EXAMPLES IN MATHEMATICS

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Abstract

This is a notes of examples. I write it mainly because I think I am not good at remembering and using examples in mathematics.

Contents

1 Analysis

1.1 Functional Analysis

$f \in C^\infty, \forall x \in \mathbb{R}, \exists n = n(x) \in \mathbb{N}^*, s.t. f^{(n)}(x) = 0.$

(From MSE:If f is infinitely differentiable then f coincides with a polynomial)

Proof. $\mathbb{R} = \bigcup_{n=1}^{\infty}[-n,n]$, hence it suffices to show f is a polynomial on $[-n,n]$.

The proof is by contradiction. Suppose f is not a polynomial on $[-n,n]$.

Let: $S_n = \{x : f^{(n)}(x) = 0\}, n \in \mathbb{N}^*$, then S_n is closed since $f^{(n)}(x)$ is continuous.

Let $X = \{x : \forall (a,b)$ containing $x, f|_{(a,b)}$ is not a polynomial.$\}$. X is a non-empty closed set without isolated points:

X is closed: $X^c = \{x; \exists (a,b)$ containing $x, f|_{(a,b)}$ is a polynomial$\}$, hence X^c is clearly open. X is non-empty: it suffices to show $X^c \neq [-n,n]$, otherwise we can find an open cover of $[-n,n]$, hence a finite subcover of $[-n,n]$ such that f is a polynomial on them respectively, then there exists a sufficiently large N, such that $f'(x) = 0$, hence f is a polynomial on $[-n,n]$, which is a contradiction. X is without isolated points: otherwise $\exists a(\text{if } \neq n, -n) \in X$, such that $\exists \epsilon > 0$, $(a-\epsilon, a) \cup (a, a+\epsilon) \subset X^c$,$(a-\epsilon,a) \cup (a, a+\epsilon)$ is bounded, so on every compact subset, f is a polynomial, hence f is a polynomial on $(a-\epsilon,a)$,$(a,a+\epsilon)$ respectively. Since $f \in C^\infty$, it is easy to show $\exists N$ sufficiently large such that $f^{(N)}(x) = 0, \forall x \in (a-\epsilon, a+\epsilon)$. Then $a \in X^c$, which is a contradiction. For $a = n, -n$, we can easily get a contradiction by the same argument.

$X = \bigcup_{n=1}^{\infty}(S_n \cap X)$, by Baire Category theorem, $\exists k \in \mathbb{N}^*$, such that $S_k \cap X$ is not a nowhere dense set, i.e. $(S_k \cap X)^\circ$ is not empty (in the induced topology of X). Hence $\exists$ interval (a,b), such that $(a,b) \cap X \subset S_k \cap X$ and $(a,b) \cap X$ is non-empty. Since X is without isolated point, every point of $(a,b) \cap X$ is an accumulation point. Hence $(a,b) \cap X \subset S_n, \forall n \geq k$(we can easily prove by the definition of derivative and S_n).

Now consider a maximal interval $(c,e) \subset (a,b) \backslash X$, then we can easily show f is a polynomial, of degree d. Then $f^{(d)} =$ const. ,$d < k$ since c or e is in X hence in S_k. So $f^{(k)} = 0$, which is a contradiction to $(a,b) \cap X$ is non-empty. □

1.2 Measure

1.2.1 Some inequality of symmetric difference

$$(i) A\Delta B \subset (A\Delta C) \cup (B \Delta C)$$

$$(ii) (\bigcup_{k\in I} A_k)\Delta(\bigcup_{k\in I} B_k) \subset \bigcup_{k\in I}(A_k \Delta B_k)$$

$$(iii) (\bigcap_{k\in I} A_k)\Delta(\bigcap_{k\in I} B_k) \subset \bigcap_{k\in I}(A_k \Delta B_k)$$

Proof. I only prove (iii):

$$(\bigcap_{k\in I} A_k)\Delta(\bigcap_{k\in I} B_k) = (\bigcap_{k\in I} A_k \backslash (\bigcap_{k\in I} B_k)) \bigcup (\bigcap_{k\in I} B_k \backslash (\bigcap_{k\in I} A_k))$$

By the symmetry, it suffices to show:

$$(\bigcap_{k\in I} A_k\backslash(\bigcap_{k\in I} B_k)) \subset \bigcap_{k\in I}(A_k\Delta B_k)$$

We have:

$$(\bigcap_{k\in I} A_k\backslash(\bigcap_{k\in I} B_k)) = \bigcap_{k\in I}\bigcap_{l\in I} A_k\backslash B_l \subset \bigcup_{k\in I} A_k\backslash B_k \subset \bigcap_{k\in I}(A_k\Delta B_k)$$

□

1.2.2 The pre-image of a Borel set by a continuous function is a Borel set

Suppose X and Y are topological space with Borel σ-algebra $\mathscr{B}_X$ and $\mathscr{B}_Y$, and f is a continuos map from X to Y. Then $f^{-1}(\mathscr{B}_Y) \subset \mathscr{B}_X$.

Proof. Let $\mathscr{S} = \{A \subset Y : f^{-1}(A) \in \mathscr{B}_X\}$, then it is easy to verify that $\mathscr{S}$ is a σ-algebra and $f^{-1}\mathscr{S} \subset \mathscr{B}_X$. Since f is continuous, open sets are contained in $\mathscr{S}$, hence $\mathscr{B}_Y \subset \mathscr{S}$, therefore $f^{-1}(\mathscr{B}_Y) \subset f^{-1}(\mathscr{S}) \subset \mathscr{B}_X$ □

1.2.3 A partition of Interval

T is measurable, μ is a finite measure, then $\forall E$ measurable

$$\lim_{n\to} \mu(T^{-n}E\backslash \bigcup_{k=0}^{n-1} T^{-k}E) = 0$$

Hint:note that, we can define

$$E_k = \{x : x \in E, x \notin T^{-1}E, ..., x \notin T^{-(k-1)}E, x \in T^{-k}E\}$$

$$E_k^* = \{x : x \notin E, x \notin T^{-1}E, ..., x \notin T^{-(k-1)}E, x \in T^{-k}E\}$$

1.2.4 Dirac measure

$(X, 2^X)$ is a measure space, δ_x is a measure:

$$\delta_x(A) = \begin{cases} 1, & x \in A \\ 0, & x \notin A \end{cases} \tag{1}$$

$$m = \sum_{n=1}^{N} \delta_{T^i x} \text{ is ergodic, where T is an endomorphism, and } T^N x = x. \tag{2}$$

1.2.5 An outer measure which is not continuous from below

$(\mathbb{Z}, 2^{\mathbb{Z}}, \mu^*)$,where:

$$\mu^*(A) = \begin{cases} 1, & A \text{ is finite and nonempty} \\ 0, & A = \emptyset \\ \infty, & A \text{ is infinite} \end{cases} \tag{3}$$

1.3 Ergodic Theory

1.3.1 Bernoulli shift is strong mixing

$Y = \{0, 1, ..., d\}, X = Y^{\mathbb{N}_0}$, $\mathcal{A}$ is the algebra generated by measurable rectangles, ${}_l[a]_k = \{x \in X : x_i = a_i, l \leq i \leq k\}$,then for all $A, B \in \sigma(\mathcal{A})$, T is the shift,then T is strong mixing, i.e. $\lim\limits_{n\to\infty} m(T^{-n}A \cap B) = m(A)m(B)$.

1.3.2 Rotation of torus

T is a measure-preserving transformation(Hint: construct an algebra which generates the Borel algebra), $m = Leb|_{[0,1)}$.

$$\begin{aligned} T : &[0, 1) \to [0, 1) \\ &x \longmapsto x + \alpha \end{aligned} \tag{4}$$

Proposition 1.1. T is ergodic if and only if α is irrational.

Proof. "$\Longrightarrow$": We prove this by contradiction, i.e. suppose $\alpha = \dfrac{q}{p}, p, q \in \mathbb{N}$, define

$$\begin{aligned} f : [0, 1) &\longmapsto [0, 1) \\ x &\longmapsto px \end{aligned} \tag{5}$$

therefore, $f \circ T = f$, by Walters(GTM 79) theorem 1.6, $f = \text{const.}$, which is a contradiction.

"$\Longleftarrow$": We prove: $\forall f \in L^2(m), f \circ T = f$ implies $f = \text{const.}$ Then by Walters(GTM 79) theorem 1.6, f is ergodic. It is easy to prove this by using the Fourier expansion of f.

Remark: $f \in L^p$, then the Fourier series of f converges almost everywhere if $1 < p < \infty$; but it is not true for $p = 1$. $\square$

1.3.3 Koopman operator

$(X, \mathcal{B}, m, T)$ is a measure preserving system, the koopman operator:

$$\begin{aligned} U_T : L^0(m) \longrightarrow &\quad L^0(m) \\ f \longmapsto &\quad f \circ T \end{aligned} \tag{6}$$

It has the following property:

$$\int f \mathrm{d}m = \int f \circ T \mathrm{d}m$$

1.3.4 Shift space is Lipschitz homeomorphic to a cantor set

1.3.5 Adding Machine

Σ^d is minimal under T, where T is an addition operator which can be seen as "+1" with number written in a inverse order. I write this because it appears in "Conformal fractals" and there is a mistake in the book.

1.3.6 Complex linear fractals(Cantor set)

This is a generalized form of cantor set.(It is from "Conformal fractals", it makes something simple look much more difficult and strange for beginners like me!)

Let $U \subset \mathbb{C}$ be an open connected set.$T_i(z) = \lambda_i z + a_i, \lambda_i \in \mathbb{C}, |\lambda| < 1, a_i \in \mathbb{C}, i \in \{1, ..., n\}$, such that $\overline{T_i(U)}$ are pairwise disjoint and contained in U.

Define the limit cantor set:

$$\Lambda = \bigcap_{k\geq 0} \bigcup_{(i_0,\ldots,i_k)} T_{i_0} \circ \ldots \circ T_{i_k}(U) = \bigcup_{(i_0,\ldots,i_k,\ldots)} \lim_{k\to\infty} T_{i_0} \circ \ldots \circ T_{i_k}(z)$$

Note that

$$\lim_{k\to\infty} T_{i_0} \circ \ldots \circ T_{i_k}(z) = \sum_{k=1}^{\infty} \lambda_{i_0} \ldots \lambda_{i_{k-1}} a_k$$

So this equality is easy to verify and the definition is independent of x.

1.4 Ordinary Differential Equation

1.4.1 An autonomous system with exactly one limit cycle

This is from Arnold's Ordinary Differential Equation P73.

$$\begin{cases} \dot{x} = y + x(1 - x^2 - y^2) \\ \dot{y} = -x + y(1 - x^2 - y^2) \end{cases} \tag{7}$$

By passing to polar coordinates, we obtain:

$$\begin{cases} \dot{r} = r(1 - r^2) \\ \dot{\theta} = -1 \end{cases} \tag{8}$$

The phase curve is as follows:

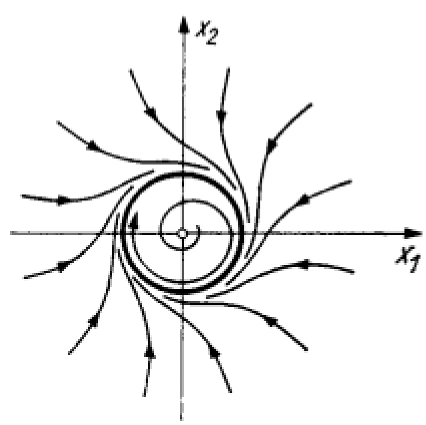

Figure 1: Integral curves in the (x,y)-plane

2 Algebra

2.1 Abstract Algebra

2.1.1 The field of Laurent series and the rational function field

The elements of field of Laurent series $k((x))$ (where k is the field) is the formal infinite sums $\sum_{n=n_0}^{+\infty} a_n x^n$, where $a_n \in k$ and $n_0 \in \mathbb{Z}$. The rational function field field can be viewed as a subfield of the field of Laurent series.

This is because any element in the rational function field $k(x)$ is of the form $\frac{f(x)}{g(x)}$, where $f(x)$, $g(x) \in k[x]$, then $\frac{f(x)}{g(x)} = f(x)[g(x)]^{-1}$.

2.1.2 Field extension of a rational function field

Suppose k is a a field, $K = k(t)$ and $F = k(t^2)$ is the field of rational functions, then $[K : F] = 2$.

Proof. We claim that: the basis of K over F is $\{1, t\}$:

Clearly this is linearly independent, and all the elements in K is of the form $a(t^2)+b(t^2)t$, where $a(x), b(x) \in k(x)$. □

2.2 Homological Algebra

2.2.1 R-module R is projective but not injective

R-module R is projective, because $\mathrm{Hom}_R(R, N) \cong R$.

R-module R is not injective, consider an exact sequence:

$$0 \longrightarrow \mathbb{Z} \xrightarrow{\times 2} \mathbb{Z} \xrightarrow{projection} \mathbb{Z}/2\mathbb{Z} \longrightarrow 0$$

$\mathrm{Hom}(\mathbb{Z}, .)$ is a left exact functor, we have the following exact sequence:

$$0 \longrightarrow \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \to \mathrm{Hom}(\mathbb{Z}, \mathbb{Z}) \xrightarrow{f} \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$$

It suffices to show f is note surjective: for $id \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$, $id \notin f(\mathrm{Hom}(\mathbb{Z}, \mathbb{Z}))$, otherwise $id(1) = f \circ g(1) \neq 1, g \in \mathrm{Hom}(\mathbb{Z}, \mathbb{Z})$.

2.2.2 R-module R is injective if R is a field

Note that when R is a field, we can consider this proposition in the category of vector space.

2.3 Commutative Algebra

2.3.1 A ring which is not Noetherian but has a Noetherian prime spectra

$R = k[x_1, x_2, ..., x_n, ...]$ is a polynomial ring with infinite indeterminates, and $I = (x_1, x_2^2, ..., x_n^n, ...)$ is an ideal. Then $S = R/I$ is a ring which is not Noetherian but has a Noetherian prime spectra.

$J=(\bar{x}_1,\bar{x}_2,...,\bar{x}_n,...)$ is and ideal of S. $S/J\cong k$, hence J is a maximal ideal; every element of J is nilpotent, hence J is contained in the nilradical, which is the intersection of all prime ideals of J. Therefore, J is the unique prime ideal of S, hence the prime spectra of S is Noetherian.There exists an strictly increasing sequence of ideals of S:$(\bar{x}_1)\subset(\bar{x}_1,\bar{x}_2)\subset ...\subset(\bar{x}_1,\bar{x}_2,...,\bar{x}_n)\subset ...$

3 Geometry

3.1 Point Set Topology

3.1.1 Projections from product space is not necessarily closed

$p:\mathbb{R}^2\to\mathbb{R}$ is not closed. For $p\{(x,y):xy=1\}=R\backslash\{0\}$.

3.1.2 A continuous map which is bijectvie but not homeomorphic

$f:\mathbb{E}^1\backslash[0,1)\to\mathbb{E}^1$

$$f(x)=\begin{cases}x, & x<0\\ x-1, & x\geq 1\end{cases} \tag{9}$$

3.1.3 A continuous map which is closed but not open

$$\begin{aligned} f:\mathbb{E}^1 &\longmapsto \mathbb{E}^1\\ x &\longmapsto 1\end{aligned} \tag{10}$$

3.1.4 A continuous map which is open but not closed

Let the open set of $\mathbb{R}$ be $\emptyset,\{1\},\mathbb{R}$.

$$\begin{aligned} f:\mathbb{E}^1 &\longmapsto \mathbb{R}\\ x &\longmapsto 1\end{aligned} \tag{11}$$

3.1.5 An example which is T_2 but not T_3, is C_1 and separable but not C_2

This example is in You's Basic Topology P44 Ex.18.
Let $S=\mathbb{R}\backslash\mathbb{Q}$,the topology of $\searrow$ is $\tau=\{U\backslash A|\text{U is open in }\mathbb{E}^1,A\subset S\}$.
(1). τ is a topology:
Clearly $\emptyset,\mathbb{R}\in\tau$.
Finite intersection:

$$U_n\in\mathbb{E}^1,A_n\subset S,n\in\{1,2,...,N\},\bigcap_{n=1}^{N}U_n\backslash A_n=(\bigcap_{n=1}^{N}U_n)\backslash(\bigcup_{n=1}^{N}A_n)\in\tau$$

Arbitrary union:

$$U_i\in\mathbb{E}^1,A_i\subset S,i\in\Lambda,\bigcup_{i\in\Lambda}(U_i\backslash A_i)\subset(\bigcup_{i\in\Lambda}U_i)\backslash(\bigcap_{i\in\Lambda}A_i)\Longrightarrow\bigcup_{i\in\Lambda}(U_i\backslash A_i)\in\tau$$

(2).$(\mathbb{R},\tau)$ is T_2 but not T_3.

This is because the open set in $\mathbb{E}^1$ is also open in τ, so it is T_2. For $(a,b)\backslash S$ and $r \in (a,b)\cap S$, we cannot find two open set separating (a,b) and $\{r\}$, hence it is not T_3.

(3).$(\mathbb{R},\tau)$ is C_1 and separable.

For the neighbourhood basis of $r\in\mathbb{R}$ is $\{(r-q_n, r+q_n)\backslash T\}_{q\in Q}$, For $r\notin\mathbb{Q}, T=S$ if $r\in\mathbb{Q}$, $T=S\backslash\{r\}$, if $r\in S$.

$(\mathbb{R},\tau)$ is separable, i.e. has a countable dense subset, because $\overline{\mathbb{Q}}=\mathbb{R}$.

(4).The induced topology of $S(\tau_S)$ from τ is discrete.

$\forall p\in S, p\in[\mathbb{R}\backslash(S\backslash\{p\})]\cap S=\{p\}$.

(5).$(\mathbb{R},\tau)$ is not C_2.

Otherwise S,τ_S would be C_2

3.1.6 Topologist's sine curve

$A=\{(x,\sin\frac{1}{x});x\in(0,1]\}, \bar{A}=\overline{\{(x,\sin\frac{1}{x});x\in(0,1]\}}$ is connected but neither path-connected nor locally connected.

$\bar{A}$ is connected: $A\cong(0,1]$ is a connected and dense subset of $\bar{A}$.

$\bar{A}$ is not locally connected: $(0,0)\in U=\bar{A}\backslash\{(x,y)|y=1\}\subset\bar{A}$ contains no connected neighbourhood.

$\bar{A}$ is not path-connected: it suffices to show ∂A is a path-connected component. Clearly ∂A is path connected, we can prove ∂A is a path-connected component of A by the above argument. Suppose a is a path such that $a(0)\in A$, $J=a^{-1}(A)$. Then J is closed and nonempty. It suffices to show J is open, then $J=I$, hence $a(I)\subset A$, then A is a path connected component. $\forall t\in J, a(t)\in A$, without loss of generosity, suppose $a(t)\neq(0,1)$, then $a(t)\in U$. There exist path-connected neighbourhood of t, such that $a(W)\subset U$, since $[0,1]$ is locally path connected. Since the continuous image of a path-connected set is path-connected, $a(W)\subset A$,therefore $W\subset a^{-1}(A)$. Hence $a^{-1}(A)$ is both open and closed. Hence A is a path connected component. Therefore $\bar{A}$ is not path-connected.

Remark: $\{(x,\sin\frac{1}{x}):x\neq0\}$ is a smooth manifold.

3.2 Algebraic Topology

4 Applied Mathematics

4.1 Asymptotic Methods and Perturbation Theory

The last problem of 1.5 from Introduction to Perturbation Methods: suppose $f=o(\phi)$, for small ϵ, where f and ϕ are continuous functions. Give an example to show that it is not necessarily true that

$$\int_0^\epsilon f = o(\int_0^\epsilon g)$$

Example:

$$f(x)=x^2|\sin(\frac{1}{x})|$$

$$\phi(x)=x\sin(\frac{1}{x})$$

$f=o(\phi)$, but $\int_0^\epsilon f\neq o(\int_0^\epsilon\phi)$:

$\int_0^\epsilon f>0,\forall\epsilon>0$, but $\int_0^\epsilon g$ has infinitely many zeros in any neighbourhood of 0.

Let $F(x) = \int_0^x x\sin(\frac{1}{x})dx$,

$$F(\frac{1}{n\pi}) = \int_0^{\frac{1}{n\pi}} x\sin(\frac{1}{x})dx = \frac{(-1)^n}{(n\pi)^3} - 3\int_0^{\frac{1}{n\pi}} x^2\cos(\frac{1}{x})dx$$

Note that $|3\int_0^{\frac{1}{n\pi}} x^2\cos(\frac{1}{x})dx| \leq 3\int_0^{\frac{1}{n\pi}} |x^2\cos(\frac{1}{x})|dx < 3\int_0^{\frac{1}{n\pi}} x^2dx = \frac{1}{(n\pi)^3}$

So $F(\frac{1}{n\pi})(-1)^n > 0$, so $o(\int_0^\epsilon g)$ has infinitely many zeros in any neighbourhood of 0, so $\int_0^\epsilon f \neq o(\int_0^\epsilon \phi)$.