B.M.I.T.

* Probability Distribution *

- The probability distribution is usually represented through either a table or a graph (usually a histogram.)

Types of PD

Discrete Probability Distribution → describe by using Probability Mass Function (PMF)

$$\boxed{\begin{array}{l} \sum P(x) = 1 \\ P(x) \geq 0 \end{array}}$$

Continuous Probability Distribution → described by using Probability density fun^n (PDF)

(1) Discrete Probability Distribution :-

* Probability Mass fu^n

E.g. (1) Random Experiment : Tossing of 3 coin

Sample space : S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}

Random variable (r.v) X : number of heads

X = 0, 1, 2, 3

Probability Distribution :

X	0	1	2	3
P(X=x)	1/8	3/8	3/8	1/8

$$\sum P(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$

$$\boxed{\sum P(x) = 1}$$

Hence, this is PMF

Eg verify whether the following fu^n can be regarded as PMF

i) $P(x) = 1/5 \quad x = 0, 1, 2, 3, 4$

$\quad = 0 \quad$ otherwise

→

x	0	1	2	3	4
$P(x)$	1/5	1/5	1/5	1/5	1/5

Here, $P(x) \geq 0$

$\sum P(x) = \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5} + \frac{1}{5}$

$\boxed{\sum P(x) = 1}$

Hence, It can be regarded as a P.M.F.

① $P(x)$ should not be negative

② $\sum P(x)$ should be 1

ii) $P(x) = \frac{x-2}{5}, \quad x = 1, 2, 3, 4, 5$

$\quad = 0, \quad$ otherwise

$P(x=1) = 1-2/5 = -1/5$

This is negative probability we got so, This can not be regarded as a PMF.

iii) $P(x) = \frac{x^2}{30} \quad x = 0, 1, 2, 3, 4$

$\quad = 0 \quad$ otherwise

→

x	0	1	2	3	4
$P(x)$	0	1/30	4/30	9/30	16/30

$P(0) = 0^2/30 = 0$

$P(1) = 1^2/30 = 1/30$
$P(2) = 2^2/30 = 4/30$
$P(3) = 3^2/30 = 9/30$
$P(4) = 4^2/30 = 16/30$

All probability are positive, so 1st condition is satisfied. $P(x) \geq 0$

$\Sigma P(x) = 0 + 1/30 + 4/30 + 9/30 + 16/30$

$\boxed{\Sigma P(x) = 1}$

2nd condition is also satisfied, $\Sigma P(x)$ is 1 //
Hence, This can be regarded as PMF

* <u>Cammulative Distributive fu^n</u> (CDF)

- The probability given by fu^n of having a value less than or equal to the argument of the fu^n for a random variable is called the cumulative Distribution fu^n

$$F(x) = P(X \leq x)$$

- CDF describes any kind of probability distribution. It can be either continuous or discrete.

For all $a <= b$
$P(X <= b) = P(X <= a) + P(a < X \leq b)$
$F(b) = F(a) + P(a < X \leq b)$
$P(a < X \leq b) = F(b) - F(a)$

- If for any discrete random variable X, we know the PMF, then using it we can calculate the CDF of X.

Example: sample space = {HHH, HHT, HTH, THH, HTT, THT, TTH, TTT}

PMF

Random variable x = Number of Head

No. of Head (X)	0	1	2	3
P(x)	1/8	3/8	3/8	1/8

lets ~~cal~~ see, if it satisfies PMF conditions

$$\Sigma P(x) = \frac{1}{8} + \frac{3}{8} + \frac{3}{8} + \frac{1}{8}$$

$$\boxed{\Sigma P(x) = 1} \checkmark$$

$$\boxed{P(x) \geq 0} \checkmark$$

Now, lets calculate cummulative distribution

$$F(x) = \begin{cases} 1/8 & , \text{when } x \leq 0 \\ 1/8 + 3/8 = 4/8 & , \text{when } x \leq 1 \\ 1/8 + 3/8 + 3/8 = 7/8 & , \text{when } x \leq 2 \\ 1/8 + 3/8 + 3/8 + 1/8 = 1 & , \text{when } x \leq 3 \end{cases}$$

This is called cummulative distribution fun.

Types of Discrete Distributions

1. Discrete Uniform Dist^n
2. Binomial Distribution
3. Negative Binomial Distribution
4. Poisson Distribution.

1. Discrete Uniform Distribution
   - i) Fixed number of outcomes
   - ii) constant probability for each outcome

e.g. For a die-roll, a random variable
X = outcome of a die-roll, ~~samp~~ sample space is
$x = \{1, 2, 3, 4, 5, 6\}$
Each of the 6 outcomes is equally likely to happen.

X	1	2	3	4	5	6
$P(x)$	1/6	1/6	1/6	1/6	1/6	1/6

Hence, X has a discrete uniform distribution.
$f(x) = 1/6 = 0.166$

e.g. Sahi sells 5 diff flavors of ice crem namely, vanalla, chocalate, strawberry, Black current, Pistachio.
Lets suppose that the next customer arriving purchases only 1 icecreme is most likely to be selected among 5 available flavors?

$S = \{$vanialla, chocolate, strawberry, black current, Pistachio$\}$
$X = \{1, 2, 3, 4, 5\}$

X	1	2	3	4	5
$P(x)$	1/5	1/5	1/5	1/5	1/5

X has a discrete uniform distribution with probability of $1/5 = 0.2$ for each value in $\{1, 2, 3, 4, 5\}$

② Binomial Distribution :- Bernoulli Trials

- The characteristics of a Binomial Random Experiment are as follows

1. Fixed number of trials
2. A success or a failure is the outcome of a single trial
3. All the trials are independent of each other
4. All the trials are same & has equal probability to occur.

The trials that satisfy the above four conditions are known as Bernoulli trials.

- A random variable X that equals the number of trials that result in success (x) from among the n Bernoulli trials, is called a binomial random variable with parameters $0<p<1$ and $n = 1, 2, 3.$

suppose,

- $n \rightarrow$ Total no. of trials
- $x \rightarrow$ No. of trials that result in success
- $n-x \rightarrow$ No. of trials that result in failure
- $p \rightarrow$ Probability of success
- $q = 1-p \rightarrow$ Probability of failure

- The probability of x successes in n trials in a random experiment is defined by the Binomial distribution.

$C(n, x) = {}^nC_x$

$$\boxed{C(n,x) = \frac{n!}{x!(n-x)!}}$$

- The probability of one of the ways the event of interest i.e. a success x can occur $p^x q^{n-x}$

$$f(x) = P(X=x) = C(n,x)p^x(1-p)^{n-x}$$

$$= C(n,x)p^x q^{n-x}$$

where,
$C(n,x)$ = Binomial coefficient
n = Total no. of Bernoulli trials
x = 0,1, ... n = no. of successes
$n-x$ = no. of failures
p = Probability of a success
q = $1-p$ = probability of a failure

Eg. 1) A balanced, six-sided die is called 3 times. What is the probability a 5 comes up exactly twice ?

→ Success : Rolling a 5
Failure : Rolling anything but a 5

Let, x represent the no. of fives in 3 rolls
x has a binomial distribution with $n=3$
& $p = 1/6$

$$P(X=x) = C(n,x)p^x(1-p)^{n-x}$$

$n = 3$, $x = 2$, $n-x = 3-2 = 1$

$$P(x=2) = C(n,x)\, p^{x} (1-p)^{n-x}$$

$$= C(3,2)\left(\tfrac{1}{6}\right)^{2}\left(1-\tfrac{1}{6}\right)^{3-2}$$

$$= \left(\frac{3!}{2!\,(3-2)!}\right)\left(\frac{1}{6}\right)^{2}\left(1-\frac{1}{6}\right)^{3-2}$$

$$= \frac{6}{2\times 1}\left(\frac{1}{6}\right)^{2}\left(1-\frac{1}{6}\right)^{3-2}$$

$$= \left(\frac{\cancel{6}^{\,3}}{\cancel{2}_{\,1}}\right)\cdot\left(\frac{1}{36}\right)\cdot\left(1-\frac{1}{6}\right)^{1}$$

$$= \left(\frac{\cancel{3}^{\,1}}{\cancel{36}}\right)\cdot\left(\frac{6-1}{6}\right)$$

$$= \left(\frac{1}{12}\right)\left(\frac{5}{6}\right)$$

$$\boxed{P(x=2) = 0.0694}$$

E.g ② According to statistics canada life tables, the probability a randomly selected 90 year old canadian male survives for at least another year is approximately 0.82

- If twenty 90-year old canadian males are randomly selected, what is the probability exactly 18 survive for atleast another year ?

→

Success : The man survives for at least one year
Failure : The man dies within one year.

let X represent the no. of men that survive for at least one year.

$n = 20$

~~$p = 0.82$~~

$x = 18$

$p = 0.82$

$$P(x=18) = C(n, x)\, p^{x} (1-p)^{n-x}$$

$$= C(20, 18) (0.82)^{18} \cdot (1-0.82)^{20-18}$$

$$= \left[\frac{20!}{18!\,(20-18!)}\right] \cdot (0.0280) \cdot (0.0324)$$

$$= \frac{\cancel{20!}}{(\cancel{18!}) * (\cancel{2!})}$$

$$= \frac{5!}{9!} \cdot (0.0280) \cdot (0.0324)$$

$$= \frac{120}{362880} \cdot (0.0280) \cdot (0.0324)$$

$$\boxed{P(x=18) = 0.173}$$

Mean $\mu_x = n * P$

variance $\sigma^2 x = n * P * (1-P)$

std deviation $(\sigma x) = \sqrt{n * p * (1-P)}$

③ Negative Binomial Distribution :-

* **PMF of negative binomial distribution :-**

$$f(x) = P(X = x) = C(x-1, r-1)\, p^r (1-p)^{x-r}$$

$$= C(x-1, r-1)\, p^r q^{x-r}.$$

where, $x = r, r+1, r+2$

p = Probability of success

$q = 1-p$ = probability of a failure

- The experiment should be performed a min. of r times to get r successes, and resultant X's range will be betn r and ∞.
- We know that for a binomial random variable
  - No. of trials is fixed
  - No. of successes is unknown (random)
- So, for a negative binomial random variable also, the no. of successes is fixed & no. of trials is random. So it could be called as the opposite or -ve of binomial random variable

for e.g.) A coin is tossed repeatedly until heads comes up for the sixth time.
what is the probability this happens on the 15th toss?

- Here, no. of trials is not fixed but No. of success is fixed
  So, this can be solved with -ve binomial distb
  $n = ?$, $r = 15^{th}$

- The geometric distribution is the distribution of the no. of trials needed to get the 1st success in repeated independent Bernoulli trials
- The negative binomial distribution is the distribution of the ~~~~ no. of trials needed to get the r^{th} success.

Negative Binomial Distribution	Binomial Distribution
i) The negative binomial distribution is the distⁿ of the no. of trials needed to get fixed no. of successes.	The binomial distribution is the distribution of the no. of successes in a fixed no. of independent Bernoulli trials.
(random variable) - X = No. of trials	- Random variable (X) = no. of successes
r = fixed no. of successes.	- n = Fixed no. of trials

E.g - A person conducting telephone surveys must get 3 more completed surveys before their job is finished

- On each randomly dialed number, there is a 9% chance of reaching an adult who will complete the survey.
- What is the probability the 3rd completed survey occurs on the 10th call?

$p = 9\% = 0.09$

$r = 3$

$x = 10$

$$P(X = x) = \binom{x-1}{r-1} p^{r} (1-p)^{x-r}$$

$$P(X=10) = \binom{10-1}{3-1} (0.09)^{3} (1-0.09)^{10-3}$$

$$= C(9,2) (0.09)^{3} \cdot (1-0.09)^{10-3}$$

$$= \frac{9!}{2!(7!)} (0.09)^{3} \cdot (1-0.09)^{10-3}$$

$$\boxed{P(X=10) = 0.01356}$$

- For the r^{th} success to occur on the x^{th} trial :

i) The first $x-1$ trials must result in $r-1$ success

$$\binom{x-1}{r-1} p^{r-1} (1-p)^{(x-1)-(r-1)}$$

ii) The x^{th} trial must be a success, which has a probability of p.

- The probability of r^{th} success occurs on the x^{th} trial is

$$P(X=x) = p \times \binom{x-1}{r-1} p^{r-1} (1-p)^{(x-1)(r-1)}$$

$$\boxed{P(X=x) = \binom{x-1}{r-1} p^{r} (1-p)^{x-r}}$$

Mean $\mu = \frac{r}{p}$

variance $\sigma^2 = \frac{r(1-p)}{p^2}$

(4) Poisson Distribution :-

The poisson probability mass function.

$$\boxed{P(X=x) = \frac{\lambda^{x} e^{-\lambda}}{x!}}$$

- We have interval of real numbers, let's assume their counts (i.e. success or an outcome of interest, x) occuring randomly throughout the interval.
- If we say that the interval could be further partitioned into sub-intervals of length small enough such that is possesses the following characteristics for each sub-intervals.

i) P(success > 1) is almost zero
ii) P(success = 1) would be the same for every subinterval and its value would be proportional to the sub-interval length
iii) n(success) in each sub-interval would be independent of other sub-intervals.

- If any random expriment satisfies the given 3 conditions would be called a Poisson process

~~Historically~~

- The random variable X that equals the no. of counts in the interval of a poisson is called Poisson random variable with parameter $0<\lambda$, and the PMF of X can be defined as follows.

where, $x = 0, 1, 2, 3,$ = no. of counts i.e. no. of total successes in the provided interval

λ = successes count we are expecting in the provided interval

e = 2.71828 i.e Euler's no