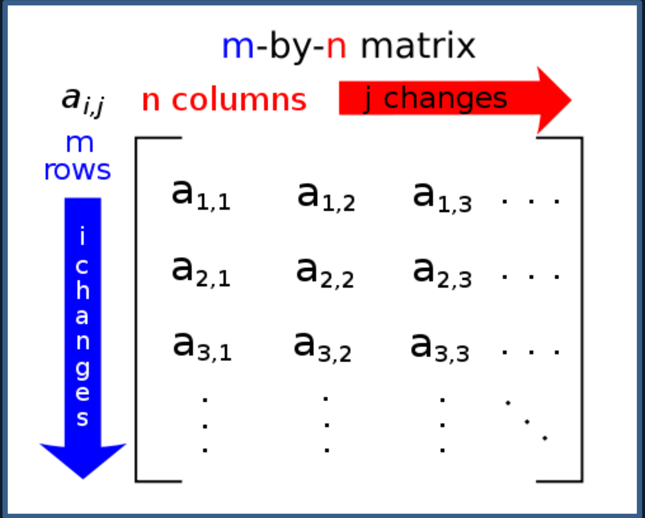
Introduction to Matrices



Lecture Contents

- What is a Matrix?
- What are they used for?
- Addition, Subtraction, Multiplication
- The Identity Matrix
- Row Major, Column Major
- Transpose, Inverse

What is a Matrix?

- A matrix is a rectangle of numbers
- They can have any number of rows and columns
- Referred to by their size, so an n x m matrix will have n rows and m columns.
- A square matrix, as you might imagine, is a matrix where n = m
- In mathematical notation a matrix is denoted by a capital letter such as M and individual numbers in the matrix are referred to by M_{ij} where i is the row and j is the column.
- So in the matrix to the right A_{11} = first row, first column = 4 and A_{32} = third row, second column = 1

$$\mathbf{A} = \begin{bmatrix} 4 & -7 & 5 & 0 \\ -2 & 0 & 11 & 8 \\ 19 & 1 & -3 & 12 \end{bmatrix}$$

This is what a 3x4 matrix looks like.

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What are Matrices used for?

- Transformations!
- This just means they are used for moving vectors around, rotating and scaling them
- They are really good at it!
- Because of this, they are one of the most fundamental and most used concepts in graphics programming.

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Addition, Subtraction, Multiplication

- The main ways matrices interact with each other.
- Some work as you might expect, some don't

Addition and Subtraction

- Super simple.
- Only works between two matrices of the same size.
- Add or subtract each corresponding number.

$$\begin{bmatrix} 15 & 8 \\ -3 & 12 \\ 4 & 6 \end{bmatrix} + \begin{bmatrix} 3 & 7 \\ 8 & 2 \\ 3 & 12 \end{bmatrix} = \begin{bmatrix} 15+3 & 8+7 \\ (-3)+8 & 12+2 \\ 4+3 & 6+12 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 15 \\ 5 & 14 \\ 7 & 18 \end{bmatrix}$$

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Multiplication

- Doesn’t quite work how you might think
- Most common matrix operation in graphics programming
- NOT communicative (A*B != B*A)
- The number of columns in the first matrix has to match the number of rows in the second
- The size of the output matrix with be:

  (number of rows in first matrix) x (number of columns in second matrix)
- So an $n_x m$ matrix * $m_x j$ matrix will result in a $n_x j$ matrix
- i.e The dimensions of the final matrix are the dimensions that don’t match each other in the input matrices

Multiplication (cont)

- Each number in the final matrix is the dot product of the corresponding row in the first matrix and the corresponding column in the second.
- This sounds kind of confusing so lets break this down.

Multiplication (cont)

- $\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$
- Dot product review:

$$x_1 * x_2 + y_1 * y_2 + z_1 * z_2 + \ldots + n_1 * n_2$$

Multiplication (cont)

- $\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$
- Lets figure out this first number.
- What row and column is it in?

Multiplication (cont)

- $\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

- First Row

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Multiplication (cont)

- $$\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

- First Row

Multiplication (cont)

- $\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

- First Row
- $v_1 = (15, 5)$

Multiplication (cont)

- $$\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} \boxed{?} & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$$

- First Row, First Column
- $v_1 = (15, 5)$

Multiplication (cont)

- $$\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} \boxed{\begin{matrix} 1 \\ 9 \end{matrix}} & \begin{matrix} 4 \\ 2 \end{matrix} & \begin{matrix} 16 \\ 7 \end{matrix} \end{bmatrix} = \begin{bmatrix} \boxed{\begin{matrix} ? \\ ? \\ ? \\ ? \end{matrix}} & \begin{matrix} ? \\ ? \\ ? \\ ? \end{matrix} & \begin{matrix} ? \\ ? \\ ? \\ ? \end{matrix} \end{bmatrix}$$

- First Row, First Column
- $v_1 = (15, 5)$

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Multiplication (cont)

- $\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

- First Row, First Column
- $v_1 = (15, 5)$ $v_2 = (1, 9)$

Multiplication (cont)

- $v_1.v_2$ $= (15 * 1) + (5 * 9)$

  $= 15 + 45$

  $= 60$

Multiplication (cont)

- $\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} \color{red}{?} & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

Multiplication (cont)

- $\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 60 & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$

Multiplication (cont)

- $\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 60 & ? & ? \\ ? & ? & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}$
- Repeat this process for each item in the final matrix

Multiplication (cont)

- $\begin{bmatrix} 15 & 5 \\ -2 & 12 \\ 3 & 8 \\ 4 & 6 \end{bmatrix} * \begin{bmatrix} 1 & 4 & 16 \\ 9 & 2 & 7 \end{bmatrix} = \begin{bmatrix} 60 & 70 & 275 \\ 106 & 16 & 52 \\ 75 & 28 & 104 \\ 58 & 28 & 106 \end{bmatrix}$
- Repeat this process for each item in the final matrix

Multiplication (cont)

- A more compact formula.
- When multiplying F by G where F is an i x m matrix and G is an m x j matrix.

$$(\mathrm{FG})_{\mathrm{ij}} = \sum_{k=1}^{m} F_{ik} G_{ki}$$

The Identity Matrix

- The Matrix equivalent of ‘1’
- Any matrix multiplied by the identity matrix doesn’t change
- It is defined as a square matrix of zeroes except for the main diagonal, which are ones.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying Vectors and Matrices

- If you treat a vector as an n x 1 matrix you can multiply it with an n x n matrix.
- This works exactly the same as the multiplication we just did.
- $\begin{bmatrix} 9 & 6 & 3 \\ 2 & 4 & 7 \\ 1 & 7 & 9 \end{bmatrix} * \begin{bmatrix} 16 \\ 1 \\ 7 \end{bmatrix}$
- This is called transforming the vector.

Transpose

- The transpose of the matrix M is called M^T
- $M^T_{ij} = M_{ji}$
- This means each of the numbers will be flipped along the main diagonal
- $M = \begin{bmatrix} 1 & 5 & 6 \\ 2 & 7 & 8 \\ 2 & 7 & 5 \end{bmatrix}$ $M^T = \begin{bmatrix} 1 & 2 & 2 \\ 5 & 7 & 7 \\ 6 & 8 & 5 \end{bmatrix}$

Inverse

- The inverse of the matrix M is called M^{-1}
- A matrix multiplied by its inverse becomes the identity matrix.
- $M * M^{-1} = I$

Transpose, Inverse

- These are operations that are applied to a single matrix
- They only work on square matrices.
- Almost all matrices you use in games are square

A note on Row Major vs Column Major

- Row major vs column major is a constant source of confusion to new students.

- There are two main issues with column vs row major
  - Memory layout
  - Row vs column vectors

Memory Layout

- We typically represent matrices in code as a 2D array of floats

```
float matrix[3][3]
```

- So this matrix could be represented like so:

- $\begin{bmatrix} 1 & 5 & 6 \\ 2 & 7 & 8 \\ 2 & 7 & 5 \end{bmatrix}$

```
float matrix[3][3] =
{
    {1,5,6},
    {2,7,8},
    {2,7,5},
};
```

- In this layout, the numbers {1,5,6} will be next to each other in memory, followed by {2,7,8}, and lastly, {2,7,5}.

Memory Layout

- However, in the way we use matrices in games and computer graphics, each column, when treated as a vector, is important.
  - We'll learn more about this in the next lecture.

- Because of this, its more convenient to store the numbers for each column next to each other in memory.

Memory Layout

- Like so:
- $$\begin{bmatrix} 1 & 5 & 6 \\ 2 & 7 & 8 \\ 2 & 7 & 5 \end{bmatrix}$$

```
float matrix[3][3] =
{
    {1,2,2},
    {5,7,7},
    {6,8,5},
};
```

- This is how ***all*** matrices are stored in memory.

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Memory Layout

- This leads to a dilemma when documenting.

- Do we write out the matrices in the traditional mathematics way, or do we write them how they're stored in computer memory?
  - The traditional way is called column major
  - The memory way is called row major

Writing them out

- Here's a matrix written in column major
- $$\begin{bmatrix} 1 & 5 & 6 \\ 2 & 7 & 8 \\ 2 & 7 & 5 \end{bmatrix}$$
- And here's the same matrix written in row major
- $$\begin{bmatrix} 1 & 2 & 2 \\ 5 & 7 & 7 \\ 6 & 8 & 5 \end{bmatrix}$$

Writing them out

- Here's a matrix written in column major
- $\begin{bmatrix} \boxed{1} & 5 & 6 \\ \boxed{2} & 7 & 8 \\ \boxed{2} & 7 & 5 \end{bmatrix}$
- And here's the same matrix written in row major
- $\begin{bmatrix} \boxed{1 \quad 2 \quad 2} \\ 5 & 7 & 7 \\ 6 & 8 & 5 \end{bmatrix}$

Writing them out

- Here's a matrix written in column major
- $$\begin{bmatrix} 1 & \boxed{5} & 6 \\ 2 & \boxed{7} & 8 \\ 2 & \boxed{7} & 5 \end{bmatrix}$$
- And here's the same matrix written in row major
- $$\begin{bmatrix} 1 & 2 & 2 \\ \boxed{5} & \boxed{7} & \boxed{7} \\ 6 & 8 & 5 \end{bmatrix}$$

Writing them out

- Here's a matrix written in column major
- $\begin{bmatrix} 1 & 5 & \boxed{6} \\ 2 & 7 & \boxed{8} \\ 2 & 7 & \boxed{5} \end{bmatrix}$
- And here's the same matrix written in row major
- $\begin{bmatrix} 1 & 2 & 2 \\ 5 & 7 & 7 \\ \boxed{6} & \boxed{8} & \boxed{5} \end{bmatrix}$

Row vs column vectors

- The second part is how we multiply our matrices by our vectors.

- The first way is like so:
  - $\begin{bmatrix} 9 & 6 & 3 \\ 2 & 4 & 7 \\ 1 & 7 & 9 \end{bmatrix} * \begin{bmatrix} 16 \\ 1 \\ 7 \end{bmatrix}$
  - Where the vector is a 3*1 matrix

Row vs Column vectors

- The second way is like so:
  - $\begin{bmatrix} 9 & 6 & 3 \\ 2 & 4 & 7 \\ 1 & 7 & 9 \end{bmatrix} * \begin{bmatrix} 16 & 1 & 7 \end{bmatrix}$
  - Where the vector is a 1*3 matrix
  - But remember! The columns in the first matrix have to match the rows in the second!
  - Here we have 3 columns in the first and 1 row in the second
  - So instead we multiply the other way around.
  - $\begin{bmatrix} 16 & 1 & 7 \end{bmatrix} * \begin{bmatrix} 9 & 6 & 3 \\ 2 & 4 & 7 \\ 1 & 7 & 9 \end{bmatrix}$

Row vs Column vectors

- But remember again!
  - A*B != B*A
- Just swapping the order wont make the same result.
- $\begin{bmatrix} 9 & 6 & 3 \\ 2 & 4 & 7 \\ 1 & 7 & 9 \end{bmatrix} * \begin{bmatrix} 16 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 171 \\ 85 \\ 86 \end{bmatrix}$
- $\begin{bmatrix} 16 & 1 & 7 \end{bmatrix} * \begin{bmatrix} 9 & 6 & 3 \\ 2 & 4 & 7 \\ 1 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 153 & 149 & 118 \end{bmatrix}$

Row vs Column vectors

- However if we transpose the matrix we do get the same result.
- $\begin{bmatrix} 9 & 6 & 3 \\ 2 & 4 & 7 \\ 1 & 7 & 9 \end{bmatrix} * \begin{bmatrix} 16 \\ 1 \\ 7 \end{bmatrix} = \begin{bmatrix} 171 \\ 85 \\ 86 \end{bmatrix}$
- $\begin{bmatrix} 16 & 1 & 7 \end{bmatrix} * \begin{bmatrix} 9 & 2 & 1 \\ 6 & 4 & 7 \\ 3 & 7 & 9 \end{bmatrix} = \begin{bmatrix} 171 & 85 & 86 \end{bmatrix}$
- TL;DR
  - Transpose the matrix and swap the multiplication order

Why do we need to know about this mess?

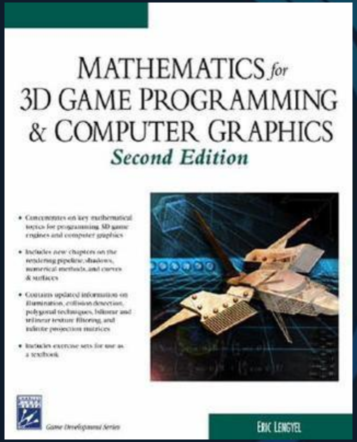
- Early in the history of computer graphics, some people decided to use row vectors and others decided to use column.

- That’s it, really.

- The biggest example is that DirectX uses row vectors and OpenGL uses column.

Further Reading

- It is *highly* recommend that you read the vector and matrix chapters of Mathematics For 3D Game Programming and Computer Graphics.
- The khan academy videos on Linear Algebra cover all of this content in a much more rigorous mathematical context.



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Conclusion

- We’ve covered:
- What a matrix is and how its used
- The basic mathematical operations that are performed on matrices (addition, subtraction, multiplication)
- The difference between row and column major
- The transpose and inverse.