FORMALIZING PICK'S THEOREM IN LEAN

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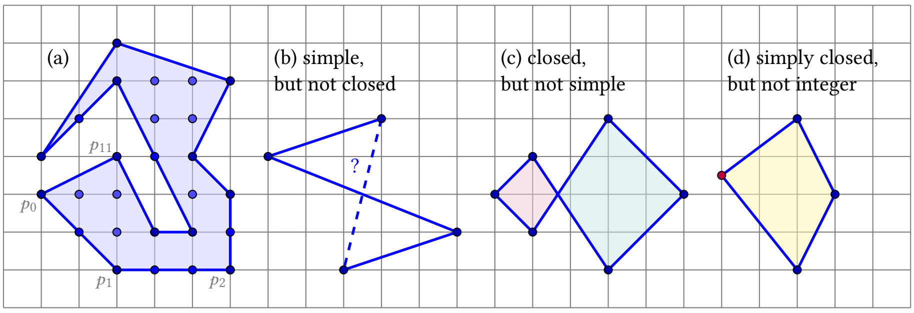
Abstract. Pick's astonishing theorem explains how to obtain the area of any integer polygon by counting lattice points. It is a notorious challenge to translate the geometric statement and intuitive reasoning into a formal statement and rigorous proof. We transform the beautiful geometry into equally elegant algebra, and then implement the algebraic proof in Lean.

1. Introduction

A *polygon* $P = (p_0, p_1, \ldots, p_n)$ is a finite sequence of points $p_i = (x_i, y_i) \in \mathbb{R}^2$. It defines the corresponding path $\gamma\colon [0,1] \to \mathbb{R}^2$ by piecewise linear interpolation of the given vertices $\gamma(i/n) = p_i$ for $i = 0, 1, \ldots, n$. We call P *closed* if $p_0 = p_n$, and *simply closed* if $\gamma(s) = \gamma(t)$ only holds for $s = t$ or $\{s, t\} = \{0, 1\}$. In this case Jordan's theorem applies: The polygonal curve $C = \gamma([0,1]) \subset \mathbb{R}^2$ separates the plane in two connected open sets A and B, so $\mathbb{R}^2 = A \sqcup B \sqcup C$, where the *exterior region* A is unbounded and the *interior region* B is bounded. Their common boundary is the curve C, so their closures are $\bar{A} = A \cup C$ and $\bar{B} = B \cup C$.



Let $\mathrm{vol}_2(B)$ denote the enclosed area. On the other hand, we can count the number of lattice points, $I := |\mathbb{Z}^2 \cap B|$ in the interior and $J := |\mathbb{Z}^2 \cap C|$ on the boundary $C = \partial B = \partial A$.

Here the magic of Pick's theorem happens:

Theorem 1.1 (Georg Pick 1899). *Let $P = (p_0, p_1, \ldots, p_n = p_0)$ be a simply closed polygon with integer vertices $p_i \in \mathbb{Z}^2$. Then $\mathrm{vol}_2(B) = I + J/2 - 1$.*

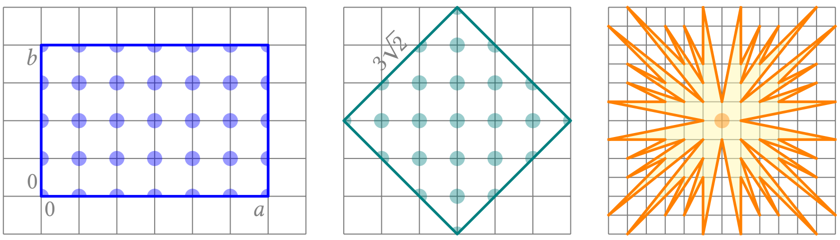


Figure 1. (a) rectangle, (b) oblique square, (c) Farey sunburst F_6

Example 1.2. (a) As a simple illustration, consider the rectangle $R = [0, a] \times [0, b]$ with $a, b \in \mathbb{N}_{\geq 1}$. We find $I = (a-1)(b-1)$ and $J = 2a + 2b$, which nicely adds up to $\mathrm{vol}_2(R) = ab$.

(b) For the oblique square Q of Figure 1(b) we find $I = 8$ and $J = 18$, whence $\mathrm{vol}_2(Q) = 16$. By Pick's theorem we can measure the area or count lattice points, whichever is simpler.

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(c) We calculate the area of the Farey sunburst F_6 shown in Figure 1(c). Its 64 vertices are given by $(x, y) \in \{-6, \dots, 6\}^2$ with $\gcd(x, y) = 1$. Closer inspection reveals 32 further boundary points. Its only inner point is $(0, 0)$. We conclude that $\mathrm{vol}_2(F_6) = 1 + 96/2 - 1 = 48$.

Remark 1.3. It is essential that the polygon P be given by integer vertices. The rectangle $R = [0, a] \times [1/4, 3/4]$, for example, has area $a/2$ but dos not contain any lattice point.

Remark 1.4. Pick's theorem is special to the plane $\mathbb{R}^2$. No such simple formula can hold in higher dimensions: The Reeve tetrahedron $T_r \subset \mathbb{R}^3$ is the convex hull of the four integer vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, r)$ with $r \in \mathbb{N}$. It has arbitrarily large volume $\mathrm{vol}_3(T_r) = r/6$, yet contains no further lattice points.

Remark 1.5. According to the formula $\mathrm{vol}_2(B) = I + J/2 - 1$, the area of any simply closed integer polygon is always integer or half integer, in brief $\mathrm{vol}_2(B) \in \frac{1}{2}\mathbb{Z}$.

Here is a nice application: Can you construct an equilateral triangle $\Delta \subset \mathbb{R}^2$ with three integer vertices $(0, 0)$, (a, b) and (c, d)? No! Its side length ℓ would satisfy $\ell^2 = a^2 + b^2 \in \mathbb{N}$ by Pythagoras, so its area $\mathrm{vol}_2(\Delta) = \sqrt{3}/4 \cdot \ell^2$ cannot be in $\frac{1}{2}\mathbb{Z}$, since $\sqrt{3}$ is irrational.

2. Formalizing Pick's theorem

We work over an ordered field $(\mathbb{K}, +, \cdot, \leq)$. The traditional choice is the field $\mathbb{R}$ of real numbers, which we consider first. It turns out, however, that the field $\mathbb{Q}$ of rational numbers suffices, and moreover is more convenient for computer implementations. By abstracting both these primary examples to ordered fields we cover all cases simultaneously.

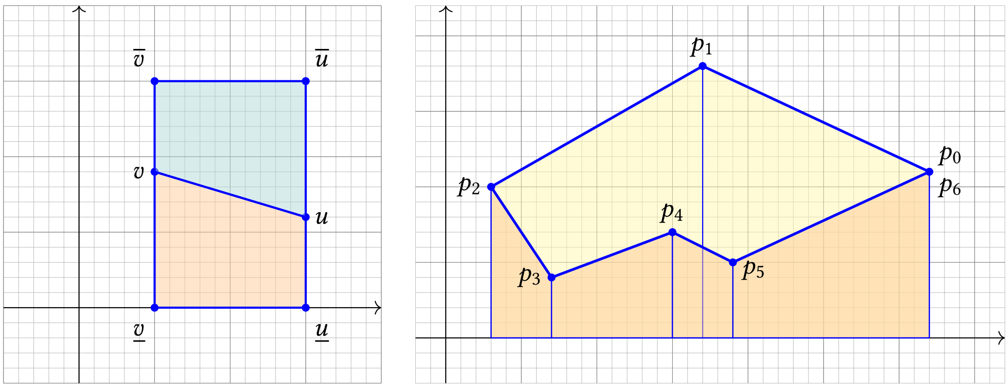


Figure 2. Area of a polygon

Definition 2.1. Given two points $u, v \in \mathbb{K}^2$ we have the (oriented) area under the trapezoid,

$$\mathrm{area}(u, v) := \frac{1}{2}(u_1 - v_1)(u_2 + v_2).$$

For every polygon $P = (p_0, p_1, \dots, p_n)$ in $\mathbb{K}^2$ we thus define its area to be

$$\mathrm{Area}(P) := \sum_{i=1}^{n} \mathrm{area}(p_{i-1}, p_i).$$

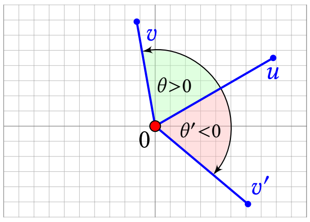
Remark 2.2. For a simply closed polygon in $\mathbb{R}^2$, this yields $|\mathrm{Area}(P)| = \mathrm{vol}_2(B) > 0$.

If $\mathrm{Area}(P) > 0$, we call our polygon P *positively oriented.* Otherwiese, if $\mathrm{Area}(P) < 0$, we reverse $P = (p_0, p_1, \dots, p_n)$ to $P' = (p_n, \dots, p_1, p_0)$. This does not change the curve C, but ensures that P' is positively oriented. This will be our standard convention.

This elegant definition of $\mathrm{Area}(P)$ clarifies the left hand side of Pick's equation. For the right hand side $I + J/2 - 1$ we have to define — and then count! — the enclosed lattice points. To this end we use the winding number of our polygon P around some point $q \in \mathbb{K}^2$.

2.1. **The euclidean angle measure.** Over $\mathbb{K} = \mathbb{R}$, we can use the *euclidean angle* $\theta = \sphericalangle(u, v)$ between vectors $u, v \in \mathbb{R}^2$: For each $u \neq 0$ and $v \notin u\,\mathbb{R}_{\leq 0}$, there exists a unique real number $\theta \in\ ]-\pi, \pi[$ such that rotation by θ aligns u with v:

$$\frac{v}{|v|} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \frac{u}{|u|}.$$



In the controversial case $v \in u\,\mathbb{R}_{<0}$, both solutions $\theta = \pm\pi$ are equally possible. Usually this case is ignored, forbidden, or abitrated. We democratically set $\sphericalangle(u, v) = 0$. This choice may seem strange at first, but turns out to be advantageous. It allows us to cover all cases uniformly, and miraculously leads to the correct point count on the boundary.

Definition 2.3. We define our *euclidean angle measure* $\operatorname{ang}\colon \mathbb{R}^2 \times \mathbb{R}^2 \to\ ]-{}^1\!/\!_2, +{}^1\!/\!_2[$ by

$$\operatorname{ang}(u, v) := \begin{cases} \theta/2\pi & \text{if } |u| \cdot |v| + u \cdot v > 0, \\ 0 & \text{if } |u| \cdot |v| + u \cdot v = 0. \end{cases}$$

By summing the angles of all edges of P, we obtain the (euclidean) *winding number*

$$\operatorname{Ang}(P) := \sum_{i=1}^{n} \operatorname{ang}(p_{i-1}, p_i).$$

Remark 2.4. If our polygon P is closed and $0 \notin C$, then $\operatorname{Ang}(P) \in \mathbb{Z}$ measures how often P winds around the origin. Likewise $\operatorname{Ang}(P - q) \in \mathbb{Z}$ measures the winding number around the point $q \in \mathbb{R}^2 \smallsetminus C$, where $P - q = (p_0 - q, p_1 - q, \dots, p_n - q)$.

Moreover, if P is simply closed and positively oriented, then its Jordan decomposition $\mathbb{R}^2 = A \sqcup B \sqcup C$ is characterized by the winding number: We have $\operatorname{Ang}(P - q) = 0$ for each exterior point $q \in A$ and $\operatorname{Ang}(P - q) = 1$ for each interior point $q \in B$. We thus obtain

$$I = \sum_{q \in \mathbb{Z}^2 \smallsetminus C} \operatorname{Ang}(P - q).$$

Our careful definition pays off for boundary points: We find $\operatorname{Ang}(P - q) = {}^1\!/\!_2$ for $q \in\ ]p_{i-1}, p_i[$ in the interior of any edge. In each vertex p_i, finally, $\operatorname{Ang}(P - p_i) = \alpha_i = \operatorname{ang}(p_{i+1} - p_i, p_{i-1} - p_i)$ measures the enclosed angle. By adding the turning angle $\beta_i = \operatorname{ang}(p_i - p_{i-1}, p_{i+1} - p_i)$ each vertex point is counted by ${}^1\!/\!_2$ as well. The sum of all turning angles is 1 by Hopf's umlaufsatz.

If P is a lattice polygon, with integer vertices $v_0, \dots, v_n \in \mathbb{Z}^2$, then

$$J/2 - 1 = \sum_{q \in \mathbb{Z}^2 \cap C} \operatorname{Ang}(P - q).$$

The right hand side $I + J/2 - 1$ of Pick's equation is defined geometrically, and usually formulated intuitively. Any attempt to define it precisely relies on Jordan's theorem. The winding number, as defined above, allows us to algebraically count this as

$$I + J/2 - 1 = \sum_{q \in \mathbb{Z}^2} \operatorname{Ang}(P - q).$$

This reduces Pick's theorem to the following algebraic statement:

Lemma 2.5 (Pick's lemma, using the euclidean angle measure). *Let $p_1, \dots, p_n = p_0 \in \mathbb{Z}^2$ be any sequence of integer points. Then the closed polygon $P = (p_0, p_1, \dots, p_n)$ satisfies Pick's equation*

$$\operatorname{Area}(P) = \sum_{q \in \mathbb{Z}^2} \operatorname{Ang}(P - q)$$

Notice that this formulation does not require P to be simple. If P is simply closed and positively oriented, then the right hand side equals $I + J/2 - 1$, by Jordan and Hopf.

2.2. **The discrete angle measure.** The euclidean angle measure requires the real numbers and transcendental functions. For computer implementation this is a burden: `Real` is good for proofs but bad for calculation, whereas `Float` ist good for approximations but bad for proofs.

For our purposes we prefer to work with the following *discrete angle measure*. This allows us to work over any ordered field $\mathbb{K}$, for example $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$.

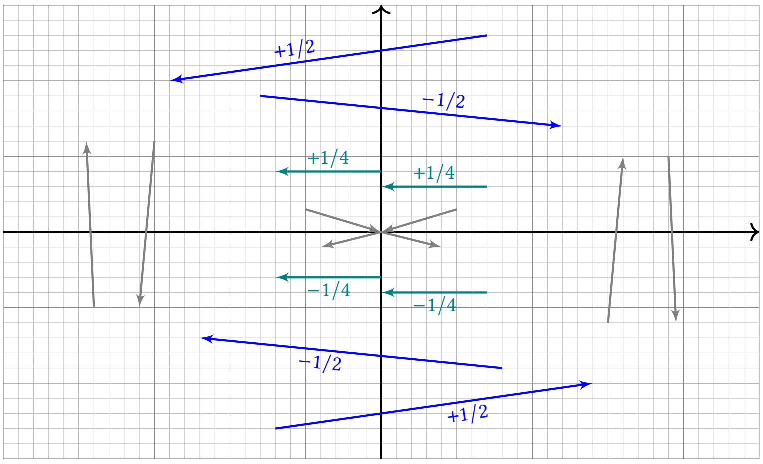


Figure 3. The discrete angle measure

Definition 2.6. For the edge between any two points $u, v \in \mathbb{K}^2$ we define the *discrete angle measure* as the number of axis crossings, as illustrated in Figure 3:

$$\operatorname{dang}(u, v) := \frac{1}{4}\left|\operatorname{sign} u_1 - \operatorname{sign} v_1\right| \cdot \operatorname{sign} \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}.$$

By summing the angles of all edges of P, we obtain the (discrete) *winding number*

$$\operatorname{Dang}(P) := \sum_{i=1}^{n} \operatorname{dang}(p_{i-1}, p_i).$$

If our polygon P is closed and $0 \notin C$, then we obtain $\operatorname{Dang}(P - q) = \operatorname{Ang}(P - q)$. The summands change, but the end result is the same. The difference is only noticable for open polygons or boundary points $q \in C$.

The theorems of Jordan and Hopf, as stated above, continue to hold with $\operatorname{Dang}(P)$ in place of $\operatorname{Ang}(P)$, see the Appendix. This generalizes and simplifies! We can thus define the elusive right hand side $I + J/2 - 1$ of Pick's equation by the *weighted sum of enclosed lattice points*

$$\operatorname{Welp}(P) := \sum_{q \in \mathbb{Z}^2} \operatorname{Dang}(P - q).$$

Lemma 2.7 (Pick's lemma, using the discrete angle measure)**.** *Let $p_1, \dots, p_n = p_0 \in \mathbb{Z}^2$ be any sequence of integer points. Then the closed polygon $P = (p_0, p_1, \dots, p_n)$ satisfies Pick's equation*

$$\operatorname{Area}(P) = \operatorname{Welp}(P).$$

Remark 2.8. At first glance this may no longer look like Pick's classical theorem. Admittedly, the quantity $I + J/2 - 1$ is geometrically more intuitive, alas notoriously vague. ("Just look!") Our formula for $\operatorname{Welp}(P)$ provides a precise definition, alas less intuitive. ("Just calculate!")

We cautiously call the above statement *Pick's lemma*. In order to arrive at *Pick's theorem*, we have to invoke Jordan's decomposition and Hopf's umlaufsatz for simply closed polygonal curves. This bridges the gap between the algebraic lemma and the geometric theorem.

Since Jordan's and Hopf's are classical theorems in their own right, we do not consider them as part of the *proof*, but rather use them as the foundation for *formulating* Pick's equation. This translation being achieved, we can now focus on Pick's lemma.

3. Proving Pick's lemma

The sum over $q \in \mathbb{Z}^2$ has finite support: We can restrict it to a sufficiently large square box $Q = \{-r, \dots, r\}^2 \subset \mathbb{Z}^2$ containing all vertices $p_1, \dots, p_n$. Each point $q \in \mathbb{Z}^2 \setminus Q$ yields $\mathrm{Dang}(P - q) = 0$. Using this, we rearrange the sum defining the right hand side:

$$\begin{aligned}\mathrm{Welp}(P) = \sum_{q \in Q} \mathrm{Dang}(P - q) &= \sum_{q \in \mathbb{Z}^2} \sum_{i=1}^{n} \mathrm{dang}(p_{i-1} - q, p_i - q) \\ &= \sum_{i=1}^{n} \underbrace{\sum_{q \in \mathbb{Z}^2} \mathrm{dang}(q_{i-1} - v, q_i - v)}_{=:\mathrm{welp}(p_{i-1}, p_i)}\end{aligned}$$

Now both sides of Pick's equation look formally very similar:

$$\begin{aligned}\mathrm{Area}(P) &= \sum_{i=1}^{n} \mathrm{area}(p_{i-1}, p_i), \\ \mathrm{Welp}(P) &= \sum_{i=1}^{n} \mathrm{welp}(p_{i-1}, p_i).\end{aligned}$$

Here another miracle happens: Both sums are not only equal, but termwise equal!

For any to lattice points $u, v \in Q$ in our square box $Q = \{-r, \dots, r\}^2 \subset \mathbb{Z}^2$ we show that

$$\mathrm{area}(u, v) = \mathrm{welp}(u, v).$$

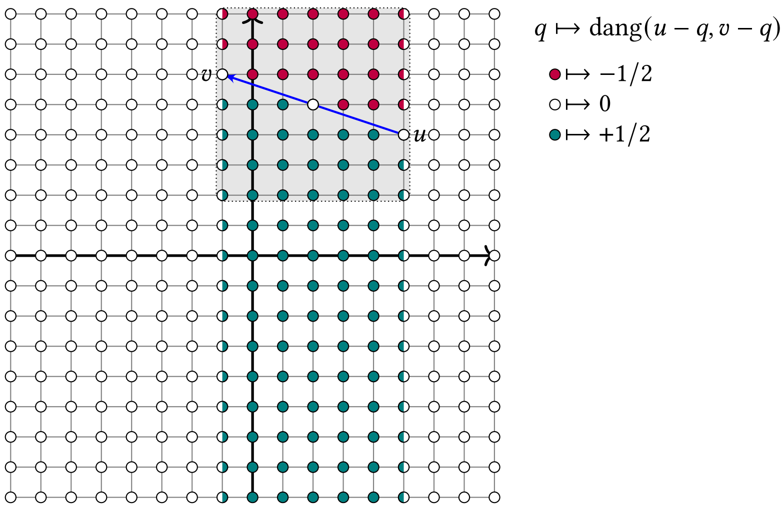


Figure 4. Proving $\mathrm{area}(u, v) = \mathrm{welp}(u, v)$

The proof is illustrated in Figure 4. The algebraic calculation proceeds as follows. We can assume $v_1 < u_1$ and $u_2 + v_2 \geq 2$; the other cases are symmetric.

(1) For $q_1 < v_1 < u_1$ we have $\mathrm{dang}(u - q, v - q) = 0$.
(2) For $q_1 > u_1 > v_1$ we have $\mathrm{dang}(u - q, v - q) = 0$.
(3) The rectangle $R = \{v_1, \dots, u_1\} \times \{u + v - r, \dots, r\}$ allows the involution $q \mapsto u + v - q$. We find $\mathrm{dang}(u - (u + v - q), v - (u + v - q)) = \mathrm{dang}(v - q, u - q) = -\mathrm{dang}(u - q, v - q)$. Thus all summands cancel pairwise, and we obtain $\sum_{q \in R} \mathrm{dang}(u - q, v - q) = 0$.
(4) The remaining points add up to $\frac{1}{2}(u_1 - v_1)(u_2 + v_2) = \mathrm{area}(u, v)$.

This proves $\mathrm{welp}(u, v) = \mathrm{area}(u, v)$ for any edge with end points $u, v \in Q$. By summing over all edges of our polygon P, we conclude $\mathrm{Area}(P) = \mathrm{Welp}(P)$.

Remark 3.1. In this algebraic form, the proof is readily implemented in Lean, or any other proof assistant.

Appendix A. Axiomatic angle measure

We can use different angle measures for the theorems of Jordan and Hopf and thus Pick. In order to cover all cases simultaneously, we extract the essential properties used in the proofs:

Definition A.1 (angle measure)**.** Let $(\mathbb{K}, +, \cdot, \leq)$ be an ordered field. An *angle measure* is a map $\mu \colon \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K}$ satisfying the following conditions:

(1) Scaling: For all $u, v \in \mathbb{K}^2$ and $\lambda \in \mathbb{K}_{>0}$ we have $\mu(u, v) = \mu(\lambda u, v) = \mu(u, \lambda v)$.
(2) Symmetry: $\mu(v, u) = -\mu(u, v)$ and $\mu(-u, -v) = \mu(u, v)$.
(3) Addition: $\mu(u, v) = \mu(u, s) + \mu(s, v)$ for $s \in [u, v]$.
(4) Normalisation: $\mu(e_1, e_2) = 1/4$.

Appendix B. Jordan's decomposition for polygons

The classical theorem, reformulated and reproven using our angle measure μ.

Appendix C. Hopf's umlaufsatz for polygons

The classical theorem, reformulated and reproven using our angle measure μ.

Appendix D. Physical plausibility: "water proof"

In every-day mathematical communication we often only sketch the idea and appeal to some degree of informal intuition. This is especially true for geometric statements, like Pick's theorem, where we often heavily rely on our visual perception. This is usually a good thing for human learning and understanding, but it notoriously hinders any sound formalization.

We sketch such a visual proof, appealing to geometric intuition as a counterpart to our algebraic formalization. We then explain how the intuition can be made precise and finally translates to our algebraic viewpoint.

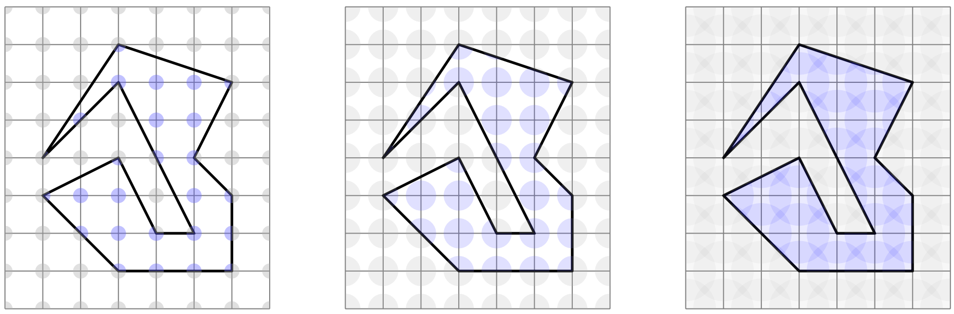


Figure 5. Πάντα ῥεῖ, everything flows. We let the water do the work.

The water proof. At each integer point $q \in \mathbb{Z}^2$ we place a unit drop of water. The water then flows evenly in all directions. Finally, the plane is uniformly covered with water, one unit of water per unit square. (To ensure finiteness, we can think of periodic boundary conditions.)

Now consider two integer points $u, v \in \mathbb{Z}^2$. Half rotation about its center $c = (u+v)/2$ defines the involution $\rho \colon z \mapsto u + v - z$. This reverses the edge and thus the flow. Since ρ maps the lattice $\mathbb{Z}^2$ to itself, the water that flows from the point $q \in \mathbb{Z}^2$ over $[u, v]$ is compensated by the water flowing from the point $\rho(q) \in \mathbb{Z}^2$ over $[u, v]$. The net flow over the edge $[u, v]$ is zero.

This shows that the amount of water in B never changes. At the start it is the share of drops that fall into B. At the end it equals the surface area of B. We thus obtain:

$$\mathrm{vol}_2(B) = I + \sum_{q \in \mathbb{Z}^2 \cap C} \alpha(q)/2\pi$$

Here $\alpha(q)$ is the inner angle at the boundary point $q \in C$. So our physical intuition, or gedankenexperiment, leads us to Pick's lemma 2.5 using the euclidean angle measure.

We once again invoke Hopf's umlaufsatz: the sum of outer angles is $\sum_{q \in C} [\pi - \alpha(q)] = 2\pi$, so we arrive at Pick's theorem $\mathrm{vol}_2(B) = I + J/2 - 1$. □

Remark D.1. This "water proof" is a wonderful example of a physical plausibility argument. To many people is seems convincing because it appeals to our physical experience and our tried-and-tested understanding of the world. But is it a proof, really? Can we actually understand the physical world? It is easy to visualize, intuitively, but hard to formalize, rigorously.

We can give the physical argument more mathematical substance. First, we want to ensure finiteness of our experimentation table, so instead of the entire plane $\mathbb{R}^2$ we consider a square $[-r,+r]^2$ with sufficiently large $r \in \mathbb{N}$ and periodic boundary conditions. Geometrically, this is a flat torus $T = \mathbb{R}^2/2r\mathbb{Z}^2$, and we keep all necessary symmetries: translations and reflections.

Second, we specify the properties of our idealized water drops. For each time $t \in \,]0,1]$ let μ_t be a continuous probability measure, ending with the uniform (Haar) measure μ_1 on T, so that $4r^2\mu_1(B)$ is the area of our inner region. Now we observe the quantity

$$f(t) := \sum_{q \in \mathbb{Z}^2/2r\mathbb{Z}^2} \mu_t(B-q).$$

We have $f(1) = \mathrm{vol}_2(B)$. For each $t > 0$ we assume μ_t to be symmetric with respect to $z \mapsto -z$ and uniform on its support. Assuming that μ_t varies continuously with t, the above symmetry argument suggests that f is constant. At the other end we look at the limit for $t \searrow 0$:

Example D.2. As illustrated in Figure 5, for small t we set $\operatorname{supp} \mu_t = D(0,t)$ to be the disc of radius t around the origin 0. For sufficiently small t thus obtain $f(t) = \sum_{q \in \mathbb{Z}^2} \mathrm{Ang}(P-q)$.

This recovers Pick's lemma 2.5 using the euclidean angle measure.

Example D.3. Alternatively, for small t we consider set $\operatorname{supp} \mu_t = [-t^2, t^2] \times [-t, t]$ to be a thin vertical rectangle. In the limit $t \searrow 0$ we thus obtain $f(t) \to \sum_{q \in \mathbb{Z}^2} \mathrm{Dang}(P-q)$.

This recovers Pick's lemma 2.7 using the discrete angle measure.

Remark D.4. This tale can be recounted by replacing water with heat. This is appealing for a physically inclined audience familiar with the heat equation or diffusion. Moreover, this provides an explicit model of the flow as partial differential equations. Within the chosen model, we can then prove that f is indeed constant and calculate the limit for $t \searrow 0$.

Remark D.5. Another variant is to think of reflective walls. Geometrically, we can take two copies of $B \cup C$ and glue them together along their common boundary C. This produces a closed surface S homeomorphic to the sphere, with volume $\mathrm{vol}_2(S) = 2\,\mathrm{vol}_2(B)$ and Euler characteristic $\chi(S) = 2$. The vertices provide the curvature, as in the Gauß–Bonnet theorem.

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