1 Introduction

This study on option pricing using GARCH is based on Chapter 12.2 Option Pricing in [Francq and Zakoïan(2010)]. It has been well known that the rate of return on stock exhibits volatility clustering, i.e. periods of large volatility tend to be followed by periods of large volatility. This phenomenon is known as heteroscedasticity, meaning that the variance or volatility of the rate of return is not constant over time. Black-Scholes model, which is well known for the explicit formula for the option price, assumes that the price of the underlying asset is driven by a geometric Brownian motion

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where μ and σ are constants and (W_t) is a standard Brownian motion. σ represents the volatility or the standard deviation of the rate of return of the asset. In Black-Scholes model, σ is assumed to be constant, which is obviously not compatible with real data. GARCH model, on the other hand, can capture the heteroscedasticity and therefore is expected to provide a more realistic estimate for option prices. Many variants of GARCH models have been proposed for pricing options.

2 The model

Suppose S_t is the asset price at time t. r is the risk-free interest rate. $\mathcal{N}(0,1)$ stands for the standard normal distribution. The GARCH model in this study is expressed as

$$\begin{cases} \log\left(S_t/S_{t-1}\right) & = r + \lambda\sigma_t - \frac{\sigma_t^2}{2} + \epsilon_t, \\ \epsilon_t & = \sigma_t\eta_t, \quad (\eta_t) \sim^{iid} N(0,1) \\ \sigma_t^2 & = \omega + \alpha\epsilon_{t-1}^2 + \beta\sigma_{t-1}^2, \quad \omega > 0, \alpha, \beta \geq 0, \end{cases} \tag{1}$$

and under the risk-neutral probability,

$$\begin{cases} \log\left(S_t/S_{t-1}\right) & = r - \frac{\sigma_t^2}{2} + \epsilon_t, \\ \epsilon_t^* & = \sigma_t\eta_t^*, \quad (\eta_t) \sim^{iid} \mathcal{N}(0,1) \\ \sigma_t^2 & = \omega + \alpha\left(\epsilon_{t-1} - \lambda\sigma_{t-1}\right)^2 + \beta\sigma_{t-1}^2. \end{cases} \tag{2}$$

The details for the model interpretation will be supplemented later. Reader can also refer to Chapter 12.2 of [Francq and Zakoïan(2010)]. Let Z_T be the payoff of the option at maturity time T. The option price is given by $e^{-r(T-t)}\mathbb{E}^Q\left[Z_T|\mathcal{F}_t\right]$, where $\mathbb{E}^Q$ is the expectation under the risk neutral probability and $\mathcal{F}_t$ is the information up to time t. Intuitively, t is interpreted as the current time.

The expectation under our model cannot be evaluated explicitly and must be evaluated by simulation. The procedures are:

1. Estimate the model parameter $(\omega, \alpha, \beta, \lambda)$ using equation (1).

2. Simulated values $S_T^{(i)}$ of S_T, the asset price at the maturity time of the option for $i = 1, \ldots, N$ are obtained by simulating, at step i, $T - t$ independent realizations $\nu_s^{(i)}$ of the $\mathcal{N}(0, 1)$ and by setting (from equation (2))

$$S_T^{(i)} = S_t \exp \left\{ (T - t) r - \frac{1}{2} \sum_{s=t+1}^{T} h_s^{(i)} + \sum_{s=t+1}^{T} \sqrt{h_s^{(i)}} \nu_s^{(i)} \right\},$$

where the $h_s^{(i)}$, $s = t + 1, \ldots, T$, are recursively computed from

$$h_s^{(i)} = \hat{\omega} + \left\{ \hat{\alpha} \left(\nu_{s-1}^{(i)} - \hat{\lambda} \right)^2 + \hat{\beta} \right\} h_{s-1}^{(i)}$$

taking, for instance, as initial value $h_t^{(i)} = \hat{\sigma}_t^2$. i represents the i-th possibility for the stock price. Hence the payoff the the option at maturity T and i-th possibility $Z_T^{(i)}$.

3. The option price at current time t is obtained by taking

$$e^{-r(T-t)} \frac{1}{N} \sum_{i=1}^{N} Z_T^{(i)}.$$

3 Application to Hang Seng Index option

5-year daily closing prices of Hang Seng Index from 2019-07-31 to 2024-07-30 are used for estimating the GARCH model parameters. The prices of those call options maturing on 2024-08-09 are available on the website of The Stock Exchange of Hong Kong Limited (hkex) and are printed as a pdf file.

The unconditional volatility (the sample standard deviation of the rate of return) is used as an input for the Black-Scholes model. The results are tabulated in Table 1. Note that the current HSI is around 17000. It can be seen that GARCH option pricing model performs much better than Black-Scholes if the options are out of money. If the option is at the money (17000), the GARCH option pricing model also performs much better than Black-Scholes. However, for strike prices 17200 and 17300, GARCH does not have an obvious advantage over Black-Scholes. In fact, the market option prices are a bit weird that the option with strike 17100 is more expensive than that with strike 17000. The reason for GARCH model not performing so well could be due to this phenomenon.

References

[Francq and Zakoïan(2010)] Francq, C. and Zakoïan, J.-M. (2010). *GARCH Models, Structure, Statistical Inference and Financial Applications.* UK: A John Wiley and Sons, Ltd., Publication.

strike	Market price	GARCH	Black-Scholes
16900	317	381.47	529.94
17000	301	334.76	467.09
17100	318	277.45	408.84
17200	307	246.19	355.31
17300	269	210.18	306.53
17400	197	168.57	262.46
17500	159	143.24	223.01
17600	127	113.26	188.01
17700	112	87.16	157.24
17800	76	83.40	130.46
17900	55	66.49	107.35

Table 1: Market prices, GARCH option prices and Black-Scholes prices