1. Maxwell's Equations

   → Gauss's Laws
   → Faraday's Laws
   → Ampere - Maxwell

2. Continuity

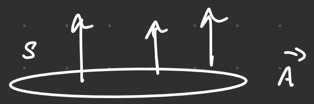
3. Wave

   → Travelling wave
   → EM wave
   → Plane Polarised

4. Propagation

   → Microscopic Ohm's Law
   → Conducting Medium

Flux

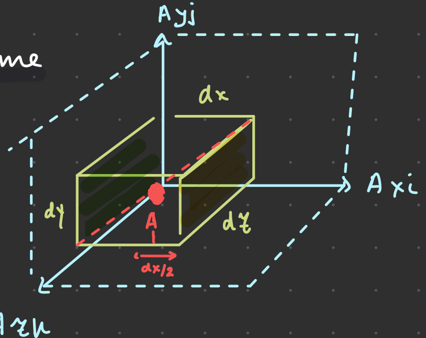


→ $d\phi$ of vector field (flux density) through infinitesimal surface $d\vec{a}$ = $d\phi = \underbrace{\vec{A}\cdot d\vec{a}}$ $d\vec{a} \to \hat{n}\, d\vec{a}$

$$\hat{n} \longrightarrow \vec{A}$$

→ $\phi = \int_S \vec{A}\cdot d\vec{a}$ (surface)

Volume



→ flux going in = $-\hat{n}$
out = $\hat{n}$

$$d\phi_R = \left[A_x + \frac{\partial A_x}{\partial x}\cdot\frac{dx}{2}\right] dx\,dy$$

$$d\phi_L = -\left[A_x - \frac{\partial A_x}{\partial x}\cdot\frac{dx}{2}\right] dx\,dy$$

$$\left[A_x + \frac{\partial A_x}{\partial x}\cdot\frac{dx}{2}\right] dx\,dy - \left[A_x - \frac{\partial A_x}{\partial x}\cdot\frac{dx}{2}\right] dx\,dy$$

$$= d\phi = \frac{\partial A_x}{\partial x}\cdot dx\,dy\,dz$$

$$A_x = \frac{\partial A_x}{\partial x}\cdot dV \Big\}$$ rate of change of field wrt x

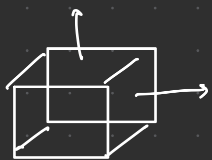
Flux

$A_y \Rightarrow \dfrac{\partial A_y}{\partial y} \cdot dV$

Net Flux:

$$d\Phi_{TOT} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dV$$

$$\Phi = \underbrace{\int_V \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) dV}_{\text{Divergence :)}}$$



Divergence $\vec{\nabla}$ → how much flux generated at any given point in space?

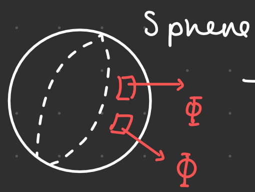
Positive $\vec{\nabla}$ ↓ source

Negative $-\vec{\nabla}$ ↓ Sink

→ rate of change of field in direction of field

$$\vec{\nabla} \cdot \vec{A} = \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right)$$

Gauss' Theorem | Divergence



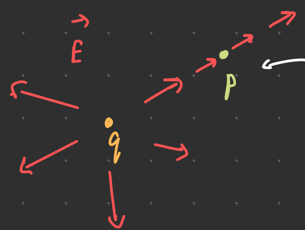
→ Total outward flux through closed surface area S = sum of normal components of $\vec{A}$ through $dx\,dy\,dz$

$$\Phi = \oint_S \vec{A} \cdot d\vec{a} \longrightarrow \underbrace{\int \vec{\nabla} \cdot \vec{A} \, dV}_{\text{Divergence of volume}}$$

→ positive divergence ⇒ more field out than in

→ negative divergence ⇒ more field in than out

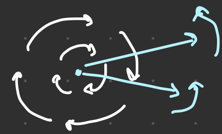
more flux out than in, positive ⟶



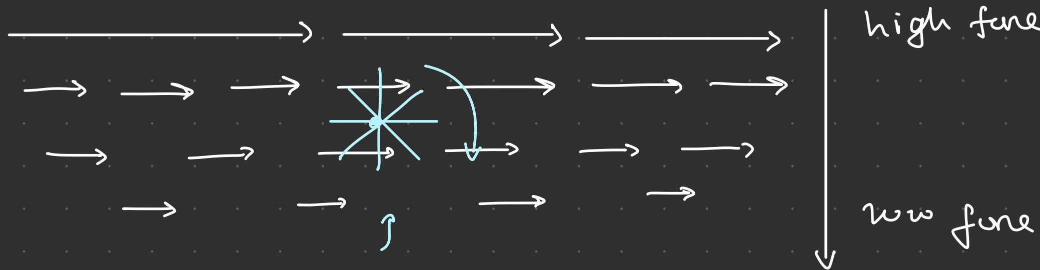
⟵ has less flux out than in, negative

Curl ($\vec{\nabla} \times$)

→ rate of change in vector field



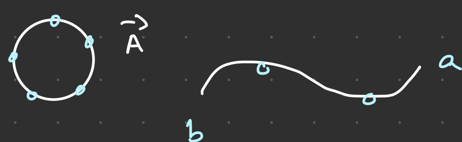
→ circulating force →



paddle wheel turn
if high force v low force

→ circulation is line integral of vector field along a path

$$\int_a^b \vec{A} \cdot d\vec{l}$$



→ if $\oint \vec{A} \cdot d\vec{l}$, closed path → conservative, line integral = 0

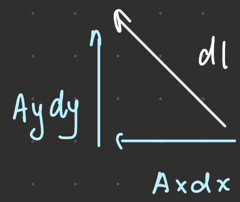
→ voltage fields are conservative

→ integrating along closed path → $\left.\int_c \vec{A} \cdot dl \right\}$ positive curl

Curl ($\vec{\nabla} \times$)

$\oint \vec{A} dl$

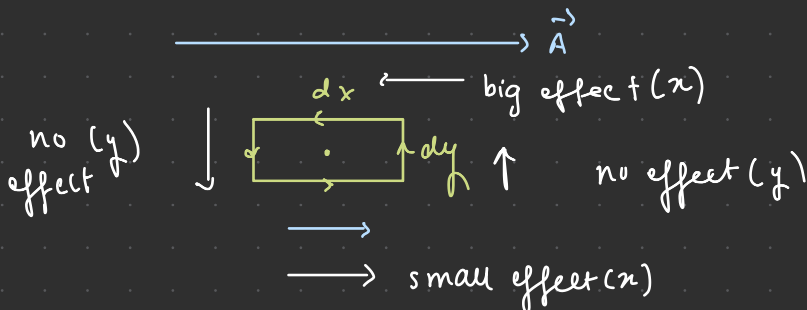
→ $\vec{dl}$ is infinitesmally small displacement along curve in space



$$\vec{A} \cdot \vec{dl} = A_x dx + A_y dy$$

→ components of a closed loop integral:

↳ rate of change of vector field at 90° to axis



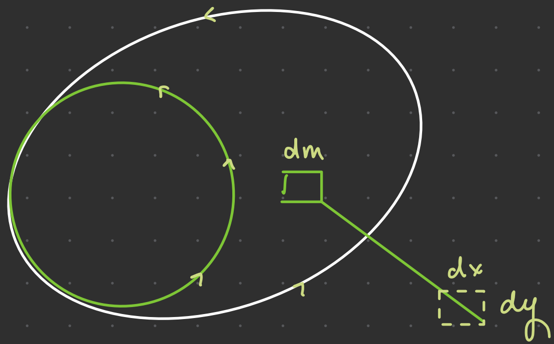
$$\oint \vec{A} \cdot dl = \left[\vec{i}\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} \right) - \vec{j}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) + \vec{k}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} \right) \right]$$

↓

$$\oint \vec{A} \cdot dl = (\vec{\nabla} \times \vec{A}) \cdot d\vec{a}$$

Stoke's Theorem

$$\oint \vec{A} \cdot dl = (\vec{\nabla} \times \vec{A})\, d\vec{a}$$



Laplacian

divergence of gradient of scalar

$$\vec{\nabla}\cdot\vec{\nabla}\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2} = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$\downarrow$

Laplacian of vector field $\vec{A}$

$$\vec{\nabla}\cdot\vec{\nabla}\vec{A} = \vec{\nabla}^2\vec{A} = \vec{\nabla}^2 A_x\vec{i} + \vec{\nabla}^2 A_y\vec{j} + \vec{\nabla}^2 A_z\vec{k}$$

Poisson's Equation: $\vec{\nabla}\cdot\vec{\nabla}\cdot\vec{A} = \vec{\nabla}^2\vec{A} = \vec{\nabla}(\vec{\nabla}\cdot\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$

$$\vec{\nabla}\vec{A} = \begin{vmatrix} \frac{\partial A_x}{\partial x} & \frac{\partial A_x}{\partial y} & \frac{\partial A_x}{\partial z} \\ \frac{\partial A_y}{\partial x} & \frac{\partial A_y}{\partial y} & \frac{\partial A_y}{\partial z} \\ \frac{\partial A_z}{\partial x} & \frac{\partial A_z}{\partial y} & \frac{\partial A_z}{\partial z} \end{vmatrix}$$

Product Rules:

- vector that is gradient of a scalar $\rightarrow$ no curl
- vector that is a curl of a scalar $\rightarrow$ no divergence

Maxwell's Equations

differential
↓
in space

integral
↓
surfaces, volumes

Parameter	Symbol	SI unit
Electric field	$\vec{E}(x,y,z)$	volt/m
Electric flux density Displacement	$\vec{D}(x,y,z)$	coulomb/m^2 Cm^{-2}
Magnetic flux	$\Phi(x,y,z)$	weber = joule/A
Magnetic flux density Magnetic inductance	$\vec{B}(x,y,z)$	tesla = weber/m^2 T = NA^{-1}m^{-1}
Magnetic field	$\vec{H}(x,y,z)$	ampere/m = Am^{-1}
Charge	Q or q	coulombs
Current	I	ampere
Charge density	$\rho(x,y,z)$	coulomb/m^3
Current density	$\vec{J}(x,y,z)$	ampere/m^2
Electro motive force	Emf, V	volts
Vector Potential	$\vec{A}(x,y,z)$	*definition*
Scalar Potential	$\phi(x,y,z)$	volt

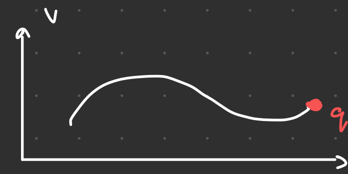
Parameter	Symbols	Free Space	SI Units
Permeability	$\mu = \mu_r \mu_0$	$\mu_0 = 4\pi \times 10^{-7}$	henry/m N/A^2
Permittivity	$\varepsilon = \varepsilon_r \varepsilon_0$	$\varepsilon_0 = 10^{-9}/(36\pi)$	farad/m
Conductivity	γ or σ	- - - -	siemens/m $(\Omega m)^{-1}$
Light Speed	$c = 1/\sqrt{\mu_0 \varepsilon_0}$	3 x 10^8	m/s
Impedance	$\eta = \sqrt{\mu_0 / \varepsilon_0}$	$120\pi = 377$	ohms, Ω

→ flux density = how field affects the body it is acting on

☆ $\vec{x} \cdot d\vec{a} \iff \vec{x} \cdot \hat{n}\, da$

↳ unit vector (normal)

Electric field $\vec{E}$



$$\vec{F}(x,y,z) = Q\vec{E}(x,y,z)$$

→ charged particle moving around closed conservative loop
has no net force = 0

→ no work is done by field on the particle

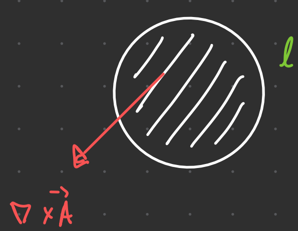
Recall Stoke's Theorem: $\oint \vec{E} \cdot ds = \int (\vec{\nabla} \times \vec{E})\, d\vec{a} = 0$

→ closed loop, electrostatic field has no curl

→ Electric field strength is measured at every point
passing through the perpendiculars

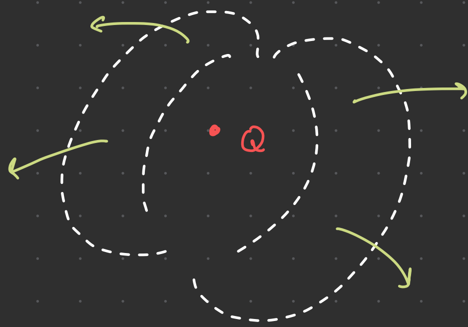
→ $\vec{E}$ field has no curl (closed loop)

Stokes Theorem relates closed line integral to the surface



Electric field

Gauss' Law for Static Electric field



→ charge in free space creates field lines

$$\varepsilon \oint_S \vec{E} \cdot \vec{da} = Q \qquad \oint \vec{D} \cdot \vec{da} = Q$$

Electric induction $\Longleftrightarrow$ $\vec{D} = \varepsilon \vec{E}$ ← $\vec{E}$ = electric field (unchanging)

$\vec{D}$ = amount of field induced on body

= flux per unit area

Electric field → Gauss' law

↳ closed surface integral ⟺ volume integral

$$\oint_S \vec{A} \cdot d\vec{a} = \int_V \vec{\nabla} \cdot \vec{A} \cdot dV$$

→ applying Gauss' Theorem:

⇒ Integral Form!

$$\oint_S \vec{D} \cdot d\vec{a} = \int_V (\vec{\nabla} \cdot \vec{D}) \cdot dV = Q = \int_V \rho \, dV$$

→ ρ = charge density scalar function

$$\rho = \vec{\nabla} \cdot \vec{D}$$

$$\int_V \vec{\nabla} \cdot \vec{D} \, dV = \int_V \rho \, dV$$

Electric Field

→ Divergence (how much field in/out) relates to how much charge

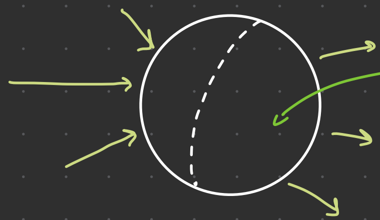
$$\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$$ ← charge density

Point charge (charge q C, distance r m)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q}{r^2}\hat{r}$
Conducting sphere (charge Q C) (perfect conductor)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (outside, r m from centre) $\vec{E} = 0$ (inside)
Uniformly charged insulating sphere (radius r_0 m)	$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}\hat{r}$ (outside) $\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Qr}{r_0^3}\hat{r}$(inside, r m from centre)
Infinite line charge (linear charge density λ C/m)	$\vec{E} = \frac{1}{2\pi\varepsilon_0}\frac{\lambda}{r}\hat{r}$ (r m from line)
Infinite flat plane (surface charge density σ C/m²)	$\vec{E} = \frac{\sigma}{2\varepsilon_0}\hat{n}$

Magnetic field $\vec{H}$

- $\vec{H}$ → field strength → force at point
- $\vec{B}$ → flux density → force on body due $\vec{H}$

→ Total magnetic flux passing through closed surface = 0



no flux generation,

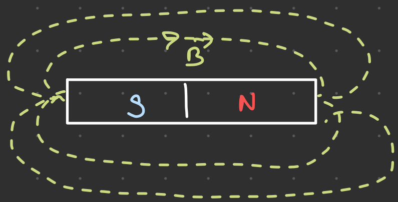
no source or sink

Net flux = 0

→ Magnetic fields have no divergence

→ No point in space generates magnetic fields

Source (− ∞) ⟶ Sink (∞)



$\vec{B} = 0$

→ $\mu_0 \mu_r = \mu$ (relative permittivity)

Magnetic fields

$$\oint_S \underbrace{\vec{B} \cdot \hat{n}}\, da = 0$$

component of magnetic field normal to surface

$\vec{B}$ = magnetic flux density

Differential form

$$\underbrace{\vec{\nabla} \cdot}_{\text{divergence}} \vec{B} = 0$$

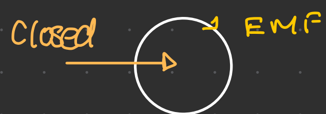
Faraday's law

Electric current induced by time-varying magnetic field

→ by moving a magnet through closed loop

emf = force to move charge around loop

$$EMF = -\frac{d\phi_m}{dt}$$ induced electromotive force



→ EMF would induce field + ↑↓

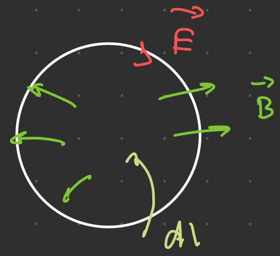
work done in moving charge over a curve $\oint_l \vec{E}\cdot\vec{dl}$

$$\oint \vec{E}\cdot\vec{dl} = -\frac{d\phi_m}{\phi t}$$

→ Flux through surface $= \int_S \vec{B}\cdot\vec{ds}$

Magnetic field

$$\oint_C \vec{E}\cdot d\vec{l} = -\int \frac{d\Phi_m}{dt}$$



→ rate of change of flux

$$\frac{d \int_S \vec{B}\cdot d\vec{a}}{dt}$$

$$\oint_C \vec{E}\cdot d\vec{l} = -\frac{d}{dt}\int \vec{B}\cdot ds = -\int \frac{\partial \vec{B}}{\partial t}\cdot \hat{n}\cdot d\vec{a}$$

$$\boxed{\oint_C \vec{E}\cdot d\vec{l} = \int_S \vec{\nabla}\times\vec{E}\cdot d\vec{a}} = -\underbrace{\int_S \frac{\partial B}{\partial t}\cdot d\vec{a}}_{\text{flux of M field, time}}$$

flux through S bounded by C

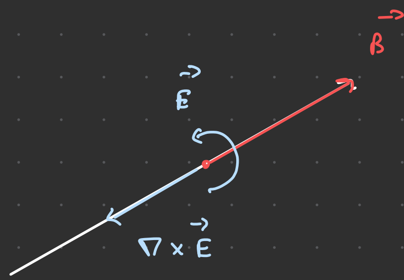
lenz' law

→ changing magnetic field creates circulating electric field

Applying Stoke's Theorem

$$\int_l \vec{E}\cdot\vec{dl} = \int_S \vec{\nabla}\times\vec{E}\cdot da = -\int_S \frac{d\vec{B}}{dt}\cdot da$$

$$\longrightarrow \quad \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{dt}$$

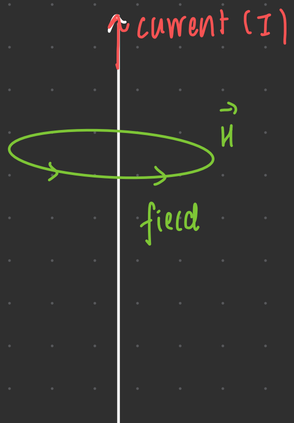


(because $\vec{\nabla}\times\vec{E} = \mu\frac{\partial \vec{H}}{\partial t}$

Ampere Maxwell's law

→ changing electric field effect on magnetic field

→ current inducing a magnetic field



$$\oint \vec{H}\cdot\vec{dl} = I$$

$\vec{J}(x,y,z)$ = current density (A/m²)

$$\int_S \vec{J}(x,y,z)\cdot\vec{da} = I$$

$$\therefore \quad \underbrace{\int \vec{J}(x,y,z)\cdot d\vec{a}}_{\text{current density}} = \underbrace{\oint \vec{H}\cdot dl}_{\text{magnetic field}} = \underbrace{I}_{\text{current}}$$

Ampere Maxwell's law

$$\oint_l \vec{H} \cdot \vec{dl} = \int_S \vec{J} \cdot \vec{da}$$

$$\boxed{\oint_C \vec{A} \cdot \vec{dl} = \int_S \nabla \times \vec{A} \cdot \vec{ds}}$$

Applying Stoke's Theorem →

$$\int_S \vec{\nabla} \times \vec{H} \cdot \vec{da} = \int_S \vec{J} \cdot \vec{da}$$

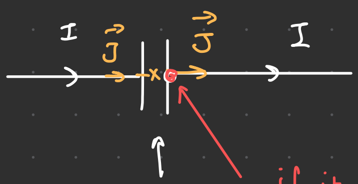
$$\vec{\nabla} \times \vec{H} = \vec{J}$$ → curl of magnetic field = charge dens.

$$\underbrace{\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{H} \right)}_{\text{div of any curl } = 0} = \vec{\nabla} \cdot \vec{J}$$

→ take divergence of both sides

→ div of any current $(\vec{J}) \neq 0$

→ i.e. capacitor



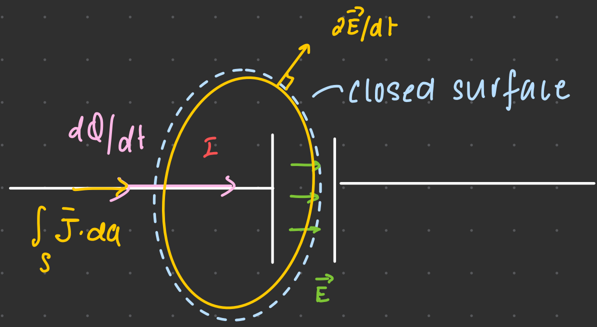
→ there is a charge between 2 plates (called displacement current)

if it looks like $\vec{J}$ is being generated, there is divergence

Displacement Current

$$\boxed{I = \frac{dQ}{dt}}$$

→ $\text{current} = \dfrac{\text{charge}}{\text{time}}$



→ charge is mounting in the volume

→ when Q is increasing (recall - Gauss' law)

$$\Phi = \int_S \vec{D}\cdot d\vec{a} = Q$$

$$I = \frac{dQ}{dt} = \frac{d\Phi}{dt}$$

$$I = \frac{d \oint_S \vec{D}\cdot d\vec{a}}{dt} = \int_S \frac{d\vec{D}}{dt}\, d\vec{a} = \int_S \vec{J}\cdot d\vec{a}$$

displacement current with current density

Ampere - Maxwell

$$-\int_S \vec{J}\cdot d\vec{a} = \int_S \frac{d\vec{D}}{dt}\cdot d\vec{a}$$

$$\underbrace{-\int_S \vec{J}\cdot d\vec{a}}_{\text{conduction (in)}} \qquad \underbrace{\int_S \frac{d\vec{D}}{dt}\cdot d\vec{a}}_{\text{displacement (current out)}}$$

Applying Gauss

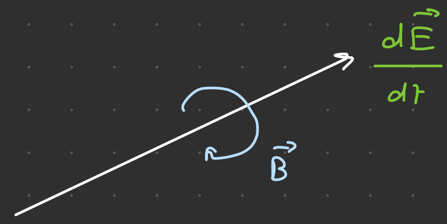
$$\int_S \left(\vec{J} + \frac{d\vec{D}}{dt}\right)\cdot d\vec{a} = 0 \longrightarrow \int_V \vec{\nabla}\cdot\left(\vec{J} + \frac{d\vec{D}}{dt}\right)\cdot d\vec{a} = 0$$

$$\downarrow$$

$$\vec{\nabla}\left(\vec{J} + \frac{d\vec{D}}{dt}\right) = 0$$

Divergence = 0

Ampere - Maxwell



Differential

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{d\vec{D}}{dt}$$

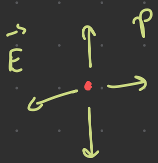
→ current → circulating magnetic field

Integral

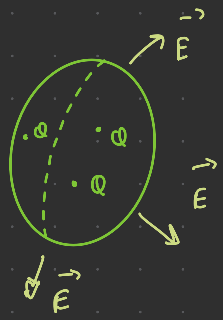
$$\oint_{l} \vec{H} \cdot dl = I_{enc} + \frac{d}{dt} \int_{S} \vec{D} \cdot d\vec{a}$$

Electric field ($\vec{E}$) | Magnetic field ($\vec{B}$)

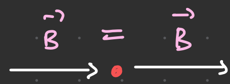
diff



int

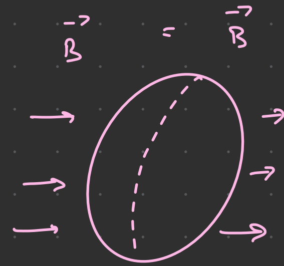


diff



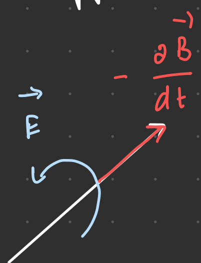
$\nabla \cdot \vec{B} = 0$

int



$\vec{B} \longrightarrow \vec{E}$

diff



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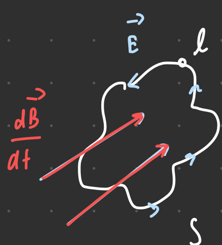
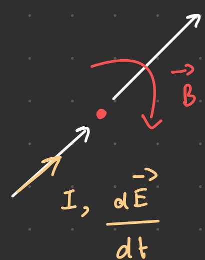


→ magnetic field through a surface

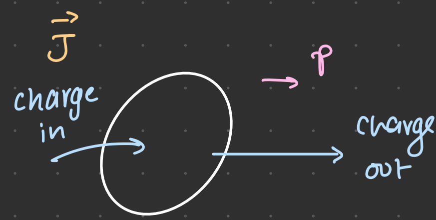
$\vec{E}, I \longrightarrow \vec{B}$

diff



int

Continuity



→ conservation of charge

$$q(t) = \int_V p(t) \cdot dV$$

→ charge in = $\int$ charge density

→ (current in - current out) through surface enclosing volume = net charge exiting body

$$\Longrightarrow \oint_S \vec{J} \cdot \vec{da}$$

→ rate of change of charge entering = net current entering

$$\frac{dq(t)}{dt} = -\oint \vec{J} \cdot \vec{da}$$

Applying Gauss'

$$\int_V \frac{dp(t)}{dt} \cdot dV = -\int_V \vec{\nabla} \cdot \vec{J} \cdot dV \quad \text{(volumes)}$$

$$\frac{dp(t)}{dt} = -\vec{\nabla} \cdot \vec{J}$$

rate of charge density

current density

Continuity Equation

$$\underbrace{\frac{d\rho(t)}{dt}}_{\substack{\text{rate of change} \\ \text{of charge} \\ \text{density}}} + \vec{\nabla} \cdot \vec{J} = 0$$

Wave Equation

→ interacting with each other as waves

Poisson's Equations

$$\underbrace{\vec{\nabla}.\vec{\nabla}\vec{A}} = \vec{\nabla}^2\vec{A} = \vec{\nabla}(\vec{\nabla}.\vec{A}) - \vec{\nabla}\times(\vec{\nabla}\times\vec{A})$$

↳ Travelling wave

- → using function in space & time → $s(x, t)$
- → describes function moving in time

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 s}{\partial t^2}$$

→ Given sin wave: $\sin(x - wt)$

$$\frac{\partial^2 s}{\partial x^2} = -\sin(x - wt)$$

$$\frac{\partial^2 s}{\partial t^2} = -w^2\sin(x - wt)$$

Wave Equation

$s(x,t) \longrightarrow \underset{+ve}{F(x-vt)} + \underset{-ve}{G(x+vt)}$

F & G are arbitrary differentiable waves

Medium

→ homogenous, isotropic

→ insulator, $\mu = \mu_0 \mu_r$, $\varepsilon = \varepsilon_0 \varepsilon_r$

Maxwell's Eqns

2 eqns
2 unknowns

$$\begin{cases} \vec{\nabla} \times \vec{E} = -\mu \dfrac{\partial \vec{H}}{\partial t} & \text{. Faraday's Law} \\ \vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \dfrac{\partial \vec{E}}{\partial t} & \text{... Ampere Maxwell} \end{cases}$$

Take curl of Ampere's law:

$$\underbrace{\vec{\nabla} \times \vec{\nabla} \times \vec{H}}_{\vec{\nabla}(\vec{\nabla}\cdot\vec{H}) - \vec{\nabla}^2 H} = \varepsilon \frac{\partial(\vec{\nabla} \times \vec{E})}{\partial t}$$

→ apply Poisson's Law $\partial J / \partial t \Rightarrow 0$

✱ remember $\vec{\nabla} \cdot \vec{H} = 0$ Always (Gauss's Law)

Ampere's law

$$-\nabla^2 \vec{H} = \varepsilon \frac{\partial(\vec{\nabla} \times \vec{E})}{\partial t}$$

from faraday's law → $\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$

$$-\nabla^2 \vec{H} = -\varepsilon\mu \frac{\partial \vec{H}}{\partial t} \cdot \frac{\partial}{\partial t}$$

$$\nabla^2 \vec{H} = \varepsilon\mu \cdot \frac{\partial^2 \vec{H}}{\partial t^2}$$

Intrinsic Impedance $\eta = \sqrt{\frac{\mu}{\varepsilon}}$

In the wave equation → $\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$

$$\vec{\nabla}^2 \vec{H} = \frac{1}{c^2} \frac{\partial^2 \vec{H}}{\partial t^2}$$

$\therefore \quad \varepsilon\mu = \frac{1}{c^2}$

$\therefore \quad c = \frac{1}{\sqrt{\varepsilon\mu}}$

Steps:

1. Take curl of both sides
2. Poisson's law
3.

3D wave

$$\vec{\nabla}^2 \vec{E} = \frac{1}{c^2} \frac{\partial^2 \vec{E}}{\partial t^2}$$

$$\vec{\nabla}^2 \vec{H} = \frac{1}{C^2} \frac{\partial^2 \vec{H}}{\partial t^2}$$

Expand this using the Laplacian: (differentiate each direction)

$$\mu\varepsilon \frac{\partial^2 E_x}{\partial t^2} = \frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2}$$

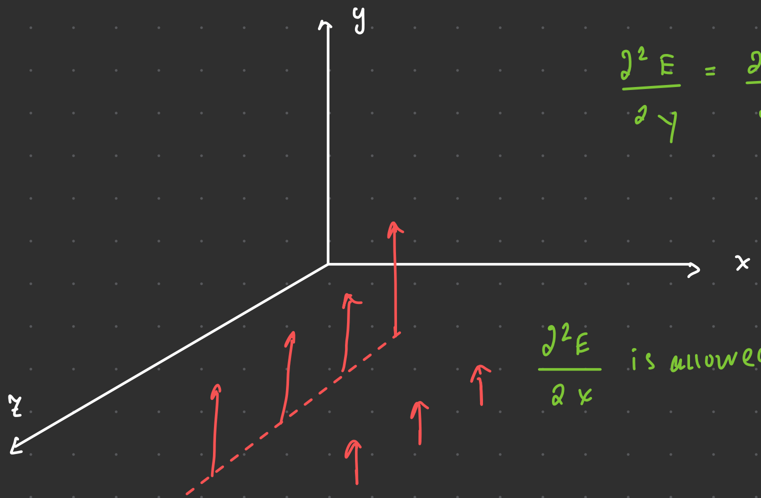
$$\mu\varepsilon \frac{\partial^2 E_y}{\partial t^2} = \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial y^2} + \frac{\partial^2 E_y}{\partial y^2}$$

$$\mu\varepsilon \frac{\partial^2 E_z}{\partial t^2} = \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial y^2} + \frac{\partial^2 E_z}{\partial y^2}$$

→ second derivatives in space → time

Plane Polarised Wave

→ reduce 3D wave → 1D equivalent



$$\frac{\partial^2 E}{\partial y} = \frac{\partial^2 E}{\partial z} = 0$$

$\frac{\partial^2 E}{\partial x}$ is allowed

→ only allow it to vary in one direction

Polarisation = $E_x = E_z = 0$

Plane of wave = $\frac{\partial E_y}{\partial y} = \frac{\partial E_y}{\partial z} = 0$

$$\frac{\partial^2 E_y}{\partial x^2} = \mu \varepsilon \underbrace{\frac{\partial^2 E_y}{\partial t^2}}$$

↳ $E_y \rightarrow F(x - vt) + G(x + vt)$

($F, G \neq$ sinusoids. any 2x differentiable functions

Fields

Remember:

$$\vec{\nabla} \cdot \vec{D} = \rho$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{E} = \frac{-\partial \vec{B}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

curl

$$\vec{\nabla} \times \vec{A} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix} = \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_y}{\partial z} \right) \hat{i} - \left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z} \right) \hat{j} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial A_y} \right) \hat{k}$$

example

$$\vec{E} = 2b(x + yz)t\,\hat{i} + 2zt\,\hat{j} + ax^2\,\hat{k}$$

$$\vec{\nabla} \times \vec{E} = (0 - 2t)\,\hat{i} - (2ax - 2byt)\,\hat{j} + (0 - 2bzt)\,\hat{k}$$

$$\vec{B} = t^2\,\hat{i} + (2axt - byt^2)\,\vec{j} + bzt^2\,\vec{k}$$

$$-\frac{\partial \vec{B}}{\partial t} = -\left((2t)\,\hat{i} + (2ax - 2byt)\,\hat{j} + 2bzt\,\hat{k}\right)$$

Find values of ρ $\Rightarrow$ remember: $\vec{\nabla} \cdot \vec{B} = 0$

$$\vec{\nabla} \cdot \vec{E} = 2bt\,\hat{i} + 0\hat{j} + 0\hat{k} = \frac{\rho}{\varepsilon}$$

$\therefore$ $\rho = 2bt \cdot \varepsilon$ (holds for $\vec{\nabla} \cdot \vec{D} = \rho$)

$\Rightarrow$ charge density $= \rho$

Find value for current density $\vec{J}$

$$\vec{J} = \nabla \times \vec{H} - \frac{\partial \vec{D}}{\partial t} \implies \frac{\nabla \times \vec{B}}{\mu} - \frac{\varepsilon\, \partial \vec{E}}{\partial t}$$

$$\underset{\text{(curl)}}{\vec{\nabla} \times \vec{B}} = (0-0)\hat{i} - (0-0)\hat{j} + (2at - 0)\hat{k}$$

$$\left\{ \begin{aligned} \vec{B} &= t^2 \hat{i} + (2axt - byt^2)\vec{j} + bzt^2 \vec{k} \\ \vec{E} &= 2b(x + yz)t\hat{i} + 2zt\hat{j} + ax^2\hat{k} \end{aligned} \right\}$$

$$\frac{\partial \vec{E}}{\partial t} = 2b(x + yz)\hat{i} + 2z\hat{j} + 0\hat{k}$$

$$\vec{J} = \frac{2at\hat{k}}{\mu} - \varepsilon\left(2b(x + yz)\hat{i} + 2z\hat{j}\right)$$

$$\therefore \quad -\varepsilon 2b(x+yz)\hat{i} - \varepsilon \cdot 2z\hat{j} + \frac{2at\hat{k}}{\mu}$$

Note that: $\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

$\rho = 2bt \cdot \varepsilon$

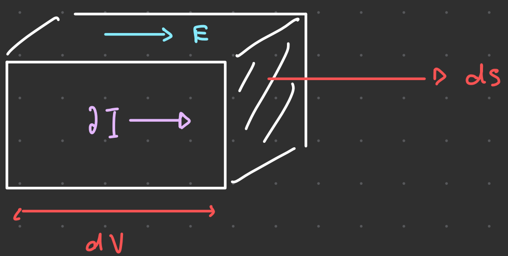
$\vec{J} = -\varepsilon 2b(x+yz)\hat{i} - \varepsilon \cdot 2z\hat{j} + \frac{2at}{\mu}\hat{k}$

$$\left.\begin{aligned} \vec{\nabla} \cdot \vec{J} &= -\varepsilon 2b \\ \frac{\partial \rho}{\partial t} &= 2b\varepsilon \end{aligned}\right\} \therefore \rightarrow -\varepsilon 2b + \varepsilon 2B = 0$$

Propagation in a conducting medium

→ in free space → $\vec{H}$ and $\vec{E}$ are in phase

Microscopic Ohm's law



$$\text{conductance} = \frac{1}{R} = \sigma\left(\frac{\text{area}}{\text{length}}\right)$$

$$dI = \frac{1}{R}\cdot dV \qquad \vec{J}\cdot\vec{ds} = \sigma\frac{|\vec{ds}|}{|\vec{dl}|}\,\vec{E}\cdot\vec{dl}$$

$$\vec{J} = \sigma\vec{E}$$

→ current density has to be in same direction as $\vec{E}$ field

Microscopic Ohm's Law

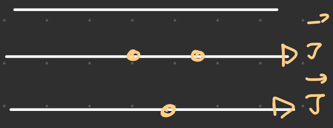
$$\vec{J} = \sigma \vec{E}$$

→ in steady state, lines of current don't terminate

→ no accumulation of charge

→ i.e. $\vec{\nabla} \cdot \vec{J} = 0$

$$\sigma . \vec{\nabla} \cdot \vec{E} = 0$$



→ single frequency, steady state

→ e.g. $E_y = E_m (\cos(\omega t - \beta x))$

Analysis in Exponential Notation

$$E_y = E_m \cdot \cos(\omega t - \beta x)$$

→ write this as a real part of a complex function

$$E_y = \text{Real}\left(E_m e^{j(\omega t - \beta x)}\right)$$

$$= \text{Real}\left(\underbrace{E_m e^{j\beta x}} \cdot \underbrace{e^{-j\omega t}}\right)$$

at $E_0 = E_m e^{-j\beta x}$

i.e. $t = 0$

$$= \text{Real}\left(E_0 e^{-j\omega t}\right)$$

complex form

$$\vec{E}(x, y, z, t) = \vec{E}_0(x, y, z)\, e^{j\omega t}$$

- Then:

$$\vec{E}(x,y,z,t) = (\alpha + j\beta)\big(cos(\omega t) + jsin(\omega t)\big)$$

- The magnitude and phase of the signal is:

$$\left|\vec{E}(x,y,z,t)\right| = |\alpha + j\beta|\, |cos(\omega t) + jsin(\omega t)|$$

$$\angle\vec{E}(x,y,z,t) = \tan^{-1}\frac{\beta}{\alpha} + \tan^{-1}\frac{\sin\omega t}{\cos\omega t}$$

- The magnitude and phase of the signal is:

$$\left|\vec{E}(x,y,z,t)\right| = |\alpha + j\beta|\, |cos(\omega t) + jsin(\omega t)|$$

$$= \sqrt{\alpha^2 + \beta^2}\,.1$$

$$\angle\vec{E}(x,y,z,t) = \tan^{-1}\frac{\beta}{\alpha} + \tan^{-1}\frac{\sin\omega t}{\cos\omega t}$$

$$= \tan^{-1}\frac{\beta}{\alpha} + \omega t$$

Analysis

$$\vec{E} = \vec{E}_0 e^{j\omega t} \longrightarrow \vec{H} = \vec{H}_0 e^{j\omega t}$$

$$\frac{\partial \vec{E}}{\partial t} = j\omega \vec{E}_0 e^{j\omega t} = j\omega \vec{E}$$

$\hookrightarrow \frac{\partial}{\partial t} = j\omega$ rate of change over time of the field

- Analyse of how field effects occur in conducting medium i.e. current flow when charge is applied

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{E}_0 j\omega t = -j\omega\mu \vec{H}_0 e^{-j\omega t}$$

$$\vec{\nabla} \times \vec{E}_0 = -j\omega\mu \vec{H}_0$$

$\Updownarrow$ divide

$$\vec{\nabla} \times \vec{H} = \sigma \vec{E} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \times \underbrace{\vec{H}_0 e^{j\omega t}}_{\text{complex form of equations}} = \sigma \vec{E}_0 e^{j\omega t} + j\omega \vec{E}_0 e^{j\omega t} \cdot \varepsilon$$

$$\vec{\nabla} \times \vec{H}_0 = (\sigma + j\omega\varepsilon) \vec{E}_0$$

To Derive the wave Equation

→ conducting media has current density

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$ ⟶ E field eqn → μ (permissivity)

$$\vec{\nabla} \times \vec{H} = \underbrace{\sigma \vec{E}}_{\vec{J} = \sigma \vec{E}} + \frac{\varepsilon \partial \vec{E}}{\partial t}$$ ⟶ H field has Epsilon (permittivity)

$$\underbrace{\vec{J} = \sigma \vec{E}}_{\text{current density}}$$

Space Equations

$$\vec{\nabla} \times \vec{E_0} = -j\omega\mu \vec{H_0}$$

$$\vec{\nabla} \times \vec{H_0} = (\sigma + j\omega\varepsilon)\vec{E_0}$$

Poisson's Equation : $\underbrace{\nabla(\nabla \cdot \vec{E})}_{=0} - \nabla^2 \vec{E} = \vec{\nabla} \times \vec{\nabla} \times \vec{E}$

→ no divergence of an electric field

→ charge is always on the outside of a conducting medium

Poisson's Equation

remember → $j\omega = \partial/\partial t$

- Solve these two simultaneous equations (Take the curl of both sides):
  $\vec{\nabla}\times\vec{\nabla}\times\vec{E}_o = -j\omega\mu(\vec{\nabla}\times\vec{H}_o\ )$

- Using poisons equation, and substitution for $\vec{\nabla}\times\vec{H}_o$:
  $-\vec{\nabla}^2\vec{E}_o = -j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}_o$

- We are left with the equation representing the propagating E field in a conducting medium:
  $\vec{\nabla}^2\vec{E}_o = j\omega\mu(\sigma + j\omega\varepsilon)\vec{E}_o$

- Let $\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)} = \boldsymbol{\gamma}$, to get:

  $\vec{\nabla}^2\vec{E}_o = \gamma^2\vec{E}_o$ - wave Equation

- $\boldsymbol{\gamma}$ is complex and is called the Propagation Constant: ($\gamma = \alpha + j\beta$)
  α = attenuation constant
  β = Phase constant

wave equation → $\underbrace{\vec{\nabla}^2 E_0}_{\text{space derivative}} = \underbrace{\gamma^2 \vec{E}_0}_{\text{time derivative}}$

$\vec{E} \Leftrightarrow \vec{H}$

→ connecting the electric & magnetic field using the wave Eqn.

Plane Polarised Wave Analysis

free space $\begin{cases} E_x = E_z = 0 \\ \dfrac{\partial E_y}{\partial y} = \dfrac{\partial E_y}{\partial z} = 0 \end{cases}$

In a conducting medium, field propagation:

$$\frac{\partial^2 E_{oy}}{\partial x^2} = \gamma^2 E_{oy}$$

γ = second order derivation

Plane polarised wave $\left(E_{oy} = E_m e^{\pm \gamma x} \right.$ ⟶ E_m is a constant

$$\frac{\partial^2 E_m e^{\pm \gamma x}}{\partial x^2} = \gamma^2 E_m e^{\pm \gamma x}$$

↓

use complex function $\vec{E} = \bar{E}_0 jwt$

$\vec{E} = E_0\, {}^{jwt}$

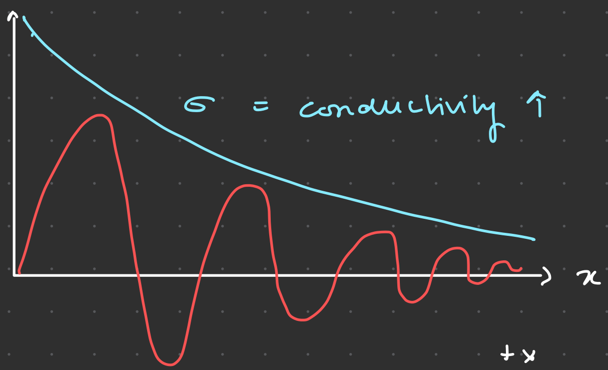
Propagating field

$$E_y = E_m e^{j\omega t} \, e^{\pm \alpha x} \, e^{\pm j\beta x}$$

sinusoidal variation

attenuation

phase shift



→ exponential attenuation $\propto$ alpha (α)

→ $\alpha \longrightarrow \gamma = \mu \varepsilon \sigma$

→ Gamma (γ) = propagation constant

if $\sigma \uparrow$, $\alpha \uparrow$, $\beta \uparrow$

To find $\vec{H}$ in a conducting medium

$$E_y = E_m e^{j\omega t} e^{-\alpha x} e^{-j\beta x}$$

$$E_{oy} = E_m e^{-\gamma x}$$ γ gamma = prop constant

Maxwell's Equation: $\vec{\nabla} \times \vec{E_0} = -j\omega\mu \vec{H_0}$

for a 1-D plane polarised wave: $\frac{\partial E_{oy}}{\partial x} \vec{k} = -j\omega\mu \vec{H_0}$

↳ this means $\vec{H_0}$ only has components in the z-direction

Plane Polarised Wave Eqns →

$$\boxed{\begin{gathered} E_z = E_x = 0 \\ \frac{\partial E_y}{\partial y} = \frac{\partial E_y}{\partial z} = 0 \end{gathered}}$$

$$\frac{\partial E_{oy}}{\partial x} \underbrace{\hat{n}}_{\text{unit vector}} = -j\omega\mu \vec{H_0} = -\gamma E_m e^{-\gamma x} = -\gamma E_{oy}$$ (1d plane polarised)

$$\therefore \frac{H_{oz}}{E_{oz}} = \frac{j\omega\mu}{\gamma}$$

- connected by a complex term
- E and H are not in phase

$$\eta = \sqrt{\frac{E_o}{\mu_o}}$$ in free space ⟶ $$\eta = \frac{j\omega\mu}{\gamma} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}}$$ (in a medium)

$$\boxed{\eta = \sqrt{\frac{j\omega\mu}{\sigma + j\omega\varepsilon}}}$$

Propagation of $\vec{H}$ in conducting medium

(same as last page) →

$$\frac{\partial E_{oy}}{\partial x} = -j\omega\mu \vec{H}_o = -\underbrace{\gamma E_m e^{-\gamma x}}_{} = -\gamma E_{oy}$$

↓

depends on γ (prop. constant)

↓

depends on conductivity of the material

→ $$\eta = \frac{j\omega\mu}{\underbrace{\gamma}_{\substack{\text{gamma} \\ \gamma}}} = \frac{j\omega\mu}{\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}}$$

What is Propagation Constant γ

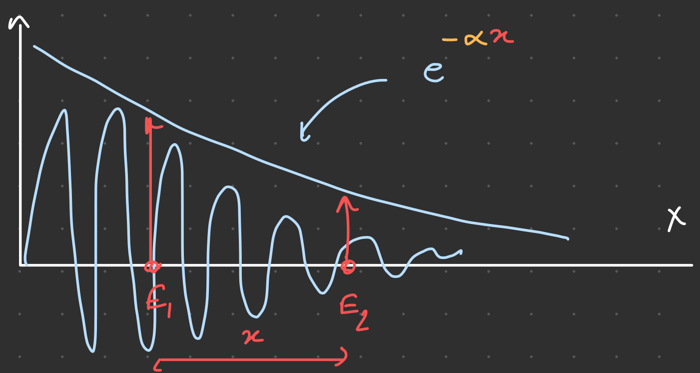
$$E_y = E_m e^{j\omega t} e^{-\alpha x} e^{-j\beta x}$$

α = attenuation constant (nepers / m)

phase constant (rads/m)

$$\boxed{\gamma = \alpha + j\beta}$$

if you had an $\vec{E}$ field through x (medium), the size of wave attenuates



$$|E_2| = |E_1| e^{-\alpha x} \Rightarrow \alpha = \frac{-\ln \frac{|E_1|}{|E_2|}}{x} \text{ nepers/m}$$

usually assume x = 1

What is Propagation Constant γ

→ nepers are dimensionless

→ $20 \log_{10} \left| \frac{E_2}{E_1} \right|$ dB

→ 1 neper = 0.866 dB

$$\frac{20 \ln \frac{|E_2|}{|E_1|}}{\ln 10} \Rightarrow \ln \frac{|E_2|}{|E_1|} \frac{20}{\ln 10} \Rightarrow -\alpha \times 8.686 \quad \text{dB}$$

1. Good Conductor → conductivity is large

$$\sigma \gg \omega \varepsilon$$

if $\gamma = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

$$= \sqrt{j\omega\mu\sigma\left(1 + \underbrace{j\frac{\omega\varepsilon}{\sigma}}\right)}$$

in a good conductor, this is negligible

In a good conductor $\gamma = \sqrt{j\omega\mu}$

1. γ in a Good Conductor

$$\gamma = \sqrt{j\omega\mu\sigma} \qquad \sqrt{j} = \left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{\omega\mu\sigma}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

$$= \sqrt{2\pi f\mu\sigma}\left(\frac{1}{\sqrt{2}} + j\frac{1}{\sqrt{2}}\right)$$

convert $\omega = 2\pi f$ to multiply across by $\sqrt{2}$

$$= \sqrt{\pi f\mu\sigma}\cdot(1+j)$$

this means $a \approx \beta$

$\gamma = \alpha + j\beta$

$$\gamma = a + j\beta = \sqrt{\pi f\mu\sigma}$$

if x is a conductor

- σ is large
- e^{-ax} is large → wave attenuates rapidly
- $v = \dfrac{\omega}{\beta} = \dfrac{\omega}{\sqrt{\pi f\mu\sigma}}$ → wave velocity is slow

remember: ω = frequency (rad/s)

β = phase constant (rads/m)

Look at different example → γ in General (2.)

$$\boxed{\text{Given: } \gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}}$$

Step 1 : Take square

$$\gamma^2 = \alpha^2 - \beta^2 + 2\alpha\beta j = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$= -\omega^2\mu\varepsilon + j\omega\mu\sigma \qquad (1)$$

→ Trying to find independent values for α and β

Step 2 : Take square of absolute value

$$|\gamma|^2 = \alpha^2 + \beta^2 = \left(\sqrt{|j\omega\mu|\,|\sigma + j\omega\varepsilon|}\right)^2$$

$$= \sqrt{\sqrt{\omega^2 u^2}\,\sqrt{\sigma^2 + \omega^2\varepsilon^2}}$$

$$= \omega\mu\sqrt{\sigma^2 + \omega^2\varepsilon^2} \qquad (2)$$

Rough Practice Test

1. $$\gamma = a + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$$

$$= a^2 + j^2\beta^2 + 2j\alpha\beta$$

$$= a^2 - \beta^2 + 2ja\beta = j\omega\mu(\sigma + j\omega\varepsilon)$$

$$= \boxed{j\omega\mu\sigma - \omega^2\mu\varepsilon}$$

2. $$|\gamma|^2 = a^2 + \beta^2 = \left|\sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}\right|^2$$

$$\sqrt{|j\omega\mu||\sigma + j\omega\varepsilon|}$$

(Take absolute square value of everything)

$$= \sqrt{\sqrt{j^2 w^2 \mu^2}\,\sqrt{\sigma^2 + j^2 w^2 \varepsilon^2}}$$

$$= \sqrt{\sqrt{w^2 \mu^2}\,\sqrt{\sigma^2 + w^2 \varepsilon^2}}$$

$$= w\mu\sqrt{\sigma^2 + w^2\sigma^2}$$

2. γ in General

Step 3: Equate Real Parts

$$(1)\quad \alpha^2 - \beta^2 = -w^2\mu\varepsilon$$

only real parts!

$\downarrow$

β^2 cancel out

$$(2)\quad \alpha^2 + \beta^2 = w\mu\sqrt{\sigma^2 + w^2\varepsilon^2}$$

$$(1) + (2) = \quad 2\alpha^2 = w\mu\sqrt{\sigma^2 + w^2\varepsilon^2} - w^2\mu\varepsilon$$

$$\alpha = \sqrt{\frac{w\mu\sqrt{\sigma^2 + w^2\varepsilon^2} - w^2\mu\varepsilon}{2}}$$

multiply everything out

$\downarrow$

divide across by $w^2\varepsilon^2$

$$= w\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \sigma^2/\varepsilon^2 w^2} - 1\right)}$$

following same steps for β:

$$\beta = w\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1 + \sigma^2/\varepsilon^2 w^2} + 1\right)}$$

Rough:

$$\sqrt{\frac{\frac{w\mu}{w}\sqrt{\frac{\sigma^2 + w^2\varepsilon^2}{w^2}} - \frac{w^2\varepsilon}{w}}{2}} = \sqrt{\frac{\mu\sqrt{\sigma^2/w^2 + \varepsilon^2}}{2} - \frac{\varepsilon \cdot w}{2}}$$

$$\alpha = \sqrt{\frac{\omega \mu \sqrt{\sigma^2 + \omega^2 \varepsilon^2} - \omega^2 \mu \varepsilon}{2}}$$

$$= \sqrt{\frac{\mu}{2}\left(\omega \sqrt{\sigma^2 + \omega^2 \varepsilon^2} - \omega^2 \varepsilon\right)}$$

$$= \sqrt{\frac{\mu \omega^2}{2}\left(\frac{1}{\omega \varepsilon}\sqrt{\frac{\sigma^2}{\omega^2 \varepsilon^2} + 1} - 1\right)}$$

$$= \omega \sqrt{\frac{\mu \varepsilon}{2}\sqrt{\sigma^2/\omega^2\varepsilon^2 + 1} - 1}$$

$$\beta = \omega \sqrt{\frac{\mu \varepsilon}{2}\sqrt{\sigma^2/\omega^2\varepsilon^2 + 1} + 1}$$

3. γ in a Good Dielectric

(Insulator)

$\sigma \ll \omega \varepsilon$

(opposite !! to good conductor)

$$\alpha = \omega \sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\sigma^2/\varepsilon^2\omega^2} - 1\right)}$$

(attenuation const).

if σ small → use binomial expansion $\left(\sqrt{1+x}\right)$

of $\left(1+\frac{\sigma^2}{\varepsilon^2\omega^2}\right)^{1/2}$ Taylor Expansion

→ by the time this reaches $1+\frac{x}{2}+\frac{x^2}{8}\ldots$,

the value becomes negligibly small

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\cdot\frac{\sigma^2}{2\varepsilon^2\omega^2}} = \omega\underbrace{\sqrt{\frac{\mu\sigma^2}{4\varepsilon\omega^2}} = \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}}$$

$$\left(1+\frac{\sigma^2}{\varepsilon^2\omega^2}\right)^{1/2} \rightarrow \frac{\sigma^2}{2\varepsilon^2\omega^2}$$

$$= \frac{\omega\sigma}{\omega 2}\sqrt{\frac{\mu}{\varepsilon}}$$

$$= \frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$$

3. γ in a Good Dielectric

$$\beta = w\sqrt{\frac{\mu\varepsilon}{2}\left(\underbrace{\sqrt{1+\sigma^2/\varepsilon^2 w^2}}\,+1\right)}$$

take binomial of this

$$\left(1+\frac{\sigma^2}{\varepsilon^2 w^2}\right)^{1/2}$$

$\downarrow$ T.S.

$$\frac{\sigma^2}{2\cdot\varepsilon^2\cdot w^2}+2$$

↳ divide across by 2

$$= w\sqrt{\frac{\mu\varepsilon}{2}\left(2+\frac{\sigma^2}{2\varepsilon^2 w^2}\right)} \approx w\sqrt{\mu\varepsilon}\left(1+\frac{\sigma^2}{8\varepsilon^2 w^2}\right)$$

$$v\ (\text{velocity}) = \frac{w}{\beta} = \frac{1}{w\sqrt{\mu\varepsilon}\left(1+\frac{\sigma^2}{8\varepsilon^2 w^2}\right)}$$

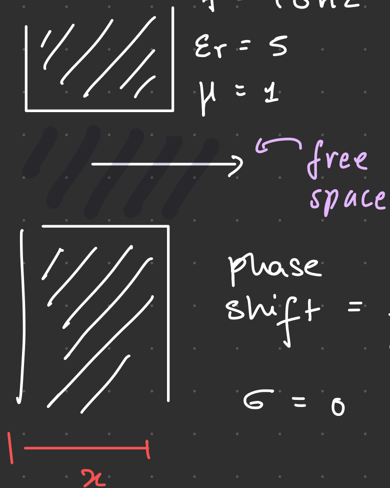
$$= \frac{1}{\mu\varepsilon}\left(1-\frac{\sigma^2}{8\varepsilon^2 w^2}\right)$$

→ velocity is high, phase shift is small

Summary

	velocity (v)	attenuation (α)	phaseshift (β)
Good conductor $\sigma \gg \omega\varepsilon$	$\frac{\omega}{\sqrt{\pi f \mu \sigma}}$	$\sqrt{\pi f \mu \sigma}$	$\sqrt{\pi f \mu \sigma}$
Good Dielectric $\sigma \ll \omega\varepsilon$	$\frac{1}{\sqrt{\mu\varepsilon}}\left(1 - \frac{\sigma^2}{8\omega^2\varepsilon^2}\right)$	$\frac{\sigma}{2}\sqrt{\frac{\mu}{\varepsilon}}$	$\omega\sqrt{\mu\varepsilon}\left(1 + \frac{\sigma^2}{8\omega^2\varepsilon^2}\right)$
Insulator $\sigma = 0$	$\frac{1}{\sqrt{\mu\varepsilon}}$	0	$\omega\sqrt{\mu\varepsilon}$

Example Q! Q1.



$f = 4\,GHz$

$\varepsilon_r = 5$

$\mu = 1$

phase shift $= \frac{\pi}{2}$

$\sigma = 0$

velocity of wave $= \omega / \beta$

velocity of wave through free space

velocity of wave through material (dielectric)

$$v_1 = \omega / \beta_1 \quad , \quad v_2 = \omega / \beta_2 \longrightarrow \beta_1 = \frac{\omega}{v_1} \quad , \quad \beta_2 = \frac{\omega}{v_2}$$

Phase difference between signals

$$\beta_2 x - \beta_1 x = \pi/2 = \frac{\omega x}{v_2} - \frac{\omega x}{v_1}$$

$$x = \frac{\pi/2}{\omega\left(\frac{1}{v_2} - \frac{1}{v_1}\right)}$$

$v_1 = v$ through free space $= c = \dfrac{1}{\sqrt{\varepsilon_0 \mu_0}}$

$v_2 = v$ through dielectric $= \dfrac{1}{\sqrt{\varepsilon_0 \varepsilon_r - \mu_0 \mu_r}}$

Q 1 cont.

$$x = \frac{\pi / 2}{w\left(1/v_2 - 1/v_1 \right)}$$

$$= \frac{\pi}{2w\left(\sqrt{\varepsilon_0 \varepsilon_r \mu_0 \mu_r} - \sqrt{\varepsilon_0 \mu_0} \right)}$$

$$= 0.152\,m \quad (\text{given } \varepsilon_0, \varepsilon_r, \mu_0, \mu_r,\ w = 2\pi f)$$

Q_2

$\sigma \gg \omega\varepsilon$ = conductor
$\sigma \ll \omega\varepsilon$ = dielectric

$\varepsilon_r = 35$

$\sigma = 0.2\,\Omega$

$v = \dfrac{\omega}{\beta}$, $f = 100\,MHz$

$\gamma = \alpha + j\beta$ → use to find if good conductor / insulator / dielectric

$$\alpha = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}-1\right)}$$ nepers / m → $\frac{dB}{m}$

$$\beta = \omega\sqrt{\frac{\mu\varepsilon}{2}\left(\sqrt{1+\frac{\sigma^2}{\varepsilon^2\omega^2}}+1\right)}$$ rads / m

using $\gamma = \alpha + j\beta = \sqrt{j\omega\mu(\sigma + j\omega\varepsilon)}$

→ direct definition

→ equate real + imaginary components

$\sqrt{-\omega^2\mu\varepsilon + j\omega\mu\sigma}$

To polar: $z = |r|(\cos\theta + j\sin\theta)$, $r = |z|$

Q2 cont...

$$\gamma = \sqrt{-\omega^2 \mu \varepsilon + j\omega\mu\sigma}$$

→ sub in values ⟶ $-\omega^2\mu\varepsilon = -153.8$, $j\omega\mu\sigma = 157.9$

→ get angle + abs value

$|\gamma| = 220.4$, $\angle\gamma = 134.2^\circ$

$$\gamma = \sqrt{220.4}\left(\cos(67.1) + j\sin(67.1)\right)$$

$$= \sqrt{r}\left(\cos\left(\frac{\theta}{2}\right) + j\sin\left(\frac{\theta}{2}\right)\right)$$

Tutorial Q

Question 2

A plane electromagnetic wave of frequency 1 *GHz* is normally incident on a slab of material with a width of .005 m, which has a relative permittivity of *15*, a relative permeability of 10 and a conductivity of 10 Ω^{-1} m^{-1}. There is a large hole in the slab and measurements made on the field that propagates through the slab and on the field that propagates through the hole. What is the phase difference and amplitude difference of the two parts of the wave (wave propagation through free space, wave propagating through the material) after the slab thickness. Do not consider boundary effects on the wave in the material.

$f = 1\,GHz$

$x = 0.005\,m$

$\varepsilon_r = 15$ (permittivity)

$\mu_r = 10$ (permeability)

$\sigma = 10$ (conductivity)

$A_{in} = 5$ Volts/m

$A - Ae^{-\alpha x}$

Know that its a good conductor because $\sigma \gg \omega\varepsilon$

$$\alpha = \frac{\sigma}{\omega\varepsilon} = \frac{10}{2\pi f(15)} =$$

$$\beta = \alpha = \sqrt{\pi f \mu \sigma} \longrightarrow \beta_1 x - \beta_2 x$$

$$\Rightarrow (5.600)(0.005) - 2\pi f\sqrt{\mu\varepsilon}$$

Amp diff $\rightarrow$ $\underline{A - Ae^{-\alpha x}}$

Tutorial

$E_m \cos\beta(x - vt)$

$E_m \cos \beta x - \beta v t$

$E_m \cdot \cos(\beta x - \omega t)$

↳ $\cos(1) = \cos(-1)$

$\omega = v\beta$

if $E_y = E_m \cos\beta(x - vt)$

↳ define $\vec{H}$ and $\vec{E}$ field

---

solution: write as Real part of a complex expression

Maxwell Eqn

$$\boxed{\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}}$$

$$\therefore \quad E_y = E_m \cos(\beta x - \omega t)$$

$$= E_m \cos(\omega t - \beta x)$$

$$= \text{Real}\left(E_m \cdot e^{j(\omega t - \beta x)}\right)$$

$$= \text{Real} \cdot E_m \cdot e^{j\omega t} \cdot e^{-\beta x j}$$

$-\vec{\nabla} \times \vec{E}$ of plane p. wave

$$= \frac{\partial E_y}{\partial x} \hat{k}$$

let $E_0 = E_m \cdot e^{-\beta x j} \quad \therefore \quad \vec{E}_0 \cdot e^{j\omega t}$

can decide this because $E_y \, (\cos\beta x - vt)$ only has $E_y(x)$

$\therefore$

This Plane Polarised Wave ⇒ only has $\boxed{\dfrac{\partial E_y}{\partial x}}$

$$\frac{\partial \vec{B}}{\partial t} = -\frac{\partial E_y(\cos(\omega t - \beta x))}{\partial x}$$ only diff by x.

$$= E_y(\sin \omega t - \beta x) \cdot \beta$$

$$\int \frac{\partial B}{\partial t} = \vec{B} = E_y \cdot \cos(\omega t - \beta x) \cdot \frac{\beta}{\omega}$$

Propagation: $\vec{J} = \sigma \vec{E}$

remember that

$\lambda = v/f$

Phase Diff $= \beta_1 x - \beta_2 x$

Amp Diff $= A - A_{in} e^{-\alpha x}$

To convert to dB →

$\times 8.686$

$20 \log_{10} \left(\frac{A_{out}}{A_{in}} \right) = $ dB gain

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$$

Maxwell's Equations

Plane Polarised wave means

$\frac{\partial E_y}{\partial x} \rightarrow$ only in $\vec{z}\ \hat{k}$

in free space,

$\vec{B} = \vec{H} \mu$

$\omega = \beta v.$

$v = \frac{1}{\sqrt{\varepsilon \mu}}$

Relationships

$\vec{B} = \frac{\beta}{\omega} E_y \cdot \hat{k}$

$\vec{B}_z = \frac{\beta}{\omega} \cdot E_y$

$\vec{B}_z = \vec{H}_z \mu_0 = \frac{\beta}{\omega} \vec{E_y}$

$\frac{H_z \mu_0 \cdot \omega}{\beta} = \vec{E_y}$

$H_z \mu_0 \cdot v = E_y$

$$\left\{ \frac{\vec{H_z} \mu_0}{\sqrt{\varepsilon \mu_0}} = \vec{E_y} \right\}$$

Relating

$H_z \longrightarrow E_y$

Propagation Revision

- $\vec{\nabla} \times \vec{E} = \mu \frac{\partial \vec{H}}{\partial t}$ curl of E = μ × partial diff of H by t

- $\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\varepsilon \cdot \partial \vec{E}}{\partial t}$

  $\vec{J} = \sigma \vec{E}$

- Poisson's Equations : $\underbrace{\vec{\nabla}(\nabla \cdot \vec{E})}_{=0} - \nabla^2 \cdot \vec{E}$

  $= \vec{\nabla} \times \vec{\nabla} \times \vec{E}$

- remember $\frac{d}{dt} = j\omega$

E.g. <u>derive propagation constant</u>

given: $E_y \Rightarrow E_m \cos \beta(x - vt)$

$= E_m \cos(\beta x - \beta v t) \longrightarrow \beta \cdot v = \omega$

$= E_m \cos(\beta x - \omega t)$

$= E_m \cos(\omega t - \beta x) \longrightarrow \cos(-1) = \cos(1)$

write as Real part of complex value in exponential form:

$= E_m e^{j(\omega t - \beta x)}$

$= E_m e^{j\omega t} e^{-\beta j x} \longrightarrow$ let $E_o = E_m e^{-\beta j x}$ for $t = 0$

$= E_o e^{j\omega t}$

if $E = E_o e^{j\omega t}$, $\dfrac{\partial \vec{E}}{\partial t} = \boxed{j\omega} \cdot E_o e^{j\omega t}$

and let $\boxed{\dfrac{d}{dt} = j\omega}$

Remember maxwell's law $\vec{\nabla} \times \vec{E} = \mu \dfrac{\partial \vec{H}}{\partial t}$ $\left(\vec{H} = H_o e^{-j\omega t}\right)$

- $\vec{\nabla} \times (E_o e^{j\omega t}) = j\omega\mu\left(H_o e^{-j\omega t}\right)$ at $t = 0$, $\vec{H_o e^{j\omega t}} = \vec{H_o}$
- $\vec{\nabla} \times \vec{E} = j\omega\mu \vec{H_o}$

Steps: start with $\vec{\nabla} \times \vec{E}$ in H at $t = 0$

Maxwell's Eqns Again!

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\varepsilon \partial \vec{E}}{\partial t} = \sigma \vec{E} + \varepsilon \cdot \frac{\partial \vec{E}}{\partial t} \quad \text{(H-field)}$$

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t} \quad \text{(E-field)}$$

Poisson:

$$\underbrace{\nabla(\nabla \cdot \vec{E})}_{\text{div of div}} - \underbrace{\nabla^2 \cdot \vec{E}}_{\nabla^2 \cdot \vec{E}} = \underbrace{\vec{\nabla} \times \vec{\nabla} \times \vec{E}}_{\text{curl of curl of } \vec{E}}$$

(but $\nabla \cdot \vec{E} = 0$)

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

using Poisson's Equation: $\vec{\nabla} \times \vec{\nabla} \times \vec{E}_0 = \vec{\nabla} \times \underbrace{\left(-\mu \frac{\partial \vec{H}}{\partial t}\right)}$

$$\int \frac{\partial \vec{H}}{\partial t} \Rightarrow \int H_0 e^{-j\omega t} = j\omega H_0 e^{-j\omega t}$$

at $t = 0 \rightarrow j\omega H_0$ $\therefore$ $\vec{\nabla} \times \vec{\nabla} \times \vec{E}_0 = \vec{\nabla} \times (\mu j\omega H_0)$

$\nabla \cdot \nabla \cdot \vec{E} = 0$, no divergence of $\vec{E}$ field $\therefore$

$$-\nabla^2 \cdot \vec{E} = \vec{\nabla} \times \vec{\nabla} \times \vec{E} = \vec{\nabla} \times (\mu j\omega H_0)$$

$$-\nabla^2 \cdot \vec{E} = \mu \cdot j\omega \, (\vec{\nabla} \times H_0)$$

Take $\vec{\nabla} \times H = \vec{J} + \dfrac{\varepsilon \partial \vec{E}}{\partial t}$

remember, if $E_0 \rightarrow E_m e^{j\omega t} e^{-j\beta x} \rightarrow E_0 e^{\cdot j\omega t} \rightarrow E_0 \cdot j\omega$

if $\quad -\nabla^2 \vec{E} = (\vec{\nabla} \times H_o)\, \mu j w$

and $\quad \vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\varepsilon \partial \vec{E}}{\partial t}$

$$= \sigma \vec{E} + \frac{\varepsilon \partial \vec{E}}{\partial t} = \sigma \vec{E} + j w \vec{E}$$

because $\quad jw = \dfrac{d}{dt}$

Remember: $-\nabla^2 E_o = \vec{\nabla} \times \vec{H}$ (because $\vec{\nabla} \cdot \vec{E} = 0$)

$$\vec{\nabla} \times \vec{H} = \vec{E}\left(\sigma + \varepsilon\left(\partial / \partial t\right)\right)$$

$$-\nabla^2 \cdot E_o = \vec{E}(\sigma + jw\varepsilon)(\mu j w)$$

$$= \mu j w (\sigma + j w \varepsilon) \cdot \vec{E}$$

$$-\nabla^2 E_{o_y} = E_{o_y}\left[\mu j w (\sigma + j w \varepsilon)\right]$$

$$\gamma^2 = \mu j w (\sigma + j w \varepsilon)$$

$$\gamma = \sqrt{\mu j w (\sigma + j w \varepsilon)}$$ (Prop constant)

Q1 Past Midterm to satisfy Eqns

$$\vec{E} = 2b(x^2+z)+\hat{i} + 2yt\hat{j} + ay^2\hat{k}$$ E-field

$$\vec{B} = -2ayt - bzt^2\hat{j} + by\hat{k}$$ Magnetic flux Density

Maxwell's Laws:

$$\vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}$$

$$\vec{\nabla}\cdot\vec{D} = \rho$$

$$\vec{\nabla}\times\vec{H} = \vec{J} + \frac{\partial\vec{D}}{\partial t}$$

$\vec{B} = \mu\vec{H}$

Steps 1: curl of $\vec{E}$

2. diff of $\vec{B}$ $\rightarrow$ easy!

$$\begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 2b(x^2+z) & \underset{2yt}{\downarrow} & ay^2t \end{vmatrix} = \hat{i}(2ya-0) - \hat{j}(0-2b) + \hat{k}(0)$$

$$= 2ay\hat{i} + 2b\hat{j} = \nabla\times\vec{E}$$

$$\frac{\partial\vec{B}}{\partial t} = -2ay - 2bz\hat{j} + 0\hat{k}$$

should equal each other.

To Determine charge and current densities

Remember

$\vec{\nabla} \cdot \vec{E} = \rho / \varepsilon$

$\vec{\nabla} \cdot \vec{D} = \rho$

$D = \varepsilon \vec{E}$

displacement electric field

$\vec{\nabla} \cdot \vec{D} = \rho$ (charge density)

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{B} = -2ayt\,\hat{\imath} - bzt^2\,\hat{\jmath} + by\,\hat{k}$$

$$\vec{\nabla} \cdot \vec{E} = 4bx\,\hat{\imath} + 2t\,\hat{\jmath} + 0\hat{k} = \rho / \varepsilon = \frac{\text{charge density}}{\text{permittivity}}$$

$$\boxed{\rho = \varepsilon (4bx\,\hat{\imath} + 2t\,\hat{\jmath})}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \partial \vec{E} / \partial t = \frac{\vec{\nabla} \times \vec{B}}{\mu}$$

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

Charge + Charge Density

$$\vec{\nabla} \times \vec{H} = J + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$B = \mu H \therefore H = B / \mu$$

$$\frac{\vec{\nabla} \times \vec{B}}{\mu} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t} \quad \therefore \quad \vec{J} = \frac{\vec{\nabla} \times \vec{B}}{\mu} - \varepsilon \frac{\partial \vec{E}}{\partial t}$$

$$\vec{\nabla} \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$$

(divergence of charge = − diff of density with t)

when these are equal, can ensure they are =

Maxwell's Eqns

Summary

- $\vec{\nabla} \times \vec{E} = \mu \dfrac{\partial \vec{H}}{\partial t} = \dfrac{\partial \vec{B}}{\partial t}$
- $\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \dfrac{\partial \vec{E}}{\partial t}$, $\vec{J} = \sigma \vec{E}$
- $\vec{\beta} = \mu \vec{H}$
- $\vec{E} \varepsilon = \vec{D} \longrightarrow \vec{\nabla} \cdot \vec{E} = \rho / \varepsilon$
- $\vec{\nabla} \cdot \vec{J} = -\dfrac{\partial \rho}{\partial t}$

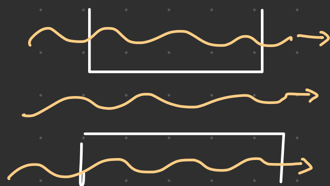
Q2 $\quad f = 4\,GHz$

$\varepsilon_r = 5$ permittivity

$\mu_r = 1$ permeability

$\sigma = 0$

To find : x



phase diff = $90°$

Remember : $\omega = v\beta$, frequency = velocity × phase shift

Phase shift ⇒ $\beta_1 x - \beta_2 x = \frac{\pi}{2}$

$$\frac{\omega x}{v_1} - \frac{\omega x}{v_2} = \frac{\pi}{2}$$

$$x\underbrace{\left(\omega/v_1 - \omega/v_2\right)}$$

v_1, v_2 through $\frac{1}{\sqrt{\varepsilon_0\mu_0}}$

$$\frac{1}{\sqrt{\varepsilon_0\varepsilon_r\mu_0\mu_r}}$$

Poynting vector

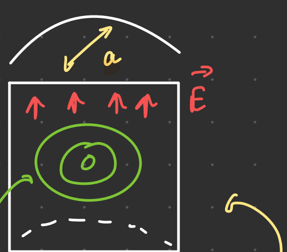
- → vector that carried power of a wave
- → direction of power transfer
- → integrate space between coaxial cables (between conductors)

Cylindrical current carrying conductor

current → proportional to $\vec{E}$ field

$$\vec{j} = \sigma \vec{E}$$

$$H_r = \frac{rI}{2\pi a^2}$$



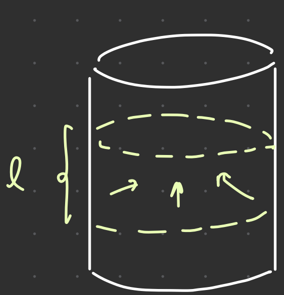
$$E_r = \frac{J}{\sigma} = \frac{I}{\pi a^2 \sigma}$$

circulating magnetic field inside the wire (anti clockwise)

$\vec{P} = \vec{E} \times \vec{H}$ ⇒ points towards centre of conductor, gets weaker towards centre as $\vec{H}$ gets weaker

Poynting vector

P at surface of conductor. $P = E_r H_r$ where $r = a$



$$= \frac{a I^2}{2\pi^2 a^4 \sigma}$$

inside of wire radially inward from surface of wire

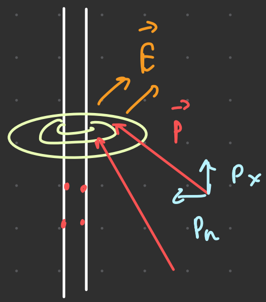
$$P_{total} = \frac{a I^2}{2\pi^2 a^4 \sigma} \cdot 2\pi a l$$

$$= \frac{I^2 l}{\pi a^2 \sigma} = \frac{l}{\pi a^2 \sigma} \cdot I^2$$

because resistance: $R = \frac{1}{\pi a^2 \sigma} = \frac{\text{length}}{\text{area}} \cdot \frac{1}{\sigma}$

$$P_{total} = R I^2$$

wire carrying current:



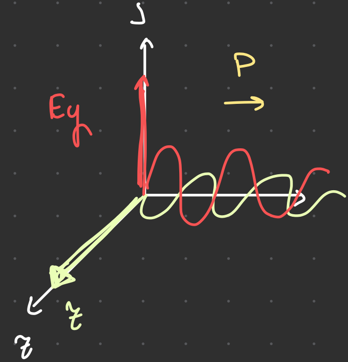
Parallel ⇒ transmitted

Normal → lost due to resistance

e.g. Plane Polarised wave

$$E_y = E_m \cos(\omega t - \beta x)$$

$$\vec{P} = \vec{E} \times \vec{H} \qquad H_z = \eta E_y$$



$$P_x = \eta E_m^2 \cos^2(\omega t - \beta x)$$

→ power transferred in same direction as wave

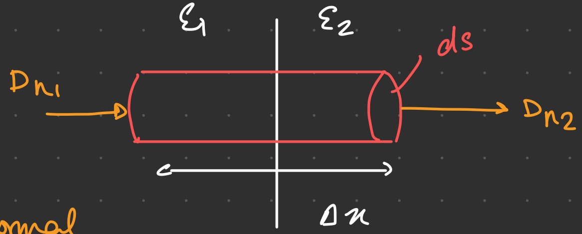
Boundary Conditions

"Skin Effects"

for field distribution to exist:

1. Be a solution to maxwell's Eqns
2. Boundary conditions

Electric Fields



D_n = flux fields normal to boundary

Gauss Law: $\oint \vec{E} ds = \frac{q_{enc}}{\varepsilon}$, $\vec{E}\varepsilon = 0$ $\Longrightarrow$ $(D_{n2} - D_{n1})\, ds = \rho_s\, ds$

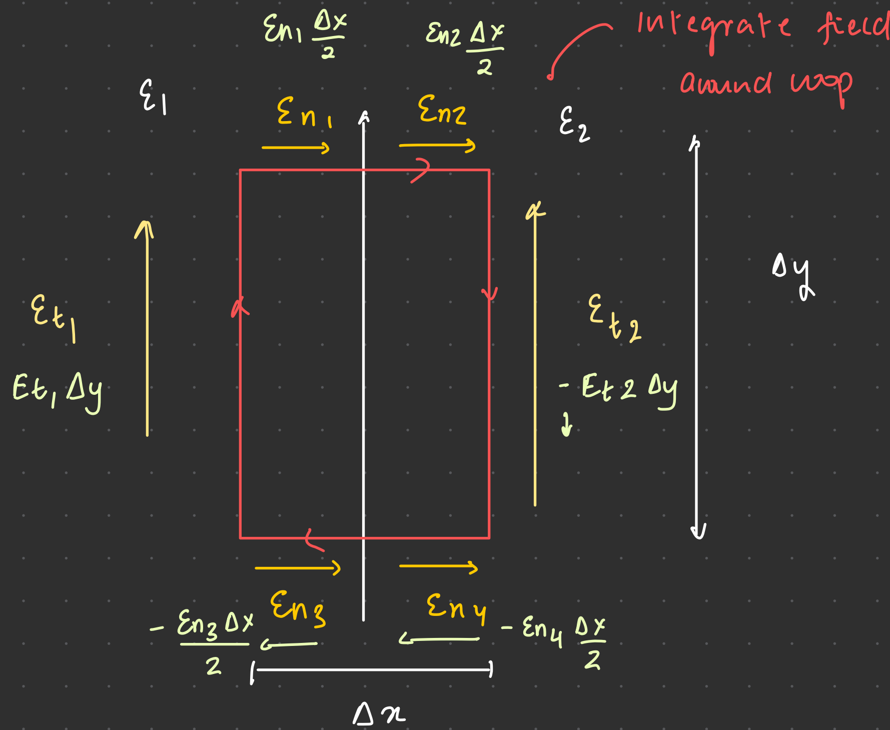
* interaction of flux and field

$\int \vec{D} ds = q_{enc}$

* no surface charge so $\rho_s = 0$, flux normal = $D_{n2} = D_{n1}$
  ↳ means normal component is continuous

Boundary Conditions

$\varepsilon_1 = \varepsilon_2$



Combining using Faraday's law: $\oint_C \vec{E} \cdot dl$

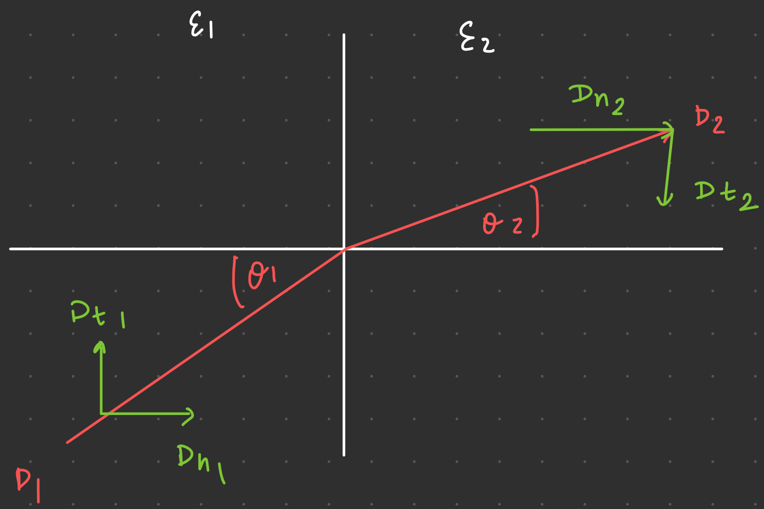
$$= -\frac{d}{dt} \int_S \underbrace{\vec{B} \cdot \hat{n}\, da}_{\varphi \text{ flux}} = -\frac{d\varphi}{dt} \text{ through surface}$$

Take limit from $\Delta x \to 0$: $\oint_C \vec{E} \cdot dl = E_{t_1} \Delta y - E_{t_2} \Delta y$

$$\therefore \boxed{E_{t_2} = E_{t_1}}$$

flux density normal = continuous, tangential $\vec{E}$ field same.

Electric Flux Refraction



$$\left.\begin{aligned} D_{n_1} &= D_1 \cos\theta_1 \\ D_{n_2} &= D_2 \cos\theta_2 \end{aligned}\right\}$$

$$E_{t_1} = \frac{D_1 \sin\theta_1}{\varepsilon_1}$$

$$E_{t_2} = \frac{D_2 \sin\theta_2}{\varepsilon_2}$$

$$\vec{E} = \frac{\vec{D}}{\varepsilon}$$

Refraction

sine $\quad \tan\theta = \dfrac{\sin\theta}{\cos\theta}$ $\left.\begin{matrix} \\ \\ \end{matrix}\right\}$ $\begin{matrix} E_{t1} = E_{t2} \\ D_{n1} = D_{n2} \end{matrix}$

$$\frac{E_{t1}}{D_{n1}} = \frac{\tan\theta_1}{\varepsilon_1} \quad \text{and} \quad \frac{E_{t2}}{D_{n2}} = \frac{\tan\theta_2}{\varepsilon_2}$$

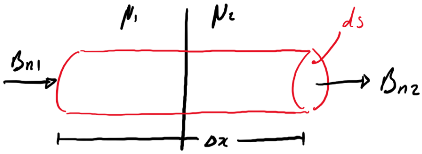
$$\frac{\tan\theta_1}{\varepsilon_1} = \frac{\tan\theta_2}{\varepsilon_2}$$

$$\frac{\tan\theta_1}{\tan\theta_2} = \frac{\varepsilon_1}{\varepsilon_2}$$

Magnetic Field at Boundary

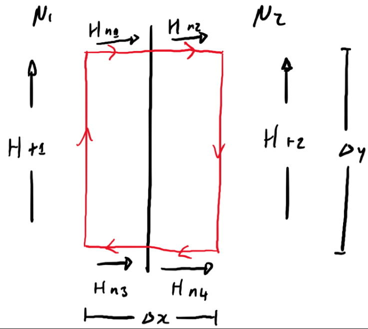
* same as electric

- Consider the magnetic field at a boundary between two dielectric materials with respective permeability μ_1, μ_2:



- From Gauss; Law for magnetic fields:
  $(B_{n2} - B_{n1})ds = 0$, since $\vec{\nabla}.\vec{B} = 0$
- Hence:
  $\boldsymbol{B_{n2} = B_{n1}}$
- The relationship dictates that the **normal component** of **Magnetic flux** at a dielectric boundary must be continuous.

- To find the relationship between the **tangential component** of the field at the boundary with respective permeability μ_1, μ_2:



Consider a loop around the boundary.
integrate $\vec{H}$ over loop.

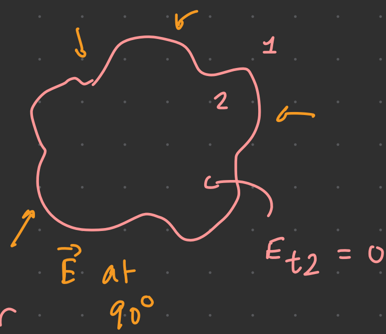
$$\oint_c \vec{H} \cdot \vec{dl} = I_t = 0 \quad \text{when } \Delta x = 0$$

$$\oint_c \vec{H} \cdot \vec{dl} = -H_{t_2}\Delta y + H_{t_1}\Delta y = 0 \quad \therefore \quad H_{t2} = H_{t1}$$

* tangential components of magnetic field are continuous

Perfect Conductor

→ no field strength inside a conductor because charges cancel



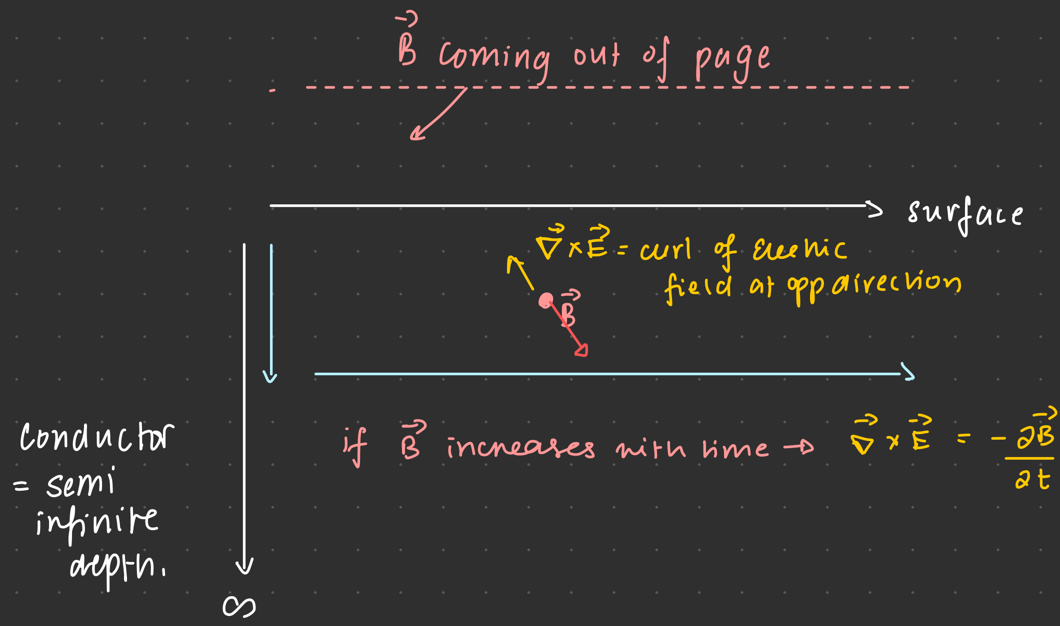
if $E_{t_2} = 0$, $E_{t_1} = 0$, no tangential component to $\vec{E}$ field at surface of conductor

Depth of penetration

δ = skin depth.

$\alpha = \sqrt{2\pi f \sigma}$

Imperfect Conductors

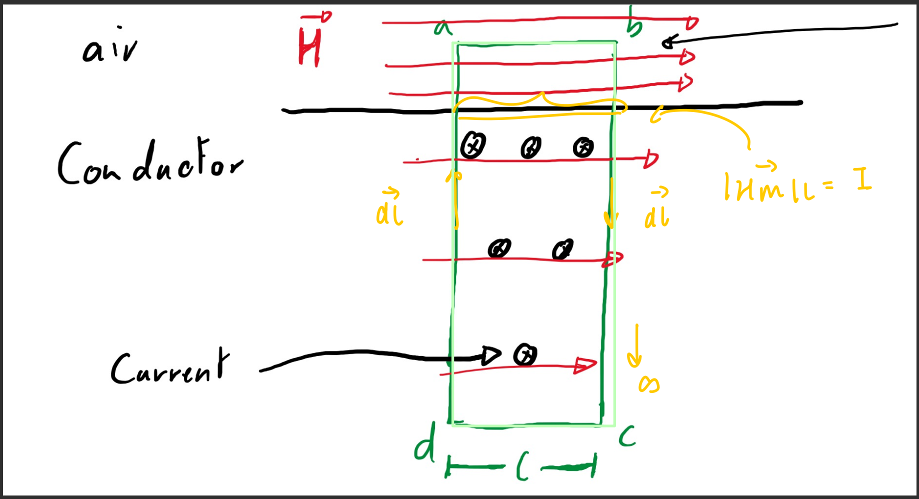


→ if there is an AC current, current will travel at "skin", very little inside

→ $\vec{B}$ decreases as moves down conductor

→ $\vec{J} = \sigma \vec{E}$, $\vec{J}$ concentrated near surface

Skin Effect



Total current enclosed:

(right at surface from abcd)

$$\oint_C \vec{H} \cdot dl = I_t$$

ab + bc + cd + da

over L = ab

$$|\vec{H}m| \ell = ab = I$$

Solution of wave in conducting medium :

- Solution of a wave in a conducting medium:

$$E_y = E_{0y}e^{j\omega t}$$
$$E_y = \underbrace{E_m}_{\text{surface}} e^{-\gamma x} e^{j\omega t}$$

- Also:

$$\vec{J} = \sigma \vec{E}$$

- So

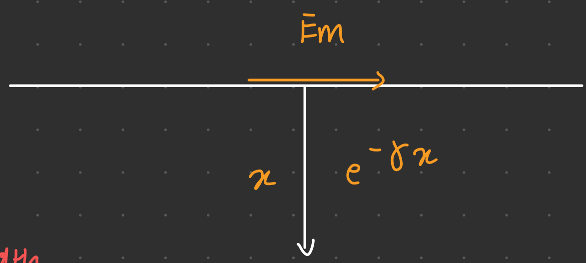
$$J_{0y} = J_m e^{-\gamma x}$$

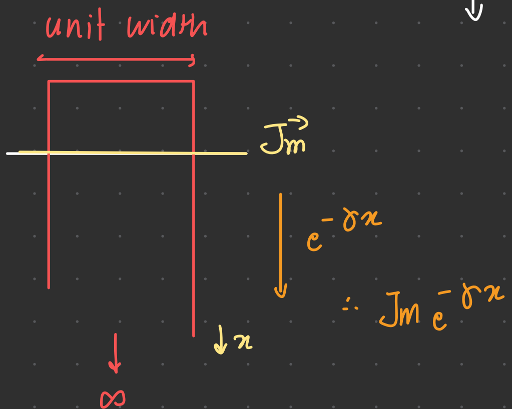
$\delta \to$ frequency dependant

- For a conductor $\alpha = \beta = \sqrt{\pi f \mu \sigma}$:

$$\gamma = \alpha + j\beta$$
$$= \frac{1+j}{\delta}$$

, δ = depth of penetration = $\frac{1}{\alpha}$ = $\frac{1}{\beta}$



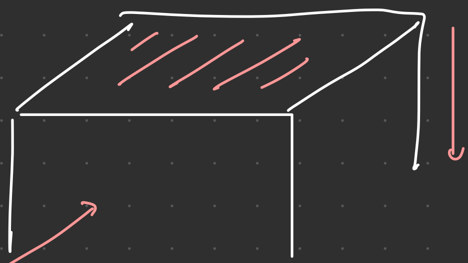


$$J_0 = \int_0^\infty J_{0y}\, dx$$
$$= \int_0^\infty J_m e^{-\gamma x}\, dx$$
$$= \left. \frac{J_m e^{-\gamma x}}{-\gamma} \right|_0^\infty \Rightarrow I_0 = \frac{J_m}{\gamma} = H_m l$$

$$\boxed{J_m = \gamma H_m}$$

Skin Effect

Higher J_m at top



$Je^{-\gamma x}$

$J_{oy} = Je^{-\gamma x}$

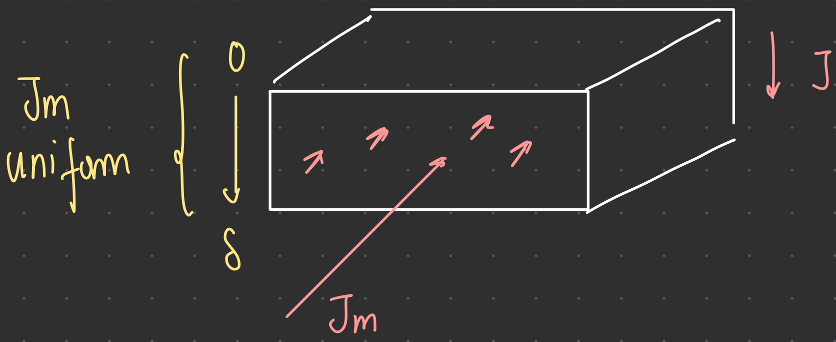
Total power dissipated per Area:

$$\frac{J_m^2 \delta}{4\sigma}$$

skin depth

current density

Model for Dissipation

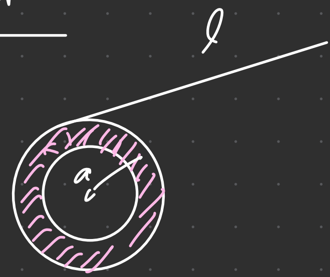


$I_0 = H_m$ (uniform for depth δ)

$$\text{Power} = \frac{J_m^2\,\delta}{4\sigma}$$

if dimensions > δ, current flow uniformly to depth δ

Cylindrical Conductor



DC resistance of length l, $R_{DC} = \frac{1}{\sigma} \cdot \frac{1}{\pi a^2}$

AC resistance: $R_{AC} = \frac{1}{\sigma} \cdot \frac{1}{\pi a^2 - \pi (a-\delta)^2}$

$$= \frac{1}{\sigma} \cdot \frac{1}{\pi (2a\delta - \delta^2)} = \frac{1}{\sigma \cdot 2\pi a \delta}$$

if $\delta \ll a$ then $\frac{R_{AC}}{R_{DC}} \approx \frac{a}{2\delta}$

$\delta = \frac{1}{\alpha} = \frac{1}{\sqrt{\sigma f \pi \mu}}$ $\uparrow f \ \downarrow \delta \ \uparrow R \ \uparrow P$

Reflections

$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

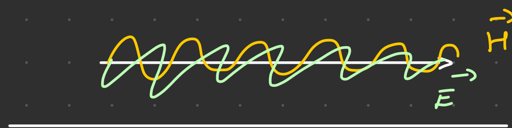
μ = permeability

ε = permittivity

Polarisation of waves

horizontal → E field varies on horizontal plane wrt surface

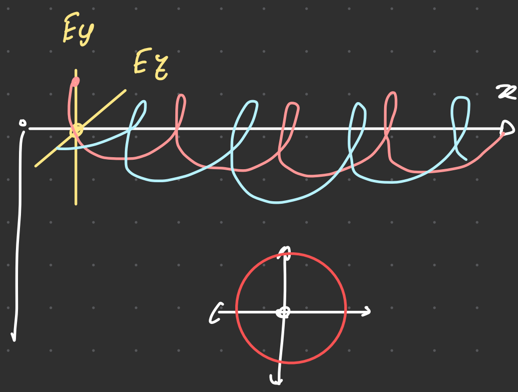
vertical:



Elliptical Polarisation:

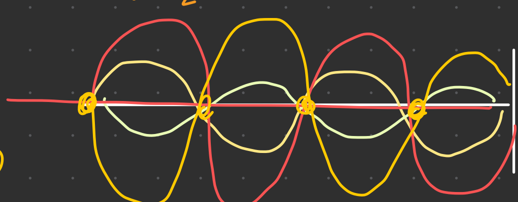
$$E_y = E_1 \sin(\omega t - \beta x)$$

$$E_z = E_2 \cos(\omega t - \beta x)$$



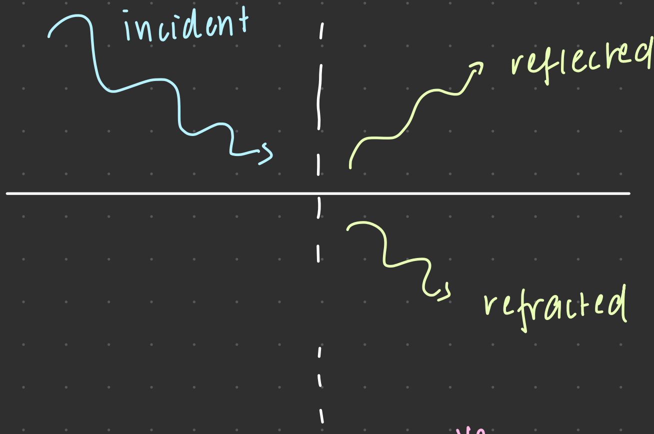
⚠ remember travelling:

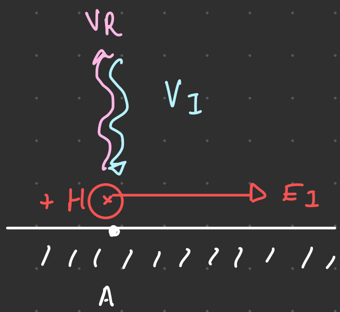
Standing wave:



reflects from wall
→ stays in phase

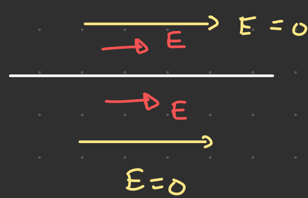
nodes always at 0





Perfect conductor:

$E_I = E_m(\cos(\omega t - \beta x))$



wave incident on perfect conductor.

$\vec{E}$ field must be continuous

$$E_y = E_I + E_R = 0 \quad \text{(in a perfect conductor)}$$

at $x = 0$

$$\left.\begin{aligned} E_y &= h_1 f(x - vt) + h_2 f(x + vt) \\ &= E_m \cos(\omega t - \beta x) + h_2 f(x + vt) \\ &= E_m \cos \beta(vt - x) + h_2 f(x + vt) \end{aligned}\right\} \quad \begin{aligned} h_2 f(vt) &= -E_m \cos \beta(vt) \\ h_2 f(vt + x) &= -E_m \cos \beta(vt + x) \end{aligned}$$

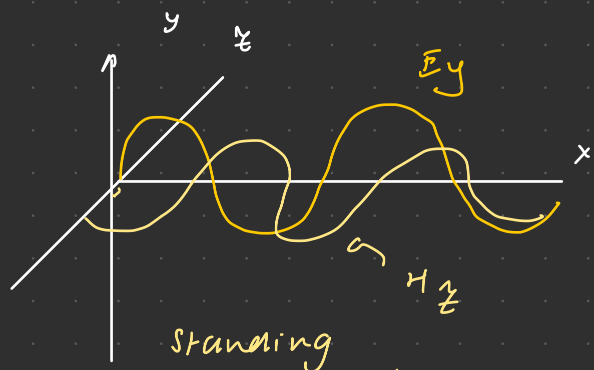
Reflection

$$E_y = \underbrace{E_m \cos(\omega t - \beta x)}_{\text{incident}} - \underbrace{E_m \cos(\omega t + \beta x)}_{\text{reflection}}$$

$$E_y = 2E_m \sin(\omega t) \sin \beta x$$

Magnetic field equivalant:

$$H_z = \frac{E_y}{\eta}$$

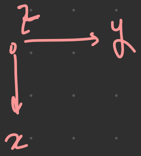


standing waves out of phase

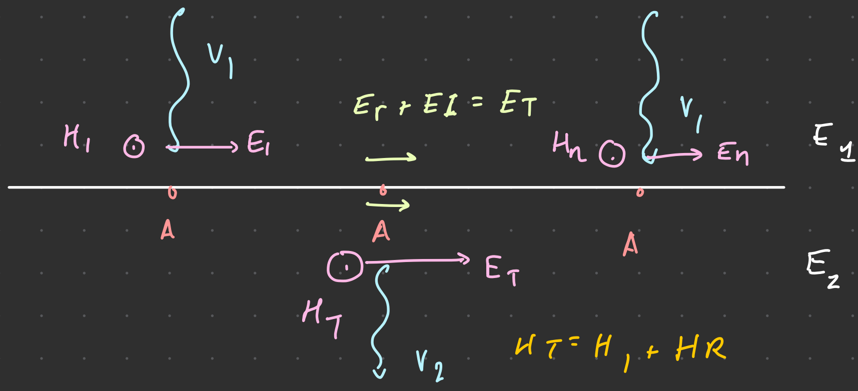
$$H_z = H_I + H_R = \frac{E_I}{\eta} - \frac{E_R}{\eta}$$

$$= \frac{E_m}{\eta}\left[\cos(\omega t - \beta x) + \cos(\omega t + \beta x)\right]$$

both are standing waves



Dielectric



To find:

$$\frac{E_t}{E_i} = \tau = \text{transmission coefficient}$$

$$\frac{E_r}{E_i} = \rho = \text{Reflection coefficient}$$

$$\frac{E_1}{H_1} = \eta, \quad \frac{E_T}{H_T} = \eta_2 \quad \underbrace{\frac{E_R}{-H_R}}_{\text{right hand rule.}} = \eta \quad \text{where } \eta = \sqrt{\frac{\mu}{\varepsilon}}$$

for $\mu_r = 1$ feris metanioul

= relative permittivity

Rewrite eqns

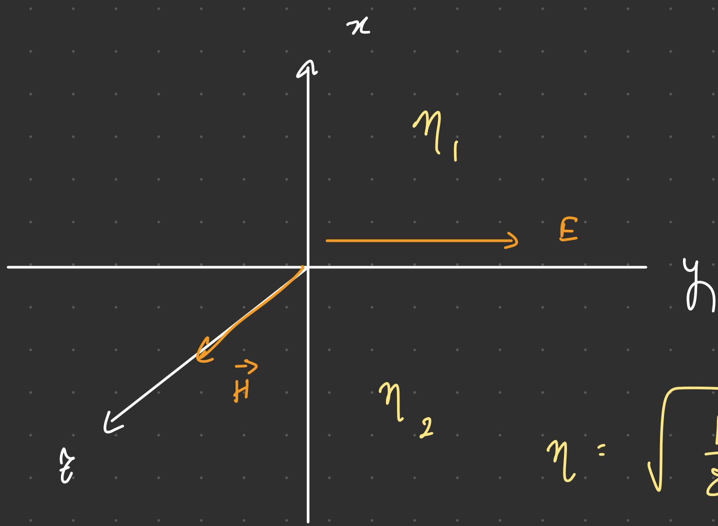
$$\frac{E_T}{\eta_2} = \frac{E_1}{\eta_1} - \frac{E_R}{\eta_1}$$

$$\Rightarrow \frac{\eta_1}{\eta_2} E_T = E_1 - E_r$$

$$\left(\begin{array}{l} E_t = E_1 + E_R \\ E_R = E_T - E_1 \end{array} \right)$$

$$= 2E_1 = E_T \left(1 + \frac{n_1}{n_2} \right)$$

$$= \frac{E_T}{E_1} = \frac{2\eta_2}{\eta_1 + \eta_2} = \tau = \text{transmission coefficient}$$



$$\eta = \sqrt{\frac{\mu}{\varepsilon}}$$

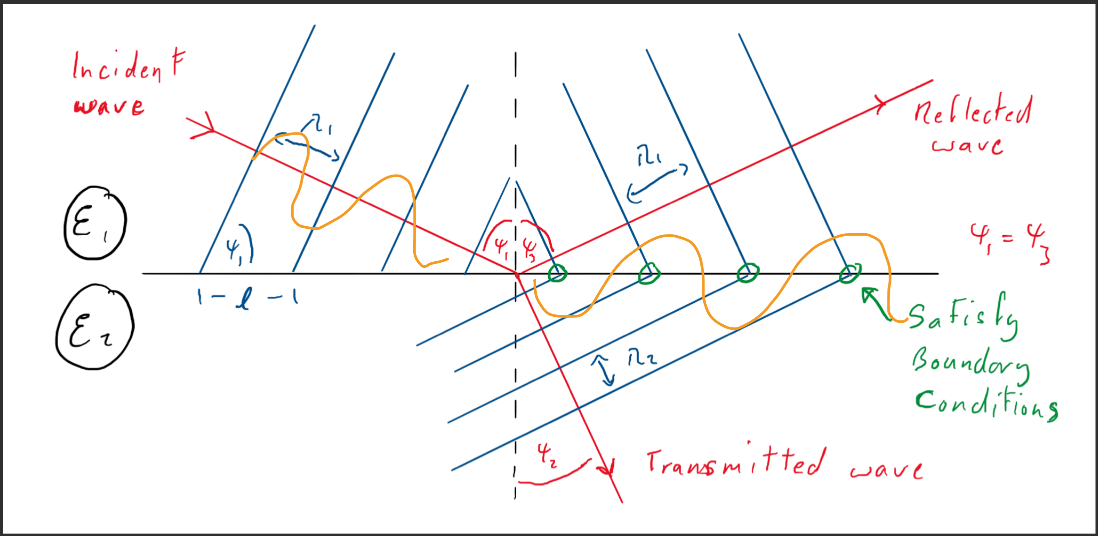
Reflection

$$E_R = E_T - E_1$$

$$= E_1 \left(\frac{2\eta_2}{\eta_1 + \eta_2} - 1 \right)$$

$$= \frac{E_R}{E_1} = \frac{\eta_2 - \eta_1}{\eta_2 + \eta_1} = p :$$ Reflection coefficient

Oblique Incidence at Boundary



$f_t = f_B$ To satisfy boundary condition

$$\sin\varphi_1 = \frac{\lambda_1}{l} \qquad \sin\varphi_2 = \frac{\lambda_2}{l} \qquad \therefore \quad \frac{\lambda_1}{\sin\varphi_1} = \frac{\lambda_2}{\sin\varphi_2}$$

$$\frac{v_1}{v_2} = \frac{\sin\varphi_1}{\sin\varphi_2}, \quad v = \sqrt{\frac{1}{\varepsilon\mu_0}}, \quad \mu_r = 1$$

$$\left.\sqrt{\frac{1}{\varepsilon_1\mu_0} \cdot \frac{\varepsilon_2\mu_0}{1}} = \sqrt{\frac{\varepsilon_2}{\varepsilon_1}}\right\} \text{Snell's Law}$$

Oblique incidence

Snell's law

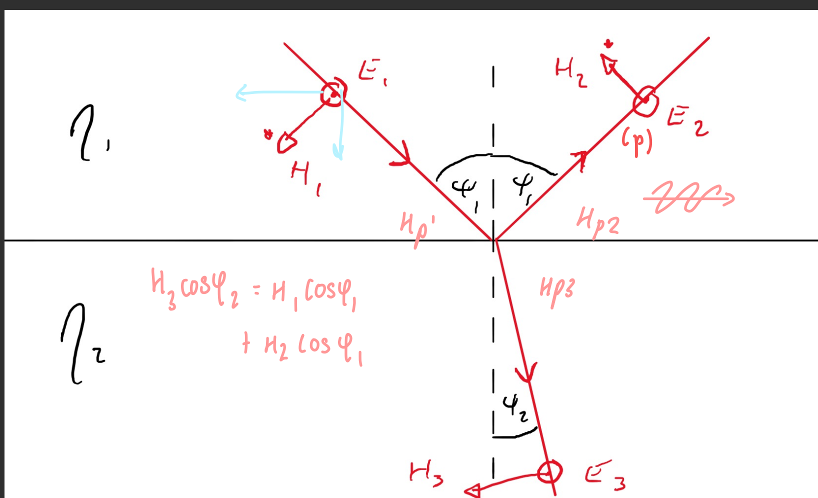
$$\vec{E} = E_m \cos(\omega t - \beta x) - E_m \cos(\omega t - \beta x)$$

same frequency $\left\{ \begin{array}{l} f_1 \\ f_2 \end{array} \right.$

$$f = \frac{v}{\eta}$$

$$v = \frac{1}{\sqrt{\mu \varepsilon}}$$

Horizontally polarised wave



Fresnel's Eqns

τ - Transmission coefficient

$$\frac{\varepsilon_1}{\eta_1}\cos\varphi_1 - \frac{\varepsilon_2}{\eta_2}\cos\varphi_2 = \frac{\varepsilon_3}{\eta_2}\cos\varphi_2$$

$$E_1 - E_2 = E_3 \frac{\eta_1}{\eta_2}\frac{\cos\varphi_2}{\cos\varphi_1}$$

relate ε_1 and ε_3

$$2E_1 = E_3\left(1 + \frac{\eta_1}{\eta_2}\frac{\cos\varphi_2}{\cos\varphi_1}\right)$$

$$\tau = \frac{\varepsilon_3}{\varepsilon_1} = \frac{2\eta_2\cos\varphi_1}{\eta_2\cos\varphi_1 + \eta_1\cos\varphi_2}$$

Reflection coefficient $= \dfrac{\mathcal{E}_2}{\mathcal{E}_1}$

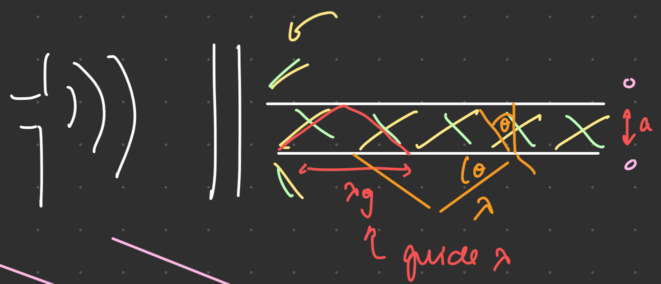
$$2E_2 = E_3\left(1 - \frac{\eta_1 \cos\varphi_2}{\eta_2 \cos\varphi_1}\right)$$

$$\boxed{\frac{\mathcal{E}_2}{\mathcal{E}_1} = \frac{\eta_2 \cos\varphi_1 - \eta \cos\varphi_2}{\eta_2 \cos\varphi_1 + \eta \cos\varphi_2} = p}$$

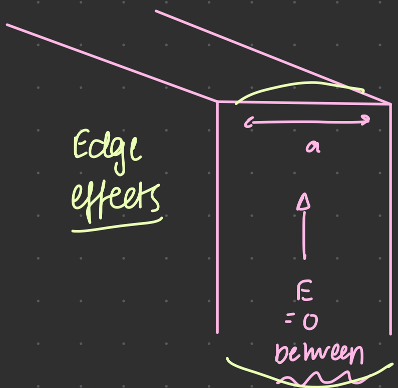
Wave Guides

Assumptions

same f, same λ, diff direction



net $\vec{E}$ field = 0 where peak + trough meet.



Parallel Plates

λ = wavelength

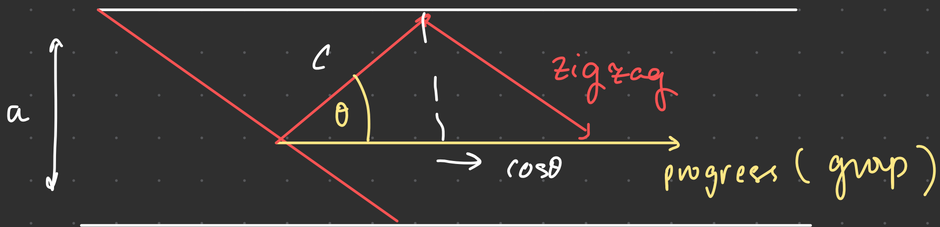
θ = oblique incidence

cutoff = lower it'll be allowed to propagate

group velocity = speed of energy in waveguide

phase velocity = speed at phase change

Progression in Guide



cut off = max λ allowed :

$$\cos\theta = \frac{\lambda}{\lambda_g} \Big\} \text{ get } \lambda_{max}$$

$$\tan\theta = \frac{\lambda_g/2}{a} = \frac{\lambda_g}{2a}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \boxed{\sin\theta \cdot \frac{\lambda_g}{\lambda} = \frac{\lambda_g}{2a}}$$

$\therefore \lambda = 2a\sin\theta$

λ_{max} when $\theta = 0$, $\sin\theta = 1$,

$$\boxed{\lambda_{max} = 2a}$$

$\lambda_c : 2a$

$$f_c = \frac{c}{\lambda_c} \Big\} \; c : \text{air}$$

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin\theta = \frac{\lambda}{2a} \qquad \cos\theta = \frac{\lambda}{\lambda g}$$

$$\left(\frac{\lambda}{2a}\right)^2 + \left(\frac{\lambda}{\lambda g}\right)^2 = 1$$

$$\boxed{\frac{1}{4a^2} + \frac{1}{\lambda g^2} = \frac{1}{\lambda^2}}$$

$2a = \lambda c$

$\therefore$

$$\frac{1}{\lambda c^2} + \frac{1}{\lambda g^2} = \frac{1}{\lambda^2}$$

Group velocity $\quad v_g = c \cdot \cos\theta$

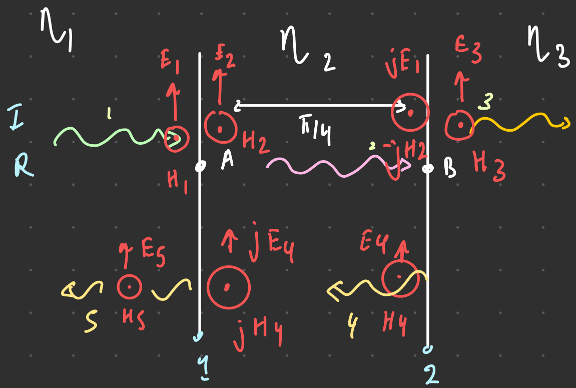
Phase velocity $\quad v_{ph} = f \cdot \lambda_g = \frac{c}{\lambda} \cdot \lambda_g$

$$= \frac{c}{\cos\theta}$$

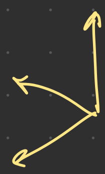
$v_{ph} > c$, $= \dfrac{c}{\sqrt{1-\sin^2\theta}}$

$$= \frac{c}{\sqrt{1-(\lambda/\lambda_c)^2}}$$

Boundary Questions



single frequency situation



1. what value for η_2 (in respect to η_1, η_3) for no reflection of wave

→ all energy propagating through η_3

any freq. wave λ

→ when $R=0$, what is I, η_1, η_2, η_3?

Incident wave has orthogonal E field, H field

Any wave $-90°$ φ lag → $e^{j\omega t} = e^{j(\omega t + \varphi)} = -je^{j\omega t}$

$e^{j\omega}e^{j\varphi}$

$$\boxed{R=0, \text{ when } E_5 = 0}$$

$$1 \begin{cases} E_1 + E_5 = E_2 - jE_4 & 1 \\ H_1 + H_5 = H_2 - jH_4 & 2 \end{cases}$$

Maxwell's Eqns

$E_1 = \eta_1 H_1$

$E_2 = \eta_2 H_2$

$E_2 = \eta_2 H_3$

$$2 \begin{cases} E_4 - jE_2 = E_3 & 3 \\ H_4 - jH_2 = H_3 & 4 \end{cases}$$

Right hand rule $\longrightarrow$ $E_4 = \eta_2(-H_4)$

$E_5 = -\eta_1 H_5$

Eqn. 2 $$\frac{E_1}{\eta_1} - \frac{E_5}{\eta_1} = \frac{E_2}{\eta_2} - j\left(-\frac{E_4}{\eta_2}\right)$$

$$\frac{E_1}{\eta_1} - \frac{E_5}{\eta_1} = \frac{E_2}{\eta_2} + j\frac{E_4}{\eta_2}$$

$$E_1 - E_5 = \frac{\eta_1}{\eta_2}(E_2 + jE_4) \quad 5.$$

$$-\frac{E_4}{\eta_2} - j\frac{E_3}{\eta_2} = \frac{E_3}{\eta_3} \quad \therefore \quad E_3 = \frac{\eta_3}{\eta_2}(-E_4 - jE_2) \quad 6.$$

Take Eqn. 2 and 5

$$E_2 + jE_4 = jE_3$$

$$E_1 - E_5 = \frac{\eta_1}{\eta_2}(E_2 + jE_4)$$

$$E_1 - E_5 = \frac{\eta_1}{\eta_2} \cdot jE_3 \qquad 7.$$

Take Eqn 1, 6

$$E_1 + E_5 = E_2 - jE_4$$

$$\left.-j\frac{E_2}{\eta_2} - \frac{E_4}{\eta_2} = \frac{E_3}{\eta_3}\right\} \text{multiply with } j = \frac{E_2}{\eta_2} - \frac{jE_4}{\eta_2} = \frac{jE_3}{\eta_3}$$

$$E_2 - jE_4 = j\frac{\eta_2}{\eta_3} \cdot E_3$$

$$E_1 + E_5 = j\frac{\eta_2}{\eta_3} \cdot E_3 \qquad 8.$$

Add Eqn 7 + 8

$$2E_1 = \frac{\eta_1}{\eta_2} jE_3 + \frac{\eta_2}{\eta_3} jE3$$

$$= jE3\left(\frac{\eta 1}{\eta 2} + \frac{\eta_2}{\eta_3}\right)$$

$$\therefore \; jE3 = \frac{2E1}{\left(\frac{\eta_1}{\eta 2} + \frac{\eta 2}{\eta 3}\right)} = \frac{2\eta_2 \eta 3}{\eta_1 \eta_3 + \eta_2^2} \quad 9.$$

Subtract Eqn 8 - 7

$$-2E5 = jE3\left(\frac{\eta_1}{\eta_2} - \frac{\eta 2}{\eta 3}\right)$$

$$jE3 = \frac{2\eta_2 \eta 3}{\eta_2^2 - \eta_1 \eta_3} \cdot E_5 \quad 10.$$

Equate Eqn 9 and 10

$$\frac{2 n_1 n_3}{n_2^2 - n_1 n_3} E_5 = \frac{2 n_2 n_3}{n_1 n_3 + n_2^2} E_1$$

$$E_5 = \frac{n_2^2 - n_1 n_3}{\underbrace{n_2^2 + n_1 n_3}_{\text{this can't be 0}}} E_1$$

setting $E_5 = 0$ $\therefore$

$$n_2^2 = n_1 n_3$$

$$n_2 = \sqrt{n_1 n_3}$$

Exam 22/23

1.3 Intrinsic Impedance

Maxwell Eqns:

$$\vec{\nabla} \times \vec{E} = -\mu \frac{\partial \vec{H}}{\partial t}$$

$$\vec{\nabla} \times \vec{H} = \vec{J} + \varepsilon \frac{\partial \vec{E}}{\partial t}$$

Take the curl:

$$\vec{\nabla} \times \vec{\nabla} \times \vec{H} = \frac{\varepsilon \cdot \partial (\vec{\nabla} \times \vec{E})}{\partial t}$$

Poisson's Equation:

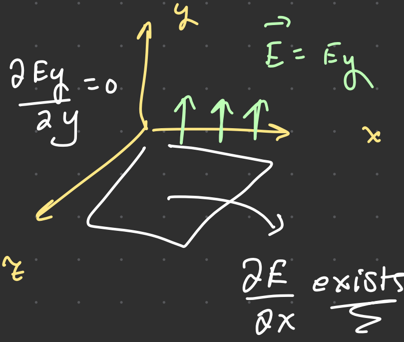
$$\vec{\nabla}(\nabla \cdot \vec{H}) - \vec{\nabla}^2 \vec{H} = \frac{\varepsilon \partial (\vec{\nabla} \times \vec{E})}{\partial t}$$

Remember $\vec{\nabla} \cdot \vec{H} = 0$

$$-\vec{\nabla}^2 \vec{H} = \frac{\varepsilon \partial \left(-\mu \frac{\partial \vec{H}}{\partial t}\right)}{\partial t} = -\mu \varepsilon \frac{\partial^2 \vec{H}}{\partial t^2}$$

☆ $$-\vec{\nabla}^2 \vec{E} = -\varepsilon \mu \frac{\partial^2 \vec{E}}{\partial t^2}$$

Polarise wave: $E_x = E_z = 0$



$$\frac{\partial E_y}{\partial y} = \frac{\partial E_y}{\partial z} = 0$$

Expand $\underbrace{\vec{\nabla}^2 \vec{E}} = \frac{\partial^2 \vec{E}}{c^2 \partial t^2}$ where $c = \frac{1}{\sqrt{\mu\varepsilon}}$

$$\frac{\partial^2 E_x}{\partial x^2} + \frac{\partial^2 E_x}{\partial y^2} + \frac{\partial^2 E_x}{\partial z^2} = \mu\varepsilon \frac{\partial^2 E_x}{\partial t^2}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

$$\frac{\partial^2 E_y}{\partial x^2} + \frac{\partial^2 E_y}{\partial^2 y^2} + \frac{\partial^2 E_y}{\partial^2 z^2} = \mu\varepsilon \frac{\partial^2 E_y}{\partial t^2}$$

$$\therefore \frac{\partial^2 E_y}{\partial x^2} = \mu\varepsilon \frac{\partial^2 E_y}{\partial t^2}$$

$E_y = E_m \cos \beta(x - vt) \rightarrow \beta v = \omega$

$= E_m \cos(\omega t - \beta x)$

Step: To find $\vec{B}$ for system.

$$\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E} \qquad \begin{vmatrix} i & j & k \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ A_x & A_y & A_z \end{vmatrix}$$

$$\frac{\partial B_x}{\partial t} = \underbrace{\frac{\partial E_z}{\partial y}}^{=0} - \frac{\partial E_y}{\partial z}$$

$$\frac{\partial B_y}{\partial t} = -\left.\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\right\} = 0$$

$$\frac{\partial B_z}{\partial t} = \cancel{\frac{\partial E_x}{\partial y}}\, 0 - \frac{\partial E_y}{\partial x}$$

$$\frac{\partial B_z}{\partial t} = -\frac{\partial E_y}{\partial x} \quad \text{where } E_y = -E_m \cos(\omega t - \beta x)$$

$$= -[-\beta E_m(-\sin(\omega t - \beta x))]$$

$$= \frac{\partial B_z}{\partial t} = -\beta E_m \sin(\omega t - \beta x)$$

Integrate: $B_z = \int \cdot \beta E_m \sin(\omega t - \beta x)\, dt$

$$B_z = \frac{\beta E_m}{\omega} \cos(\omega t - \beta x)$$

→ $E_y = E_m(\cos(\omega t - \beta x))$

→ $B_z = \underbrace{\frac{\beta}{\omega}}_{\downarrow\ \frac{\text{rads}}{\text{s}}} (E_y)$ rads/m

$\therefore \frac{\beta}{\omega} = \frac{m}{s} = \frac{1}{v}$

$\boxed{\frac{1}{v} = \text{phase velocity}}$

→ $\mu H_z = \frac{1}{v} E_y \Rightarrow \mu \cdot v \cdot H_z = E_y$

Know that $v = \frac{1}{\sqrt{\mu \varepsilon}}$ $\therefore$ $\mu \cdot \frac{1}{\sqrt{\mu \varepsilon}} \cdot H_z = E_y$

$$\boxed{E_y = \sqrt{\frac{\mu}{\varepsilon}} \cdot H_z}$$

Q2. <u>Energy Densities</u>

To find: Total power emanating from closed surface:

Electric field: $\frac{\vec{D}\cdot\vec{E}}{2}$ J/m^3

Magnetic field: $\frac{\vec{B}\cdot\vec{H}}{2}$ J/m^3

Closed surface: Power out = - rate of change of stored energy.

$$\oint_S \vec{P}\cdot d\vec{s} = -\frac{\partial}{\partial t}\int_V \left(\text{E field density} + \text{magnetic field density}\right)\cdot dv$$

$$= \oint_S \vec{P}\cdot d\vec{s} = -\frac{\partial}{\partial t}\int_V \left(\frac{\vec{D}\cdot\vec{E}}{2} + \frac{\vec{B}\cdot\vec{H}}{2}\right) dv$$

Remember:

$$\begin{cases} \mu\vec{H} = \vec{B} \\ \varepsilon\vec{E} = \vec{D} \end{cases}$$

$$\underbrace{H\vec{H}\cdot\vec{H} = H\vec{H}^2}_{\text{chain rule.}}$$

$$\Rightarrow -\int_V \left(\frac{\partial\vec{H}}{\partial t}\cdot\mu\vec{H} + \frac{\partial\vec{E}}{\partial t}\cdot\varepsilon\vec{E}\right)\cdot dv$$

$$= \int_V \left(\frac{\partial\vec{H}}{\partial t}\cdot\vec{H} + \frac{\partial\vec{E}}{\partial t}\cdot\vec{E}\right)\cdot dv$$

$$= \int_V \left(\vec{H}\cdot(\nabla\times\vec{E}) - \vec{E}\cdot(\nabla\times\vec{H})\right)\cdot dv$$

$$\nabla\cdot(\vec{M}\times\vec{N}) = \vec{N}(\nabla\times\vec{M}) - \vec{M}(\nabla\times\vec{N}) \therefore \int_V \nabla\cdot(\vec{E}\times\vec{H})\cdot dv$$

Gauss Theorem: $\boxed{\oint \vec{P}\cdot d\vec{s} = \oint_S (\vec{E}\times\vec{H})\cdot d\vec{s}}$

Q2.2 Cylindrical Conductor

$r = 4 \times 10^{-3}$ m

$\sigma = 5.8 \times 10^{7}\ \Omega^{-1} m^{-1}$

$l = 1.5$ m

$V = 12 \times 10^{-3}$ V

In a cylinder

$$\boxed{H_r = \frac{rI}{2\pi a^2} \qquad E_r = \frac{J}{\sigma} = \frac{I}{\pi a^2 \sigma}}$$

Poynting vector = $\vec{E} \times \vec{H}$

E field = $\frac{V}{l} = \frac{12 \times 10^{-3}}{1.5} = 8 \times 10^{-3} = \frac{8mV}{m}$

Current Density : $\vec{J} = \sigma \vec{E}$ ∴

$(5.8 \times 10^{7})(8 \times 10^{-3}) = 464{,}000\ A/m^2$

Trying to Calculate: $\vec{E} \times \vec{H}$ if $R = \frac{\rho}{lA} \Rightarrow R = \frac{l}{2\pi a^2}$

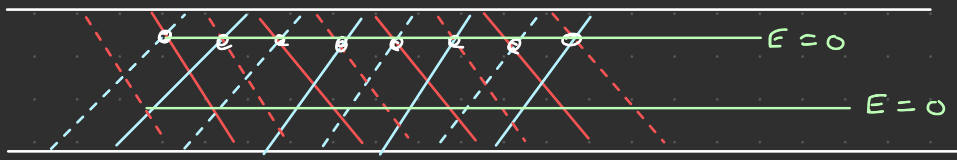
$$\left.\vec{H} = \frac{rJ}{2} = \frac{rI}{2\pi a^2}\right\}$$ This varies → r = from centre

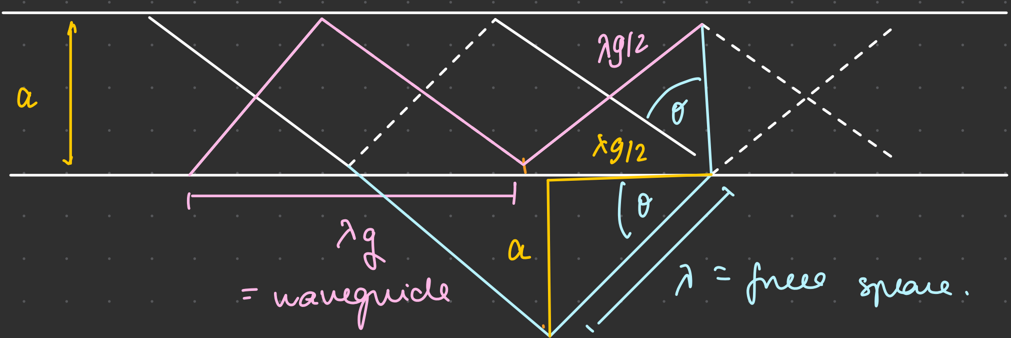
at $r = 4mm \rightarrow \frac{rJ}{2} \Rightarrow 928$

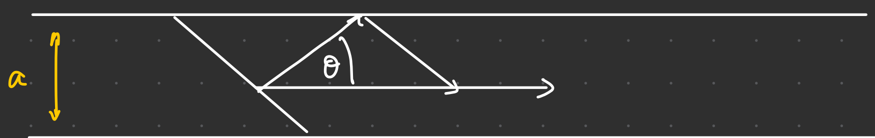
$r = 2mm \rightarrow$

$r = 0mm \rightarrow$

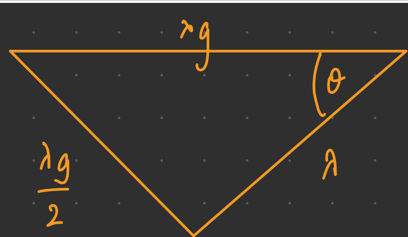
Waveguides 4.1





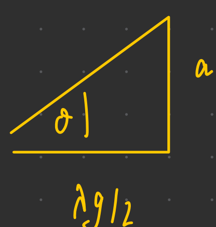


$\cos\theta = \dfrac{a}{h} =$



$\Rightarrow \dfrac{\lambda}{\lambda g}$

$\tan\theta = \dfrac{o}{a}$



$= \dfrac{\lambda g/2}{a} = \dfrac{\lambda g}{2a}$

$\cos\theta = \dfrac{\lambda}{\lambda_g}$, $\tan\theta = \dfrac{\lambda_g}{2a}$

$\sin\theta$ in terms of $\cos\theta, \tan\theta$:

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \sin\theta \cdot \frac{\lambda}{\lambda_g} = \frac{\lambda_g}{2a}$$

$$\therefore \frac{\lambda_g}{2a} = \frac{\lambda_g}{\lambda} \cdot \sin\theta \Rightarrow \frac{1}{2a} = \frac{\sin\theta}{\lambda} \Rightarrow \sin\theta = \frac{\lambda}{2a}$$

$$\Rightarrow \lambda = 2a\sin\theta$$

λ_c when $\sin\theta = 1$, $\boxed{\lambda = 2a}$

if $a = 0.03m \therefore 2(0.03) = 0.06m$

$$f_c = \frac{c}{\lambda_c} = \frac{3\times10^8}{0.06} = 50\times10^8 \text{ Hz}$$

4.2 $\lambda, \lambda_g, \lambda_c$:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin\theta = \frac{\lambda}{2a} \qquad \cos\theta = \frac{\lambda_g}{\lambda}$$

$$\left(\frac{\lambda}{2a}\right)^2 + \left(\frac{\lambda}{\lambda_g}\right)^2 = 1$$

* Divide across by λ^2

$$\underbrace{\left(\frac{1}{2a}\right)^2} + \left(\frac{1}{\lambda_g}\right)^2 = \frac{1}{\lambda^2}$$

Recall that this $= \lambda_c$ $\therefore \left(\frac{1}{\lambda_c}\right)^2 + \left(\frac{1}{\lambda_g}\right)^2 = \frac{1}{\lambda^2}$

4.3 θ component zigzag:

given that $f = 8 \times 10^{12}$ Hz

$$\lambda = \frac{c}{f} = \frac{3 \times 10^8}{8 \times 10^{12}} = \boxed{3.75 \times 10^{-5} \text{ m}}$$

Knowing $\sin\theta = \frac{\lambda}{2a}$ ∴ $a = 0.03$m ∴

$$= \frac{(3.75 \times 10^{-5})}{0.03}$$

$\sin^{-1}(\text{ans}) = 36.86°$

4.4

$$V_g = c \cdot \cos\theta = (3 \times 10^8)(\cos(38.86))$$

$$= 2.34 \times 10^8 \text{ m/s}$$

$$V_{ph} = \frac{c}{\cos\theta} = \frac{3 \times 10^8}{\cos(38.86)} = 3.84 \times 10^8 \text{ m/s}$$

Expectation:

$$\left.\begin{array}{l} V_g < c \\ V_{ph} > c \end{array}\right\} \; V_g \times V_{ph} = c^2$$

To verify Maxwell's Eqns

1. $\vec{\nabla} \cdot \vec{B} = 0$

2. $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\varepsilon}$

3. $\vec{\nabla} \times \vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$

4. $\vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\partial \vec{B}}{\partial t}$

find ρ and $\vec{J}$

<u>Eqn. 4</u> :