习　题　答　案

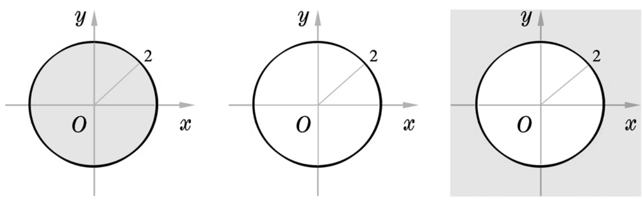
第　一　章

1. (1) $\dfrac{1}{25}(-8+6\mathrm{i})$,　　(2) $2\sum\limits_{k=0}^{[n/2]}(-1)^k\dfrac{n!}{(2k)!(n-2k)!}$.

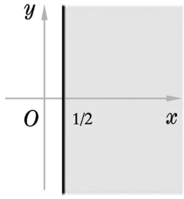
2. 在本题中，$k=0,\pm1,\pm2,\cdots$.

	实　部	虚　部	模	辐　角	附　　注
(1)	1	$\sqrt{3}$	2	$\dfrac{\pi}{3}+2k\pi$	
(2)	$\cos(\sin x)$	$\sin(\sin x)$	1	$\sin x+2k\pi$	
(3)	$\mathrm{e}^{-y}\cos x$	$\mathrm{e}^{-y}\sin x$	e^{-y}	$x+2k\pi$	$x=\operatorname{Re}z,\ y=\operatorname{Im}z$
(4)	$\mathrm{e}^{x}\cos y$	$\mathrm{e}^{x}\sin y$	e^{x}	$y+2k\pi$	$x=\operatorname{Re}z,\ y=\operatorname{Im}z$
(5)	$\cos\phi(x)$	$\sin\phi(x)$	1	$\phi(x)+2k\pi$	
(6)	$1-\cos\alpha$	$\sin\alpha$	$2\sin\dfrac{\alpha}{2}$	$\dfrac{\pi-\alpha}{2}+2k\pi$	

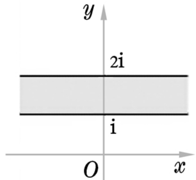
3.



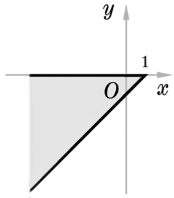
(1)



(2)



(3)



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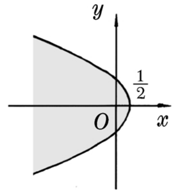

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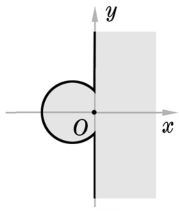

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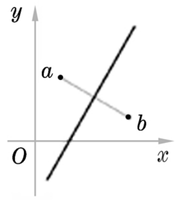
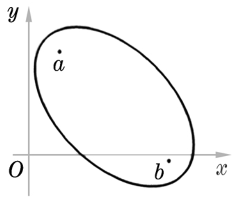


(7)



(8)

4. (1) $\dfrac{\sin(n\phi/2)}{\sin(\phi/2)}\sin\dfrac{n+1}{2}\phi$, (2) $\dfrac{\sin(n\phi/2)}{\sin(\phi/2)}\cos\dfrac{n+1}{2}\phi$.

5.

	聚 点	极 限	上极限	下极限
(1)	$\pm 1/2$	无	$1/2$	$-1/2$
(2)	0	0	0	0
(3)	∞	∞	无	无
(4)	∞	∞	无	无
(5)	$0, \pm 1/2, \pm\sqrt{3}/2, \pm 1$	无	无	无
(6)	$\pm 1/2, \pm 1$	无	1	-1

第　二　章

1. (1) 全平面不可导，不解析；　(2) 全平面不可导，不解析；

   (3) 只在 $z=0$ 点可导，$(z\,\mathrm{Re}\,z)'\big|_{z=0}=0$；全平面不解析；

   (4) 在 $(-1,-1)$ 点可导，导数值为 $(-2,-2)$；除此点外，处处不可导，全平面不解析；

   (5) 只在抛物线 $x=y^2$ 上可导，导数值为 $(6x,0)$；全平面不解析；

   (6) 只在直线 $y=x-1$ 上可导，导数值为 $(2,2)$；全平面不解析.

3. (1) $z^2+z+\mathrm{i}C$;　(2) $\dfrac{1}{z}+\mathrm{i}C$;　(3) $\mathrm{e}^{-\mathrm{i}z}+\mathrm{i}C$;　(4) $\cos z+\mathrm{i}C$.

4. (1) $1-\mathrm{i}$;　(2) $\cos z$.

5. $-\mathrm{i}z^3+(1+\mathrm{i})C$.

6. (1) $z=\dfrac{\pi}{4}+2n\pi+\dfrac{\mathrm{i}}{2}\ln 2,\ \dfrac{3\pi}{4}+2n\pi-\dfrac{\mathrm{i}}{2}\ln 2,\quad n=0,\pm1,\pm2,\cdots$;

   (2) $z=2n\pi\pm\mathrm{i}\ln\left(4+\sqrt{15}\right),\quad n=0,\pm1,\pm2,\cdots$;

   (3) 无解；

   (4) $z=\left(2n\pm\dfrac{1}{3}\right)\pi\mathrm{i},2n\pi\mathrm{i},\ n=0,\pm1,\pm2,\cdots$.

7. (1) $z=1-\mathrm{e}^{2k\pi\mathrm{i}/n}=-2\mathrm{i}\mathrm{e}^{k\pi\mathrm{i}/n}\sin\dfrac{k\pi}{n},\ k=0,1,2,\cdots,n-1$.

8. (1) 多值；　(2) 多值；　(3) 多值；　(4) 单值；

   (5) 单值；　(6) 多值；　(7) 多值；　(8) 单值.

9. (1) 分支点为 a 和 b；　(2) 分支点为 a, b, ∞；

   (3) 分支点为 1, $\mathrm{e}^{2\pi\mathrm{i}/3}$, $\mathrm{e}^{4\pi\mathrm{i}/3}$, ∞;　(4) 分支点为 1, $\mathrm{e}^{2\pi\mathrm{i}/3}$, $\mathrm{e}^{4\pi\mathrm{i}/3}$;

   (5) 分支点为 $\pm\mathrm{i}$ 和 ∞；　(6) 分支点为 $\pm\dfrac{2n+1}{2}\pi$, $n=0,\pm1,\pm2,\cdots$.

10. 当割线上岸 $\arg(z-2)=0$ 时，在割线下岸 $w(3)=3\mathrm{e}^{2\pi\mathrm{i}/3}$.

    共有三个单值分支．对于另外两个单值分支，可分别规定在割线上岸 $\arg(z-2)=2\pi$ 或 4π，在割线下岸 $w(3)$ 分别为 $3\mathrm{e}^{4\pi\mathrm{i}/3}$ 和 3.

11. 逆时针移动通过 x 轴时 $w(2)=\sqrt{2}$；回到原点时 $w(0)=-\sqrt{2}$.

12. 图 2.10(a)：$w(3)=3\ln 2+\mathrm{i}\pi$;

    图 2.10(b)：$w(3)=3\ln 2-\mathrm{i}\pi$;

    图 2.10(c)：在割线上岸 $w(3)=3\ln 2-\mathrm{i}\pi$，在割线下岸 $w(3)=3\ln 2+\mathrm{i}\pi$.

13. 按照题中有关 $\ln z$ 单值分支的规定，等式 $\ln\left(z^2\right)=2\ln z$ 只在 $-\pi/2<\arg z<\pi/2$ 的条件下成立.

14. (1) $2n\pi\mathrm{i},\ n=0,\pm1,\pm2,\cdots$；$\left(2n+\dfrac{1}{2}\right)\pi\mathrm{i},\ n=0,\pm1,\pm2,\cdots$；

$(2n+1)\pi\mathrm{i},\ n=0,\pm1,\pm2,\cdots$；

$\dfrac{1}{2}\ln 2+\left(2n+\dfrac{1}{4}\right)\pi\mathrm{i},\ n=0,\pm1,\pm2,\cdots$；

(2) $\exp\left\{-2n\pi+\mathrm{i}\ln 2\right\},\ n=0,\pm1,\pm2,\cdots$；

$\exp\left\{-(2n+1/2)\pi\right\},\ n=0,\pm1,\pm2,\cdots$；

$\exp\left\{(2n+1)\pi\right\},\ n=0,\pm1,\pm2,\cdots$；

$\exp\left\{-\left(2n+\dfrac{1}{4}\right)\pi+\dfrac{\mathrm{i}}{2}\ln 2\right\},\ n=0,\pm1,\pm2,\cdots.$

15. $f(\mathrm{i})=2^{p/2}\mathrm{e}^{-3p\pi\mathrm{i}/4},\quad f(-\mathrm{i})=2^{p/2}\mathrm{e}^{-5p\pi\mathrm{i}/4},\quad f(\infty)=\mathrm{e}^{-\mathrm{i}\pi p}.$

16. $w'(2)=1/5$. 实际上，在任意一个确定的单值分支内，都有同样大小的导数值，这是因为

$$\left(\arctan z\right)'=\frac{1}{1+z^2}$$

单值. 规定单值分支是为了保证可求导数.

第　三　章

1. (1) (i) $2+2\mathrm{i}$；　(ii) $2+\mathrm{i}$；

   (2) (i) $2\mathrm{i}-2$；　(ii) $-2\mathrm{i}-2$.

2. (1) $2\pi\mathrm{i}$；　(2) 0；　(3) 0；　(4) 2π.

3. (1) (i) 0；　(ii) $\dfrac{\sqrt{2}}{2}\pi\mathrm{i}$；　(iii) $\sqrt{2}\pi\mathrm{i}$；　(iv) $\sqrt{2}\pi\mathrm{i}$；

   (2) (i) $\dfrac{\pi}{\mathrm{e}}$；　(ii) $-2\pi\sinh 1$；　(iii) $-2\pi\sinh 1$；　(iv) $-4\pi\sinh 1$.

4. (1) $2\pi\mathrm{i}$；　(2) 0；　(3) $2\pi\mathrm{i}\sin 1$；　(4) $4\pi\mathrm{i}$.

   (5) $2\pi\mathrm{i}$；　(6) $4\pi\mathrm{i}$；　(7) $-\dfrac{1}{3}\pi\mathrm{i}$；　(8) 0.

5. (1) $\pi\mathrm{i}$；　(2) $a=-2$.

6. (1) $\begin{cases} 16\pi\mathrm{i}, & z=0, \\ \dfrac{32\pi\mathrm{i}}{z^2}(\mathrm{e}^z-1-z), & |z|<2, \text{且 } z\neq 0, \\ -\dfrac{32\pi\mathrm{i}}{z^2}(1+z), & |z|>2. \end{cases}$

   (2) $-2\pi\mathrm{i}\ln\dfrac{3}{5}$.

第 四 章

1. (1) 收敛但不绝对收敛； (2) 收敛但不绝对收敛.

2. 级数和 $S=\begin{cases}\dfrac{1}{(1-z)^2}, & |z|<1,\\ \dfrac{1}{z(1-z)^2}, & |z|>1.\end{cases}$

3. (1) $\ln 2-\dfrac{1}{2}$； (2) $\dfrac{\pi}{8}-\dfrac{1}{2}\ln 2$；

   (3) $\dfrac{1}{4}\ln 2$； (4) $\dfrac{\pi}{8}$.

4. (1) $|z|<1$； (2) $\operatorname{Re} z>-\dfrac{1}{2}$；

   (3) $\left|z^2+2z+1\right|<1$； (4) 在全平面收敛.

5. 级数和 $S=\begin{cases}2z-\ln(1+z), & |z|<1,\\ 2-2\ln 2, & z=1.\end{cases}$

7. (1) $-\ln\left(2\sin\dfrac{\theta}{2}\right),\ \dfrac{1}{2}(\pi-\theta)$； (2) $\dfrac{1}{2}\ln\cot\dfrac{\theta}{2},\ \dfrac{\pi}{4}\operatorname{sgn}\theta$；

   (3) $\dfrac{\pi}{4}\theta$； (4) $\dfrac{\pi}{2\sqrt{3}}$.

8. (1) $R=\infty$； (2) $R=\infty$； (3) $R=\mathrm{e}$； (4) $R=\infty$；

   (5) $R=1$； (6) $R=2$； (7) $R=\infty$； (8) $R=1$.

第　五　章

1. (1) $-2(z-1)-(z-1)^2$, $|z-1|<\infty$；　(2) $\sum\limits_{k=0}^{\infty}\frac{(-)^{n+k}}{(2k+1)!}(z-n\pi)^{2k+1}$, $|z-n\pi|<\infty$；

(3) $\sum\limits_{n=0}^{\infty}\frac{\sin 2(n+1)\pi/3}{\sin 2\pi/3}z^n$, $|z|<1$；　(4) $\sum\limits_{n=1}^{\infty}\left[\sum\limits_{k=0}^{[(n-1)/2]}\frac{(-)^k}{(2k+1)!}\right]z^n$, $|z|<1$；

(5) $\mathrm{e}\left(1+z+\frac{3}{2}z^2+\frac{13}{6}z^3+\frac{73}{24}z^4+\cdots\right)=\sum\limits_{n=0}^{\infty}\frac{1}{n!}\left.\frac{\mathrm{d}^n\left(z^{n-1}\mathrm{e}^z\right)}{\mathrm{d}z^n}\right|_{z=1}z^n,\quad |z|<1$；

2. (1) $\frac{\pi\mathrm{i}}{2}-\sum\limits_{n=1}^{\infty}\frac{\mathrm{i}^n}{n}(z-\mathrm{i})^n$, $|z-\mathrm{i}|<1$；　(2) $-\frac{3}{2}\pi\mathrm{i}-\sum\limits_{n=1}^{\infty}\frac{\mathrm{i}^n}{n}(z-\mathrm{i})^n$, $|z-\mathrm{i}|<1$；

(3) $\sum\limits_{n=0}^{\infty}\frac{(-)^n}{2n+1}z^{2n+1}$, $|z|<1$；　(4) $(2k+1)\pi\mathrm{i}+\sum\limits_{n=0}^{\infty}\frac{2}{2n+1}z^{-(2n+1)}$, $|z|>1$.

3. (1) $\frac{1}{2}\ln\frac{1+z}{1-z}=\operatorname{arc\,tanh} z$, $|z|<1$, 规定 $\left.\ln\frac{1+z}{1-z}\right|_{z=0}=0$；

(2) $\cosh z$，$|z|<\infty$；

(3) $\frac{1}{1-z}$，收敛区域为 $|z|<2$ 与 $\operatorname{Re} z<1$ 的公共区域；

(4) $\frac{1}{1-z}$，收敛区域为 $|z|<3$ 与 $\operatorname{Re} z<3/2$ 及 $|z-2|<1$ 的公共区域.

4. (1) $\sum\limits_{n=-1}^{\infty}(-)^{n+1}(n+2)(z-1)^n$, $0<|z-1|<1$；

(2) $\sum\limits_{n=3}^{\infty}z^{-n}$;

(3) $-\sum\limits_{n=-1}^{-\infty}z^n-\sum\limits_{n=0}^{\infty}\frac{1}{2^{n+1}}z^n$;　(4) $\sum\limits_{n=2}^{\infty}\left(2^{n-1}-1\right)z^{-n}$;

(5) $1-\frac{3}{2}\sum\limits_{n=0}^{\infty}\frac{1}{2^{2n}}z^n-2\sum\limits_{n=-1}^{-\infty}\frac{1}{3^{n+1}}z^n$;　(6) $1+\sum\limits_{n=1}^{\infty}\left(3\times 2^{2n-1}-2\times 3^{n-1}\right)z^{-n}$.

5. (1) $\sum\limits_{n=1}^{\infty}\left[\sum\limits_{k=1}^{2n-1}\frac{(-)^{k-1}}{k(2n-k)}\right]z^{2n}=\sum\limits_{n=1}^{\infty}\frac{1}{n}\left[\sum\limits_{k=1}^{2n-1}\frac{(-)^{k-1}}{k}\right]z^{2n},\quad |z|<1$；

(2) $\sum\limits_{n=1}^{\infty}(-)^{n-1}\left[\sum\limits_{k=0}^{n-1}\frac{1}{(2k+1)(n-k)}\right]z^{2n+1}=2\sum\limits_{n=1}^{\infty}\frac{(-)^{n-1}}{2n+1}\left[\sum\limits_{k=1}^{2n}\frac{1}{k}\right]z^{2n+1}$, $|z|<1$;

6. (1) $z=\pm a\mathrm{i}$，一阶极点；

(2) $z=0$，二阶极点；$z=\infty$，本性奇点；

(3) $z=0$，可去奇点；$z=\infty$，本性奇点；

(4) $z=0$，可去奇点；$z=\infty$，本性奇点；

(5) $z=0$，本性奇点；

(6) $z=0$，可去奇点；　$z=(n\pi)^2,\ n=1,2,\cdots$，一阶极点；　$z=\infty$，非孤立奇点；

(7) $z=1$ 在 $\ln z\big|_{z=1}=0$ 的单值分支内是二阶极点，在其它单值分支内是一阶极点；

(8) $z=\infty$，本性奇点.

7. (1) 二阶极点；　(2) $z=\infty$ 解析；　(3) 本性奇点；　(4) 非孤立奇点；

(5) 本性奇点；　(6) $z=\infty$ 解析；　(7) 为非孤立奇点；　(8) 一阶极点.

第 六 章

1. (1) e; (2) 2e; (3) 0; (4) $\dfrac{1}{6}$;

   (5) 0; (6) $(-)^n(2n+1)\pi$.

2. (1) $\operatorname{res} f(0)=1$, $\operatorname{res} f(\pm 1)=-1/2$;

   (2) $\operatorname{res} f(\pm \mathrm{i})=\mp \mathrm{i}\dfrac{(2m)!}{(m!)^2\, 2^{2m+1}}$;

   (3) $\operatorname{res} f(2n\pi)=2$, $n=0, \pm 1, \pm 2, \cdots$;

   (4) $\operatorname{res} f(0)=0$ ($z=0$为可去奇点), $\operatorname{res} f\big(-n^2\pi^2\big)=(-)^{n+1}2(n\pi)^2$, $n=1,2,\cdots$;

   (5) $\operatorname{res} f(0)=-\mathrm{J}_1(1)$, $\operatorname{res} f(\infty)=\mathrm{J}_1(1)$;

   (6) $\operatorname{res} f(0)=-\dfrac{1}{2}$;

   (7) $\ln z\big|_{z=1}=0$ 时，$\operatorname{res} f(1)=\dfrac{1}{2}$;

   $\ln z\big|_{z=1}=2n\pi\mathrm{i}$ 时，$\operatorname{res} f(1)=\dfrac{1}{2n\pi\mathrm{i}}$, $n=\pm 1, \pm 2, \cdots$;

   (8) $\operatorname{res} f(0)=n+1$, $\operatorname{res} f(-1)=-n$;

3. (1) $\operatorname{res} f(\infty)=-1$; (2) $\operatorname{res} f(\infty)=-1$;

   (3) $z=\infty$ 是非孤立奇点，无留数可言； (4) $\operatorname{res} f(\infty)=0$;

   (5) $\operatorname{res} f(\infty)=0$; (6) $\operatorname{res} f(\infty)=\pm\dfrac{1}{8}$，符号视单值分支而定.

4. (1) $-\dfrac{\pi\mathrm{i}}{\sqrt{2}}$; (2) 0; (3) $\dfrac{\pi\mathrm{i}}{\sqrt{2}}$; (4) $\sqrt{2}\pi\mathrm{i}$;

   (5) $-4n\mathrm{i}$; (6) 0; (7) $\pi\mathrm{i}$; (8) $2n+1$.

5. (1) $\dfrac{(2n-1)!!}{(2n)!!}2\pi=\dfrac{(2n)!}{(n!)^2}\dfrac{\pi}{2^{2n-1}}$; (2) $\dfrac{2a\pi}{\big(a^2-b^2\big)^{3/2}}$;

   (3) $\dfrac{\pi}{\sqrt{2}}$; (4) $\dfrac{3\pi}{4\sqrt{2}}$.

6. (1) $\dfrac{\pi}{\sqrt{2}}$; (2) $\dfrac{(2n-1)!!}{(2n)!!}\pi=\dfrac{(2n)!}{(n!)^2}\dfrac{\pi}{2^{2n}}$;

   (3) $\dfrac{\pi}{n\sin\dfrac{2m+1}{2n}\pi}$; (4) $2\ln 2$.

7. (1) $\dfrac{\pi}{2\sqrt{2}}\mathrm{e}^{-1/\sqrt{2}}\Big(\sin\dfrac{1}{\sqrt{2}}+\cos\dfrac{1}{\sqrt{2}}\Big)=\dfrac{\pi}{2}\mathrm{e}^{-1/\sqrt{2}}\sin\Big(\dfrac{1}{\sqrt{2}}+\dfrac{\pi}{4}\Big)$;

   (2) $\dfrac{7\pi}{16\mathrm{e}}$; (3) $\dfrac{\pi}{\mathrm{e}}\big(\sin 1+\cos 1\big)$; (4) $\pi\,\mathrm{e}^{-a-1}\dfrac{1-\mathrm{e}^{-2n}}{1-\mathrm{e}^{-2}}$.

8. (1) 0; (2) $\dfrac{\pi}{4a}\sin 2a$; (3) $\dfrac{\pi}{2}\left(\dfrac{1}{2}-\dfrac{1}{\mathrm{e}}\right)$; (4) $\pi(\cot p\pi-\cot q\pi)$;

9. (1) $\pi\cot\pi s$; (2) $\dfrac{\pi}{4}\dfrac{1-s}{\cos(\pi s/2)}$; (3) $-\pi^2\dfrac{\sin\pi a}{\cos^2\pi a}$; (4) $\dfrac{1}{2}\dfrac{\ln^2 b-\ln^2 a}{b-a}$.

第 七 章

1\. (1) $2^n\Gamma(n+1)$; (2) $\dfrac{\Gamma(2n+1)}{2^n\Gamma(n+1)}$; (3) $\dfrac{\Gamma(n+\nu+1)}{\Gamma(\nu+1)}$;

(4) $\dfrac{\Gamma(n+\nu+2)}{\Gamma(\nu+1)}\dfrac{\Gamma(n-\nu+1)}{\Gamma(-\nu)}=-\dfrac{\sin\pi\nu}{\pi}\Gamma(n+\nu+2)\Gamma(n-\nu+1)$.

2\. (1) $\Gamma(1-\alpha)\cos\dfrac{\alpha\pi}{2}$, $\Gamma(1-\alpha)\sin\dfrac{\alpha\pi}{2}$; (2) $\Gamma(\alpha)\cos\alpha\theta$, $\Gamma(\alpha)\sin\alpha\theta$.

4\. (1) $2^{p+q+1}\mathrm{B}(p+1,q+1)$; (2) $\dfrac{1}{2}\mathrm{B}\left(\dfrac{1-\alpha}{2},\dfrac{1+\alpha}{2}\right)=\dfrac{\pi}{2\cos(\pi\alpha/2)}$.

5\. (1) $\dfrac{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma)}{\Gamma(\alpha+\beta+\gamma+1)}$; (2) $\dfrac{1}{pqr}\dfrac{\Gamma\left(\dfrac{\alpha}{p}\right)\Gamma\left(\dfrac{\beta}{q}\right)\Gamma\left(\dfrac{\gamma}{r}\right)}{\Gamma\left(\dfrac{\alpha}{p}+\dfrac{\beta}{q}+\dfrac{\gamma}{c}+1\right)}$.

6\. (1) $2\ln 2-1$; (2) $\dfrac{\pi}{2}\coth\pi+\dfrac{\pi^2}{2}\dfrac{1}{\sinh^2\pi}$.

第 八 章

1. (1) $\dfrac{n!}{p^{n+1}}$; (2) $\dfrac{\Gamma(\alpha+1)}{p^{\alpha+1}}$; (3) $\dfrac{\omega}{(p+\lambda)^2+\omega^2}$; (4) $\arctan\dfrac{\omega}{p}$;

   (5) $-\dfrac{p}{2}\ln\dfrac{p^2+\omega^2}{p^2}+\omega\arctan\dfrac{\omega}{p}$; (6) $\dfrac{1}{2p}\ln\left(1+p^2\right)$.

3. (1) $\dfrac{\omega}{p^2+\omega^2}\coth\dfrac{p\pi}{2\omega}$; (2) $\dfrac{1}{p^2}-\dfrac{a}{p}\dfrac{\mathrm{e}^{-pa}}{1-\mathrm{e}^{-pa}}$.

4. (1) $\left[1-\mathrm{e}^{-at}\left(1+at+\dfrac{1}{2}a^2t^2\right)\right]\eta(t)$; (2) $\dfrac{1}{\omega}(1-\cos\omega t)\,\eta(t)$;

   (3) $\left(1+\dfrac{5}{3}\mathrm{e}^{-t}+\dfrac{1}{3}\mathrm{e}^{t/2}-3\mathrm{e}^{-t/2}\right)\eta(t)$; (4) $t\cos\omega t\,\eta(t)$;

   (5) $(t-\tau)\,\eta(t-\tau)$; (6) $\displaystyle\sum_{n=1}^{\infty}\eta(t-na)=\left[\frac{t}{a}\right]\eta(t)$.

5. (1) $\begin{cases}\dfrac{E_0\omega(\cos\omega_1 t-\cos\omega t)}{L\left(\omega^2-\omega_1^2\right)}, & \omega\neq\omega_1,\\[2mm] \dfrac{E_0}{2L}t\sin\omega t, & \omega=\omega_1;\end{cases}$ 其中 $\omega_1=1/\sqrt{LC}$.

   (2) $\begin{cases}\dfrac{E}{R}\left(1+A\mathrm{e}^{-\gamma_1 t}-B\mathrm{e}^{-\gamma_2 t}\right), & k<1,\\[2mm] \dfrac{E}{R}\left(1-\mathrm{e}^{-\alpha t}-\dfrac{\alpha t}{2}\mathrm{e}^{-\alpha t}\right), & k=1,\\[2mm] \dfrac{E}{R}\left\{1-\mathrm{e}^{-\alpha t}\left[\cos\omega t+\dfrac{\alpha-\beta}{\omega}\sin\omega t\right]\right\}, & k>1,\end{cases}$

   其中

   $$\alpha=1/(2RC),\qquad \beta=R/L,\qquad k=2\beta/\alpha,\qquad \omega=\alpha\sqrt{1-k},$$
   $$\gamma_1=\alpha(1-\sqrt{1-k}),\qquad \gamma_2=\alpha(1+\sqrt{1-k}),\qquad A=\frac{\beta-\gamma_2}{\alpha_2-\alpha_1},\qquad B=\frac{\beta-\gamma_1}{\alpha_2-\alpha_1};$$

   (3) $y(t)=at\mathrm{e}^{-t}$; (4) $f(t)=5\mathrm{e}^{2t}+4\mathrm{e}^{-t}-6t\mathrm{e}^{-t}$.

6. (1) $\dfrac{1}{2}\ln\dfrac{b^2+c^2}{a^2+c^2}$; (2) $\dfrac{\pi}{2}b$; (3) $\dfrac{\pi}{2}\left(1-\mathrm{e}^{-|t|}\right)\operatorname{sgn}t$.

7. (1) $\cosh\omega t\,\eta(t)$;

   (2) $\dfrac{1}{4\omega^3}\left[\cosh\omega(t-\tau)\sin\omega(t-\tau)-\sinh\omega(t-\tau)\cos\omega(t-\tau)\right]\eta(t-\tau)$;

   (3) $\eta(t-\alpha)$;

   (4) $\displaystyle\left\{1-\frac{\pi}{4}\sum_{n=0}^{\infty}\frac{1}{2n+1}\sin\frac{2n+1}{2l}\pi x\exp\left[-\left(\frac{2n+1}{2l}\pi\right)^2 t\right]\right\}\eta(t)$

$$= \left[\sum_{n=0}^{\infty} (-)^n \operatorname{erfc} \frac{2nl + x}{2\sqrt{t}} - \sum_{n=1}^{\infty} (-)^n \operatorname{erfc} \frac{2nl - x}{2\sqrt{t}} \right] \eta(t).$$

8. (1) $\frac{1}{3}\left(\frac{\pi}{\sqrt{3}} + \ln 2\right)$;

(2) $\frac{1}{4\sqrt{2}}\left[2\ln\left(1+\sqrt{2}\right) + \pi\right]$;

(3) $\frac{2}{3}\ln 2 - \frac{\pi}{6\sqrt{3}}$;

(4) $\frac{1}{4}\left(\frac{\pi}{\sqrt{3}} - \ln 3\right)$.

第 九 章

1. (1) $(z-1)w''-zw'+w=0$; (2) $z^4w''-(1-2z)z^2w'-2w=0$;

   (3) $z^4w''+2z^3w'+a^2w=0$; (4) $z^2\big(z^2-1\big)w''+2z\big(z^2+1\big)w'-2w=0$.

2. (1) $w_1(z)=\sum\limits_{n=0}^{\infty}\frac{\Gamma(3/4)}{n!\,\Gamma(n+3/4)}\left(\frac{z}{2}\right)^{4n}$, $w_2(z)=\sum\limits_{n=0}^{\infty}\frac{\Gamma(5/4)}{n!\,\Gamma(n+5/4)}\left(\frac{z}{2}\right)^{4n+1}$;

   (2) $w_1(z)=\sum\limits_{n=0}^{\infty}\frac{\Gamma(2/3)}{n!\,\Gamma(n+2/3)}\frac{z^{3n}}{3^{2n}}$, $w_2(z)=\sum\limits_{n=0}^{\infty}\frac{\Gamma(4/3)}{n!\,\Gamma(n+4/3)}\frac{z^{3n+1}}{3^{2n}}$;

   (3) $w_1(z)=z$, $w_2(z)=\sqrt{1-z^2}$.

   (4) $w_1(z)=\frac{1}{1+z+z^2}$, $w_2(z)=\frac{z}{1+z+z^2}$.

3. (1) $w_1(z)=\frac{1}{z}\ln(1-z)+\frac{1}{1-z}$, $w_2(z)=\frac{1}{z}$;

   (2) $w_1(z)=z^{1/3}\sin z^2$, $w_2(z)=z^{1/3}\cos z^2$;

   (3) $w_1(z)=z$, $w_2(z)=z\ln z-1+\sum\limits_{n=2}^{\infty}\frac{1}{(n-1)\,n!}z^n$;

   (4) $w_1(z)=z^2\mathrm{e}^{-z}$,

   $$w_2(z)=z^2\mathrm{e}^{-z}\ln z-1-z+z^2-z^2\sum_{n=2}^{\infty}(-)^n\left(\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)\frac{z^n}{n!}.$$

4. $w_1(z)=\frac{\sin mz}{mz}=\sum\limits_{n=0}^{\infty}\frac{(-)^k}{(2k+1)!}(mz)^{2k}$; $w_2(z)=\frac{\cos mz}{mz}=\sum\limits_{n=0}^{\infty}\frac{(-)^k}{(2k)!}(mz)^{2k-1}$.

5. $w_1(z)=\mathrm{J}_0(\mathrm{i}mz)$;

   $$w_2(z)=\mathrm{J}_0(\mathrm{i}mz)\ln z-\sum_{n=1}^{\infty}\left[\frac{1}{(n!)^2}\left(\frac{1}{n}+\frac{1}{n-1}+\cdots+1\right)\left(\frac{mz}{2}\right)^{2n}\right].$$

第 十 章

2. (1) $\dfrac{1}{k}\sinh k(x-t)\,\eta(x-t)$;

   (2) $2\big[w_2(x)w_1(t)-w_1(x)w_2(t)\big]\eta(x-t)$，

   其中 $w_1(x)$, $w_2(x)$ 见第 9 章习题第 2 (1) 题. 在计算中要用到 $\begin{vmatrix} w_1(x) & w_2(x) \\ w_1'(x) & w_2'(x) \end{vmatrix}=\dfrac{1}{2}$.

   (3) $\dfrac{x-t}{1+x+x^2}\eta(x-t)$.

3. $A\cos kt+\dfrac{B}{k}\sin kt+\dfrac{1}{k}\displaystyle\int_0^t \sin k(t-\tau)f(\tau)\mathrm{d}\tau$;

下面第 3, 4, 8 题的答案中，用到下列简写记号:

$$D_1(t,x)\equiv\begin{vmatrix} w_1(t) & w_2(t) \\ w_1(x) & w_2(x) \end{vmatrix},\qquad D_2(t,x)\equiv\begin{vmatrix} w_1(t) & w_2(t) \\ w_1'(x) & w_2'(x) \end{vmatrix},$$

4. (1) $A\cosh kx+\dfrac{B}{k}\sinh kx+\dfrac{1}{k}\displaystyle\int_0^x \sinh k(x-t)f(t)\mathrm{d}t$;

   (2) $2AD_2(x,0)-2BD_1(x,0)+2\displaystyle\int_0^x D_1(t,x)f(t)\mathrm{d}t$.

5. (1) $-\dfrac{1}{k}\dfrac{\sinh k(1-t)}{\sinh k}\sinh kx+\dfrac{1}{k}\sinh k(x-t)\,\eta(x-t)$;

   (2) $-\dfrac{2w_2(x)}{w_2(1)}D_1(t,1)+2D_1(t,x)\eta(x-t)$;

   (3) $-\dfrac{l-t}{l}\dfrac{x}{1+x+x^2}+\dfrac{x-t}{1+x+x^2}\eta(x-t)$.

6. $-\dfrac{1}{k}\mathrm{e}^{-k\xi}\sinh kx+\dfrac{1}{k}\sinh k(x-\xi)\,\eta(x-\xi)=\begin{cases}-\dfrac{1}{k}\mathrm{e}^{-k\xi}\sinh kx, & 0<x<\xi,\\[2mm] -\dfrac{1}{k}\mathrm{e}^{-kx}\sinh k\xi, & x>\xi.\end{cases}$

7. (1) $-\dfrac{1}{k}\mathrm{e}^{-k\xi}\cosh kx+\dfrac{1}{k}\sinh k(x-\xi)\,\eta(x-\xi)=\begin{cases}-\dfrac{1}{k}\mathrm{e}^{-k\xi}\cosh kx, & 0<x<\xi,\\[2mm] -\dfrac{1}{k}\mathrm{e}^{-kx}\cosh k\xi, & x>\xi;\end{cases}$

   (2) $-\dfrac{1}{k\cos k(b-a)}\big[\cos k(b-\xi)\,\sin k(x-a)+\sin(b-a+x-\xi)\,\eta(x-\xi)\big]$

$$=\begin{cases}-\dfrac{\cos k(b-\xi)}{k\cos k(b-a)}\,\sin k(x-a), & a<x<\xi,\\[2mm] -\dfrac{\sin k(\xi-a)}{k\cos k(b-a)}\,\cos k(b-x), & \xi<x<b.\end{cases}$$

8. (1) $B\dfrac{\sin kx}{\sin k}+A\dfrac{\sin k(1-x)}{\sin k}-\dfrac{\sin kx}{k\sin k}\displaystyle\int_0^1 \sin k(1-t)f(t)\mathrm{d}t+\dfrac{1}{k}\int_0^x \sin k(x-t)f(t)\mathrm{d}t$;

(2) $B\dfrac{\sinh kx}{\sinh k}+A\dfrac{\sinh k(1-x)}{\sinh k}-\dfrac{\sinh kx}{k\sinh k}\displaystyle\int_0^1 \sinh k(1-t)f(t)\mathrm{d}t+\dfrac{1}{k}\int_0^x \sinh k(x-t)f(t)\mathrm{d}t$;

(3) $B\dfrac{w_2(x)}{w_2(1)}+2A\dfrac{D_1(x,1)}{w_2(1)}-\dfrac{2w_2(x)}{w_2(1)}\displaystyle\int_0^1 D_1(t,1)f(t)\mathrm{d}t+2\int_0^x D_1(t,x)f(t)\mathrm{d}t$.

第 十 一 章

1. 存在阻尼时，波动方程变为 $\rho\dfrac{\partial^2 u}{\partial t^2}+\alpha\dfrac{\partial u}{\partial t}=T\dfrac{\partial^2 u}{\partial x^2}$,

   若同时存在阻尼与弹性恢复力，则方程为 $\rho\dfrac{\partial^2 u}{\partial t^2}+\alpha\dfrac{\partial u}{\partial t}=T\dfrac{\partial^2 u}{\partial x^2}-ku$.

2. $\dfrac{\partial^2 u}{\partial t^2}-a^2\dfrac{\partial^2 u}{\partial x^2}=0$,

   $u\Big|_{x=0}=0,\qquad \dfrac{\partial u}{\partial x}\Big|_{x=l}=0$,

   $u\Big|_{t=0}=\dfrac{F}{ES}x,\quad \dfrac{\partial u}{\partial t}\Big|_{t=0}=0$.

3. $\dfrac{\partial u}{\partial t}-\dfrac{\kappa}{\rho c}\dfrac{\partial^2 u}{\partial x^2}=\dfrac{I^2R}{\rho cS}$,

   $\left(\dfrac{\partial u}{\partial x}-\dfrac{H}{k}u\right)_{x=0}=-\dfrac{H}{k}u_0,\qquad \left(\dfrac{\partial u}{\partial x}+\dfrac{H}{k}u\right)_{x=0}=+\dfrac{H}{k}u_0$,

   $u\Big|_{t=0}=u_0\left(1-\dfrac{2x}{l}\right)^3$.

4. $\dfrac{\partial u}{\partial t}=D\nabla^2 u+\alpha u$，$D$ 是扩散率.

5. $\dfrac{\partial u}{\partial x}\Big|_{x=0}=-\dfrac{q_1}{k},\qquad \dfrac{\partial u}{\partial x}\Big|_{x=l}=\dfrac{q_2}{k}$.

6. 采用球坐标系，坐标原点在球心，极轴指向太阳，则边界条件为

$$\left(\frac{\partial u}{\partial r}+\frac{H}{k}u\right)_{r=a}=\begin{cases}\dfrac{M}{k}\cos\theta, & 0\leqslant\theta\leqslant\dfrac{\pi}{2},\\ 0, & \dfrac{\pi}{2}<\theta\leq\pi,\end{cases}$$

   其中 H 是牛顿冷却定律中的比例常数.

第 十 二 章

1. (1) $u=f(3x+y)+g(x-y)$;

(2) $u=f(x+y+\mathrm{i}x)+g(x+y-\mathrm{i}x)$;

(3) $u=f(x+y)+g(y)$;

(4) $u=\dfrac{1}{r}\big[f(r+ct)+g(r-ct)\big]$;

(5) $u=f(x-(a+b)t)+g(x-(a-b)t)$;

(6) $u=f_1(x+y)+f_2(x-y)+f_3(x+\mathrm{i}y)+f_4(x-\mathrm{i}y)$.

2. (1) $u=f(x+\mathrm{i}y)+g(x-\mathrm{i}y)+\dfrac{1}{12}x^4+\dfrac{1}{6}x^3y$;

(2) $u=f(x+y)+g(x-y)+\dfrac{1}{6}x^3(y-1)$;

(3) $u=f(x+y)+xg(x+y)+\dfrac{1}{12}x^4+\dfrac{1}{6}y^3$.

3. (1) $u=f(xy)\ln x+g(xy)$;

(2) $u=f(y+x)+g(y-x)+\sin xy$.

4. $u(x,t)=\begin{cases}\dfrac{1}{2}\Big[\phi(x+at)-\phi(at-x)\Big]+\dfrac{1}{2a}\displaystyle\int_{at-x}^{x+at}\psi(\xi)\mathrm{d}\xi, & 0<x<at,\\ \dfrac{1}{2}\Big[\phi(x+at)+\phi(x-at)\Big]+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi, & at\leqslant x<\infty.\end{cases}$

5. 通解 $u(x,t)=\dfrac{1}{l-x}\big[f(x+at)+g(x-at)\big]$；由初条件可定出

$$u(x,t)=\frac{1}{l-x}\left\{\frac{1}{2}\big[(l-x+at)\phi(x+at)+(l-x-at)\phi(x-at)\big]+\frac{1}{2a}\int_{x-at}^{x+at}(l-\xi)\psi(\xi)\mathrm{d}\xi\right\}.$$

第 十 三 章

1. $u(x,t)=\dfrac{8Fl}{ES\pi^2}\sum_{n=0}^{\infty}\dfrac{(-)^n}{(2n+1)^2}\sin\dfrac{2n+1}{2l}\pi x\cos\dfrac{2n+1}{2l}\pi at.$

2. $u(x,t)=\dfrac{2hl^2}{\pi^2 c(l-c)}\sum_{n=1}^{\infty}\dfrac{1}{n^2}\sin\dfrac{n\pi}{l}c\,\sin\dfrac{n\pi}{l}x\,\cos\dfrac{n\pi}{l}at.$

3. $u(x,t)=\dfrac{8b}{\pi^3}\sum_{n=0}^{\infty}\dfrac{1}{(2n+1)^3}\sin\dfrac{2n+1}{l}\pi x\,\exp\left\{-\left(\dfrac{2n+1}{l}\pi\right)^2\kappa t\right\}.$

4. $u(x,y,t)=\dfrac{64Al^4}{\pi^6}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\dfrac{1}{(2n+1)^3(2m+1)^3}\sin\dfrac{2n+1}{l}\pi x\,\sin\dfrac{2m+1}{l}\pi y\,\cos\omega_{nm}t,$

$$\omega_{nm}=\sqrt{(2n+1)^2+(2m+1)^2}\,\frac{\pi}{l}a.$$

5. $u(x,y)=u_0\left(1-\dfrac{x}{2a}\right)-\dfrac{48u_0}{\pi^4}\sum_{n=0}^{\infty}\dfrac{1}{(2n+1)^4}\dfrac{\sinh\dfrac{2n+1}{b}\pi x}{\sinh\dfrac{2n+1}{b}\pi a}\cos\dfrac{2n+1}{b}\pi y.$

6. $u(x,t)=\dfrac{8bl^4}{\pi^5 a^2}\sum_{n=0}^{\infty}\dfrac{1}{(2n+1)^5}\left(1-\cos\dfrac{2n+1}{l}\pi at\right)\sin\dfrac{2n+1}{l}\pi x$

$$=\frac{b}{12a^2}x^2(l-x)^2+\frac{bl^2}{12a^2}x(l-x)-\frac{8bl^4}{\pi^5a^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\sin\frac{2n+1}{l}x\,\cos\frac{2n+1}{l}\pi at.$$

7. (1) $u(x,y)=\dfrac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\dfrac{1}{(2n+1)^3}\left\{1-\dfrac{\cosh\dfrac{2n+1}{a}\pi y}{\cosh\dfrac{2n+1}{2a}\pi b}\right\}\sin\dfrac{2n+1}{a}\pi x$

$$=x(a-x)-\frac{8a^2}{\pi^3}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^3}\frac{\cosh\dfrac{2n+1}{a}\pi y}{\cosh\dfrac{2n+1}{2a}\pi b}\sin\frac{2n+1}{a}\pi x;$$

(2) $u(x,y)=\dfrac{2a^4}{\pi^3}\sum_{n=1}^{\infty}\dfrac{(-)^{n-1}}{n^3}\left\{y-\dfrac{b}{2}\dfrac{\sinh\dfrac{n\pi}{a}y}{\sinh\dfrac{n\pi}{2a}b}\right\}\sin\dfrac{n\pi}{a}x$

$$-\frac{8a^4}{\pi^5}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\left\{y-\frac{b}{2}\frac{\sinh\dfrac{2n+1}{a}\pi y}{\sinh\dfrac{2n+1}{2a}\pi b}\right\}\sin\frac{2n+1}{a}\pi x$$

$$=\frac{1}{12}xy\left(a^3-x^3\right)+\frac{a^4b}{\pi^3}\sum_{n=1}^{\infty}\frac{(-)^n}{n^3}\frac{\sinh\dfrac{n\pi}{a}y}{\sinh\dfrac{n\pi}{2a}b}\sin\frac{n\pi}{a}x$$

$$+\frac{4a^4b}{\pi^5}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^5}\frac{\sinh\dfrac{2n+1}{a}\pi y}{\sinh\dfrac{2n+1}{2a}\pi b}\sin\frac{2n+1}{a}\pi x.$$

8. $u(x,t)=\dfrac{Aa}{ES\omega}\dfrac{\sin\dfrac{\omega}{a}x}{\cos\dfrac{\omega}{a}l}\sin\omega t+\dfrac{4A\omega}{ES\pi a}\displaystyle\sum_{n=0}^{\infty}\frac{(-)^n}{2n+1}\frac{\sin\dfrac{2n+1}{2l}\pi x}{\left(\dfrac{\omega}{a}\right)^2-\left(\dfrac{2n+1}{2l}\pi\right)^2}\sin\frac{2n+1}{2l}\pi at.$

如果 ω 正好是杆的某一个固有频率，例如 $\omega=\omega_m=(2m+1)\pi a/(2l)$，则应将上式中级数内的 $n=m$ 项和齐次化函数合并，再利用洛必达法则求极限，即可化为

$$\begin{aligned}u(x,t)=&\frac{6Al}{ES}\frac{(-)^m}{(2m+1)^2\pi^2}\sin\frac{2m+1}{2l}\pi x\sin\frac{2m+1}{2l}\pi at\\&+\frac{2A}{ES}\frac{(-)^{m+1}}{(2m+1)\pi}\left[x\cos\frac{2m+1}{2l}\pi x\sin\frac{2m+1}{2l}\pi at+at\sin\frac{2m+1}{2l}\pi x\cos\frac{2m+1}{2l}\pi at\right]\\&+\frac{8(2m+1)Al}{ES\pi^2}\sum_{n=0}^{\infty}{}'\frac{(-)^n}{2n+1}\frac{1}{(2m+1)^2-(2n+1)^2}\sin\frac{2n+1}{2l}\pi x\sin\frac{2n+1}{2l}\pi at,\end{aligned}$$

其中 $\sum\limits_{n=0}^{\infty}{}'$ 表示和式中不含 $n=m$ 项.

9. $u(x,t)=\cos\dfrac{\pi}{l}x\cos\dfrac{\pi}{l}at+\dfrac{2l}{\pi a}\sin\dfrac{\pi}{2l}x\sin\dfrac{\pi}{2l}at.$

10. $u(x,t)=A\dfrac{\sinh(1+\mathrm{i})\alpha(l-x)}{\sinh(1+\mathrm{i})\alpha l}\mathrm{e}^{\mathrm{i}\omega t}-2\kappa A\displaystyle\sum_{n=1}^{\infty}\frac{n\pi}{(n\pi)^2\kappa+\mathrm{i}\omega}\sin\frac{n\pi}{l}x\,\mathrm{e}^{-(n\pi/l)^2\kappa t},$

其中 $\alpha=\sqrt{\dfrac{\omega}{2\kappa}}$.

11.
$$\begin{aligned}u(x,t)=&A\frac{\sin\alpha(l-x)}{\sin\alpha l}\mathrm{e}^{-\alpha^2\kappa t}+B\frac{\sin\beta x}{\sin\beta l}\mathrm{e}^{-\beta^2\kappa t}\\&+\sum_{n=1}^{\infty}2n\pi\left[\frac{A}{(\alpha l)^2-(n\pi)^2}-\frac{(-)^nB}{(\beta l)^2-(n\pi)^2}\right]\sin\frac{n\pi}{l}x\exp\left\{-\left(\frac{n\pi}{l}\right)^2\kappa t\right\}\\=&2A\pi\sum_{n=1}^{\infty}\frac{n}{(\alpha l)^2-(n\pi)^2}\sin\frac{n\pi}{l}x\left\{\exp\left[-\left(\frac{n\pi}{l}\right)^2\kappa t\right]-\exp\left[-\alpha^2\kappa t\right]\right\}\\&-2B\pi\sum_{n=1}^{\infty}\frac{(-)^nn}{(\beta l)^2-(n\pi)^2}\sin\frac{n\pi}{l}x\left\{\exp\left[-\left(\frac{n\pi}{l}\right)^2\kappa t\right]-\exp\left[-\beta^2\kappa t\right]\right\}.\end{aligned}$$

12. 临界厚度 $l_c=\pi\sqrt{\dfrac{D}{\alpha}}$.

第 十 四 章

1. 取平面极坐标系，使边界条件为 $u(r,\phi)\big|_{r=a} = V\,\mathrm{sgn}\,(\sin\phi)$，则柱内电势分布为

$$u(r,\phi) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\left(\frac{r}{a}\right)^{2n+1}\sin(2n+1)\phi = \frac{2V}{\pi}\arctan\frac{2ar\sin\phi}{a^2-r^2}.$$

2. $$u(r,\phi) = A_0\frac{\ln b-\ln r}{\ln b-\ln a} + \sum_{m=1}^{\infty}\frac{\left(\dfrac{r}{b}\right)^m-\left(\dfrac{b}{r}\right)^m}{\left(\dfrac{a}{b}\right)^m-\left(\dfrac{b}{a}\right)^m}\big(A_m\cos m\phi + B_m\sin m\phi\big)$$

$$-C_0\frac{\ln a-\ln r}{\ln b-\ln a} - \sum_{m=1}^{\infty}\frac{\left(\dfrac{r}{a}\right)^m-\left(\dfrac{a}{r}\right)^m}{\left(\dfrac{a}{b}\right)^m-\left(\dfrac{b}{a}\right)^m}\big(C_m\cos m\phi + D_m\sin m\phi\big).$$

其中 A_m, B_m 和 C_m, D_m 是 $f(\phi)$ 和 $g(\phi)$ 的展开系数，

$$f(\phi) = A_0 + \sum_{m=1}^{\infty}\big(A_m\cos m\phi + B_m\sin m\phi\big), \quad g(\phi) = C_0 + \sum_{m=1}^{\infty}\big(C_m\cos m\phi + D_m\sin m\phi\big).$$

3. $$u(r,\phi) = \frac{M}{kh\pi} + \frac{1}{2k}\frac{Ma}{ha+1}\frac{r}{a}\sin\phi - \frac{2Ma}{k\pi}\sum_{m=1}^{\infty}\frac{1}{(ha+2m)\big(4m^2-1\big)}\left(\frac{r}{a}\right)^{2m}\cos 2m\phi.$$

注：不妨假设阳光垂直于柱轴．取柱坐标系，z 轴即为柱轴，阳光照射的半个柱面取为 $0\leqslant\phi\leqslant\pi$，则边界条件为

$$\left(\frac{\partial u}{\partial r}+hu\right)_{r=a} = \begin{cases}\dfrac{M}{k}\sin\phi, & 0\le\phi\le\pi,\\ 0, & \pi<\phi<2\pi.\end{cases}$$

4. (1) $u(r,\phi) = a^2-r^2$;

(2) $u(r,\phi) = \dfrac{1}{2}\big(a^2-r^2\big)r\sin\phi$;

(3) $u(r,\phi) = \dfrac{1}{3}\big(a^2-r^2\big)r^2\sin 2\phi$;

(4) $u(r,\phi) = \dfrac{1}{2}\big(a^2-r^2\big)r(\sin\phi+\cos\phi)$.

5. $$u(r,\phi) = \frac{4V}{\pi}\phi + \frac{2V}{\pi}\sum_{n=1}^{\infty}\frac{(-)^n}{n}\frac{\left(\dfrac{r}{a}\right)^{4n}-\left(\dfrac{a}{r}\right)^{4n}+\left(\dfrac{b}{r}\right)^{4n}-\left(\dfrac{r}{b}\right)^{4n}}{\left(\dfrac{b}{a}\right)^{4n}-\left(\dfrac{a}{b}\right)^{4n}}\sin 4n\phi$$

$$= \frac{4V}{\pi}\sum_{n=0}^{\infty}\frac{1}{2n+1}\frac{\sinh\dfrac{2n+1}{\ln b-\ln a}\pi\phi}{\sinh\dfrac{2n+1}{\ln b-\ln a}\dfrac{\pi^2}{4}}\sin\frac{\ln r-\ln a}{\ln b-\ln a}(2n+1)\pi.$$

6. $u(r,t)=A\dfrac{\sin p\pi r}{r\sin p\pi}\mathrm{e}^{-(p\pi)^2\kappa t}+\dfrac{2A}{\pi}\sum\limits_{n=1}^{\infty}(-)^n\dfrac{n}{n^2-p^2}\dfrac{\sin n\pi r}{r}\mathrm{e}^{-(n\pi)^2\kappa t}.$

如果 $p=$ 正整数 m，则为

$$u(r,t)=(-)^m A\left\{\cos m\pi r+\frac{\sin m\pi r}{2m\pi r}\left[1-(2m\pi)^2\kappa t\right]\right\}\mathrm{e}^{-(m\pi)^2\kappa t}+\frac{2A}{\pi}\sum_{n=1}^{\infty}{}'\frac{(-)^n n}{n^2-m^2}\frac{\sin n\pi r}{r}\mathrm{e}^{-(n\pi)^2\kappa t},$$

其中 $\sum\limits_{n=1}^{\infty}{}'$ 表示和式中不含 $n=m$ 项.

第 十 五 章

2. $$\begin{cases}\dfrac{2^{k+1}(k!)^2}{(k-l)!(k+l+1)!}, & k\geq l,\\[2ex] 0, & k<l.\end{cases}$$

3. (1) $$\begin{cases}2(\ln 2-1), & l=0,\\ -\dfrac{2}{l(l+1)}, & l=1,2,3,\cdots;\end{cases}$$

(2) $2^{1-\alpha}\dfrac{\Gamma(1-\alpha)\Gamma(l+\alpha)}{\Gamma(\alpha)\Gamma(l-\alpha+2)}$.

(3) $$\begin{cases}\dfrac{(-)^{m+n}}{(2m+1)(2m+2)-2n(2n+1)}\dfrac{(2m+1)!(2n)!}{2^{2m+2n}(m!n!)^2}, & k+l=\text{ 奇数，不妨设 } k=2n,\ l=2m+1,\\[2ex] \dfrac{1}{2l+1}\delta_{kl}, & k+l=\text{ 偶数};\end{cases}$$

(4) $\dfrac{2(l+1)}{(2l+1)(2l+3)}$;

(5) $\dfrac{2(l+1)(l+2)}{(2l+1)(2l+3)(2l+5)}$;

(6) $\dfrac{4l(l+1)-2}{(2l-1)(2l+1)(2l+3)}$.

4. (1) $x^2=\dfrac{1}{3}\mathrm{P}_0(x)+\dfrac{2}{3}\mathrm{P}_2(x)$;

(2) $\sqrt{1-2xt+t^2}=\displaystyle\sum_{k=0}^{\infty}\left(\frac{t^{k+2}}{2k+3}-\frac{t^k}{2k-1}\right)\mathrm{P}_k(x)$;

(3) $|x|=\displaystyle\sum_{k=0}^{\infty}(-)^{k-1}\frac{(2k)!}{(2^k k!)^2}\frac{4k+1}{2(2k-1)(k+1)}\mathrm{P}_{2k}(x)$;

(4) $\dfrac{1}{2}\big[x+|x|\big]=\dfrac{1}{2}\mathrm{P}_1(x)+\displaystyle\sum_{k=0}^{\infty}(-)^{k-1}\frac{(2k)!}{(2^k k!)^2}\frac{4k+1}{2^2(2k-1)(k+1)}\mathrm{P}_{2k}(x)$.

5. $u(r,\theta)=\dfrac{b-3a}{3(b-a)}u_0+\dfrac{2b}{3(b-a)}\dfrac{a}{r}u_0+\dfrac{2}{3}\dfrac{b^3a^2}{b^5-a^5}\left[\left(\dfrac{r}{a}\right)^2-\left(\dfrac{a}{r}\right)^3\right]u_0\mathrm{P}_2(\cos\theta)$.

6. $u(r,\theta)=\dfrac{M}{2}\dfrac{1}{Ha+k}r\mathrm{P}_1(\cos\theta)+\dfrac{Ma}{2}\displaystyle\sum_{l=0}^{\infty}\frac{(-)^{l+1}}{Ha+2lk}\frac{(2l)!}{(2^l l!)^2}\frac{4l+1}{(2l-1)(2l+2)}\left(\frac{r}{a}\right)^{2l}\mathrm{P}_{2l}(\cos\theta)$.

7. $u(r,\theta)=u_0\displaystyle\sum_{l=0}^{\infty}(-)^l\frac{4l+3}{2l+2}\frac{(2l)!}{(2^l l!)^2}\left(\frac{r}{a}\right)^{2l+1}\mathrm{P}_{2l+1}(\cos\theta)$.

8. $\dfrac{u_0}{a}r\cos\theta$.

9. $u(x,t)=\sum_{k=0}^{\infty}\left[A_k\cos\sqrt{(k+1)(2k+1)}\,\omega t+B_k\sin\sqrt{(k+1)(2k+1)}\,\omega t\right]\mathrm{P}_{2k+1}\left(\frac{x}{l}\right)$;

$A_k=\frac{4k+3}{l}\int_0^l\phi(x)\mathrm{P}_{2k+1}\left(\frac{x}{l}\right)\mathrm{d}x,\ B_k=\frac{4k+3}{\omega l\sqrt{(k+1)(2k+1)}}\int_0^l\psi(x)\mathrm{P}_{2k+1}\left(\frac{x}{l}\right)\mathrm{d}x.$

10. $u(r,\theta)=\begin{cases}\frac{Q}{a}\sum_{l=0}^{\infty}\frac{(-)^l(2l)!}{(2^l l!)^2}\left[\left(\frac{r}{a}\right)^{2l}-\left(\frac{a}{b}\right)^{2l+1}\left(\frac{r}{b}\right)^{2l}\right]\mathrm{P}_{2l}(\cos\theta), & r<a,\\ \frac{Q}{r}\sum_{l=0}^{\infty}\frac{(-)^l(2l)!}{(2^l l!)^2}\left[\left(\frac{a}{r}\right)^{2l}-\left(\frac{a}{b}\right)^{2l}\left(\frac{r}{b}\right)^{2l+1}\right]\mathrm{P}_{2l}(\cos\theta), & r>a.\end{cases}$

11. (1) $\sin^2\theta\cos^2\phi=\frac{2}{3}\sqrt{\pi}\,\mathrm{Y}_0^0(\theta,\phi)-\frac{2}{3}\sqrt{\frac{\pi}{5}}\,\mathrm{Y}_2^0(\theta,\phi)+\sqrt{\frac{2\pi}{15}}\left[\mathrm{Y}_2^2(\theta,\phi)+\mathrm{Y}_2^{-2}(\theta,\phi)\right]$;

(2) $-\sqrt{\frac{2\pi}{3}}\left[\mathrm{Y}_1^1(\theta,\phi)+\mathrm{Y}_1^{-1}(\theta,\phi)\right]-\sqrt{\frac{2\pi}{15}}\left[\mathrm{Y}_2^1(\theta,\phi)+\mathrm{Y}_2^{-1}(\theta,\phi)\right]$.

12. (1) $u(r,\theta,\phi)=\frac{r}{a}\mathrm{P}_1^1(\cos\theta)\cos\phi=-\frac{r}{a}\sin\theta\cos\phi$;

(2) $u(r,\theta,\phi)=-\frac{1}{3}\left(\frac{r}{a}\right)^2\mathrm{P}_2^1(\cos\theta)\cos\phi=\left(\frac{r}{a}\right)^2\cos\theta\sin\theta\cos\phi$.

13. $$\begin{aligned}u(r,\theta,\phi)&=\frac{1}{6}A\left(r^2-a^2\right)-\frac{1}{21}Br^2\left(r^2-a^2\right)\mathrm{P}_2^1(\cos\theta)\cos\phi\\&=\frac{1}{6}A\left(\frac{r}{a}\right)^2+\frac{1}{14}Br^2\left(r^2-a^2\right)\sin2\theta\cos\phi.\end{aligned}$$

第 十 六 章

2. (1) $\dfrac{\pi}{a}\mathrm{J}_2(a)$; (2) $\dfrac{1}{a}\exp\left\{-\dfrac{b}{4a}\right\}$;

(3) $\dfrac{\Gamma(\nu+3/2)}{\sqrt{\pi}}\dfrac{2a(2b)^{\nu}}{(a^2+b^2)^{\nu+3/2}}$; (4) $\dfrac{b^{\nu}}{(2a^2)^{\nu+1}}\exp\left\{-\dfrac{b^2}{4a^2}\right\}$.

4. (1) $\dfrac{1}{2^n n!}-x^{-n}\mathrm{J}_n(x)$; (2) $a^3\mathrm{J}_1(a)-2a^2\mathrm{J}_2(a)$;

(3) $2\sin\dfrac{t}{2}$; (4) $2^{-n}\sqrt{\pi}\,t^{n+1/2}\mathrm{J}_{n+1/2}\left(\dfrac{t}{2}\right)$.

6. $u(r,t)=8A\displaystyle\sum_{i=1}^{\infty}\frac{1}{\mu_i^3\mathrm{J}_1(\mu_i)}\mathrm{J}_0\left(\frac{\mu_i}{R}r\right)\cos\frac{\mu_i}{R}at$, μ_i 是 $\mathrm{J}_0(x)$ 的第 i 个正零点，$i=1,2,3,\cdots$.

7. $u(r,\theta,t)=4u_0\sin 2\theta\displaystyle\sum_{i=1}^{\infty}\frac{1}{\mu_i^4}\frac{4-(\mu_i^2+4)\mathrm{J}_0(\mu_i)}{\mathrm{J}_0^2(\mu_i)}\mathrm{J}_2\left(\frac{\mu_i}{a}r\right)\exp\left\{-\left(\frac{\mu_i}{a}\right)^2\kappa t\right\}$,

μ_i 是 $\mathrm{J}_2(x)$ 的第 i 个正零点，$i=1,2,3,\cdots$.

8. $u(r,z,t)=\sin nz\displaystyle\sum_{i=1}^{\infty}B_i\mathrm{J}_0(\mu_i r)\exp\left\{-(\mu_i^2+n^2)\kappa t\right\}$, $B_i=\dfrac{2}{\mathrm{J}_1^2(\mu_i)}\displaystyle\int_0^1 f(r)\mathrm{J}_0(\mu_i r)r\mathrm{d}r$,

μ_i 是 $\mathrm{J}_0(x)$ 的第 i 个正零点，$i=1,2,3,\cdots$.

9. $u(r,t)=\pi u_0\displaystyle\sum_{i=1}^{\infty}\frac{\mathrm{J}_0(\mu_i b)}{\mathrm{J}_0(\mu_i a)+\mathrm{J}_0(\mu_i b)}\left[\mathrm{J}_0(\mu_i r)\mathrm{N}_0(\mu_i a)-\mathrm{J}_0(\mu_i a)\mathrm{N}_0(\mu_i r)\right]\mathrm{e}^{-\mu_i^2\kappa t}$,

μ_i 是超越方程 $\mathrm{J}_0(\mu b)\mathrm{N}_0(\mu a)-\mathrm{J}_0(\mu a)\mathrm{N}_0(\mu b)=0$ 的第 i 个正根，$i=1,2,3,\cdots$.

10. (1) $u(r,t)=2AR^2\displaystyle\sum_{i=1}^{\infty}\frac{1}{(\mu_i a)^2-\omega^2R^2}\frac{1}{\mu_i\mathrm{J}_1(\mu_i)}\mathrm{J}_0\left(\frac{\mu_i}{R}r\right)\left(\sin\omega t-\frac{\omega R}{\mu_i a}\sin\frac{\mu_i}{R}at\right)$,

μ_i 是 $\mathrm{J}_0(x)$ 的第 i 个正零点，$i=1,2,3,\cdots$.

当 ω 正好是圆膜的某一本征频率 (例如，$\omega=\mu_j a/R$) 时，可用洛必达法则求其极限值，

$$\begin{aligned}u(r,t)=&-\frac{AR}{a}\frac{1}{\mu_j^2\mathrm{J}_1(\mu_j)}\mathrm{J}_0\left(\frac{\mu_j}{R}r\right)\left(t\cos\frac{\mu_j}{R}at-\frac{R}{\mu_j a}\sin\frac{\mu_j}{R}at\right)\\&+\frac{2AR^2}{a^2}\sum_{i=1}^{\infty}{}'\frac{1}{\mu_i^2(\mu_i^2-\mu_j^2)}\frac{1}{\mathrm{J}_1(\mu_i)}\mathrm{J}_0\left(\frac{\mu_i}{R}r\right)\left(\mu_i\sin\frac{\mu_j}{R}at-\mu_j\sin\frac{\mu_i}{R}at\right),\end{aligned}$$

其中 $\displaystyle\sum_{i=1}^{\infty}{}'$ 表示和式中不含 $i=j$ 项.

(2) $u(r,t)=8AR^2\displaystyle\sum_{i=1}^{\infty}\frac{1}{(\mu_i a)^2-\omega^2R^2}\frac{1}{\mu_i^3\mathrm{J}_1(\mu_i)}\mathrm{J}_0\left(\frac{\mu_i}{R}r\right)\left(\sin\omega t-\frac{\omega R}{\mu_i a}\sin\frac{\mu_i}{R}at\right)$,

μ_i是$\mathrm{J}_0(x)$的第i个正零点，$i=1,2,3,\cdots$.

若 ω 正好是圆膜的某一本征频率(例如，$\omega=\mu_j a/R$) 时，可用洛必达法则求其极限值，

$$\begin{aligned}u(r,t)=&-\frac{4AR}{a^2}\frac{1}{\mu_j^5\mathrm{J}_1(\mu_j)}\mathrm{J}_0\left(\frac{\mu_j}{R}r\right)\left(\frac{\mu_j}{R}at\cos\frac{\mu_j}{R}at-\sin\frac{\mu_j}{R}at\right)\\&+\frac{8AR^2}{a^2}\sum_{i=1}^{\infty}{}'\frac{1}{\mu_i^4(\mu_i^2-\mu_j^2)}\frac{1}{\mathrm{J}_1(\mu_i)}\mathrm{J}_0\left(\frac{\mu_i}{R}r\right)\left(\mu_i\sin\frac{\mu_j}{R}at-\mu_j\sin\frac{\mu_i}{R}at\right),\end{aligned}$$

其中 $\sum\limits_{i=1}^{\infty}{}'$ 表示和式中不含 $i=j$ 项.

11. (1) $\dfrac{1}{\sqrt{2b}}\dfrac{1}{\sqrt{a^2+b^2}}\sqrt{b+\sqrt{a^2+b^2}}$, $\dfrac{1}{\sqrt{2b}}\dfrac{1}{\sqrt{a^2+b^2}}\dfrac{a}{\sqrt{b+\sqrt{a^2+b^2}}}$;

(2) $\dfrac{1}{\alpha^2+\beta^2}$.

12. $u(r,z)=u_0\dfrac{\mathrm{I}_0(2\pi r/h)}{\mathrm{I}_0(2\pi a/h)}\sin\dfrac{2\pi}{h}z$.

13. $$\begin{aligned}u(r,t)&=2u_0\sum_{n=1}^{\infty}(-)^{n-1}\mathrm{j}_0\left(\frac{n\pi}{a}r\right)\exp\left\{-\left(\frac{n\pi}{a}\right)^2\kappa t\right\}\\&=\frac{2u_0a}{\pi r}\sum_{n=1}^{\infty}\frac{(-)^{n-1}}{n}\sin\frac{n\pi}{a}r\exp\left\{-\left(\frac{n\pi}{a}\right)^2\kappa t\right\}.\end{aligned}$$

14. 临界半径分别是 $\mu_1\sqrt{\dfrac{D}{\beta}}$ 和 $\pi\sqrt{\dfrac{D}{\beta}}$，其中 $\mu_1=2.4048\cdots$ 是 $\mathrm{J}_0(x)$ 的最小正零点.

15. $\mathrm{e}^{r\cos\theta}\mathrm{J}_0(r\sin\theta)=\sum\limits_{l=0}^{\infty}\dfrac{r^l}{l!}\mathrm{P}_l\left(\cos\theta\right)$.

第 十 七 章

1. (1) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[x^2\dfrac{\mathrm{d}y}{\mathrm{d}x}\right]+\left(\lambda x+x^2\right)y=0$;

(2) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[x^a(1-x)^{b-a}\dfrac{\mathrm{d}y}{\mathrm{d}x}\right]+\lambda x^{a-1}(1-x)^{b-a-1}y=0$;

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[x\mathrm{e}^{-x}\dfrac{\mathrm{d}y}{\mathrm{d}x}\right]+\lambda\mathrm{e}^{-x}y=0$;

(4) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{-x^2}\dfrac{\mathrm{d}y}{\mathrm{d}x}\right]+2\lambda\mathrm{e}^{-x^2}y=0$.

2. $\lambda_n=\left(\dfrac{2n+1}{\ln b-\ln a}\dfrac{\pi}{2}\right)^2,\quad R_n(r)=\sin\left(\dfrac{\ln r-\ln a}{\ln b-\ln a}\dfrac{2n+1}{2}\pi\right),\quad n=0,1,2,\cdots$.

4. $\lambda_n=\dfrac{2n\pi-\theta}{b-a},\quad y_n(x)=\exp\left(-\mathrm{i}\dfrac{2n\pi-\theta}{b-a}x\right),\quad n=0,\pm1,\pm2,\cdots$.

5. α,β,γ 需满足 $\sin(\alpha+\beta)=\sin\gamma\cos\gamma$.

7. $u(x,y)=\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{A_{mn}}{(m\pi/a)^2+(n\pi/b)^2}\sin\frac{m\pi}{a}x\,\sin\frac{n\pi}{b}y$,

$A_{mn}=\dfrac{4}{ab}\displaystyle\int_0^a\sin\frac{m\pi}{a}x\,\mathrm{d}x\int_0^b f(x,y)\sin\frac{n\pi}{b}y\,\mathrm{d}y$.

8. (1) 本征值 λ_n 是方程 $\left(\dfrac{a}{c}\right)^2\sqrt{\lambda}\tan\sqrt{\lambda}l=1$ 的第 n 个正根，$n=1,2,3,\cdots$；本征函数为 $X_n(x)=\sin\sqrt{\lambda_n}x$；

(2) $$\int_0^l X_n(x)X_m(x)\mathrm{d}x=\begin{cases}-\left(\dfrac{a}{c}\right)^2\sin\sqrt{\lambda_n}l\,\sin\sqrt{\lambda_m}l, & m\neq m,\\[2ex] \dfrac{l}{2}-\dfrac{1}{2}\dfrac{a^2c^2}{a^4\lambda_n+c^4}, & n=m.\end{cases}$$

(3) $$\int_0^l X_n'(x)X_m'(x)\mathrm{d}x=\lambda_n\left(\frac{l}{2}+\frac{1}{2}\frac{a^2c^2}{a^4\lambda_n+c^4}\right)\delta_{nm}.$$

第 十 八 章

1. $U(x,p)=\dfrac{u_0}{p}\exp\left\{-\sqrt{\dfrac{p}{\kappa}}x\right\},\qquad u(x,t)=u_0\,\mathrm{erfc}\dfrac{x}{2\sqrt{\kappa t}}.$

2. $U(x,p)=\begin{cases}\dfrac{u_0}{2p}\exp\left\{\sqrt{\dfrac{p}{\kappa}}x\right\},\\ \dfrac{u_0}{p}-\dfrac{u_0}{2p}\exp\left\{-\sqrt{\dfrac{p}{\kappa}}x\right\},\end{cases}\qquad u(x,t)=\begin{cases}\dfrac{u_0}{2}\mathrm{erfc}\Big(-\dfrac{x}{2\sqrt{\kappa t}}\Big), & x<0,\\ u_0-\dfrac{u_0}{2}\mathrm{erfc}\Big(\dfrac{x}{2\sqrt{\kappa t}}\Big), & x>0.\end{cases}$

3. $U(x,p)=\dfrac{A}{p+\alpha^2\kappa}\dfrac{\sinh\sqrt{p/\kappa}\,(l-x)}{\sinh\sqrt{p/\kappa}\,l}+\dfrac{B}{p+\beta^2\kappa}\dfrac{\sinh\sqrt{p/\kappa}\,x}{\sinh\sqrt{p/\kappa}\,l}.$

   反演而得到的 $u(x,t)$ 和习题 13.11 的形式相同.

4.

5. $u(x,t)=\dfrac{1}{2}\big[\phi(x+at)+\phi(x-at)\big]+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi+\dfrac{1}{2a}\int_0^t\left[\int_{x-a(t-\tau)}^{x+a(t-\tau)}f(\xi,\tau)\mathrm{d}\xi\right]\mathrm{d}\tau.$

6. $u(r,z)=\dfrac{2V_0}{\pi}\displaystyle\int_0^\infty\dfrac{\mathrm{I}_0(kr)}{\mathrm{I}_0(ka)}\dfrac{\sin kz}{k}\mathrm{d}k.$

第 十 九 章

1. $u(r,\theta,\phi)=\dfrac{q_0}{4\pi\varepsilon_0}\left(\dfrac{1}{r}-\dfrac{1}{a_0}\mathrm{e}^{-2r/a_0}-\dfrac{1}{r}\mathrm{e}^{-2r/a_0}\right)$

2. (1) $G(\boldsymbol{r};\boldsymbol{r}')=\dfrac{1}{4\pi\varepsilon_0}\left[\dfrac{1}{|\boldsymbol{r}-\boldsymbol{r}'|}-\dfrac{a}{r'}\dfrac{1}{|\boldsymbol{r}-(a/r')^2\boldsymbol{r}'|}\right]$;

   (2) 取点电荷所在方向为极轴方向，球面上的感生电荷密度为

$$\sigma(\theta,\phi)=\varepsilon_0\frac{\partial G}{\partial r}\bigg|_{r=a}=-\frac{1}{4\pi a}\frac{a^2-r'^2}{\left(a^2-2ar'\cos\theta+r'^2\right)^{3/2}}.$$

3. 取球坐标系，坐标原点位于球心，点电荷位于 $(r,\theta,\phi)=(r',0,0)$ 处，$r'>a$.

$$G(\boldsymbol{r};\boldsymbol{r}')=\frac{1}{4\pi\varepsilon_0}\left(\frac{1}{\sqrt{r^2+r'^2-2rr'\cos\theta}}-\frac{a}{\sqrt{r^2r'^2+a^4-2a^2rr'\cos\theta}}\right),\quad r>a.$$

4. 见习题 13.6.

5. $G(x,t;x_0,t_0)=\dfrac{I}{2\rho a}\eta\left(t-t_0-\dfrac{|x-x_0|}{a}\right)$.

6. $u(x,t)=\dfrac{1}{2}\big[\phi(x-at)+\phi(x+at)\big]+\dfrac{1}{2a}\displaystyle\int_{x-at}^{x+at}\psi(\xi)\mathrm{d}\xi$.

7. $G(\boldsymbol{r},t;\boldsymbol{r}',t')=\dfrac{1}{8[\pi\kappa(t-t_0)]^{3/2}}\exp\left\{-\dfrac{|\boldsymbol{r}-\boldsymbol{r}'|^2}{4\kappa(t-t')}\right\}\eta(t-t')$.

9. $$u(\boldsymbol{r},t)=\frac{1}{8(\pi\kappa)^{3/2}}\int_0^t\mathrm{d}t'\iiint\frac{1}{(t-t')^{3/2}}\mathrm{e}^{-\left|\boldsymbol{r}-\boldsymbol{r}'\right|^2/4\kappa(t-t')}f(\boldsymbol{r}',t')\,\mathrm{d}\boldsymbol{r}'$$
$$+\frac{1}{8(\pi\kappa t)^{3/2}}\iiint\mathrm{e}^{-\left|\boldsymbol{r}-\boldsymbol{r}'\right|^2/4\kappa t}\phi(\boldsymbol{r}')\,\mathrm{d}\boldsymbol{r}'.$$

第 二 十 章

1. (1) $yy''+y'^2=0,\ y^2=ax+b$;

(2) $y''-y=0,\ y=a\,\mathrm{e}^x+b\,\mathrm{e}^{-x}$;

(3) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\dfrac{x}{x+y'^2}\right]=0,\ y=ax^{3/2}-\dfrac{1}{2}x^2+b$;

(4) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[\sqrt{1+x}\dfrac{y'}{\sqrt{1+y'^2}}\right]=0,\ y=2a\sqrt{1+x-a^2}+b.$

2. $z\cos\dfrac{\theta+b}{\sqrt{2}}=a.$

3. $az+b\theta=c$. 作为它的特殊情形，包括圆 $z=$ 常数、直线 $\theta=$ 常数以及特殊的螺线 $az+b\theta=0$.

4. $u(r,\theta,\phi)=\dfrac{a}{\mathrm{j}_1(ka)}\mathrm{j}_1(kr)\mathrm{P}_l\,(\cos\theta)$.

5. (1) $x+1=a\cosh\left(\dfrac{y}{a}-b\right)$;

(2) $y=\left(\dfrac{x+b}{2a}\right)^2+a^2$;

(3) $(y+b)^2+4a^4(2x+3)^2=4a^2$;

(4) $(x-a)^2+y^2=b^2$;

(5) $\mathrm{e}^y\cos(x-a)=b$;

(6) $(x-y-a)^2=4b^2\big(x+y-b^2\big)$;

(7) $\theta=\arcsin\left(1-\dfrac{2a}{r}\right)+b$;

(8) $r=a\mathrm{e}^{\theta}$.

6.

7. 相应的泛函极值问题是条件极值问题

$$J[u]=\iiint\limits_V(\nabla u)^2\mathrm{d}\boldsymbol{r}+\frac{\alpha}{\beta}\iint\limits_\Sigma u^2\mathrm{d}\varSigma,$$

$$\left(\alpha u+\beta\frac{\partial u}{\partial n}\right)_\Sigma=0,\qquad\iiint\limits_V u^2\mathrm{d}\boldsymbol{r}=1.$$

8. (1) $\lambda_1=2.5,\ \lambda_2=10.5$;

(2) $\lambda=14\mp\sqrt{133}=2.467, 25.533.$

第二十一章

1. (1) 当 $y>0$ 时为椭圆型．令 $\xi=x,\eta=2\sqrt{y}$，方程化为$\dfrac{\partial^2u}{\partial\xi^2}+\dfrac{\partial^2u}{\partial\eta^2}=0$;

当 $y<0$ 时为双曲型．令 $\xi=x,\eta=2\sqrt{-y}$，方程化为 $\dfrac{\partial^2u}{\partial\xi^2}-\dfrac{\partial^2u}{\partial\eta^2}=0$;

(2) 椭圆型．令 $\xi=\operatorname{arcsinh}x,\eta=\operatorname{arcsinh}y$，方程化为 $\dfrac{\partial^2u}{\partial\xi^2}+\dfrac{\partial^2u}{\partial\eta^2}=0$;

(3) 抛物型．令 $\xi=y\sin x,\eta=y\cos x$，方程化为 $\left(\xi^2+\eta^2\right)\dfrac{\partial^2u}{\partial\eta^2}-\xi\dfrac{\partial u}{\partial\xi}+\eta\dfrac{\partial u}{\partial\eta}=0$;

(4) 双曲型．令 $\xi=x+y-\cos x,\ \eta=x-y+\cos x$，方程化为 $\dfrac{\partial^2u}{\partial\xi\partial\eta}=0$.

2. (2) 作变换 $u(x,y)=\mathrm{e}^{-ax+by}v(x,y)$，即可将方程化为 $\dfrac{\partial^2v}{\partial x^2}-\dfrac{\partial^2v}{\partial y^2}+(b^2-a^2)v=0$.

3. $u(x,t)=\phi\left(\dfrac{x+at}{2}\right)+\psi\left(\dfrac{x-at}{2}\right)-\phi(0)$.