Minimum Bounds on the Number of Electromagnets Required for Remote Magnetic Manipulation

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***Abstract*—Numerous magnetic-manipulation systems have been developed to control objects in relatively large workspaces. These systems vary in their number of electromagnets, their configuration, and their limitations. To date, no attempt has been made to rigorously quantify how many electromagnets are required to perform a given magnetic manipulation task. For some tasks, such as controlling the field at a point, the answer is clear: the same number as dimension of control. For tasks that apply magnetic forces on an object, the answer is less clear, and some systems, which have more control magnets than kinematic degrees of freedom (DOFs), have demonstrated unexpected singularities that only arise at specific object orientations. This paper provides a general analysis for static electromagnetic systems rooted in the governing magnetic equations and proves an unintuitive result. That is, if only magnetic fields and forces are used to control an unconstrained magnetic object, four magnetic sources are required for 3-DOF force control and eight magnetic sources are required for orientation-independent 5-DOF force and heading control.**

***Index Terms*—Magnetic manipulation, mechanism design, medical robots and systems, micro/nano robots.**

I. Introduction

Over the past 25 years, numerous magnetic-manipulation systems have been developed to control objects in relatively large workspaces. These systems vary in the number of electromagnets, in the configuration, and in their limitations. Magnetic manipulation of an untethered object in an unrestricted workspace has been pursued for applications ranging from wind-tunnel model stabilization [1], to medical device control [2], to microrobotic manipulation [3]. The magnetic control approaches discussed in the literature fall into three categories: heading only, force only, and combined heading and force. Despite the large research effort expended on manipulating objects magnetically, there has never been an analysis to quantify the minimum number of magnets necessary to perform heading only, force only, or combined heading and force control of an object.

Control of only the heading is based on the observation that an unconstrained magnetic object will attempt to align with a magnetic field. In 1996, Honda *et al.* exploited this characteristic to propel a helical device with an embedded magnet through fluid by applying a rotating magnetic field [4]. Since then, several groups have investigated how to leverage this effect for position control of a device [5]–[8] and for quasi-static heading control of a device [9], [10]. As these systems demonstrate, and as expected, for 3-D field control, at least three electromagnets are required.

A nonuniform magnetic field will apply a force on a magnetized object, which can be used to control an object's position. The most common force-based position control setup applies a static field in one direction, to magnetize or align the object, and uses three gradient fields to apply forces for position manipulation [11]–[15]. These designs have been realized with as few as four magnets and as many as eight magnets. It has been demonstrated in [16] that only three magnets are necessary to perform force-based control of an object's position, but the method presented requires a nonmagnetic restoring force, e.g., gravity, to stabilize the system, has a limited workspace, and has nonlinear constraints on the forces that can be applied to the object. It has remained unclear if 3-D force-based position control can be achieved with only three magnets and without the limitations associated with the method described by Petruska *et al.* [16].

Starting with a large effort at the University of Virginia in the 1990s, there have been several groups focused on designing systems capable of controlling both the heading and position of objects with magnetic fields and forces. These independent efforts have resulted in fixed-magnet designs ranging from six magnets [17] to eight magnets [18], [19] to 12 magnets [20]. The six-magnet design used by the University of Virginia had unexpected orientation-dependent singularities [21], which forced the project to switch from a heading and force-based control approach to only heading control [22]. The orientation-dependent singularity described in [21] has also been identified in a more recent study examining optimal magnetic configurations [23]. However, this orientation-dependent singularity does not appear in the eight- and 12-magnet systems previously mentioned, indicating that six-magnet systems may be fundamentally insufficient to independently control the 3-D force acting on and 2-D heading of an object without singularities.

Conversely, Mahoney and Abbott demonstrated control of both the heading and position of a permanent-magnet object with a position- and heading-controlled permanent-magnet manipulator. Thus, only five inputs, i.e., changes in the control magnet's position and heading, are required to achieve heading and force-based control with a nonstatic magnetic system [24]. Because this method always has a magnetically attractive force between the object and manipulator, it requires a nonmagnetic restoring force, e.g., gravity, to stabilize the system.

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To date, there has been no attempt to rigorously quantify how many stationary magnets are required to perform a given manipulation task. For some tasks, such as field control, the answer is clear, i.e., the same number as the dimension of control. For tasks involving forces, the answer is less clear, and some systems, which initially seem to have enough magnets, have demonstrated unexpected singularities in their workspace. To address this issue, in Section II, we take an analysis approach similar to the one outlined in [20] for analyzing the field and gradient generated by the system, and we show that the force capability of the system can be analyzed independently from the object's orientation. Using this orientation-independent approach, in Section III, we prove the unintuitive result that, if there is no external restoring force, four magnets are necessary for force-based control and eight magnets are necessary for heading and force-based control without orientation-dependent singularities. In Section IV, we discuss the implications of this result, how nonstatic magnetic systems, such as [24], fit into this context, and how this analysis approach can be extended to compare magnetic system performance.

In this paper, bold font will be used to represent vectors (e.g., $\mathbf{a}$, $\mathbf{A}$), capitalized blackboard font to represent matrices (e.g., $\mathbb{B}$), and the calligraphic font to represent functions that repack vectors into matrices and matrices into vectors (e.g., $\mathcal{S}(\mathbf{m})$). The 2-norm of a vector will be expressed as $\|\mathbf{a}\|$; the inner product of two vectors will be expressed as $\mathbf{a} \cdot \mathbf{b}$; the cross product of two vectors will either be expressed as $\mathbf{a} \times \mathbf{b}$ or in the skew-symmetric matrix form $\mathcal{S}(\mathbf{a})\mathbf{b}$; the normalized direction of a vector will be written as $\hat{\mathbf{a}}$; and the transpose of a vector or matrix will be expressed as $\mathbf{a}^\top$.

II. Magnetic Manipulation Background

The quasi-static magnetic field used for magnetic manipulation are described by Maxwell's equations as

$$\nabla \cdot \mathbf{B} = 0 \tag{1}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{J} \tag{2}$$

where $\mathbf{B}$ is the magnetic-vector field, $\mathbf{J}$ is the current-density vector field (normally zero in a manipulation space and nonzero in the windings of a magnet), and $\mu_0 = 4\pi \times 10^{-7}$ Tm·A^{-1} is the magnetic permeability of free space [25]. Together with the appropriate boundary conditions, these equations completely describe a quasi-static magnetic field. The force $\mathbf{F}$ and torque $\boldsymbol{\mathcal{T}}$ acting on an object with dipole moment $\mathbf{m}$ when placed in a magnetic field $\mathbf{B}$ are

$$\mathbf{F} = (\mathbf{m} \cdot \nabla)\mathbf{B} \tag{3}$$

$$\boldsymbol{\mathcal{T}} = \mathbf{m} \times \mathbf{B}. \tag{4}$$

In current free space, (1) constrains the gradient matrix of the vector field $\mathbf{B}$ to have zero trace, and (2) constrains the gradient matrix of the vector field $\mathbf{B}$ to be symmetric. Using these requirements, (3) can be rearranged into a matrix form

$$\mathbf{F} = \begin{bmatrix} \frac{\partial \mathrm{B}_x}{\partial x} & \frac{\partial \mathrm{B}_x}{\partial y} & \frac{\partial \mathrm{B}_x}{\partial z} \\ \frac{\partial \mathrm{B}_x}{\partial y} & \frac{\partial \mathrm{B}_y}{\partial y} & \frac{\partial \mathrm{B}_y}{\partial z} \\ \frac{\partial \mathrm{B}_x}{\partial z} & \frac{\partial \mathrm{B}_y}{\partial z} & -\left(\frac{\partial \mathrm{B}_y}{\partial y} + \frac{\partial \mathrm{B}_x}{\partial x}\right) \end{bmatrix} \mathbf{m}. \tag{5}$$

A manipulation system has direct control over the field and gradient it produces through the applied currents and indirect control of the object's dipole moment. Rearranging (5), the force can be written as

$$\begin{aligned} \mathbf{F} &= \begin{bmatrix} m_x & m_y & m_z & 0 & 0 \\ 0 & m_x & 0 & m_y & m_z \\ -m_z & 0 & m_x & -m_z & m_y \end{bmatrix} \begin{bmatrix} \frac{\partial \mathrm{B}_x}{\partial x} \\ \frac{\partial \mathrm{B}_x}{\partial y} \\ \frac{\partial \mathrm{B}_x}{\partial z} \\ \frac{\partial \mathrm{B}_y}{\partial y} \\ \frac{\partial \mathrm{B}_y}{\partial z} \end{bmatrix} \\ &= \mathcal{F}(\mathbf{m})\,\mathcal{G}\left(\nabla \mathbf{B}^\top\right) = \mathcal{F}(\mathbf{m})\,\mathbf{G} \end{aligned} \tag{6}$$

and the torque can be written as

$$\begin{aligned} \boldsymbol{\mathcal{T}} &= \begin{bmatrix} 0 & -m_z & m_y \\ m_z & 0 & -m_x \\ -m_y & m_x & 0 \end{bmatrix} \begin{bmatrix} \mathrm{B_x} \\ \mathrm{B}_y \\ \mathrm{B}_z \end{bmatrix} \\ &= \mathcal{S}(\mathbf{m})\,\mathbf{B}. \end{aligned} \tag{7}$$

The function $\mathcal{F}(\mathbf{m})$ packs the dipole moment of the object into the necessary matrix form. The function $\mathcal{G}\left(\nabla \mathbf{B}^\top\right)$ packs the field-gradient matrix (5) into a five-element field-gradient vector, which for compactness will be written as $\mathbf{G}$. The function $\mathcal{S}(\mathbf{m})$ maps the vector cross-product in (4) to its matrix form. Combining (6) and (7), the wrench on a dipole can be expressed as

$$\begin{bmatrix} \boldsymbol{\mathcal{T}} \\ \mathbf{F} \end{bmatrix} = \begin{bmatrix} \mathcal{S}(\mathbf{m}) & \mathbb{O} \\ \mathbb{O} & \mathcal{F}(\mathbf{m}) \end{bmatrix} \begin{bmatrix} \mathbf{B} \\ \mathbf{G} \end{bmatrix} \tag{8}$$

where $\mathbb{O}$ is an appropriately sized matrix of zeros. Equation (8) shows that the six mechanical degrees of freedom (DOFs), i.e., $\boldsymbol{\mathcal{T}}$ and $\mathbf{F}$, and are a function of the eight magnetic DOF, i.e., $\mathbf{B}$ and $\mathbf{G}$.

From (2), the magnetic field and, thus, its gradients are linear with the applied current I, which is related to $\mathbf{J}$ by the cross-sectional area of the wire used to construct the magnet. This linear mapping is given by the following vector equations:

$$\mathbf{B}(\mathbf{p}, \mathbf{I}) = \mathbb{B}(\mathbf{p})\mathbf{I} \tag{9}$$

$$\mathbf{G}(\mathbf{p}, \mathbf{I}) = \mathbb{G}(\mathbf{p})\mathbf{I} \tag{10}$$

where the current I for each magnet is packed into the column-vector $\mathbf{I}$, and $\mathbf{p}$ is the position of interest in the workspace. The columns of $\mathbb{B}$ and $\mathbb{G}$, respectively, correspond to the current-normalized field and gradient gains attributed to each magnet. Let there be N magnets; then, $\mathbb{B}$ is a $3 \times N$ matrix and $\mathbb{G}$ is a $5 \times N$ matrix. Both $\mathbb{B}$ and $\mathbb{G}$ are a function of the location of the workspace position $\mathbf{p}$ and can be found using a standard field modeling technique, e.g., direct numerical integration, finite element modeling, or *in situ* field measurements.

The algebraic system of equations that maps the magnet currents to the applied force and torque can be viewed as a two-step process. First, the currents generate a field and gradient at the object's location. Second, the field and the gradient interact with the object's dipole moment to produce both a torque and force. This two-step process can be expressed as

$$\begin{bmatrix} \boldsymbol{\mathcal{T}} \\ \mathbf{F} \end{bmatrix} = \begin{bmatrix} \mathcal{S}(\mathbf{m}) & \mathbb{O} \\ \mathbb{O} & \mathcal{F}(\mathbf{m}) \end{bmatrix} \begin{bmatrix} \mathbb{B}(\mathbf{p}) \\ \mathbb{G}(\mathbf{p}) \end{bmatrix} \mathbf{I}. \tag{11}$$

Because $\mathbf{m}$ is not always sensed, it is more convenient to linearize the system by specifying the desired field (which maps to a desired dipole moment) and the desired force, as in [13], [18], [21]

$$\begin{aligned} \begin{bmatrix} \mathbf{B} \\ \mathbf{F} \end{bmatrix} &= \begin{bmatrix} \mathbb{I} & \mathbb{O} \\ \mathbb{O} & \mathcal{F}(\mathcal{M}(\mathbf{B})) \end{bmatrix} \begin{bmatrix} \mathbb{B}(\mathbf{p}) \\ \mathbb{G}(\mathbf{p}) \end{bmatrix} \mathbf{I} \\ &= \mathcal{C}(\mathcal{M}(\mathbf{B})) \begin{bmatrix} \mathbb{B}(\mathbf{p}) \\ \mathbb{G}(\mathbf{p}) \end{bmatrix} \mathbf{I} \end{aligned} \tag{12}$$

where the function $\mathcal{M}(\mathbf{B})$ maps the applied field to the object's dipole moment, and $\mathbb{I}$ is an appropriately sized identity matrix. The convenience of (12) is that an unrestrained magnetic object will align with the applied field direction for quasi-static manipulations, i.e., heading command changes that are slow compared to the rotational time constant of the object-field alignment. If the object is made from a permanent-magnetic material, then (12) is linear with the applied current. However, if the object is made of soft-magnetic material (e.g., Nickel–Iron), (12) becomes quadratic in current, because the dipole moment $\mathbf{m}$ is a function of the applied field strength. For ellipse like shapes, the function $\mathcal{M}(\mathbf{B})$ has been characterized in [26] and [27].

A magnetic-manipulation system generates a field and gradient that can be controlled to apply a desired force and torque on an object. By analyzing the fields and gradients required to perform a task, i.e., by studying the matrices $\mathcal{F}$ and $\mathcal{C}$, the analysis of a magnetic-manipulation system can be decoupled from the magnetic device being manipulated. In the following section, we examine $\mathcal{F}$ and $\mathcal{C}$ to determine the requirements for the field and gradient vector to produce a field and force generally. Once these requirements are defined, any magnetic system can be evaluated to determine if it is capable of performing a task, independently from the object being manipulated, solely by examining the $\mathbb{B}$ and $\mathbb{G}$ matrices.

III. Consequences for Static Magnet Configurations

The manipulation of magnetic devices can be divided into three categories. The first is 3-DOF heading or rotational control in which the object's dipole-moment direction is controlled and the applied forces are neglected. The second is 3-DOF force-based control, in which the object's dipole moment is assumed to be aligned with the field or constrained. The third is 5-DOF heading and force-based control of an unconstrained object, in which object's dipole moment is controlled by an applied magnetic field and the position is controlled by an independently applied magnetic force. Full (6-DOF) control of a magnetic object's orientation and position has been demonstrated for systems with more than one embedded permanent magnet [28], [29], but their specific case does not alter the analysis here. From an algebraic perspective, at least three control inputs are necessary to achieve 3-DOF control and five inputs are required for 5-DOF control. This section addresses how many inputs are required from a magnetic perspective.

A. Field Control

Field control agrees with algebraic intuition because the field is linear with the current. Thus, to have full control of the field vector, $\mathbb{B}(\mathbf{p})$ must be equal in rank to the dimension of the workspace at every point in the workspace. For a 2-D workspace, two independent current inputs are required, and for a 3-D workspace, three current inputs are required.

B. General Force Control

Force control is more complicated to analyze than field control because $\mathcal{F}(\mathbf{m})$ in (6), which relates the five-element gradient vector to the resulting three-element force vector, has a nontrivial null-space that is a function of the object's dipole moment. At any instant, the gradients necessary to produce a desired force can be obtained from (6) with the pseudoinverse of the matrix $\mathcal{F}(\mathbf{m})$ (i.e., $\mathcal{F}(\mathbf{m})^\top \left(\mathcal{F}(\mathbf{m})\mathcal{F}(\mathbf{m})^\top\right)^{-1}$)

$$\begin{bmatrix} \dfrac{\partial \mathrm{B}_x}{\partial x} \\ \dfrac{\partial \mathrm{B}_x}{\partial y} \\ \dfrac{\partial \mathrm{B}_x}{\partial z} \\ \dfrac{\partial \mathrm{B}_y}{\partial y} \\ \dfrac{\partial \mathrm{B}_y}{\partial z} \end{bmatrix} = \begin{bmatrix} \dfrac{(\mathbf{m}^\top\mathbf{m})\, m_x}{\alpha_1} & \dfrac{-m_x^2 m_y}{\alpha_1} & \dfrac{-m_z}{\alpha_2} \\ \dfrac{m_y^3 + m_y m_z^2}{\alpha_1} & \dfrac{m_x^3 + m_x m_z^2}{\alpha_1} & 0 \\ \dfrac{(\mathbf{m}^\top\mathbf{m})\, m_z}{\alpha_1} & \dfrac{-m_x m_z m_z}{\alpha_1} & \dfrac{m_x}{\alpha_2} \\ \dfrac{-m_x m_y^2}{\alpha_1} & \dfrac{(\mathbf{m}^\top\mathbf{m})\, m_y}{\alpha_1} & \dfrac{-m_z}{\alpha_2} \\ \dfrac{-m_x m_z m_z}{\alpha_1} & \dfrac{(\mathbf{m}^\top\mathbf{m})\, m_z}{\alpha_1} & \dfrac{m_y}{\alpha_2} \end{bmatrix} \mathbf{F} \tag{13}$$

$$\begin{aligned} \alpha_1 &= \left(\mathbf{m}^\top\mathbf{m}\right)^2 - (m_x m_y)^2 \\ \alpha_2 &= \mathbf{m}^\top\mathbf{m} + m_z^2. \end{aligned}$$

To understand how the system is affected by object's heading (i.e., the direction of $\mathbf{m}$), we examine the form of (13) for the following three cases:

$$1)\ \mathbf{m} = m\begin{bmatrix}1\\0\\0\end{bmatrix} \Rightarrow \begin{bmatrix}\frac{\partial \mathrm{B}_x}{\partial x}\\ \frac{\partial \mathrm{B}_x}{\partial y}\\ \frac{\partial \mathrm{B}_x}{\partial z}\\ \frac{\partial \mathrm{B}_y}{\partial y}\\ \frac{\partial \mathrm{B}_y}{\partial z}\end{bmatrix} = \frac{1}{m}\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\\0&0&0\end{bmatrix}\mathbf{F}$$

$$2)\ \mathbf{m} = m\begin{bmatrix}0\\1\\0\end{bmatrix} \Rightarrow \begin{bmatrix}\frac{\partial \mathrm{B}_x}{\partial x}\\ \frac{\partial \mathrm{B}_x}{\partial y}\\ \frac{\partial \mathrm{B}_x}{\partial z}\\ \frac{\partial \mathrm{B}_y}{\partial y}\\ \frac{\partial \mathrm{B}_y}{\partial z}\end{bmatrix} = \frac{1}{m}\begin{bmatrix}0&0&0\\1&0&0\\0&0&0\\0&1&0\\0&0&1\end{bmatrix}\mathbf{F}$$

$$3)\ \mathbf{m} = m\begin{bmatrix}0\\0\\1\end{bmatrix} \Rightarrow \begin{bmatrix}\frac{\partial \mathrm{B}_x}{\partial x}\\ \frac{\partial \mathrm{B}_x}{\partial y}\\ \frac{\partial \mathrm{B}_x}{\partial z}\\ \frac{\partial \mathrm{B}_y}{\partial y}\\ \frac{\partial \mathrm{B}_y}{\partial z}\end{bmatrix} = \frac{1}{m}\begin{bmatrix}0&0&-\frac{1}{2}\\0&0&0\\1&0&0\\0&0&-\frac{1}{2}\\0&1&0\end{bmatrix}\mathbf{F}.$$

In each case, only three gradients are required to produce a desired force. Moreover, since the coordinate system can always be chosen to correspond with one of these cases, only three gradients are ever required to generate a desired force given a specific object dipole-moment direction. Note that, case 3 has the additional flexibility of allowing combinations of $\partial \mathrm{B}_x/\partial x$ and $\partial \mathrm{B}_y/\partial y$ to produce a z-direction force, with the minimum 2-norm solution provided.

Given an object's dipole moment, only three gradients are required to generate the desired force. However, all the gradients need to be controllable if the dipole-moment direction changes during the control task. For example, consider the task of applying force to a rotating object that does not change position. Let the object's dipole moment initially correspond to case 1, at this moment only $\frac{\partial B_x}{\partial x}$, $\frac{\partial B_x}{\partial y}$, and $\frac{\partial B_x}{\partial z}$ are required. Now, let the object rotate about the z-direction until it corresponds to case 2. Both $\frac{\partial B_y}{\partial y}$ and $\frac{\partial B_y}{\partial z}$ are required to apply any force, while both $\frac{\partial B_x}{\partial x}$ and $\frac{\partial B_x}{\partial z}$ no longer contribute to the force. Between these two configurations, the necessary gradients will change continuously. Although only three independent linear combinations of the five gradients are required at any interim dipole-moment direction, all five gradients are required at some point along the trajectory.

These cases require that all five terms are controllable to generate forces in the three object headings shown, but do not preclude the possibility of linear dependencies between $\partial \mathrm{B}_x/\partial x$ and $\partial \mathrm{B}_y/\partial y$, $\partial \mathrm{B}_x/\partial z$ and $\partial \mathrm{B}_y/\partial y$, or $\partial \mathrm{B}_x/\partial x$ and $\partial \mathrm{B}_y/\partial z$. These potentially acceptable linear combinations need to be further evaluated to determine if a dipole-moment direction can be found that prevents a force direction from being produced.

If $\partial \mathrm{B}_x/\partial x$ and $\partial \mathrm{B}_y/\partial y$ are linearly dependent, (6) becomes

$$\mathbf{F} = \begin{bmatrix} m_x & m_y & m_z & \cdot & 0\\ \alpha m_y & m_x & 0 & \cdot & m_z\\ -(1+\alpha)m_z & 0 & m_x & \cdot & \mathrm{m}_y\end{bmatrix}\begin{bmatrix}\frac{\partial \mathrm{B}_x}{\partial x}\\ \frac{\partial \mathrm{B}_x}{\partial y}\\ \frac{\partial \mathrm{B}_x}{\partial z}\\ \cdot\\ \frac{\partial \mathrm{B}_y}{\partial z}\end{bmatrix}$$
$$= \mathcal{F}_{\mathrm{a}}(\mathbf{m})\,\mathbf{G}_{\mathrm{a}} \tag{14}$$

where "·" has been placed in the position of the removed linearly dependent quantities, and α is the coefficient describing the dependence. $\mathcal{F}_{\mathrm{a}}$ maps dipole moment to force while taking into account the linear dependences in $\mathbf{G}$, which is reduced to $\mathbf{G}_{\mathrm{a}}$ by removing one of the linearly dependent terms. Note that when $\alpha = -1$, no force in the F_z-direction can be produced when $m_x = m_y = 0$. Taking the configuration where $m_x = \beta m_y$ and $m_z = 0$ yields

$$\mathcal{F}_{\mathrm{a}}\left(\mathrm{m}\begin{bmatrix}1\\ \beta\\ 0\end{bmatrix}\right) = m\begin{bmatrix}1 & \beta & 0 & \cdot & 0\\ \alpha\beta & 1 & 0 & \cdot & 0\\ 0 & 0 & 1 & \cdot & \beta\end{bmatrix}. \tag{15}$$

If α is positive, the configuration cannot set F_x independently from F_y for any heading where $\beta = \sqrt{\frac{1}{\alpha}}$. For α negative, the case where $m_x = \beta m_z$ and $m_y = 0$ yields

$$\mathcal{F}_{\mathrm{a}}\left(\mathrm{m}\begin{bmatrix}1\\ 0\\ \beta\end{bmatrix}\right) = m\begin{bmatrix}1 & 0 & \beta & \cdot & 0\\ 0 & 1 & 0 & \cdot & \beta\\ -(1+\alpha)\beta & 0 & 1 & \cdot & 0\end{bmatrix}. \tag{16}$$

This is singular, i.e., F_x and F_z are not independently controllable when $\alpha < -1$ and $\beta = \sqrt{\frac{-1}{1+\alpha}}$. Finally, the case where

$m_y = \beta m_z$ and $m_x = 0$ yields

$$\mathcal{F}_{\mathrm{a}}\left(m\begin{bmatrix}0\\1\\\beta\end{bmatrix}\right) = m\begin{bmatrix}0 & 1 & \beta & \cdot & 0\\ \alpha & 0 & 0 & \cdot & \beta\\ -(1+\alpha)\beta & 0 & 0 & \cdot & 1\end{bmatrix} \tag{17}$$

and shows that F_y and F_z are not independently controllable when $-1 < \alpha < 0$ and $\beta = \sqrt{\frac{-\alpha}{1+\alpha}}$. Therefore, if the $\partial \mathrm{B}_x / \partial x$ and $\partial \mathrm{B}_y / \partial y$ rows of $\mathbb{G}$ are linearly dependent, then there always exists a direction $\hat{\mathbf{m}}$ for which $\mathcal{F}(\hat{\mathbf{m}})\,\mathbb{G}$ has a rank less than 3.

Next, we need to analyze the system when $\partial \mathrm{B}_x / \partial z$ and $\partial \mathrm{B}_y / \partial y$ are linearly dependent. Taking a similar approach

$$\mathcal{F}_{\mathrm{a}}(\mathbf{m}) = \begin{bmatrix} m_x & m_y & m_z & \cdot & 0\\ 0 & m_x & \alpha m_y & \cdot & m_z\\ -m_z & 0 & m_x - \alpha m_z & \cdot & m_y\end{bmatrix}. \tag{18}$$

Analyzing the situation when $m_z = \beta m_x$ and $m_y = 0$ yields

$$\mathcal{F}_{\mathrm{a}}\left(m\begin{bmatrix}1\\0\\\beta\end{bmatrix}\right) = m\begin{bmatrix}1 & 0 & \beta & \cdot & 0\\ 0 & 1 & 0 & \cdot & \beta\\ -\beta & 0 & (1-\alpha\beta) & \cdot & 0\end{bmatrix} \tag{19}$$

which is singular when $\beta = \frac{1}{2}\left(\alpha \pm \sqrt{\alpha^2 - 4}\right)$ and $|\alpha| \geq 2$. In this configuration, it is not possible to generate a force in the $\begin{bmatrix}\beta_+ & 0 & 1\end{bmatrix}^\top$ or $\begin{bmatrix}1 & 0 & \beta_-\end{bmatrix}^\top$ directions, respectively. Next, when $m_x = m_z$ and $m_y = \beta m_x$, (18) becomes

$$\mathcal{F}_{\mathrm{a}}\left(m\begin{bmatrix}1\\\beta\\1\end{bmatrix}\right) = m\begin{bmatrix}1 & \beta & 1 & \cdot & 0\\ 0 & 1 & \alpha\beta & \cdot & 1\\ -1 & 0 & (1-\alpha) & \cdot & \beta\end{bmatrix} \tag{20}$$

and is singular when the rank of $\mathcal{F}_{\mathrm{a}}(\mathbf{m})\,\mathcal{F}_{\mathrm{a}}(\mathbf{m})^\top$ is less than 3, i.e.,

$$\det\left(\mathcal{F}_{\mathrm{a}}(\mathbf{m})\,\mathcal{F}_{\mathrm{a}}(\mathbf{m})^\top\right) = 0$$

which occurs if $\beta = \sqrt{\frac{2-\alpha}{\alpha}}$ and $0 < \alpha \leq 2$. In this configuration, no force can be applied in the $\begin{bmatrix}1 & -\beta & 1\end{bmatrix}^\top$ direction. Similarly, if $m_x = -m_z$ and $m_y = \beta m_x$, then the system is singular when $\beta = \sqrt{-\frac{2+\alpha}{\alpha}}$ and $-2 \leq \alpha < 0$, and no force can be applied in the $\begin{bmatrix}1 & -\beta & -1\end{bmatrix}^\top$ direction. Therefore, if the $\partial \mathrm{B}_x / \partial z$ and $\partial \mathrm{B}_y / \partial y$ rows of $\mathbb{G}$ are linearly dependent, then there always exists a direction $\hat{\mathbf{m}}$ for which $\mathcal{F}(\hat{\mathbf{m}})\,\mathbb{G}$ has a rank less than 3.

Finally, analyzing the possible linear dependence between $\partial \mathrm{B}_x / \partial x$ and $\partial \mathrm{B}_y / \partial z$ transforms $\mathcal{F}_{\mathrm{a}}(\mathbf{m})$ into

$$\mathcal{F}_{\mathrm{a}}(\mathbf{m}) = \begin{bmatrix} m_x & m_y & m_z & 0 & \cdot\\ \alpha m_z & m_x & 0 & m_y & \cdot\\ \alpha m_y - m_z & 0 & m_x & -m_z & \cdot\end{bmatrix}. \tag{21}$$

When $m_y = \beta m_z$ and $m_x = 0$, this becomes

$$\mathcal{F}_{\mathrm{a}}\left(m\begin{bmatrix}0\\\beta\\1\end{bmatrix}\right) = m\begin{bmatrix}0 & \beta & 1 & 0 & \cdot\\ \alpha & 0 & 0 & \beta & \cdot\\ \alpha\beta - 1 & 0 & 0 & -1 & \cdot\end{bmatrix} \tag{22}$$

which is singular when $\beta = \frac{1}{2\alpha}\left(1 \pm \sqrt{1 - 4\alpha^2}\right)$ and $-\frac{1}{2} \leq \alpha \leq \frac{1}{2}, \alpha \neq 0$. In this configuration, F_y and F_z are linearly dependent. When $\beta m_x = m_y = m_z$, the system is singular if $\beta = \sqrt{2\alpha - 1}$ and $\alpha \geq \frac{1}{2}$, and when $\beta m_x = m_y = -m_z$, the system is singular if $\beta = \sqrt{-2\alpha - 1}$ and $\alpha \leq -\frac{1}{2}$. These two singularities can be verified by examining the determinant of $\mathcal{F}(\mathbf{m})\,\mathcal{F}(\mathbf{m})^\top$ and correspond to configurations that cannot apply force in the $\begin{bmatrix}\beta & 1 & 1\end{bmatrix}^\top$ or $\begin{bmatrix}\beta & 1 & -1\end{bmatrix}^\top$ directions, respectively. Therefore, if the $\partial \mathrm{B}_x / \partial x$ and $\partial \mathrm{B}_y / \partial z$ rows of $\mathbb{G}$ are linearly dependent, then there always exists a direction $\hat{\mathbf{m}}$ for which $\mathcal{F}(\hat{\mathbf{m}})\,\mathbb{G}$ has a rank less than 3.

Thus, at any instant, only three linearly independent gradient combinations are required to apply an arbitrary force, but as many as five are required if the direction of the dipole moment changes. Therefore, for a system to control the force on a dipole in any orientation, it must be capable of controlling each of the gradient terms independently. This requires $\mathbb{G}(\mathbf{p})$ to be full row rank at every position $\mathbf{p}$ in the workspace.

Since the magnetic force on a dipole is produced by local magnetic-field gradients, it is not possible to apply a magnetic force in the absence of a global magnetic field. This magnetic field will cause an unconstrained object to rotate until the object's dipole moment is aligned with the field because of the magnetic torque (4). For a magnetic-manipulation system to continuously apply a desired force, it must be capable of either sensing or controlling the object's orientation and updating the magnetic gradients faster than the dipole-moment direction of the object can change. For an unconstrained object, this is most easily accomplished by controlling the field as well as the gradients at the object.

C. *Combined Field and Force Control*

For a magnetic system to control the force on a dipole in any orientation, it must be capable of controlling each of the gradient terms independently. For a magnetic system to control an object's dipole moment, it must be capable of specifying a field in any direction. Thus, a necessary condition to perform singularity-free field and force control is that both $\mathbb{B}$ and $\mathbb{G}$ are full rank, but the consequence of a linear dependence between the rows of $\mathbb{B}$ and $\mathbb{G}$ must be analyzed.

To analyze the effects of a linear dependence between the field and gradient, define the combined matrix

$$\mathcal{C}(\mathbf{m}) = \begin{bmatrix}\mathbb{I} & \mathbb{O}\\ \mathbb{O} & \mathcal{F}(\mathbf{m})\end{bmatrix} \tag{23}$$

which maps the stacked field and gradient vector to field and force as a function of dipole moment $\mathbf{m}$. Now, modify $\mathcal{C}(\mathbf{m})$ into an augmented form $\mathcal{C}_{\mathrm{a}}(\mathbf{m})$ by replacing one of the first three columns and one of the last five columns with a single column that is a linear combination of removed columns, i.e., define the system where there is a linear dependence between the field and the gradient similar to (14). $\mathcal{C}_{\mathrm{a}}(\mathbf{m})$ is a 6×7 matrix, while $\mathcal{C}(\mathbf{m})$ is a 6×8 matrix.

For every linear combination of field and gradient, there is a nonzero value of $\mathbf{m}$ that satisfies the equation

$\det\left(\mathcal{C}_{\mathrm{a}}\left(\mathbf{m}\right)\mathcal{C}_{\mathrm{a}}\left(\mathbf{m}\right)^{\top}\right)=0$, i.e., there is always a dipole-moment direction for which the field cannot be specified independently from the force, regardless of whether the dipole moment and the magnetic field are assumed to be aligned. This result can be deduced from above. A linear dependence between a field component and a gradient component requires the gradient to be zero when the field is zero, but all gradients are required to produce forces if the orientation of the object is arbitrary. Therefore, there must exist an object orientation for which it is not possible to achieve any force and any field when a linear dependence between field and gradient exists. To independently control the field and the force at a point, regardless of the object orientation, e.g., while performing wrench control, the rows of $\mathbb{G}$ and $\mathbb{B}$ must be linearly independent and both matrices must be full row rank.

D. Heading and Force Control

Open-loop control of the heading of the object, i.e., assuming the object is always aligned with the field, constrains the possible orientations of the object and could allow for a nonlinear control approach to avoid the singular combinations of applied field, desired force, and dipole-moment direction discussed previously. Specifying the field and the force on an object of arbitrary heading requires that all combinations of all three vectors (field, force, dipole moment direction) are possible. Specifying only a heading direction, assuming the object is aligned with the field, reduces the possible number of combinations. Therefore, it is possible that the singularities arising from a field and gradient linear dependence do not overlap with the admissible configurations, under the assumption that the field and dipole moment are always aligned. There are 15 possible linear dependences between the field and the gradient that must be considered.

Of the 15 possible linear combinations, only two result in singularities of $\mathcal{C}_{\mathrm{a}}\left(\mathbf{m}\right)\mathcal{C}_{\mathrm{a}}\left(\mathbf{m}\right)^{\top}$ that are physically achievable with the assumption that the object and the field are stably aligned while producing a force in any direction. The physically achievable singularities occur when there is a linear dependence between B_x and $\frac{\partial \mathrm{B}_x}{\partial x}$ or a linear dependence between B_y and $\frac{\partial \mathrm{B}_y}{\partial y}$. The other 13 possible linear dependences have unreachable force states under the assumption that the object is stably aligned with the field. For example, in the case of a linear dependence between B_x and $\frac{\partial \mathrm{B}_y}{\partial y}$

$$\mathcal{C}_{\mathrm{a}}\left(\mathbf{m}\right)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & m_x & m_y & m_z & \cdot & 0 \\ \alpha m_y & 0 & 0 & 0 & m_x & 0 & \cdot & m_z \\ -\alpha m_z & 0 & 0 & -m_z & 0 & m_x & \cdot & m_y \end{bmatrix}. \tag{24}$$

Take the configuration where the object's dipole moment is aligned with the y-direction, i.e., when $\mathrm{B}_x=\mathrm{B}_y=m_x=m_z=0$. In this configuration, (24) becomes

$$\mathcal{C}_{\mathrm{a}}\left(\begin{bmatrix} 0 \\ \mathrm{m_y} \\ 0 \end{bmatrix}\right)=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & m_y & 0 & \cdot & 0 \\ \alpha m_y & 0 & 0 & 0 & 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \cdot & m_y \end{bmatrix}. \tag{25}$$

Now, consider the first and fifth rows of (25). A y-direction force can only be produced in conjunction with an x-direction field, as their scaling factors share the first column and no other columns contribute to either. However, it is not possible to apply an x-direction field without violating the underlying constraint that the object is aligned with the y-direction. Physically, the result of a y-directed force would be to rotate the dipole moment of tool away from the y-direction in proportion to the force. Thus, the reachable force space for the tool in this scenario is reduced to the x- and z-directions. The same approach can be applied to the remaining twelve singular combinations, and there is always a dipole-moment direction that reduces the achievable forces from a 3-D space to a 2-D space.

The remaining two cases are analogous to each other. Take, for example, the case when B_x and $\frac{\partial \mathrm{B}_x}{\partial x}$ are linearly dependent. In this case

$$\mathcal{C}_{\mathrm{a}}\left(\mathbf{m}\right)=\begin{bmatrix} 1 & 0 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & \cdot & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \cdot & 0 & 0 & 0 & 0 \\ \alpha \mathrm{m}_x & 0 & 0 & \cdot & m_y & m_z & 0 & 0 \\ 0 & 0 & 0 & \cdot & m_x & 0 & m_y & m_z \\ -\alpha m_z & 0 & 0 & \cdot & 0 & m_x & m_z & m_y \end{bmatrix}. \tag{26}$$

Inspection of the rows of $\mathcal{C}_{\mathrm{a}}\left(\mathbf{m}\right)$ shows that F_x is the only force direction that must be produced in linear combination to the field (F_z does not share this problem, because m_z appears twice on its row allowing for both a z-directed dipole moment and a z-directed force independent from the x-field). In this case, F_x and B_x are not linearly independent when the object is oriented solely in the x-direction, which is consistent with the underlying assumptions.

This singularity has two ramifications. Foremost and the reason the row rank of $\mathcal{C}_{\mathrm{a}}\left(\mathbf{m}\right)$ has been reduced to 5, the sign of the force generated from this combination can never change. This can be seen because a negative input will both negate the gradient and flip the alignment of the object. Thus, the product of the two will be sign invariant. In this case, a nonmagnetic restoring force in the x-direction is required, so that the sum of the magnetic force and the restoring force can change sign. Second, the magnitude of the field, with which the object must align, will be directly proportional to the force required in the dependent direction. This results in a system more susceptible to orientation disturbances when small forces are applied, which will lead to a breakdown in the fundamental assumption that the field and dipole moment are aligned. If both B_x and $\frac{\partial \mathrm{B}_x}{\partial x}$ are linearly dependent and B_y and $\frac{\partial \mathrm{B}_y}{\partial y}$ are linearly dependent, then the system becomes singular because a z-directed dipole moment precludes a z-directed force. Therefore, if there is an appropriately directed nonmagnetic restoring force and if the one linear dependence is between B_x and $\frac{\partial \mathrm{B}_x}{\partial x}$, B_y and $\frac{\partial \mathrm{B}_y}{\partial y}$, or, by rotational equivalence, B_z and $\frac{\partial \mathrm{B}_x}{\partial x} + \frac{\partial \mathrm{B}_y}{\partial y}$, a magnetic-manipulation system can be designed such that only seven electromagnets are required to control the object's heading and the applied force.

E. Wrench Control

Because no torque can be produced on a dipole about the dipole axis, a linear dependence between B_x and $\frac{\partial \mathrm{B}_x}{\partial x}$, for example, will not affect the ability of the system to apply a wrench on a permanent-magnetic object. Thus, only seven inputs are necessary for wrench control of a permanent magnet, as long as the only linear dependence in the magnetic system is between B_x and $\frac{\partial \mathrm{B}_x}{\partial x}$, B_y and $\frac{\partial \mathrm{B}_y}{\partial y}$, or B_z and $\frac{\partial \mathrm{B}_x}{\partial x} + \frac{\partial \mathrm{B}_y}{\partial y}$. Caution should be exercised when performing wrench control with such systems, because the sign and magnitude of the field generated in the dipole-moment direction will be uncontrollable when operating in a field-force control singularity. As the necessary force changes sign, the field in that linearly dependent direction will also change sign. This will result in a situation where a permanent-magnetic object will transition between a dynamically stable orientation and a dynamically unstable orientation due to the force applied. If the dynamics of the magnetic system that change its field inputs are slower than the rotational dynamics of the object, then the tool heading can become unstable resulting in undesired instabilities in the control system. An eight coil system can avoid this problem.

IV. Discussion

A. Force-Based Position Control

In many cases, such as the manipulation of beads and ferrofluid droplets [12], [13], only the forces acting on the object are specified and the object's dipole-moment direction is free to change as necessary. To realize force control with only three electromagnets that is robust to orientation disturbances, it is necessary to either constrain the object's dipole moment or to design the system to be capable of generating the necessary gradients when the object is aligned with the field as in [16]. As discussed previously, there are two consequences of designing a three coil system that cannot decouple the applied force from the applied field, with which the object will align. First, the force generated in at least one direction can never be reversed. In such a system, a nonmagnetic force, such as gravity or from a tether, is required to provide an offset, so that the total force on the object can be controlled even though the magnetic force can never change sign. Second, the magnitude of the applied field, which results as a consequence of applying a desired force, and which passively controls the object's heading, scales with the magnitude of the applied force. As the magnitude of desired magnetic force tends to zero, disturbances that affect the object's heading can easily affect the actual force applied because the assumption that the object is aligned with the applied field becomes invalid.

To avoid these complications, the heading of the object must be constrained and, for practical reasons, this is often done magnetically. To magnetically constrain the heading of the object in addition to applying force in any direction, at least four magnets are required. One linear combination of the magnets is responsible for the field to align the object with the desired direction. The current in this linear combination can remain unchanged during the procedure and could be realized with permanent magnets or superconducting magnets to reduce the electrical power consumed. The other three linear combinations are used to control the force on the object with their currents changing as necessary to apply the desired force. This approach has successfully been implemented to perform 3-DOF manipulation of beads in MRI systems, e.g., [12].

B. Force-Based Position and Heading Control

A static magnetic system requires eight independent inputs to control the force on an object and the field at the object location regardless of object orientation. If only heading and force control are required, then, in certain geometries, it is possible to achieve a singularity free 5-DOF workspace with seven inputs, but only if there is an appropriately directed restoring force. This restoring force could be provided by a tether or gravity, for example. If only wrench control is required, then only seven inputs are required, but the dynamics of the control system must be fast enough to stabilize the unstable tool orientations when operating in the heading-force singularity. This explains why orientation-dependent singularities have been noted for magnetic manipulation systems with six electromagnets [21], [23], but have been avoided by eight electromagnet systems [18], [19], [30].

To build a system with less than eight inputs that can control heading and force, it is necessary to first design it such that only the admissible singularities exist in the desired workspace. Due to the nonlinear spatial nature of magnetic fields, this is not trivial. A nonlinear control method that takes into account the linear dependence's effect on force must also be developed, such as was done in [16] for a special 3-DOF case. If successful, such a system design will be more susceptible to disturbances than an eight input system, because it is not possible to control the strength of the aligning field (equivalent to a rotational stiffness) independent of force. Moreover, some trajectories cannot be executed due to nonlinear constraints imposed by the dependences. For example, a trajectory that wishes to apply no force on the object, e.g., to drop as quickly as possible in a highly viscous environment, while rotating toward the linearly dependent direction cannot be executed because applying the field to achieve the desired rotation will simultaneously apply an undesired force.

The requirement for a system to have at least eight electromagnets for field-force-based manipulation is necessary, but not sufficient. The sufficient condition for singularity-free control for field and force is for the stacked field and gradient matrix $\left[\mathbb{B}(\mathbf{p})^\top \quad \mathbb{G}(\mathbf{p})^\top\right]^\top$ to have a rank of 8 at every location $\mathbf{p}$ in the desired workspace. For example, a system comprising three orthogonal Helmholtz-coil pairs (to control field direction) and three orthogonal Maxwell-coil pairs (to control gradients) constitutes a total of 12 magnets if operated independently, but 5-DOF manipulation is not possible because the off-diagonal gradients ($\frac{\partial \mathrm{B}_x}{\partial y}$, $\frac{\partial \mathrm{B}_x}{\partial z}$, $\frac{\partial \mathrm{B}_y}{\partial z}$) are identically zero at the center of such a system. To achieve 5-DOF manipulation using the standard coil pairs (Helmholtz, Maxwell, and gradient-saddle), it is necessary to use eight pairs, for example, three orthogonal Helmholtz-coil pairs (to control B_x, B_y, and B_z), two orthogonal Maxwell-coil pairs (to control $\frac{\partial B_x}{\partial x}$ and $\frac{\partial B_y}{\partial y}$), and three gradient-saddle-coil pairs (to control $\frac{\partial B_x}{\partial y}$, $\frac{\partial B_x}{\partial z}$, and $\frac{\partial B_y}{\partial z}$). Alternatively, two of the Helmholtz-coil pairs, for example, the ones responsible for controlling B_x and B_y, could be replaced by independently controlling the four saddle coils responsible for producing the $\frac{\partial B_x}{\partial y}$ and $\frac{\partial B_y}{\partial z}$ gradients, respectively. Their configuration still results in eight independently controlled sources and a stacked-field-and-gradient matrix with full rank, but reduces the number of physical coils required and, *ceteris paribus,* reduces the system performance (since the four saddle coils are now responsible for producing both a field and a gradient).

When the configuration is not static, the linearity assumption between inputs and outputs is invalid. In such a case, the effect of an input can change both the shape and magnitude of the magnetic field, which reduces the number of inputs required for control. For example, Mahoney and Abbott have demonstrated 5-DOF position and heading control of an object using a permanent magnet manipulated by a 6-DOF robotic arm [24]. Similar to the 3-DOF field-aligned positioning case discussed in [16], the force between the object and the permanent-magnet manipulator is always attractive and a restoring force is required to stabilize the system. Also similar to [16], the field magnitude is proportional to the force applied, and small applications of magnetic force, to move downward for example, result in less robust heading control. Their reconfigurable system requires fewer control DOFs than a static magnet arrangement because $\mathbb{G}(\mathbf{p})$ is no longer a constant matrix. Instead, it is a local linearization of the mapping between changes in inputs, e.g., position, and changes in gradients. As such, it is not necessary for $\mathbb{G}(\mathbf{p})$ to be full row rank for every change in control input; instead, it is only necessary for $\mathcal{F}(\mathbf{m})\,\mathbb{G}(\mathbf{p})$ to be full rank for every change in input where both $\mathbf{m}$ and $\mathbb{G}$ depend on the current control state. The minimum number of inputs to independently control the field and force at a point is reduced from eight for the static case to six for the reconfigurable case. Likewise, the minimum number of inputs required for 5-DOF heading and position control, in the presence of an appropriately directed restoring force, is reduced from seven for the static case to five for the reconfigurable case.

C. System Comparisons

Different coil-system designs will have different performance characteristics. Quantifying those differences is necessary for one system to be compared against another. Since the combined field and gradient matrix must be full rank for field and force control at every point, the capability of a magnetic system can be quantified independently from the task or device being manipulated. Let $\mathbb{N}_{\mathbb{G}}$ be the normalized null space of the $\mathbb{G}$ matrix, i.e., $\mathbb{N}_{\mathbb{G}} = \mathbb{I} - \mathbb{G}\left(\mathbb{G}^\top\mathbb{G}\right)^{-1}\mathbb{G}^\top$, and let $\mathbb{N}_{\mathbb{B}}$ be the similarly defined normalized null space of the $\mathbb{B}$ matrix. The minimum singular value of the product $\mathbb{B}\mathbb{N}_{\mathbb{G}}$ is the value of the smallest field gain given the capability to independently specify any gradient, which is proportional to the worst-case torque that can be achieved. Likewise, the minimum singular value of the product $\mathbb{G}\mathbb{N}_{\mathbb{B}}$ is the minimum gradient gain that can be achieved given the capability to independently specify any field, which is proportional to the worst-case force on a permanent magnet that can be achieved. Finally, the product of the minimum singular values of $\mathbb{B}\mathbb{N}_{\mathbb{G}}$ and $\mathbb{G}\mathbb{N}_{\mathbb{B}}$ is proportional to the worst-case force on a soft-magnetic tool that can be achieved. By comparing these three metrics between systems, multiple systems can be compared at a magnetic level without specific object information or a comparison of multiple possible object orientations. Moreover, any given manipulation task, be it controlling a soft-magnetic ellipsoid, e.g., [31], or a permanent-magnetic capsule, e.g., [20], can be defined in terms of the minimum singular values for field, gradient, or field-gradient product required to achieve the motions desired. These requirements can be directly compared with a system's capabilities, in terms of the minimum field and gradient singular values, without performing task-specific simulations.

V. Conclusion

An analysis of the consequences of Maxwell's equations on the configuration of a magnetic-manipulation system has been provided. As expected, only three magnets are required to control the 3-D field at a point. Similarly, only three magnets are required to control the force at a point, but the three magnet solution requires either a nonmagnetic restoring torque or a nonmagnetic restoring force to stabilize the system. To control the force on a dipole with only magnetic inputs, four magnets are required. Combined field and force control requires eight stationary magnets, but heading and force control or wrench control can be achieved with only seven stationary magnets. The seven magnet cases require either an appropriately directed restoring force to counter the force direction that cannot change sign in heading control mode, or appropriately fast dynamics to stabilize the heading in wrench control mode. This is because the applied field can oscillate between a rotationally stable and unstable configuration. Reconfigurable magnetic manipulation systems can achieve similar control authority to static systems with fewer inputs. Only five are required for heading and force control, because the field shape in the workspace can be modified by changing the location or orientation of the control magnets during the magnetic manipulation task.

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