# FORMALIZING PICK'S THEOREM, EFFICIENTLY

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Abstract. Pick's astonishing theorem explains how to obtain the area of any integer polygon by counting lattice points. It is a notoriously difficult challenge to translate the geometric statement and intuitive reasoning into a formal statement and rigorous proof. We transform the beautiful geometry into equally elegant algebra, and then implement the algebraic proof in Lean.

1. Introduction: Pick's wonderful theorem

A *polygon* $P = (p_0, p_1, \ldots, p_n)$ is a finite sequence of points $p_i = (x_i, y_i) \in \mathbb{R}^2$. It defines the corresponding path $\gamma \colon [0, 1] \to \mathbb{R}^2$ by piecewise linear interpolation of the given vertices $\gamma(i/n) = p_i$ for $i = 0, 1, \ldots, n$. We call P *closed* if $p_0 = p_n$, and *simply closed* if $\gamma(s) = \gamma(t)$ only holds for $s = t$ or $\{s, t\} = \{0, 1\}$. In this case Jordan's theorem applies: The polygonal curve $C = \gamma([0, 1]) \subset \mathbb{R}^2$ separates the plane in two connected open sets A and B, so $\mathbb{R}^2 = A \sqcup B \sqcup C$, where the *exterior region* A is unbounded and the *interior region* B is bounded. Their common boundary is the curve C, so their closures are $\bar{A} = A \cup C$ and $\bar{B} = B \cup C$.
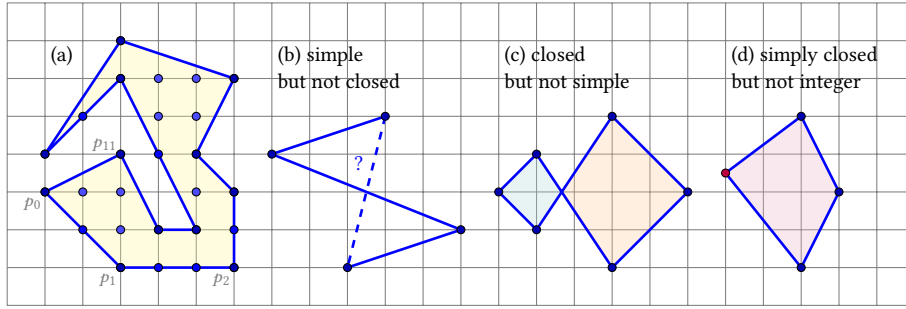
Let $\mathrm{vol}_2(B)$ denote the enclosed area. On the other hand, we can count the number of lattice points, $I := |\mathbb{Z}^2 \cap B|$ in the interior and $J := |\mathbb{Z}^2 \cap C|$ on the boundary $C = \partial B = \partial A$.

Here the magic of Pick's theorem happens:

Theorem 1.1 (Georg Pick 1899 [1]). *Let $P = (p_0, p_1, \ldots, p_n = p_0)$ be a simply closed polygon with integer vertices $p_i \in \mathbb{Z}^2$. Then $\mathrm{vol}_2(B) = I + J/2 - 1$.*
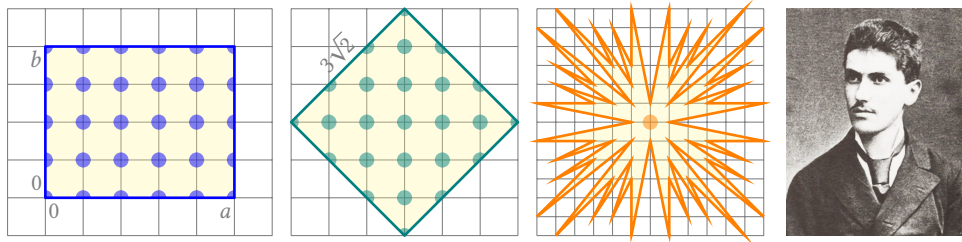

Figure 1. (a) rectangle, (b) oblique square, (c) Farey sunburst F_6, (d) Georg Pick

Example 1.2. (a) As a simple illustration, consider the rectangle $R = [0, a] \times [0, b]$ with $a, b \in \mathbb{N}_{\geq 1}$. We find $I = (a - 1)(b - 1)$ and $J = 2a + 2b$, which nicely adds up to $\mathrm{vol}_2(R) = ab$.

(b) For the oblique square Q of Figure 1(b) we find $I = 13$ and $J = 12$, whence $\mathrm{vol}_2(Q) = 18$. By Pick's theorem we can measure the area or count lattice points, whichever is simpler.

(c) We calculate the area of the Farey sunburst F_6 shown in Figure 1(c). Its 64 vertices are given by $(x, y) \in \{-6, \ldots, 6\}^2$ with $\gcd(x, y) = 1$. Closer inspection reveals 32 further boundary points. Its only inner point is $(0, 0)$. We conclude that $\mathrm{vol}_2(F_6) = 1 + 96/2 - 1 = 48$.

Remark 1.3. It is essential that the polygon P be given by integer vertices. The rectangle $R = [0, a] \times [1/4, 3/4]$, for example, has area $a/2$ but dos not contain any lattice point.

Remark 1.4. Pick's theorem is special to the plane $\mathbb{R}^2$. No such simple formula can hold in higher dimensions: The Reeve tetrahedron $T_r \subset \mathbb{R}^3$ is the convex hull of the four integer vertices $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$ and $(1, 1, r)$ with $r \in \mathbb{N}$. It has arbitrarily large volume $\mathrm{vol}_3(T_r) = r/6$, yet contains no further lattice points.

Remark 1.5. According to the formula $\mathrm{vol}_2(B) = I + J/2 - 1$, the area of any simply closed integer polygon is always integer or half integer, in brief $\mathrm{vol}_2(B) \in \frac{1}{2}\mathbb{Z}$.

Here is a nice application: Can you construct an equilateral triangle $\Delta \subset \mathbb{R}^2$ with three integer vertices $(0, 0)$, (a, b) and (c, d)? No! Its side length ℓ would satisfy $\ell^2 = a^2 + b^2 \in \mathbb{N}$ by Pythagoras, so its area $\mathrm{vol}_2(\Delta) = \sqrt{3}/4 \cdot \ell^2$ cannot be in $\frac{1}{2}\mathbb{Z}$, since $\sqrt{3}$ is irrational.

2. How to formalize Pick's theorem?

We work over an ordered field $(\mathbb{K}, +, \cdot, \leq)$. The traditional choice is the field $\mathbb{R}$ of real numbers, which we consider first. It turns out, however, that the field $\mathbb{Q}$ of rational numbers suffices, and moreover is more convenient for computer implementations. By abstracting both these primary examples to ordered fields we cover all cases simultaneously.
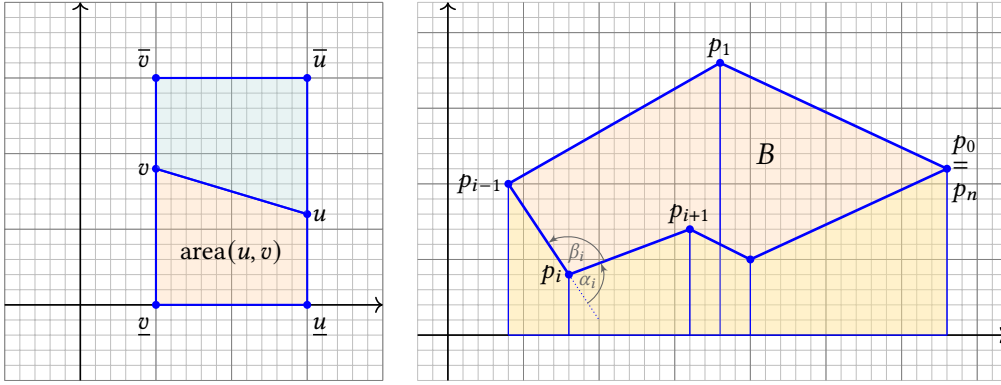

Figure 2. Oriented area of a trapezoid and of a polygon

Definition 2.1. Given two points $u, v \in \mathbb{K}^2$ we have the (oriented) area of the trapezoid,

$$(2.1) \qquad \mathrm{area}(u, v) := \frac{1}{2}(u_1 - v_1)(u_2 + v_2).$$

For every polygon $P = (p_0, p_1, \ldots, p_n)$ in $\mathbb{K}^2$ we thus define its area to be

$$(2.2) \qquad \mathrm{Area}(P) := \sum_{i=1}^{n} \mathrm{area}(p_{i-1}, p_i).$$

Remark 2.2. For a simply closed polygon in $\mathbb{R}^2$, this yields $|\mathrm{Area}(P)| = \mathrm{vol}_2(B) > 0$.

If $\mathrm{Area}(P) > 0$, we call our polygon P *positively oriented*. Otherwiese, if $\mathrm{Area}(P) < 0$, we reverse $P = (p_0, p_1, \ldots, p_n)$ to $P' = (p_n, \ldots, p_1, p_0)$. This does not change the curve C, but ensures that P' is positively oriented. This will be our standard convention.

This elegant definition of $\mathrm{Area}(P)$ clarifies the left hand side of Pick's equation. For the right hand side $I + J/2 - 1$ we have to define — and then count! — the enclosed lattice points. To this end we use the winding number of our polygon P around some point $q \in \mathbb{K}^2$.

2.1. **The euclidean angle measure.** Over $\mathbb{K} = \mathbb{R}$, we can use the *euclidean angle* $\theta = \sphericalangle(u, v)$ between vectors $u, v \in \mathbb{R}^2$: For each $u \neq 0$ and $v \notin u\,\mathbb{R}_{\leq 0}$, there exists a unique real number $\theta \in \,]-\pi, \pi[$ such that rotation by θ aligns u with v:

(2.3)
$$\frac{v}{|v|} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \frac{u}{|u|}.$$
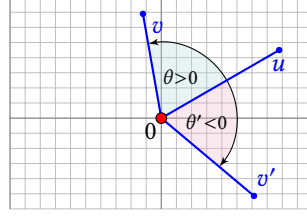
In the controversial case $v \in u\,\mathbb{R}_{<0}$, both solutions $\theta = \pm\pi$ are equally possible. Usually this case is ignored, forbidden, or abitrated. We democratically set $\sphericalangle(u, v) = 0$. This choice may seem strange at first, but turns out to be advantageous. It allows us to cover all cases uniformly, and miraculously leads to the correct point count on the boundary.

Definition 2.3. We define our *euclidean angle measure* ang: $\mathbb{R}^2 \times \mathbb{R}^2 \to \,]-1/2, +1/2[$ by

(2.4)
$$\mathrm{ang}(u, v) := \begin{cases} \theta/2\pi & \text{if } |u| \cdot |v| + u \cdot v > 0, \\ 0 & \text{if } |u| \cdot |v| + u \cdot v = 0. \end{cases}$$

By summing the angles of all edges of P, we obtain the (euclidean) *winding number*

(2.5)
$$\mathrm{Ang}(P) := \sum_{i=1}^{n} \mathrm{ang}(p_{i-1}, p_i).$$

Remark 2.4. If our polygon P is closed and $0 \notin C$, then $\mathrm{Ang}(P) \in \mathbb{Z}$ measures how often P winds around the origin. Likewise $\mathrm{Ang}(P - q) \in \mathbb{Z}$ measures the winding number around the point $q \in \mathbb{R}^2 \setminus C$, where $P - q = (p_0 - q, p_1 - q, \ldots, p_n - q)$.

If P is simply closed and positively oriented, then its Jordan decomposition $\mathbb{R}^2 = A \sqcup B \sqcup C$ is characterized by the winding number: We have $\mathrm{Ang}(P - q) = 0$ for each exterior point $q \in A$ and $\mathrm{Ang}(P - q) = 1$ for each interior point $q \in B$. We thus obtain

(2.6)
$$I = \sum_{q \in \mathbb{Z}^2 \setminus C} \mathrm{Ang}(P - q).$$

Our careful definition pays off: $\mathrm{Ang}(P - q)$ measures the interior angle at each boundary point $q \in C$. For $q \in \,]p_{i-1}, p_i[$ in the interior of any edge, we find $\mathrm{Ang}(P - q) = 1/2$. In each vertex, we find $\mathrm{Ang}(P - p_i) = \beta_i := \mathrm{ang}(p_{i+1} - p_i, p_{i-1} - p_i)$. By adding the turning angle $\alpha_i = \mathrm{ang}(p_i - p_{i-1}, p_{i+1} - p_i)$ each vertex point is counted by $1/2$ as well. (This is illustrated in Figure 2.) The sum of all turning angles is 1 by Hopf's umlaufsatz.

If P is a simply closed lattice polygon, with integer vertices $v_0, \ldots, v_n \in \mathbb{Z}^2$, then

(2.7)
$$J/2 - 1 = \sum_{q \in \mathbb{Z}^2 \cap C} \mathrm{Ang}(P - q).$$

The right hand side $I + J/2 - 1$ of Pick's equation is defined geometrically, and usually formulated intuitively. Any attempt to define it precisely relies on Jordan's theorem. The winding number, as defined above, allows us to algebraically count this as

(2.8)
$$I + J/2 - 1 = \sum_{q \in \mathbb{Z}^2} \mathrm{Ang}(P - q).$$

This reduces Pick's theorem to the following algebraic statement:

Lemma 2.5 (Pick's lemma, using the euclidean angle measure). *Let $p_1, \ldots, p_n = p_0 \in \mathbb{Z}^2$ be any sequence of integer points. Then the closed polygon $P = (p_0, p_1, \ldots, p_n)$ satisfies Pick's equation*

(2.9)
$$\mathrm{Area}(P) = \sum_{q \in \mathbb{Z}^2} \mathrm{Ang}(P - q)$$

Notice that this formulation does not require P to be simple. If P is simply closed and positively oriented, then the right hand side equals $I + J/2 - 1$, by Jordan and Hopf.

2.2. **The discrete angle measure.** The euclidean angle measure requires real numbers and transcendental functions. For computer implementation this is a heavy burden: Real is good for proofs but bad for calculation, whereas Float ist good for approximations but bad for proofs.

For our purposes we prefer to work with the following *discrete angle measure*. This allows us to work over any ordered field $\mathbb{K}$, for example $\mathbb{Q} \subseteq \mathbb{K} \subseteq \mathbb{R}$.
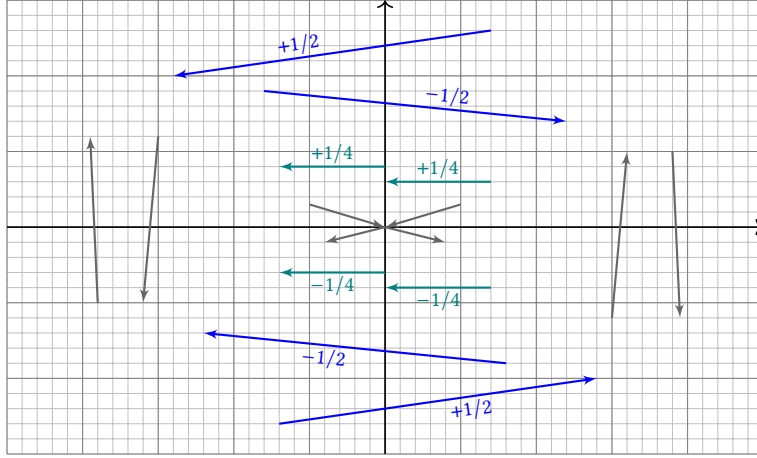

Figure 3. The discrete angle measure

Definition 2.6. For the edge between any two points $u, v \in \mathbb{K}^2$ we define the *discrete angle measure* as the number of axis crossings, as illustrated in Figure 3:

$$(2.10) \qquad \operatorname{dang}(u, v) := \frac{1}{4} |\operatorname{sign} u_1 - \operatorname{sign} v_1| \cdot \operatorname{sign} \det \begin{bmatrix} u_1 & v_1 \\ u_2 & v_2 \end{bmatrix}.$$

By summing the angles of all edges of P, we obtain the (discrete) *winding number*

$$(2.11) \qquad \operatorname{Dang}(P) := \sum_{i=1}^{n} \operatorname{dang}(p_{i-1}, p_i).$$

If our polygon P is closed and $0 \notin C$, then $\operatorname{Dang}(P) = \operatorname{Ang}(P)$ measures the winding number around 0. The summands change, but the overall sum is the same; the difference is only noticeable for boundary points $q \in C$ or open polygons.

The theorems of Jordan and Hopf, as stated above, continue to hold with $\operatorname{Dang}(P)$ in place of $\operatorname{Ang}(P)$, see the Appendix. We can thus define the elusive right hand side $I + J/2 - 1$ of Pick's equation by the *weighted sum of enclosed lattice points*

$$(2.12) \qquad \operatorname{Welp}(P) := \sum_{q \in \mathbb{Z}^2} \operatorname{Dang}(P - q).$$

Lemma 2.7 (Pick's lemma, using the discrete angle measure). *Let $p_1, \ldots, p_n = p_0 \in \mathbb{Z}^2$ be any sequence of integer points. Then the closed polygon $P = (p_0, p_1, \ldots, p_n)$ satisfies Pick's equation*

$$(2.13) \qquad \operatorname{Area}(P) = \operatorname{Welp}(P).$$

Remark 2.8. Notice that this formulation does not require P to be simple. If P is simply closed and positively oriented, then the right hand side equals $I + J/2 - 1$, by Jordan and Hopf.

At first glance Pick's lemma may no longer look like Pick's classical theorem. Admittedly, the quantity $I + J/2 - 1$ is geometrically more intuitive, alas notoriously vague. ("Just look!") Our formula for $\operatorname{Welp}(P)$ provides a precise definition, alas less intuitive. ("Just calculate!")

We cautiously call the above statement *Pick's lemma*. In order to arrive at Pick's theorem, we have to invoke Jordan's decomposition and Hopf's umlaufsatz for simply closed polygonal curves. This bridges the gap between the algebraic lemma and the geometric theorem.

Since Jordan's and Hopf's are classical theorems in their own right, we do not consider them as part of the *proof* of Pick's theorem, but rather as the foundation for precisely *formulating* the statement. This translation being achieved, we can now focus on proving Pick's lemma.

3. Proof of Pick's lemma

The sum (2.12) over $q \in \mathbb{Z}^2$ has finite support: We can restrict it to a sufficiently large square box $Q = \{-r, \dots, r\}^2 \subset \mathbb{Z}^2$ containing all vertices $p_1, \dots, p_n$. Each point $q \in \mathbb{Z}^2 \smallsetminus Q$ yields $\mathrm{Dang}(P - q) = 0$. Using this restriction, we can swap the double sum:

$$(3.1) \qquad \mathrm{Welp}(P) := \sum_{q \in \mathbb{Z}^2} \mathrm{Dang}(P - q) = \sum_{q \in Q} \sum_{i=1}^{n} \mathrm{dang}(p_{i-1} - q, p_i - q)$$

$$= \sum_{i=1}^{n} \underbrace{\sum_{q \in Q} \mathrm{dang}(p_{i-1} - q, p_i - q)}_{=:\mathrm{welp}(p_{i-1}, p_i)}$$

Now both sides of Pick's equation look formally very similar:

$$(3.2) \qquad \mathrm{Area}(P) = \sum_{i=1}^{n} \mathrm{area}(p_{i-1}, p_i),$$

$$(3.3) \qquad \mathrm{Welp}(P) = \sum_{i=1}^{n} \mathrm{welp}(p_{i-1}, p_i).$$

Here another miracle happens: Both sums are not only equal, but termwise equal!

For any to lattice points $u, v \in Q$ in our square box $Q = \{-r, \dots, r\}^2 \subset \mathbb{Z}^2$ we show that

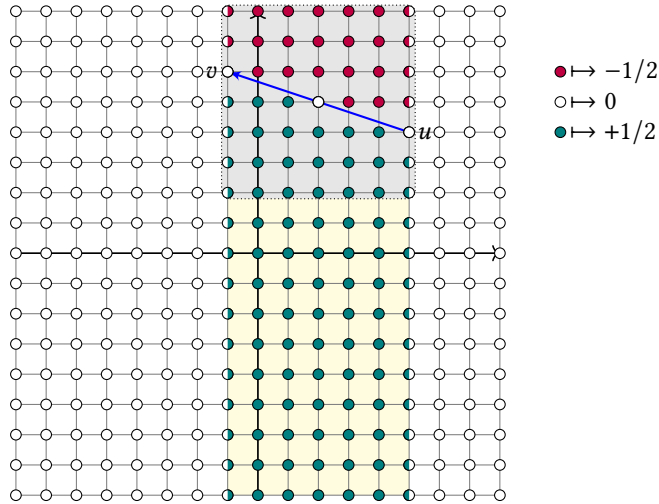$$(3.4) \qquad \mathrm{area}(u, v) = \mathrm{welp}(u, v).$$

Figure 4. Proving $\mathrm{welp}(u, v) = \mathrm{area}(u, v)$ by inspecting $q \mapsto \mathrm{dang}(u - q, v - q)$

The proof is illustrated in Figure 4 by plotting $q \mapsto \mathrm{dang}(u - q, v - q)$. The algebraic calculation proceeds as follows. We assume $v_1 < u_1$ and $u_2 + v_2 \geq 2$; the other cases are symmetric.

(1) For $q_1 < v_1$ or $q_1 > u_1$ we have $\mathrm{dang}(u - q, v - q) = 0$.
(2) The rectangle $R = \{v_1, \dots, u_1\} \times \{u + v - r, \dots, r\}$ allows the involution $q \mapsto u + v - q$. We find $\mathrm{dang}(u - (u + v - q), v - (u + v - q)) = \mathrm{dang}(v - q, u - q) = -\mathrm{dang}(u - q, v - q)$. Thus all summands cancel pairwise, and we obtain $\sum_{q \in R} \mathrm{dang}(u - q, v - q) = 0$.
(3) The remaining summands add up to $\frac{1}{2}(u_1 - v_1)(u_2 + v_2) = \mathrm{area}(u, v)$. More precisely, we find $\mathrm{dang}(u - q, v - q) = \tfrac{1}{2}$ for $q \in \{u_1 + 1, \dots, v_1 - 1\} \times \{-r, \dots, u_2 + v_2 - r - 1\}$ and $\mathrm{dang}(u - q, v - q) = \tfrac{1}{4}$ for $q \in \{u_1, v_1\} \times \{-r, \dots, u_2 + v_2 - r - 1\}$.

This proves $\mathrm{welp}(u, v) = \mathrm{area}(u, v)$ for any edge with end points $u, v \in Q$. By summing over all edges of our polygon P, we conclude $\mathrm{Area}(P) = \mathrm{Welp}(P)$, as claimed.

Remark 3.1. In this algebraic form, the proof is readily implemented in Lean, or any other proof assistant. This completes the promised formalization.

Appendix A. Axiomatic angle measure

We can use different angle measures for the theorems of Jordan and Hopf and finally Pick. In order to cover all cases simultaneously, we extract the essential properties used in the proofs:

Definition A.1 (angle measure)**.** Let $(\mathbb{K}, +, \cdot, \leq)$ be an ordered field. An *angle measure* is a map $\mu\colon \mathbb{K}^2 \times \mathbb{K}^2 \to \mathbb{K}$ satisfying the following conditions:

(1) Scaling: For all $u, v \in \mathbb{K}^2$ and $\lambda \in \mathbb{K}_{>0}$ we have $\mu(u, v) = \mu(\lambda u, v) = \mu(u, \lambda v)$.
(2) Symmetry: $\mu(v, u) = -\mu(u, v)$ and $\mu(-u, -v) = \mu(u, v)$.
(3) Addition: $\mu(u, v) = \mu(u, s) + \mu(s, v)$ for $s \in [u, v]$.
(4) Normalisation: $\mu(e_1, e_2) = 1/4$.

Examples include the euclidean angle measure ang and the discrete angle measure dang.

Appendix J. Jordan's decomposition for polygons

Jordan's decomposition for polygons can be formulated and proven using any angle measure. We work over any ordered field $(\mathbb{K}, +, \cdot, \leq)$ and continue to use the notation of the introduction. Let $P = (p_0, p_1, \ldots, p_n)$ be a polygon in $\mathbb{K}^2$ with corresponding path $\gamma\colon [0, 1] \to \mathbb{K}^2$ and image curve $C = \gamma([0, 1]) \subset \mathbb{K}^2$. Using the angle measure μ, we define the winding number

$$\mu(P) := \sum_{i=1}^{n} \mu(p_{i-1}, p_i).$$

Theorem J.1. *Let $P = (p_0, p_1, \ldots, p_n)$ be a polygon in $\mathbb{K}^2$, simply closed and positively oriented.*
(1) The complement $X := \mathbb{K}^2 \smallsetminus C = A \sqcup B$ decomposes into two connected components, namely

$$A = \{ q \in X \mid \mu(P - q) = 0 \},$$
$$B = \{ q \in X \mid \mu(P - q) = 1 \}.$$

(2) The curve C is the boundary of A and of B within the plane $\mathbb{K}^2$.
(3) At each boundary point $q \in C$, the value $\mu(P - q)$ measures the interior angle:
(3a) At each point $q \in \,]p_{i-1}, p_i[$ in the interior of an edge, we find $\mu(P - q) = 1/2$.
(3b) At each vertex $q = p_i$ we find $\mu(P - p_i) = \beta_i := \mu(p_{i+1} - p_i, p_{i-1} - p_i)$.

Remark J.2. We assume positive orientation, so that area$(P) > 0$. Geometrically this ensures that, when traveling along the path γ, the interior region B will be on the left of our curve C.

(1) The usual topological notions over $\mathbb{R}$ now aquire an algebraic flavour over $\mathbb{K}$: Two points $a, b \in X$ are *connected* in X, if they can be connected by a polygonal path within X.

(2) The fact that C is the common boundary means that each point $q \in C$ has arbitrarily close neighbours in A and in B. These can be reached by the two bisectors of the angle at q.

(3) The extension to boundary points $q \in C$ follows from (1) by cutting the corner at q. It is not only oddly satisfying but also practically useful. In particular it leads us to Pick's theorem.

A pen-and-paper proof can be established as in the classical setting, and then implemented in a proof assistant. This foundational work usually requires "augmenting the existing geometry libraries" [5]. It merits further exploration beyond the scope of the modest article.

Appendix H. Hopf's umlaufsatz for polygons

Hopf's umlaufsatz for polygons can be formulated and proven using any angle measure μ.

Theorem H.1. *Let $P = (p_0, p_1, \ldots, p_n)$ be a simply closed polygon in $\mathbb{K}^2$ and positively oriented. At each vertex p_i, cyclically indexed by integers modulo n, we define the turning angle*

$$\alpha_i := \mu(p_{i+1} - p_i, p_i - p_{i-1}).$$

Then the total tangent turning number (or umlaufzahl) adds up to $\sum_{i=1}^{n} \alpha_i = 1$.

Remark H.2. The turning angle α_i and the interior angle β_i add up to $1/2$. Summing over all vertices, we obtain $n/2 = \sum_{i=1}^{n} \alpha_i + \beta_i = 1 + \sum_{i=1}^{n} \beta_i$, whence $\sum_{i=1}^{n} \beta_i = n/2 - 1$.

The simplest case $n = 3$ is the familiar statement that, in any triangle, the interiour angles add up to $\beta_1 + \beta_2 + \beta_3 = 1/2$, correspoding to $\pi/2$ or $180°$.

Together with Jordan's theorem, we obtain the desired point count $\sum_{q \in \mathbb{Z}^2 \cap C} \mu(P - q) = J/2 - 1$.
We have used the theorems of Jordan and Hopf in order to motivate our *formulation* of Pick's lemma; our *proof*, however, is self-contained and does not logically require them.

H.1. **Formalizing Hopf's umlaufsatz.** A proof of Hopf's umlaufsatz is straight-forward for triangles. It immediately extends to convex polygons and more generally to star-shaped polygons. It seems intuitively plausible for any simply closed polygon, but in this general case a proof is far from obvious. It becomes false for non-simply closed polygons.
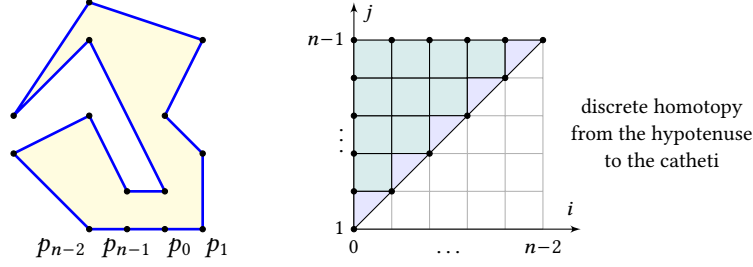

Figure 5. Proof of Hopf's theorem

Hopf first formulated his umlaufsatz [2] for simply closed paths $\gamma \colon [0,1] \to \mathbb{R}^2$ that are continuously differentiable. (He then showed that it suffices to require this condition piecewise.) He proved his theorem by an ingenious homotopy argument. For future reference, we sketch an algebraic version for polygons over any ordered field $(\mathbb{K}, +, \cdot, \leq)$, using any angle measure μ. The key idea is adapted from Hopf's original and illustrated in Figure 5.

Hopf's homotopy proof. We consider the vertices $p_1, \ldots, p_n = p_0$ with $p_i = (x_i, y_i) \in \mathbb{K}^2$. After renumbering, we can assume that y_0 is minimal. After translation we have $p_0 = (0,0)$ and $y_i \geq 0$ for all i. By inserting small edges around p_0, we can achieve $0 = y_0 = y_1 = y_{n-1} = y_{n-2}$ und $x_{n-2} < x_{n-1} < x_0 < x_1$. (This is shown on the left of Figure 5.)

For $0 \leq i < j \leq n-1$ we consider the secant $s_{i,j} := p_j - p_i$. From $s_{0,1}$ via $s_{0,j}$ to $s_{0,n-1}$ we perform a first half turn, then via $s_{i,n-1}$ to $s_{n-2,n-1}$ we perform a second half turn.

If, moreover, our polygon P is simply closed, then this movement corresponds to the umlaufzahl. More precisely, we prove the following equation by a discrete homotopy:

$$(\text{H.1}) \qquad \sum_{k=1}^{n-2} \mu(s_{k-1,k}, s_{k,k+1}) \;=\; \sum_{j=1}^{n-2} \mu(s_{0,j}, s_{0,j+1}) + \sum_{i=1}^{n-2} \mu(s_{i-1,n-1}, s_{i,n-1})$$

(1) Triangles: For $0 \leq i \leq n-2$ and $j = i+1$ we find

$$\mu(s_{i-1,i}, s_{i,i+1}) = \mu(s_{i-1,i}, s_{i-1,i+1}) + \mu(s_{i-1,i+1}, s_{i,i+1}).$$

This essentially uses that our polygon is simple, so the triangle is non-degenerate.

(2) Squares: For $0 \leq i < n-2$ and $i+1 < j \leq n-1$ we find

$$0 = \mu(s_{i,j}, s_{i,j+1}) + \mu(s_{i,j+1}, s_{i+1,j+1}) - \mu(s_{i,j}, s_{i+1,j}) + \mu(s_{i+1,j}, s_{i+1,j+1}).$$

This essentially uses that our polygon is simple, so $[p_i, p_{i+1}] \cap [p_j, p_{j+1}] = \emptyset$.

The left hand side of (H.1) is the total tangent turning number. We replace each summand using (1) and then add all terms (2). Most of the summands cancel pairwise. The remaining terms yield the right hand side of (H.1). This equals 1, as explained above.   □

Appendix W. A "water proof" plausiblity argument

In every-day mathematical communication we often only sketch the idea and appeal to some degree of informal intuition. This is especially true for geometric statements, like Pick's theorem, where we often heavily rely on our visual perception. This is usually a good thing for human learning and understanding, but may also inhibit the desire for a sound formalization.

We sketch such a visual proof, appealing to geometric intuition as a counterpart to our algebraic formalization. We then explain how the intuition can be made precise and finally translates to our algebraic viewpoint, closing this circle of ideas.
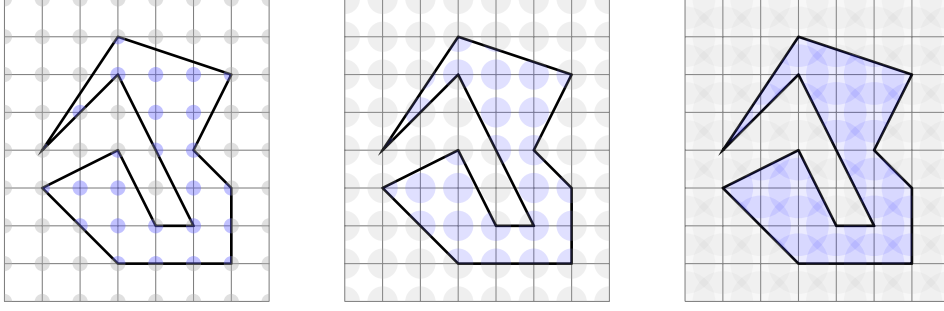
Figure 6. Πάντα ῥεῖ, everything flows. We let the water do the work.

The water proof. At each integer point $q \in \mathbb{Z}^2$ we place a unit drop of water. The water then flows evenly in all directions. Finally, the plane is uniformly covered with water, one unit of water per unit square. (To ensure finiteness, we can think of periodic boundary conditions.)

Now consider two integer points $u, v \in \mathbb{Z}^2$. Half rotation about its center $c = (u+v)/2$ defines the involution $\rho \colon z \mapsto u + v - z$. This reverses the edge and thus the flow. Since ρ maps the lattice $\mathbb{Z}^2$ to itself, the water that flows from the point $q \in \mathbb{Z}^2$ over $[u, v]$ is compensated by the water flowing from the point $\rho(q) \in \mathbb{Z}^2$ over $[u, v]$. The net flow over the edge $[u, v]$ is zero.

This shows that the total amount of water in B never changes. At the start it is the share of drops that fall into B. At the end it equals the surface area of B. We thus obtain:

$$\mathrm{vol}_2(B) = I + \sum_{q \in \mathbb{Z}^2 \cap C} \beta(q)/2\pi$$

Here $\beta(q)$ is the interior angle at the boundary point $q \in C$. So our physical intuition, or gedankenexperiment, leads us to Pick's lemma 2.5 using the euclidean angle measure.

We once again invoke Hopf's umlaufsatz: the sum of turning angles is $\sum_{q \in C}[\pi - \beta(q)] = 2\pi$, so we arrive at Pick's theorem $\mathrm{vol}_2(B) = I + J/2 - 1$.   □

Remark W.1. This "water proof" is a wonderous example of a physical plausibility argument. To many people is seems convincing because it appeals to our physical experience and our tried-and-tested understanding of the world. But is it a proof, really? Can we actually understand the physical world? It is easy to visualize, intuitively, but hard to formalize, rigorously.

We can give the physical argument more mathematical substance. First, we want to ensure finiteness of our experimentation table, so instead of the entire plane $\mathbb{R}^2$ we consider a square $[-r, +r]^2$ with sufficiently large $r \in \mathbb{N}$ and periodic boundary conditions. Geometrically, this is a flat torus $T = \mathbb{R}^2/2r\mathbb{Z}^2$, and we keep all necessary symmetries: translations and reflections.

Second, we specify the properties of our idealized water drops. For each time $t \in \,]0, 1]$ let μ_t be a continuous probability measure, ending with the uniform (Haar) measure μ_1 on T, so that $4r^2\mu_1(B)$ is the area of our inner region. Now we observe the quantity

$$f(t) := \sum_{q \in \mathbb{Z}^2/2r\mathbb{Z}^2} \mu_t(B - q).$$

We have $f(1) = \mathrm{vol}_2(B)$. For each $t > 0$ we assume μ_t to be symmetric with respect to $z \mapsto -z$ and uniform on its support. Assuming that μ_t varies continuously with t, the above symmetry argument suggests that f is constant. At the other end we look at the limit for $t \searrow 0$:

Example W.2. As illustrated in Figure 6, for small t we set $\mathrm{supp}\,\mu_t = D(0, t)$ to be the disc of radius t around the origin 0. For sufficiently small t thus obtain $f(t) = \sum_{q \in \mathbb{Z}^2} \mathrm{Ang}(P - q)$.

This recovers Pick's lemma 2.5 using the euclidean angle measure.

Example W.3. Alternatively, for small t we consider set $\mathrm{supp}\,\mu_t = [-t^2, t^2] \times [-t, t]$ to be a thin vertical rectangle. In the limit $t \searrow 0$ we thus obtain $f(t) \to \sum_{q \in \mathbb{Z}^2} \mathrm{Dang}(P - q)$.

This recovers Pick's lemma 2.7 using the discrete angle measure.

Remark W.4. This tale can be recounted by replacing water with heat. This is appealing for a physically inclined audience familiar with the heat equation or diffusion. Moreover, this provides

an explicit model of the flow as partial differential equations, this time for $t \in]0, \infty[$. Within the chosen model, we can then prove that f is indeed constant and calculate the limit for $t \searrow 0$.

Remark W.5. Another variant is to think of reflective walls. Geometrically, we can take two copies of $B \cup C$ and glue them together along their common boundary C. This produces a closed surface S homeomorphic to the sphere, with volume $\mathrm{vol}_2(S) = 2\,\mathrm{vol}_2(B)$ and Euler characteristic $\chi(S) = 2$. The vertices provide the curvature, as in the Gauß–Bonnet theorem.

Appendix Q. Quotes

Pick's theorem has been formally implemented, with considerable effort, first in 2011 by John Harrison in HOL Light [4], then in 2024 by Sage Binder and Katherine Kosaian in Isabelle [5].

> The proof turned out to be somewhat harder work than expected. [...] Charming and surprising as Pick's theorem is, one would normally consider it a very straight-forward result [...]. What accounts for the difficulty of formalizing Pick? Of course, it could be just a reflection of our own ineptitude. However we prefer to believe that there is an intrinsic difficulty in formalizing many geometric proofs in stating and proving in a formal way various properties that seem intuitively obvious 'by eye'. (John Harrison [4])

We close with a heartwarming anecdote concerning practical applicability:

> Some years ago, the Northwest Mathematics Conference was held in Eugene, Oregon. To add a bit of local flavor, a forester was included on the program, and those who attended his session were introduced to a variety of nice examples which illustrated the important role that mathematics plays in the forest industry. One of his problems was concerned with the calculation of the area inside a polygonal region drawn to scale from field data obtained for a stand of timber by a timber cruiser. The standard method is to overlay a scale drawing with a transparency on which a square dot pattern is printed. Except for a factor dependent on the relative sizes of the drawing and the square grid, the area inside the polygon is computed by counting all of the dots fully inside the polygon, and then adding half of the number of dots which fall on the bounding edges of the polygon. Although the speaker was not aware that he was essentially using Pick's formula, I was delighted to see that one of my favorite mathematical results was not only beautiful, but even useful. (Duane DeTemple, cited in [3])

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