the collision frequency scales as $\nu \propto 1/\sqrt{m}$, which implies that both D and μ scale as $D, \mu \propto 1/\sqrt{m}$. Consequently, electrons are much more mobile and will tend to leave a high density plasma region much quicker than the ions. This leads to a charge separation, which creates an electric field, since the plasma prefers to be quasi-neutral and have $n_e \approx n_i = n$. Thus even if there is no electric field originally, a field will be set up by the separation process. Clearly this field is directed in such a way that it tends to decrease the diffusion velocity of the electrons and increase the diffusion velocity of the ions. In steady state, the electric field has adjusted the electron and ion fluxes, $\mathbf{\Gamma}_\mathrm{e}$ and $\mathbf{\Gamma}_\mathrm{i}$, respectively, so that they are equal, i.e., $\mathbf{\Gamma}_\mathrm{e} = \mathbf{\Gamma}_\mathrm{i} \equiv \mathbf{\Gamma}_a$. This implies that the regulating so-called ambipolar field, $\mathbf{E}_a$, can be obtained from the equation

$$\mu_e n\mathbf{E}_a - D_e\nabla n = \mu_i n\mathbf{E}_a - D_i\nabla n, \tag{5.14}$$

from which $\mathbf{E}_a$ can be solved to be

$$\mathbf{E}_a = \frac{D_i - D_e}{\mu_i - \mu_e} \frac{\nabla n}{n}. \tag{5.15}$$

The common particle flux becomes

$$\mathbf{\Gamma}_{\rm a} = -\frac{\mu_i D_e - \mu_e D_i}{\mu_i - \mu_e} \nabla n \equiv -D_a \nabla n, \tag{5.16}$$

where we have introduced the *ambipolar diffusion constant*, D_a, given by

$$D_a \equiv \frac{\mu_i D_e - \mu_e D_i}{\mu_i - \mu_e} = \frac{\mu_i D_e + |\mu_e| D_i}{\mu_i + |\mu_e|}. \tag{5.17}$$

When the ion and electron temperatures are equal, the ambipolar diffusion constant can be simplified using $\mu_i \ll |\mu_e|$ to

$$D_a \approx \frac{\mu_i D_e + |\mu_e| D_i}{|\mu_e|} = 2D_i \ll D_e, \tag{5.18}$$

where the definitions (5.7) and (5.8) have been used. Thus the ambipolar diffusion rate is primarily determined by the heavy ions. The evolution of the (common) particle density, n, is then determined by the ambipolar diffusion equation

$$\frac{\partial n}{\partial t} = D_a \nabla^2 n, \tag{5.19}$$

which is obtained using Eqs. (5.13) and (5.16).

5.2.2 The diffusion equation

We will here give a short review of the basic properties of the diffusion equation

$$\frac{\partial n}{\partial t} = D \nabla^2 n. \tag{5.20}$$

A first simple approach to determine the dynamical behaviour of the solutions of the diffusion equation is to look for solutions of the form $n(t, x) = n_0 \sin(kx) e^{-\gamma t}$, where for simplicity we have assumed a one-dimensional geometry. Inserting this ansatz into the diffusion equation, we obtain the simple equation

$$-\gamma n = -D k^2 n, \tag{5.21}$$

which gives the decay constant $\gamma = Dk^2$. This implies that perturbations with wave number k, which have a characteristic dimension $L = 1/k$, decay with a characteristic time scale $\tau = 1/\gamma = L^2/D$. We emphasise the qualitative picture implied by this result: Density modulations with scale length L decay with a time constant τ, which is proportional to the scale length squared. Thus, modulations with short scale lengths diffuse away and disappear more rapidly than those with long scale lengths.

A more elaborate analysis can be done using the method of separation of variables. For this purpose we write $n(t, \mathbf{x}) = S(\mathbf{x})T(t)$. Inserting this into the diffusion equation and dividing the equation by $n = S(\mathbf{x})T(t)$ we obtain a separated set of equations

$$\frac{1}{T}\frac{dT}{dt} = D\frac{\nabla^2 S}{S} = \text{const} = -\frac{1}{\tau}, \tag{5.22}$$

where the separation constant has been written as $-1/\tau$. The equation for the time variation, can be directly solved to give $T(t) = e^{-t/\tau}$, which shows the physical importance of τ as the decay time. Note however that τ is still unknown. For the spatial variation we obtain the equation

$$\nabla^2 S + \frac{1}{D\tau} S = 0. \tag{5.23}$$

It is instructive to introduce the parameter $\kappa^2 = 1/(D\tau)$, which emphasises the fact that the equation for $S(\mathbf{x})$ is really an eigenvalue problem. The solution (the eigenfunction) and the eigenvalue must both be determined. A crucial role in all eigenvalue problems is played by the imposed boundary conditions. In the present case we will assume a simple one-dimensional situation with $S(0) = S(L) = 0$, i.e., the plasma is in contact with a conducting wall at $x = 0$ and $x = L$.

The corresponding solution of Eq. (5.23) is of the form

$$S(x) = A\sin(\kappa x) + B\cos(\kappa x), \tag{5.24}$$

where, $S(0) = 0$ implies that $B = 0$. In order to assure the vanishing of $S(x)$ at $x = L$, we must have

$$\kappa L = m\pi, \quad m = 1, 2, 3, \ldots \quad \Rightarrow \quad \kappa = \frac{m\pi}{L}. \tag{5.25}$$

The decay time, τ_m, of the eigenmode m is then determined by

$$\tau_m = \frac{L^2}{m^2\pi^2 D}. \tag{5.26}$$

It is clear that the higher order modes disappear most rapidly and that eventually only the lowest order mode remains and decays with the time constant

$$\tau_0 = \frac{L^2}{\pi^2 D}. \tag{5.27}$$

This behaviour is illustrated in Fig. 5.1.

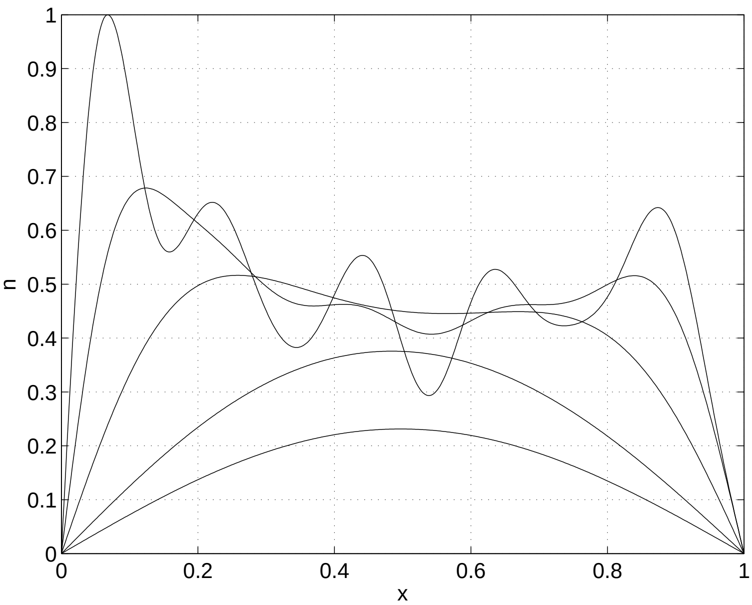


Figure 5.1. The particle density at five different times (not equally spaced). Notice the rapid vanishing of the small scale density variations and the slow decay of the longest scale variation.

5.2.3 Ambipolar diffusion in TR-switches

Ambipolar diffusion plays an interesting and important role in TR-switch operation (see the introduction in chapter 4). If the microwave input power into the switch is increased slowly, the transmitted power first increases linearly with input power, but when breakdown occurs the transmitted power drops to a low level, see Fig. 4.8. The electron density in the TR-switch is determined by the competition between ionisation, which increases the electron density, and diffusion, which makes electrons diffuse to the walls where they are lost. The corresponding continuity equation for the electron density reads

$$\frac{\partial n}{\partial t} = D\nabla^2 n + \nu_i n. \tag{5.28}$$

Using the fact that the diffusion term can be modelled in terms of a decay time, τ, or equivalently, a loss frequency, ν_D, we can simplify the continuity equation to read

$$\frac{dn}{dt} = -\nu_D n + \nu_i n. \tag{5.29}$$

This implies that the breakdown condition can be written in the physically obvious form $\nu_i = \nu_D$, i.e., the ionisation frequency equals the loss frequency. In Argon, which is the conventional filling gas in TR-switches, the ionisation frequency is proportional to E^2, i.e., proportional to the microwave power, P. Since the stationary state of the electron density equation corresponds to $P = P_b$, where P_b is the breakdown power, it might at first seem like the transmitted power should be clamped at the level $P = P_b$ for increasing power, rather than fall to a level $P = P_s$, much lower than P_b. Furthermore, when the incident power is decreased from above the breakdown limit, the transmitted power does not jump back to the value $P \approx P_b$ when the incident power passes P_b. Instead the transmitted power stays at the low level, P_s, until the incident power reaches $P = P_s$. Only then does the transmitted power decrease linearly with incident power. The large drop in transmitted power as well as the hysteresis effect (seen in Fig. 4.8) are both due to the phenomenon of ambipolar diffusion. When breakdown occurs, the generated electrons rapidly start to leave the breakdown region by diffusion. The corresponding charge separation transforms the diffusion flow from free (electron) diffusion (with diffusion constant D_e) to ambipolar diffusion (with diffusion constant D_a). Since $\nu_i \propto \nu_D \propto D$, we infer that $P_b \propto D_e$ whereas the sustainment power $P_s \propto D_a$. In a situation where the electron and ion temperatures are very different (which is the case in the breakdown plasma), Eq. (5.17) can be used to relate D_e and D_a as

$$D_a \approx D_e \sqrt{\frac{m_e T_e}{m_i T_i}}, \tag{5.30}$$

which implies that the ratio of the breakdown power to the sustainment power should be given by

$$\frac{P_s}{P_b} \sim \sqrt{\frac{m_e T_e}{m_i T_i}}. \tag{5.31}$$

For Argon $m_i/m_e \approx 40 \times 1840$ and the temperatures can be estimated as $T_i \approx 1000$ K, whereas $T_e \approx T_{\rm ion}/2$, where $T_{\rm ion}$ denotes the ionisation temperature of Argon ($T_{\rm ion} \approx 10^5$ K). This implies that $P_b - P_s \approx 15$ dB, in good agreement with the observed drop in transmitted power, see Fig. 4.8. Physically this means that the microwave power needed to create a breakdown plasma is larger than that needed to sustain it, once it is created.

5.3 Plasma diffusion in a magnetised weakly ionised gas

We will now consider the effect of a magnetic field on the diffusion in a weakly ionised gas. First of all it is clear that the magnetic field will not affect the diffusional motion along the field itself. This diffusion process will proceed as described in the previous paragraphs. However, for the diffusion across the magnetic field, and along $\mathbf{E}$ and ∇n, the collisions between the plasma particles and neutral background particles will play a completely

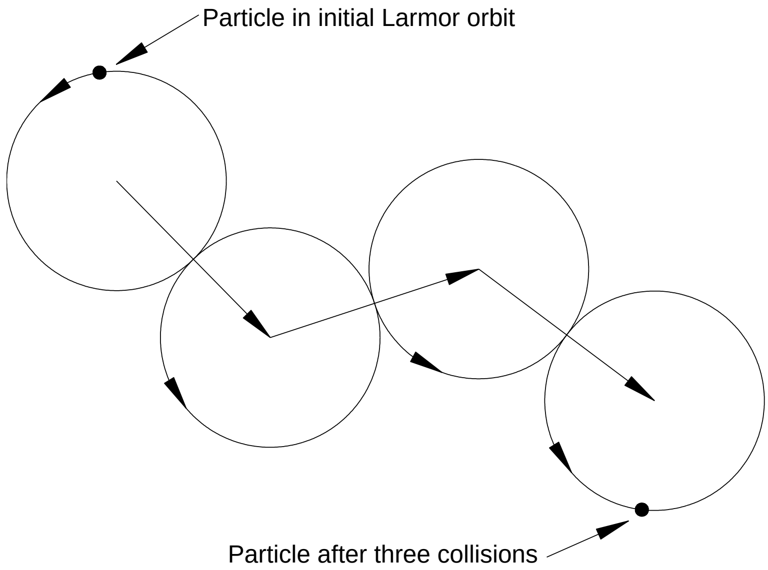


Figure 5.2. Qualitative plot of the particle motion in a magnetised plasma in the presence of collisions.

different role. In fact if there were no collisions, no diffusion or transport would take place across the magnetic field lines in the direction of $\mathbf{E}$ and ∇n, since the particles would be bound to the field lines moving in their Larmor orbits. However, if a collision occurs, the particle (or more precisely the guiding centre of the particle orbit) may jump to another field line, see Fig. 5.2. This implies that the particles now diffuse due to the collisions and not in spite of the collisions as in a non-magnetised plasma.

When particles diffuse they can be considered to perform a random walk in space. In classical random walk theory, the diffusion constant, D, is given by

$$D \approx \nu(\Delta x)^2, \tag{5.32}$$

where ν is the collision frequency and Δx is the step length in the diffusion process. For non-magnetised plasmas we have found that $D = k_B T/(m\nu)$, but since $k_B T/m \approx v^2$ and $v/\nu \approx \lambda_m$, we have $D \approx \nu\lambda_m^2$ in accordance with the general expression with the step length equal to the mean free path between collisions.

In the case of a magnetised plasma, the step length in the stochastic random walk process should be the Larmor radius, r_L, and we expect that the diffusion constant for diffusion perpendicular to the magnetic field, $D_\perp$, should be given by

$$D_\perp \approx \nu r_L^2, \tag{5.33}$$

provided that $r_L \ll \lambda_m$.

In order to investigate the diffusion in magnetised plasmas more quantitatively we start from the equation of motion, i.e.,

$$mn\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = nq(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - k_B T\nabla n - mn\nu\mathbf{v}. \tag{5.34}$$

As before, in the non-magnetised case, we consider the stationary drift velocity (across the field) that arises as a consequence of the force balance, i.e., the velocity, $\mathbf{v}$, which is determined by

$$0 = nq(\mathbf{E} + \mathbf{v}\times\mathbf{B}) - k_B T\nabla n - mn\nu\mathbf{v}. \tag{5.35}$$

Solving for $\mathbf{v}$ as before we obtain

$$\mathbf{v} = \frac{q}{m\nu}\mathbf{E} - \frac{k_B T}{m\nu}\frac{\nabla n}{n} + \frac{q}{m\nu}\mathbf{v}\times\mathbf{B}. \tag{5.36}$$

Operating with $\times\mathbf{B}$ on the last equation we obtain

$$\mathbf{v}\times\mathbf{B} = \mu\mathbf{E}\times\mathbf{B} - D\frac{\nabla n}{n}\times\mathbf{B} - \mu B^2\mathbf{v}. \tag{5.37}$$

If the latter result is used to replace the $\mathbf{v}\times\mathbf{B}$-term in Eq. (5.35), we can express $\mathbf{v}$ as a function of $\mathbf{E}$, $\mathbf{B}$ and ∇n as follows

$$\mathbf{v} = \mu_\perp\mathbf{E} - D_\perp\frac{\nabla n}{n} + \frac{\mu^2 B^2}{1+\mu^2 B^2}(\mathbf{v}_{\mathbf{E}\times\mathbf{B}} + \mathbf{v}_{\mathrm{diam}}), \tag{5.38}$$

where the mobility and the diffusion constant in the direction orthogonal to the magnetic field, $\mu_\perp$ and $D_\perp$, respectively, are given by

$$\mu_\perp = \frac{\mu}{1+\mu^2 B^2}, \tag{5.39}$$

$$D_\perp = \frac{D}{1+\mu^2 B^2}. \tag{5.40}$$

Furthermore, $\mathbf{v}_{\mathbf{E}\times\mathbf{B}}$ and $\mathbf{v}_{\mathrm{diam}}$ denote the $\mathbf{E}\times\mathbf{B}$ and diamagnetic drifts,

$$\mathbf{v}_{\mathbf{E}\times\mathbf{B}} = \frac{\mathbf{E}\times\mathbf{B}}{B^2}, \tag{5.41}$$

$$\mathbf{v}_{\mathrm{diam}} = -\frac{k_B T}{q}\frac{\nabla n\times\mathbf{B}}{nB^2}. \tag{5.42}$$

The diamagnetic drift is obtained from Eq. (2.25), where the force is due to the pressure per particle, i.e., $\mathbf{F} = -\nabla p/n$. This result implies that the drift motion consists of two parts: (i) a diffusional motion along $\mathbf{E}$ and ∇n and (ii) a drift motion perpendicular to $\mathbf{B}$ (in the directions of $\mathbf{E}\times\mathbf{B}$ and $\nabla n\times\mathbf{B}$). However, the coefficients of these drifts are determined by the simultaneous presence of collisions and magnetic field. The parameter

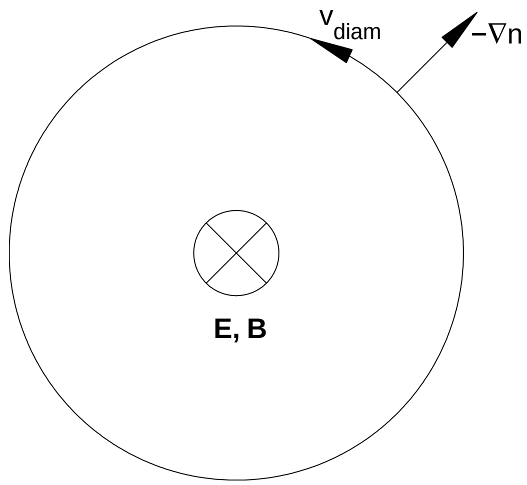


Figure 5.3. Qualitative plot showing the (harmless) poloidal flow caused by the diamagnetic drift in a cylindrical geometry.

$\mu^2 B^2$ determines the relative importance of the modifications. This crucial parameter can be written in physically more pregnant forms as follows

$$\mu^2 B^2 = \frac{q^2 B^2}{m^2 \nu^2} = \frac{\omega_c^2}{\nu^2} = \frac{\omega_c^2}{v^2}\frac{v^2}{\nu^2} = \frac{\lambda_m^2}{r_L^2}, \tag{5.43}$$

i.e., the magnitude of $\mu^2 B^2$ is determined by the ratio between the cyclotron frequency and the collision frequency, or equivalently, by the ratio of the mean free path between collisions and the Larmor radius.

In the limit when $\omega_c/\nu \to \infty$, or equivalently $\lambda_m/r_L \to \infty$, the parameter $\mu B \to \infty$, the expression for the drift velocity reduces to the usual collisionless drifts across the magnetic field

$$\mathbf{v} = \mathbf{v}_{\mathbf{E}\times\mathbf{B}} + \mathbf{v}_{\mathrm{diam}}. \tag{5.44}$$

In a cylindrical plasma, where the density gradients tend to be directed radially and the electric fields driving the current are directed along $\mathbf{B}$, these drifts are harmless in the sense that they only cause poloidal drifts within the plasma and no plasma transport to the walls, Fig. 5.3. The presence of the collisions will give rise to a finite value of μB, which reduces these drift velocities by multiplying by the factor $\mu^2 B^2/(1 + \mu^2 B^2) < 1$. Correspondingly, when $\omega_c/\nu \to 0$, or equivalently $\lambda_m/r_L \to 0$, the parameter $\mu B \to 0$, we regain the previous collisional result for the non-magnetised case, i.e., $\mathbf{v}_D = \mu\mathbf{E} - D\nabla n/n$.

We conclude that the presence of a magnetic field implies that the diffusion flow along the directions of $\mathbf{E}$ and, in particular, along the direction of

∇n tends to be suppressed. This fact is an important feature in all schemes for magnetic confinement of plasmas.

5.4 Collisions in a magnetised plasma

In a fully ionised plasma, the collisions take place between charged particles. This implies that the collision dynamics will be quite different from those where charged particles collide with a neutral particle. The primary reason for this is that a collision between two charged particles involves long range Coulomb interactions.

5.4.1 Collisions between like particles

We will first show that collisions between like particles, i.e., between electrons or between ions do not lead to any diffusion flow at all in a magnetised plasma.

The equation of motion for a charged particle in the presence of only a magnetic field is

$$m\frac{d\mathbf{v}}{dt} = q\mathbf{v} \times \mathbf{B}_0. \tag{5.45}$$

The total position vector of the particle, $\mathbf{r}$, is composed by the position of the guiding centre, $\mathbf{r}_0$, and the vector describing the Larmor rotation, $\mathbf{r}_L$, i.e., $\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_L$. Assuming there is no drift velocity of the guiding centre, we have $\mathbf{v} = d\mathbf{r}_L/dt$, and Eq. (5.45) can be integrated once to read

$$m\mathbf{v} = q\mathbf{r}_L \times \mathbf{B}_0. \tag{5.46}$$

Solving for $\mathbf{r}_L$ by taking the cross product of this equation with $\mathbf{B}_0$ we obtain

$$\mathbf{r}_L = -\frac{m\mathbf{v} \times \mathbf{B}_0}{qB_0^2}. \tag{5.47}$$

The total position vector is consequently given by

$$\mathbf{r} = \mathbf{r}_0 + \mathbf{r}_L = \mathbf{r}_0 - \frac{\mathbf{p} \times \mathbf{B}_0}{qB_0^2}, \tag{5.48}$$

where $\mathbf{p}$ denotes the momentum of the particle. Assume now that two particles collide, exchanging a momentum $\Delta\mathbf{p}$. During the collision, the position vectors of both the colliding particles must be continuous, i.e., $\Delta\mathbf{r} = 0$. This implies that the guiding centre position must change an amount $\Delta\mathbf{r}_0$ determined by

$$\Delta\mathbf{r} = \Delta\mathbf{r}_0 - \frac{\Delta\mathbf{p} \times \mathbf{B}_0}{qB_0^2} = 0, \tag{5.49}$$

i.e.,

$$\Delta\mathbf{r}_0 = \frac{\Delta\mathbf{p} \times \mathbf{B}_0}{qB_0^2}. \tag{5.50}$$

The centre of gravity of the system consisting of the two colliding particles, denoted by $\mathbf{r}_T$, is given by

$$\mathbf{r}_T = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}, \tag{5.51}$$

and the corresponding time average is

$$\langle\mathbf{r}_T\rangle = \frac{m_1\mathbf{r}_{0,1} + m_2\mathbf{r}_{0,2}}{m_1+m_2}. \tag{5.52}$$

During the collision, the exchange of momentum ($\Delta\mathbf{p}_1 = -\Delta\mathbf{p}_2 = \Delta\mathbf{p}$) leads to a change in mean position vector given by

$$\Delta\langle\mathbf{r}_T\rangle = \frac{m_1\Delta\mathbf{r}_{0,1} + m_2\Delta\mathbf{r}_{0,2}}{m_1+m_2} = \frac{\Delta\mathbf{p}\times\mathbf{B}_0}{(m_1+m_2)B_0^2}\left(\frac{m_1}{q_1} - \frac{m_2}{q_2}\right). \tag{5.53}$$

However, we infer from this that for collisions between like particles where $m_1/q_1 = m_2/q_2$, the centre of gravity for the two particle system does not change, i.e., no diffusion occurs. However, collisions between unlike particles will lead to a change of the centre of gravity and diffusion.

5.4.2 Plasma resistivity

Collisions between electrons and ions will lead to an exchange of momentum between the electron and ion fluids. This effect gives rise to an extra force, a friction force, in the equations of motion, Eq. (4.12), for the fluids according to

$$m_i n_i \left[\frac{\partial\mathbf{v}_i}{\partial t} + (\mathbf{v}_i\cdot\nabla)\mathbf{v}_i\right] = e n_i(\mathbf{E} + \mathbf{v}_i\times\mathbf{B}) - \nabla p_i + \mathbf{P}_{ie}, \tag{5.54}$$

$$m_e n_e \left[\frac{\partial\mathbf{v}_e}{\partial t} + (\mathbf{v}_e\cdot\nabla)\mathbf{v}_e\right] = -e n_e(\mathbf{E} + \mathbf{v}_e\times\mathbf{B}) - \nabla p_e + \mathbf{P}_{ei}, \tag{5.55}$$

where $\mathbf{P}_{ie}$ and $\mathbf{P}_{ei}$ represents the momentum transfer from the electron fluid to the ion fluid, and from the ion fluid to the electron fluid, respectively. Clearly this implies that $\mathbf{P}_{ei} = -\mathbf{P}_{ie}$. The momentum transfer can be written

$$\mathbf{P}_{ei} = m_e n(\mathbf{v}_i - \mathbf{v}_e)\nu_{ei}, \tag{5.56}$$

where we have assumed that $n = n_e \approx n_i$ and also introduced the electron-ion collision frequency, ν_{ei}. It is clear that the momentum transfer must have the following characteristics

$$\mathbf{P} \propto \begin{cases} \mathbf{v}_i - \mathbf{v}_e, & \text{relative velocity} \\ e^2, & \text{Coulomb collisions} \\ n_e n_i \approx n^2. & \end{cases} \tag{5.57}$$

The remaining constant of proportionality is the *plasma resistivity*, η, which is defined from

$$\mathbf{P}_{ei} = \eta n^2 e^2 (\mathbf{v}_i - \mathbf{v}_e). \tag{5.58}$$

Comparing Eq. (5.58) with the expression for the momentum transfer term in Eq. (5.56), we obtain

$$\nu_{ei} = \frac{n^2 e^2}{m_e n}\eta = \epsilon_0 \frac{ne^2}{m_e \epsilon_0}\eta = \epsilon_0 \omega_{p,e}^2 \eta. \tag{5.59}$$

Finally we emphasise that the relation $\mathbf{P}_{ie} = m_i n(\mathbf{v}_e - \mathbf{v}_i)\nu_{ie}$, together with the fact that $\mathbf{P}_{ei} = -\mathbf{P}_{ie}$, implies the (at first sight) strange relation

$$m_e \nu_{ei} = m_i \nu_{ie}. \tag{5.60}$$

This violation of reciprocity is due to the very different nature of the collision dynamics when a heavy particle collides with a light particle and vice versa. Since this is an important aspect of collisions we will analyse this point in some more detail in the coming sections.

5.4.3 Comments on the concept of collision frequency

A first indication of the subtlety involved in the concept of collision frequency and an explanation of the seemingly strange relation between the electron-ion and ion-electron collision frequencies can be given as follows: The friction forces in the equations of motion for the electron and ion fluids can be written symmetrically as

$$\mathbf{P}_{ei} = m_e n(\mathbf{v}_i - \mathbf{v}_e)\nu_{ei}, \tag{5.61}$$
$$\mathbf{P}_{ie} = m_i n(\mathbf{v}_e - \mathbf{v}_i)\nu_{ie}. \tag{5.62}$$

As mentioned before, for elastic collisions $\mathbf{P}_{ei} = -\mathbf{P}_{ie}$, which directly implies that $m_e \nu_{ei} = m_i \nu_{ie}$. How can the collision frequency between electrons and ions be different from that between ions and electrons when only the relative velocity between the particles is involved? The explanation lies in what is meant by a collision. The collision time (i.e., the inverse of the collision frequency) is the time after which the momentum of the particle has changed significantly, i.e., the change in momentum is comparable to the original momentum itself ($|\Delta\mathbf{p}| \sim |\mathbf{p}|$).

Consider the detailed dynamics involved when a light particle (e.g., an electron) of mass m_e collides (one-dimensionally) with a heavy particle (e.g., an ion) of mass m_i. During the collision (assumed elastic), total energy and momentum of the two-particle system are conserved. This implies that

$$m_e u = m_e u' + m_i w', \tag{5.63}$$
$$\frac{m_e u^2}{2} = \frac{m_e u'^2}{2} + \frac{m_i w'^2}{2}. \tag{5.64}$$

The energy relation can be written as

$$m_e(u^2 - u'^2) = m_e(u - u')(u + u') = m_i w'^2, \tag{5.65}$$

which, using the momentum relation $m_e(u - u') = m_i w'$, gives the simple result $u + u' = w'$. Consequently there are two relations expressing w' in terms of u and u',

$$w' = \frac{m_e}{m_i}(u - u') = u + u'. \tag{5.66}$$

The velocity of the light particle after the collision can be expressed as

$$u' = -u\frac{1 - m_e/m_i}{1 + m_e/m_i} \approx -u + 2\frac{m_e}{m_i}u. \tag{5.67}$$

The corresponding change of momentum, Δp_e, is then

$$\Delta p_e = m_e(u' - u) \approx -2m_e u = -2p_e. \tag{5.68}$$

Thus, when an electron collides with an ion, the change in momentum is directly of the order of the momentum itself, i.e., a single encounter is enough to cause a "collision".

The situation is completely different when an ion collides with an electron. In order to study this situation we can use our previous expressions and just interchange m_e and m_i and the meaning of u and w. The velocity of the ion after the encounter is then

$$u' = -u\frac{1 - m_i/m_e}{1 + m_i/m_e} = u\frac{1 - m_e/m_i}{1 + m_e/m_i} \approx u - 2\frac{m_e}{m_i}u. \tag{5.69}$$

The corresponding change of the momentum of the ion, Δp_i, is then

$$\Delta p_i = m_i(u' - u) \approx -2m_e u = -2\frac{m_e}{m_i}p_i. \tag{5.70}$$

At the encounter, the ion momentum only changes by a fraction $2m_e/m_i$! This small change does not represent a collision in the sense of our definition. In order to change the ion momentum by such an amount that $|\Delta p_i| \approx 2|p_i|$, a large number, N, of encounters is needed. In fact, N can be estimated from the relation $N|\Delta p_i| \approx 2p_i$. The true collision time for ions on electrons, τ_{ie}, is then

$$\tau_{ie} = N\tau_{ei} = \frac{2|p_i|}{|\Delta p_i|}\tau_{ei} = \frac{m_i}{m_e}\tau_{ei}. \tag{5.71}$$

Clearly this implies the previously stated result

$$m_e\nu_{ei} = m_i\nu_{ie}. \tag{5.72}$$

This asymmetry of collisions can be qualitatively summarised by saying that when a light particle encounters a heavy particle, it "bounces off" with a corresponding large change of momentum, but a heavy particle has to "run over" a large number of light particles before its momentum has been significantly changed.

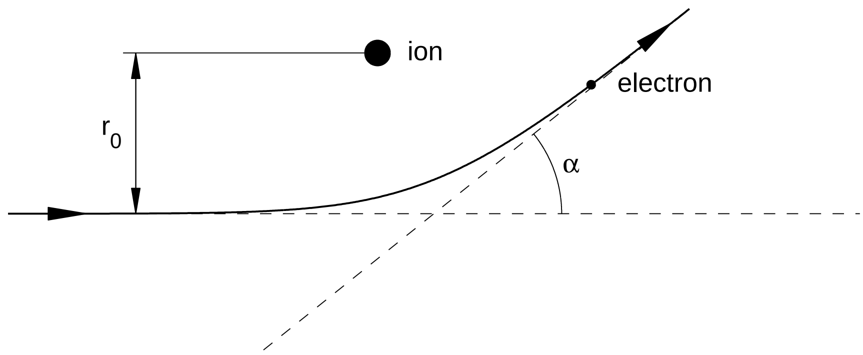


Figure 5.4. Qualitative picture of the Coulomb interaction between an electron and an ion.

5.5 Simple model for Coulomb collisions

In collisions involving charged and neutral particles, the collision cross section is rather well defined since the interaction forces rapidly decay with distance outside a certain characteristic radius. This also means that such collisions are essentially two-particle processes. However, for collisions between charged particles, which interact by means of the Coulomb force, the interaction force decays rather slowly with distance of separation. This fact has two consequences: It is less obvious how to define the collision cross section and collisions tend to involve many particles simultaneously.

A simple model that gives a qualitatively correct result for the collision cross section and conveys some of the characteristic properties of Coulomb collisions can be given as follows: Consider an electron approaching an ion, which, in the absence of Coulomb interactions, would have passed a distance r_0 (impact distance) from the ion, see Fig. 5.4. Under the influence of the Coulomb force, $\mathbf{F} = -e^2/(4\pi\epsilon_0 r^2)\hat{r}$, the electron will be attracted towards the ion and leaves the interaction region on a trajectory making an angle α with the incident direction. Clearly this deflection angle depends on the impact parameter r_0. Very roughly we can say that the Coulomb force, F, acts during a time, t_{pass}, approximately given by $t_{\text{pass}} \approx r_0/v$, where v is the initial velocity of the particle. The corresponding change of electron momentum is then

$$\Delta(m_e v) \approx |F t_{\text{pass}}| \approx \frac{e^2}{4\pi\epsilon_0 r_0 v}. \tag{5.73}$$

As before we define a true collision as an event where the change of momentum is of the order of the momentum itself, i.e., the deflection angle should be of the order of 90°. This implies that the impact parameter must be sufficiently small, so that

$$\Delta(m_e v) \approx \frac{e^2}{4\pi\epsilon_0 r_0 v} \sim m_e v. \tag{5.74}$$

The cross section for the Coulomb collision, σ, can then be estimated as

$$\sigma \approx \pi r_0^2 \approx \frac{\pi e^4}{(4\pi\epsilon_0 m_e v^2)^2}. \tag{5.75}$$

The corresponding collision frequency, ν_{ei}, is then given by

$$\nu_{ei} = n\sigma v \approx \frac{n\pi e^4}{(4\pi\epsilon_0 m_e)^2 v^3}. \tag{5.76}$$

We note the inverse dependence on particle velocity, which is an artifact caused by the properties of the Coulomb interaction process. This is in stark contrast to the situation of collisions between charged particles and neutrals where the collision frequency increases with particle velocity.

The resistivity, η, in a fully ionised plasma can now be evaluated, using Eq. (5.59), to be

$$\eta = \frac{m_e}{ne^2}\nu_{ei} \approx \frac{\pi e^2}{(4\pi\epsilon_0)^2 m_e v^3}. \tag{5.77}$$

Replacing v^2 with the averaged value over the (assumed) Maxwellian velocity distribution ($m_e v^2 \rightarrow k_B T_e$), we finally obtain

$$\eta \approx \frac{\pi e^2 \sqrt{m_e}}{(4\pi\epsilon_0)^2 (k_B T_e)^{3/2}}. \tag{5.78}$$

This derivation of the collision frequency and the resistivity is based on the two particle collision picture with a strong deflection as a result of the interaction. However, the long range nature of the Coulomb force makes the cumulative effect of many "small-angle collisions" important. It can be shown by a more detailed analysis that the expressions for ν_{ei} and η derived here are essentially correct, but should be multiplied by the so-called Coulomb logarithm, $\ln\Lambda$, where $\Lambda = \langle\lambda_D/r_0\rangle$ and $\langle\rangle$ denotes averaging over the Maxwellian velocity distribution. The Coulomb logarithm includes the effect of many small-angle collisions corresponding to large impact parameters r_0, but also takes into account the fact that for $r_0 > \lambda_D$, the Coulomb force is screened out and no significant collisional interactions take place. The logarithmic dependence involved in the Coulomb logarithm implies a rather weak further dependence on plasma density and temperature, and typically $\ln\Lambda \approx 10$–20.

5.6 The plasma resistivity

The physical significance of the parameter η as a resistivity was not obvious when it was introduced. Some complementary comments on this issue will clarify the physical importance of η. Consider for simplicity the case when the forces balance each other in the equation of motion for the electron fluid, Eq. (5.55), and when also $\mathbf{B}_0 = 0$ and $p_e = 0$, i.e.,

$$0 = -en\mathbf{E} + \mathbf{P}_{ei}. \tag{5.79}$$

Combining this with the previous expression for $\mathbf{P}_{ei}$, Eq. (5.58), we obtain

$$\mathbf{P}_{ei} = \eta n^2 e^2 (\mathbf{v}_i - \mathbf{v}_e) = \eta en\mathbf{j} = en\mathbf{E}. \tag{5.80}$$

This clearly implies

$$\mathbf{E} = \eta\mathbf{j}. \tag{5.81}$$

which is simply Ohm's law with η playing the role of specific resistivity. There are several important features worth emphasising in the expression for the resistivity of a fully ionised plasma.

The resistivity is independent of the plasma density, n.

This implies the somewhat unexpected feature that the current driven by an applied electric field, $\mathbf{E}$, in a fully ionised plasma, is independent of the plasma density. For a weakly ionised plasma, the current was given by

$$\mathbf{j} = -en\mathbf{v} = -en\mu\mathbf{E} = \eta\mathbf{E}, \tag{5.82}$$

where $\eta = |en\mu| = ne^2/(m_e\nu) \propto n$, since in this case the collision frequency is independent of the plasma density (being rather proportional to the density of the neutrals). However in a fully ionised plasma, the number density of scatterers is equal to the plasma density and $\nu \propto n$. Consequently the dependencies on n in the numerator and the denominator in the expression for η cancel, and η becomes independent of n.

The resistivity decreases with increasing plasma temperature, T_e

The expression for the resistivity, Eq. (5.78), implies that a fully ionised plasma becomes an increasingly good conductor as the temperature increases. In fact the resistivity can be expressed in physical units as

$$\eta \approx 5.2 \times 10^{-5} \frac{Z_i \ln \Lambda}{T_e^{3/2}\,[\mathrm{eV}]} \quad [\Omega\mathrm{m}], \tag{5.83}$$

where Z_i denotes the charge number of the plasma ions. Thus for a hydrogen plasma with $T_e = 100$ eV, we have $\eta \approx 5{\times}10^{-7}$ Ωm, whereas for $T_e = 10$ keV, we have $\eta \approx 5 \times 10^{-10}$ Ωm. For comparison we note that the resistivity of metallic conductors like stainless steel and copper are 7×10^{-7} Ωm and 2×10^{-8} Ωm, respectively. Thus, the resistivity of a plasma is very low at high temperatures. This implies that hot plasmas of temperatures $T_e > 10$ keV can in many situations be considered as collisionless.

Electron runaway

Another interesting artifact of the velocity dependence of the collision frequency ($\nu_{ei} \propto 1/v^3$) is that if a strong electric field is suddenly applied to a fully ionised plasma, high velocity electrons (in the tail of the velocity distribution) may experience that the friction force is not able to balance the accelerating force of the electric field. This implies that the electrons start to increase their velocity leading to an even smaller friction force. The result of this process is an unimpeded acceleration of the electrons to extremely high energies where other (non-collisional) processes become important and affect the electron dynamics. This process is called *electron runaway* and is a potential problem in future Tokamak reactors, since the runaway electron may loose its magnetic confinement and strike the walls of the reactor.

An estimate of the importance of this effect can be obtained by considering the equation of motion of a typical electron under the influence of an applied electric field, $\mathbf{E} = -E\hat{x}$, and the collisional friction force

$$m_e \frac{dv_e}{dt} = eE - m_e \nu_{ei} v_e. \tag{5.84}$$

Using the expression for ν_{ei}, we can write

$$m_e \frac{dv_e}{dt} = eE \left(1 - \frac{v_c^2}{v_e^2}\right), \tag{5.85}$$

where the critical velocity, v_c, is given by

$$v_c^2 \equiv \frac{\pi n e^3 \ln \Lambda}{(4\pi\epsilon_0)^2 m_e E}. \tag{5.86}$$

Clearly, if the initial velocity of the electron, $v_e(0)$, is smaller than v_c, the velocity will reach a balance between acceleration and deceleration. However if $v_e(0) > v_c$, we get $dv_e(0)/dt > 0$ and the electron starts to accelerate. This implies that the friction force decreases and the acceleration can continue with ever increasing speed. Thus, the electron "runs away".

From a practical point of view, the importance of this phenomenon depends on the relative magnitude of v_c and the thermal velocity of the electrons, v_{th}. If $v_{th} \ll v_c$, only a few electrons high up in the velocity distribution are affected by the runaway phenomenon. However, if $v_c \approx v_{th}$, a large portion of the electron population may be accelerated leading to a "slide away" phenomenon where a significant part of the electron population experiences runaway with subsequent serious consequences for the plasma confinement. The critical electric field at which $v_c = v_{th}$ is called the *Dreicer field*, E_c, and is determined by

$$E_c = \frac{\pi n e^3 \ln \Lambda}{(4\pi\epsilon_0)^2 k_B T_e}. \tag{5.87}$$

For a plasma temperature $T_e \approx 1$ keV and a plasma density $n \approx 10^{19}$ m^{-3}, the corresponding Dreicer field is $E_c \approx 2$ V/m. Typical "loop" voltages in large Tokamaks are $V_{\text{loop}} \approx 1$ V, which with a major radius of 1 m implies an electric field of $E \approx 0.2$ V/m, which is well below the Dreicer limit. However, in connection with so-called *disruptions*, events where the plasma is suddenly cooled down, large inductive electric fields are produced (when the plasma is trying to preserve the plasma current in spite of the strongly increased resistivity). These disruption fields may become of the order or even larger than the Dreicer field and may cause significant runaway electron production and serious damage on the walls of the containment vessel.

5.6.1 Ohmic heating of plasmas

We will here give an application illustrating the importance of the temperature dependence of the resistivity by analysing the method of ohmic plasma heating. The principle is very simple: By passing a current through a plasma, the ohmic losses due to the finite resistivity will heat the plasma. The dissipated ohmic power is primarily going to the electrons, which leads to an increase of the electron energy content, which can be modelled as follows:

$$\frac{d}{dt}\left(\frac{3nk_BT_e}{2}\right) = jE = \eta j^2. \tag{5.88}$$

Since $\eta \propto T_e^{-3/2}$, we can write

$$\frac{dT_e(t)}{dt} = CT_e^{-3/2}, \tag{5.89}$$

where the constant C is given by $C \approx 4\times 10^{15} j^2/n$ (provided T_e is measured in eV). Eq. (5.89) is easily solved to give

$$T_e(t) = T_e(0)\left(1 + \frac{t}{t_0}\right)^{2/5}, \tag{5.90}$$

where the characteristic time constant t_0 is given by $t_0 = 2T_e(0)^{5/2}/(5C) \propto T_e(0)^{5/2} j^2/n$. Two important physical points should be emphasised in connection with this result: (i) The time constant for the heating process, t_0, increases rapidly for increasing temperatures and (ii) the heating rate becomes very slow ($T_e(t) \approx (5Ct/2)^{2/5} \propto (j^2 t/n)^{2/5}$). Thus there are three ways of obtaining high final temperatures in Tokamak plasmas.

Long heating time

This is not a viable alternative since the heating time is limited by the energy confinement time. For heating times longer than the energy confinement time, the heating power will just disappear to the walls of the containment vessel.

High current

This is neither a good alternative since the current is limited by the so-called Kruskal-Shafranov criterion, which determines the critical current at which the plasma becomes unstable and looses its good confinement properties.

Low density

Again this option is closed by several practical constraints, which all favour high densities: The Lawson criterion implies that the density-temperature product should be large and furthermore a small density leads to other undesirable features as, e.g., electron runaway.

As an illustration we consider what can be achieved using ohmic heating in the JET Tokamak. The energy confinement time is of the order of 1 s and the density is typically $n \approx 5 \times 10^{19}$ m^{-3}. The cross section of JET is elliptical with half axis $a = 1.25$ m and $b = 2.10$ m. In order to reach 1 keV in 1 second, the necessary total current, I, is then

$$I = \pi abj \approx 10^{-8}\sqrt{\frac{nT_e^{5/2}}{t}} \approx 3 \times 10^6 \quad \text{(A)}. \tag{5.91}$$

The associated "loop voltage", V, is determined by

$$V = 2\pi R_0 \eta j = \eta \frac{2\pi R_0}{\pi ab} I, \tag{5.92}$$

where $R_0 \approx 3$ m is the major radius. For temperatures of the order of 1 keV, the resistivity is $\eta \approx 3 \times 10^{-8}$, which implies that V is of the order of 1 V. The corresponding ohmic heating power is then of the order of 1 MW.

JET is constructed to carry currents of the order of 2–4 MA, which implies that using only ohmic current as heating source in JET, it would be difficult to reach temperatures exceeding 1 keV. This explains why several other heating sources are used in JET to increase the temperature above 1 keV. The most important ones are neutral beam injection (NBI) and radio wave frequency heating. In neutral beam heating, a beam of high energy (of the order of 100 keV) neutral particles is injected into the plasma, where the particles become ionised and trapped by the magnetic field. During slowing down by collisions with the plasma particles they transfer their energy to the plasma, thus increasing the temperature. In radio frequency heating a wave beam is launched into the plasma. The wave type and the frequency are chosen such that the wave can propagate into the plasma centre where it is absorbed by resonance (ion cyclotron resonance heating, ICRH). The heating powers associated with NBI and ICRH, typically 10 MW, are much larger than the ohmic heating power. In this way it has been possible to raise the plasma temperature to the order of 10 keV.

5.7 Diffusion in a fully ionised plasma

In a weakly ionised plasma, the plasma particles collide primarily with neutral gas particles. The cross section for this type of collision process is essentially determined by the geometrical dimension of the atom/molecule. In a fully ionised plasma, collisions occur between charged particles, interacting via the long-range Coulomb force. This difference in collisional dynamics has important consequences for the diffusion process.

5.7.1 Diffusion constants

The one-fluid equations of section 3.5 (the equation of motion and Ohm's law) can be used to derive the diffusion constant in a fully ionised plasma. In stationary state and with the gravity force neglected the equations can be shown to read

$$0 = \mathbf{j} \times \mathbf{B} - \nabla p, \tag{5.93}$$

$$\eta \mathbf{j} = \mathbf{E} + \mathbf{v} \times \mathbf{B}. \tag{5.94}$$

The drift velocity across the magnetic field, $\mathbf{v}_\perp$, is obtained by taking the usual cross product with $\mathbf{B}$. This yields

$$\mathbf{v}_\perp = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \eta \frac{\mathbf{j} \times \mathbf{B}}{B^2} = \frac{\mathbf{E} \times \mathbf{B}}{B^2} - \eta \frac{\nabla p}{B^2}. \tag{5.95}$$

As emphasised before, from the point of view of plasma confinement, the most serious term here is the last one, which in a cylindrical plasma with a radially directed pressure gradient would give rise to a particle flow to the walls determined by

$$\mathbf{\Gamma}_\perp = n\mathbf{v}_\perp = -\frac{\eta n \sum_j k_B T_j}{B^2} \nabla p \equiv -D_{cl} \nabla n, \tag{5.96}$$

where the *classical diffusion constant*, D_{cl}, for a fully ionised plasma is given by

$$D_{cl} = \frac{\eta n \sum_j k_B T_j}{B^2}. \tag{5.97}$$

There are several important properties of D_{cl}, that should be emphasised.

Scaling with B

The diffusion constant scales as $D_{cl} \propto 1/B^2$, i.e., analogously to the situation in a weakly ionised plasma. The basic origin of this property is of course the fact that the step length of the diffusion process is $r_L \propto 1/B$.

Scaling with density

The diffusion constant is proportional to the plasma density ($D_{cl} \propto n$) in contrast to the case of a weakly ionised plasma where the diffusion constant was independent of the plasma density (but proportional to the density of neutral particles). The physical explanation of this scaling is the fact that the density of scatterers now is the plasma density.

Scaling with temperature

Since the resistivity scales with temperature according to $\eta \propto T^{-3/2}$, we obtain $D \propto T^{-1/2}$. In a weakly ionised plasma $D \propto T^{1/2}$. This difference is due to the velocity dependence of the cross section for Coulomb collisions ($\sigma \propto v^{-4} \propto T^{-2}$).

5.7.2 Agreement with experiments

Summarising the different dependencies we have the following scaling of the classical diffusion constant with magnetic field, plasma density and temperature

$$D_{cl} \propto \frac{n}{B^2 T^{1/2}}. \tag{5.98}$$

However, in experiments, in particular in the early plasma experiments, the diffusion was empirically found to obey a completely different scaling, the so-called Bohm scaling where

$$D_B \approx \frac{1}{16}\frac{k_B T}{eB} \propto \frac{T}{B}. \tag{5.99}$$

Thus, instead of decreasing with increasing temperature, the diffusion constant was found to increase and although D did decrease with increasing magnetic field, the decrease was much slower than predicted. What was even worse was that the Bohm diffusion was three or four orders of magnitude larger than the value predicted by the classical theory.

The physical background of this anomalous diffusion, and ways to eliminate it, has been the subject of numerous investigations since the Bohm diffusion was discovered in the end of the 40's. A qualitative explanation of the empirically found anomalous Bohm diffusion can be given as follows: In our analysis of the classical diffusion we neglected the $\mathbf{E} \times \mathbf{B}$ drift, tacitly assuming that it did not have a radial component. However, different types of plasma waves and so-called "convective cells" may involve electrostatic electric fields in the poloidal direction, which together with the strong toroidal magnetic field would give rise to drift motions in the "dangerous" (from the point of view of plasma confinement) radial direction. This loss flow can be estimated as

$$\Gamma_\perp \approx n v_\perp \approx n\frac{E}{B}. \tag{5.100}$$

The electric fields that can arise in a plasma are determined by the Debye screening properties which implies that the maximum potential energy disturbance, $e\phi$, must be of the order of the thermal energy, $k_B T$. Assuming the radial extension of the plasma to be of the order of R, the electric field corresponding to the potential ϕ is $E \approx \phi/R$. Putting all this information together we have

$$\Gamma \approx n\frac{E}{B} \approx n\frac{k_B T}{eR}\frac{1}{B} \approx \frac{k_B T}{eB}|\nabla n| = D|\nabla n|, \tag{5.101}$$

where

$$D \approx \frac{k_B T}{eB} \propto \frac{T}{B}, \tag{5.102}$$

in reasonable agreement with the observed Bohm diffusion.

During the years a good understanding of the physics responsible for the Bohm diffusion (and other "anomalous" diffusion processes) has been obtained and in present day experiments the observed diffusion rate is more in agreement with theoretical predictions (although not necessarily classical), both with respect to scaling and magnitude.

5.8 The diffusion equation of a fully ionised plasma

In a fully ionised plasma, the diffusion constant is

$$D = \frac{\eta n \sum_j k_B T_j}{B^2} \equiv An, \tag{5.103}$$

where we have stressed the dependence on the plasma density. The fact that the diffusion constant is proportional to the density implies that the corresponding diffusion equation becomes nonlinear

$$\frac{\partial n}{\partial t} = \nabla\cdot(D\nabla n) = A\nabla\cdot(n\nabla n). \tag{5.104}$$

This equation cannot be solved using Fourier transformation, but the method of separation of variables still works and again writing $n(\mathbf{r},t) = S(\mathbf{r})T(t)$, the one-dimensional form of the diffusion equation can be separated into the two equations

$$\frac{1}{T^2}\frac{dT}{dt} = \frac{A}{S}\frac{d}{dx}\left(S\frac{dS}{dx}\right) = -\frac{1}{\tau}. \tag{5.105}$$

The time dependence is easily solved to give

$$T(t) = \frac{1}{1 + t/\tau}. \tag{5.106}$$

However, the equation determining the space dependence is a more complicated equation

$$\frac{d}{dx}\left(S\frac{dS}{dx}\right) + \frac{1}{A\tau}S = 0. \tag{5.107}$$

This equation should be solved under the conditions $S(\pm L/2) = 0$. Concentrating on the lowest order mode, we have $dS(0)/dx = 0$ and $S(0) = n_0$. It is convenient to normalise the equation by letting $S(x)/S(0) \to S(x)$, $x/(L/2) \to x$, and write it as

$$\frac{d}{dx}\left(S\frac{dS}{dx}\right) + \kappa^2 S = 0, \tag{5.108}$$

where $\kappa^2 = L^2/(4A\tau n_0)$, and the new boundary conditions become $S(\pm 1) = 0$, $dS(0)/dx = 0$, and $S(0) = 1$. Again this equation constitutes an eigenvalue problem, where we should determine the (normalised) density profile as well as the eigenvalue, κ^2, which gives the decay time, τ. This problem is however much more complicated than the previous diffusion equation since the present problem is a nonlinear eigenvalue problem.

The one-dimensional form of the diffusion equation can be solved exactly in terms of elliptic functions, but the solution is very complicated and gives no clear picture of the properties of the diffusion profile. Instead we will give a simple approximate solution. It is clear that the nonlinear diffusion equation can be written

$$\frac{1}{2}\frac{d^2S^2}{dx^2} + \kappa^2 S = 0. \tag{5.109}$$

For small x, we know that $S(x) \approx 1$ and we can solve the equation asymptotically for small x by approximating

$$\frac{1}{2}\frac{d^2S^2}{dx^2} \approx -\kappa^2, \tag{5.110}$$

which implies the solution procedure

$$\frac{dS^2(x)}{dx} \approx C_1 - 2\kappa^2 x = -2\kappa^2 x, \qquad \text{since } \frac{dS(0)}{dx} = 0, \tag{5.111}$$

$$S^2(x) \approx C_2 - \kappa^2 x^2 = 1 - \kappa^2 x^2, \qquad \text{since } S(0) = 1, \tag{5.112}$$

and using the fact that $S(\pm 1) = 0$, we infer the following simple solution of the nonlinear eigenvalue problem

$$S(x) \approx \sqrt{1 - x^2}, \quad \kappa^2 \approx 1. \tag{5.113}$$

This approximate solution is in very good agreement with the exact solution and the approximate eigenvalue, $\kappa^2 \approx 1$, is also in good agreement with the exact one, $\kappa^2 \approx 1.14$, which is given by a complicated expression involving elliptic functions.

If an approximate solution is known for an eigenvalue problem, a standard approach for estimating the eigenvalues is to apply direct variational methods or to use different integrated quadratic forms of the equation. To illustrate this approach we multiply Eq. (5.109) with S^2, integrate over the interval $[0, 1]$, and partially integrate the diffusion term. This yields an expression from which the eigenvalue can be solved as

$$\kappa^2 = \frac{1}{2}\int_0^1 \left(\frac{dS^2}{dx}\right)^2 dx \Bigg/ \int_0^1 S^3\, dx. \tag{5.114}$$

Inserting the approximate density profile $S(x) \approx \sqrt{1 - x^2}$ into this expression, the eigenvalue is easily found to be $\kappa^2 \approx 32/(9\pi) \approx 1.13$, in excellent agreement with the exact result.

Once the eigenvalue is determined, the confinement time is obtained as

$$\tau = \frac{L^2}{4A\kappa^2 n_0} \approx 0.22\frac{L^2}{D(n_0)}. \tag{5.115}$$

The confinement time is again determined by the characteristic dimension of the plasma container and the diffusion constant, which in this case depends on the density itself—a typically nonlinear feature. There is but a small quantitative difference between the linear and nonlinear diffusion equations. (In the linear case when the diffusion does not depend on the density we obtained $\tau = L^2/(\pi^2 D) \approx 0.10L^2/D$.)

5.9 Banana diffusion

As mentioned earlier, diffusion can be viewed as a random walk process. This implies that the diffusion constant, D, can be expressed in terms of the step length, Δx, and the mean time between collisions, τ, as

$$D = \frac{(\Delta x)^2}{\tau}. \tag{5.116}$$

For diffusion in a weakly ionised plasma this implied that

$$D = \nu\lambda_m^2, \tag{5.117}$$

where ν is the collision frequency and λ_m is the mean free path between collisions. In a strongly magnetised plasma where $r_L \ll \lambda_m$, the step length in the diffusion process is the Larmor radius, r_L, and the diffusion constant becomes

$$D = \nu r_L^2 \ll \nu\lambda_m^2. \tag{5.118}$$

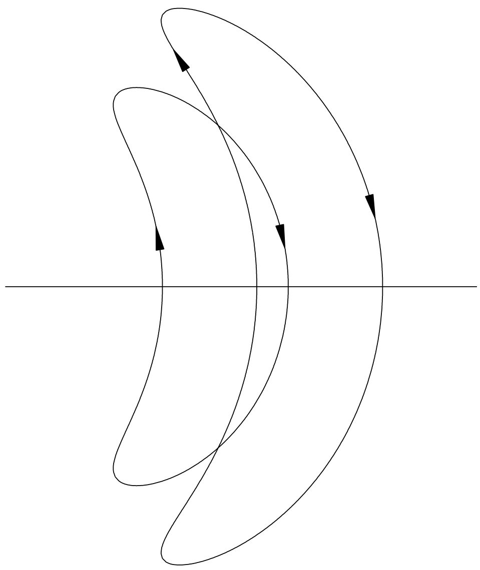


Figure 5.5. The diffusion of banana orbits.

As an application of this view of the diffusion process, we will consider the diffusion of particles trapped in banana orbits in Tokamaks, i.e., particles trapped in the shallow magnetic mirrors on the outside of the torus ($R > R_0$), see Fig. 2.22. However, when banana particles collide, they will jump from one banana orbit to another, see Fig. 5.5. The step length is here the typical width of the banana orbit, Δ, which is much larger than the Larmor radius. Consequently, this diffusion could potentially give rise to a large loss flow to the walls. The total diffusion flow of plasma particles can be written as

$$\mathbf{\Gamma} = -D_f \nabla n_f - D_b \nabla n_b, \tag{5.119}$$

where indices f and b refer to free (circulating) and banana (trapped) particles, respectively. As previously shown in chapter 2 we have $n_b/n \approx 1 - n_f/n \approx \sqrt{2a/R_0}$, where a and R_0 denote the minor and major radii, respectively, and n is the total plasma density. If this relation is used in Eq. (5.119) we obtain

$$\mathbf{\Gamma} \approx -D_f \nabla n - \sqrt{\frac{2a}{R_0}} D_b \nabla n = -\left(D_f + \sqrt{\frac{2a}{R_0}} D_b\right) \nabla n \equiv -D \nabla n, \tag{5.120}$$

where the total diffusion coefficient, D, is given by

$$D = D_f + \sqrt{\frac{2a}{R_0}} D_b. \tag{5.121}$$

The diffusion constant D_f is known to be $D_f = \nu r_L^2$, and using the arguments given above we can write for the diffusion constant of the banana

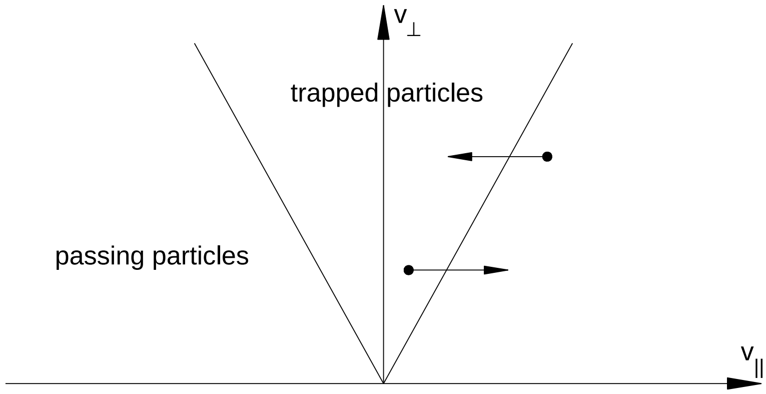


Figure 5.6. Velocity space regions of trapped and passing particles and the collisional transformation between these regions.

particles

$$D = \nu^* \Delta^2, \tag{5.122}$$

where the width of the banana orbit was estimated in chapter 2 as

$$\Delta \approx \frac{v_{\|}^{\max}}{\omega_c^p}, \tag{5.123}$$

where ω_c^p is the cyclotron frequency in the poloidal magnetic field and $v_{\|}^{\max}$ is the maximum parallel velocity of the banana trapped particles, which was found to be

$$v_{\|}^{\max} \approx v_{th} \frac{2a}{R_0} \tag{5.124}$$

where v_{th} is the thermal velocity of the plasma particles.

The evaluation of the collision frequency, ν^*, requires some care: As emphasised earlier the collision frequency, ν, for collisions between charged particles is determined by "strong interactions" where the scattering angle is of the order of 90°. Since the "opening angle", θ_t, of the trapped particle region in velocity space is very small, typically $\theta_t \approx \sqrt{2a/R_0}$, we anticipate that $\nu^* \ll \nu$—even a weak interaction between two charged particles may turn a trapped particle into a passing particle and vice versa, see Fig. 5.6. The collision frequency is obtained as

$$\nu = v_{th} n \sigma \propto v_{th} n r_0^2, \tag{5.125}$$

where r_0 is the impact parameter. It is clear that large impact parameters give rise to small-angle deflections and in this limit, Rutherford's scattering formula, $r_0 \propto \cot(\theta/2)$, implies that $r_0 \propto 1/\theta$. A good qualitative estimate of the collision frequency for the trapped particles, ν^*, should then be

$$\frac{\nu^*}{\nu} \approx \frac{(\pi/2)^2}{\theta_t^2} \approx \frac{R_0}{2a}. \tag{5.126}$$

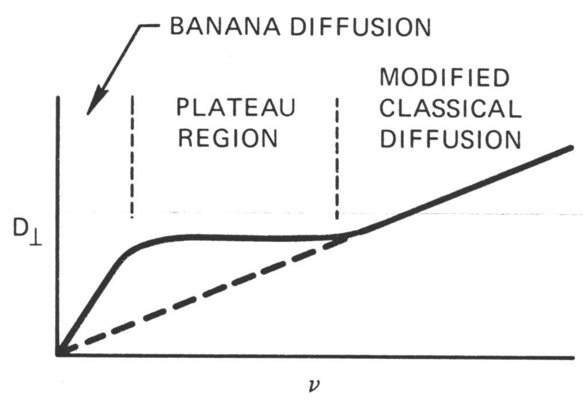


Figure 5.7. Qualitative plot showing the variation of the diffusion constant, D, with collision frequency, ν. The characteristic transition frequencies are approximately ω_b (the lower one) and ω_c (the upper one) [from *Plasma Physics and Controlled Fusion*, F. F. Chen (Plenum Press, New York, 1984)].

We have now estimated both the collision frequency and the step length of the diffusion process for the banana particles and the result can be summarised as

$$D_b = \nu^* \Delta^2 \approx \nu \frac{R_0}{2a} \frac{4v_{th}^2 a}{R_0 (\omega_c^p)^2} = 2\nu r_L^2 \left(\frac{B_T}{B_p}\right)^2 = 2\left(\frac{B_T}{B_p}\right)^2 D_f. \qquad (5.127)$$

The total diffusion constant can now be written as

$$D = D_f + \sqrt{\frac{2a}{R_0}} D_b = \left[1 + 2\sqrt{\frac{2a}{R_0}} \left(\frac{B_T}{B_p}\right)^2\right] D_f. \qquad (5.128)$$

The factor $2\sqrt{2a/R_0}(B_T/B_p)^2$ is usually much greater than unity, which implies that $D \ll D_f$ and that the plasma experiences a strongly enhanced diffusion due to the presence of banana trapped particles. On the other hand, we realise that diffusion of trapped particles can only occur provided the collision frequency, ν^*, is smaller than the bounce frequency of the trapped particles, ω_b. Otherwise there will be no banana orbits at all. The diffusion constant can consequently be expected to vary with collision frequency as shown in Fig. 5.7. Note the three characteristic regions: The rapid rise of D for small collision frequencies (the banana regime), which is terminated at the frequency when the banana orbits are destroyed by too frequent collisions, the flat region (the plateau regime) which finally joins the classical diffusion region where the collision frequency becomes greater than the cyclotron frequency.

Chapter 6

Magnetohydrodynamics

The theory of magnetohydrodynamics (MHD) has been used to describe many basic phenomena in plasma physics ranging from magnetic fields within stellar and planetary interiors through astrophysical phenomena to stability of magnetically confined fusion plasmas. MHD is the study of macroscopic interaction of electrically conducting liquids and gases with magnetic fields. The separate identities of the positively and negatively charged particles do not feature in the formulation, so the conducting medium is a magnetised and continuous fluid. As we have seen in chapter 3, we have to assume a number of conditions to be satisfied in order to take as the starting point of the MHD description of a plasma the macroscopic continuum equations of fluid mechanics together with electromagnetic theory applied to an electrically conducting fluid. First of all the single-fluid description is only valid provided the plasma is collision dominated. The collision dominance requires that the distribution functions of the particle species are locally Maxwellian, which in turn implies that the MHD time scale must be sufficiently long for there to be adequately many collisions between particles. It has to be much longer than the time required for light to traverse the plasma. This allows the displacement current to be neglected from the Maxwell's equations. A condition for collision dominance is associated with the MHD scale length, a, which must be large compared with the mean free path between collisions. The scale length should also be large compared to the Debye length, which means that the fluid is quasi-neutral, i.e., the densities of electrons and ions are approximately equal. Furthermore, if a is large compared with the ion Larmor radius, then the diamagnetic and Hall effects which appear in the electron momentum equation, Eq. (3.81), can be neglected. Another condition for the single-fluid picture to be valid is that the electrons and ions should have identical temperatures, which when combined with quasi-neutrality implies that the pressures are also the same. Even within the simplified structure of MHD, further approximations are possible and made from time to time in order to simplify a problem or to approximate a physical situation. For example, sometimes we are able to solve a problem with the assumptions that the plasma has finite electrical conductivity; on other occasions, however, it is necessary to consider only a perfectly conducting plasma if analytical results are to be obtained. In much of what follows, we shall neglect transport processes. Thus, we will suppose electrical resistivity to be zero. This will be a reasonable approximation for plasma phenomena with a time-scale short compared with that required for a magnetic field to diffuse through the plasma. However, we will also discuss processes where the finite plasma resistivity has to be taken into account leading the phenomenon of magnetic field diffusion.

6.1 MHD equations and conservation properties

According to chapter 3, the equations of MHD are

$$\frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{v}) = 0, \qquad \text{(mass continuity)}, \tag{6.1}$$

$$\frac{d}{dt}\left(\frac{p}{\rho_m^\gamma}\right) = 0, \qquad \text{(adiabatic equation of state)}, \tag{6.2}$$

$$\rho_m \frac{d\mathbf{v}}{dt} = \mathbf{j} \times \mathbf{B} - \nabla p, \qquad \text{(momentum equation)}, \tag{6.3}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}, \qquad \text{(Ampère's law)}, \tag{6.4}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \text{(Faraday's law)}, \tag{6.5}$$

$$\nabla \cdot \mathbf{B} = 0, \tag{6.6}$$

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}, \qquad \text{(resistive Ohm's law)}, \tag{6.7}$$

where γ is the ratio of the specific heats, η is the electrical resistivity of the plasma and $d/dt \equiv \partial/\partial t + \mathbf{v} \cdot \nabla$ is the convective derivative. It may be the case that the plasma is sufficiently collision dominated for the collision picture to apply, but yet sufficiently “collisionless” for the resistivity in Eq. (6.7) to be neglected. In this case the right-hand side of Eq. (6.7) may be neglected to yield

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = 0, \qquad \text{(ideal Ohm's law)}. \tag{6.8}$$

The Eqs. (6.1)–(6.6) with Eq. (6.8) constitute the ideal MHD equations, while the inclusion of Eq. (6.7) is described as resistive MHD.

Ideal MHD possesses a number of basic conservation laws enjoyed by fluid mechanics and electromagnetism, namely

$$\int \rho_m \, d\mathbf{r} = \text{const}, \qquad \text{(conservation of plasma mass)}, \tag{6.9}$$

$$\int \rho_m \mathbf{v} \, d\mathbf{r} = \text{const}, \qquad \text{(conservation of plasma momentum)}, \tag{6.10}$$

$$\int (K + W) \, d\mathbf{r} = \text{const}, \qquad \text{(conservation of total energy)}, \tag{6.11}$$

where

$$K = \int \frac{\rho_m v^2}{2} \, d\mathbf{r} \tag{6.12}$$

is the plasma kinetic energy and

$$W = \int \left(\frac{p}{\gamma - 1} + \frac{B^2}{2\mu_0} \right) d\mathbf{r} \tag{6.13}$$

is the plasma potential energy. Furthermore, in the next section we will show that also the magnetic flux is conserved within the ideal MHD. Note that the above relations have been derived using the boundary condition that the plasma extends to a stationary and perfectly conducting wall and that the integrals are taken over the plasma volume. This means that the tangential electric field and normal magnetic field vanish at the conducting wall. The fact that the ideal MHD conserves mass, momentum, energy, and magnetic flux is of importance for studying dynamical plasma properties since these conservation laws apply to general nonlinear, time-dependent, and multi-dimensional systems.

6.2 Conservation of magnetic flux

As a consequence of the perfect conductivity assumed by Ohm's law within the ideal MHD, the magnetic flux passing through any arbitrary surface element moving with the plasma is constant. To show this, consider a closed loop C of surface area S drawn in the fluid. A magnetic field $\mathbf{B}$ passes through the loop and the magnetic flux Φ linking the loop is

$$\Phi = \int_S \mathbf{B} \cdot \hat{n} \, dS, \tag{6.14}$$

where $\hat{n}$ is the unit vector normal to the surface. Then the time rate of change of the flux through the surface moving with the velocity $\mathbf{u}$ is given by

$$\frac{d\Phi}{dt} = \int_S \frac{\partial \mathbf{B}}{\partial t} \cdot \hat{n} \, dS - \oint_C \mathbf{u} \times \mathbf{B} \cdot d\mathbf{l}, \tag{6.15}$$

which is a well known result, that can easily be obtained from Faraday's law of induction.

If the ideal Ohm's law, Eq. (6.8), is substituted into Faraday's law, Eq. (6.5), we obtain

$$\frac{\partial \mathbf{B}}{dt} = \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{6.16}$$

We then substitute Eq. (6.16) into Eq. (6.15) and employ Stoke's theorem to convert the surface integral to a line integral

$$\frac{d\Phi}{dt} = -\oint_C (\mathbf{u} - \mathbf{v}) \times \mathbf{B} \cdot d\mathbf{l}. \tag{6.17}$$

Clearly, if the loop moves with the plasma velocity so that $\mathbf{u} = \mathbf{v}$, we have that

$$\frac{d\Phi}{dt} = 0, \tag{6.18}$$

i.e., the flux, Φ, is conserved. Since the above proof applies to any arbitrary surface area, it follows immediately, by considering the whole plasma cross section, that the total flux in an ideal MHD plasma is constant. This leads to the intuitive picture that, in an ideal MHD plasma, the plasma is "tied" to the magnetic field, or conversely that the magnetic field lines are "frozen" into plasma.

6.3 Hydromagnetic equilibrium

The determination of equilibria is one of the primary functions of MHD. A general investigation of the equilibrium states of a plasma is very complicated and beyond the scope of this book. However, some simple considerations on this issue will be discussed. In a stationary case, the MHD equations (6.3) and (6.4) become

$$\nabla p = \mathbf{j} \times \mathbf{B}, \tag{6.19a}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}. \tag{6.19b}$$

The first of these equations, resulting from the momentum equation, expresses a balance between the pressure force, ∇p, and the Lorentz force, $\mathbf{j} \times \mathbf{B}$. A qualitative picture of the force balance in a magnetised plasma cylinder, where ∇p is directed radially inwards, is shown in Fig. 6.1. In order to balance the pressure force directed outwards, there must be an azimuthal current as indicated in the figure. The magnitude of this current is determined by

$$\mathbf{j} = \frac{\mathbf{B} \times \nabla p}{B^2} = k_B (T_e + T_i) \frac{\mathbf{B} \times \nabla n}{B^2}, \tag{6.20}$$

where we have assumed that the temperature of electrons, T_e, and of ions, T_i, are constant in space. It follows from the above expression that the considered current is diamagnetic. This current is created by the fact that the average of the particle velocities associated with the Larmor rotation is

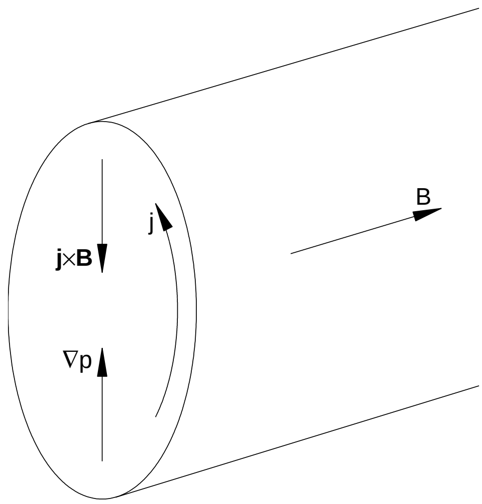


Figure 6.1. Qualitative picture of the geometry involved in the force balance of a magnetised plasma.

nonzero in the presence of a pressure gradient. We note that both **j** and **B** are perpendicular to ∇p, i.e., **j** and **B** must lie on surfaces of constant p. For constant plasma temperature, the equilibrium state has the property that the plasma density is constant along the magnetic field lines. Rewriting the MHD equations by eliminating the current, one obtains

$$\nabla p = \frac{1}{\mu_0}(\nabla \times \mathbf{B}) \times \mathbf{B} = \frac{1}{\mu_0}\left[(\mathbf{B}\cdot\nabla)\mathbf{B} - \frac{\nabla B^2}{2}\right]. \tag{6.21}$$

In situations where the magnetic field is homogeneous in the direction of the field, i.e., when $\mathbf{B}\cdot\nabla = 0$, Eq. (6.21) implies the important and very useful result

$$p + \frac{B^2}{2\mu_0} = \text{const}, \tag{6.22}$$

which is simply the statement that the sum of the particle pressure, p, and the magnetic pressure, $B^2/(2\mu_0)$, is constant. The ratio between these two pressures is called the β-value

$$\beta = \frac{nk_B T}{B^2/(2\mu_0)}, \tag{6.23}$$

which characterises the magnitude of the diamagnetic effect in the plasma.

6.4 Magnetic pinches

The relation between the particle and magnetic pressures, discussed in the previous section, has a number of important applications. Many of the

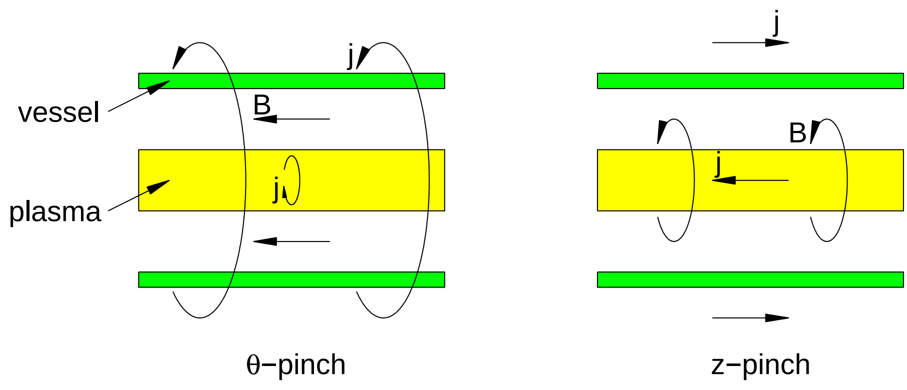


Figure 6.2. The geometries of the theta and linear pinches.

early plasma experiments used this property for confining and heating of the plasma in the form of so-called *plasma pinches*. Consider a non-magnetised plasma contained in a metallic cylinder. In the theta-pinch, a strong azimuthal current is suddenly applied to the (conducting) plasma container, see Fig. 6.2. We assume for simplicity that the plasma is an infinitely good conductor, i.e., $\eta = 0$. The magnetic field induced by the current exerts a "pinching" pressure on the plasma which compresses until the plasma pressure balances the magnetic pressure, i.e., $p \approx B^2/(2\mu_0)$.

A simple variant of this scheme is the linear pinch, where the roles of **j** and **B** have been reversed, i.e., a strong longitudinal current is here induced in the plasma. The induced magnetic field is now azimuthal, but the Lorentz force is still directed radially inwards and compresses the plasma, see Fig. 6.2.

Let us analyse the corresponding equilibrium in some more detail. The equilibrium equations (6.19) become in the case of radial symmetry

$$\frac{dp}{dr} = -jB, \tag{6.24}$$

$$\frac{1}{\mu_0 r}\frac{d}{dr}(rB) = j, \tag{6.25}$$

or eliminating the current we have

$$\frac{dp}{dr} = -\frac{B}{\mu_0 r}\frac{d}{dr}(rB). \tag{6.26}$$

Multiplying this equation by r^2, and integrating from zero to the pinch radius, r_0, we obtain

$$\int_0^{r_0} r^2 \frac{dp}{dr}\,dr = -\frac{1}{\mu_0}\int_0^{r_0} rB\frac{d(rB)}{dr}\,dr, \tag{6.27}$$

which, by using partial integration, implies

$$\left[r^2 p\right]_0^{r_0} - 2\int_0^{r_0} rp\,dr = -\frac{1}{2\mu_0}\left[(rB)^2\right]_0^{r_0}. \tag{6.28}$$

We can assume that $p(r_0) = 0$ and that $T = T_e \approx T_i \approx$ const. Eq. (6.28) then reduces to

$$4k_B T\int_0^{r_0} rn\,dr = \frac{1}{2\mu_0}\left[(rB)^2\right]_0^{r_0}. \tag{6.29}$$

Introducing the total number of particles per unit length of the pinch, N, as

$$N \equiv 2\pi\int_0^{r_0} rn\,dr, \tag{6.30}$$

Eq. (6.29) can be rewritten as

$$2Nk_B T = \frac{\pi}{2\mu_0}\left[(rB)^2\right]_0^{r_0}. \tag{6.31}$$

From Ampère's law relating the current to the magnetic field we have

$$[rB]_0^{r_0} = \frac{\mu_0}{2\pi}\int_0^{r_0} j2\pi r\,dr = \frac{\mu_0}{2\pi}I, \tag{6.32}$$

where I denotes the total plasma current. Using now Eq. (6.32) in Eq. (6.31), we finally obtain Bennet's relation for a linear stationary pinch

$$I^2 = \frac{16\pi}{\mu_0}Nk_B T. \tag{6.33}$$

This expression shows that the total current is related to the plasma temperature and the total number of particles (per unit length of the pinch). In particular, the plasma temperature is proportional to the square of the plasma current, and inversely proportional to the line density, N.

6.5 Magnetic pressure and sunspots

A beautiful application of the pressure balance involving particle and magnetic pressures is the explanation given by H. Alfvén of the sunspot phenomenon. He suggested that the sunspots, i.e., the dark regions observed on the surface of the sun, should be regions where cylinders of magnetic fields "burst through" the suns surface, see Fig. 6.3. Since the sunspots are observed to be comparatively long-lived phenomena, it is natural to assume that the equilibrium is determined by a balance between particle and magnetic pressure as described by Eq. (6.22). Assuming that the magnetic pressure outside the sun spot is negligible and that the particle pressures in and outside the sun spot are p_0 and p_1 respectively, we obtain

$$p_1 = p_0 + \frac{B^2}{2\mu_0} \quad \Rightarrow \quad B^2 = 2\mu_0(p_1 - p_0). \tag{6.34}$$

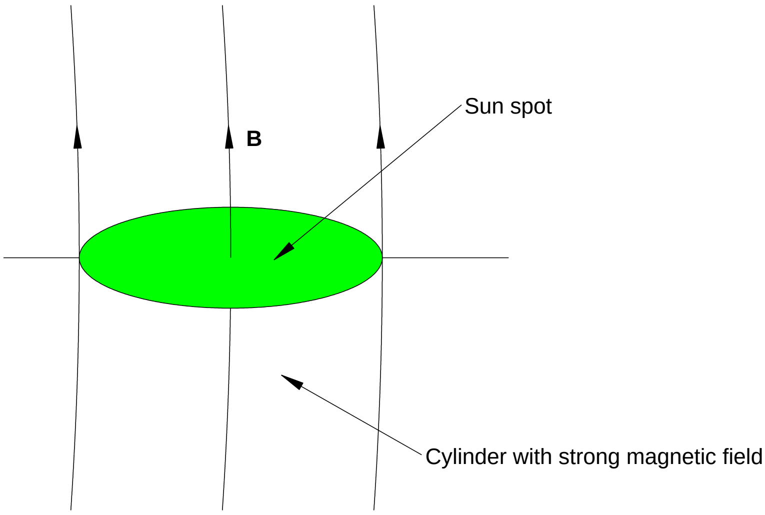


Figure 6.3. The geometry of a sun spot according to H. Alfvén with a magnetic cylindrical tube coming out of the sun surface.

The particle density is approximately constant over the Sun's surface (and is known to be of the order of $n \approx 10^{24}$ m^{-3}), and consequently the temperature in the sun spot, T_0, must be lower than in the surrounding regions, T_1. From intensity measurements it has been found that a temperature difference indeed exists and is of the order of 1000 K. Using these numbers in Eq. (6.34) we obtain a magnetic field $B \approx 1$ kG $= 0.1$ T in rather good agreement with other independent measurements of the magnetic field strength in the sun spot.

6.6 Magnetic pressure and galactic spiral arms

S. Chandrasekhar and E. Fermi have suggested that magnetic fields might play an important role in determining the equilibrium of the spiral arms of the galaxies. It is found that the gravitational attraction force, which tends to make the spiral arms contract, is so strong that the particle pressure would not alone be able to balance the gravitation. Chandrasekhar and Fermi drew the conclusion that there must be a magnetic field along the spiral arms and that the magnetic pressure together with the particle pressure would be able to balance the gravitation, see Fig. 6.4.

We begin to consider the problem (following Chandrasekhar and Fermi) by finding the gravitational pressure. The gravitation force per unit volume, $\mathbf{F}_g$ is given by $\mathbf{F}_g = \rho_m \mathbf{a}_g$, where ρ_m is the mass density in the spiral arm

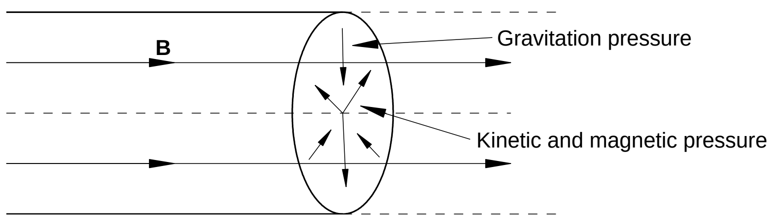


Figure 6.4. The geometry of the spiral arm with the longitudinal magnetic field and the different pressure forces.

and $\mathbf{a}_g$ is the gravitational acceleration, which is given by

$$\mathbf{a}_g = -G\frac{2M}{r}\hat{r}, \tag{6.35}$$

where G is the gravitational constant and r is the radius. Furthermore, $M = M(r)$ is the total mass per unit length inside the radius r, i.e., $M = \pi r^2 \rho_t$, where ρ_t is the total density (including the contribution from the stars). The gravitational force can now be rewritten as

$$\mathbf{F}_g = \rho_m \left(-G\frac{2\pi r^2 \rho_t}{r}\right)\hat{r} = -2\pi G \rho_m \rho_t r \hat{r} = -\nabla(\pi G \rho_m \rho_t r^2). \tag{6.36}$$

Thus, the inward-directed gravitational pressure, given by $p_g = \pi G \rho_m \rho_t r^2$, must be balanced by the magnetic and kinetic particle pressures, p_m and p_k, respectively, which are given by

$$p_k = \frac{\rho_m v^2}{3}, \quad p_m = \frac{B^2}{2\mu_0}. \tag{6.37}$$

The pressure balance condition on the boundary of the spiral arm, where $r = R$, reads $p_g(R) = p_k + p_m$. Relevant typical figures for the spiral arms are $R \approx 250$ pc $\approx 7.7 \times 10^{18}$ m, $\rho_m \approx 2 \times 10^{-21}$ kg/m^3, $\rho_t \approx 3\rho_m$ and $G \approx 6.7 \times 10^{-11}$ m^3/kg^2. This yields a gravitational pressure of $p_g \approx 1.5 \times 10^{-13}$ N/m^2. The turbulent velocity in the spiral arms is $v \approx 5 \times 10^3$ m/s, which gives a kinetic pressure of $p_k \approx 2 \times 10^{-14}$ N/m^2. The difference between the gravitational and the kinetic pressure must then be due to the magnetic pressure, i.e., $B = [2\mu_0(p_g - p_k)]^{1/2} \approx 6 \times 10^{-10}$ T. We note that this result is in rather good agreement with our previous estimate obtained from the observed fluctuations of the plane of polarisations in the light emitted by the pulsars in the galactic spiral arms.

6.7 Magnetic diffusion

The finite resistivity of a plasma leads to a diffusion of the magnetic field. This effect plays an important role in most applications of plasmas and in particular in astrophysics and fusion plasma confinement. Consider the MHD equations

$$\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{j}, \tag{6.38a}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \tag{6.38b}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j}. \tag{6.38c}$$

A single equation for $\mathbf{B}$ can be obtained by eliminating $\mathbf{j}$ and $\mathbf{E}$ from these equations. The result is

$$\frac{\eta}{\mu_0} \nabla \times (\nabla \times \mathbf{B}) = -\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{v} \times \mathbf{B}), \tag{6.39}$$

where we have assumed the resistivity to be a constant. Since $\nabla \cdot \mathbf{B} = 0$, Eq. (6.39) can be simplified to read

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 \mathbf{B} + \nabla \times (\mathbf{v} \times \mathbf{B}). \tag{6.40}$$

The first term on the right hand side represents a diffusion and the second a convection of the magnetic field. In order to concentrate on the diffusion process we consider a non-moving plasma, i.e., we assume $\mathbf{v} = 0$. Eq. (6.40) then reduces to a diffusion equation of the same form that we have analysed previously

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{\eta}{\mu_0} \nabla^2 \mathbf{B}. \tag{6.41}$$

If we introduce the characteristic length scale for the magnetic field variation, L, the time scale for the penetration of the magnetic field into the plasma, τ, can be estimated as

$$\tau \approx \frac{\mu_0 L^2}{\eta}. \tag{6.42}$$

Since the time τ can be seen as the decay time of the magnetic field, it can also be interpreted as the time for the magnetic energy to dissipate into Joule heating of the plasma.

From the MHD equation relating current and magnetic field, Eq. (6.38), we infer that $j \approx B/(\mu_0 L)$, which implies that the dissipated Joule energy density, W_j, can be estimated as

$$W_j \approx \eta j^2 \tau \approx \eta \left(\frac{B}{\mu_0 L} \right)^2 \frac{\mu_0 L^2}{\eta} = \frac{B^2}{\mu_0} \approx W_B, \tag{6.43}$$

where W_B denotes the energy density stored in the magnetic field.

6.8 MHD description of Alfvén waves

One of the most interesting aspects of an electrically conducting fluid in a magnetic field is the great variety of waves which can exist. As an example, we will concentrate on the hydromagnetic waves known as Alfvén waves. Actually, H. Alfvén, by using the analogy with elastic strings, was the first to recognise in 1942 that transverse waves propagating along the magnetic force lines might be generated in the ideal MHD magnetised plasma. The discussion presented below follows the one given by C. Wahlén in 1944. Let us start by assuming that the plasma is incompressible and perfectly conducting. Then using the fact that

$$\frac{d\rho_m}{dt} = \frac{\partial \rho_m}{\partial t} + \mathbf{v} \cdot \nabla \rho_m = 0. \tag{6.44}$$

Eq. (6.1) implies that

$$\nabla \cdot \mathbf{v} = 0. \tag{6.45}$$

Consequently, Eq. (6.16) can be written as

$$\frac{\partial \mathbf{B}}{\partial t} = (\mathbf{B} \cdot \nabla)\mathbf{v} - (\mathbf{v} \cdot \nabla)\mathbf{B}. \tag{6.46}$$

Substituting Eq. (6.4) into Eq. (6.3), the momentum equation becomes

$$\rho_m \frac{d\mathbf{v}}{dt} = -\frac{1}{\mu_0}\mathbf{B} \times (\nabla \times \mathbf{B}) - \nabla p. \tag{6.47}$$

Assume now that the flow velocity of the unperturbed plasma is zero and that a constant magnetic field $\mathbf{B}_0$ exists throughout the plasma. Then, consider a small transverse perturbation of the equilibrium situation, namely

$$\mathbf{B} = \mathbf{B}_0 + \mathbf{B}_1, \tag{6.48a}$$

$$\mathbf{v} = \mathbf{v}_1, \tag{6.48b}$$

$$p = p_0 + p_1, \tag{6.48c}$$

where the index 1 denotes perturbed quantities. Neglecting the second order terms, Eq. (6.47) becomes

$$\begin{aligned} \rho_0 \frac{\partial \mathbf{v}_1}{\partial t} &= -\frac{1}{\mu_0}\mathbf{B}_0 \times (\nabla \times \mathbf{B}_1) - \nabla p_1 = \\ &= -\nabla\left(p_1 + \frac{1}{\mu_0}\mathbf{B}_0 \cdot \mathbf{B}_1\right) + \left(\frac{1}{\mu_0}\mathbf{B}_0 \cdot \nabla\right)\mathbf{B}_1. \end{aligned} \tag{6.49}$$

Since $\nabla \cdot \mathbf{v}_1 = 0$, $\nabla \cdot \mathbf{B}_1 = 0$, and the equation of state, Eq. (6.2), implies that $dp_1/dt = 0$, it follows that

$$\nabla\left(p_1 + \frac{1}{\mu_0}\mathbf{B}_0 \cdot \mathbf{B}_1\right) = 0. \tag{6.50}$$

Eq. (6.49) becomes simply

$$\rho_0 \frac{\partial \mathbf{v}_1}{\partial t} = \left(\frac{1}{\mu_0} \mathbf{B}_0 \cdot \nabla \right) \mathbf{B}_1. \tag{6.51}$$

Also, using Eqs. (6.48), Eq. (6.46) becomes

$$\frac{\partial \mathbf{B}_1}{\partial t} = (\mathbf{B}_0 \cdot \nabla) \mathbf{v}_1, \tag{6.52}$$

which, combined with Eq. (6.51), gives

$$\rho_0 \frac{\partial^2 \mathbf{v}_1}{\partial t^2} = \frac{1}{\mu_0} (\mathbf{B}_0 \cdot \nabla)^2 \, \mathbf{v}_1. \tag{6.53}$$

For $\mathbf{B}_0$ oriented along the z-axis, the above equation becomes

$$\frac{\partial^2 \mathbf{v}_1}{\partial t^2} = v_A^2 \frac{\partial^2 \mathbf{v}_1}{\partial z^2}. \tag{6.54}$$

Similarly,

$$\frac{\partial^2 \mathbf{B}_1}{\partial t^2} = v_A^2 \frac{\partial^2 \mathbf{B}_1}{\partial z^2}, \tag{6.55}$$

where

$$v_A^2 = \frac{B_0^2}{\mu_0 \rho_0}, \tag{6.56}$$

defines the *Alfvén velocity.* From the above wave equations it follows that $\mathbf{v}_1$ and $\mathbf{B}_1$ are parallel, and dynamically the system resembles transverse waves on a stretched elastic string. These waves are known as the Alfvén waves. The existence of the Alfvén waves means that energy may be transported with the velocity v_A from its source region even though there is no bulk flow velocity, i.e., $\mathbf{v}_0 = 0$.

6.9 Stability of MHD plasmas

The stability of plasmas in magnetic fields is one of the primary research subjects in the area of controlled thermonuclear fusion and space plasma physics.

A plasma consists of many moving charged particles and has many MHD degrees of freedom. When a certain mode of perturbation grows, it enhances diffusion. One feature common to plasma confinement devices, e.g. Tokamaks, is that their range of operation is limited by instabilities. These instabilities may be silent leading to abruptly termination of the plasma discharge and damaging the walls when an attempt is made to operate with higher current or plasma densities. For these reasons it is important to be able to predict the properties of large-scale instabilities in confined plasmas.

Many features of these instabilities are predicted by the MHD model, which treats the plasma as a perfectly conducting fluid, being acted upon by magnetic and pressure-driven forces. It has been used to treat a number of instability phenomena in hot plasma experiments as well as in astrophysical observations.

6.9.1 General remarks on the problem of stability

As we have seen in the previous sections, there is no difficulty in constructing systems in MHD equilibrium. The usefulness of such an equilibrium configuration depends, however, on its stability. Investigation of the stability of a system consists of inquiry as to how a configuration in equilibrium responds to a disturbance. If the disturbance dies out, the system is stable, while if it the amplitude grows with time, the system is unstable. We may understand the nature of an instability by considering a plasma equilibrium which has been formed by prescribing magnetic and electric fields such that the particle orbits are regular and well confined. In the presence of a small perturbation of the fields, the particle orbits will be modified with the result that perturbed currents and charges will arise. These can serve as sources for electromagnetic fields, which may either oppose or reinforce the initial perturbation. In the latter case, an instability arises. A familiar mechanical analogy is the system shown in Fig. 6.5. A ball resting at the bottom of a valley is an example of a stable system, i.e., any perturbation will produce only oscillatory motion of the ball around its equilibrium position. A ball resting on the top of a hill is clearly an unstable system, i.e., the amplitude of a perturbation will increase continuously as the ball falls down the hill. A ball located on a saddle is another example of an unstable system; the is, however, a class of perturbations with respect to which the system is stable and will oscillate.

Fig. 6.5d represents a system that is stable only against perturbations of limited amplitude (a nonlinear theory is needed to describe the instability). Finally, Fig. 6.5e illustrates an unstable system of restricted amplitude. A more relevant fluid analogy is the Rayleigh-Taylor instability, which will be considered in the next subsection.

There are two main approaches to the analysis of stability of a system: The method of normal modes and the energy method. The method of normal modes involves solving the linearised equation of motion for small deviations from the equilibrium state. If the solutions will increase indefinitely with time, the system is unstable. The second method involves searching for perturbations producing a decrease of the potential energy of the system as compared with its energy at equilibrium. If no such perturbation exists, the system is stable. In order to demonstrate both methods, we consider the one-dimensional equation of motion. If the coordinate of the equilibrium is x_0 and the force is $F(x)$, one may expand the force in the immediate

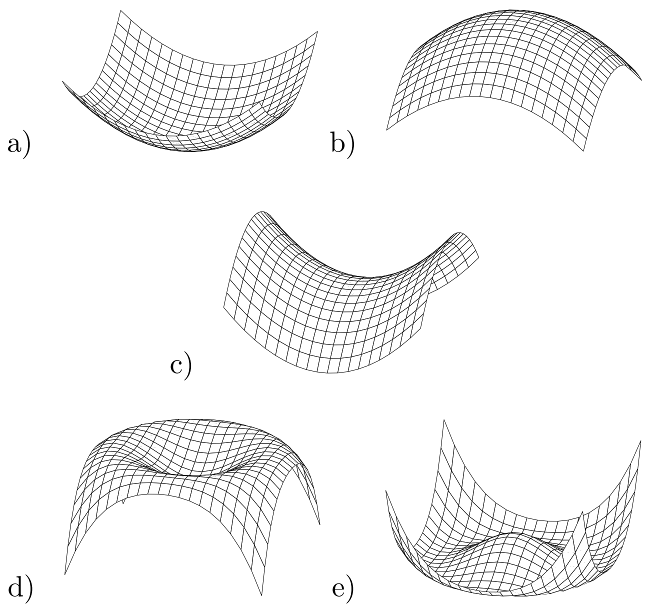


Figure 6.5. Examples of systems that are: stable (a), unstable (b, c), stable for small displacements only (d), and unstable for small displacements but stable for large displacements.

neighbourhood of the equilibrium and obtain

$$m\frac{d^2x}{dt^2} = F(x) = F(x_0) + F'(x_0)(x - x_0) + \dots \tag{6.57}$$

Since x_0 corresponds to an equilibrium position, we have that $F(x_0) = 0$. For small displacements the higher-order terms in the Taylor expansion can be neglected. Introducing $x - x_0 = \xi$, we obtain

$$m\frac{d^2\xi}{dt^2} = F'(x_0)\xi, \tag{6.58}$$

which yields the solution

$$\xi = \xi_0 \exp(i\omega t), \tag{6.59}$$

where

$$\omega^2 = -\frac{F'(x_0)}{m}. \tag{6.60}$$

It is clear that for $F'(x_0) > 0$, ω is imaginary and the displacement grows exponentially in time, while for $F'(x_0) < 0$, ω is real and the solution is oscillatory. Hence, the stability can be characterised by the sign of ω^2; $\omega^2 > 0 \Rightarrow$ stability, $\omega^2 < 0 \Rightarrow$ instability, and $\omega^2 = 0 \Rightarrow$ neutral. Another way of expressing the criterion for stability is by using $F(x) = -V'(x)$. If $V''(x_0)$ is positive so is ω^2, and stability follows, while a negative $V''(x_0)$ indicates instability. This can also be seen directly from the energy principle. Since the system is conservative, the sum of kinetic and potential energy is constant. If V'' is positive at an equilibrium it follows that a displacement increases the potential energy, hence decreases the kinetic energy. Therefore, the ball in static equilibrium (zero kinetic energy) cannot climb out of the well without external help. On the top of a hill, however, the farther the ball rolls the more kinetic energy it acquires and the faster it runs away.

6.9.2 The Rayleigh-Taylor instability

A very important instability is the well known Rayleigh-Taylor instability, which occurs when a heavy fluid is supported against gravity by a light fluid. This instability is involved in convection and may be related to solar prominences. Many other instabilities can be considered as essentially Rayleigh-Taylor instabilities if gravity is replaced by an acceleration, and a less dense fluid is pushing against a denser one. Also, instabilities are produced by curved magnetic field lines. In this case gravity is replaced by the centrifugal acceleration, which particles suffer in their motion along the magnetic field lines. In this context the instabilities are usually called interchange or flute instabilities and they are important in fusion research.

In order to give a simple demonstration of the Rayleigh-Taylor instability, let us consider the case of an incompressible and unmagnetised plasma with the equilibrium density ρ_0 and constant gravity acting in the negative x-direction, $\mathbf{g} = -g\hat{x}$. A nonuniform density in an incompressible fluid can be obtained in practice by preparing stratified solutions of salt in water, for example.

The hydrodynamic equations for an ideal fluid in the presence of a gravitational force are

$$\rho_m \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v}\right] = -\nabla p - g\rho_m\hat{x}, \tag{6.61}$$

$$\frac{\partial \rho_m}{\partial t} = -\mathbf{v}\cdot\nabla\rho_m, \tag{6.62}$$

$$\nabla\cdot\mathbf{v} = 0. \tag{6.63}$$

The incompressibility condition, $\nabla\cdot\mathbf{v} = 0$, means that the fluid flows into any fixed volume as fast as it leaves. If the equilibrium is steady-state $(\partial/\partial t = 0)$ and without any flows, we have

$$\frac{\partial p}{\partial x} = -\rho_0 g, \tag{6.64}$$

which means that the pressure at any level is determined by the weight of all the fluid above. If the system is perturbed so that perturbed quantities vary as

$$A_1 = a_1(x)\exp(\gamma t - iky), \tag{6.65}$$

the linearised equations are obtained as

$$\rho_0\gamma\mathbf{v}_1 = -\nabla p_1 - \rho_1 g\hat{x}, \tag{6.66}$$

$$\gamma\rho_1 = -\mathbf{v}_1\cdot\nabla\rho_0, \tag{6.67}$$

$$\nabla\cdot\mathbf{v}_1 = 0. \tag{6.68}$$

The perturbed variables may be eliminated by algebraic means to derive a single governing equation

$$\frac{\partial}{\partial x}\left(\rho_0\gamma^2\frac{\partial v_{1x}}{\partial x}\right) = k^2\left(\rho_0\gamma^2 - g\frac{d\rho_0}{dx}\right)v_{1x}. \tag{6.69}$$

Assume now that the equilibrium density changes slowly compared with the wavelength of the perturbations, i.e., that $1/k \ll |H|$, where the scale height of the density, H, is given by $H^{-1} = d(\ln\rho_0)/dx$. Then the left-hand side of Eq. (6.69) is negligible and we obtain

$$\gamma^2 = \frac{1}{\rho_0}\frac{d\rho_0}{dx}g = \frac{g}{H}. \tag{6.70}$$

Thus the system is unstable if $(d\rho_0/dx) > 0$, i.e., $\gamma^2 > 0$. Note that $\nabla\rho_m$ and $\mathbf{g}$ oppose each other for instability.

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