The Trigonometric Fractals

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May 11, 2019

The inscribed circle problem involves placing a circle in a regular N-sided polygon. A set of N circles can be placed between this central circle and each vertex. This pattern can continue indefinitely by placing successive circles between the previous circle and the vertex. This article provides the solution for the radii of each circle, for any regular polygon.

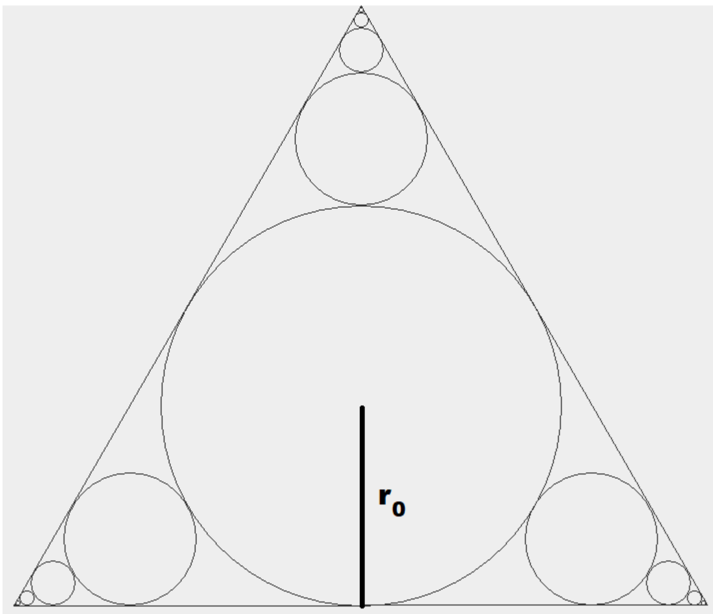
We first tackle, the triangle ($N = 3$), followed by the square ($N = 4$) from which we determine a generalized procedure for all N. The final expression simplifies and inspires a second pass over the derivation for a more intuitive explanation. This makes tackling the *excribed* circle problem much easier, where the polygon is instead place within the circle, and it too is solved. We will find that the ratios of sucessive radii are associated with tan, and sin, respectively. This implies there is a third cos-fractal, which I have not been able to find, perhaps the reader can attempt to try to construct it!

A very useful quantity is the interior angle of a N-gon and is given by,

$$\theta = \frac{N-2}{N}\pi. \tag{1}$$

1 N=3

We will begin with $N = 3$, the triangle as it is the simplest polygon. With the interest of generalizing, we choose our standard unit to be the radius of the central circle, r_0.



In general, we start by determining the relationship between the initial and adjacent radius. There are many different approaches to this problem. The path you take will impact the simplicity of this problem. This was my first attempt. In the case of the triangle, naturally we turn to the inradius equation for a triangle (derived from *Heron's Formula*):

$$r = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s}, s = (a+b+c)/2, \tag{2}$$

where a, b, and c are the side lengths of the triangle. In our case, our equilateral triangle gives $a = b = c$, giving $s = 3a/2$. Now,

$$
\begin{aligned}
r &= \frac{\sqrt{s(s-a)^3}}{s} \\
&= \frac{\sqrt{(3a/2)(\frac{3a}{2}-a)^3}}{(3a/2)} \\
&= \sqrt{2}a\frac{\sqrt{3(\frac{1}{2})^3}}{3} \\
&= a\frac{\sqrt{2}}{\sqrt{2^3}}\frac{1}{\sqrt{3}} \\
r &= \frac{a}{2\sqrt{3}}
\end{aligned}
$$

Using Equation 1 we determine that the height, h, of the triangle in terms of the inscribed central circle radius is,

$$
\begin{aligned}
h &= a\sin\left(\frac{N-2}{N}\pi\right), N = 3 \\
&= 2\sqrt{3}r\sin\left(\frac{\pi}{3}\right) \\
&= 3r
\end{aligned}
$$

It is easy to see that the first non-central circle is a distance $h - 2r = r$ away from the vertex. Drawing a horizontal line at the base of this circle gives us a similar triangle to the original. Well, we just worked out what the radius was given the height of the enclosing triangle. So, our first radius, r_1 can be determined,

$$
\begin{aligned}
h_0 &= 3r_0 \\
h_1 &= 3r_0 - 2r_0 \\
&= r_0 \\
r_1 &= r_0/3
\end{aligned}
$$

Visually, we can see that this pattern can be repeated,

$$
r_n = r_{n-1}/3
$$

This recurrence relation is very easy to solve and leads to,

$$
\boxed{r_n = \left(\frac{1}{3}\right)^n r_0}
$$

2 N=4

We tackle the square in a similar fashion to the triangle. The idea again is to figure out the distance between the central circle and a vertex. In this case, it is very simple. We have a square of side length $2r_0$, so the distance from the center to a vertex, L_0, is,

$$
\begin{aligned}
L_0 &= \frac{\sqrt{(2r_0)^2 + (2r_0)^2} - 2r_0}{2} \\
L_0 &= (\sqrt{2}-1)r_0
\end{aligned}
$$

With this, we can form a triangle of height L_0 that closes off the corner of any given vertex. Unlike the $N = 3$ case however, this forms an isosceles triangle. Since for $N = 4$, $\theta = \pi/2$, we have that $a = b = c/\sqrt{2}$. Plugging this into Equation 2 gives

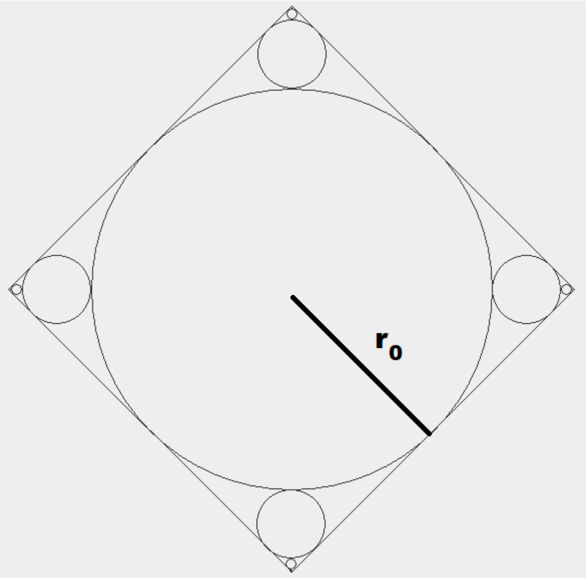
$$\begin{aligned}
s &= a(2+\sqrt{2})/2 \\
s-a &= a/\sqrt{2} \\
s-c &= (2-\sqrt{2})a/2 \\
r &= \frac{\sqrt{s(s-a)^2(s-c)}}{s} \\
r &= \frac{a^2}{2}\frac{\sqrt{(2+\sqrt{2})(2-\sqrt{2})/2}}{s} \\
r &= a\frac{\sqrt{(4-2)/2}}{2+\sqrt{2}} \\
r &= \frac{a}{2+\sqrt{2}}
\end{aligned}$$

We can now calculate the radius given the side length, but above we calculated the height, L, so we need that $a = Lsec(\theta/2)$, giving

$$\begin{aligned}
r &= \frac{h\sec(\theta/2)}{2+\sqrt{2}}, \theta = \pi/2 \\
r_1 &= r_0\sqrt{2}\frac{\sqrt{2}-1}{2+\sqrt{2}} \\
r_1 &= r_0\frac{2-\sqrt{2}}{2+\sqrt{2}}
\end{aligned}$$

Just like with $N = 3$, this process can be repeated indefinitely giving,

$$\boxed{r_n = \left(\frac{2-\sqrt{2}}{2+\sqrt{2}}\right)^n r_0}$$



3 Generalized

Now for the generalization, the central idea in both proofs was to determine the height, L_1 of the triangle enclosed between the vertex and central circle. In general, this is the difference between the distance from the center to a vertex, R, and the radius, r,

$$L = R - r.$$

These two can be related according to two well-known area formulas,

$$A = \frac{1}{2} N a r,$$

and,

$$A = \frac{R}{2} N a \sqrt{1 - \frac{a^2}{4R^2}}.$$

Equating these gives,

$$\begin{aligned} r &= R\sqrt{1 - \frac{a^2}{4R^2}} \\ \left(\frac{r}{R}\right)^2 + \frac{a^2}{4R^2} &= 1 \\ r^2 + \frac{a^2}{4} &= R^2 \end{aligned}$$

However, we would like to eliminate the side length, a. A simple triangle reveals that the side length is related to the outer radius as,

$$a = 2R\sin(\pi/N).$$

This gives,

$$\begin{aligned} r^2 + R^2\sin^2(\pi/N) &= R^2 \\ r^2 &= R^2\left(1 - \sin^2\left(\frac{\pi}{N}\right)\right) \\ r &= R\cos\left(\frac{\pi}{N}\right) \implies R = r\sec\left(\frac{\pi}{N}\right) \end{aligned}$$

Finally, we arrive at,

$$L = \left(\sec\left(\frac{\pi}{N}\right) - 1\right) r.$$

Given that height, we go back to Equation 2 for the isosceles triangle. A height of L gives,

$$\begin{aligned} a &= L\sec(\theta/2) \\ b &= L\sec(\theta/2) = a \\ c &= 2L\tan(\theta/2) \end{aligned}$$

Now,

$$\begin{aligned}
s &= \frac{L}{\cos(\theta/2)}(1+\sin(\theta/2)) \\
s-a &= \frac{L}{\cos(\theta/2)}\sin(\theta/2) \\
s-c &= \frac{L}{\cos(\theta/2)}(1-\sin(\theta/2)) \\
r &= \sqrt{s(s-a)^2(s-c)}/s \\
&= \sqrt{\frac{L}{\cos(\theta/2)}(1+\sin(\theta/2))\frac{L^2}{\cos^2(\theta/2)}\sin^2(\theta/2)\frac{L}{\cos(\theta/2)}(1-\sin(\theta/2))}/s \\
&= \frac{L\sin(\theta/2)\sec(\theta/2)}{\sqrt{2}}\frac{\sqrt{(1+\sin(\theta/2))(1-\sin(\theta/2))}}{1+\sin(\theta/2)} \\
&= L\sin(\theta/2)\sec(\theta/2)\frac{\cos(\theta/2)}{1+\sin(\theta/2)} \\
&= \frac{L\sin(\theta/2)}{1+\sin(\theta/2)} \\
&= \frac{L}{1+\csc(\theta/2)} \\
&= \frac{L}{1+\csc(\frac{N-2}{2N}\pi)} \\
&= \frac{L}{1+\csc(\frac{\pi}{2}-\frac{\pi}{N})} \\
r &= \frac{L}{1+\sec(\frac{\pi}{N})}
\end{aligned}$$

Plugging in for L gives,

$$\begin{aligned}
r_1 &= \frac{\sec\left(\frac{\pi}{N}\right)-1}{\sec\left(\frac{\pi}{N}\right)+1}r_0 \\
&= \frac{1-\cos\left(\frac{\pi}{N}\right)}{1+\cos\left(\frac{\pi}{N}\right)}r_0 \\
&= \frac{2\sin^2\left(\frac{\pi}{2N}\right)}{2\cos^2\left(\frac{\pi}{2N}\right)}r_0 \\
&= \tan^2\left(\frac{\pi}{2N}\right)r_0.
\end{aligned}$$

In the third equality, the half-angle formulas weere used. And just as before, this process can be done indefinitely yielding,

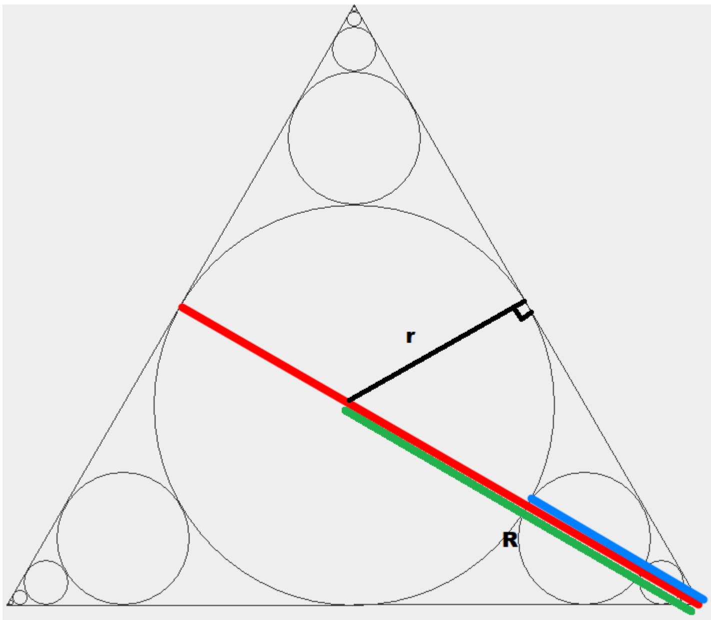
$$\boxed{r_n = \left(\tan^2\left(\frac{\pi}{2N}\right)\right)^n r_0} \tag{3}$$

as the n'th radius inscribed in an $N-$sided polygon.

4 Simplified Proof

As with most proofs, the first iteration can be very clunky and difficult to workout even though the final result may be simple and obvious in retrospect. In this case, motivated by the general solution, we can see an simple way to this answer.

Inspired by the general solution, we look for the ratio found in Equation 3.



In Figure 4, we can see that the ratio between the blue line, and the red line determines the ratio between the enclosing triangle and equivalently the ratio of the inscribed circles.

First, the green line, R, is given by $r = Rcos(\pi/N)$, which we found earlier, but is now very easy to see. This tells us that the length of the red line is,

$$L_{red} = r(1 + \sec(\pi/N))$$

and the blue is,

$$\begin{aligned} L_{blue} &= L_{red} - 2r \\ &= r(\sec(\pi/N) - 1) \end{aligned}$$

Their ratio, γ gives,

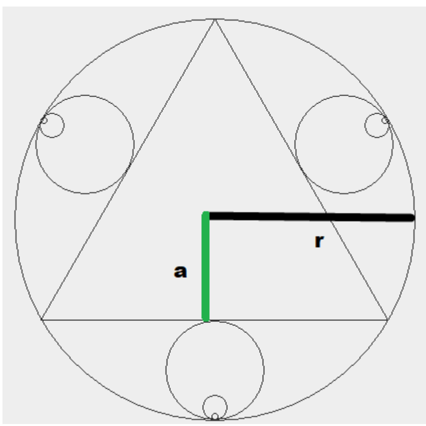
$$\gamma = \frac{\sec(\pi/N) - 1}{\sec(\pi/N) + 1}$$

which can be arranged to match Equation 3,

$$\gamma = \tan^2\left(\frac{\pi}{2N}\right).$$

5 Excribed Circle

Using this approach, it become easy to tackle the twin problem where the circle encloses the polygon.



The height of the circle is (by definition), $2r$. And the height for the next iteration is $2r-(r+a)=r-a$. We can determine a to be,

$$a = rcos(\theta)$$

This gives us,

$$\begin{aligned}\gamma &= \frac{1-\cos\left(\frac{\pi}{N}\right)}{2}\\ &= \sin^2\left(\frac{\pi}{2N}\right),\end{aligned}$$

by using the half-angle formula. As always, this is done indefinitely yeilding,

$$\boxed{r_n = \left(\sin^2\left(\frac{\pi}{2N}\right)\right)^n r_0}.$$

6 Conclusion

We approached the inscribed problem by solving the $N=3$ and $N=4$ cases explicitly, then extending the procedure to general N. We found a geometric series for the radii of the inscribed circles of the form,

$$r_n = \gamma_N^n r_0.$$

Along with this longer derivation, we found that for the inscribed circles,

$$\gamma_N = \tan^2\left(\frac{\pi}{2N}\right).$$

Upon seeing the reasoning for the proble, we were inspired to revisit the derivation, with the hope of a simpler approach. This reduced the length of the derivation by a quarter. That simplified derivation was used to solve the exscribed circle problem giving the radii as,

$$\gamma_N = \sin^2\left(\frac{\pi}{2N}\right).$$

Now we have two ratios associated with tan, and sin. This heavily implies that there exists a third fractal for which the ratio is associated with cos. However I have been unable to find it - *erhem*, I mean, this is trivial to find, and is left as an exercise for the reader.