Chapter 4

2D TRANSFORMATION

2D Transformation means a change in either position or orientation or size or shape of graphics objects like line, circle, arc, ellipse, rectangle, polygon, polylines etc. Now all such graphics objects are constructed with finite no of points – which are theoretically[1] the simplest geometric objects or the basic building blocks that cannot be defined in terms of anything simpler. A point itself has no geometric or analytic properties other than its position or place. It has no size, orientation, length, area or volume. It has no inside or outside nor any other geometric characteristics, but from the point of view of object transformation in computer graphics, these points matter the most. Because all sorts of object transformation – scaling, translation, rotation, reflection etc. results from simultaneous transformation or change of position of all or some of the constituent points of an object. By manipulating the matrices which defines the points, we control the position of such points.

4.1 REPRESENTATION OF POINT AND OBJECT

A point's coordinates can be expressed as elements of a matrix. Two matrix formats are used: one is row matrix and other is column matrix format.

For a 2D point (x, y) the row matrix format is a 1-row, 2-column matrix $[x\ y]$.

For a 3D point (x, y, z) it's a 1-row, 3-column matrix $[x\ y\ z]$. The same points in the column matrix format can be expressed as 1-column, 2-row matrix, $\begin{pmatrix} x \\ y \end{pmatrix}$ and 1-column 3-row matrix $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ respectively.

In this book we will follow column matrix convention. As all graphic objects are built with finite no. of points each such object can be uniquely represented by a optimum no. of definition points lying on the object. Though there may be thousands of different adjacent points on a line between its two end points, this particular line segment can be uniquely expressed by the end points only. For example in column matrix format, $\begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$ may represent one and only one line between points (1, 2) and (3, 4).

[1] Though theoretically points have no dimension, the points plotted on graph paper or, LC displays have a finite size - however small the size may be. In the graphic display devices like the computer screen such points are nothing but illuminated phosphor dots or pixels. Higher the resolution of the screen, better it approximates the theoretical point.

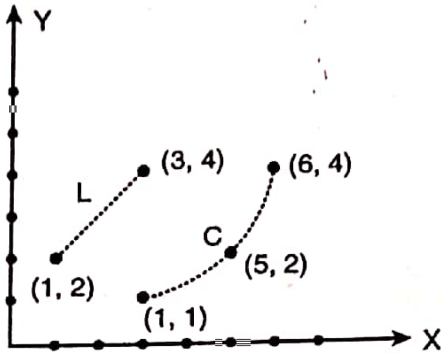

Fig. 4.1 A closer look at the line will reveal that it is basically a series of adjacent points, maintaining the straight line connectivity between the definition points (1, 2) & (3, 4). Similarly the arc segment 'C' is uniquely represented by 3 points (1, 1) – (5, 2) – (6, 4) in column-matrix format $C = \begin{pmatrix} 1 & 5 & 6 \\ 1 & 2 & 4 \end{pmatrix}$

A triangle having vertices at (1, 1, 1), (3, 5, 5) and (5, 4, 6) in 3D space (ref. Fig. 4.3) can be represented uniquely by

$$\begin{pmatrix} 1 & 3 & 5 \\ 1 & 5 & 4 \\ 1 & 5 & 6 \end{pmatrix}$$

A 5 × 3 2D rectangle having its lower left vertex at (1, 1) in Fig. 4.2 can be represented by

$$\begin{pmatrix} 1 & 6 & 6 & 1 \\ 1 & 1 & 4 & 4 \end{pmatrix}$$

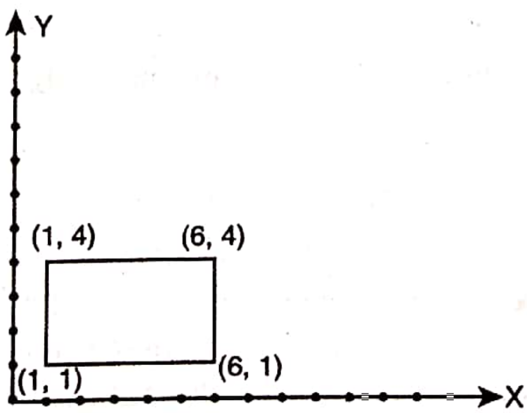

Fig. 4.2

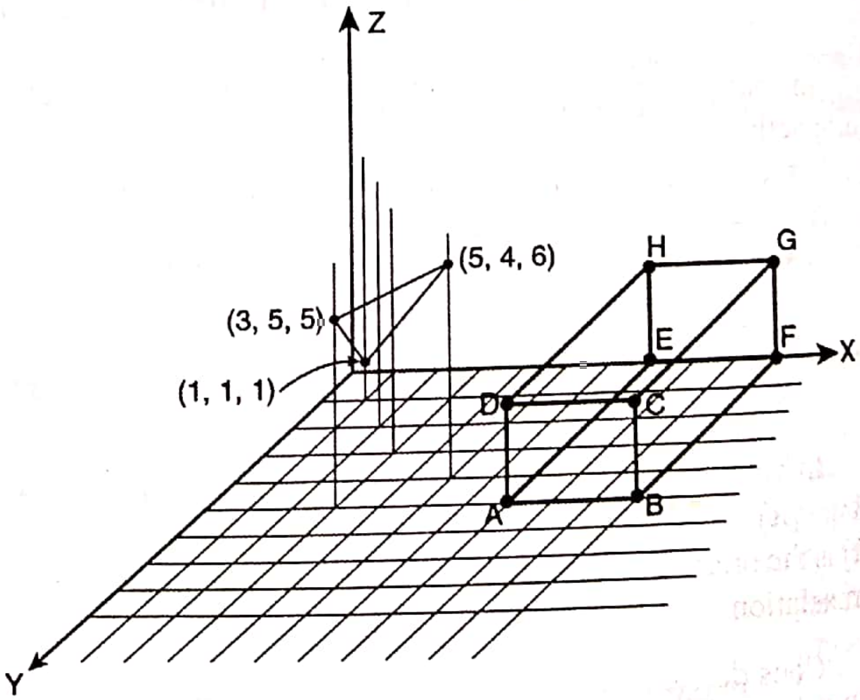

Fig. 4.3

The rectangular parallelopiped $ABCDEFGH$ in 3D space (ref. Fig. 4.3) may be defined by the matrix

$$\begin{array}{cccccccc} \begin{pmatrix} 7 & 10 & 10 & 7 & 7 & 10 & 10 & 7 \\ 5 & 5 & 5 & 5 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3 & 3 & 0 & 0 & 3 & 3 \end{pmatrix} \\ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow\ \ \downarrow \\ A\ \ B\ \ C\ \ D\ \ E\ \ F\ \ G\ \ H \end{array}$$

Now in the following sections, we will see how the coordinates of these object definition points changes to represent the transformed object.

4.2 TRANSLATION

Let us think of a point P in a 2D plane. Assume also that the current position or location of P is depicted by its coordinate (x, y) w.r.t. a reference frame.

Now if we force P to move Δx distance horizontally and at the same time Δy distance vertically then the changed location of P becomes $(x+\Delta x, y+\Delta y)$.

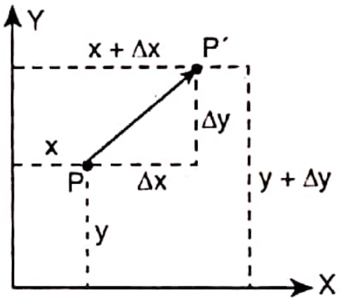

Fig. 4.4 Translation of a point

In terms of object transformation we can say that the original point object $P(x, y)$ has been translated to become P' $(x'\ y')$ and amount of translation applied is the vector $\overrightarrow{PP'}$, where $|\overrightarrow{PP'}| = \sqrt{(\Delta x)^2 + (\Delta y)^2}$

Vectorially we can express this transformation as, $P' = P + \overrightarrow{PP'}$

Algebraically, $x' = x + \Delta x$

$y' = y + \Delta y$

In matrix formulation the above relation can be more compactly expressed as,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix} \qquad (1)$$

$\begin{pmatrix} \Delta x \\ \Delta y \end{pmatrix}$ is called *Translation vector* or *Shift vector.*

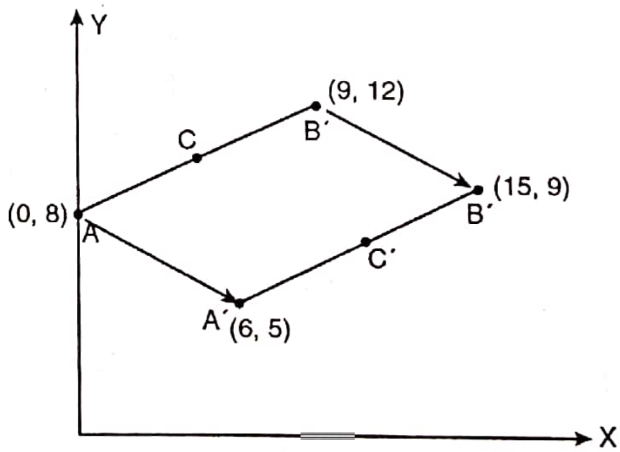

Fig. 4.5 Translation of a line

In general, the above equation (1) may be expressed as $[X'] = [X] + [T_T]$, where $[X']$ is the transformed object matrix, $[X]$ is the original object matrix and $[T_T]$ is the transformation (translation) matrix.

Consider the line (shown in Fig. 4.5) with end points A (0,8) and B (9,12). If we have to move this line AB to $A'B'$ we have to apply equal translation to each of the endpoints A & B and then redraw the line between the new end points deleting the old line AB. The actual operation of drawing a line between two endpoints depends on the display device used and the draw–algorithm followed. Here, we consider only the mathematical operations on the position vectors of the endpoints.

Here, A (0, 8) becomes A' (6, 5), implying $\Delta x = 6,\ \Delta y = -3$

B (9,12) becomes B' (15, 9), implying $\Delta x = 6,\ \Delta y = -3$

So we can say, $\begin{pmatrix} 6 \\ 5 \end{pmatrix} = \begin{pmatrix} 0 \\ 8 \end{pmatrix} + \begin{pmatrix} 6 \\ -3 \end{pmatrix}$ and $\begin{pmatrix} 15 \\ 9 \end{pmatrix} = \begin{pmatrix} 9 \\ 12 \end{pmatrix} + \begin{pmatrix} 6 \\ -3 \end{pmatrix}$

Combining these two $\Rightarrow \begin{pmatrix} 6 & 15 \\ 5 & 9 \end{pmatrix} = \begin{pmatrix} 0 & 9 \\ 8 & 12 \end{pmatrix} + \begin{pmatrix} 6 & 6 \\ -3 & -3 \end{pmatrix}$

It is to be noted that *whatever amount of translation we apply to a straight line, the length and orientation (slope) of the translated line remains same as that of the original line.*

It is implied from the Fig. 4.5 that this $\begin{pmatrix} 6 \\ -3 \end{pmatrix}$ translation matrix is theoretically applied to (i.e. added to) all the points forming the line AB. This can be tested with any intermediate point AB between A & B. Think of the midpoint; before transformation it is

$$C = \left(\frac{9+0}{2}, \frac{12+8}{2} \right) = (4.5, 10)$$

After transformation it is,

$$C' = \left(\frac{6+15}{2}, \frac{5+9}{2} \right) = (10.5, 7)$$

So, $\Delta x = 10.5 - 4.5 = 6$

$\Delta y = 7 - 10 = -3$

And this is the reason why we can express transformations in terms of any constituent point of the object concerned.

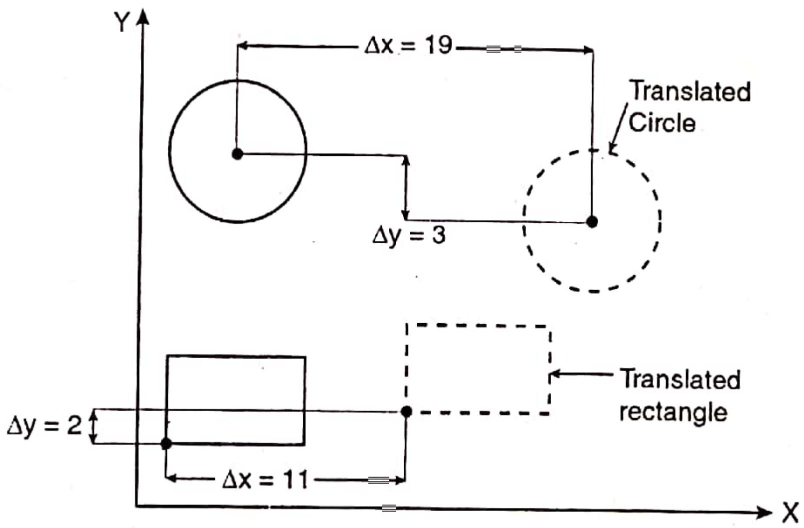

***Fig. 4.6** The translated rectangle's point – coordinates are changed by* $\begin{pmatrix} 11 \\ 2 \end{pmatrix}$ *w.r.t. the corresponding point – coordinates of the original rectangle.*

Similarly the circle is displaced by $\begin{pmatrix} 19 \\ -3 \end{pmatrix}$

For changing the position of a circle or ellipse, we translate the centre coordinates and redraw the figure in the new location.

Note from Fig. 4.5 and Fig. 4.6 that we are transforming the objects without distorting the original shape size and orientation.

Thus we can define *Translation as a rigid body transformation that moves objects without deformation. Every point on the object is translated by the same amount and there exists a one to one correspondence between the transformed points and original points.*

4.3 ROTATION

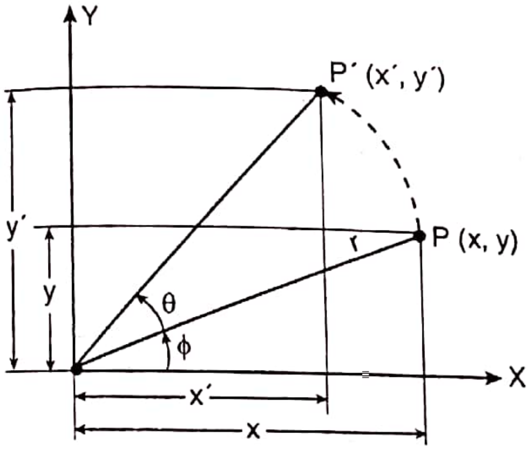

Fig 4.7 Rotation of a point about the origin

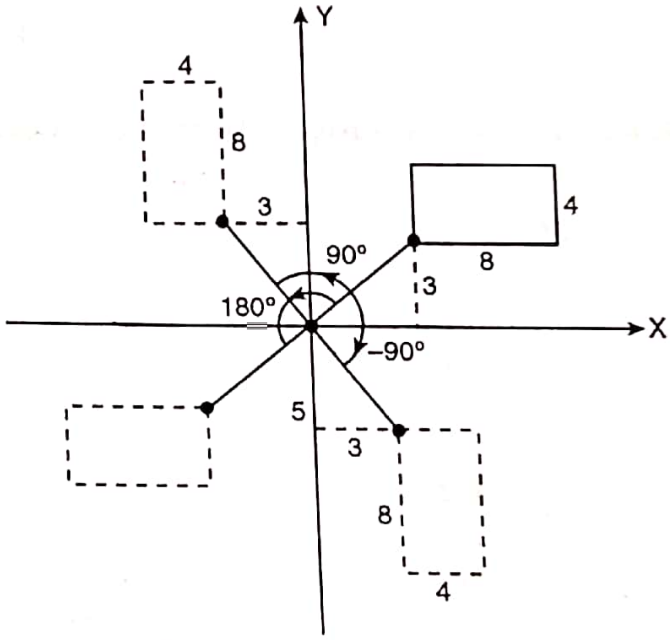

Fig 4.8 Rotation of a rectangle about the origin

This transformation is used to rotate objects about any point in a reference frame. Unlike translation rotation brings about changes in position as well as orientation. The point about which the object is rotated is called the *pivot point or rotation point.* Conventionally anti-clockwise rotation about the pivot point is represented by positive angular value. Such transformation can also be visualized as rotation about an axis that is perpendicular to the reference plane and passes through the pivot point.

4.3.1 Rotation about Origin

Consider a trial case where the pivot point is the origin as shown in Fig. 4.7. Then the point to be rotated $P(x, y)$ can be represented as

$$x = r\cos\phi,\quad y = r\sin\phi$$

where $(r,\ \phi)$ is the polar coordinate of P. When this point P is rotated through an angle θ in anti- clockwise direction, the new point $P'(x', y')$ becomes,

$$x' = r\cos\ (\theta + \phi)\quad y' = r\sin\ (\theta + \phi)$$

Rewriting the above equations using laws of sines and cosines from trigonometry,

$$\left.\begin{aligned} x' &= r\cos\theta\cos\phi - r\sin\theta\sin\phi \\ y' &= r\sin\theta\cos\phi + r\cos\theta\sin\phi \end{aligned}\right\} \quad (2)$$

Replacing $r\cos\phi$ and $r\sin\phi$ with x and y respectively in (2) we get the simplified form,

$$x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$

In matrix rotation the above relation can be more compactly expressed as,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \quad (3)$$

Symbolically $[X'] = [T_R]\,[X]$ where $[T_R]$ is the transformation matrix for rotation.

4.3.2 Rotation about an Arbitrary Pivot Point

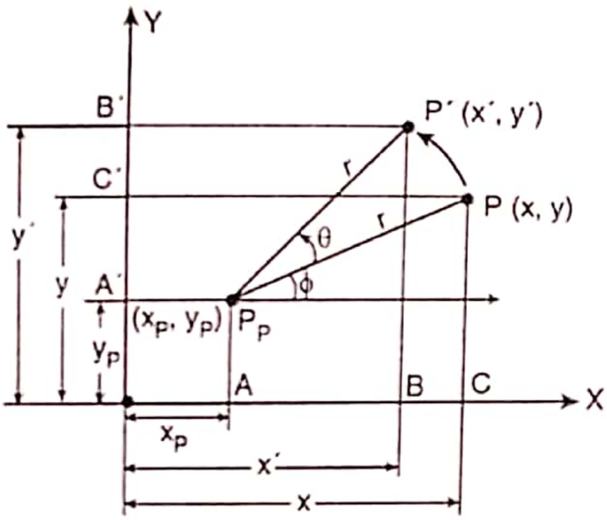

Fig. 4.9 CCW rotation of point P about P_p

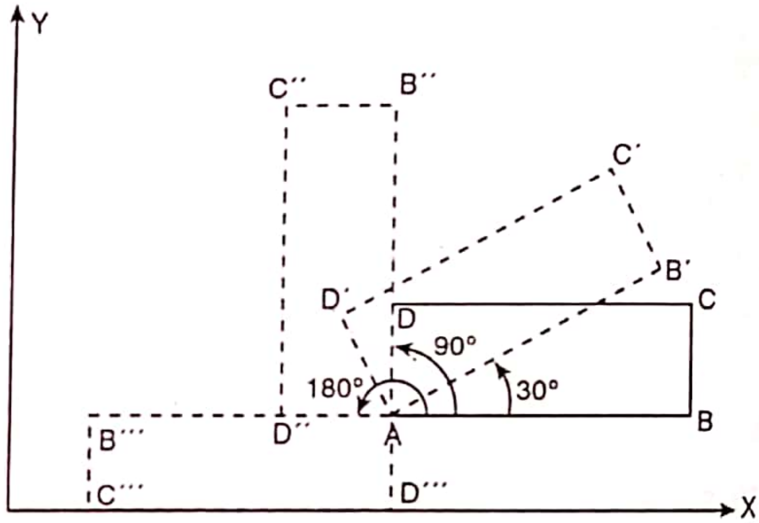

Fig. 4.10 CCW rotation of rectangle ABCD about one of its corner point A

Eqn. (3) is applicable only for rotation about the origin. But in many applications the pivot point will not be the origin. It may be any point lying on the object(s) to be rotated or any point outside the object simply anywhere in the same 2D plane. For example consider the cases when we want to rotate any st.line about one of its end points or any rectangle about one of the corner points or any object lying on a circle about its centre.

Refer Fig. 4.9. The Pivot Point is an arbitrary point P_p having coordinates (x_p, y_p). After rotating $P(x, y)$ through a positive θ angle its new location is $x'y'$ (P')

$$\begin{aligned} \text{Here}\quad x' &= OB \\ &= OA + AB \\ &= x_p + r\cos(\theta+\phi) \\ &= x_p + r\cos\phi\cos\theta - r\sin\phi\sin\theta \\ &= x_p + (x - x_p)\cos\theta - (y - y_p)\sin\theta \end{aligned}$$

$$\because \quad r\cos\phi = AC$$
$$= OC - OA$$
$$= x - x_P$$

and
$$r\sin\phi = A'C'$$
$$= OC' - OA'$$
$$= y - y_P$$

Also,
$$y' = OB'$$
$$= OA' + A'B'$$
$$= y_P + r\sin(\theta + \phi)$$
$$= y_P + r\cos\phi\sin\theta + r\sin\phi\cos\theta$$
$$= y_P + (x - x_P)\sin\theta + (y - y_P)\cos\theta$$

Now,
$$\left.\begin{aligned} x' &= x_P + (x - x_P)\cos\theta - (y - y_P)\sin\theta \\ y' &= y_P + (x - x_P)\sin\theta + (y - y_P)\cos\theta \end{aligned}\right\} \quad (4)$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x - x_P \\ y - y_P \end{pmatrix} + \begin{pmatrix} x_P \\ y_P \end{pmatrix}$$

If we rearrange eqn. (4) by grouping the (x_P, y_P) and (x, y) related terms we get

$$x' = (x_P - x_P\cos\theta + y_P\sin\theta) + (x\cos\theta - y\sin\theta)$$
$$= \{x_P(1 - \cos\theta) + y_P\sin\theta\} + (x\cos\theta - y\sin\theta)$$

Similarly,

$$y' = \{(-x_P\sin\theta) + y_P(1 - \cos\theta)\} + (x\sin\theta + y\cos\theta)$$

This grouping allows us to express x' y' matrix, in terms of x_P, y_P matrix and x, y matrix as,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 - \cos\theta & \sin\theta \\ -\sin\theta & 1 - \cos\theta \end{pmatrix} \begin{pmatrix} x_P \\ y_P \end{pmatrix}$$

Symbolically $[X'] = [T_R]\ [X] + [T_P]$ (5)

Thus we see that the general equation for rotation i.e. eqn. (5) differs from (3) by an additive term[2] $[T_P]$ involving pivot point coordinates. Later we will show how to convert such an expression into a more convenient format $[X'] = [T]\ [X]$ involving no additive terms.

Evaluation of the determinant of the general rotation matrix

$$[T_R] = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \text{ yields } \det[T_R] = \cos^2\theta + \sin^2\theta = 1$$

In general transformations with a determinant identically equal to + 1 yield pure rotation.

2 $$[T_P] = \begin{bmatrix} 1 - \cos\theta & \sin\theta \\ -\sin\theta & 1 - \cos\theta \end{bmatrix} \begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} (1 - \cos\theta)x_P + \sin\theta\, y_P \\ (-\sin\theta)x_P + (1 - \cos\theta)y_P \end{bmatrix}$$

In Fig. 4.7 P is transformed to P' through a (+) ve θ rotation. If we wish to bring back P' to P, we have to apply an inverse transformation i.e. a (–) ve θ rotation. According to eqn. (3) the required transformation matrix to obtain P from P' is,

$$\begin{pmatrix} \cos(-\theta) & -\sin(-\theta) \\ \sin(-\theta) & \cos(-\theta) \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

If we symbolize the above matrix as $[T_R]^{\text{inv}}$, then we find

$$[T_R][T_R]^{inv} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

$$= \begin{pmatrix} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta \\ \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = [\mathrm{I}] \text{ where } [\mathrm{I}] \text{ is the identity matrix.}$$

This implies that the inverse rotation matrix $[T_R]^{\text{inv}}$ for pure rotation is identical to the inverse of the rotation matrix i.e. $[T_R]^{-1}$

The above is true because from matrix properties we know only $[T_R]\ [T_R]^{-1} = [I]$

Interestingly enough the transpose of the rotation matrix $[T_R]$

$$\text{i.e. } [T_R]^T = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = [T_R]^{-1}$$

• Thus we can infer that *the inverse of any pure rotation matrix, i.e. one with a determinant equal to +1, is its transpose.*

4.4 SCALING

4.4.1 Scaling with Respect to the Origin

Scaling is a transformation that changes the size or shape of an object. Scaling w.r.t. origin can be carried out by multiplying the coordinate values (x, y) of each vertex of a polygon, or each endpoint of a line or arc or the center point and peripheral definition points of closed curves like a circle by scaling factors s_x and s_y respectively to produce the coordinates $(x'\ y')$.

The mathematical expression for pure scaling is,

$$\left.\begin{aligned} x' &= s_x\, x \\ y' &= s_y\, y \end{aligned}\right\} \Rightarrow \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{6}$$

or symbolically, $[X'] = [T_S]\ [X]$

s_x expands or shrinks object dimensions along X direction whereas s_y affects dimensions along Y direction.

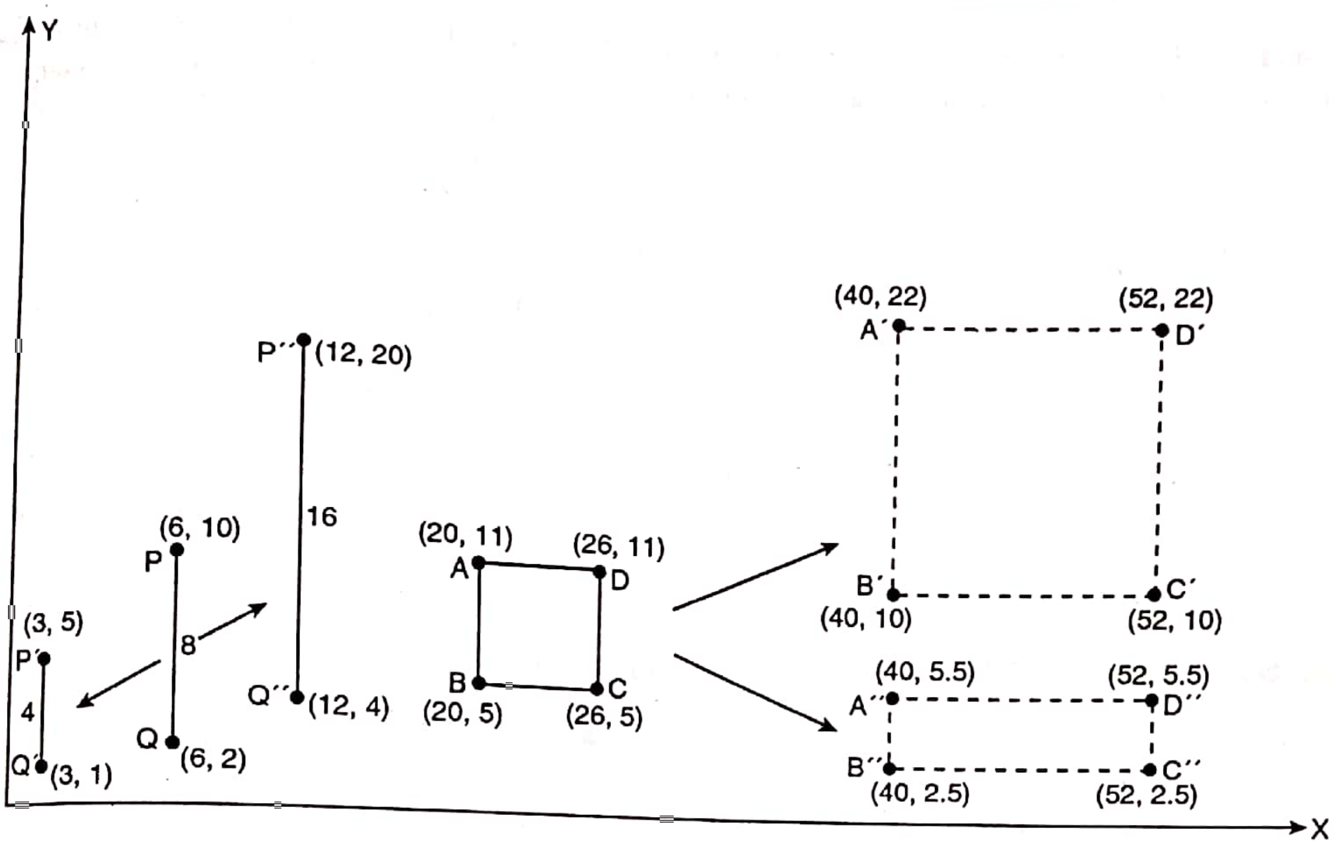

Fig. 4.11 Line PQ scaled to P′Q′ ($s_x = s_y = 0.5$) and to P′′Q′′ ($s_x = s_y = 2$). Square ABCD (each side 6 units) uniformly scaled ($s_x = s_y = 2$) producing a bigger square A′B′C′D′ with each side 12 units. But with non-uniform scaling ($s_x = 2$, $s_y = 0.5$) the square changes to a 12 by 3 rectangle A′′B′′C′′D′′.

Fig. 4.12 shows the effect of scaling transformation on curves. The shapes of the curves C_1 and C_2 are identical. But curve C_2 is larger by a factor of $s = 2$ since the coordinates of any point P' on C_2 are two times larger than the respective coordinates of corresponding point P and C_1. The magnitude of the tangent vector at any point on C_2 is also twice as that at the corresponding point on C_1 though their direction in space is identical.

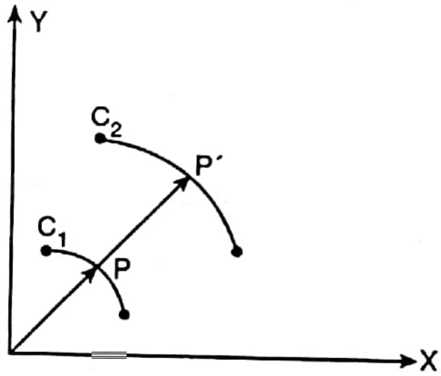

Fig. 4.12

We can represent the scaling transformation carried out on square $ABCD$ in Fig. 4.11 as,

$$[A'B'C'D'] = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} [A\,B\,C\,D]$$

and $$[A''B''C''D''] = \begin{pmatrix} 2 & 0 \\ 0 & 0.5 \end{pmatrix} [A\,B\,C\,D]$$

Notice that in the second case where $s_x \neq s_y$, a distortion in shape has occurred - the square has transformed into a rectangle.

In general, for ***uniform scaling***, if $s_x = s_y > 1$, a uniform expansion occurs; i.e. the object becomes larger. If $s_x = s_y < 1$, then a uniform compression occurs; i.e. the object gets smaller. Non-uniform expansions or compressions occur, depending on whether s_x and s_y are individually > 1 or < 1 but unequal, such scaling is also known as ***differential scaling***. While the basic object shape remains unaltered in uniform scaling, the shape and size both changes in differential scaling. Another interesting point to be noted is that, there is always a translation associated with scaling. Look at the figures, irrespective of the nature of scaling & the scale factors the scaled objects have substantially moved from their respective original positions. This is easily understood if we recall that during scaling transformation, the position vectors of the definition points are actually scaled with respect to the origin. For example consider the position vector of point Q (6, 2) of line PQ. Its magnitude with respect to origin is $\sqrt{6^2+2^2}=2\sqrt{10}$. The magnitude of the position vectors of the scaled points Q' (3, 1), Q'' (12, 4) are respectively $\sqrt{10}$ and $4\sqrt{10}$ implying uniform scaling with scale factors 0.5 and 2 respectively. Also note that the direction of the position vectors of the original point $\left(\tan^{-1}\frac{2}{6}\right)$ and the scaled points $\left(\tan^{-1}\frac{1}{3} \text{ and } \tan^{-1}\frac{4}{12}\right)$ remains same. Thus, quite obviously, *pure uniform scaling with factors < 1 moves objects closer to the origin while factors > 1 moves objects farther from origin, at the same time decreasing or increasing the object size.*

4.4.2 Scaling with Respect to any Arbitrary Point

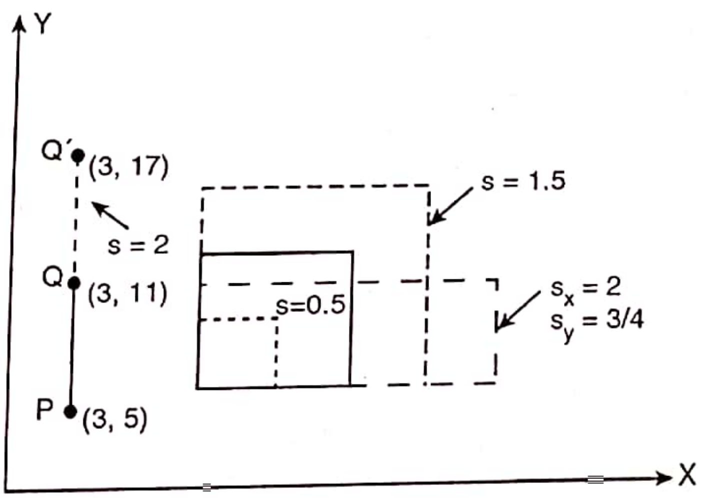

Fig. 4.13

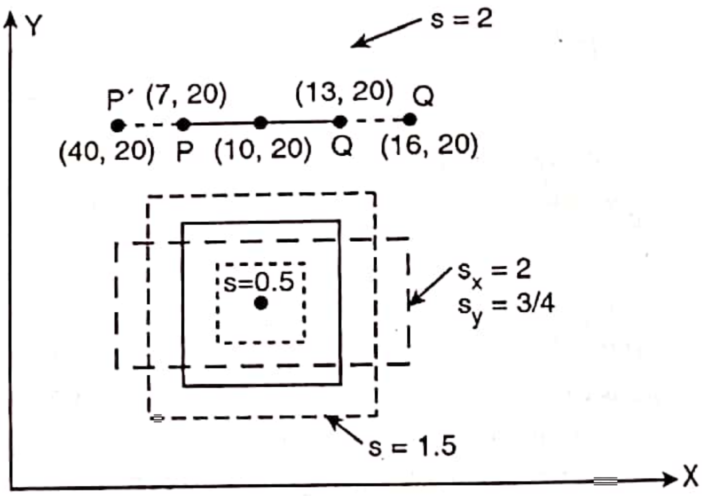

Fig. 4.14

By scaling an object we generally expect it to grow or shrink in size or shape based on its original position (as shown in Fig. 4.13 and Fig. 4.14). Apart from scaling, the movement of the object (which is intrinsically associated in scaling) with respect to origin, is mostly unwanted and can be eliminated by scaling the object with respect to a point conveniently chosen on the object itself. The point so chosen, called the *fixed point* remains unaltered in position after the scaling transformation when the scale factors are applied on the objects dimensions relative to that fixed point. As a result the object seems to expand or shrink in size or change in shape without any displacement of the object as a whole.

For example consider the scaling transformation of PQ to PQ' (not $P'Q'$) in Fig. 4.13. Here P is considered as the fixed point. So, instead of multiplying the scale factor ($s = 2$) to both P & Q position vectors it is multiplied with Q - P and then added with P to obtain shifted Q' and then the line PQ' is reconstructed.

$$Q' = P + s(Q-P) = (3,5) + 2\{(3-3),\ (11-5)\}$$
$$= (3,5) + (0,12)$$
$$= (3,\ 17)$$

Note that $Q - P$ is the dimension of the line PQ relative to P. Now compare the result with the scaling of line PQ as shown in Fig. 4.11 Instead of endpoint P if we consider the midpoint of PQ as the fixed point it will expand symmetrically in both direction forming $P'Q'$ (Fig. 4.14), the midpoint remaining unchanged in position.

For scaling of a rectangle shown in Fig. 4.14 the fixed point is the centroid of the rectangle. Thus the fixed point need not necessary lie on the object – it can be any point either on, within or outside the object. Here we derive the most generalized expression for scaling any object (P) coordinate say (x, y) with respect to any arbitrary point say, $P_f(x_f, y_f)$.

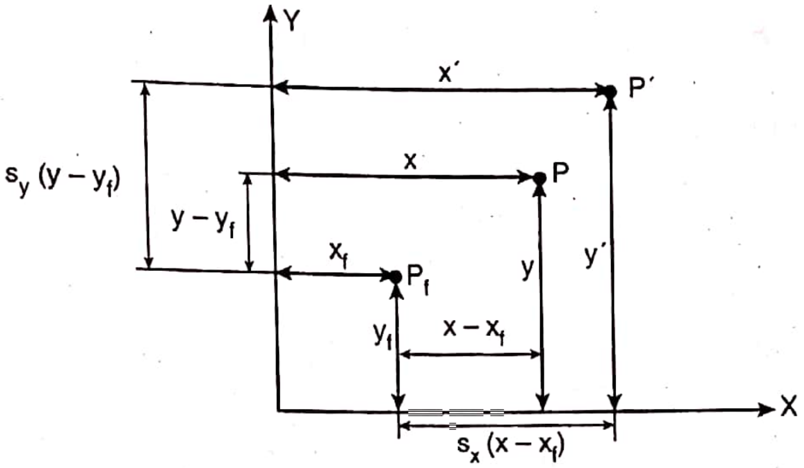

Fig. 4.15 *Scaling of a point with respect to a fixed point s_x, $s_y > 1$*

As in this case the distance between the point in question $P(x, y)$ and the fixed point $P_f(x_f, y_f)$ is scaled we can write,

$$x' = x_f + (x - x_f)s_x$$
$$y' = y_f + (y - y_f)s_y$$

where $(x'\ y')$ are the scaled coordinates and s_x, s_y are the scale factors.

We can rewrite these transformation equations to separate the multiplicative and additive terms:

$$x' = s_x x + (1 - s_x)x_f$$
$$y' = s_y y + (1 - s_y)y_f$$

In matrix notation,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} s_x & 0 \\ 0 & s_y \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 - s_x & 0 \\ 0 & 1 - s_y \end{pmatrix} \begin{pmatrix} x_f \\ y_f \end{pmatrix}$$

Symbolically[3],

$$[X'] = [T_s]\ \ [X] + [T_f] \qquad (7)$$

s_x and s_y can be equal or unequal and can be >1, <1, equal to 1 or even negative integer or fraction but never equal to zero.

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[3] $[T_f] = \begin{pmatrix} 1 - s_x & 0 \\ 0 & 1 - s_y \end{pmatrix} \begin{pmatrix} x_f \\ y_f \end{pmatrix} = \begin{pmatrix} (1 - s_x)x_f \\ (1 - s_y)y_f \end{pmatrix}$

Comparing eqn. (7) with (5) we find that coordinates for a *fixed point* feature in the scaling equations similar to the coordinates for a *pivot point* in the rotation equations.

4.5 REFLECTION

A reflection is a transformation that produces a mirror image of an object. In 2D reflection we consider any line in 2D plane as the mirror; the original object and the reflected object are both in the same plane of the mirror line. However we can visualize a 2D reflection as equivalent to a 3D rotation of 180° about the mirror line chosen. The rotation path being in the plane perpendicular to the plane of mirror line. Here we will study some standard 2D reflection cases characterized by the mirror line.

4.5.1 Reflection about X Axis

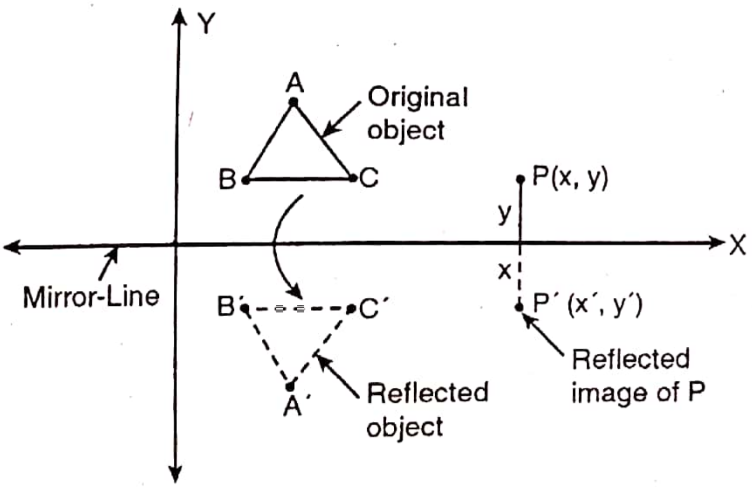


Fig. 4.16 Reflection about X axis

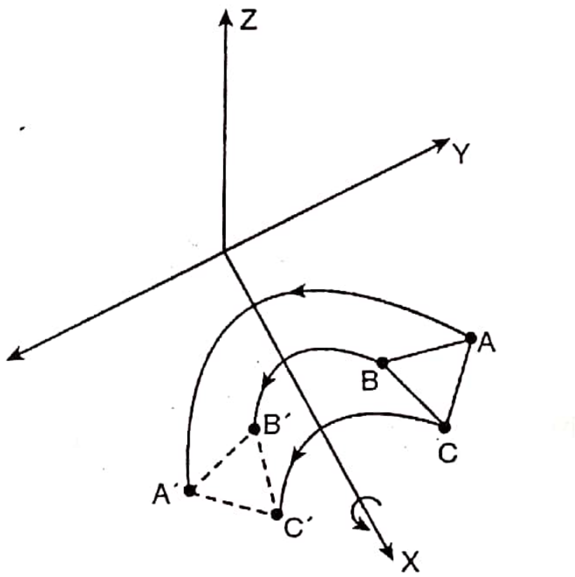


Fig. 4.17 2D reflection about X axis as 3D rotation(180^0) about X axis

The basic principles behind reflection transformation are

(1) The image of an object is formed on the side opposite to where the object is lying, with respect to the mirror line.

(2) The perpendicular distance of the object (i.e. the object points) from the mirror line is identical to the distance of the reflected image (i.e. the corresponding image points) from the same mirror line. Once again, the two perpendiculars must be along the same straight line.

Therefore the relation between the point P (x, y) and its image P' (x', y') about X axis is simply,

$$\begin{aligned} x' &= x \\ y' &= -y \end{aligned} \quad \text{(Refer fig. 4.16)}$$

So the transformation matrix for reflection about X axis or $y = 0$ axis is,

$[T_M]_{y=0} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and the transformation is represented as,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{8}$$

i.e. $[X'] = [T_M]_{y=0}\,[X]$

Consider the 2D reflection of the triangle ABC about X axis (in Fig. 4.16) forming the image $\Delta\ A'B'C'$. In the other way we can interpret this event as a 3D rotation- we can think of the ABC triangle moving out of the xy plane and rotating 180° in 3D space about the X axis and back into the xy plane on the other side of the X axis.

4.5.2 Reflection about Y Axis

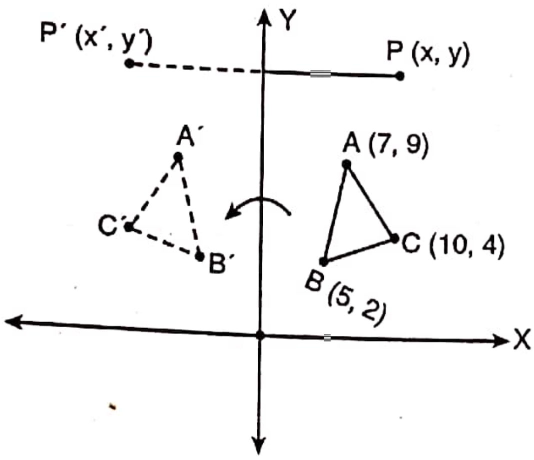


Fig. 4.18 Reflection about Y axis

A reflection about Y axis flips x coordinates while y coordinates remains the same. For reflection of $P(x, y)$ to $P'(x', y')$ in Fig. 4.18,

$$x' = -x$$
$$y' = y$$

This transformation is identified by the reflection –transformation matrix

$$[T_M]_{x=0} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \qquad (9)$$

The transformed new vertices $A'B'C'$ of triangle ABC are given by

$$\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 7 & 5 & 10 \\ 9 & 2 & 4 \end{pmatrix} = \begin{pmatrix} -7 & -5 & -10 \\ 9 & 2 & 4 \end{pmatrix}$$

4.5.3 Reflection about the Straight Line $y = x$

To find the relation between a point's coordinates (x, y) and those of its image (x', y'), reflected through $y = x$ straight line, look at Fig. 4.19

Here $PK = P'K$ and both PK and $P'K$ are perpendicular to $y = x$ or OK.

From simple geometry you will find $\Delta\ POK \cong \Delta\ P'OK$

$\therefore\ OP = OP'$ and $\angle\ POK = \angle\ P'OK$ implying also

$\angle\ POM = \angle\ P'ON$ because $y = x$ straight line makes 45° angle with both the axes.

That in turn implies $\Delta\, POM \cong \Delta P'ON$

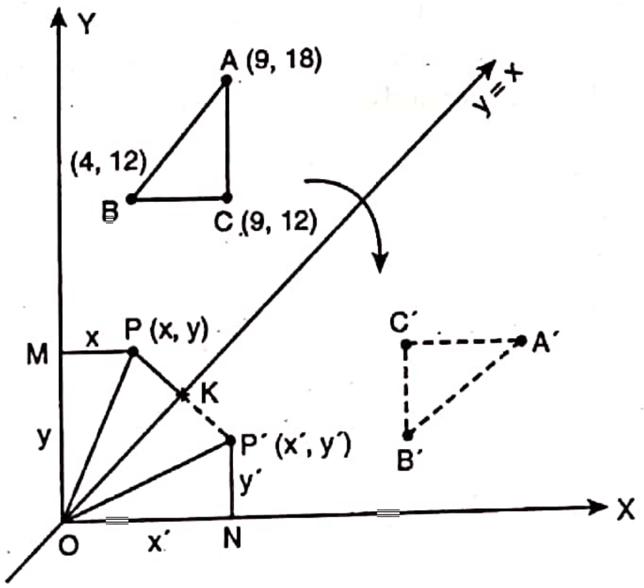


Fig. 4.19

So, $PM = P'N$ and $OM = ON$
but $PM = x$, $P'N = y'$, $OM = y$, $ON = x'$

Hence $\quad x' = y$
$\qquad\qquad y' = x$

This can be represented by,

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \qquad (10)$$

where $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the transformation matrix $[T_M]_{y=x}$

Thus the transformed vertices $A'B'C'$ of triangle ABC in Fig. 4.19 are given by

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 9 & 4 & 9 \\ 18 & 12 & 12 \end{pmatrix} = \begin{pmatrix} 18 & 12 & 12 \\ 9 & 4 & 9 \end{pmatrix}$$

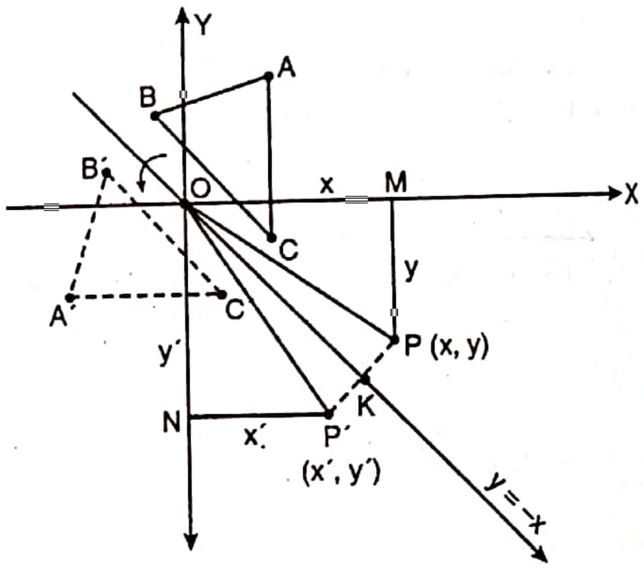


Fig. 4.20

4.5.4 Reflection about the Straight Line $y = -x$

Considering the experimental point $P(x, y)$, we draw a geometric Fig. 4.20 similar to that in Fig. 4.19. Similarly we can prove

$$\left.\begin{aligned} PM &= P'N \\ OM &= ON \end{aligned}\right\}$$

But this time we can't simply state $x' = y$ and $y' = x$ from the above relation, because that is only the equations of magnitude. Looking at the figure we find that P and its image P' both being in the 4th quadrant, x and x' are (+)ve whereas y and y' are –ve. So considering this fact we should write,

$$x' = -y$$

This can be represented by,

$$y' = -x$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

where $\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$ is the transformation matrix $[T_M]_{y=-x}$

Thus we can infer that unlike in the case of reflection about diagonal axis $y = x$, in reflections about the other diagonal $y = -x$, the coordinate values are interchanged with their signs reversed.

Notice the changes of the vertices of triangle ABC to $A'B'C'$ in Fig. 4.20 are given by,

$$\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 4 & -2 & 4 \\ 6 & 4 & -2 \end{pmatrix} = \begin{pmatrix} -6 & -4 & 2 \\ -4 & 2 & -4 \end{pmatrix}$$

4.5.5 Reflection Relative to the Origin

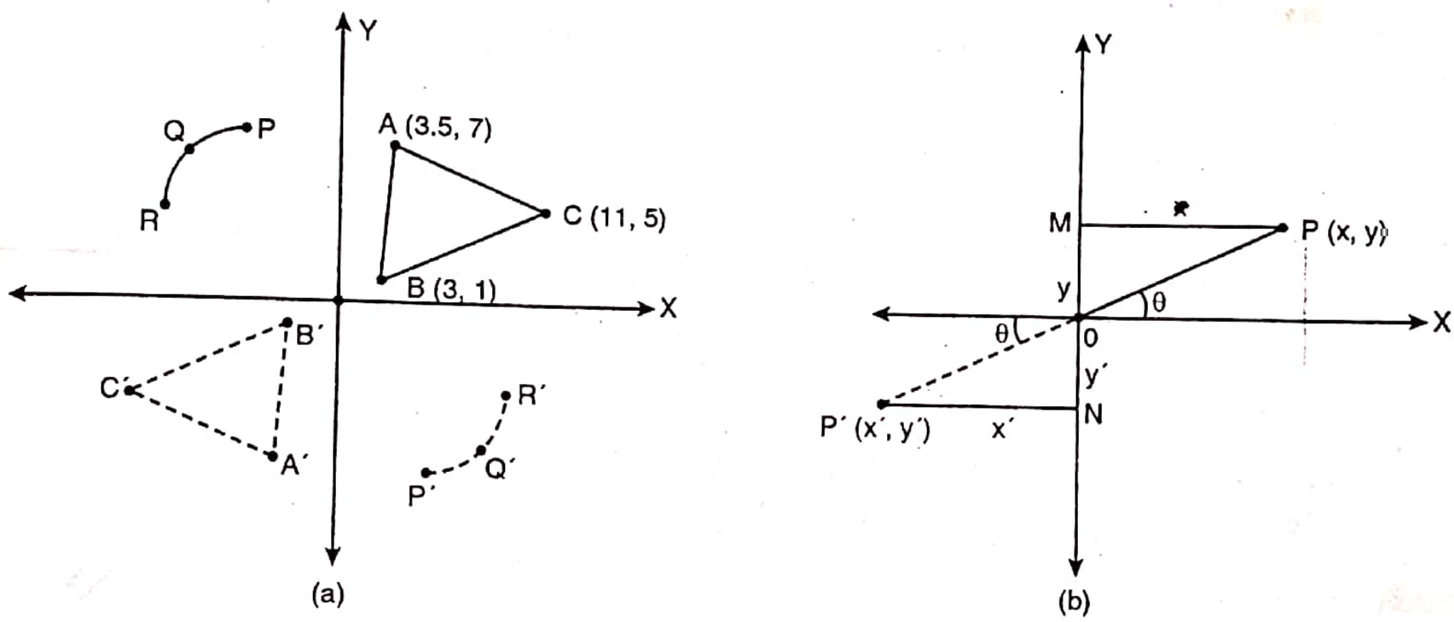


Fig. 4.21

In this case we actually choose the mirror-line as the axis perpendicular to the xy plane and passing through the origin. After reflection both the x and y coordinate of the object point is flipped, i.e. x' becomes $-x$ and y' becomes $-y$. This can be easily proved from geometry as shown in Fig. 4.21(b). The coordinate signs are just opposite because the image is always formed in the quadrant opposite to that of the object.[4]

Thus $x' = -x$

$y' = -y$, which can be represented in matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \text{ where}$$

4 Interested readers may refer to *Mathematical Elements for Computer Graphics* by Rogrers & Adams, MGH-1989.

$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ is the transformation matrix $[T_M]_{y=x=0}$

The change of the triangle ABC to $A'B'C'$ in Fig. 4.21(a) is given by,

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 3.5 & 3 & 11 \\ 7 & 1 & 5 \end{pmatrix} = \begin{pmatrix} -3.5 & -3 & -11 \\ -7 & -1 & -5 \end{pmatrix}$$

Similarly the curve PQR is changed to $P'Q'R'$ represented by,

$$\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} -5 & -8 & -9 \\ 8 & 7 & 4 \end{pmatrix} = \begin{pmatrix} 5 & 8 & 9 \\ -8 & -7 & -4 \end{pmatrix}$$

Notice that the radial distances of all the object points, before and after reflection, remains unaltered.

In the previous sections we have discussed reflection transformation about axes or lines passing through origin. The question, which now comes to mind, is how to find the transformation equations for reflections about any arbitrary line $y = mx + c$ in 2D xy plane. We cannot derive those equations as easily as we did for rotation about any arbitrary pivot point or scaling about any arbitrary fixed point. However, we can accomplish such reflections with a combination of basic translate-rotate-reflect transformations, which may be easier for you to understand after we establish a convenient method of combining consecutive basic transformations on an object in the next section.

Before that it is important to note the fact that all the standard reflection transformations discussed so far could have been achieved, alternatively, by scaling with appropriate choice of scale factors (+ve, or -ve). In that sense reflection is not a basic transformation. But interestingly all these reflection matrices have a determinant equal to -1.

In general, if the determinant of a transformation matrix is identically -1, then the transformation produces a pure reflection.

4.6 HOMOGENEOUS COORDINATES AND COMBINATION OF TRANSFORMATION

We have seen how the shape, size, position and orientation of 2D objects can be controlled by performing matrix operation on the position vectors of the object definition points. In some cases, however a desired orientation of an object may require more than one transformation to be applied successively. For illustration let us consider a case which requires 90° rotation of a point $\begin{pmatrix} x \\ y \end{pmatrix}$ about origin followed by reflection through the line $y = x$.

After rotation, $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ x \end{pmatrix}$

This $\begin{pmatrix} x' \\ y' \end{pmatrix}$ then undergoes reflection to produce $\begin{pmatrix} x'' \\ y'' \end{pmatrix}$

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -y \\ x \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

These successive matrix operations can be symbolically expressed as,

$$[X''] = [T_M]_{y=x}[X']$$

i.e. $[X''] = [T_M]_{y=x}\{[T_R]_{90^\circ}[X]\}$

As we know matrix multiplication is associative we can first perform $[T_M]_{y=x}\,[T_R]_{90^\circ}$ instead of performing $[T_R]_{90^\circ}\,[X]$ first (but not $[T_R]_{90^\circ}\,[T_M]_{y=x}$, thereby maintaining the right to left order of succession) to form a *resultant transformation matrix*. This matrix when multiplied with the original point coordinates will yield the ultimate transformed coordinates.

$$\text{Thus, } [T_M]_{y=x}[T_R]_{90^\circ} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$= [T_{COMB}] \ (say)$$

$$\text{Now, } [T_{COMB}][X] = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ -y \end{pmatrix}$$

Which shows the same result as before.

This process of calculating the product of matrices of a number of different transformations in a sequence is known as *concatenation* or, *combination* of transformations and the resultant product matrix is referred to as *composite or concatenated transformation matrix*. Application of concatenated transformation matrix on the object coordinates eliminates the calculation of intermediate coordinate values after each successive transformation.

But the real problem arises when there is a translation or rotation or scaling about an arbitrary point other than the origin involved among several successive transformations. The reason being the general form of expression of such transformations is not simply, $[X'] = [T]\,[X]$ involving the 2 by 2 array $[T]$ containing multiplicative factors, rather it is in the form, $[X'] = [T_1]\,[X] + [T_2]$, where $[T_2]$ is the additional two element column matrix containing the translational terms. Such transformations cannot be combined to form a single resultant representative matrix. This problem can be eliminated if we can combine $[T_1]$ and $[T_2]$ into a single transformation matrix. This can be done by expanding the usual 2×2 transformation matrix format into 3×3 form. The general 3×3 form will be something like,

$$\begin{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} & \begin{matrix} m \\ n \end{matrix} \\ \begin{matrix} 0 & 0 \end{matrix} & 1 \end{pmatrix}$$ where the elements a, b, c, d of the upper left 2×2 sub matrix are the multiplicative factors of $[T_1]$ and m, n are the respective x, y translational factors of $[T_2]$.

But such 3×3 matrices are not conformable for multiplication with 2×1 2D position vector matrices. Herein lies the need to include a *dummy coordinate* to make 2×1 position vector matrix $\begin{pmatrix} x \\ y \end{pmatrix}$ to a 3×1 matrix $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ where the third coordinate is dummy.

Now if we multiply, $\begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$ with a non-zero scalar 'h' then the matrix it forms is

$\begin{pmatrix} xh \\ yh \\ h \end{pmatrix}$ or symbolically, say, $\begin{pmatrix} x_h \\ y_h \\ h \end{pmatrix}$ which is known as the *homogeneous coordinates* or homogeneous position vector of the same point $\begin{pmatrix} x \\ y \end{pmatrix}$ in 2D plane.

$$\begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} x_h \\ y_h \\ h \end{pmatrix} = h \cdot \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}$$

The extra coordinate h is known as a weight, which is homogeneously applied to the cartesian components.

Thus a general homogeneous coordinate representation of any point P (x, y) is (x_h, y_h, h) or (xh, yh, h) that implies, $x = \dfrac{x_h}{h}$, $y = \dfrac{y_h}{h}$

As 'h' can have any non-zero value, there can be infinite number of equivalent homogeneous representation of any given point in space. For example (6, 4, 2), (12, 8, 4), (3, 2, 1), (1/2, 1/3, 1/6), (–3, –2, –1) all represent the physical point (3, 2).

But so far as geometric transformation is concerned our choice is simply $h = 1$ and the corresponding homogeneous coordinate triple $(x, y, 1)$ for representation of point positions (x, y) in xy-plane. Other values of parameter 'h' are needed frequently in matrix formulation of three dimensional viewing transformation. Though we have introduced homogeneous coordinates just as a tool for making our transformation operations easier, they have their own significance as the coordinates of points in projective space (not a subject we will pursue here)[5].

Expressing positions in homogeneous coordinates allows us to represent all geometric transformation equations uniformly as matrix multiplication. Coordinates are represented with three element column vectors and transformation operations are written in form of 3 by 3 matrices. Thus, for translation, we now have,

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \Delta x \\ 0 & 1 & \Delta y \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \qquad (11)$$

or,

Symbolically $[X'] = [T_T]\,[X]$

Compare eqn.(11) with eqn. (1) both yield the same result, $x' = x + \Delta x$ and $y' = y + \Delta y$

Equations of rotation and scaling with respect to coordinate origin (derived earlier as (3) & (6) respectively) may be modified as,

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{12}$$

and

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{13}$$

respectively.

Similarly the modified general expression for reflection may be

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{14}$$

where values of a, b, c, d depend upon choice of coordinate axes or diagonal axes as mirror line.

The simplest $[X'] = [T]\,[X]$ form representing rotation about any arbitrary pivot point (x_P, y_P) as modified from eqn. (5) is,

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta & (1-\cos\theta)x_P + \sin\theta\, y_P \\ \sin\theta & \cos\theta & (1-\cos\theta)y_P - \sin\theta\, x_P \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{15}$$

and for scaling with respect to any arbitrary fixed point (x_f, y_f) the modified form of eqn. (7) is

$$\begin{pmatrix} x' \\ y' \\ 1 \end{pmatrix} = \begin{pmatrix} s_x & 0 & (1-s_x)\,x_f \\ 0 & s_y & (1-s_y)\,y_f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ 1 \end{pmatrix} \tag{16}$$

It is now for you to do some matrix multiplication to compare results of eqn. (15) & (16) with those of eqn. (5) & eqn. (7) respectively for complete satisfaction. However we will show how to derive (15) & (16) using the theory of composite transformation in the next section.

But before running to the next section just pause few minutes. You must have noticed that in all the above 3 × 3 transformation matrices the bottom corner element on the diagonal line is always 1 as the bottom row is always [0 0 1]. Now what happens if we force this corner element to be anything other than 1, say 's'? The effect is interesting. If the diagonal becomes [1 1 s] and all the other elements are zero i.e. if

$$[T]=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & s\end{pmatrix}$$

then $$[T][X]=[T]=\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & s\end{pmatrix}\begin{pmatrix}x\\ y\\ 1\end{pmatrix}=\begin{pmatrix}x\\ y\\ s\end{pmatrix}=\begin{pmatrix}x'\\ y'\\ h\end{pmatrix}=[X']$$

Here $x' = x$, $y' = y$, $h = s$. Normalizing this yields the transformed coordinates as x/s and y/s which implies that overall scaling has occurred, s being the scale factor.

If $s < 1$, then an uniform expansion occurs whereas

If $s > 1$ an uniform compression occurs.

4.7 COMPOSITE TRANSFORMATION

Since we are now aware of representing all kinds of transformation, in a homogeneous manner, we can reduce a long sequence of transformations to a series of matrix multiplications, with little difficulty. No matrix addition is required. If $[T_1]$, $[T_2]$, $[T_3]$ be any three transformation matrices then the matrix product, $[T_1]\,[T_2]\,[T_3] = ([T_1]\,[T_2])\,[T_3] = [T_1]\,([T_2]\,[T_3])$ which implies we can evaluate matrix product using either a left to right or a right to left associative grouping. But we must be extremely careful about the order in which transformations actually takes place and the order we follow while multiplying the corresponding transformation matrices. *For column-matrix representation of coordinate positions, we form composite transformations by multiplying matrices in order from right to left.* Different order of multiplication of matrices will give different results. Because we all know that matrix product may not be commutative always, i e

$[T_1]\,[T_2] \neq [T_2]\,[T_1]$.

For example if we first rotate an object (counter-clockwise 90° about origin) and then translate the rotated object (by 2,1) then the representative matrix expression is,

$$[X'] = [T_T]_{2,1}\,[T_R]_{90^\circ}\,[X]$$

$$=\begin{pmatrix}1 & 0 & 2\\ 0 & 1 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}0 & -1 & 0\\ 1 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2 & 3 & 3 & 2\\ 1 & 1 & 2 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}$$ (Here $[X]$ represents the rectangle shown in Fig. 5.22(a))

$$=\begin{pmatrix}0 & -1 & 2\\ 1 & 0 & 1\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}2 & 3 & 3 & 2\\ 1 & 1 & 2 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}=\begin{pmatrix}1 & 1 & 0 & 0\\ 3 & 4 & 4 & 3\\ 1 & 1 & 1 & 1\end{pmatrix}$$

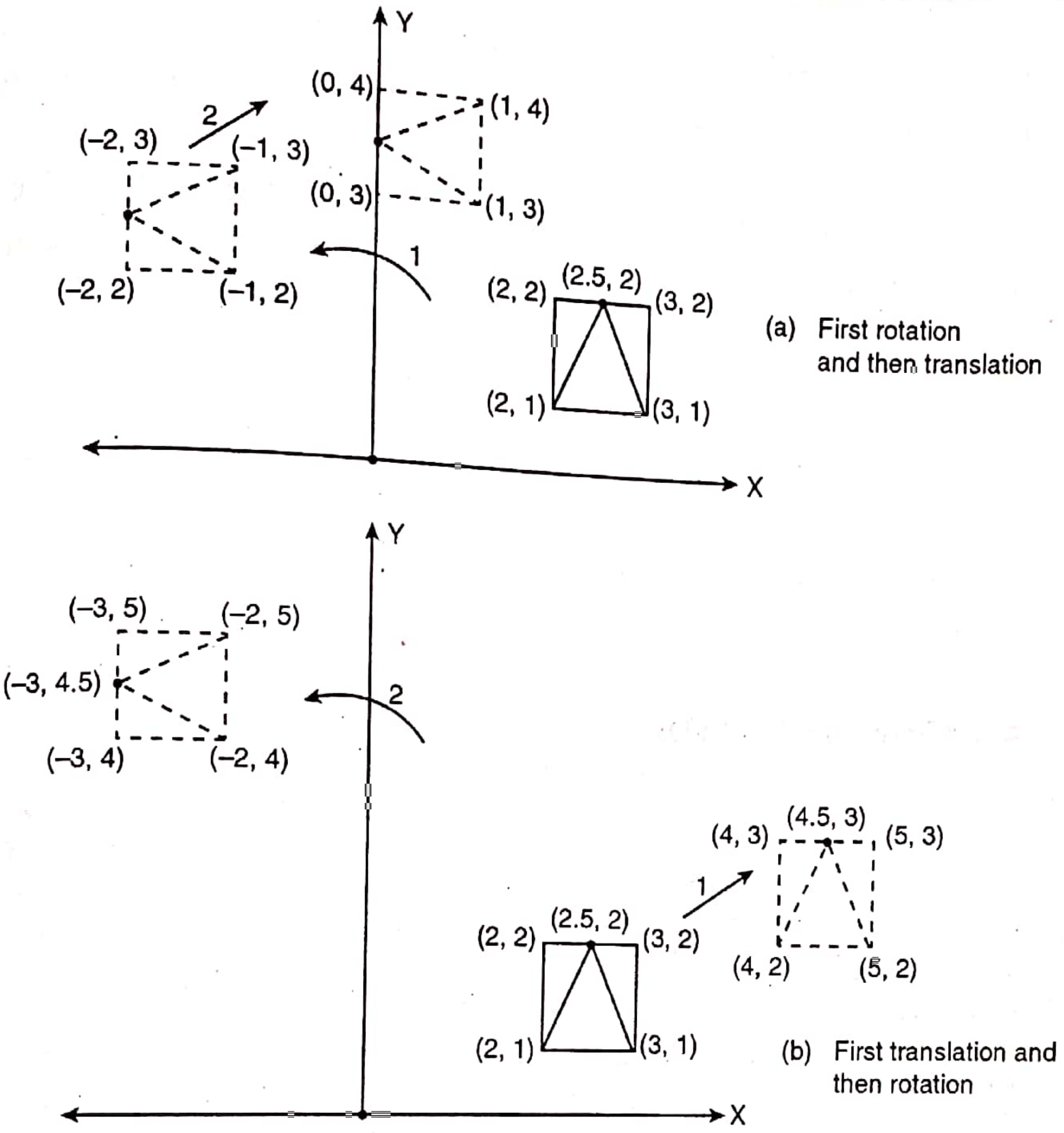


Fig. 4.22

Now if we reverse the order of multiplication of $[T_T]_{2,1}$ and $[T_R]_{90°}$ implying first translation and then rotation, then,

$$[X'] = [T_R]_{90°}\,[T_T]_{2,1}\,[X]$$

$$= \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 3 & 2 \\ 1 & 1 & 2 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & -2 & -3 & -3 \\ 4 & 5 & 5 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix}$$

The transformed object matrix $[X']$ is different in the above two cases.

However, *multiplication of transformation matrices is commutative for a sequence of transformations, which are of the same kind.*

As an example two successive rotations could be performed in either order and the final result would be the same. If the successive rotations are θ_1 and θ_2 about the origin to finally transform the point P to P' then,

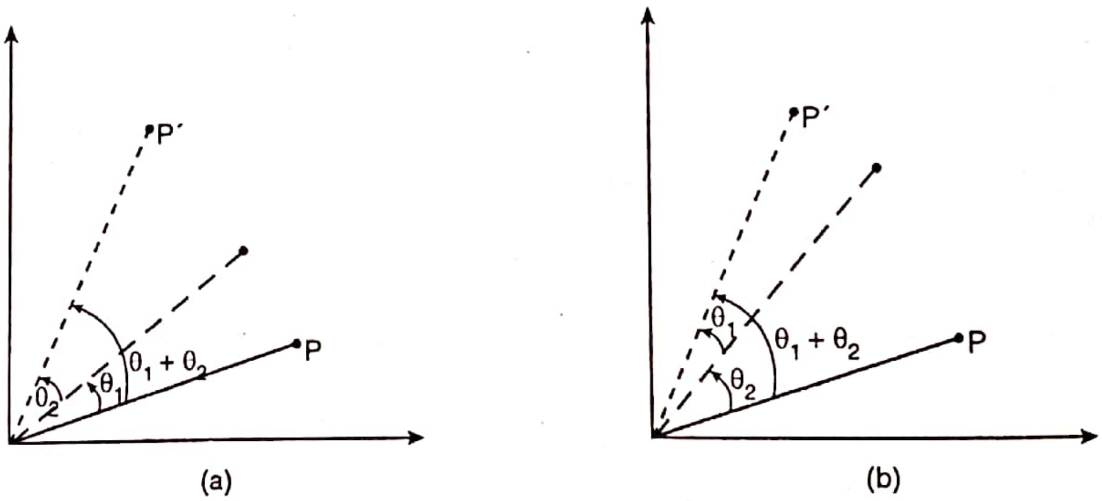
$$P' = [T_R]_{\theta_2}[T_R]_{\theta_1} P$$

$$= \begin{pmatrix} \cos\theta_2 & -\sin\theta_2 & 0 \\ \sin\theta_2 & \cos\theta_2 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta_1 & -\sin\theta_1 & 0 \\ \sin\theta_1 & \cos\theta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} P$$

$$= \begin{pmatrix} (\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) & -(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2) & 0 \\ (\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2) & (-\sin\theta_1\sin\theta_2 + \cos\theta_1\cos\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} P$$

$$= \begin{pmatrix} \cos(\theta_1+\theta_2) & -\sin(\theta_1+\theta_2) & 0 \\ \sin(\theta_1+\theta_2) & \cos(\theta_1+\theta_2) & 0 \\ 0 & 0 & 1 \end{pmatrix} P$$

$$= [T_R]_{\theta_1}[T_R]_{\theta_2} P = [T_R]_{\theta_1+\theta_2} P$$



Successive rotation: first by θ_1 & then by θ_2

First θ_2 rotation followed by θ_1 rotation; result same

Fig. 4.23

The commutative property also holds true for successive translations or scaling or successive reflection about coordinate axes (excluding the diagonals).

For two successive translations of $(\Delta x_1, \Delta y_1)$ and $(\Delta x_2, \Delta y_2)$

$$[T_T]_{\Delta x_2,\Delta y_2}[T_T]_{\Delta x_1,\Delta y_1} = [T_T]_{\Delta x_1,\Delta y_1}[T_T]_{\Delta x_2,\Delta y_2}$$

$$= \begin{pmatrix} 1 & 0 & \Delta x_1 \\ 0 & 1 & \Delta y_1 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & \Delta x_2 \\ 0 & 1 & \Delta y_2 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & \Delta x_1 + \Delta x_1 \\ 0 & 1 & \Delta y_1 + \Delta y_2 \\ 0 & 0 & 1 \end{pmatrix} = [T_T]_{\Delta x_1 + \Delta x_2,\ \Delta y_1 + \Delta y_2}$$

For two successive scaling by (s_{x_1}, s_{y_1}) and (s_{x_2}, s_{y_2}),

$$[T_s]_{s_{x_2}, s_{y_2}} \, [T_s]_{s_{x_1}, s_{y_1}} = \begin{pmatrix} s_{x_2} & 0 & 0 \\ 0 & s_{y_2} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_{x_1} & 0 & 0 \\ 0 & s_{y_1} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s_{x_1} s_{x_2} & 0 & 0 \\ 0 & s_{y_1} s_{y_2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= [T_s]_{s_{x_1}, s_{y_1}} \, [Ts]_{s_{x2}, s_{y_2}}$$

$$= [T_s]_{(s_{x_1},\, s_{x_2}),\, (s_{y_1},\, s_{y_2})}$$

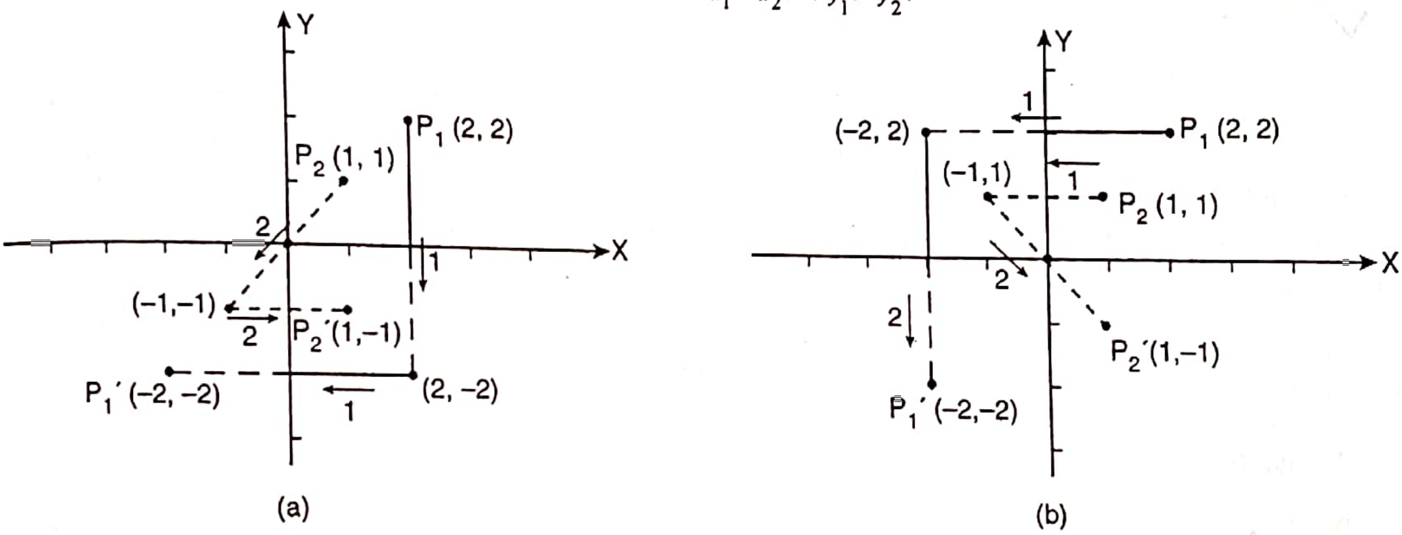


Fig. 4.24 Two commutative pair of reflection transformation $P_1 \rightarrow P'_1$: about X axis & Y axis, $P_2 \rightarrow P'_2$: about origin & Y axis, operation sequences are shown by arrows & numbers

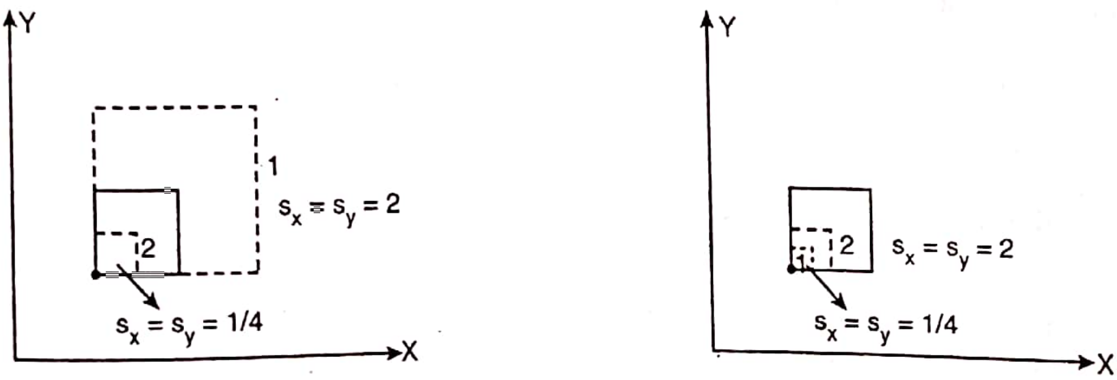
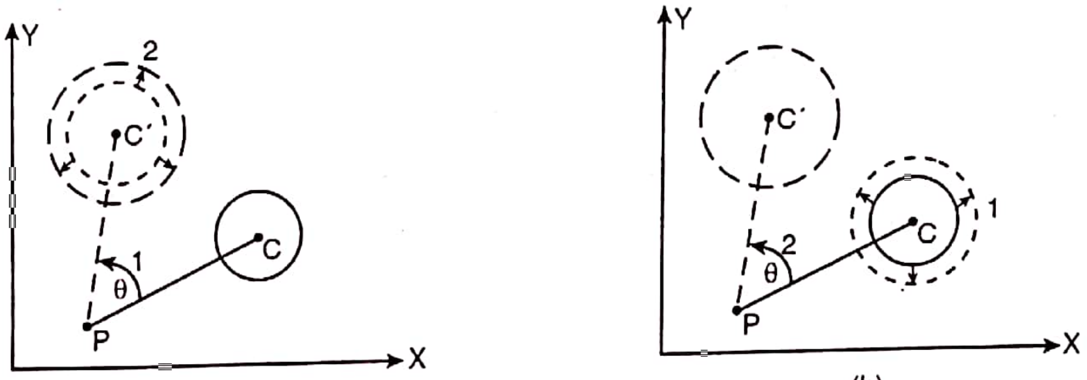


Fig. 4.25 Commutative pair of fixed point scaling; irrespective of the order of application of scale factors 2 & ¼ the final size of the square is 2 × ¼ i.e. half the size of the original square

One more example of commutative pair of operations is rotation and uniform scaling



(a) *First rotation of circle about P then scaling of circle w.r.t. its centre*

(b) *First scaling about centre, C followed by rotation about P; the result is same*

Fig. 4.26

4.7.1 Inverse Transformation

For each geometric transformation there exists an inverse transformation which describes just the opposite operation of that performed by the original transformation. Any transformation followed by its inverse transformation keeps an object unchanged in position, orientation, size and shape. For example if an object is translated 3 units in the (+)ve X direction and then 3 units in the (–)ve X direction, the object comes back to its initial position implying no resultant transformation. We will use inverse transformation to nullify the effects of already applied transformation.

We will designate inverse of any transformation $[T]$ as $[T]^{-1}$. Listed below are the notations of inverse transformation matrices of some standard transformations.

Translation : $[T_T]^{-1}{}_{\Delta x, \Delta y} = [T_T]_{-\Delta x, -\Delta y}$

Rotation : $[T_R]^{-1}{}_{\theta} = [T_R]_{-\theta}$

Scaling : $[T_S]^{-1}{}_{S_x, S_y} = [T_S]_{1/S_x,\, 1/S_y}$

Reflection : $[T_M]^{-1}{}_{y=0} = [T_M]_{y=0}$ and $[T]^{-1}{}_{x=0} = [T_M]_{x=0}$

In the following sections we will derive the transformation matrices for general pivot point rotation, general fixed point scaling and general reflection (about any line), using the basic matrices for pure transformation, (such as rotation and scaling with respect to coordinate origin, reflection about coordinate axes) and translation. The technique here is based on the concept of composite transformation and inverse transformation.

4.7.2 General Pivot Point Rotation

Apart from solving a problem from the most basic theory another alternative approach is to derive the solution with reference to the familiar solution of a simpler problem of its kind. The standard solution (transformation matrix) of rotation about origin is much simpler and easy to remembers compared to that of rotation about an arbitrary point other than origin. Hence in an alternative approach we will convert our present problem to a simple problem of rotation about the origin with the help of some additional translations. The sequential steps to be performed in this regard are as follows:

1. Translate the pivot point (x_p, y_p) and the object by the same amount such that the pivot point moves to the coordinate origin while the relative distances between the pivot point and the object points remains unchanged.
2. Rotate the object about the origin (by θ).
3. Translate the rotated object by an amount such that the pivot point comes back to its original position.

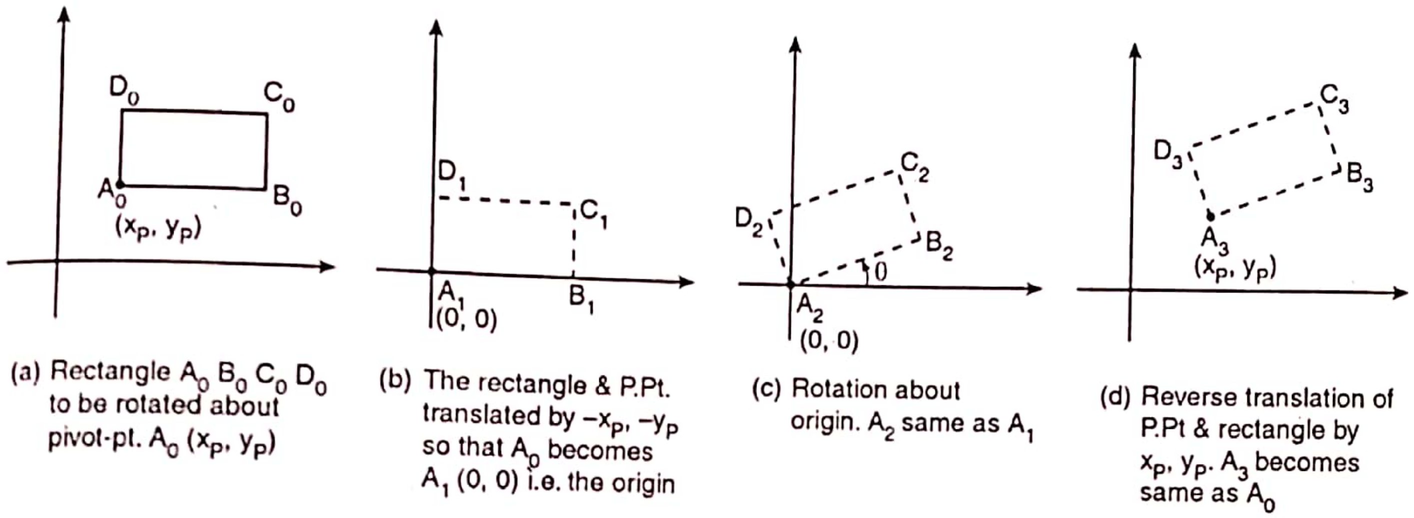


Fig. 4.27

Now combining the matrices corresponding to above transformations sequence wise the resultant transformation matrix is, $[T_{\text{COMB}}] = [T_T]^{-1}_{-x_P,\,-y_P}\,[T_R]_\theta\,[T_T]_{-x_P,-y_P}$

$$= \begin{pmatrix} 1 & 0 & x_P \\ 0 & 1 & y_P \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_P \\ 0 & 1 & -y_P \\ 0 & 0 & 1 \end{pmatrix}$$

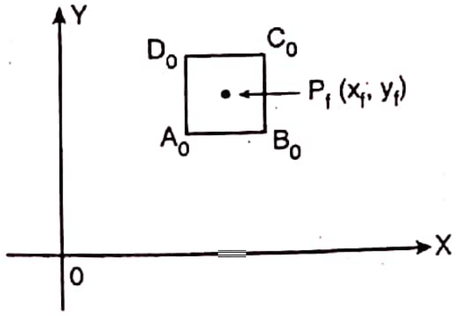
$$= \begin{pmatrix} \cos\theta & -\sin\theta & (1-\cos\theta)x_P + \sin\theta\ y_P \\ \sin\theta & \cos\theta & (1-\cos\theta)y_P - \sin\theta\ x_P \\ 0 & 0 & 1 \end{pmatrix}$$

This is the same transformation matrix we obtained for general pivot point rotation in eqn. (15).

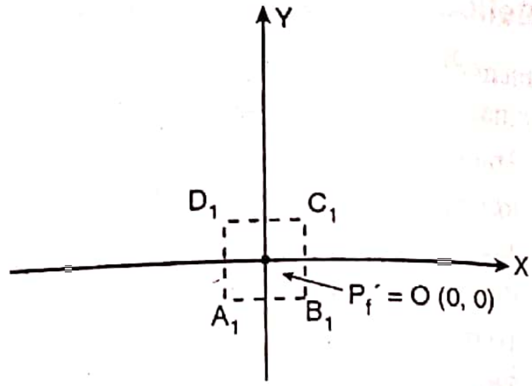
The same principle,

Reducing a complex transformation problem to its simplest basic form at the cost of some additional to and fro translational or rotational movement is followed while deriving the transformation matrix for general fixed-point scaling or general reflection.

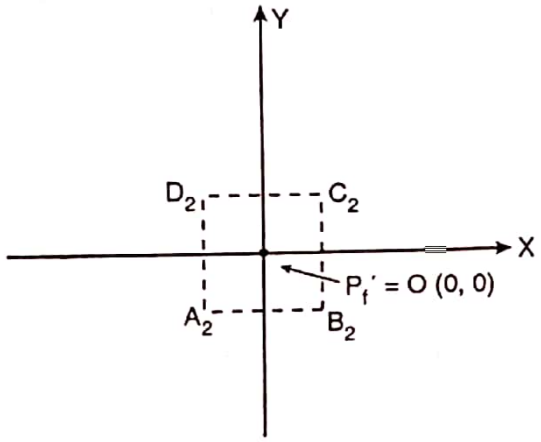
4.7.3 General Fixed Point Scaling



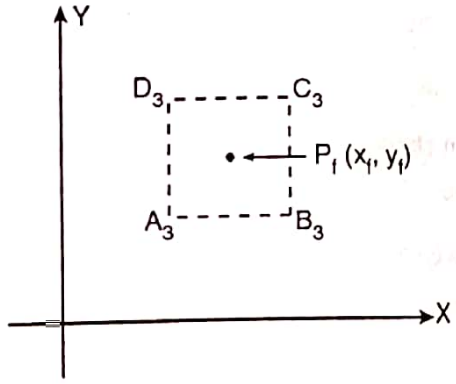
(a) Original position of object $A_0 B_0 C_0 D_0$ – to be scaled about its centroid P_f



(b) Translation of object so that P_f coincides with origin



(c) Scaling of object w.r.t. origin



(d) Reverse translation of scaled object such that P_f returns to its initial position

Fig. 4.28

Here the problem is to scale an object with respect to a fixed point other than the origin. We will reduce this to the basic and simplest form of scaling with respect to origin through the following steps. (ref Fig. 4.28)

1. Translate the object & the fixed point (x_f, y_f) equally so that the fixed point coincides with the coordinate origin.
2. Scale the translated object with respect to the coordinate origin (with scale factors s_x, s_y).
3. Use the inverse translation of step(1) to return the object & the fixed point to their original positions.

The composite transformation matrix resulting from combination of transformations as per above sequence is given by $[T_{\text{COMB}}] = [T_T]^{-1}_{-x_f,-y_f}[T_s]_{s_x,s_y}[T_T]_{-x_f,-y_f}$

$$= \begin{pmatrix} 1 & 0 & x_f \\ 0 & 1 & y_f \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -x_f \\ 0 & 1 & -y_f \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} s_x & 0 & (1-s_x)x_f \\ 0 & sy & (1-s_y)y_f \\ 0 & 0 & 1 \end{pmatrix}$$

Compare this with eqn. (16).

4.7.4 Reflection through an Arbitrary Line

The problem of reflection of an object through a line that neither passes through the origin nor is parallel to the coordinate axes can be solved using the following steps.

1. Translate the line and the object so that the line passes through the origin.
2. Rotate the line and the object about the origin until the line is coincident with one of the coordinate axes about which we are familiar to perform reflection.
3. Reflect the object about that coordinate axis.
4. To the objects, apply the inverse rotation about the origin
5. Translate the object back to the original location.

In matrix notation $[T_{COMB}] = [T_T]^{-1}\,[T_R]^{-1}\,[T_M]\,[T_R]\,[T_T]$

If the arbitrary mirror line is given by $y = mx + c$ where m = slope of the line = $\tan\theta$, $\theta = \tan^{-1}(m)$ being the angle the line makes with the X axis.

c = intercept on the Y axis made by the line (implying $(0, c)$ is a point on the line).

Then the amount of translation in step (1) should be $(0, -c)$ whereas the amount of rotation in step (2) should be $-\theta$ i.e. $-\tan^{-1}(m)$ about the origin if we prefer to merge the line with the X axis.

So we can say

$$[T_{\text{COMB}}] = [T_T]^{-1}_{0,-c}\,[T_R]^{-1}_{-\theta}\,[T_M]_{y=0}\,[T_R]_{-\theta}\,[T_T]_{0,-c}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} (\cos^2\theta - \sin^2\theta) & 2\sin\theta\cos\theta & -2c\sin\theta\cos\theta \\ 2\sin\theta\cos\theta & (\sin^2\theta - \cos^2\theta) & c(\cos^2\theta - \sin^2\theta) + c \\ 0 & 0 & 1 \end{pmatrix} \tag{17}$$

Put $\theta = \tan^{-1}(m)$

Note that eqn. (17) doesn't hold good for a mirror line parallel to the Y axis, say, $x = k$ straight line where $k \neq 0$. The reason is simple; we assumed that the mirror line $y = mx + c$ makes a finite intercept 'c' on the Y axis, but the line $x = k$ doesn't intersect the Y axis at all. For such cases we will first translate the object and mirror line $-k$ units in X direction to merge the mirror line with Y axis, then reflect the object about Y axis and finally translate the reflected object k units in X direction.

Accordingly, $[T_M]_{x=k} = [T_T]^{-1}_{-k,0}\,[T_M]_{x=0}\,[T_T]_{-k,0}$

$$= \begin{pmatrix} 1 & 0 & k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & -k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 0 & 2k \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

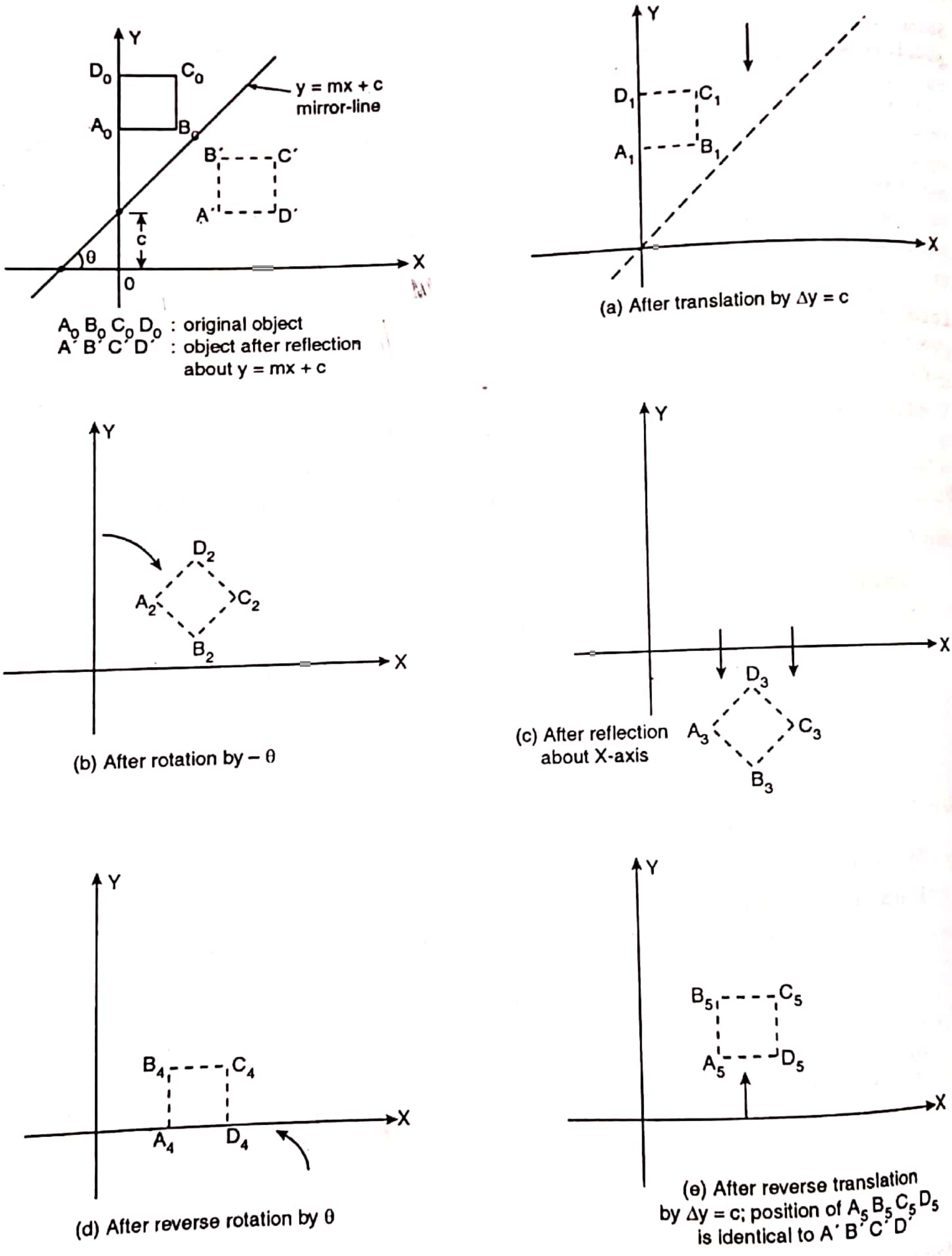

Y
X
D_0
C_0
A_0
B_0
y = mx + c
mirror-line
B′
C′
A′
D′
c
θ
0
$A_0\ B_0\ C_0\ D_0$: original object
A′ B′ C′ D′ : object after reflection about y = mx + c
D_1
C_1
A_1
B_1
(a) After translation by $\Delta y = c$
D_2
A_2
C_2
B_2
(b) After rotation by $-\theta$
D_3
A_3
C_3
B_3
(c) After reflection about X-axis
B_4
C_4
A_4
D_4
(d) After reverse rotation by θ
B_5
C_5
A_5
D_5
(e) After reverse translation by $\Delta y = c$; position of $A_5\ B_5\ C_5\ D_5$ is identical to A′ B′ C′ D′

Fig. 4.29