Forewords

Hi, I'm Chee Xiang, also known as Pallon. I have created this help sheet for my own study, and it is the treasure and culmination of my efforts. However, for some reason, I have decided to share this help sheet with other students, including you. One of the main reasons is that I hope it can help you in your studies. Another fun reason is to observe how my little help sheet can spread throughout our community. You can think of it as me performing a Breadth-First Search (BFS) with myself as the root node, trying to find the maximum depth.

I'm sure it will also be fun to keep a record of who is in our community. Therefore, I have created a contributor list that records those who have progressively contributed to the improvement of this help sheet (of course, it starts with only me). If you're bored or interested, you can also check their LinkedIn profiles (if they have provided them) to connect with them. You can find the contributor list here:
https://docs.google.com/spreadsheets/d/1r2SgdThPMcMoWAEyFPwaqrq1CjQ87ZfKuwDy0lADPJM/edit#gid=0

Feel free to click on this link to provide me with feedback on this help sheet:
https://docs.google.com/forms/d/1taARJqNnGrxbzEunaVD6IhviVO5hTVbMv6D66nSnaJw/edit

Lastly, please note that this help sheet is not perfect, and it could contain errors or undergo changes in versions, despite all the contributors and me doing our best to maintain its correctness. The latest version of the help sheet can be retrieved at https://github.com/PallonCX/CS1231S-Helpsheet. Please use it at your own risk.

Wishing you all the best in your studies!

Lecture 1: Speaking Mathematically

Important Sets:
$\mathbb{N}$ - Natural numbers (includes 0) | $\mathbb{Z}$ - Integers | $\mathbb{Q}$ - Rational numbers | $\mathbb{R}$ - Real numbers [$\mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R}$]

Superscripts (Top) & Subscripts (Bottom):
$\mathbb{Z}^+$: Positive integers | $\mathbb{R}^-$: Negative real numbers | $\mathbb{Z}_{\geq 12}$: Integers greater than or equal to 12 (0 is neither – or +)

Terminology:
Definition:
A precise and unambiguous description of the meaning of a mathematical term. It characterizes the meaning of a word by giving all the properties and only those properties that must be true.
Axiom / Postulate:
A statement that is assumed to be true without proof.
Theorem:
A mathematical statement that is proved using rigorous mathematical reasoning.
Lemma:
A small theorem.
Corollary:
A result that is a simple deduction from a theorem.
Conjecture:
A statement believed to be true, but for which there is no proof (yet).

Basic Properties of Integers $\boxed{\text{Lecture \#1 Slides \#26}}$ $\forall\ x, y, z \in \mathbb{Z}$:
Closure under + and x: $x + y \in \mathbb{Z}$ and $xy \in \mathbb{Z}$
Commutativity: $x + y = y + x$ and $xy = yx$
Associativity: $x + y + z = (x + y) + z = x + (y + z)$ and $xyz = (xy)z = x(yz)$.
Distributivity: $x(y + z) = xy + xz$ and $(y + z)x = yx + zx$.
Trichotonomy: Exactly one of the following is true: $x = y$, or $x < y$, or $x > y$.

Definition: **Even and Odd integers** $\boxed{\text{Lecture \#1 Slides \#27}}$:
If n is an integer, then
n is even $\Leftrightarrow \exists$ an integer k such that $n = 2k$.
n is odd $\Leftrightarrow \exists$ an integer k such that $n = 2k + 1$.

Assumption 1 $\boxed{\text{Lecture \#1 Slides \#27}}$:
Every integer is even or odd, but not both.

Definition: **Divisibility** $\boxed{\text{Lecture \#1 Slides \#32}}$:
If n, $d \in \mathbb{Z}$ and $d \neq 0$: $d \mid n \Leftrightarrow \exists k \in \mathbb{Z}$ such that $n = dk$.
Thus, "n is a multiple of d", or "d is a factor of n", or "d is a divisor of n" or "d divides n".
*$a|b$ is a statement, which is evaluated to true or false. It is not a numerical value.

Irrationality of $\sqrt{2}$ $\boxed{\text{Theorem 4.7.1 (5th: 4.8.1)}}$: $\sqrt{2}$ is irrational.

Definition: **Rational and irrational numbers** $\boxed{\text{Lecture \#1 Slides \#37}}$:
r is rational $\Leftrightarrow \exists\ a, b \in \mathbb{Z}$ s.t. $r = \frac{a}{b}$ and $b \neq 0$.
A real number that is not rational is irrational.

Definition: **Fraction in lowest term** $\boxed{\text{Lecture \#1 Slides \#37}}$:
A quotient of two integers with a nonzero denominator is also commonly known as a fraction. A fraction a/b (where $b \neq 0$) is said to be in lowest terms if the largest integer that divides both a and b is 1.

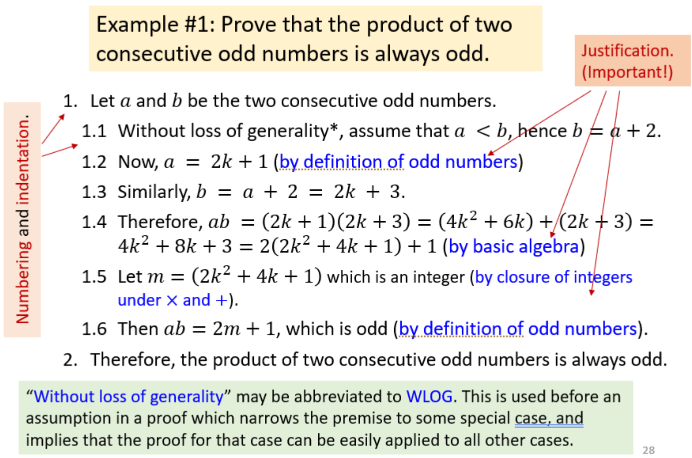
Assumption 2 $\boxed{\text{Lecture \#1 Slides \#37}}$:
Every rational can be reduced to a fraction in its lowest term.

$\boxed{\text{Proposition 4.6.4 (5th: 4.7.4)}}$: For all integers n, if n^2 is even then n is even.

Tutorial #1 Question #10: Let n be an integer. Then n^2 is odd if and only if n is odd.

Definition: **Colorful** (Only for CS1231S) Lecture #1 Slides #44:
An integer n is said to be colorful if there exists some integer k such that $n = 3k$.

General pattern of proof:



Some tips:
1. Write the draft proof by
(1) Start of the proof: Let
(2) End of the proof: Thus (Conclusion)
2. Continue the remaining parts with logical arguments, definitons or properties

Proofs:

Type of Proof	Pattern	Usage
Proof by construction (Direct proof)	1. Let / Consider 2. Note that (The thing let above have some property) 3. Also, (With the property, we can conclude that)	-When a specific value with some properties is required to suffice the proof -To proof existential statement (Find a specific value or give directions to find)
Disproof by counter example	1. Let 2. Therefore, (With the property of the thing let above, the statement is not true)	-To show a statement is false (Often for universal statement) -One counter-example is sufficient
Proof by exhaustion / Proof by cases / Proof by brute force	1. Let (Specific values from cases, such as 0, 1) 1.1 (Show that everything let above suffice the statement) 2. Therefore, (Conclude)	-Number of cases is finite
Proof by deduction (Direct proof)	1. Let (An abstract value, such as n, k) 1.1 (Show that the thing let above suffice the statement) 2. Therefore, (Conclude)	-Number of cases is infinite -General problem
Proof by contradiction (Indirect proof)	1. Suppose not, that is, (The negation of the statement) 1.1 (Show that we can deduce something that contradicts the assumption) 2. Therefore, the assumption that (The negation of the statement) is false. 3. Hence (The statement we want to prove)	-The statement to prove is absence of form -Direct proof is difficult
Proof by contraposition	1. Contrapositive statement: (The contraposition of statement) 2. (Prove the contrapositive statement) 3.Hence, (The original statement is true)	-Conditional statement that is hard to prove in the original direction
Proof by Mathematical Induction	1. Let $P(n) \equiv$ ... (Set up predicate.) 2. Basis step: … 3. Assume $P(k)$ is true for some k (Inductive Hypothesis) 4. Inductive step: (Start from k + 1, and use inductive hypothesis to support the proof) 5.Therefore, (Conclude statement)	

Lecture 2: The Logic of Compound Statements (aka Propositional Logic)

Definition: **Statement** $\boxed{\text{Definition 2.1.1}}$:
A statement (or proposition) is a sentence that is true or false, but not both.

Definition: **Negation** $\boxed{\text{Definition 2.1.2}}$:
If p is a statement variable, the negation of p is "not p" or "it is not the case that p" and is denoted ~ p.

Definition: **Conjunction** $\boxed{\text{Definition 2.1.3}}$:
If p and q are statement variables, the conjunction of p and q is "p and q", denoted $p \wedge q$.

Definition: **Disjunction** $\boxed{\text{Definition 2.1.4}}$:
If p and q are statement variables, the disjunction of p and q is "p or q", denoted $p \vee q$.

Definition: **Statement Form / Propositional Form** $\boxed{\text{Definition 2.1.5}}$:
A statement form (or propositional form) is an expression made up of statement variables and logical connectives that becomes a statement when actual statements are substituted for the component statement variables.

Definition: **Logical Equivalence** $\boxed{\text{Definition 2.1.6}}$:
Two statement forms are called logically equivalent if, and only if, they have identical truth values for each possible substitution of statements for their statement variables. The logical equivalence of statement forms P and Q is denoted by $P \equiv Q$.
*To show logical equivalence: Show that truth table have identical truth values
*To show non-equivalence:
(1) Truth table method: Show that truth table have at least one row where truth values differ
(2) Counter-example method: Find a counter example such that one is true another is false.

Definition: **Tautology** $\boxed{\text{Definition 2.1.7}}$:
A tautology is a statement form that is always true regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a tautology is a tautological statement.

Definition: **Contradiction** $\boxed{\text{Definition 2.1.8}}$:
A contradiction is a statement form that is always false regardless of the truth values of the individual statements substituted for its statement variables. A statement whose form is a contradiction is a contradictory statement.

Logical Equivalence $\boxed{\text{Theorem 2.1.1}}$: *Just quote the law name

Given any statement variables *p*, *q* and *r*, a tautology **true** and a contradiction **false**:

1	Commutative laws	$p \wedge q \equiv q \wedge p$	$p \vee q \equiv q \vee p$
2	Associative laws	$p \wedge q \wedge r$ $\equiv (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	$p \vee q \vee r$ $\equiv (p \vee q) \vee r \equiv p \vee (q \vee r)$
3	Distributive laws	$p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$
4	Identity laws	$p \wedge$ **true** $\equiv p$	$p \vee$ **false** $\equiv p$
5	Negation laws	$p \vee {\sim}p \equiv$ **true**	$p \wedge {\sim}p \equiv$ **false**
6	Double negative law	${\sim}({\sim}p) \equiv p$	
7	Idempotent laws	$p \wedge p \equiv p$	$p \vee p \equiv p$
8	Universal bound laws	$p \vee$ **true** $\equiv$ **true**	$p \wedge$ **false** $\equiv$ **false**
9	De Morgan's laws	${\sim}(p \wedge q) \equiv {\sim}p \vee {\sim}q$	${\sim}(p \vee q) \equiv {\sim}p \wedge {\sim}q$
10	Absorption laws	$p \vee (p \wedge q) \equiv p$	$p \wedge (p \vee q) \equiv p$
11	Negation of **true** and **false**	~**true** $\equiv$ **false**	~**false** $\equiv$ **true**

*Implication law: $p \rightarrow q \equiv {\sim}p \vee q$
*Variant absorption laws: $p \wedge ({\sim}p \vee q) \equiv p \wedge q \mid p \vee ({\sim}p \wedge q) \equiv p \vee q$

Definition: **Conditional** Definition 2.2.1:
If p and q are statement variables, the conditional of q by p is "if p then q" or "p implies q", denoted $p \rightarrow q$. It is false when p is true and q is false; otherwise it is true. We called p the hypothesis (or antecedent) of the conditional and q the conclusion (or consequent).
* A conditional statement that is true when its hypothesis is false is often called vacuously true or true by default.

Definition: **Contrapositive** Definition 2.2.2:
The contrapositive of $p \rightarrow q$ is $\sim q \rightarrow \sim p$.

Definition: **Converse** Definition 2.2.3:
The converse of $p \rightarrow q$ is $q \rightarrow p$.

Definition: **Inverse** Definition 2.2.4:
The inverse of $p \rightarrow q$ is $\sim p \rightarrow \sim q$.

*Note that:
(1) Conditional statement ≡ Its contrapositive | (2) Its converse ≡ its inverse | (3) Statement ≢ Converse (Normally)

Definition: **Only if** Definition 2.2.5:
"p only if q" means "if not q then not p" or "$\sim q \rightarrow \sim p$". Or, equivalently, "if p then q" or "$p \rightarrow q$".

Definition: **Biconditional** Definition 2.2.6:
Given statement variables p and q, the biconditional of p and q is "p if, and only if, q" and is denoted $p \leftrightarrow q$. It is true if both p and q have the same truth values and is false if p and q have opposite truth values. The words if and only if are sometimes abbreviated iff.
*$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

Definition: **Necessary and Sufficient Conditions** Definition 2.2.7:
If r and s are statements, "r is a sufficient condition for s" means "if r then s" or "$r \rightarrow s$",
"r is a necessary condition for s" means "if not r then not s" or "if s then r" or "$s \rightarrow r$".
* r is a necessary and sufficient condition for s means "r if and only if s" or "$r \leftrightarrow s$".

<table>
<tr><td>Logical connectives</td><td>Not / Negation: ~</td><td>If-then / Implies: →</td><td>If and only if: ↔</td><td>And: ∧
*We use "but" in English sometimes</td><td>Or: ∨</td><td>Exclusive-or (Special)</td></tr>
<tr><td>Truth tables</td>
<td><table><tr><th>p</th><th>~p</th></tr><tr><td>T</td><td>F</td></tr><tr><td>F</td><td>T</td></tr></table></td>
<td><table><tr><th>p</th><th>q</th><th>p → q</th></tr><tr><td>T</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>T</td></tr><tr><td>F</td><td>F</td><td>T</td></tr></table></td>
<td><table><tr><th>p</th><th>q</th><th>p ↔ q</th></tr><tr><td>T</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>F</td></tr><tr><td>F</td><td>F</td><td>T</td></tr></table></td>
<td><table><tr><th>p</th><th>q</th><th>p ∧ q</th></tr><tr><td>T</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>F</td></tr><tr><td>F</td><td>T</td><td>F</td></tr><tr><td>F</td><td>F</td><td>F</td></tr></table></td>
<td><table><tr><th>p</th><th>q</th><th>p ∨ q</th></tr><tr><td>T</td><td>T</td><td>T</td></tr><tr><td>T</td><td>F</td><td>T</td></tr><tr><td>F</td><td>T</td><td>T</td></tr><tr><td>F</td><td>F</td><td>F</td></tr></table></td>
<td>$(p \vee q) \wedge \sim (p \wedge q)$</td></tr>
<tr><td>Order or operation</td><td>Performed first</td><td colspan="2">Coequal (Performed last)</td><td colspan="2">Coequal (After negation)
*Use parentheses to disambiguate</td><td>-</td></tr>
</table>

Definition: **Argument** Definition 2.3.1:
An argument (argument form) is a sequence of statements (statement forms). All statements in an argument (argument form), except for the final one, are called premises (or assumptions or hypothesis). The final statement (statement form) is called the conclusion. The symbol •, which is read "therefore", is normally placed just before the conclusion. To say that an argument form is valid means that no matter what particular statements are substituted for the statement variables in its premises, if the resulting premises are all true, then the conclusion is also true.
*Testing validity of argument form (Critical row method):
(1) Identify premises and conclusion
(2) Construct truth table

(3) Find critical row (All premises are true), if conclusion in every critical row is true, then the argument form is valid

* Tutorial #1 Additional Notes Testing validity of argument form:

Given an argument:

$p1$

$p2$

:

pk

$\therefore q$

where $p_1, p_2, \ldots, p_k$ are the k premises and q the conclusion, we can say that "the argument is valid if and only if $(p_1 \wedge p_k \wedge \cdots \wedge p_k) \longrightarrow q$ is a tautology".

Term: **Syllogism** Lecture #2 Slides #57:
An argument form consisting of two premises and a conclusion.

Term: **Rule of inference** Lecture #2 Slides #61:
A form of argument that is valid.

*Just quote the rule name

Rule of inference		
Modus Ponens	$p \to q$ p • q	
Modus Tollens	$p \to q$ $\sim q$ • $\sim p$	
Generalization	p • $p \vee q$	q • $p \vee q$
Specialization	$p \wedge q$ • p	$p \wedge q$ • q
Conjunction	p q • $p \wedge q$	

Rule of inference		
Elimination	$p \vee q$ $\sim q$ • p	$p \vee q$ $\sim p$ • q
Transitivity	$p \to q$ $q \to r$ • $p \to r$	
Proof by Division Into Cases	$p \vee q$ $p \to r$ $q \to r$ • r	
Contradiction Rule	$\sim p \to$ **false** • p	

Term: **Fallacy** Lecture #2 Slides #69:
An error in reasoning that results in an invalid argument.
*Three common fallacies:
(1) Using ambiguous premises (2) Circular reasoning (Assume the conclusion) (3) Jumping to a conclusion
*Example:

Converse error / Fallacy of affirming the consequence	Inverse error	Valid argument with a false premise (Logical problem in premise)
$p \to q$ q • p	$p \to q$ $\sim p$ • $\sim q$	

Definition: **Sound and Unsound Arguments** Definition 2.3.2:
An argument is called sound if, and only if, it is valid and all its premises are true. An argument that is not sound is called unsound.

Some common questions:
(1) Which of the following statements is/are logically equivalent to … ?
(i) Try to transform the option into the given statement (ii) Substitute the statement variables with true and false

(2) What is/are the missing premise(s) to make the following argument valid?
(i) Try to transform the conclusion into a form that is the conjunction of all premises

Common questions:
(1) Simplify proposition – Change “implies” to “and” and “or”, remember to use ≡
(2) Mastermind –
(i) If 3 colour is correct, we can deduct the 4th colour is among those which’s not inside.
(ii) If those who have the 4th colour are all in the sequence, then only one among them have correct colour
(iii) Once get all the correct colour, refer to sequence which have only sink without hit to know some position
(iv) Lastly, use the answer to check all (Optional)

Lecture 3: The Logic of Quantified Statements (aka Predicate Logic)

Definition: **Predicate** $\boxed{\text{Definition 3.1.1}}$:

A predicate is a sentence that contains a finite number of predicate variables and becomes a statement when specific values are substituted for the variables. The domain of a predicate variable is the set of all values that may be substituted in place of the variable.

*Ways to change predicate into statement:

(1) Assign specific values to all predicate variables

(2) Add quantifiers

Definition: **Truth set** $\boxed{\text{Definition 3.1.2}}$:

If P(x) is a predicate and x has domain D, the truth set is the set of all elements of D that make P(x) true when they are substituted for x. The truth set of P(x) is denoted $\{x \in D \mid P(x)\}$.

* In set theory, the symbol | is used to mean "such that".

*Sometimes we can narrow the domain to truth set in quantified statements.

Symbol: **Universal quantifier** / ∀ denotes "for all" (or "for any", "for every", "for each")

Definition: **Universal Statement** $\boxed{\text{Definition 3.1.3}}$:

Let Q(x) be a predicate and D the domain of x. A universal statement is a statement of the form "$\forall x \in D$, Q(x)" (May omit commas). It is defined to be true iff Q(x) is true for every x in D. It is defined to be false iff Q(x) is false for at least one x in D. A value for x for which Q(x) is false is called a counterexample.

*To check the truth and falsity:

(1) True: Method of exhaustion (Try every value in the domain) / Prove

(2) False: Find a counterexample

*Vacuous truth:

In general, a statement of the form $\forall x \in D\ (P(x) \to Q(x))$ is called vacuously true or true by default if, and only if, P(x) is false for every x in D.

As a special case, $\forall a \in X,\ P(a)$ is vacuously true if X is an empty set.

*Usually in the form $\forall x\ (P(x) \to Q(x))$, not $\forall x\ (P(x) \wedge Q(x))$

Symbol: **Existential quantifier** / ∃ denotes "there exists", "there is a", "we can find a", "there is at least one", "for some", and "for at least one".

* The words "such that" or "s.t." are inserted just before the predicate. (May omit)

*∃! is used to denote "there exists a unique" or "there is one and only one".

Definition: **Existential Statement** $\boxed{\text{Definition 3.1.4}}$:

Let Q(x) be a predicate and D the domain of x. An existential statement is a statement of the form "$\exists x \in D$ such that Q(x)". It is defined to be true iff Q(x) is true for at least one x in D. It is defined to be false iff Q(x) is false for all x in D.

*To check the truth and falsity:

(1) True: Find an example

(2) False: Prove

*Usually in the form $\exists x\ (P(x) \wedge Q(x))$, not $\exists x\ (P(x) \to Q(x))$

Negation of a Universal Statement $\boxed{\text{Theorem 3.2.1}}$:

$\sim(\forall x \in D, P(x)) \equiv \exists x \in D$ such that $\sim P(x)$

Negation of an Existential Statement $\boxed{\text{Theorem 3.2.2}}$:

$\sim(\exists x \in D$ such that $P(x)) \equiv \forall x \in D, \sim P(x)$

Relation among ∀, ∃, ∧, and ∨ $\boxed{\text{Lecture \#3 Slides \#35}}$:

$\forall x \in D,\ Q(x) \equiv Q(x_1) \wedge Q(x_2) \wedge \ldots \wedge Q(x_n)$

$\exists x \in D,\ Q(x) \equiv Q(x_1) \vee Q(x_2) \vee \ldots \vee Q(x_n)$

Definition: **Contrapositive, converse, inverse** $\boxed{\text{Definition 3.2.1}}$:

Consider a statement of the form: $\forall x \in D\ (P(x) \to Q(x))$. Its contrapositive is: $\forall x \in D\ (\sim Q(x) \to \sim P(x))$. Its converse is:

$\forall x \in D$ (Q(x) → P(x)). Its inverse is: $\forall x \in D$ (~P(x) → ~Q(x)).
*It follows that $\forall x \in D\ (P(x) \to Q(x)) \equiv \forall x \in D\ (\sim Q(x) \to \sim P(x))$ and $\forall x \in D\ (P(x) \to Q(x)) \not\equiv \forall x \in D\ (Q(x) \to P(x))$

Definition: **Necessary and Sufficient conditions, Only if** Definition 3.2.2:
"∀ x, r(x) is a sufficient condition for s(x)" means "∀ x (r(x) → s(x))". "
∀ x, r(x) is a necessary condition for s(x)" means "∀ x (~r(x) → ~s(x))" or, equivalently, "∀x (s(x) → r(x))".
"∀ x, r(x) only if s(x)" means "∀x (~s(x) → ~r(x))" or, equivalently, "∀x (r(x) → s(x))" .

Negations of Multiply-Quantified Statements Lecture #3 Slides #57:
~(∀x ∈ D, ∃y∈E such that P(x, y)) ≡ ∃x∈D such that ∀y∈E, ~P(x, y)
~(∃x∈D such that ∀y∈ E, P(x, y)) ≡ ∀x∈D, ∃y∈E such that ~P(x, y)
*Basically change the ∀ to ∃ and ∃ to ∀, and negates any predicate.

Order of Quantifier Lecture #3 Slides #61:
In a statement containing both ∀ and ∃, changing the order of the quantifiers usually changes the meaning of the statement. However, if one quantifier immediately follows another quantifier of the same type, then the order of the quantifiers does not affect the meaning.

Definition: **Valid Argument Form** Definition 3.4.1:
No matter what particular predicates are substituted for the predicate symbols in its premises, if the resulting premise statements are all true, then the conclusion is also true.

Universal Modus Ponens	Universal Modus Tollens	Converse Error (Quantified Form)	Inverse Error (Quantified Form)	Universal Transitivity
$\forall x\ (P(x) \to Q(x))$. $P(a)$ for a particular a. • $Q(a)$.	$\forall x\ (P(x) \to Q(x))$. $\sim Q(a)$ for a particular a. • $\sim P(a)$.	$\forall x\ (P(x) \to Q(x))$. $Q(a)$ for a particular a. • $P(a)$.	$\forall x\ (P(x) \to Q(x))$. $\sim P(a)$ for a particular a. • $\sim Q(a)$.	$\forall x\ (P(x) \to Q(x))$. $\forall x\ (Q(x) \to R(x))$. • $\forall x\ (P(x) \to R(x))$.

Rule of Inference for quantified statements	Name
$\forall x \in D\ P(x)$ $\therefore P(a)$ if $a \in D$	Universal instantiation
$P(a)$ for every $a \in D$ $\therefore \forall x \in D\ P(x)$	Universal generalization
$\exists x \in D\ P(x)$ $\therefore P(a)$ for some $a \in D$	Existential instantiation
$P(a)$ for some $a \in D$ $\therefore \exists x \in D\ P(x)$	Existential generalization

Tutorial #2 Question #3:
(a) Integers are closed under division. (Disproved)
(b) Rational numbers are closed under addition. (Proved by deduction)
(c) Rational number are closed under division. (Disproved)

Tutorial #2 Question #7: $\forall x \in \mathbb{R}\ ((x^2 > x) \to (x < 0) \vee (x > 1))$. (Proof by deduction)

Tutorial #2 Question #10: If n is a product of two positive integers a and b, then $a \le n^{1/2}$ or $b \le n^{1/2}$ (Proof by contraposition / contradiction)

Lecture 4: Methods of Proof

Definition: **Prime and Composite** Lecture #4 Slides #6:

n is prime: $(n>1)\wedge\forall r,s\in\mathbb{Z}^+, (n=rs\rightarrow(r=1\wedge s=n)\vee(r=n\wedge s=1))$.

n is prime: $(n>1)\wedge(\forall r,s\in\mathbb{Z}\ ((r>1)\wedge(s>1)\rightarrow rs\neq n))$.

n is composite: $\exists r,s\in\mathbb{Z}^+\ (n=rs\wedge(1<r<n)\wedge(1<s<n))$.

Definition: **Prime** Lecture #4 Slides #7:

n is prime: $(x\neq 1)\wedge\forall y,z\ (x=yz\rightarrow((y=x)\vee(y=1)))$

Proving Existential Statement	Constructive Proof
Disproving Universal Statement	Counterexample (For conditional statement, find a value in domain for which the hypothesis is true but the conclusion is false.
Proving Universal Statement	(Finite) Method of exhaustion (Infinite) Generalizing from the generic particular (To show that every element of a set satisfies a certain property, suppose x is a particular but arbitrarily chosen element of the set, and show that x satisfies the property.)

Theorem 4.2.1 (5^{th}: 4.3.1): Every integer is a rational number. (Constructive Proof)

Theorem 4.2.2 (5^{th}: 4.3.2): The sum of any two rational numbers is rational. (Constructive Proof)

Corollary 4.2.3 (5^{th}: 4.2.3): The double of a rational number is rational.

A Positive Divisor of a Positive Integer Theorem 4.3.1 (5^{th}: 4.4.1):

For all positive integers a and b, if $a \mid b$, then $a\leq b$. (Constructive Proof)

Divisors of 1 Theorem 4.3.2 (5^{th}: 4.4.2):

The only divisors of 1 are 1 and -1. (Proof by division into cases)

Transitivity of Divisibility Theorem 4.3.3 (5^{th}: 4.4.3):

For all integers a, b and c, if $a \mid b$ and $b \mid c$, then $a \mid c$. (Constructive Proof)

Theorem 4.6.1 (5^{th}: 4.7.1): There is no greatest integer. (Proof by contradiction)

Proposition 4.6.4 (5^{th}: 4.7.4): For all integers n, if n^2 is even than n is even. (Proof by contraposition)

Lecture 5: Set Theory

Definition: **Set** Lecture #5 Slides #6:
Set is a unordered collection of objects. The objects are called members or elements of the set. Order and duplicate do not matter, which means an element is only counted once regardless of the number of duplicate in the set.

Definition: **Membership of a Set (Notation: ∈)** Lecture #5 Slides #7:
If S is a set, the notation $x \in S$ means that x is an element of S. ($x \notin S$ means x is not an element of S.)

Definition: **Cardinality of a Set (Notation: $|S|$)** Lecture #5 Slides #7:
The cardinality of a set S, denoted as $|S|$, is the size of the set, that is, the number of elements in S.

Term: **Set-Roster Notation** Lecture #5 Slides #6:
A set may be specified by writing all of its elements between braces. Examples: {1, 2, 3}, {1, 2, 3, …, 100}, {1, 2, 3, …}. (The symbol ⋯ is called an ellipsis and is read "and so forth".)

Term: **Set-Builder Notation** Lecture #5 Slides #11:
Let U be a set and $P(x)$ be a predicate over U. Then the set of all elements $x \in U$ such that $P(x)$ is true is denoted $\{x \in U : P(x)\}$ or $\{x \in U \mid P(x)\}$ which is read as "the set of all x in U such that $P(x)$ (is true)".
*An object z is an element of the set $S = \{x \in U : P(x)\}$ only if $z \in U$ and $P(z)$ is true.

Term: **Replacement Notation** Lecture #5 Slides #12:
Let A be a set and $t(x)$ be a term in a variable x. Then the set of all objects of the form $t(x)$ where x ranges over the elements of A is denoted $\{t(x): x \in A\}$ or $\{t(x) \mid x \in A\}$ which is read as "the set of all $t(x)$ where $x \in A$".
* An object z is an element of $S=\{t(x) : x \in A\}$ only if there is an $x \in A$ such that $t(x)=z$.

Definition: **Subset** Lecture #5 Slides #13:
$A \subseteq B$ iff $\forall x\ (x \in A \Rightarrow x \in B)$. "$A$ is a subset of B" is same meaning as "A is contained in B". We may also write $B \supseteq A$ which reads as "B contains A" or "B includes A".
* $A \nsubseteq B \Leftrightarrow \exists x\ (x \in A \wedge x \notin B)$.

Definition: **Proper Subset** Lecture #5 Slides #13:
$A \subsetneq B$, iff $A \subseteq B$ and $A \neq B$. In this case, we may say that the inclusion of A in B is proper or strict.

Definition: **Empty Set** Lecture #5 Slides #14:
A set with no element is an empty set, denoted as $\emptyset$ or $\{\}$.

Theorem 6.2.4: An empty set is a subset of every set, i.e. $\emptyset \subseteq A$ for all sets A.

Definition: **Singleton** Lecture #5 Slides #14:
A set with exactly one element is called a singleton.

Definition: **Ordered Pair** Lecture #5 Slides #16:
An ordered pair is an expression of the form (x,y). Two ordered pairs (a,b) and (c,d) are equal iff $a=c$ and $b=d$.
Symbolically: $(a,b)=(c,d) \Leftrightarrow (a=c) \wedge (b=d)$.

Definition: **Cartesian Product** Lecture #5 Slides #17:
Given sets A and B, the Cartesian product of A and B, denoted $\boldsymbol{A} \times \boldsymbol{B}$ and read "A cross B", is the set of all ordered pairs (a,b) where a is in A and b is in B. Symbolically: $A \times B=\{(a,b) : a \in A \wedge b \in B\}$.

Definition: **Set equality** Lecture #5 Slides #19:
Given sets A and B, A equals B, written $\boldsymbol{A}=\boldsymbol{B}$ iff every element of A is in B and every element of B is in A.
Symbolically: $A=B \Leftrightarrow A \subseteq B \wedge B \subseteq A$ or $A=B \Leftrightarrow \forall x\ (x \in A \Leftrightarrow x \in B)$ Lecture #5 Slides #21
*To prove set equality:
[S1] Let sets X and Y be given. To prove $X = Y$: [S2] (⊆) Prove that $X \subseteq Y$. (Take any element of X, prove it's in Y)
[S3] (⊇) Prove that $Y \subseteq X$ (or $X \supseteq Y$). [S4] From (2) and (3), conclude that $X=Y$.

Term: **Universal set** Lecture #5 Slides #24:
In a certain situation within some context, all sets being considered as a specific sets, for example sets of real numbers, thus the sets of real numbers would be called universal set or a universe of discourse for the discussion.

Definition: **Union, intersection, difference and complement** Lecture #5 Slides #25:
Let A and B be subsets of a universal set U.
The union of A and B, denoted $\boldsymbol{A \cup B}$, is the set of all elements that are in at least one of A or B.
$A \cup B = \{x \in U : x \in A \vee x \in B\}$
The intersection of A and B, denoted $\boldsymbol{A \cap B}$, is the set of all elements that are common to both A and B.
$A \cap B = \{x \in U : x \in A \wedge x \in B\}$
The difference of B minus A (or relative complement of A in B), denoted $\boldsymbol{B - A}$, or $\boldsymbol{B \setminus A}$, is the set of all elements that are in B and not A. $B \setminus A = \{x \in U : x \in B \wedge x \notin A\}$
The complement of A, denoted $\bar{A}$, is the set of all elements in U that are not in A. (Note: Epp uses the notation A^c.)
$\bar{A} = \{x \in U \mid x \notin A\}$.
*Sometimes U is omitted in the definition
*$\bar{X} = U \setminus X$ Lecture #5 Slides #26

Notation: **Interval of real numbers** Lecture #5 Slides #27:
Given real numbers a and b with $a \leq b$:
$(a,b) = \{x \in \mathbb{R} : a < x < b\}$, $[a,b] = \{x \in \mathbb{R} : a \leq x \leq b\}$, $(a,b] = \{x \in \mathbb{R} : a < x \leq b\}$, $[a,b) = \{x \in \mathbb{R} : a \leq x < b\}$.
The symbols ∞ and $-\infty$ are used to indicate intervals that are unbounded either on the right or on the left:
$(a,\infty) = \{x \in \mathbb{R} : x > a\}$, $[a,\infty) = \{x \in \mathbb{R} : x \geq a\}$, $(-\infty,b) = \{x \in \mathbb{R} : x < b\}$, $(-\infty,b] = \{x \in \mathbb{R} : x \leq b\}$.

Definition: **Union & Intersection for more than two sets** Lecture #5 Slides #28:
$\bigcup_{i=0}^{n} A_i = A_0 \cup A_1 \cup \cdots \cup A_n$
$\bigcap_{i=0}^{n} A_i = A_0 \cap A_1 \cap \cdots \cap A_n$

> • **Definition**
>
> **Unions and Intersections of an Indexed Collection of Sets**
> Given sets $A_0, A_1, A_2, \ldots$ that are subsets of a universal set U and given a nonnegative integer n,
>
> $$\bigcup_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for at least one } i = 0, 1, 2, \ldots, n\}$$
> $$\bigcup_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for at least one nonnegative integer } i\}$$
> $$\bigcap_{i=0}^{n} A_i = \{x \in U \mid x \in A_i \text{ for all } i = 0, 1, 2, \ldots, n\}$$
> $$\bigcap_{i=0}^{\infty} A_i = \{x \in U \mid x \in A_i \text{ for all nonnegative integers } i\}.$$

Definition: **Disjoint** Lecture #5 Slides #29:
Two sets are disjoint iff they have no elements in common. Symbolically: A and B are disjoint iff $A \cap B = \emptyset$.

Definition: **Mutually disjoint** Lecture #5 Slides #29:
Sets $A_1, A_2, A_3, \cdots$ are **mutually disjoint** (or **pairwise disjoint** or **nonoverlapping**) iff no two sets A_i and A_j with distinct subscripts have any elements in common, i.e. for all $i, j = 1,2,3,\cdots$, $A_i \cap A_j = \emptyset$ whenever $i \neq j$.

Definition: **Partition**:
Lecture #5 Slides #29 Division of set into nonoverlapping (or disjoint) pieces.
Lecture #5 Slides #30 If A is called a union of mutually disjoint subsets A_1, A_2, A_3, and A_4, then the collection of sets {A1, A2, A3, A4} is said to be a partition of A.

The Quotient-Remainder Theorem Theorem 4.4.1:
Given any integer n and positive integer d, there exist unique integers q and r such that $n = dq + r$ and $0 \leq r < d$.
*Usage: By the quotient-remainder theorem, every integer n can be written in exactly one of the three forms: n = 3k, or n = 3k+1, or n = 3k + 2 for some integer k.

Definition: **Power set** Lecture #5 Slides #33:
Given a set A, the power set of A, denoted $\mathcal{P}(A)$, is the set of all subsets of A.

Theorem 6.3.1: Suppose A is a finite set with n elements, then $\mathcal{P}(A)$ has 2^n elements. In other words, $|\mathcal{P}(A)| = 2^{|A|}$.
(Proof by mathematical induction)

Definition: **Ordered $\boldsymbol{n}$-tuples** Lecture #5 Slides #36:
Let $n \in \mathbb{Z}^+$ and let $x_1, x_2, \cdots, x_n$ be (not necessarily distinct) elements. An **ordered $\boldsymbol{n}$-tuple** is an expression of the

form $(x_1, x_2, \cdots, x_n)$. An **ordered pair** is an ordered 2-tuple; an **ordered triple** is an ordered 3-tuple. Equality of two ordered n-tuples: $(x_1, x_2, \cdots, x_n) = (y_1, y_2, \cdots, y_n) \Leftrightarrow x_1 = y_1, x_2 = y_2, \cdots, x_n = y_n$.

Definition: **Cartesian product** Lecture #5 Slides #36:
Given sets $A_1, A_2, \cdots, A_n$, the **Cartesian product** of $A_1, A_2, \cdots, A_n$, denoted $A_1 \times A_2 \times \cdots \times A_n$, is the set of all ordered n-tuples $(a_1, a_2, \cdots, a_n)$ where $a_1 \in A_1,\ a_2 \in A_2, \cdots, a_n \in A_n$. $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \cdots, a_n) : a_1 \in A_1 \wedge a_2 \in A_2 \wedge \cdots \wedge a_n \in A_n\}$. If A is a set, then $A^n = A \times A \times \cdots \times A$ (n many A's).

Some Subset Relations Theorem 6.2.1:
Inclusion of Intersection: For all sets A and B, (a)$A \cap B \subseteq A$ (b) $A \cap B \subseteq B$
Inclusion in Union: For all sets A and B, (a)$A \subseteq A \cup B$ (b)$B \subseteq A \cup B$
Transitive Property of Subsets: For all sets A, B and C, $A \subseteq B \wedge B \subseteq C \rightarrow A \subseteq C$.

Procedural Versions of Set Definitions Lecture #5 Slides #40:
Let X and Y be subsets of a universal set U and suppose a and b are elements of U.
(1) $a \in X \cup Y \Leftrightarrow a \in X \vee a \in Y$ (2) $a \in X \cap Y \Leftrightarrow a \in X \wedge a \in Y$ (3) $a \in X - Y \Leftrightarrow a \in X \wedge a \notin Y$
(4)$a \in \bar{X} \Leftrightarrow a \notin X$ (5) $(a, b) \in X \times Y \Leftrightarrow a \in X \wedge b \in Y$

Set Identities Theorem 6.2.2: *Just quote law name

Let all sets referred to below be subsets of a universal set U.

1. *Commutative Laws*: For all sets A and B,
   (a) $A \cup B = B \cup A$ and (b) $A \cap B = B \cap A$.
2. *Associative Laws*: For all sets A, B and C,
   (a) $(A \cup B) \cup C = A \cup (B \cup C)$ and (b) $(A \cap B) \cap C = A \cap (B \cap C)$.
3. *Distributive Laws*: For all sets A, B and C,
   (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and
   (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
4. *Identity Laws*: For all sets A,
   (a) $A \cup \emptyset = A$ and (b) $A \cap U = A$.
5. *Complement Laws*: For all sets A,
   (a) $A \cup \bar{A} = U$ and (b) $A \cap \bar{A} = \emptyset$.
6. *Double Complement Law*: For all sets A,
   $\bar{\bar{A}} = A$.
7. *Idempotent Laws*: For all sets A,
   (a) $A \cup A = A$ and (b) $A \cap A = A$.
8. *Universal Bound Laws*: For all sets A,
   (a) $A \cup U = U$ and (b) $A \cap \emptyset = \emptyset$.
9. *De Morgan's Laws*: For all sets A and B,
   (a) $\overline{A \cup B} = \bar{A} \cap \bar{B}$ and (b) $\overline{A \cap B} = \bar{A} \cup \bar{B}$.
10. *Absorption Laws*: For all sets A and B,
    (a) $A \cup (A \cap B) = A$ and (b) $A \cap (A \cup B) = A$.
11. *Complements of U and $\emptyset$*:
    (a) $\bar{U} = \emptyset$ and (b) $\bar{\emptyset} = U$.
12. *Set Difference Law*: For all sets A and B,
    $A \setminus B = A \cap \bar{B}$.

Tutorial #3 Question #5: For all sets A,B,C, $A \cap (B \setminus C) = (A \cap B) \setminus C$.

Tutorial #3 Question #6: For all sets A,B,C, $A \setminus (B \setminus C) = (A \setminus B) \cup (A \cap C)$.

Tutorial #3 Question #8: Let A and B be set. $A \subseteq B$ if and only if $A \cup B = B$.

Assignment #1 Question #6:
(a) $A \subseteq B \Leftrightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$. (Proved)
(b) $(A \cup B = A \cup C) \Rightarrow B = C$. (Disproved)
(c) $(A \cap B = A \cap C) \Rightarrow B = C$. (Disproved)
(d) $(A \cup B = A \cup C) \wedge (A \cap B = A \cap C) \Rightarrow B = C$. (Proved)

Assignment #2 Question #4: $A = (A \setminus B) \cup (A \cap B)$.

Common questions:
1. Find power set – Just list out, can check by number of element
2. Prove set equality – Noted does the question mention set operation (also called element method, which is the definition of sets and some propositional logic) or set identities
3. Find partition – Non-empty subsets + Mutually disjoint + Every element in exactly one of the component

Lecture 6: Relations

Definition: **Relation** Lecture #6 Slides #6:
Let A and B be sets. A (binary) relation from $\boldsymbol{A}$ to $\boldsymbol{B}$ is a subset of $\boldsymbol{A}\times\boldsymbol{B}$. Given an ordered pair (x,y) in $A\times B$, $\boldsymbol{x}$ is related to $\boldsymbol{y}$ by $\boldsymbol{R}$, or $\boldsymbol{x}$ is $\boldsymbol{R}$-related to $\boldsymbol{y}$, written $\boldsymbol{x}\ \boldsymbol{R}\ \boldsymbol{y}$, iff $(\boldsymbol{x},\boldsymbol{y})\in\boldsymbol{R}$.
* $x\ R\ y$ means $(x,y)\in R$, $x \not R y$ means $(x,y)\notin R$

Definition: **Domain, Co-domain, Range** Lecture #6 Slides #9:
Let A and B be sets and R be a relation from A to B. The domain of R, $Dom(R)$, is the set $\{a\in A : aRb \text{ for some } b\in B\}$. The co-domain of R, $coDom(R)$, is the set B. The range of R, $Range(R)$, is the set $\{b\in B: aRb \text{ for some } a\in A\}$.

Definition: **Inverse of a Relation** Lecture #6 Slides #12:
Let R be a relation from A to B. The **inverse relation** R^{-1} from B to A is: $R^{-1} = \{(y,x)\in B\times A : (x,y)\in R\}$.
* $\forall x\in A, \forall y\in B\ \big((y,x)\in R^{-1} \Leftrightarrow (x,y)\in R\big)$

Definition: **Relation on a Set** Lecture #6 Slides #12:
A relation on a set $\boldsymbol{A}$ is a relation from A to A. In other words, a relation on a set A is a subset of $A\times A$ (A^2).
* In general, we may write A^n for $A\times \dots \times A$ (n times).

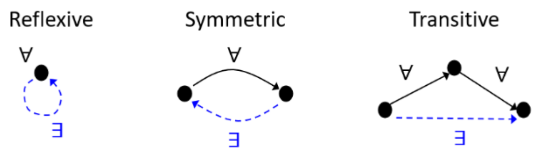
Definition: **Composition of Relations** Lecture #6 Slides #16:
Let A, B and C be sets. Let $R\subseteq A\times B$ be a relation. Let $S\subseteq B\times C$ be a relation. The composition of R with S, denoted $S\circ R$, is the relation from A to C such that: $\forall x\in A, \forall z\in C(x\ S\circ R\ z \Leftrightarrow (\exists y\in B\ (xRy \wedge ySz)))$
* There is a "path" from x to z via some intermediate element $y\in B$ in the arrow diagram.

Proposition: **Composition is Associative** Lecture #6 Slides #18 Tutorial #4 Question #6:
Let A, B, C, D be sets. Let $R\subseteq A\times B,\ S\subseteq B\times C$ and $T\subseteq C\times D$ be relations. $T\circ(S\circ R) = (T\circ S)\circ R = T\circ S\circ R$

Proposition: **Inverse of Composition** Lecture #6 Slides #18:
Let A, B and C be sets. Let $R\subseteq A\times B$ and $S\subseteq B\times C$ be relations. $(S\circ R)^{-1} = R^{-1}\circ S^{-1}$

Definition: **$\boldsymbol{n}$-ary Relation** Lecture #6 Slides #19:
Given n sets $A_1, A_2, \cdots, A_n$, an **$\boldsymbol{n}$-ary relation $\boldsymbol{R}$** on $A_1\times A_2\times\cdots\times A_n$ is a subset of $A_1\times A_2\times\cdots\times A_n$. The special cases of 2-ary, 3-ary and 4-ary relations are called **binary**, **ternary** and **quaternary relations** respectively.

Definition: **Reflexivity, Symmetry, Transitivity** Lecture #6 Slides #19:
Let R be a relation on a set A.
R is **reflexive** iff $\forall x\in A\ (xRx)$. *Prove: Let a and prove $a\ R\ a$. *If A has n elements, R has at least n elements
R is **symmetric** iff $\forall x,y\in A\ (xRy \Rightarrow yRx)$. *Prove: Let $a\ R\ b$ and prove $b\ R\ a$.
R is **transitive** iff $\forall x,y,z\in A\ (xRy\wedge yRz \Rightarrow xRz)$. *Prove: Let $a\ R\ b$ and $b\ R\ c$, then prove $a\ R\ c$.
*Properties of a relation, not properties of members of the set.



Definition: **Transitive Closure** Lecture #6 Slides #31:
Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties: (1) R^t is transitive. (2) $R\subseteq R^t$. (3) If S is any other transitive relation that contains R, then $R^t\subseteq S$.
*The relation obtained by adding the least number of ordered pairs to ensure transitivity is called the transitive closure of the relation.

Definition: **Partition** Lecture #6 Slides #35:
C is a **partition** of a set A if the following hold:
(1) C is a set of which all elements are non-empty subsets of A, i.e., $\emptyset \neq S\subseteq A$ for all $S\in$ C.
(2) Every element of A is in exactly one element of C, i.e., $\forall x\in A\ \exists S\in \mathrm{C}\ (x\in S)$ and $\forall x\in A\ \exists S_1, S_2, \in \mathrm{C}\ (x\in S_1 \wedge x\in S_2 \Rightarrow S_1 = S_2)$.
*Shorter definition: A partition of set A is a set C of non-empty subsets of A such that $\forall x\in A\ \exists! S\in \mathrm{C}\ (x\in S)$.
* Lecture #6 Slides #34 A partition of a set A is a finite or infinite collection of nonempty, mutually disjoint subsets whose union is A.
*Elements of a partition are called **components** of the partition.

Definition: **Relation Induced by a Partition** Lecture #6 Slides #38:
Given a partition C of a set A, the relation R induced by the partition is defined on A as follows: $\forall x,y \in A$, $xRy \Leftrightarrow \exists$ a component S of C s.t. $x,y \in S$.

Relation Induced by a Partition Theorem 8.3.1:
Let A be a set with a partition and let R be the relation induced by the partition. Then R is reflexive, symmetric, and transitive.

Definition: **Equivalence Relation (Symbol ~)** Lecture #6 Slides #40:
Let A be a set and R a relation on A. R is an **equivalence relation** iff R is reflexive, symmetric and transitive.

Definition: **Equivalence Class** Lecture #6 Slides #43:
Suppose A is a set and $\sim$ is an equivalence relation on A. For each $a \in A$, the **equivalence class** of a, denoted $[\boldsymbol{a}]$ and called the **class of $\boldsymbol{a}$** for short, is the set of all elements $x \in A$ s.t. a is $\sim$-related to x. $[a]_{\sim} = \{x \in A : a{\sim}x\}$

Equivalence Classes Lemma Rel.1:
Let ~ be an equivalence relation on a set A. The following are equivalent for all $x,y \in A$.
(i) $x{\sim}y$ (ii) $[x]=[y]$ (iii) $[x] \cap [y] \neq \emptyset$.

The Partition Induced by an Equivalence Relation Theorem 8.3.4:
If A is a set and R is an equivalence relation on A, then the distinct equivalence classes of R form a partition of A; that is, the union of the equivalence classes is all of A, and the intersection of any two distinct classes is empty.

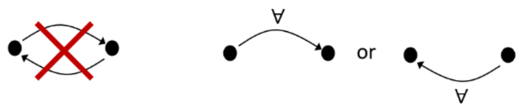
Definition: **Congruence** Lecture #6 Slides #53:
Let $a, b \in \mathbb{Z}$ and $n \in \mathbb{Z}^+$. Then a is congruent to b modulo n iff $a - b = nk$ for some $k \in \mathbb{Z}$. In other words, $n \mid (a - b)$. In this case, we write $a \equiv b \pmod{n}$.

Proposition: Lecture #6 Slides #54:
Congruence-mod n is an equivalence relation on $\mathbb{Z}$ for every $n \in \mathbb{Z}^+$.
*It will form n distinct equivalence classes.

Definition: **Set of equivalence classes** Lecture #6 Slides #56:
Let A be a set and ~ be an equivalence relation on A. Denote by $A/{\sim}$ the set of all equivalence classes with respect to ~, i.e., $A/{\sim} = \{[x]\_{\sim} : x \in A\}$. We may read $A/{\sim}$ as "the quotient of A by ~".
*Which is the partition induced by equivalence relation.

Equivalence classes form a partition Theorem Rel.2:
Let ~ be an equivalence relation on a set A. Then $A/{\sim}$ is a partition of A.

Definition: **Antisymmetry** Lecture #6 Slides #63:
Let R be a relation on a set A. R is **antisymmetric** iff $\forall x, y \in A\ (x\,R\,y \wedge y\,R\,x \Rightarrow x = y)$.
*R is ***not*** **antisymmetric** iff $\exists x, y \in A\ (x\,R\,y \wedge y\,R\,x \wedge x \neq y)$.
*Different with not symmetric



Definition: **Partial Order Relation (Symbol ≼ (Curly less than or equal to))** Lecture #6 Slides #68:
Let R be a relation on a set A. Then R is a partial order relation (or simply partial order) iff R is reflexive, antisymmetric and transitive.
*May view as a set of tasks

Definition: **Partially Ordered Set** Lecture #6 Slides #68:
A set A is called a partially ordered set (or poset) with respect to a partial order relation R on A, denoted by (A,R).

Definition: **Comparability** Lecture #6 Slides #79:
Suppose $\preccurlyeq$ is a partial order relation on a set A. Elements a and b of A are said to be comparable iff either $a \preccurlyeq b$ or $b \preccurlyeq a$. Otherwise, a and b are noncomparable.

Definition: **Maximal, minimal, largest, smallest** Lecture #6 Slides #80:
Let a set A be partially ordered with respect to a relation $\preccurlyeq$ and $c\in A$.
(1) c is a maximal element of A iff $\forall x\in A$, either $x\preccurlyeq c$, or x and c are not comparable. Alternatively, c is a maximal element of A iff $\forall x\in A\ (c\preccurlyeq x\Rightarrow c=x.)$
(2) c is a minimal element of A iff $\forall x\in A$, either $c\preccurlyeq x$, or x and c are not comparable. Alternatively, c is a minimal element of A iff $\forall x\in A\ (x\preccurlyeq c\Rightarrow c=x)$. (Nothing is below)
(3) c is the largest element / greatest element / maximum of A iff $\forall x\in A\ (x\preccurlyeq c)$.
(4) c is the smallest element / least element / minimum of A iff $\forall x\in A\ (c\preccurlyeq x)$. (Everything is above)

Proposition: **A smallest element is minimal / A largest element is maximal.** Lecture #6 Slides #83:
Consider a partial order $\preccurlyeq$ on a set A. Any smallest element is minimal and any largest element is maximal.

Definition: **Total Order Relations** Lecture #6 Slides #86:
If R is a partial order relation on a set A, and for any two elements x and y in A, either $x\ R\ y$ or $y\ R\ x$, then R is a total order relation (or simply total order) on A. R is a total order iff R is a partial order and $\forall x,y\in A\ (x\ R\ y \vee y\ R\ x)$.
*Hasse diagram of total order is one single line (chain)

Definition: **Linearization of a partial order** Lecture #6 Slides #87:
Let $\preccurlyeq$ be a partial order on a set A. A **linearization** of $\preccurlyeq$ is a total order $\preccurlyeq^*$ on A such that $\forall x, y \in A\ (x \preccurlyeq y \Rightarrow x \preccurlyeq^* y)$.
*Linearization of a total order is the total order itself.
*If two elements are not comparable, their order can be interchangeable as long as does not violate the conditions.

Kahn's Algorithm (1962)

Input: A finite set A and a partial order $\preccurlyeq$ on A.

1. Set $A_0 := A$ and $i := 0$.
2. Repeat until $A_i = \emptyset$
   2.1. find a minimal element c_i of A_i wrt $\preccurlyeq$
   2.2. set $A_{i+1} = A_i \setminus \{c_i\}$
   2.3. set $i := i + 1$

Output: A linearization $\preccurlyeq^*$ of $\preccurlyeq$ defined by setting, for all indices i, j,
$$c_i \preccurlyeq^* c_j \Leftrightarrow i \le j.$$

Definition: **Well-Ordered Set** Lecture #6 Slides #91:
Let $\preccurlyeq$ be a total order on a set A. A is **well-ordered** iff every non-empty subset of A contains a smallest element.
Symbolically, $\forall S \in \mathcal{P}(A),\ S \neq \emptyset \Rightarrow \big(\exists x \in S\ \forall y \in S\ (x \preccurlyeq y)\big)$

Tutorial #4 Question #2: The following are logically equivalent:
(1) R is symmetric, i.e. $\forall x,y\in A\ (x\ R\ y \Rightarrow y\ R\ x)$. (2) $\forall x,y\in A\ (x\ R\ y \Leftrightarrow y\ R\ x)$ (3) $R=R^{-1}$

Tutorial #4 Question #9:
(a) if $x\in S\in \mathcal{C}$, then $[x]=S$. (b) $A/{\sim}= C$.

Definition: **Reflexive closure** Tutorial #5 Question #5:
Let A be a set and R a relation on A. It is the smallest relation on A that is reflexive and contains R as a subset.

Definition: **Asymmetry** Tutorial #5 Question #6:
$\forall x,y\in A\ (x\ R\ y\Rightarrow y\ \not R\ x)$.

Tutorial #5 Question #6: Every asymmetric relation is antisymmetric. (asymmetry property forces the antisymmetry property to be vacuously true.)

Tutorial #5 Question #7: Consider a set A and a total order $\preccurlyeq$ on A. Show that all minimal elements are smallest.

Definition: **Comparable, compatible** Tutorial #5 Question #8:
We say a,b are comparable iff $a\preccurlyeq b$ or $b\preccurlyeq a$. We say a,b are compatible iff there exists $c\in A$ such that $a\preccurlyeq c$ and $b\preccurlyeq c$.

Tutorial #5 Question #10: In all partially ordered sets, any two comparable elements are compatible.

Common questions:
1. Find relation – Use definition of relation, definition of inverse relation, definition of composition of relations
2. Find equivalence class – Use definition of equivalence class, definition of ~
3. Proof equivalence – Proof Reflexivity, Symmetry, Transitivity

Type of relation	Type	Irreflexive	Reflexive	Symmetric	Antisymmetric	Asymmetric	Transitive	Antitransitive
Equal Relation on $\mathbb{Q}$	Equivalence	No	Yes	Yes	Yes	No	Yes	No
Congruence-mod n	Equivalence	No	Yes	Yes	No	No	Yes	No
Relation Induced by a Partition	Equivalence	No	Yes	Yes	No	No	Yes	No
Cardinality	Equivalence	No	Yes	Yes	No	No	Yes	No
Divisibility (For all positive integers)	Partial order (Not total order)	No	Yes	No	Yes	No	Yes	No
Divisibility (For all integers)		No	Yes	No	No	No	Yes	No
Less than or equal to (For natural numbers or any lower bounded set)	Partial order (Total order) (Well-ordered)	No	Yes	No	Yes	No	Yes	No
Less than or equal to (For any non-lower bounded set)	Partial order (Total order) (Not well-ordered)	No	Yes	No	Yes	No	Yes	No
Less than (Rational numbers)		Yes	No	No	Yes	Yes	Yes	No
Subset	Partial order (Not total order)	No	Yes	No	Yes	No	Yes	No
Proper Subset		Yes	No	No	Yes	Yes	Yes	No
Empty relation (on empty set)	Equivalence & partial order (Total order) (Well-ordered)	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Empty relation (on non-empty set)		Yes	No	Yes	Yes	Yes	Yes	Yes

Arrow diagram	Directed graph	Hasse Diagrams
Represent the elements of A as points in one region and the elements of B as points in another region. For each	Instead of representing A as two separate sets of points, represent A only once, and draw an arrow from each point of A to its related point.	For all distinct $x, y, m \in A$: If $x \preccurlyeq y$ and no $m \in A$ is such that $x \preccurlyeq m \preccurlyeq y$, then x is placed below y

$x \in A$ and $y \in B$, draw an arrow from x to y iff xRy.	If a point is related to itself, a loop is drawn that extends out from the point and goes back to it.	with a line joining them, else no line joins x and y.
For relation on two different sets	For relation on a set	For relation on a set (Partial order)