Real and Functional Analysis

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For Mathematical Engineering

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Real analysis

1 Set Theory

Lezione 1 (13/09/23)

Let X be a set, then $\mathcal{P}(X) := \{Y | Y \subseteq X\}$ is a **power set**

Let I be an index set, then $\{E_i\}_{i \in I}$ with $E_i \subseteq X \ \ \forall\, i \in I$ is a **family**, or collection, indexed by I

If $I = \mathbb{N}$, then $\{E_n\}_{n \in \mathbb{N}}$ is a **sequence**

Definition.

A family $\{E_i\}_{i \in I}$ is **disjoint** if $E_j \cap E_k = \emptyset \ \ \forall\, j, k \in I, \ j \neq k$

Definition.

A sequence $\{E_n\} \subseteq \mathcal{P}(X)$ is said to be **monotone increasing** (ascending) if $E_n \subseteq E_{n+1} \ \ \forall\, n \in \mathbb{N}$

Monotone decreasing (descending) if $E_n \supseteq E_{n+1} \ \ \forall\, n \in \mathbb{N}$

Example:

$$\bigcap_{n=1}^{\infty} \left(a - \frac{1}{n}, b + \frac{1}{n}\right) = [a, b] \qquad \bigcup_{n=1}^{\infty} \left[a + \frac{1}{n}, b - \frac{1}{n}\right] = (a, b)$$

Revision:

Let $\{x_n\} \subseteq \mathbb{R}$, then

$$\underset{n \to \infty}{limsup}\ x_n := \inf_{k \geq 1} \{\sup_{n \geq k} x_n\}$$

$$\underset{n \to \infty}{liminf}\ x_n := \sup_{k \geq 1} \{\inf_{n \geq k} x_n\}$$

Definition.

Let $\{E_n\} \subseteq \mathcal{P}(X)$

$$\limsup_{n\to\infty} E_n := \bigcap_{k=1}^{\infty}\left[\bigcup_{n=k}^{\infty} E_n\right]$$

$$\liminf_{n\to\infty} E_n := \bigcup_{k=1}^{\infty}\left[\bigcap_{n=k}^{\infty} E_n\right]$$

If $\underset{n\to\infty}{limsup} E_n = \underset{n\to\infty}{liminf} E_n =: F$ then $F = \underset{n\to\infty}{lim} E_n$

Examples (1.1, 1.2)

- Ex. i) $x \in \limsup\limits_{n\to\infty} E_n \Leftrightarrow x \in E_n$ for infinitely many $n \in \mathbb{N}$

  ii) $x \in \liminf\limits_{n\to\infty} E_n \Leftrightarrow \exists\, k \in \mathbb{N}$ s.t. $x \in E_n \quad \forall n \geq k$.

  iii) $\left(\liminf\limits_{n\to\infty} E_n\right)^c = \limsup\limits_{n\to\infty} \left(E_n^c\right)$

- Ex. i) $\{E_n\} \nearrow \Rightarrow \lim\limits_{n\to\infty} E_n = \bigcup\limits_{n=1}^{\infty} E_n$

  ii) $\{E_n\} \downarrow \Rightarrow \lim\limits_{n\to\infty} E_n = \bigcap\limits_{n=1}^{\infty} E_n$

Definition.

A family of sets $\{E_i\}_{i\in I}$ is called a **cover** (or covering) of X if $X \subseteq \bigcup\limits_{i\in I} E_i$

A subfamily of a cover ($\Leftrightarrow \{E_i\}_{i\in J}, J \subseteq I$) which itself forms a cover is called a **subcover**

Definition.

Let $E \subseteq X$, the function $\chi_E : X \to \mathbb{R}$ $\quad \chi_E(x) := \begin{cases} 1 & \text{if } x \in E \\ 0 & \text{if } x \notin E \end{cases}$ is called **characteristic function**

Examples (1.3, 1.4)

• Ex. Let $E_1, E_2 \subseteq X$

$$\chi_{E_1 \cap E_2} = \chi_{E_1} \cdot \chi_{E_2}$$

$$\chi_{E_1 \cup E_2} = \chi_{E_1} + \chi_{E_2} - \chi_{E_1} \cdot \chi_{E_2}$$

• Ex. $\{E_n\} \subseteq \mathcal{P}(X)$, $P := \limsup_{n\to\infty} E_n$

$Q := \liminf_{n\to\infty} E_n$

Then

$$\chi_P = \limsup_{n\to\infty} \chi_{E_n}$$

$$\chi_Q = \liminf_{n\to\infty} \chi_{E_n}$$

1.1 Relations

Let X be a set. A **relation** in X is a subset $R \subseteq X \times X$

Obs. x is said to be in relation with y if $(x, y) \in R$

Definition.

R is an **equivalence relation** $(x, y) \in R \leftrightsquigarrow xRy$ if:

i) $(x, x) \in R \;\; \forall x \in X$ reflexivity

ii) $(x, y) \in R \implies (y, x) \in R$ symmetry

iii) $(x, y) \in R, \, (y, z) \in R \implies (x, z) \in R$ transitivity

Definition.

$E_x := \{y \in X \,:\, yRx\}$ $(x \in X)$ is the **equivalence class** of x

Remark:

$X = \bigcup_{x \in X} E_x$ disjoint union

Definition.

$X/R := \{E_x \,:\, x \in X\}$ **quotient set**

Obs. The cardinality of a set X can be finite and infinite, which can be countable ($X \leftrightarrow \mathbb{N}$) or uncountable.

Remark:

i) $\mathbb{Q}$ is countable

ii) $\mathbb{R}$ is not countable (cardinality of continuum)

iii) countable union of countable sets is countable

iv) X, Y countable $\implies$ $X \times Y$ is countable

2 Measure

2.1 σ-algebra

Lezione 2 (14/09/23)

Definition.

X set. A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to be an **algebra** if:

i) $\emptyset \in \mathcal{A}$ *ii)* $E \in \mathcal{A} \implies E^c \in \mathcal{A}$ *iii)* $A, B \in \mathcal{A} \implies A \cup B \in \mathcal{A}$

Remark:

i) $X \in \mathcal{A} \Leftarrow \emptyset^c = X$

ii) $A, B \in \mathcal{A} \implies A \cap B \in \mathcal{A}$ because $A \cap B = (A^c \cup B^c)^c$

Definition.

A family $\mathcal{A} \subseteq \mathcal{P}(X)$ is said to be a σ**-algebra** if

i) $\emptyset \in \mathcal{A}$ *ii)* $E \in \mathcal{A} \implies E^c \in \mathcal{A}$

iii) $\{E_k\}_{k \in \mathbb{N}} \in \mathcal{A} \implies (\bigcup_{k=1}^{\infty} E_k) \in \mathcal{A}$ we passed from a finite family to a countable family

Remark:

$\{E_k\} \subseteq \mathcal{A} \implies \bigcap_{k=1}^{\infty} E_k = [\bigcup_{k=1}^{\infty} E_k^c]^c \in \mathcal{A}$

Definition.

$(X, \mathcal{A})$ is called **measurable space**.

The elements of $\mathcal{A}$ are called measurable sets.

Obs. σ stands for countable family.

Obs. We now see the following theorem about σ-algebras built from subsets.

Theorem.

Let $S \subseteq \mathcal{P}(X)$. Then there exists a σ-algebra $\sigma_0(S)$ s.t.

i) $S \subseteq \sigma_0(S)$

ii) $\forall\, \sigma$-algebra $\mathcal{A} \subseteq \mathcal{P}(X)$ s.t. $S \subseteq \mathcal{A}$ we have $\sigma_0(S) \subseteq \mathcal{A}$

Definition.

$\sigma_0(S)$ is called **minimal σ-algebra** generated by S

Proof. (sketch)

$$\mathcal{V} := \{\mathcal{A} \subseteq \mathcal{P}(X) \,:\, S \subseteq \mathcal{A},\ \mathcal{A} \text{ is } \sigma\text{-algebra}\}$$

$$\sigma_0(S) := \bigcap \{\mathcal{A} \,:\, \mathcal{A} \in \mathcal{V}\}$$

□

2.2 Borel sets

Definition.

Let (X, d) be a metric space. Let $\mathcal{G}$ be the family of open sets of X

$\sigma_0(\mathcal{G})$ is called **Borel σ-algebra**. That will be denoted by $\mathcal{B}(X)$.

The elements of $\sigma_0(\mathcal{G})$ are called Borel sets

Remark:

i) The following sets are Borel:

• open sets • closed sets • countable intersections of open sets • countable union of closed sets

ii) $\mathcal{B}(\mathbb{R}) = \sigma_0(I)$ with:

$$I = \{(a, b) \,:\, a, b \in \mathbb{R},\ a \leq b\}$$

$$I = \{[a, b] \,:\, a, b \in \mathbb{R},\ a \leq b\}$$

$$I = \{(a, b] \,:\, a, b \in \mathbb{R},\ a \leq b\}$$

$$I = \{(a,b] : -\infty \leq a < b < \infty\} \cup \{(a,\infty) : a \in \mathbb{R}\}$$

$$I = \{(a,\infty) : a \in \mathbb{R}\}$$

iii) $\mathcal{B}(\mathbb{R}^N) = \sigma_0(K)$ with:

$$K = \{ \text{ n-dim closed rectangles } \}$$

$$K = \{ \text{ n-dim open rectangles } \}$$

iv) $\mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(I)$ with:

$$I = \{(a,b) : a,b \in \mathbb{R},\ a < b\} \cup \{[-\infty,b) : b \in \mathbb{R}\} \cup \{(a,+\infty] : a \in \mathbb{R}\}$$

$$I = \{(a,+\infty] : a \in \mathbb{R}\}$$

Definition.

Let X be a set, $\mathcal{C} \subseteq \mathcal{P}(X)$, $\emptyset \in \mathcal{C}$.

A function $\mu : \mathcal{C} \to \overline{\mathbb{R}}_+ \equiv [0,\infty]$ is a **measure** on $\mathcal{C}$ if

i) $\mu(\emptyset) = 0$

ii) $\sigma-$additivity $\quad \forall \{E_k\}_{k\in\mathbb{N}} \subseteq \mathcal{C}$ disjoint s.t. $\bigcup\limits_{k=1}^{\infty} E_k \in \mathcal{C} \quad \mu\left[\bigcup\limits_{k=1}^{\infty} E_k\right] = \sum\limits_{k=1}^{\infty} \mu(E_k)$

Remark:

If $\mathcal{C}$ is a σ-algebra, then $\bigcup\limits_{k=1}^{\infty} E_k \in \mathcal{C}$

Definition.

μ is **finite** if $\mu(X) < \infty$

μ is **σ-finite** if there exists $\{E_k\} \subseteq \mathcal{C}$ s.t. $X = \bigcup\limits_{k=1}^{\infty} E_k$, $\mu(E_k) < \infty \ \ \forall k \in \mathbb{N}$

Definition.

Let $\mathcal{A} \subseteq \mathcal{P}(X)$ be a σ-algebra, $\mu : \mathcal{A} \to \overline{\mathbb{R}}_+$ a measure.

$(X,\mathcal{A},\mu)$ is called a **measure space**

If $\mu(X) = 1$, $(X,\mathcal{A},\mu)$ is a **probability space** and μ is a **probability measure** (or a probability)

Example of measure: (2.1, 2.2, 2.3)

(i) X set, $\mathcal{C} \subseteq \mathcal{P}(X)$ s.t. $\emptyset \in \mathcal{C}$

$\mu(\emptyset) := 0$

$\mu(E) := \infty \quad \forall E \in \mathcal{C}, E \neq \emptyset$

(ii) $\mu : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$

$$\mu(E) := \begin{cases} \overbrace{|E|}^{\text{number of elements of } E}, & \text{if } E \text{ is finite} \\ \infty, & \text{otherwise} \end{cases}$$

is a measure, called counting measure

$\mu^{\#}$

$\mu^{\#}$ is finite iff X is finite,

$\mu^{\#}$ is σ-finite iff X is countable.

iii) $X \neq \emptyset$, $x_0 \in X$

$\delta_{x_0} : \mathcal{P}(X) \to \mathbb{R}_+$

$$\delta_{x_0}(E) := \begin{cases} 1, & \text{if } x_0 \in E \\ 0, & \text{otherwise} \end{cases}$$

δ_{x_0} is a measure, called Dirac measure concentrated at x_0.

THEOREM: Properties of a measure.

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then,

i) (additivity) $\forall \{E_1, ..., E_n\} \subset \mathcal{A}$ a finite disjoint family $\quad \mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k)$

ii) (monotonicity) $\forall E, F \in \mathcal{A},\ E \subseteq F \implies \mu(E) \leq \mu(F)$

iii) (σ-subadditivity) $\forall \{E_k\} \subseteq \mathcal{A} \quad \mu\left(\bigcup_{k=1}^{\infty} E_k\right) \leq \sum_{k=1}^{\infty} \mu(E_k)$

iv) (continuity) $\forall \{E_k\} \subseteq \mathcal{A} \nearrow \implies \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k)$

v) (continuity) $\forall \{E_k\} \subseteq \mathcal{A} \searrow,\ \mu(E_1) < \infty \implies \mu\left(\bigcap_{k=1}^{\infty} E_k\right) = \lim_{k\to\infty} \mu(E_k)$

Proof.

i) $E_1, ..., E_n$, $n \in \mathbb{N}$ becomes a set ($\{E_k\}_{k\in\mathbb{N}}$) when unite with $E_{n+1} = E_{n+2} = ... = \emptyset$

$$\mu\left(\bigcup_{k=1}^{n} E_k\right) = \mu\left(\bigcup_{k=1}^{n} E_k \cup \bigcup_{k=n+1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} E_k\right) \overset{\sigma-add}{=} \sum_{k=1}^{\infty} \mu(E_k) = \sum_{k=1}^{n} \mu(E_k) + \underbrace{\sum_{k=n+1}^{\infty} \mu(E_k)}_{0} = \sum_{k=1}^{n} \mu(E_k)$$

ii) Let $\mu(F \backslash E) = \mu(F) - \mu(E)$

$$F = E \cup (F\backslash E)\ ,\ E \cap (F\backslash E) = \emptyset \implies \mu(F) \overset{(i)}{=} \mu(E) + \underbrace{\mu(F\backslash E)}_{\geq 0} \geq \mu(E)$$

iii) $\begin{cases} F_1 := E_1 \\ F_n := E_n \backslash \left(\bigcup_{k=1}^{n-1} E_k\right) \quad n \geq 2\ n \in \mathbb{N} \end{cases}$

$$\{F_n\} \subseteq \mathcal{A} \text{ disjoint, } F_k \subseteq E_k \quad \forall k \in \mathbb{N}$$

$$\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k \implies \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \overset{\sigma\text{-add}}{=} \sum_{k=1}^{\infty} \underbrace{\mu(F_k)}_{\leq \mu(E_k)} \leq \sum_{k=1}^{\infty} \mu(E_k)$$

iv) $F_k := E_k \backslash E_{k-1} \quad k \in \mathbb{N} \quad (E_0 := \emptyset)$

$$\implies \bigcup_{k=1}^{n} F_k = E_n\ ,\ \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k\ ,\ \{F_k\} \text{ disjoint}$$

$$\implies \mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \mu\left(\bigcup_{k=1}^{\infty} F_k\right) \overset{\sigma-\text{add}}{=} \sum_{k=1}^{\infty} \mu(F_k) = \lim_{n\to\infty} \sum_{k=1}^{n} \mu(F_K) = \lim_{n\to\infty} \mu\left(\bigcup_{k=1}^{n} F_k\right) = \lim_{n\to\infty} \mu(E_n)$$

v) $F_k := E_1 \backslash E_k$ $(k \in \mathbb{N})$ $\{F_k\} \nearrow$

$$\mu\left(\bigcup_{k=1}^{\infty} F_k\right) \overset{(iv)}{=} \lim_{k\to\infty} \mu(F_k) = \lim_{k\to\infty} [\mu(E_1) - \mu(E_k)] = \mu(E_1) - \lim_{k\to\infty} \mu(E_k)$$

$$\text{Moreover, } \bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} (E_1 \cap E_k^c) = E_1 \cap \left(\bigcup_{k=1}^{\infty} E_k^c\right) = E_1 \backslash \left(\bigcap_{k=1}^{\infty} E_k\right)$$

$$\implies \mu\left(\bigcup_{k=1}^{\infty} F_k\right) = \mu(E_1) - \mu\left(\bigcap_{k=1}^{\infty} E_k\right) \implies \lim_{k\to\infty} \mu(E_k) = \mu\left(\bigcap_{k=1}^{\infty} E_k\right)$$

$\square$

Lezione 3 (20/09/23)

Remark:

(v) fails if $\mu(E_1) = \infty$. In fact, consider $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^*)$ and E_n the integers bigger than n.

$$E_n := \{k \in \mathbb{N} : k \geq n\} \ (n \in \mathbb{N}) \qquad \bigcap_{n=1}^{\infty} E_n = \emptyset \implies \mu^*\left(\bigcap_{n=1}^{\infty} E_n\right) = 0$$

$$\mu^*(E_n) = \infty \quad \forall\, n \in \mathbb{N} \implies \lim_{n\to\infty} \mu^*(E_n) = \infty \neq 0$$

2.3 Sets of zero measure

Let $(X, \mathcal{A}, \mu)$ be a measure space

Definition.

$N \subseteq X$ is said to be a set of **zero measure** if $N \in \mathcal{A},\ \mu(N) = 0$

$E \subseteq X$ is said to be **negligible** if $\exists\, N \in \mathcal{A}$ s.t. $E \subseteq N,\ \mu(N) = 0$

$\mathcal{N}_\mu$ = collection of sets of zero measure

$\mathcal{T}_\mu$ = collection of negligible sets

Example (3.1)

$X = \{a, b, c\}$

$\mathcal{A} = \{\emptyset, \{a\}, \{b, c\}, X\}$

$\mu(X) = \mu(\{a\}) = 1$ $\qquad$ μ is a measure,

$\mu(\emptyset) = \mu(\{b, c\}) = 0$

$N = \{b, c\} \in \mathcal{N}_\mu$

$\{b\}, \{c\} \notin \mathcal{A}$, $\qquad \{b\} \subseteq N$, $\{c\} \subseteq N$

$\{b\}, \{c\} \in \mathcal{T}_\mu \setminus \mathcal{N}_\mu$.

Definition.

A property P on X is said to be true **almost everywhere** if $\{x \in X \, : \, P(x) \text{ is false}\} \in \mathcal{N}_\mu$

Example (3.2)

<u>Examples</u> (i) $f, g : X \to \overline{\mathbb{R}}$ are <u>equal a.e.</u> if

$\{x \in X \, : \, f(x) \neq g(x)\} \in \mathcal{N}_\mu$

(ii) $f : X \to \overline{\mathbb{R}}$ is <u>finite a.e.</u> if

$\{x \in X : f(x) = \pm\infty\} \in \mathcal{N}_\mu$

(iii) $f : \overset{\in \mathcal{A}}{D} \to \overline{\mathbb{R}}$ is <u>defined a.e. in X</u>

if $D^c \in \mathcal{N}_\mu$.

Remark:

$\overset{a.e.}{=}$ is an equivalence relation in the set of functions $f : X \to \overline{R}$

2.4 Complete measure

Definition.

A measure space $(X, \mathcal{A}, \mu)$ is said to be **complete** if $\mathcal{T}_\mu \subseteq \mathcal{A}$. In such case, μ is a complete measure, $\mathcal{A}$ is a complete σ-algebra.

Remark:

$\mathcal{N}_\mu = \mathcal{T}_\mu \iff (X, \mathcal{A}, \mu)$ is complete

Theorem.

Let $(X, \mathcal{A}, \mu)$ be a measure space.

Let $\overline{\mathcal{A}} := \{E \subseteq X \mid \ \exists\, F, G \in \mathcal{A} \ s.t. \ F \subseteq E \subseteq G \, , \, \mu(G \backslash F) = 0\}$ $\quad \overline{\mu} : \overline{\mathcal{A}} \to \mathbb{R}_+ \quad \overline{\mu}(E) := \mu(F)$. Then,

i) $\overline{\mathcal{A}}$ is a σ-algebra and $\overline{\mathcal{A}} \supseteq \mathcal{A}$

ii) $\overline{\mu}$ is a complete measure, $\overline{\mu}|_{\mathcal{A}} = \mu$ and $(X, \overline{\mathcal{A}}, \overline{\mu})$ is a complete measure space.

More precisely, it is the smallest (w.r.t. inclusion) complete measure space which contains $(X, \mathcal{A}, \mu)$

2.5 Outer measure

Let X be a set.

Definition.

A function $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ is said to be an **outer measure** on X, if

i) $\mu^*(\emptyset) = 0$

ii) $E_1 \subseteq E_2 \implies \mu^*(E_1) \leq \mu^*(E_2)$

iii) $\mu^* \left(\bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n=1}^{\infty} \mu^*(E_n)$

Remark:

If μ is a measure on $\mathcal{P}(X)$, then μ is an outer measure

Definition.

Let $K \subseteq \mathcal{P}(X)$, $\emptyset \in K$. Let $\nu : K \to \overline{\mathbb{R}}_+$ a function s.t. $\nu(\emptyset) = 0$

We call **outer measure generated** by (K, ν) a function $\mu^* : \mathcal{P}(X) \to \overline{\mathbb{R}}_+$ s.t. for $E \subseteq X$

$\mu^*(E) := inf \left\{ \sum_{n=1}^{\infty} \nu(I_n) \;\middle|\; E \subseteq \bigcup_{n=1}^{\infty} I_n \,,\, \{I_n\} \subseteq K \right\}$ if E can be covered by a countable $I_n \subseteq K$.

$\mu^*(E) := \infty$ otherwise.

Remark:

If $I \in K$ then $\mu^*(I) \leq \nu(I)$

Lezione 4 (21/09/23)

Theorem: Generation of outer measure.

μ^* is an outer measure on X

Proof.

i)

$$\emptyset \in K \implies \mu^*(\emptyset) \leq \nu(\emptyset) = 0 \implies \mu^*(\emptyset) = 0$$

ii) $E_1 \subseteq E_2$

If there exists a countable cover of E_2 it is also a countable cover of E_1

From the definition of μ^* it follows that $\mu^*(E_1) \leq \mu^*(E_2)$

If E_2 does not have a countable cover, then $\mu^*(E_1) \leq \mu^*(E_2) = \infty$

iii) It is obvious if $\sum_{n=1}^{\infty} \mu^*(E_n) = \infty$

On the contrary, suppose that $\sum\limits_{n=1}^{\infty} \mu^*(E_n) < \infty$ thus $\mu^*(E_n) < \infty \ \ \forall n \in \mathbb{N}$
For the definition of μ_* (and inf)

$$\forall \varepsilon > 0 \ \ \forall n \in \mathbb{N} \ \ \exists \{I_{n,k}\} \subseteq K \ s.t. \ E_n \subseteq \bigcup_{k=1}^{\infty} I_{n,k} \quad \mu^*(E_n) + \frac{\varepsilon}{2^n} > \sum_{k=1}^{\infty} \nu(I_{n,k})$$

Since $\bigcup_{n=1}^{\infty} E_n \subseteq \bigcup_{n,k=1}^{\infty} I_{n,k} \subseteq K$ it follows that

$$\mu^* \left(\bigcup_{n=1}^{\infty} E_n \right) \leq \sum_{n,k=1}^{\infty} \nu\left(I_{n,k}\right) < \sum_{n=1}^{\infty} \left[\mu^*(E_n) + \frac{\varepsilon}{2^n} \right] = \sum_{n=1}^{\infty} \mu^*(E_n) + \varepsilon$$

Where $\varepsilon > 0$ is arbitrary, thus the thesis follows.

□

Obs. We constructed an outer measure, but our goal is a measure, so now we go from one to the other.

2.6 Generation of a measure

Let μ^* be an outer measure of X.

Definition.

$E \subseteq X$ is said to be μ^***-measurable** if the Caratheodory condition is satisfied:

$$\mu^*(Z) = \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \quad \forall Z \subseteq X$$

Obs. The Caratheodory condition is simply derived from the measure of the union of the two addendi, since they are obviously disjoint.

Lemma:

$E \subseteq X$ is μ^*-measurable $\Longleftrightarrow \mu^*(Z) \geq \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \ \ \forall Z \subseteq X$

Obs. We have to show that this condition is equivalent to the Caratheodory condition and so to the equality, to do that we only have to show that the opposite condition $\leq$ is always satisfied.

Proof.

It is enough to show that $\ \ \forall E \subseteq X \ \ \mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \ \ \forall Z \subseteq X$

$$Z = Z \cap X = Z \cap (E \cup E^c) = (Z \cap E) \cup (Z \cap E^c)$$

by σ-subadditivity (iii) of μ^*, we obtain $\mu^*(Z) \leq \mu^*(Z \cap E) + \mu^*(Z \cap E^c)$

□

Lemma:

If $\mu^*(E) = 0$ then E is μ^*-measurable

Proof.

$\forall\, Z \subseteq X \quad \mu^*(Z \cap E) + \mu^*(Z \cap E^c) \leq \mu^*(E) + \mu^*(Z)$ by monotonicity (ii) since $Z \cap E \subseteq E$ and $Z \cap E^c \subseteq Z$

Since $\mu^*(E) = 0$ we can apply the preceding lemma and the thesis follows. $\square$

Theorem.

Let $\mathcal{L} := \{E \subseteq X \,:\, E \text{ is } \mu^*\text{-measurable}\}$ and μ^* be an outer measure on X. Then,

i) $\mathcal{L}$ is a σ-algebra

ii) $\mu^*|_{\mathcal{L}}$ is a complete measure on $\mathcal{L}$

Obs. So the restriction of the outer measure on the sets satisfying the Caratheodory condition is a measure.

Summary:

X set, $(K, \nu) \rightsquigarrow \mu^*$ outer measure, C-condition $\rightsquigarrow \mathcal{L}$, $\mu := \mu^*|_{\mathcal{L}}$, $(X, \mathcal{L}, \mu)$ is a complete measure space.

Obs. We don't see the proof, but we discuss the completeness.

$N \in \mathcal{L}$ s.t. $\mu(N) = 0 = \mu^*(N)$ since μ is the restriction

Let $E \subseteq N$, since μ^* is monotone, then $\mu^*(E) \leq \mu^*(N) = 0 \implies \mu^*(E) = 0$ for the lemma E is μ^*-measurable so $E \in \mathcal{L}$ and we can conclude that the measure is complete (negligible is measurable)

2.7 Lebesgue measure

Let $\mathcal{I}$ be a family of open bounded intervals

$$\mathcal{I} := \{(a, b) \,:\, a, b \in \mathbb{R},\; a \leq b\} \qquad \text{Obs. } \emptyset \in \mathcal{I} \; (a = b)$$

$$\lambda_0 : \mathcal{I} \to \mathbb{R}_+ \qquad \lambda_0(\emptyset) := 0 \qquad \lambda_0((a, b)) := b - a$$

Let $X = \mathbb{R}$ and $(K, \nu) = (\mathcal{I}, \lambda_0) \rightsquigarrow \lambda^*$ is an outer measure on $\mathbb{R}$

Definition.

λ^* generated by $(\mathcal{I}, \lambda_0)$ is called **outer Lebesgue measure**

λ^*-measurable sets are called Lebesgue measurable sets

The σ-algebra is $\mathcal{L}(\mathbb{R})$ and $\lambda := \lambda^*|_{\mathcal{L}(\mathbb{R})}$ is called **Lebesgue measure**

Theorem.

Any countable subset $E \subseteq \mathbb{R}$ is Lebesgue measurable and $\lambda(E) = 0$

Proof.

Let $a \in \mathbb{R}$

$$\{a\} \subseteq (a-\varepsilon, a+\varepsilon) \;\; \forall \varepsilon > 0 \implies \lambda^*(\{a\}) = 0 \implies \{a\} \in \mathcal{L}$$

$$E = \bigcup_{n=1}^{\infty} \{a_n\} \text{ with } a_n \in \mathbb{R} \;\; \forall n \in \mathbb{N} \;\text{ since } \{a_n\} \in \mathcal{L} \implies E \in \mathcal{L}$$

$$0 \leq \lambda(E) \leq \sum_{n=1}^{\infty} \lambda(\{a_n\}) = 0$$

□

Theorem.

$\mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R}) \quad \big(\mathcal{B}(\mathbb{R}) \subsetneq \mathcal{L}(\mathbb{R})\big)$

Proof.

Since $\mathcal{B}(\mathbb{R}) = \sigma_0((a,\infty))$ it's enough to show that $(a,\infty) \in \mathcal{L}(\mathbb{R})$

Let $A \subseteq \mathbb{R}$ be any set. We assume that $a \notin A$ otherwise we replace A by $A \backslash \{a\}$ (this leaves the outer measure unchanged). We must show that

$$(1) \qquad \lambda^*(A_1) + \lambda^*(A_2) \leq \lambda^*(A)$$

Where $A_1 = A \cap (-\infty, a)$ and $A_2 = A \cap (a, \infty)$, from Caratheodory ($Z \cap E^c$ and $Z \cap E$).

We can substitute the outer measure of A with his definition, so (1) is equivalent to the following:

$$(2) \qquad \lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I_k)$$

For any countable collection $\{I_n\}$ of open bounded intervals that covers A, since $\lambda^*(A)$ it's defined as inf.

$$\forall\, k \in \mathbb{N} \;\; I'_k := I_k \cap (-\infty, a), \;\; I''_k := I_k \cap (a, \infty) \overset{\text{are disjoint}}{\implies} \lambda_0(I_k) = \lambda_0(I'_k) + \lambda_0(I''_k)$$

$\{I'_k\}$ is a cover of A_1 and $\{I''_k\}$ is a cover of A_2

$$\implies \lambda^*(A_1) \leq \sum_{k=1}^{\infty} \lambda_0(I'_k) \qquad \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I''_k) \qquad \implies (2)$$

$$\text{since } \lambda^*(A_1) + \lambda^*(A_2) \leq \sum_{k=1}^{\infty} \lambda_0(I'_k) + \sum_{k=1}^{\infty} \lambda_0(I''_k) = \sum_{k=1}^{\infty} [\lambda_0(I'_k) + \lambda_0(I''_k)] = \sum_{k=1}^{\infty} \lambda_0(I_k)$$

□

Obs. It's possible to verify the following relation between measurable spaces:

$$(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda) \text{ is the completion of } (\mathbb{R}, \mathcal{B}(\mathbb{R}), \lambda|_{\mathcal{B}(\mathbb{R})})$$

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Theorem.

Let $E \subseteq \mathbb{R}$ the following statements are equivalent:

i) $E \in \mathcal{L}(\mathbb{R})$

ii) $\forall \varepsilon > 0 \;\; \exists\, A \subseteq \mathbb{R}$ open s.t. $E \subseteq A,\; \lambda^*(A \backslash E) < \varepsilon$

iii) $\exists\, G \subseteq \mathbb{R}$ of class $\mathcal{G}_\delta$ (countable intersection of open sets) s.t. $E \subseteq G,\; \lambda^*(G \backslash E) = 0$

iv) $\forall \varepsilon > 0 \;\; \exists\, C \subseteq \mathbb{R}$ closed s.t. $C \subseteq E,\; \lambda^*(E \backslash C) < \varepsilon$

v) $\exists\, F \subseteq \mathbb{R}$ of class $\mathcal{F}_\delta$ (countable union of closed sets) s.t. $F \subseteq E,\; \lambda^*(E \backslash F) = 0$

Where (ii) and (iii) are about outer approximation and (iv) and (v) about inner approx.

Lemma: Excision property

If $A \in \mathcal{L}(\mathbb{R})$ and $\lambda^*(A) < \infty$, then

$$A \subseteq B \implies \lambda^*(B \backslash A) = \lambda^*(B) - \lambda^*(A)$$

Proof.

Since $A \in \mathcal{L}(\mathbb{R})$, $B \cap A = A$ and $B \cap A^c = B \backslash A$. Then,

$$\lambda^*(B) = \lambda^*(B \cap A) + \lambda^*(B \cap A^c) = \lambda^*(A) + \lambda^*(B \backslash A)$$

$\square$

Proof. (Partial demonstration of the theorem: (i) $\implies$ (ii) $\implies$ (iii) $\implies$ (i))

(i) $\implies$ (ii) $\quad E \in \mathcal{L}(\mathbb{R})\;,\; \lambda(E) < \infty$

By definition of outer measure, $\;\forall\, \varepsilon > 0 \;\; \exists \{I_k\}$ (open bounded intervals) which covers E and for which

$$\sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon$$

Define $O := \bigcup_{k=1}^{\infty} I_k$. Then O is open and $E \subseteq O$

$$\lambda^*(O) \leq \sum_{k=1}^{\infty} \lambda_0(I_k) < \lambda^*(E) + \varepsilon \implies \lambda^*(O) - \lambda^*(E) < \varepsilon$$

$E \in \mathcal{L}\;,\; \lambda^*(E) < \infty \overset{\text{excision}}{\implies} \lambda^*(O \backslash E) = \lambda^*(O) - \lambda^*(E) < \varepsilon$

(ii) $\implies$ (iii) $\quad \forall\, k \in \mathbb{N}$ choose $O_k \supseteq E$ open for which $\lambda^*(O_k \backslash E) < \dfrac{1}{k}$

Define $G = \bigcap_{k=1}^{\infty} O_k$. Then G is a $\mathcal{G}_\delta$ set and $G \supseteq E$

Moreover $\quad \forall\, k \in \mathbb{N} \;\; G\backslash E \subseteq O_k \backslash E$

By monotonicity, $\lambda^*(G\backslash E) \leq \lambda^*(O_k\backslash E) < \dfrac{1}{k}$

$$\overset{k\to\infty}{\Longrightarrow} \lambda^*(G\backslash E) = 0$$

(iii) $\implies$ (i) $\quad G\backslash E \in \mathcal{L}(\mathbb{R})$ since $\lambda^*(G\backslash E) = 0$

$G \in \mathcal{L}(\mathbb{R})$ since $G \in \mathcal{G}_\delta \subseteq \mathcal{B}(\mathbb{R}) \subseteq \mathcal{L}(\mathbb{R})$

$$\implies E = G \cap (G\backslash E)^c \in \mathcal{L}(\mathbb{R})$$

Obs. Since the complementary of a measurable is meas. and the intersection of meas. is meas.

Obs. This is the proof regarding only inner sets.

□

2.8 Non measurable sets

Theorem: Vitali.

Any measurable set $E \in \mathcal{L}(\mathbb{R})$ with $\lambda^*(E) > 0$ contains a subset that fails to be measurable.

Theorem.

There are disjoint sets $A, B \subset \mathbb{R}$ for which $\quad \lambda^*(A \cup B) < \lambda^*(A) + \lambda^*(B)$

Proof.

Assume, by contradiction, that $\quad \lambda^*(A \cup B) = \lambda^*(A) + \lambda^*(B)$ for any $A, B \subset \mathbb{R},\; A \cap B = \emptyset$

$$\forall\, E, Z \subseteq \mathbb{R} \;\; \lambda^*(\underbrace{Z \cap E}_{A}) + \lambda^*(\underbrace{Z \cap E^c}_{B}) = \lambda(\underbrace{(Z \cap E) \cup (Z \cap E^c)}_{A \cup B}) = \lambda^*(Z)$$

So $\forall E \subseteq \mathbb{R}$, E fulfills the C-condition. But if any E satisfies C-condition, then any E is measurable. This contradicts the preceding theorem.

□

2.9 Lebesgue measure in N dimensions

An N-dimensions interval is the cartesian product of N intervals

$$\mathcal{I}^N := \left\{ \prod_{k=1}^{N} (a_k, b_k) \ : \ a_k, b_k \in \mathbb{R} \ , \ a_k \leq b_k \ k = 1, ..., N \right\}$$

$$\lambda_0^N(\emptyset) := 0 \qquad \lambda_0^N\left(\prod_{k=1}^{N}(a_k, b_k)\right) = \prod_{k=1}^{N}(b_k - a_k)$$

Let $X = \mathbb{R}^N$, $(K, \nu) = (\mathcal{I}^N, \lambda_0^N) \rightsquigarrow \lambda^{*,N}$ outer measure in $\mathbb{R}^N$

$\lambda^{*,N}$ and C-condition $\rightsquigarrow \mathcal{L}(\mathbb{R}^N)$

$(\mathbb{R}^N, \mathcal{L}(\mathbb{R}^N), \lambda^n)$ is a complete measure space, where $\lambda^N := \lambda^{*,N}|_{\mathcal{L}(\mathbb{R}^N)}$ is the N-dim Lebesgue measure.

All the properties that are valid in the one dimensions are also valid in N-dim.

3 Measurable functions

Definition.

Let $(X, \mathcal{A}),\ (X', \mathcal{A}')$ be two measurable spaces.

A function $f : X \to X'$ is said to be **measurable** if $f^{-1}(E) \in \mathcal{A} \ \ \forall\, E \in \mathcal{A}'$

Proposition:

$(X, \mathcal{A}),\ (X', \mathcal{A}'),\ (X'', \mathcal{A}'')$ measurable spaces

Let $f : X \to X'$ and $g : X' \to X''$ be measurable functions.

Then $g \circ f : X \to X''$ is measurable

Obs. Is important that the codomain of f and the domain of g have the same σ-algebra, otherwise the proposition is not valid

Proof.

$$\forall\, E \in \mathcal{A}' \ f^{-1}(E) \in \mathcal{A} \qquad \forall\, F \in \mathcal{A}'' \ g^{-1}(F) \in \mathcal{A}'$$

$$\forall\, F \in \mathcal{A}'' \ \ (g \circ f)^{-1}(F) = f^{-1}[\underbrace{g^{-1}(F)}_{=E \in \mathcal{A}'}] \in \mathcal{A}$$

□

Theorem.

$(X, \mathcal{A}),\ (X', \mathcal{A}')$ measurable spaces. Let $\mathcal{C}' \subseteq \mathcal{P}(X')$ s.t. $\sigma_0(\mathcal{C}') = \mathcal{A}'$. Then,

$$f : X \to X' \text{ measurable } \iff f^{-1}(E) \in \mathcal{A} \ \ \forall E \in \mathcal{C}'$$

Obs. Instead of verifying the property for all sets you only need to verify it for a set of generators.

Proof.

If f is measurable, then $\mathcal{C}' \subseteq \mathcal{A}' \implies$ thesis

Instead, set $\Sigma := \{E \subseteq X' \ : \ f^{-1}(E) \in \mathcal{A}\}$. It is easily seen that Σ is a σ-algebra.

$f^{-1}(E) \in \mathcal{A} \ \ \forall E \in \mathcal{C}' \implies \mathcal{C}' \subseteq \Sigma$. Thus, $\mathcal{A}' = \sigma_0(\mathcal{C}') \subseteq \Sigma \implies$ thesis

$\square$

Definition.

$(X, \mathcal{L})$ measurable space, (X', d') metric space and $(X', \mathcal{B}')$ measurable space.

$f : X \to X'$ measurable is called **Lebesgue measurable**.

Obs. To consider the Borel σ-algebra we need the open sets, but to define those we need a metric space.

Definition.

$(X, d),\ (X, \mathcal{B});\ \ (X', d'),\ (X', \mathcal{B}')$

$f : X \to X'$ measurable is called **Borel measurable**.

Corollary:

$(X, \mathcal{L})$ measurable space, (X', d') metric space and $(X', \mathcal{B}')$

$f : X \to X'$ is (Lebesgue) measurable $\iff$ $\forall$ open set $E \subseteq X'$ $\ f^{-1}(E) \in \mathcal{L}$

Proof.

It is a direct consequence of the preceding theorem, since $\sigma_0(\mathcal{C}') = \mathcal{B}'$ where $\mathcal{C}' = \{\text{open sets of } X'\}$

$\square$

Proposition:

$f : X \to X'$ continuous is Borel measurable (where X and X' are metric spaces, to define continuity)

Proof.

$\mathcal{C}$ open sets in X and $\mathcal{C}'$ open sets in X'

$$f \text{ continuous} \iff \forall\, E \in \mathcal{C}' \; f^{-1}(E) \in \mathcal{C}$$

$\mathcal{C} \subseteq \mathcal{B}$ and $\mathcal{B}' = \sigma_0(\mathcal{C}') \overset{\text{thm}}{\implies} f$ is Borel measurable

$\square$

Remark:

f Borel measurable $\implies f$ Lebesgue measurable, because $\mathcal{B} \subseteq \mathcal{L}$

Corollary:

f continuous $\implies$ f $\mathcal{L}$-measurable

Proposition:

$f : X \subseteq \mathbb{R} \to \mathbb{R}$ $\mathcal{L}$-measurable and $g : \mathbb{R} \to \mathbb{R}$ continuous $\implies g \circ f : X \to \mathbb{R}$ $\mathcal{L}$-measurable

Proof.

$$X \subseteq \mathbb{R} \;,\; (X, \mathcal{L}(\mathbb{R})) \;,\; (\mathbb{R}, \mathcal{B}(\mathbb{R}))$$

$$\forall E' \subseteq \mathbb{R} \text{ open } \; f^{-1}(E') \in \mathcal{L}(\mathbb{R})$$

$$\forall E \subseteq \mathbb{R} \text{ open } \; g^{-1}(E) \text{ is open, since g is continuous}$$

$$\implies \forall E \subseteq \mathbb{R} \;\; (g \circ f)^{-1}(E) = f^{-1}[\; \underbrace{g^{-1}(E)}_{\text{is open}=E'} \;] \in \mathcal{L}(\mathbb{R})$$

$\square$

Remark:

(i) Instead, if we assumed that $g : \mathbb{R} \to \mathbb{R}$ is $\mathcal{L}$-measurable, then codomain of f is $(X, \mathcal{B}(\mathbb{R}))$ and domain of g is $(\mathbb{R}, \mathcal{L}(\mathbb{R}))$. But $\mathcal{L}(\mathbb{R}) \supsetneq \mathcal{B}(\mathbb{R})$ and $f^{-1}[\underbrace{g^{-1}(E)}_{\in \mathcal{L}(\mathbb{R})}] \notin \mathcal{L}(\mathbb{R})$

(ii) f, g $\mathcal{L}$-measurable $\not\Rightarrow g \circ f$ is $\mathcal{L}$-measurable

Theorem: Lusin.

Let $f : E \subseteq \mathbb{R} \to \mathbb{R}$ be $\mathcal{L}$-measurable. Then $\forall \varepsilon > 0$ there are a continuous function $g : \mathbb{R} \to \mathbb{R}$ and $F \subset E$ closed for which $f = g$ in F, $\lambda(E \backslash F) < \varepsilon$

Theorem.

Let $f, g : \mathbb{R} \to \mathbb{R}$. If f is $\mathcal{L}$-measurable and $f = g$ a.e., then g is $\mathcal{L}$-measurable

3.1 Real valued functions

$\mathcal{M}(X, \mathcal{A}) := \{f : X \to \overline{\mathbb{R}} \text{ measurable}\}$ where domain and codomain are $(X, \mathcal{A})$, $(\overline{\mathbb{R}}, \mathcal{B}(\overline{\mathbb{R}}))$

$\mathcal{M}_+(X, \mathcal{A}) := \{f : X \to \overline{\mathbb{R}} \text{ measurable}, \ f \geq 0\}$

$\forall \alpha \in \mathbb{R}$ let

$$\{f > \alpha\} := \{x \in X \ : \ f(x) > \alpha\} = f^{-1}((+\alpha, +\infty])$$

$$\{f \geq \alpha\} := f^{-1}([\alpha, +\infty]) \qquad \{f < \alpha\} := f^{-1}([-\infty, \alpha)) \qquad \{f \leq \alpha\} := f^{-1}([-\infty, \alpha])$$

Theorem.

Let $(X, \mathcal{A})$ be a measurable space, $f : X \to \overline{\mathbb{R}}$

The following statements are equivalent:

(i) f is measurable

(ii) $\{f > \alpha\} \in \mathcal{A} \ \ \forall \alpha \in \mathbb{R}$

(iii) $\{f \geq \alpha\} \in \mathcal{A} \ \ \forall \alpha \in \mathbb{R}$

(iv) $\{f < \alpha\} \in \mathcal{A} \ \ \forall \alpha \in \mathbb{R}$

(v) $\{f \leq \alpha\} \in \mathcal{A} \ \ \forall \alpha \in \mathbb{R}$

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Proof.

(i)$\Longleftrightarrow$(ii)

$$\mathcal{A}' = \mathcal{B}(\overline{\mathbb{R}}) = \sigma_0(\{\underbrace{(\alpha, +\infty] \ : \ \alpha \in \mathbb{R}}_{\mathcal{C}'}\}) \qquad \text{(verified at exercise class)}$$

f is measurable iff (if and only if) $f^{-1}(\underbrace{(\alpha, +\infty]}_{E}) \in \mathcal{A} \ \ \forall \alpha \in \mathbb{R}$

(ii) $\Longrightarrow$ (iii)

$$\{f \geq \alpha\} = \bigcap_{n=1}^{\infty} \underbrace{\left\{f > \alpha - \frac{1}{n}\right\}}_{\in \mathcal{A} \text{ for (ii)}} \in \mathcal{A}$$

(iii) $\Longrightarrow$ (iv)

$$\{f < \alpha\} = \underbrace{\{\, f \geq \alpha \,\}}_{\in \mathcal{A} \text{ for (iii)}}^{c} \in \mathcal{A}$$

(iv) $\Longrightarrow$ (v)

$$\{f \leq \alpha\} = \bigcup_{n=1}^{\infty} \underbrace{\left\{f < \alpha + \frac{1}{n}\right\}}_{\in \mathcal{A} \text{ for (iv)}} \in \mathcal{A}$$

(v) $\Longrightarrow$ (ii)

$$\{f > \alpha\} = \underbrace{\{\, f \leq \alpha \,\}}_{\in \mathcal{A} \text{ for (v)}}^{c} \in \mathcal{A}$$

$\square$

Theorem.

Let $f, g : X \to \overline{\mathbb{R}}$ $\;f, g \in \mathcal{M}(X, \mathcal{A})$. Then,

(i) $\{f < g\} \in \mathcal{A}$

(ii) $\{f \leq g\} \in \mathcal{A}$

(iii) $\{f = g\} \in \mathcal{A}$

Proof.

$$(i) \qquad \{f < g\} = \bigcup_{r \in \mathbb{Q}} [\{\underbrace{f < r}_{\in \mathcal{A}}\} \cap \{\underbrace{r < g}_{\in \mathcal{A}}\}] \in \mathcal{A}$$

Here we used rational numbers, because the σ-algebra is closed only for countable unions. But with $\mathbb{Q}$ we can cover all the codomain, because $\mathbb{Q}$ is dense

$$(ii) \qquad f, g \in \mathcal{M} \overset{(i)}{\Longrightarrow} \{f > g\} \in \mathcal{A} \Longrightarrow \{f > g\}^c = \{f \leq g\} \in \mathcal{A}$$

$$(iii) \qquad \{f = g\} = \{f \geq g\} \cap \{f \leq g\} \in \mathcal{A}$$

$\square$

Theorem.

Let $\{f_n\} \subseteq \mathcal{M}(X, \mathcal{A})$. Then,

$$\sup_{n \in \mathbb{N}} f_n \, , \; \inf_{n \in \mathbb{N}} f_n \in \mathcal{M}(X, \mathcal{A})$$

Proof.

$$\forall\, \alpha \in \mathbb{R} \quad \{\sup_{n\in\mathbb{N}} f_n > \alpha\} = \bigcup_{n=1}^{\infty} \{f_n > \alpha\} \in \mathcal{A} \implies \sup_{n\in\mathbb{N}} f_n \in \mathcal{M}$$

$$\inf_{n\in\mathbb{N}} f_n = -\sup_{n\in\mathbb{N}} \{-f_n\} \in \mathcal{M}$$

$\square$

Corollary:

$$f, g \in \mathcal{M}(X, \mathcal{A}) \implies max\{f, g\}\,,\, min\{f, g\} \in \mathcal{M} \quad \text{and} \quad f_{\pm} \in \mathcal{M}$$

Obs. Because the maximum (minimum) is the supremum (infimum) between two functions and $f_+(_-)$ is the max (min) of f and 0

Theorem.

$\{f_n\} \subseteq \mathcal{M}(X, \mathcal{A})$. Then $\limsup_{n\to\infty} f_n\,,\, \liminf_{n\to\infty} f_n \in \mathcal{M}$

Proof.

$$\limsup_{n\to\infty} f_n = \inf_{k\geq 1}\left(\sup_{n\geq k} f_n\right) \in \mathcal{M} \qquad \liminf_{n\to\infty} f_n = -\limsup_{n\to\infty}\,(-f_n) \in \mathcal{M}$$

$\square$

Theorem.

$f, g\,:\, X \to \mathbb{R}$ and $f, g \in \mathcal{M}(X, \mathcal{A})$. Then,

$$f + g \in \mathcal{M} \qquad f \cdot g \in \mathcal{M}$$

Proof.

$$\varphi : X \to \mathbb{R}^2 \quad \varphi(x) := (f(x), g(x)) \;\; \forall\, x \in X \qquad \psi : \mathbb{R}^2 \to \mathbb{R} \quad \psi(s,t) := s + t \qquad \chi : \mathbb{R}^2 \to \mathbb{R} \quad \chi(s,t) := s \cdot t$$

$$\implies \begin{cases} \psi \circ \varphi = f + g \\ \chi \circ \varphi = f \cdot g \end{cases}$$

Claim: φ is measurable. In fact

$$\varphi : (X, \mathcal{A}) \to (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \text{ is measurable} \iff \forall\, E \subseteq \mathbb{R}^2 \text{ open } \varphi^{-1}(E) \in \mathcal{A}$$

Let $E = R := (a,b) \times (c,d)$

$$\varphi^{-1}(R) = \{x \in X \,:\, (f(x), g(x)) \in R\} = \{x \in X \,:\, f(x) \in (a,b)\} \cap \{x \in X \,:\, g(x) \in (c,d)\} =$$

$$= \underbrace{f^{-1}((a,b))}_{\in \mathcal{A}} \cap \underbrace{g^{-1}((c,d))}_{\in \mathcal{A}} \in \mathcal{A}$$

$$\forall\, E \subseteq \mathbb{R}^2 \text{ open} \qquad E = \bigcup_{k=1}^{\infty} R_k \text{ where } R_k = (a_k, b_k) \times (c_k, d_k) \qquad \varphi^{-1}(E) = \bigcup_{k=1}^{\infty} \varphi^{-1}(R_k) \in \mathcal{A}$$

$\psi, \chi \in C^0(\mathbb{R}^2)$ and $\psi, \chi : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R})) \implies \psi, \chi$ are $\mathcal{B}$-measurable.

Therefore, since the hypothesis of the theorem of composition are satisfied, $\psi \circ \varphi$ and $\chi \circ \varphi$ are measurable.

$\square$

Corollary:

Let $f : X \to \mathbb{R}$

(i) $f \in \mathcal{M}(X, \mathcal{A}) \iff f_\pm \in \mathcal{M}_+(X, \mathcal{A})$

(ii) $f \in \mathcal{M}(X, \mathcal{A}) \implies |f| \in \mathcal{M}(X, \mathcal{A})$

Proof.

(i) The first implication $(\implies)$ has already been done

$(\impliedby)$ $f = f_+ - f_- \in \mathcal{M}$ for the previous theorem

(ii) $f \in \mathcal{M} \overset{(i)}{\implies} f_+ , f_- \in \mathcal{M} \overset{thm}{\implies} |f| = f_+ + f_- \in \mathcal{M}$

$\square$

Lemma:

Let $A \subseteq X$. Then $\chi_A \in \mathcal{M} \iff A \in \mathcal{A}$

Proof.

We need a characterization in terms of superlevel: $$\{\chi_A > \alpha\} = \begin{cases} X & \text{if } \alpha < 0 \\ A & \text{if } \alpha \in [0,1) \\ \emptyset & \text{if } \alpha \geq 1 \end{cases}$$

Since X and $\emptyset$ always belong to $\mathcal{A}$ then $\chi_A \in \mathcal{M} \iff \{\chi_A > \alpha\} \in \mathcal{M} \iff A \in \mathcal{A}$ $\square$

Remark:

$$|f| \in \mathcal{M} \nRightarrow f \in \mathcal{M}$$

Example:

$$f(x) := \chi_E(x) - \chi_{E^c}(x) = \begin{cases} 1 & x \in E \\ -1 & x \in E^c \end{cases} \qquad \text{where } E \subseteq X \text{ , } E \notin \mathcal{A}$$

$|f| \equiv 1 \ \in \mathcal{M}(X, \mathcal{A})$ but $\{f > \frac{1}{2}\} = E \notin \mathcal{A} \implies f \notin \mathcal{M}$ because it exists a superlevel not included in the σ-algebra

3.2 Simple functions

Obs. With measurable functions we can define a new type of integral, but we need a few steps to do that

Definition.

Let X be a set. A function $S : X \to \mathbb{R}$ is said to be a **simple function** if its image $S(X)$ is finite:

$$S(X) = \{c_1, ..., c_n\} \ \ n \in \mathbb{N} \quad c_k \neq c_l \text{ for } k \neq l \quad c_i \in \mathbb{R} \ \forall i = 1, ..., n$$

$S = \sum_{k=1}^{n} c_k \chi_{E_k}$ is called canonical form of S, where $\{E_k\}_{k=1}^{n}$ is a partition of X:
$E_k := \{x \in X \ : \ S(x) = c_k\} \ \ k = 1, ..., n \qquad X = \bigcup_{k=1}^{n} E_k$ s.t. $E_k \cap E_l = \emptyset$ for $k \neq l$

Remark:

$S \in \mathcal{M}(X, \mathcal{A}) \iff E_k \in \mathcal{A} \ \ \forall k = 1, ..., n$

$\mathcal{S}(X, \mathcal{A}) := \{\text{measurable simple functions } f : X \to \mathbb{R}\} \subseteq \mathcal{M}$

$\mathcal{S}_+(X, \mathcal{A}) := \{\text{measurable simple functions } f : X \to \mathbb{R} \text{ , } f \geq 0 \text{ in } X\} \subseteq \mathcal{M}_+$

Remark:

If E_k are intervals, then $f : I \to \mathbb{R}$ is a **step function**

$$f := \sum_{l=0}^{n-1} c_l \, \chi_{[x_l, x_{l+1})} \ \text{ where } a_0 = x_0 < x_1 < ... < x_n = a_1 \text{ and } c_i \in \mathbb{R} \ \ i = 0, ..., n-1$$

Theorem: The simple approximation theorem.

Let $(X, \mathcal{A})$ be a measurable space, $f : X \to \overline{\mathbb{R}}$. Then there exists a sequence $\{S_n\}$ of simple functions s.t.

$$S_n \xrightarrow[n\to\infty]{} f \text{ in } X$$

Furthermore,

(i) if $f \in \mathcal{M}(X, \mathcal{A})$, then $\{S_n\} \subseteq \mathcal{S}(X, \mathcal{A})$

(ii) $f \geq 0 \implies \{S_n\} \nearrow, \ 0 \leq S_n \leq f$

(iii) f bounded $\implies S_n \xrightarrow[n\to\infty]{} f$ uniformly in X

Proof. (sketch)

$f \geq 0, \ f$ bounded, $\ 0 \leq f(x) \leq 1 \ \ \forall x \in X, \quad f : X \to [0, 1]$

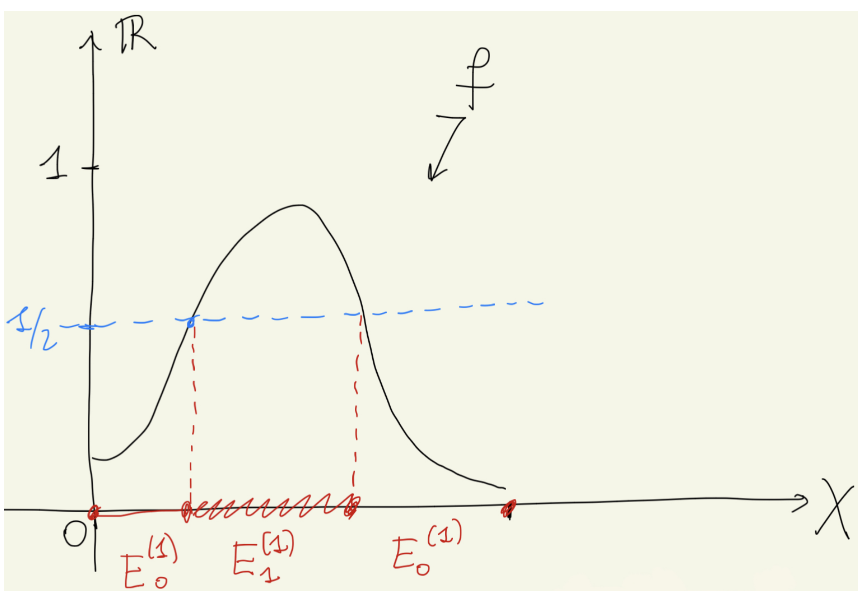
Divide $[0, 1]$ in 2^n intervals of length $2^{-n} \ \ \forall n \in \mathbb{N}$. Then define

$$E_k^{(n)} := \left\{ x \in X \ : \ \frac{k}{2^n} \leq f(x) < \frac{k+1}{2^n} \right\} \qquad k = 0, ..., 2^n - 1$$

Define $S_n := \sum_{k=0}^{2^n-1} \frac{k}{2^n} \chi_{E_k^{(n)}}$ $\quad \{S_n\}$ has the desired properties

For example with $n = 1$ we have $2^1 - 1 = 1$ so $k = 0, 1$

$$E_0^{(1)} = \{x \in X \ : \ 0 \leq f(x) < \tfrac{1}{2}\} \qquad E_1^{(1)} = \{x \in X \ : \ \tfrac{1}{2} \leq f(x) < 1\}$$



$\square$

3.3 Essentially bounded functions

Lezione 7 (28/09/23)

$(X, \mathcal{A}, \mu)$ measure space

$\forall\, N \in \mathcal{N}_\mu$ (collection of null sets). We define $\alpha_N := \sup\limits_{x \in N^c} f(x)$

$$N_1, N_2 \in \mathcal{N}_\mu,\ N_2 \subseteq N_1 \implies \alpha_{N_1} \leq \alpha_{N_2}$$

Definition.

Essential sup $\quad \underset{X}{ess\,sup}\, f := inf \left\{ \sup\limits_{x \in N^c} f(x) \ : \ N \in \mathcal{N}_\mu \right\}$

Essential inf $\quad \underset{X}{ess\,inf}\, f := sup \left\{ \inf\limits_{x \in N^c} f(x) \ : \ N \in \mathcal{N}_\mu \right\}$

Proposition:

Let $f \in \mathcal{M}(X, \mathcal{A})$. Then $\exists\, N \in \mathcal{N}_\mu$ s.t.

$$\underset{X}{ess\,sup}\, f = \sup_{x \in N^c} f(x)$$

Moreover,

$$f(x) \leq \underset{X}{ess\,sup}\, f \quad a.e. \text{ (almost everywhere) } \ \text{in } X$$

Proof.

Suppose $\underset{X}{ess\,sup}\, f < \infty$

$$\forall\, k \in \mathbb{N} \quad \exists\, N_k \in \mathcal{N}_\mu \ \text{ s.t. } \sup_{x \in N_k^c} f < \underset{X}{ess\,sup}\, f + \frac{1}{k}$$

Define $N := \bigcup_{k=1}^{\infty} N_k$ countable union of measurable and null sets $\implies N \in \mathcal{N}_\mu$

$N^c = \bigcap_{k=1}^{\infty} N_k^c \subseteq N_k^c \ \ \forall\, k \in \mathbb{N}$

$$\implies \underset{X}{ess\,sup}\, f \leq \sup_{N^c} f \leq \sup_{N_k^c} f < \underset{X}{ess\,sup}\, f + \frac{1}{k}$$

$$\implies \sup_{N^c} f = \underset{X}{ess\,sup}\, f$$

$$\overline{N} := \{x \in X \ : \ f(x) > \underset{X}{ess\,sup}\, f\} \subseteq N \quad \text{because } f \leq \sup_{N^c} f \ in \ N^c$$

$$\implies \overline{N} \in \mathcal{N}_\mu \implies f \leq \underset{X}{ess\,sup}\, f \ a.e. \ in \ X, \text{ or rather in } \overline{N}^c$$

□

Proposition:

$f \in \mathcal{M}(X, \mathcal{A})$. Then,

i) $\underset{X}{ess\,sup}\, f = -\underset{X}{ess\,inf}\,(-f)$

ii) $\underset{X}{ess\,sup}\,(kf) = k\,\underset{X}{ess\,sup}\, f \quad \forall\, k \geq 0$

Proposition:

$f, g \in \mathcal{M}(X, \mathcal{A})$. Then,

i) $f \leq g \ a.e. \implies \underset{X}{ess\,sup}\, f \leq \underset{X}{ess\,sup}\, g$

ii) $\underset{X}{ess\,sup}\,(f + g) \leq \underset{X}{ess\,sup}\, f + \underset{X}{ess\,sup}\, g$

iii) $f = g \ a.e. \ in \ X \implies \underset{X}{ess\,sup}\, f = \underset{X}{ess\,sup}\, g$

iv) $g \geq 0 \ a.e. \ in \ X \implies fg \leq (\underset{X}{ess\,sup}\, f) g \ a.e. \ in \ X$

Remark:

$A \in \mathcal{L}(\mathbb{R}) \quad f : A \subseteq \mathbb{R} \to \mathbb{R}$ continuous $\implies \underset{A}{ess\,sup}\, f = \sup_A f$

Definition.

We say that $f \in \mathcal{M}(X, \mathcal{A})$ is **essentially bounded** in X, if $\underset{X}{ess\,sup}\, |f| < \infty$

$\mathcal{L}^\infty(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} \ s.t. \ f$ is essentially bounded$\}$

Obs. Remember that $L^\infty \neq \mathcal{L}^\infty$ the first is a set of classes of equivalence, the second a set of functions

Remark:

(i) $f \in \mathcal{L}^\infty \implies f$ finite a.e. in X because $|f| \leq \underset{X}{ess\,sup}\, |f| < \infty$ a.e. in X

(ii) f is finite in X $\not\Longrightarrow f \in \mathcal{L}^\infty$

Counter example: $f : \mathbb{R} \to \overline{\mathbb{R}} \quad f(x) := \begin{cases} \dfrac{1}{|x|}\ , \ x \neq 0 \\ +\infty\ , \ x = 0 \end{cases}$

f is finite in $E = \mathbb{R} \backslash \{0\}$ and since $\lambda(\{0\}) = 0$ then f is finite a.e. in $\mathbb{R}$

But $\underset{X}{ess\,sup}\, f = +\infty \implies f \notin \mathcal{L}^\infty$

3.4 The Lebesgue integral

3.4.1 Non negative measurable simple functions

Obs. We define the integral in general, not only on $\mathbb{R}$.

Let $(X, \mathcal{A}, \mu)$ be a measure space.

$s \in \mathcal{S}_+(X, \mathcal{A}) \quad s = \sum_{k=1}^{n} c_k \chi_{E_k}$ canonical form $c_1, ..., c_n \in \mathbb{R}_+$, $\{E_k\}$ partition of X $\quad s = \begin{cases} c_1 & in\ E_1 \\ \vdots \\ c_n & in\ E_n \end{cases}$

Definition.

The **integral** of a non negative measurable **simple function** is:

$$\int_X s \, d\mu := \sum_{k=1}^{n} c_k \mu(E_k)$$

If $E \in \mathcal{A}$, we set $\displaystyle\int_E s \, d\mu := \int_X s\chi_E \, d\mu$

Remark:

i) $s\chi_E = \sum_{k=1}^{n} c_k \chi_{E_k \cap E} \implies \displaystyle\int_E s \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k \cap E)$

ii) $\displaystyle\int_X \chi_E \, d\mu = \mu(E) \quad \forall\, E \in \mathcal{A}$

iii) $\displaystyle\int_N s \, d\mu = 0 \quad \forall\, N \in \mathcal{N}_\mu$

Proof.

(ii)

$E_1 = E,\ E_2 = E^c \quad c_1 = 1,\ c_2 = 0 \quad \chi_E(x) = \sum_{k=1}^{2} c_k \chi_{E_k}(x)$

$$\implies \int_X \chi_E \, d\mu = c_1\mu(E_1) + c_2\mu(E_2) = \mu(E)$$

(iii)

$$\int_N s \, d\mu = \sum_{k=1}^{n} c_k \mu(E_k \cap N) = 0 \quad \text{because } E_k \cap N \subseteq N \text{ and } E_k \cap N \in \mathcal{A}$$

$\square$

Proposition:

i) Let $s \in \mathcal{S}_+(S, \mathcal{A})$, $c \geq 0$. Then,

$$\int_X c\, s \, d\mu = c \int_X s \, d\mu$$

ii) $s, t \in \mathcal{S}_+(X, \mathcal{A}) \implies \displaystyle\int_X (s + t) \, d\mu = \int_X s \, d\mu + \int_X t \, d\mu$

iii) $s, t \in \mathcal{S}_+(X, \mathcal{A})$, $s \leq t \implies \displaystyle\int_X s \, d\mu \leq \int_X t \, d\mu$

iv) $s \in \mathcal{S}_+(X, \mathcal{A})$, $E, F \subseteq X$ $s.t.$ $E \subseteq F \implies \displaystyle\int_E s \, d\mu \leq \int_F s \, d\mu$

Proposition:

Let $s \in \mathcal{S}_+(X, \mathcal{A})$, $\forall\, E \in \mathcal{A}$

$$\varphi(E) := \int_E s\, d\mu \quad \varphi : \mathcal{A} \to \overline{\mathbb{R}}_+ \text{ is a measure}$$

Proof.

$$\varphi(\emptyset) = \int_\emptyset s\, d\mu = 0 \quad \text{since } \mu(\emptyset) = 0$$

σ-additivity: let $E_k \in \mathcal{A}$ disjoint (with $k \in \mathbb{N}$) $\quad s = \sum\limits_{l=1}^{m} d_l \chi_{F_l}$ $(F_l \in \mathcal{A})$

$$\varphi(E) = \sum_{l=1}^{m} d_l \mu(F_l \cap E) \overset{\mu \text{ measure}}{=} \sum_{l=1}^{m} \sum_{k=1}^{\infty} d_l \mu(F_l \cap E_k) = \sum_{k=1}^{\infty} \sum_{l=1}^{m} d_l \mu(F_l \cap E) = \sum_{k=1}^{\infty} \int_{E_k} s\, d\mu$$

□

3.4.2 Non negative measurable functions

Obs. To generalize to measurable functions is important to use the simple approximation theorem, any measurable function can be approximated with simple functions. In this way we can calculate the integral.

Definition.

Given $f : X \to \overline{R}_+$ *s.t.* $f \in \mathcal{M}_+(X, \mathcal{A})$. The **integral** of a non negative **measurable function** is:

$$\int_X f\, d\mu := \sup_{s \in \mathcal{S}_f} \int_X s\, d\mu$$

Where $\mathcal{S}_f := \{s \in \mathcal{S}_+(X, \mathcal{A}) \,:\, s \le f \text{ in } X\}$

If $E \in \mathcal{A}$, we set $\displaystyle\int_E f\, d\mu := \int_X f \chi_E\, d\mu$

Remark:

$\mathcal{S}_f \neq \emptyset$ for the simple approximation theorem $\exists\, \{s_n\} \subseteq \mathcal{S}_f$ *s.t.* $s_n \le s_{n+1}\ \forall n \in \mathbb{N}$ and $s_n \xrightarrow[n\to\infty]{} f \text{ in } X$

Remark:

It is also possible to define $\displaystyle\int_X f\, d\mu := \lim_{n\to\infty} \int_X s_n\, d\mu \quad \left(\text{where } \int_X f\, d\mu \text{ is independent of } \{s_n\} \right)$

Properties:

- All the properties of the simple functions integrals

- $\displaystyle\int_N f\,d\mu = 0 \;\; \forall f \in \mathcal{M}_+(X,\mathcal{A}) \;\; \forall N \in \mathcal{N}_\mu$ because $\displaystyle\int_N s\,d\mu = 0 \;\; \forall s \in \mathcal{S}_+(X,\mathcal{A})$

Proposition: Chebyshev inequality

Let $f \in \mathcal{M}_+(X,\mathcal{A})$. Then $\forall\, c > 0$

$$\mu(\{f \geq c\}) \leq \frac{1}{c}\int_{\{f\geq c\}} f\,d\mu \leq \frac{1}{c}\int_X f\,d\mu$$

Proof.

$E_c := \{f \geq c\} \in \mathcal{A}$

$$c\chi_{E_c} \leq f\chi_{E_C}$$

$$\int_X f\,d\mu \overset{f\text{ non neg.}}{\geq} \int_{E_c} f = \int_X f\chi_{E_c}\,d\mu \geq \int_X c\chi_{E_c}\,d\mu = c\int_X \chi_{E_c}\,d\mu = c\mu(E_c)$$

□

Proposition:

Let $f \in \mathcal{M}_+(X,\mathcal{A})$ s.t. $\displaystyle\int_X f\,d\mu < \infty$ (is finite). Then f is finite a.e. in X

Proof.

$\mu(\{f = \infty\}) = 0 \iff$ thesis

$$\{f = \infty\} = \bigcap_{n=1}^{\infty}\{f > n\} \qquad \text{Let } E_n := \{f > n\}$$

a) $\{E_n\} \searrow$

b) $\mu(E_n) \overset{Cheb.\,in.}{\leq} \dfrac{1}{n}\displaystyle\int_X f\,d\mu \;\; \forall n \in \mathbb{N}$

(b) $\implies \mu(E_1) < \infty$ and together with (a) $\implies \mu(E_n) < \infty$ this is necessary to swap limit and intersection.

$$\mu(\{f = \infty\}) = \mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty}\mu(E_n) \overset{(b)}{=} 0$$

□

Lezione 8 (04/10/23)

Lemma: Vanishing lemma

Let $f \in \mathcal{M}_+(X,\mathcal{A})$ be s.t. $\displaystyle\int_X f\,d\mu = 0$. Then $f = 0$ a.e. in X

Proof.

Thesis $\iff \mu(\{f>0\})=0$

$$\{f>0\}=\bigcup_{n=1}^{\infty}\{f\geq\frac{1}{n}\}\qquad\text{Let } F_n:=\{f\geq\frac{1}{n}\}$$

$$\{F_n\}\nearrow\qquad \frac{1}{n}\chi_{F_n}\leq f\chi_{F_n}$$

$$\mu(F_n)\overset{Cheb.\,in.}{\leq}\frac{1}{\frac{1}{n}}\int_X f\,d\mu=0\ \ \forall\,n\in\mathbb{N}$$

$$\text{Therefore,}\ \ \mu(\{f>0\})=\mu\left(\bigcup_{n=1}^{\infty}F_n\right)=\lim_{n\to\infty}\mu(F_n)=0$$

□

3.4.3 Monotone convergence theorem

Theorem: Monotone convergence theorem, Beppo Levi.

Let $\{f_n\}\subseteq\mathcal{M}_+(X,\mathcal{A}),\ f:X\to\overline{\mathbb{R}}_+$ be s.t.

i) $f_n\leq f_{n+1}\ in\ X\ \ \forall\,n\in\mathbb{N}$

ii) $f_n\xrightarrow[n\to\infty]{} f$ pointwise in X

Then,

$$\lim_{n\to\infty}\int_X f_n\,d\mu=\int_X\left(\lim_{n\to\infty}f_n\right)\,d\mu\quad\left[=\int_X f\,d\mu\right]$$

Obs. Here we can appreciate the importance of the Lebesgue integral, for which you can pass to the limit under integral without needing uniform convergence.

Proof.

$f\in\mathcal{M}_+(X,\mathcal{A})$

$$\int_X f_n\,d\mu\leq\int_X f_{n+1}\,d\mu\leq\int_X f\,d\mu\quad\text{by monotonicity}$$

So we know that this is an increasing, bounded sequence, it exists the limit

$$\alpha:=\lim_{n\to\infty}\int_X f_n\,d\mu\ \ \text{and}\ \ \alpha\leq\int_X f\,d\mu$$

Claim: $\alpha\geq\int_X f\,d\mu$

In fact, $\forall\,\varepsilon\in(0,1)$, $\forall\,s\in\mathcal{S}_f$ let $E_n:=\{(1-\varepsilon)s\leq f_n\}\ \ \forall\,n\in\mathbb{N}$

We have that:

(a) $\{E_n\}\subseteq\mathcal{A}$

(b) $\{E_n\}\nearrow$ because $\{f_n\}\nearrow$

(c) $X=\bigcup_{n=1}^{\infty}E_n$ (discussed below)

Indeed, $\bigcup_{n=1}^{\infty}E_n\subseteq X$ and we need to show that $X\subseteq\bigcup_{n=1}^{\infty}E_n$. Let $x\in X$.

If $f(x) = +\infty$, then by definition of limit

$$\exists \overline{n} \in \mathbb{N} \; s.t. \; \forall n > \overline{n} \;\; (1-\varepsilon)s(x) < f_n(x) \implies x \in E_n \;\; \forall \, n > \overline{n} \implies x \in \bigcup_{n=1}^{\infty} E_n$$

Instead if $f(x) < +\infty$, then

$$\exists \overline{n} \in \mathbb{N} \; s.t. \; \forall n > \overline{n} \;\; (1-\varepsilon)s(x) \overset{s \in \mathcal{S}_f}{\leq} (1-\varepsilon)f(x) < f_n \implies x \in E_n \;\; \forall \, n > \overline{n} \implies x \in \bigcup_{n=1}^{\infty} E_n$$

Thus $X \subseteq \bigcup_{n=1}^{\infty} E_n$ and $X = \bigcup_{n=1}^{\infty} E_n$

It follows that

$$(1-\varepsilon) \int_{E_n} s \, d\mu \leq \int_{E_n} f_n \, d\mu \leq \int_X f \, d\mu$$

$$\text{for } n \to \infty \; E_n \to X \implies (1-\varepsilon) \int_X s \, d\mu \leq \lim_{n \to \infty} \int_X f_n \, d\mu = \alpha$$

$\varepsilon > 0$ was arbitrary, hence we can let ε go to zero

$$\int_X s \, d\mu \leq \alpha \implies \sup_{s \in \mathcal{S}_f} \int_X s \, d\mu \leq sup \, \alpha \implies \int_X f \, d\mu \leq \alpha$$

$\square$

Remark:

$\varphi(E_n) := \int_{E_n} s \, d\mu \quad$ with $E_n \nearrow$

$$\lim_{n \to \infty} \varphi(E_n) = \varphi\left(\bigcup_{n=1}^{\infty} E_n\right) = \varphi(X) = \int_X s \, d\mu$$

Obs. If you have a decreasing sequence the demonstration is different, we'll see it in an exercise.

Lemma: Fatou's lemma

Let $\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$. Then,

$$\underset{n \to \infty}{liminf} \int_X f_n \, d\mu \geq \int_X \left(\underset{n \to \infty}{liminf} \, f_n \right) d\mu$$

Proof.

$$\underset{n \to \infty}{liminf} \, f_n \in \mathcal{M}_+(X, \mathcal{A})$$

$$g_k : X \to \overline{\mathbb{R}}_+ \quad g_k := \inf_{n \geq k} f_n$$

Properties of $\{g_k\}$:

(a) $\{g_k\} \subseteq \mathcal{M}_+(X, \mathcal{A})$, $g_k \nearrow$

(b) $g_k \leq f_k \quad \forall\, k \in \mathbb{N}$

(c) $\lim\limits_{k\to\infty} g_k = \sup\limits_{k\geq 1} g_k = \operatorname*{liminf}\limits_{n\to\infty} f_n$

$$(b) \implies \int_X g_k \, d\mu \leq \int_X f_k \, d\mu \quad \forall\, k \in \mathbb{N}$$

$$\implies \operatorname*{liminf}_{k\to\infty} \int_X g_k \, d\mu \leq \operatorname*{liminf}_{k\to\infty} \int_X f_k \, d\mu$$

Since $\{g_k\}$ is increasing then also the sequence $\left\{\int_X g_k \, d\mu\right\}$ is increasing so

$$\operatorname*{liminf}_{k\to\infty} \int_X g_k \, d\mu = \lim_{k\to\infty} \int_X g_k \, d\mu \overset{MCT}{=} \int_X \left(\lim_{k\to\infty} g_k\right) \, d\mu \overset{(c)}{=} \int_X \left(\operatorname*{liminf}_{n\to\infty} f_n\right) \, d\mu$$

$$\implies \operatorname*{liminf}_{k\to\infty} \int_X f_k \, d\mu \leq \int_X \left(\operatorname*{liminf}_{n\to\infty} f_n\right) \, d\mu$$

□

Remark: In some cases you can get the strict inequality. Example:

$$(X, \mathcal{A}, \mu) = (\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^*) \quad f_n := \chi_{\{n\}} \quad f_n(m) = \begin{cases} 1 & m = n \\ 0 & n \neq m \end{cases} \quad \forall\, m \in \mathbb{N}$$

$$\lim_{n\to\infty} f_n = 0 \implies \int_{\mathbb{N}} \left(\operatorname*{liminf}_{n\to\infty} f_n\right) \, d\mu^* = 0$$

$$\forall\, n \in \mathbb{N} \quad \int_{\mathbb{N}} f_n \, d\mu^* = 1 \implies \operatorname*{liminf}_{n\to\infty} \int_{\mathbb{N}} f_n \, d\mu^* = 1 > 0$$

Theorem: Integration of series.

$\{f_n\} \subseteq \mathcal{M}_+(X, \mathcal{A})$. Then,

$$\int_X \left(\sum_{n=1}^{\infty} f_n\right) \, d\mu = \sum_{n=1}^{\infty} \int_X f_n \, d\mu$$

Obs. This is a corollary of Beppo Levi easy to demonstrate.

Theorem.

Let $f \in \mathcal{M}_+(X, \mathcal{A})$. Then,

i) $\nu : \mathcal{A} \to \overline{\mathbb{R}}_+ \quad \nu(E) := \int_E f \, d\mu \quad (E \in \mathcal{A})$ is a measure.

ii) Let $g \in \mathcal{M}_+(X, \mathcal{A})$. Then (we can see f as the density of the measure)

$$\int_X g \, d\nu = \int_X gf \, d\mu$$

Proof.

i) $\nu(\emptyset) = 0$, since $\mu(\emptyset) = 0$

Let $\{E_k\} \subseteq \mathcal{A}$ disjoint, $E = \bigcup_{k=1}^{\infty} E_k$. Then,

$$\nu(E) = \int_X f\chi_E \, d\mu = \int_X f \sum_{k=1}^{\infty} \chi_{E_k} \, d\mu \overset{MCT}{=} \sum_{k=1}^{\infty} \int_X f\chi_{E_k} \, d\mu = \sum_{k=1}^{\infty} \nu(E_k)$$

ii) Let $g \equiv s \in \mathcal{S}_+(X, \mathcal{A}) \quad s = \sum\limits_{k=1}^{n} c_k \chi_{F_k} \quad \{F_k\} \subseteq \mathcal{A}$ disjoint, $X = \bigcup_{k=1}^{n} F_k$. Then,

$$\int_X s \, d\nu = \sum_{k=1}^{n} c_k \, \nu(F_k) = \sum_{k=1}^{n} c_k \int_{F_k} f \, d\mu = \int_X \left(\sum_{k=1}^{n} c_k f \chi_{F_k} \right) d\mu = \int_X sf \, d\mu$$

If $g \in \mathcal{M}_+(X, \mathcal{A})$, then we get the thesis by approximation.

$\square$

Remark:

i) $\forall \, E \in \mathcal{A} \quad \mu(E) = 0 \implies \nu(E) = 0$

ii) $''d\nu = f \, d\mu''$

3.4.4 Null sets and integrals

Theorem.

$f, g \in \mathcal{M}_+(X, \mathcal{A})$ s.t. $f = g$ *a.e. in* X. Then,

$$\int_X f \, d\mu = \int_X g \, d\mu$$

Proof.

$$N := \{f \neq g\} \quad N \in \mathcal{A}, \; \mu(N) = 0$$

$$\int_N f \, d\mu = \int_N g \, d\mu = 0$$

$$\int_X f \, d\mu = \underbrace{\int_N f \, d\mu}_{0} + \int_{N^c} f \, d\mu = \int_{N^c} f \, d\mu = \int_{N^c} g \, d\mu = \underbrace{\int_N g \, d\mu}_{0} + \int_{N^c} g \, d\mu = \int_X g \, d\mu$$

$\square$

Corollary:

$f \in \mathcal{M}_+(X, \mathcal{A})$. Then,

(i) $\displaystyle\int_N f \, d\mu = 0 \iff$ (ii) $f = 0 \; a.e. \; in \; X$

Proof.

(i) $\implies$ (ii) already done

(ii) $\implies$ (i)

We take $g = 0 \; in \; X \implies \displaystyle\int_X g \, d\mu = 0$ and $f = g \; a.e. \; in \; X \quad \implies \displaystyle\int_X f \, d\mu = \int_X g \, d\mu = 0$

□

3.4.5 Integrable functions

Lezione 9 (05/10/23)

Let $(X, \mathcal{A}, \mu)$ be a measure space and $f : X \to \overline{\mathbb{R}}$

Definition.

$f : X \to \overline{\mathbb{R}}$ is said to be **integrable** on X, if $f \in \mathcal{M}(X, \mathcal{A})$ and $\displaystyle\int_X f_+ \, d\mu < \infty$, $\displaystyle\int_X f_- \, d\mu < \infty$

Where $f = f_+ - f_-$ and $f_+, f_- \in \mathcal{M}_+$

$\mathcal{L}^1(X, \mathcal{A}, \mu) := \{f : X \to \overline{\mathbb{R}} \text{ integrable in } X\}$

Definition.

Let $f \in \mathcal{L}^1$. Then the **Lebesgue integral** of f on X is

$$\int_X f \, d\mu := \int_X f_+ \, d\mu - \int_X f_- \, d\mu$$

Moreover, if $E \in \mathcal{A}$, we set

$$\int_E f \, d\mu := \int_X f \chi_E \, d\mu = \int_E f_+ \, d\mu - \int_E f_- \, d\mu = \int_X f_+ \chi_E \, d\mu - \int_X f_- \chi_E \, d\mu$$

Proposition:

$f : X \to \overline{\mathbb{R}}$. Then,

i) $f \in \mathcal{L}^1 \iff f_\pm \in \mathcal{L}^1$

ii) $f \in \mathcal{L}^1 \iff f \in \mathcal{M}$ and $|f| \in \mathcal{L}^1$

iii) $f \in \mathcal{L}^1 \implies \left| \displaystyle\int_X f \, d\mu \right| \leq \int_X |f| \, d\mu$

Proof.

i) $f \in \mathcal{L}^1 \overset{def}{\iff} f \in \mathcal{M}\ ,\ \int_X f_\pm \, d\mu < \infty$

$f_\pm \in \mathcal{L}^1 \overset{def}{\iff} f_\pm \in \mathcal{M}\ ,\ \int_X f_\pm \, d\mu < \infty$

(these are the integrals of $(f_+)_+$ and $(f_-)_-$ and the other integrals are given since $(f_+)_- = (f_-)_+ = 0$)

Since $f \in \mathcal{M} \iff f_\pm \in \mathcal{M}$ we have the thesis.

ii) $f \in \mathcal{L}^1 \overset{def}{\iff} f \in \mathcal{M}\ ,\ \int_X f_\pm \, d\mu < \infty$

$f \in \mathcal{M} \implies |f| \in \mathcal{M}$

$|f| \in \mathcal{L}^1 \iff \int_X |f| \, d\mu < \infty \quad \text{since } |f|_+ = |f| \text{ and } |f|_- = 0$

$(\implies)$

$$f \in \mathcal{L}^1 \implies \int_X |f| \, d\mu = \underbrace{\int_X f_+ \, d\mu}_{<\infty} + \underbrace{\int_X f_- \, d\mu}_{<\infty} < \infty$$

$$\implies f \in \mathcal{M}\ ,\ |f| \in \mathcal{L}^1$$

$(\impliedby)$

$f \in \mathcal{M}, |f| \in \mathcal{L}^1$

$$\implies \int_X f_+ \, d\mu + \int_X f_- \, d\mu = \int_X |f| \, d\mu < \infty \implies \int_X f_+ \, d\mu < \infty\ ,\ \int_X f_- \, d\mu < \infty \implies f \in \mathcal{L}^1$$

iii) $$\left| \int_X f \, d\mu \right| = \left| \int_X f_+ \, d\mu - \int_X f_- \, d\mu \right| \leq \int_X f_+ \, d\mu + \int_X f_- \, d\mu = \int_X \underbrace{f_+ + f_-}_{|f|} \, d\mu = \int_X |f| \, d\mu$$

$\square$

With property (ii) we can define $\mathcal{L}^1 := \{ f : X \to \overline{\mathbb{R}} \ :\ f \text{ is measurable and } \int_X |f|^1 \, d\mu < \infty \}$

Proposition:

$\mathcal{L}^1(X, \mathcal{A}, \mu)$ is a vector space

Proof.

Let $f, g \in \mathcal{L}^1$, $\lambda \in \mathbb{R}$

$\implies f_\pm\ ,\ g_\pm$ finite a.e. in X, since they are non negative functions

$\implies f\ ,\ g$ finite a.e. in X

$\implies h := f + \lambda g$ can be defined a.e. in X (it couldn't if we had non finite functions: $\infty - \infty$ is not defined)

$$\int_X |h| \, d\mu \leq \int_X |f| \, d\mu + |\lambda| \int_X |g| \, d\mu < \infty \implies h \in \mathcal{L}^1$$

$\square$

Theorem.

$f, g \in \mathcal{L}^1$, $\lambda \in \mathbb{R}$.Then,

i) $\displaystyle\int_X (f+g)\,d\mu = \int_X f\,d\mu + \int_X g\,d\mu$

ii) $\displaystyle\int_X \lambda f = \lambda \int_X f\,d\mu$

Proposition:

Let $f \in \mathcal{L}^1(X, \mathcal{A}, \mu)$ be s.t.

$$\int_E f\,d\mu = 0 \ \ \forall E \in \mathcal{A} \quad \Longrightarrow \ f = 0 \ a.e. \ in \ X$$

Obs. Compared to the corollary we saw previously, here we took out the sign request ($f \in \mathcal{M}_+$), but we are asking the integral to be null over any set, that is a stronger condition.

Proof.

$E_+ := \{f \geq 0\}$, $E_- := \{f \leq 0\} \ \in \mathcal{A}$

$$\int_{E_+} f\,d\mu = 0 \ , \ \int_{E_-} f\,d\mu = 0 \ \ \overset{\text{vanishing lemma}}{\Longrightarrow} \ f = 0 \ a.e. \ in \ E_+ \text{ and } E_-$$

$$\Longrightarrow \ f = 0 \ a.e. \ in \ E_+ \cup E_- = X$$

$\square$

Theorem.

Let $f \in \mathcal{L}^1$, $g \in \mathcal{M}$, $f = g \ a.e. \ in \ X$. Then,

$$g \in \mathcal{L}^1 \ \text{ and } \ \int_X f\,d\mu = \int_X g\,d\mu$$

Proof.

$f_+ = g_+ \geq 0$, $f_- = g_- \geq 0 \quad a.e. \ in \ X$

$$\Longrightarrow \ \int_X f_+\,d\mu = \int_X g_+\,d\mu \ \ , \ \ \int_X f_-\,d\mu = \int_X g_-\,d\mu$$

$\square$

3.4.6 Dominated convergence theorem

Theorem: Lebesgue (or dominated) convergence theorem.

Let $\{f_n\} \subseteq \mathcal{M}(X,\mathcal{A}),\ f \in \mathcal{M}(X,\mathcal{A})$ s.t. $f_n \xrightarrow[n\to\infty]{} f$ *a.e. in* X

Suppose that there exists $g \in \mathcal{L}^1(X,\mathcal{A},\mu)$ s.t. $|f_n| \le g$ *a.e. in* X $\forall\, n \in \mathbb{N}$. Then,

$$f_n,\, f \in \mathcal{L}^1 \quad \text{and} \quad \int_X |f_n - f|\, d\mu \xrightarrow[n\to\infty]{} 0$$

In particular,

$$\lim_{n\to\infty} \int_X f_n\, d\mu = \int_X \left(\lim_{n\to\infty} f_n\right) d\mu = \int_X f\, d\mu$$

Proof.

$|f_n| \le g$ *a.e. in* X $\forall\, n \in \mathbb{N} \implies \int_X |f_n|\, d\mu \le \int_X g\, d\mu < \infty$

$|f_n| \le g$ *a.e. in* X $\forall\, n \in \mathbb{N} \overset{n\to\infty}{\implies} |f| \le g$ *a.e. in* $X \implies \int_X |f|\, d\mu \le \int_X g\, d\mu < \infty$

$$\implies f_n\ ,\ f \in \mathcal{L}^1 \implies f_n\ ,\ f \quad \text{finite } \textit{a.e. in } X$$

Define $g_n := 2g - |f_n - f|$ $(n \in \mathbb{N})$

$|f_n - f| \le |f_n| + |f| \le 2g$ *a.e. in* X , $\forall\, n \in \mathbb{N}$

So, $g_n \ge 0$ *a.e. in* X , $\forall\, n \in \mathbb{N}$

$$\implies g_n \in \mathcal{M}_+ \quad \text{(sum of measurable functions)}$$

$$2\int_X g\, d\mu = \int_X \underbrace{\left(\lim_{n\to\infty} g_n\right)}_{\text{since } f_n \to f} d\mu \overset{\text{Fatous's lemma}}{\le} \liminf_{n\to\infty} \int_X g_n\, d\mu =$$

$$= \liminf_{n\to\infty} \int_X [2g - |f_n - f|]\, d\mu = \int_X 2g\, d\mu + \liminf_{n\to\infty}\left(-\int_X |f_n - f|\, d\mu\right)$$

$$= \int_X 2g\, d\mu - \limsup_{n\to\infty} \int_X |f_n - f|\, d\mu \implies \limsup_{n\to\infty} \int_X |f_n - f|\, d\mu \le 0$$

$$\int_X |f_n - f|\, d\mu \ge 0 \implies \liminf_{n\to\infty} \int_X |f_n - f|\, d\mu \ge 0$$

$$\implies \lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0$$

$$0 \le \left|\int_X f_n\, d\mu - \int_X f\, d\mu\right| \le \left|\int_X f_n - f\, d\mu\right| \le \int_X |f_n - f|\, d\mu \xrightarrow[n\to\infty]{} 0$$

$$\implies \int_X f_n\, d\mu \xrightarrow[n\to\infty]{} \int_X f\, d\mu$$

$\square$

Remark:

If $\begin{cases} \mu(X) < \infty \\ \exists\, M > 0 \,:\, |f_n| \le M \text{ } a.e. \text{ } in \text{ } X\,, \quad \forall n \in \mathbb{N} \end{cases}$ Then $g := M$

Indeed, $g \in \mathcal{M}$, $\displaystyle\int_X |g|\,d\mu = \int_X M\,d\mu = M\mu(X) < \infty \implies g \in \mathcal{L}^1$

Theorem: Integretion of series.

Let $\{f_n\} \subseteq \mathcal{L}^1$ s.t $\sum\limits_{n=1}^{\infty} \displaystyle\int_X |f_n|\,d\mu < \infty$

Then the series $\sum\limits_{n=1}^{\infty} f_n$ converges a.e. in X and $\displaystyle\int_X \left(\sum_{n=1}^{\infty} f_n\right) d\mu = \sum_{n=1}^{\infty} \int_X f_n\,d\mu$

3.4.7 Comparisons

Peano-Jordan measure and Lebesgue measure

Riemann integral and Lebesgue integral

Theorem.

$E \subseteq \mathbb{R}^n$ E is P-J measurable. Then $E \in \mathcal{L}(\mathbb{R}^n)$ and $m_{pj}(E) = \lambda(E)$

Obs. We can see the Lebesgue measure as a generalization of the P-J measure.

Obs. Remember that the Riemann integral is constructed by approximation of step functions (division of domain) and the function is integrable if the approximation from above (outer) is equal to the approximation from below (inner).

Remark:

$E := [0,1] \cap \mathbb{Q}$ is not P-J measurable because the outer measure is 1 and the inner measure is 0, the reason is that you have a countable set of discontinuity points

$E \in \mathcal{L}(\mathbb{R})$ and $\lambda(E) = 0$

$\implies PJ \subsetneq \mathcal{L}$

Theorem.

$I = [a,b],\ f \in \mathcal{R}(I)$ (integrable according to Riemann). Then $f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$ and $\displaystyle\int_I f\,d\lambda = \int_a^b f(x)\,dx$

Obs. Not surprising because the step functions are simple functions.

Obs. Important theorem because when we have to calculate the Lebesgue integral we often calculate the Riemann integral and apply this theorem.

Remark:

$I = [0,1]$, $f = \chi_{I\cap\mathbb{Q}}$

$$f \notin \mathcal{R}(I)$$

$$f \in \mathcal{L}^1 \qquad \int_I f\, d\lambda = \lambda(I \cap \mathbb{Q}) = 0$$

The relation is different when we consider generalized integrals.

$I = (\alpha, \beta)$ $\quad \mathcal{R}^i(I) = \{f : I \to \mathbb{R}$ integrable in the generalized (improper) sense$\}$

In this case ($\alpha = -\infty$, $\beta = +\infty$) are allowed.

Theorem.

$f \in \mathcal{R}^i(I) \implies f \in \mathcal{M}(I, \mathcal{L}(I))$

Furthermore, $f, |f| \in \mathcal{R}^i(I) \implies f \in \mathcal{L}^1(I, \mathcal{L}(I), \lambda)$ and

$$\int_I f\, d\lambda = \int_\alpha^\beta f(x)\, dx$$

Remark:

$$f : [0,\infty) \to \mathbb{R} \qquad f(x) := \begin{cases} \dfrac{sin\, x}{x} & x \neq 0 \\ 1 \quad x = 0 \end{cases}$$

$$f \in \mathcal{R}^i(I) \qquad \int_0^\infty \frac{sin\, x}{x}\, dx = \frac{\pi}{2}$$

$$\int_{\mathbb{R}_+} \left| \frac{sin\, x}{x} \right| d\lambda = \int_0^\infty \left| \frac{sin\, x}{x} \right| dx = \infty \implies f \notin \mathcal{L}^1$$

3.5 L^1 and L^∞

Lezione 10 (11/10/23)

Let $(X, \mathcal{A}, \mu)$ be a measure space. Then $fRg \overset{\text{def}}{\iff} f = g$ *a.e. in* X is an equivalence relation.

$[f] := \{g | fRg\}$ is an equivalence class.

$$L^1(X, \mathcal{A}, \mu) := \mathcal{L}^1(X, \mathcal{A}, \mu)/R \quad \text{and } [f] \in L^1$$

But we simplify and we say $f \in L^1$.

Lemma:

L^1 is a metric space, with $d(f,g) := \displaystyle\int_X |f-g|\,d\mu \;\; \forall\, f,g \in L^1$

Proof.

$$d : L^1 \times L^1 \to \mathbb{R}$$

$$\forall\, f,g \in L^1 \quad \int_X |f-g|\,d\mu \leq \int_X |f|\,d\mu + \int_X |g|\,d\mu < \infty$$

$$d(f,g) \geq 0 \;\; \forall\, f,g \in L^1 \quad d(f,f) = 0 \;\; \forall\, f \in L^1$$

$$d(f,g) = 0 \iff \int_X |f-g|\,d\mu = 0 \implies |f-g| = 0 \; a.e. \; in \; X \implies f = g \; a.e. \; in \; X \iff f = g \; in \; L^1$$

Obs. The last equivalence wouldn't be true in the space $\mathcal{L}^1$

$$d(f,g) = d(g,f) \;\; \forall\, f,g \in L^1$$

$$d(f,g) \leq d(f,h) + d(h,g) \;\; \forall\, f,g,h \in L^1$$

$\square$

Remark:

i) $\mathcal{L}^1$ is not a metric space

ii) L^1 is a vector space

$L^\infty(X,\mathcal{A},\mu) := \mathcal{L}^\infty(X,\mathcal{A},\mu)/R$

Lemma:

L^∞ is a metric space with $d(f,g) := \underset{X}{esssup}\,|f-g|$

3.6 Types of convergence

Let $\{f_n\} \subseteq \mathcal{M}(X,\mathcal{A})$, $f_n : X \to \mathbb{R}$ and $f : X \to \overline{\mathbb{R}}$

- **Pointwise convergence**

$$f_n \xrightarrow[n\to\infty]{} f \text{ pointwise } in \; X \overset{\text{def}}{\iff} \forall\, x \in X \;\; f_n(x) \xrightarrow[n\to\infty]{} f(x)$$

- **Uniform convergence**

$$f_n \xrightarrow[n\to\infty]{} f \text{ uniformly } in\ X \overset{\text{def}}{\iff} \sup_{x\in X} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$$

- **Convergence almost everywhere**

$$f_n \xrightarrow[n\to\infty]{} f\ a.e.\ in\ X \overset{\text{def}}{\iff} \{x \in X\ :\ f_n(x) \xrightarrow[n\to\infty]{} f(x)\}^c \in \mathcal{N}_\mu$$

- **Convergence in $\mathbf{L}^1$**

Since it's a convergence in measure space we need: $\{f_n\} \subseteq L^1,\ f \in L^1$

$$f_n \xrightarrow[n\to\infty]{} f\ in\ L^1 \overset{\text{def}}{\iff} d(f_n, f) \xrightarrow[n\to\infty]{} 0 \iff \int_X |f_n - f|\, d\mu \xrightarrow[n\to\infty]{} 0$$

This is called convergence in mean and is part of the thesis of Beppo Levi theorem.

Obs. Therefore, we can use the hypothesis of the theorem to verify the convergence in L^1.

- **Convergence in $\mathbf{L}^\infty$**

Since it's a convergence in measure space we need: $\{f_n\} \subseteq L^\infty,\ f \in L^\infty$

$$f_n \xrightarrow[n\to\infty]{} f\ in\ L^\infty \overset{\text{def}}{\iff} d(f_n, f) \xrightarrow[n\to\infty]{} 0 \iff \underset{x\in X}{esssup}\, |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$$

- **Convergence in measure**

$$f_n \xrightarrow[n\to\infty]{} f\ in\ measure \overset{\text{def}}{\iff} \forall \varepsilon > 0\ \mu(\{|f_n - f| \geq \varepsilon\}) \xrightarrow[n\to\infty]{} 0$$

Theorem.

Let $\mu(X) < \infty$ and $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e in X. Then,

$$f_n \xrightarrow[n\to\infty]{} f\ a.e.\ in\ X \implies f_n \xrightarrow[n\to\infty]{} f\ in\ measure$$

Remark:

In the case of $\mu(X) = \infty$ convergence a.e. in X $\not\Longrightarrow$ convergence in measure. Counter example:

$$f_n : \mathbb{R} \to \mathbb{R} \quad f_n := \chi_{[n,\infty)} \quad \text{we have } \lambda(\mathbb{R}) = \infty$$

$$f_n \xrightarrow[n\to\infty]{} 0\ in\ \mathbb{R}$$

$$\mu(\{f_n \geq \frac{1}{2}\}) = \infty \quad \forall n \in \mathbb{N} \implies f_n \nrightarrow 0\ in\ measure$$

Remark:

Convergence in measure $\not\Longrightarrow$ convergence a.e.

This is seen with the example of $f_n := \chi_{I_n}$ the Rademachen sequence (exercise session)

Theorem.

Let $f_n, f \in \mathcal{M}(X, \mathcal{A})$ be finite a.e. in X. Then,

$$f_n \xrightarrow[n\to\infty]{} f \textit{ in measure} \implies \exists \text{ a subsequence } \{f_{n_k}\} \textit{ s.t. } f_{n_k} \xrightarrow[k\to\infty]{} f \textit{ a.e. in } X$$

Theorem.

Let $f_n, f \in L^1(X, \mathcal{A}, \mu)$. Then,

$$f_n \xrightarrow[n\to\infty]{} f \textit{ in } L^1 \implies f_n \xrightarrow[n\to\infty]{} f \textit{ in measure}$$

Proof.

Suppose, by contradiction, that $f_n \underset{n\to\infty}{\not\rightarrow} f$ in measure.

Then $\exists \varepsilon > 0 \textit{ s.t. } \mu(|f_n - f| \geq \varepsilon) \geq \sigma$ for infinitely many $n \in \mathbb{N}$ (contrary of a limit)

$$\int_X |f_n - f| \, d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} |f_n - f| \, d\mu \geq \int_{\{|f_n - f| \geq \varepsilon\}} \varepsilon \, d\mu = \varepsilon \, \mu(\{|f_n - f| \geq \varepsilon\}) \geq \varepsilon \cdot \sigma$$

This is valid for infinitely many $n \in \mathbb{N}$, this means that $f_n \not\rightarrow f$ in L^1 $\square$

Remark:

$f_n \xrightarrow[n\to\infty]{} f \textit{ in measure} \not\Longrightarrow f_n \xrightarrow[n\to\infty]{} f \textit{ in } L^1$

Counter example: $f_n(x) = n\chi_{[0,\frac{1}{n}]}(x) \quad x \in [0,1]$

$$f_n \xrightarrow[n\to\infty]{} 0 \textit{ a.e. in } [0,1] \quad \lambda([0,1]) = 1 < \infty \implies f_n \xrightarrow[n\to\infty]{} 0 \textit{ in measure}$$

$$\int_0^1 |f_n - 0| \, d\lambda = \int_0^1 f_n \, d\lambda = \int_0^{\frac{1}{n}} n \, d\lambda = n \cdot \frac{1}{n} = 1 \quad \forall n \in \mathbb{N} \quad \implies f_n \not\rightarrow 0 \textit{ in } L^1$$

Obs. From this example we also see that convergence a.e. doesn't imply convergence in L^1.

Corollary:

If $f_n \xrightarrow[n\to\infty]{} f$ in L^1, then there exists a subsequence $\{f_{n_k}\}$ s.t. $f_{n_k} \xrightarrow[n\to\infty]{} f$ a.e. in X

Examples (10.1, 10.2, 10.3, 10.4, 10.5, 10.6)

• Ex. $f_n(x) = \frac{1}{n} e^{-nx}$, $x \in [0,1]$

$$\lim_{n\to\infty} \int_{[0,1]} f_n \, d\lambda$$

$$f_n \xrightarrow[n\to\infty]{} 0 \quad \text{in } [0,1]$$

f_n is meas., $|f_n| \le \underset{g \,\in\, L^1([0,1])}{\overset{\shortparallel}{1}}$ $\quad \forall x \in [0,1], \forall n \in \mathbb{N}$

$$\overset{DCT}{\Longrightarrow} \quad \lim_{n\to\infty} \int_{[0,1]} f_n \, d\lambda = \int_{[0,1]} \underbrace{\left(\lim_{n\to\infty} f_n\right)}_{=0} d\lambda = 0.$$

• Ex. $f_n(x) = \frac{\operatorname{arctg}(nx)}{n(1+x^2)}$, $x \in [1,\infty)$.

$$\lim_{n\to\infty} \int_{[1,\infty)} f_n \, d\lambda$$

$$f_n \xrightarrow[n\to\infty]{} 0 \quad \text{in } [1,\infty)$$

f_n is meas.

$$|f_n| \le \frac{\pi}{2} \frac{1}{1+x^2} =: g \in L^1([1,\infty))$$

$$\overset{DCT}{\Longrightarrow} \quad \lim_{n\to\infty} \int_{[1,\infty)} f_n \, d\lambda = \int_{[1,\infty)} \underbrace{\left(\lim_{n\to\infty} f_n\right)}_{=0} d\lambda = 0.$$

— · —

• Ex. $f_n(x) := \min\{e^x, n\}$, $x \in [0,5]$

$$\lim_{n\to\infty} \int_{[0,5]} f_n \, d\lambda$$

$$f_n(x) \xrightarrow[n\to\infty]{} e^x =: f(x), \quad \forall x \in [0,5]$$

f_n is meas. , $f_n \ge 0$

$f_n \le f_{n+1} \quad \forall n \in \mathbb{N}, \ \forall x \in [0,5]$

$$\text{BL} \implies \lim_{n\to\infty} \int_{[0,5]} f_n \, d\lambda = \int_{[0,5]} \left(\lim_{n\to\infty} f_n\right) d\lambda = \int_{[0,5]} e^x \, d\lambda$$

$$= \int_0^5 e^x \, dx = e^5 - 1.$$

• Ex.

$$\lim_{n\to\infty} \int_{(0,\infty)} \underbrace{\frac{1}{\left(1+\frac{x}{n}\right)^n x^{1/n}}}_{=: f_n(x)} \, d\lambda$$

$$f_n(x) \xrightarrow[n\to\infty]{} f(x) := e^{-x} \quad \forall x \in (0,\infty).$$

$\forall n \in \mathbb{N}$ f_n is meas. in $(0,\infty)$

$$\left(1+\frac{x}{n}\right)^n = 1 + x + \frac{n(n-1)}{n^2}\frac{x^2}{2} + \dots \overset{n>2}{>} \frac{x^2}{4}$$

$$\forall x \ge 1 \qquad 0 < f_n(x) = \frac{1}{\left(1+\frac{x}{n}\right)^n \underbrace{x^n}_{\ge 1}} \le \frac{4}{x^2} \qquad \forall n \in \mathbb{N}$$

$$\forall x \in (0,1) \qquad 0 < f_n(x) = \frac{1}{\underbrace{\left(1+\frac{x}{n}\right)^n}_{\ge 1} \underbrace{x^n}_{\ge \sqrt{x}}} \le \frac{1}{\sqrt{x}}$$

$$g(x) := \begin{cases} \dfrac{4}{x^2}, & x \ge 1 \\[2mm] \dfrac{1}{\sqrt{x}}, & x \in (0,1) \end{cases} \in L^1((0,\infty))$$

$$|f_n(x)| \le g(x) \quad \forall x \in (0,\infty),\ \forall n \in \mathbb{N}$$

$$\overset{\text{DCT}}{\Longrightarrow} \quad \lim_{n\to\infty} \int_{(0,\infty)} f_n \, d\lambda = \int_{(0,\infty)} \lim_{n\to\infty} f_n \, d\lambda = \int_{(0,\infty)} e^{-x} dx = 1.$$

———— . ————

• Ex

$f_n : X \to \mathbb{R}$

$f_n \ge 0$, meas.

$\lim_{n\to\infty} f_n = f$ in X

$f_n \le f \quad \forall n \in \mathbb{N}$ in X

$$\Longrightarrow \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X \left(\lim_{n\to\infty} f_n\right) d\mu$$

$$\int_X f_n \, d\mu \le \int_X f \, d\mu \quad \forall n \in \mathbb{N}$$

$$\Downarrow$$

$$\limsup_{n\to\infty} \int_X f_n \, d\mu \le \cancel{\limsup_{n\to\infty}} \int_X f \, d\mu$$

$$\liminf_{n\to\infty} \int_X f_n \, d\mu \ge \int_X \underbrace{\left(\liminf_{n\to\infty} f_n\right)}_{= f} d\mu$$

$$\int_X f \, d\mu \le \liminf_{n\to\infty} \int_X f_n \, d\mu \le \limsup_{n\to\infty} \int_X f_n \, d\mu \le \int_X f \, d\mu$$

$$\Longrightarrow \lim_{n\to\infty} \int_X f_n \, d\mu = \int_X f \, d\mu.$$

———— . ————

3.7 Functions of bounded variation

Lezione 11 (12/10/23)

Let $I = [a, b]$, $f : I \to \mathbb{R}$.
Let $\mathcal{P} := \{a = x_0 < x_1 < ... < x_n = b\}$ be a partition of I.
Define $v_a^b(f; \mathcal{P}) := \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})|$
Let $\underline{\mathrm{P}}$ be the collection of partitions of I.

Definition.

$$V_a^b(f) := \sup_{\mathcal{P} \in \underline{\mathrm{P}}} v_a^b(f; \mathcal{P})$$

If $V_a^b(f) < \infty$, then we say that f is a **function of bounded variation** in $[a, b]$

$$BV([a, b]) := \{f : [a, b] \to \mathbb{R} \mid V_a^b(f) < \infty\}$$

Obs. For a constant function the variation is always the same, but a function that "moves" a lot has different variations in different partitions

Remark:
i) If $f : [a, b] \to \mathbb{R}$ is monotone, then $V_a^b(f) = |f(b) - f(a)| < \infty \implies f \in BV([a, b])$
ii) $f \in BV([a, b]) \implies f$ is bounded in $[a, b]$ and $\sup_{x \in [a,b]} |f(x)| \le |f(a)| + V_a^b(f)$
iii) $f(x) := \begin{cases} x \sin\left(\frac{1}{x}\right) & x \in \left(0, \frac{2}{\pi}\right] \\ 0 \quad x = 0 \end{cases}$ even though is uniformly continuous, $f \notin BV([0, \frac{2}{\pi}])$

Proof. (iii)
We take the partition switched from x_n to x_0, because computations are easier,

$$P_n = \{x_n = 0, x_{n-1}, ..., x_1, x_0 = \tfrac{2}{\pi}\} \quad x_l = \frac{2}{(2l + 1)\pi} \text{ for } l = 0, ..., n - 1 \text{ and } x_n = 0$$

$$|f(x_{l+1}) - f(x_l)| \sim \frac{C}{l} \quad (C > 0)$$

$$v_a^b(f; \mathcal{P}_n) = \sum_{l=0}^{n-1} |f(x_{l+1}) - f(x_l)| \xrightarrow[n\to\infty]{} +\infty$$

$$\implies \sup_{\mathcal{P} \in \underline{\mathrm{P}}} v_a^b(f; \mathcal{P}) = +\infty \implies f \notin BV([0, \tfrac{2}{\pi}]) \text{ but } f \in UC([0, \tfrac{2}{\pi}])$$

$\square$

Theorem.

$f : [a, b] \to \mathbb{R}$. the following statements are equivalent:

i) $f \in BV([a, b])$

ii) $\exists\, \varphi, \psi : [a, b] \to \mathbb{R}$ increasing s.t. $f = \varphi - \psi$ (called Jordan decomposition)

Theorem.

Let $f : I \to \mathbb{R}$ be a monotone function. Then f is differentiable a.e. in I

Theorem.

Let $f : I \to \mathbb{R}$ be increasing. Then $f' \in L^1(I)$ and $\displaystyle\int_I f'\, d\lambda \le f(b) - f(a)$

Corollary:

If $f \in BV([a, b])$. Then f' exists a.e. in $[a, b]$ and $f' \in L^1((a, b))$

Thus obviously $f' \notin L^1((a, b)) \implies f \notin BV((a, b))$. This is a useful application.

3.8 Absolutely continuous functions

Let $J = [a, b]$, $f : J \to \mathbb{R}$ and

$$\mathcal{F}(J) = \{\text{finite collections of closed intervals } \subseteq J \text{ without interior points in common}\}$$

Definition.

A function $f : J \to \mathbb{R}$ is said to be **absolutely continuous** in J, if $\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 \; s.t.$

$$\forall\, \{[a_k, b_k]\}_{k=1}^n \in \mathcal{F}(J) \text{ for which } \sum_{k=1}^n (b_k - a_k) < \delta \qquad \sum_{k=1}^n |f(b_k) - f(a_k)| < \varepsilon$$

$AC([a, b]) := \{f : [a, b] \to \mathbb{R} \text{ absolutely continuous}\}$

Oss. Absolutely continuous functions ensure the usual continuity and also control the rate of change in a stronger sense.

Remark:

i) k=1 $\{[a_k, b_k]\} = \{[x, y]\}$ if $y \ge x$ or $\{[y, x]\}$ if $x > y$

$$\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0 : \quad \forall\, x, y \in [a, b] \;\; |x - y| < \delta \implies |f(y) - f(x)| < \varepsilon$$

This is the definition of uniformly continuous in $[a,b]$ for f, so

$$f \in AC \implies f \in UC \implies f \in C^0$$

ii) In general $f \in UC \nRightarrow f \in AC$ let's see a counter example:

$$f(x) = \begin{cases} x \sin\left(\dfrac{1}{x}\right) & x \in [-1,1]\backslash\{0\} \\ 0 \quad x = 0 \end{cases} \qquad f \in UC([-1,1])\,,\ f \notin AC([-1,1]) \text{ (since } f \notin BV([-1,1]))$$

iii) f Lipschitz in $[a,b] \implies f \in AC$

$$\sum_{k=1}^{n} |f(b_k) - f(a_k)| \le \sum_{k=1}^{n} L|b_k - a_k| = L\sum_{k=1}^{n}(b_k - a_k) < L\delta < \varepsilon \quad \text{provided that } \delta = \frac{\varepsilon}{L}$$

iv) $f \in AC \nRightarrow f$ Lipschitz (counter example)

$$f(x) = \sqrt{x}\ ,\ x \in [0,1] \quad f \notin Lipschitz$$

$$f(x) = \sqrt{x} = \int_0^x \underbrace{\frac{1}{2\sqrt{t}}}_{\in L^1} dt \implies f \in AC \quad \text{(we'll see a corollary below for this implication)}$$

Theorem.

Let $(X, \mathcal{A}, \mu)$ be a measure space, $f \in \mathcal{M}_+(X, \mathcal{A})$ be s.t. $\displaystyle\int_X f\,d\mu < \infty$. Then,

$$\forall\, \varepsilon > 0 \ \ \exists\, \delta > 0 \ s.t. \ \ \forall\, E \in \mathcal{A} \text{ with } \mu(E) < \delta \text{ we have } \int_E f\,d\mu < \varepsilon$$

Proof.

Set $F_n := \{f < n\}$ $(n \in \mathbb{N})$ $F_n \in \mathcal{A}$, $\{F_N\} \nearrow$

$$X = \{f = \infty\} \cup \{f < \infty\} = \{f = \infty\} \cup \left[\bigcup_{n=1}^{\infty} F_n\right]$$

$$\int_X f\,d\mu < \infty \implies f \text{ finite a.e. in } X \iff \mu(\{f = \infty\}) = 0$$

$$\implies \int_X f\,d\mu = \lim_{n\to\infty} \int_{F_n} f\,d\mu$$

$$\forall\, \varepsilon > 0 \ \ \exists\, \overline{n} \in \mathbb{N} \ s.t. \ \ \forall\, n > \overline{n} \quad \left|\int_X f\,d\mu - \int_{F_n} f\,d\mu\right| < \frac{\varepsilon}{2} \quad \text{(definition of the limit from the line above)}$$

Since $\displaystyle\int_X f\,d\mu - \int_{F_n} f\,d\mu = \int_{F_n^C} f\,d\mu$. Therefore, for a fixed $n > \overline{n}$ we have

$$\int_E f\,d\mu = \int_{E\cap F_n} f\,d\mu + \int_{E\cap F_n^C} f\,d\mu < n\,\mu(E) + \frac{\varepsilon}{2} < n\frac{\varepsilon}{2n} + \frac{\varepsilon}{2} = \varepsilon$$

Since $\int_{E\cap F_n} f\,d\mu < \int_{E\cap F_n} n\,d\mu < \int_E n\,d\mu = n\,\mu(E) < n\,\delta$ and provided that $\delta = \frac{\varepsilon}{2n}$

□

Corollary:

Let $I = [a,b]$, $f \in L^1(I)$. Then,

$$F(x) := \int_{[a,x]} f\,d\lambda \quad \text{is absolutely continuous in I}$$

Proof.

Let $E := \bigcup_{k=1}^n [a_k, b_k]$ with $\{[a_k, b_k]\} \in \mathcal{F}(I)$ $\quad \lambda(E) = \sum_{k=1}^n (b_k - a_k)$

$$\sum_{k=1}^n |F(b_k) - F(a_k)| = \sum_{k=1}^n \left| \int_{a_k}^{b_k} f\,d\lambda \right| \le \sum_{k=1}^n \int_{[a_k,b_k]} |f|\,d\lambda = \int_E |f|\,d\lambda$$

$f \in L^1(I) \implies |f| \in \mathcal{M}_+$, $\int_I |f|\,d\lambda < \infty$. So by the previous theorem,

$$\forall\,\varepsilon > 0 \ \ \exists\,\delta > 0 : \quad \forall\,E \in \mathcal{L}\,,\ \lambda(E) < \delta \quad \implies \int_E |f|\,d\lambda < \varepsilon$$

Therefore, $F \in AC(I)$ (by reorganizing the implication we get the AC definition)

□

Theorem.

$$f \in AC([a,b]) \implies f \in BV([a,b])$$

3.9 Fundamental theorems of calculus

Theorem: First FTC.

If $f \in L^1((a,b))$, then $F(x) := \int_{[a,x]} f\,d\lambda$ is differentiable a.e. in (a,b)

Definition.

$x_0 \in [a,b]$ is a **Lebesgue point** for f if

$$\lim_{h\to 0} \frac{1}{h} \int_{x_0}^{x_0+h} |f(t) - f(x_0)|\,d\lambda = 0$$

Theorem.

If $f \in L^1((a,b))$, then a.e. $x_0 \in [a,b]$ is a Lebesgue point for f

Proof. (1° FTC)

Let $x \in [a,b]$ be a Lebesgue point for f and $h \neq 0$ s.t. $x+h \in [a,b]$

$$\frac{F(x+h)-F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h} [f(t)-f(x)]\,dt$$

$$\left|\frac{F(x+h)-F(x)}{h} - f(x)\right| \le \frac{1}{|h|}\left|\int_x^{x+h} [f(t)-f(x)]\,dt\right| \xrightarrow[h\to 0]{} 0$$

$$\frac{F(x+h)-F(x)}{h} \xrightarrow[h\to 0]{} F'(x) \implies F'(x) = f(x)$$

By the preceding theorem, $F' = f$ a.e. in (a,b) $\square$

Theorem: second FTC.

Let $\varphi : [a,b] \to \mathbb{R}$. Then the following statements are equivalent:

i) $\varphi \in AC([a,b])$

ii) φ is differentiable a.e. in $[a,b]$ with $\varphi' \in L^1(a,b)$ and $\varphi(x) - \varphi(a) = \displaystyle\int_{[a,x]} \varphi'\,d\lambda \quad \forall\, x \in [a,b]$

3.10 Derivative of measures

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Definition.

Let $(X, \mathcal{A})$ be a measurable space and μ, ν measures.

A function $\varphi \in \mathcal{M}_+(X, \mathcal{A})$ is said to be Radon-Nikodym derivative of ν w.r.t. μ, if

$$\nu(E) = \int_E \varphi\,d\mu \quad \forall\, E \in \mathcal{A}$$

We write $\varphi = \dfrac{d\nu}{d\mu}$ and $d\nu = \varphi d\mu$

Definition.

We say that ν is **absolutely continuous** w.r.t. μ, if

$$\mu(E) = 0 \implies \nu(E) = 0$$

We write $\nu << \mu$

Theorem: Radon-Nikodym.

Let $(X, \mathcal{A})$ be a measurable space and μ, ν measures.

Suppose that $\nu << \mu$ and that μ is σ-finite.

Then $\dfrac{d\nu}{d\mu}$ exists and is unique.

Functional analysis

4 Metric spaces

4.1 Separability

$X \equiv (X, d)$ with d a metric.

Definition.
$A \subset X$ is **dense** in X if $\overline{A} = X$ (closure of A).

Definition.
X is **separable** if there exists $A \subset X$ countable and dense in X.

Obs. This means that any element of the set is a limit of a sequence of A. If X is separable, then you can approximate X by means of countable sequences.

Example: $X = \mathbb{R},\ A = \mathbb{Q}$.

Theorem: Stone-Weierstrass.
The set of polynomials is dense in $C^0([a, b])$.

Corollary:
$C^0([a, b])$ is separable.

Proof.
By S.-W. theorem, for any $f \in C^0([a, b]),$ given $\varepsilon > 0,$ $\exists\, p =$ polynomial s.t. $d(f, p) < \dfrac{\varepsilon}{2}$ (density)
We can find a polynomial r with <u>rational coefficients</u> s.t. $d(p, r) < \dfrac{\varepsilon}{2}$ (since $\mathbb{Q}$ is dense in $\mathbb{R}$)

$$\implies d(f, r) < d(f, p) + d(p, r) < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Therefore the set of all polynomials with rational coefficients is dense in $C^0([a,b])$.

Moreover, such set is countable and this yields the thesis. $\square$

Theorem: Baire.

Let X be a complete metric space.

The intersection of a countable family of open sets dense in X is a set dense in X.

So, if a sequence $\{A_n\}_{n\in\mathbb{N}}$ is such that A_n is open and $\overline{A_n} = X \;\; \forall\, n \in \mathbb{N}$. Then $\overline{\bigcap_{n=1}^{\infty} A_n} = X$.

Theorem: Baire, version II.

Let X be a complete metric space. Let $\{A_n\}_{n\in\mathbb{N}}$ be a sequence of closed sets s.t. $\bigcup_{n=1}^{\infty} A_n = X$.

Then $\exists\, n_0 \in \mathbb{N}$ s.t. $Int\,(A_{n_0}) \neq \emptyset$

4.2 Compactness

Definition.

X **compact** $\overset{\text{def}}{\iff}$ from any open cover of X we can extract a finite open cover.

Obs. An open cover of X is a collection of open sets such that X is a subset of the union of those sets.

Definition.

X is **sequentially compact**

$\Updownarrow$ def

From any sequence $\{x_n\} \subset X$ we can extract a subsequence which converges to some $x_0 \in X$

Definition.

X is said to be **totally bounded** if $\;\forall\,\varepsilon > 0 \;\; \exists\, A \subset X$ finite s.t.

$$\forall x \in X \;\; dist(x, A) := \int_{y\in A} d(x,y) < \varepsilon$$

Theorem.

Let X be a metric space. The following statements are equivalent:

i) X is compact

ii) X is sequentially compact

iii) X is complete and totally bounded

COROLLARY:

$E \subseteq X$ is compact $\implies$ E is closed and bounded.

REMARK:

$X = \mathbb{R}^n$ then $E \subseteq X$ is compact $\iff$ E is closed and bounded.

Compactness in $\mathbf{C^0(X)}$

Let X be a compact metric space.

$$C^0(X) := \{f : X \to \mathbb{R} \text{ continuous}\} \quad \text{with} \quad d(f, g) := \sup_{x \in X} |f(x) - g(x)|$$

$$C^0(X) \text{ is a complete metric space.}$$

DEFINITION.

$A \subset C^0(X)$ is an **equicontinuous set** if $\forall \varepsilon > 0 \ \exists \delta_\varepsilon > 0 : \forall f \in A \ , \ \forall x, y \in X \ , \ d(x, y) < \delta_\varepsilon$

$$|f(x) - f(y)| < \varepsilon$$

Obs. We talk about equicontinuity because δ depends only on ε since it's $\forall f$ $(\delta = \delta(\varepsilon, \cancel{f}))$.

So, even though, this is an infinite dimension space, it's not really infinite, since the function is not important.

DEFINITION.

$E \subseteq X$ is **relatively compact** if $\overline{E}$ is compact.

THEOREM: Ascoli-Arzela.

$F \subset C^0(X)$ is bounded and equicontinuous $\iff$ F is relatively compact

So, $F \subset C^0(X)$ is bounded, closed and equicontinuous $\iff$ F is compact

Tipically, we have $F = \{f_n\} \subset C^0(X)$ and $X = [a, b]$ or a compact $X = K \subseteq \mathbb{R}^n$

Let's see equicontinuity, boundedness and compactness applied to this case:

• $\delta = \delta(\varepsilon, \cancel{n})$ equicontinuity

•• $\exists M > 0 : \sup_X |f_n| \leq M \quad \forall n \in \mathbb{N}$

Obs. All f_n bounded it's not enough, so f_n are uniformly bounded, that means $M = M(\cancel{n})$

••• F is relatively compact $\iff$ $\overline{F}$ is compact $\iff$ $\overline{F}$ is sequentially compact

$\iff \quad \forall \{f_{n_j}\} \subset \overline{\{f_n\}} = \overline{F} \quad \exists \{f_{n_{j_k}}\} \subset \overline{F} \ , \ f \in C^0 \, (f \in \overline{F}) \ | \ f_{n_{j_k}} \xrightarrow[j \to \infty]{} f \ in \ C^0(X)$ (unif. conv. in X)

$\iff \quad \forall \{f_{n_j}\} \subset \{f_n\} = F \quad \exists \{f_{n_{j_k}}\} \subset F \ , \ f \in C^0 \ | \ f_{n_{j_k}} \to f \ in \ C^0(X)$

$\implies \{f_{n_j}\} \equiv \{f_n\} \quad \exists \{f_{n_k}\} \subset F \ , \ f \in C^0 \ \mid \ f_{n_k} \xrightarrow[k\to\infty]{} f \in C^0(X)$

Obs. This is the definition of compactness: for every sequence $\{f_{n_j}\}$ of F, there exists a subsequence $\{f_{n_{j_k}}\}$ with a limit belonging to F.
At the end we get an implication, because we only consider one subsequence (the element $\{f_n\}$).

Corollary:
$\{f_n\} \subset C^0(X)$ bounded and equicontinuos

$$\implies \quad \exists \{f_{n_k}\} \subset \{f_n\} \ , \ f \in C^0(X) \mid f_{n_k} \xrightarrow[k\to\infty]{} f \ in \ C^0(X)$$

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Corollary: of Ascoli-Arzela
$\{f_n\} \subset C^1([a,b]) \ , \ \ \exists C > 0$ s.t.

$$(i) \ \sup_{[a,b]} |f_n| \leq C \ \ \forall n \in \mathbb{N} \qquad\qquad (ii) \ \sup_{[a,b]} |f_n'| \leq C \ \ \forall n \in \mathbb{N}$$

$$\implies \quad \exists \ \{f_{n_k}\} \subset \{f_n\} \ , \ f \in C^0 \ : \ f_{n_k} \xrightarrow[k\to\infty]{} f \ in \ C^0$$

Proof.

$$(i) \iff \{f_n\} \text{ is bounded in } C^0$$

$$\forall \, x, y \in [a,b] \ \ f_n(y) - f_n(x) = f_n'(\xi_n)(y-x) \quad \text{for some } \xi_n \text{ in between } x \text{ and } y$$

$$|f_n(y) - f_n(x)| = |f'(\xi_n)||y-x| \overset{(ii)}{\leq} C|y-x| \ \ \forall n \in \mathbb{N}$$

$$\iff \{f_n\} \text{ is equi-Lipschitz in } [a,b]$$

Obs. Which means that every f_n is Lipschitz and C doesn't depend on n.

$$\implies \{f_n\} \text{ is equicontinuous}$$

By A.-A. theorem, the thesis follows.

□

5 Banach spaces

5.1 Normed spaces

Definition.

Let X be a vector space. A **norm** on X is a function

$$\|\cdot\|: X \to [0, +\infty) \; s.t.$$

i) $\|x\| = 0 \iff x = 0$

ii) $\forall \alpha \in \mathbb{R}\,,\; x \in X \quad \|\alpha x\| = |\alpha| \|x\|$

iii) $\forall x, y \in X \quad \|x + y\| \leq \|x\| + \|y\|$

$(X, \|\cdot\|)$ is a **normed space**.

Remark:

$(X, \|\cdot\|)$ normed space $\implies$ (X, d) metric space with $d(x, y) := \|x - y\|$

Example:

i) $\mathbb{R}^n$ with $dim\,\mathbb{R}^n = n < \infty$

$\|x\|_p := \left(\sum_{i=1}^{n} |x_i|^p\right)^{\frac{1}{p}}$, $\; p \in [1, +\infty)$

$\|x\|_\infty := \max_{i=1\ldots n} |x_i|$

ii) $C^0([a, b]) \quad \|f\|_{C^0} \equiv \|f\|_\infty := \max_{[a,b]} |f|$

iii) $L^1(X, \mathcal{A}, \mu) \quad \|f\|_1 := \int_X |f|\, d\mu$

iv) $L^\infty(X, \mathcal{A}, \mu) \quad \|f\|_\infty := ess\,sup_X\, |f|$

v) $C^k([a, b]) \quad \|f\|_{\infty,k} := \sum_{i=0}^{k} \|f^{(i)}\|_\infty \quad$ with $f^{(0)} \equiv f$

vi) $BV([a, b]) \quad \|f\|_{BV} := \begin{cases} |f(a)| + V_a^b(f) \\ \|f\|_1 + V_a^b(f) \end{cases}$ (the cases mean there are two possible choices)

vii) $AC([a, b]) \quad \|f\|_{AC} := \begin{cases} |f(a)| + \|f'\|_1 \\ \|f\|_1 + \|f'\|_1 \end{cases}$

viii) l^p , l^∞ with $p \in [1,\infty)$. Elements in l^p are sequences: $x = \{x^{(k)}\}_{k\in\mathbb{N}}$ or $x = (x^{(1)}, ..., x^{(k)}, ...)$

$\|x\|_p := \left(\sum_{k=1}^{\infty} |x^{(k)}|^p\right)^{\frac{1}{p}} < \infty \quad p \in [1,\infty)$ $\quad l^p$:={x is a sequence of real numbers such that $\|x\|_p < \infty$}

$\|x\|_\infty = \sup_{k\in\mathbb{N}} |x^{(k)}|$ $\quad l^\infty$:={x is a sequence of real numbers such that $\|x\|_\infty < \infty$}

Example (13.1)

• Ex. $\{x_n\} \subset \ell^2$

x_1 sequence $\{x_1^{(k)}\}$

x_2 = $\{x_2^{(k)}\}$

$\vdots$

x_n = $\{x_n^{(k)}\}$

$$x_n \equiv x_n^{(k)} := \frac{1}{n+k}$$

$\forall n \in \mathbb{N}$

$$\|x_n\|_{\ell^2} = \left(\sum_{k=1}^{\infty} |x_n^{(k)}|^2\right)^{1/2} = \left(\sum_{k=1}^{\infty} \frac{1}{(n+k)^2}\right)^{1/2} < \infty$$

$\Rightarrow \forall n \in \mathbb{N} \quad \underline{x_n \in \ell^2} \iff \underline{\{x_n\} \subset \ell^2}$.

5.2 Sequences and series

Definition.

Let $(X, \|\cdot\|)$ be a normed space, $\{x_n\} \subset X$, $x \in X$

$$x_n \xrightarrow[n\to\infty]{} x \iff \|x_n - x\| \xrightarrow[n\to\infty]{} 0$$

Remark:

$x_n \to x \implies \|x_n\| \to \|x\|$

In fact, $x_n \to x \iff \|x_n - x\| \to 0 \implies \|x_n\| \to \|x\|$ for triangular inequality: $\big|\, \|x_n\| - \|x\| \,\big| \leq \|x_n - x\|$

Definition.

$\{x_n\} \subset X$ is a **Cauchy sequence** if $\forall \varepsilon > 0 \;\; \exists \overline{n} \in \mathbb{N}$ s.t.

$$\|x_m - x_n\| < \varepsilon \;\; \forall\, m, n > \overline{n}$$

Remark:

$\{x_n\}$ convergent $\underset{\not\Longleftarrow}{\Longrightarrow}$ $\{x_n\}$ is Cauchy

Definition.

$\{x_n\}$ is **bounded** if $\exists M > 0$ s.t. $\|x_n\| < M \;\; \forall\, n \in N$

Remark:

$\{x_n\}$ is Cauchy $\Longrightarrow$ $\{x_n\}$ is bounded

Definition.

$\{x_n\} \subset X \quad s_n := x_0 + x_1 + ... + x_n = \sum\limits_{k=0}^{n} x_k$

$\{s_n\}$ is called **sequence of partial sums** or series.

The series $\{s_n\}$ is said to be **convergent** if there exists $x \in X$ s.t.

$$s_n \xrightarrow[n\to\infty]{} x \iff \|s_n - x\| \xrightarrow[n\to\infty]{} 0$$

$x = \sum\limits_{k=0}^{\infty} x_k$ is the sum of the series or simply series

Remark:

$\sum\limits_{n=0}^{\infty} \|x_n\|$ convergent, with $\|x_n\| \in \mathbb{R}_+ \not\Longrightarrow \sum\limits_{n=0}^{\infty} x_n$ convergent

5.3 Completeness

Definition.

$(X, \|\cdot\|)$ is **complete** $\overset{\text{def}}{\iff} (X, d)$ with $d(x, y) := \|x - y\|$ is complete $\iff$

$\iff$ every Cauchy sequence in X is convergent.

Definition.

A complete normed space is called **Banach space**.

Example:

All the examples we have seen above are valid also here.

Theorem: Criterion for completeness.

i) Let X be a Banach space, $\{x_n\} \subset X$.

If $\sum\limits_{n=1}^{\infty} \|x_n\|$ converges, then $\sum\limits_{n=1}^{\infty} x_n$ converges.

ii) Let X be a normed space.

If for any $\{x_n\} \subset X$ s.t. $\sum\limits_{n=1}^{\infty} \|x_n\|$ converges, one has also that $\sum\limits_{n=1}^{\infty} x_n$ converges, then X is Banach.

Definition.

Let X be a vector space. If $\forall n \in \mathbb{N}$ there exists a set of n linearly independent vectors in X, then we say that X has **infinite dimension**.

Let X be a normed space and Y a vector subspace (v.s.s.) of X

• $\dim Y < \infty \implies Y$ is closed

•• $\dim Y = \infty \not\implies Y$ is closed

Definition.

Let $(X, \|\cdot\|)$, $(X, \|\cdot\|_{\#})$ be normed spaces. $\|\cdot\|$, $\|\cdot\|_{\#}$ are **equivalent** if

$$\exists\, m > 0\ ,\ \ M > 0 \ s.t. \ m\|x\| \leq \|x\|_{\#} \leq M\|x\| \ \ \forall x \in X$$

Theorem.

X normed space, $\dim X < \infty \implies$ all norms on X are equivalent.

Remark:

$(C^0([a, b]), \|\cdot\|_{\infty})$ is a Banach space.

$(C^0([a, b]), \|\cdot\|_1)$ is a normed space, but it's not complete.

Dim$C^0 = \infty$. Indeed $\|\cdot\|_{\infty}$, $\|\cdot\|_1$ are not equivalent.

5.4 Compactness in normed spaces

Let X be a normed space, $x_0 \in X$, $r > 0$

Define (open) ball $B_r(x_0) := \{x \in X \mid \|x - x_0\| < r\}$

Define closed ball $\overline{B}_r(x_0) := \{x \in X \mid \|x - x_0\| \leq r\}$

Closure of $B_r(x_0) \equiv \overline{B_r(x_0)}$

In the case of a vector structure we can say $\overline{B}_r(x_0) = \overline{B_r(x_0)}$

But, in general, for a metric space $\overline{B_r(x_0)} \subsetneq \overline{B}_r(x_0)$

Lezione 14 (25/10/23)

Lemma: Riesz

Let X be a normed space, $E \subsetneq X$ a <u>closed</u> vector subspace. Then,

$$\forall \varepsilon > 0 \;\; \exists x \in X \;\, s.t. \; \|x\| = 1 \text{ and } dist(x, E) = \inf_{\xi \in E} \|x - \xi\| \geq 1 - \varepsilon$$

Obs. For all ε you want to find an element far enough. This is possible because the closure of E is not X.

Proof.

Let $y \in X \backslash E$. $d := dist(y, E) > 0$ since E is closed.

$$d = dist(y, E) = \inf_{\xi \in E} \|\xi - y\|$$

Let $\varepsilon \in (0, 1)$. For the definition of infimum, we can find $\zeta \in E$ s.t.

$$d \leq \|y - \zeta\| \leq \frac{d}{1 - \varepsilon}$$

Let $x := \dfrac{y - \zeta}{\|y - \zeta\|}$. Clearly $\|x\| = 1$ and $\forall \xi \in E$

$$\|x - \xi\| = \left\| \frac{y - \zeta}{\|y - \zeta\|} - \xi \right\| = \frac{1}{\|y - \zeta\|} \big\| y - \zeta - \xi \|y - \zeta\| \big\| = \frac{1}{\|y - \zeta\|} \big\| y - \underbrace{(\zeta + \xi\|y - \zeta\|)}_{\in E} \big\| \geq \frac{d}{\|y - \zeta\|} \geq 1 - \varepsilon$$

$$\implies dist(x, E) \geq 1 - \varepsilon$$

$\square$

Remark:

If $dimX < \infty$ (or X is Hilbert)

Then $\zeta = Proj_E\, y$ is called almost orthogonal element and $d = \|y - \zeta\|$.

So for $\|x\| = 1$, we have $dist(x, E) \geq 1$

Theorem: Riesz.

Let X be a normed space. If the closed ball $\overline{B_1}(0)$ is compact, then $dimX < \infty$

Proof.

Let $x_1 \in \overline{B_1}(0)$ and $Y_1 := Span\{x_1\}$

Y_1 is a v.s.s. of X and $dimY_1 = 1 < \infty$

$\implies Y_1$ is closed, since is a v.s.s of finite dimension.

If $X = Y_1$, then $dimX < \infty$ and the thesis is given.

If $X \neq Y_1$, then we can use the Riesz lemma with $\varepsilon = \frac{1}{2}$ to find $x_2 \in \overline{B_1}(0)$ s.t.

$$\|x_2 - \underbrace{x_1}_{\in Y_1}\| \geq \frac{1}{2}$$

Let $Y_2 = Span\{x_1, x_2\}$, which is closed (v.s.s. finite dim).

If $X = Y_2$, then $dimX < \infty$.

If $X \neq Y_2$, we can use again the Riesz lemma to find $x_3 \in \overline{B_1}(0)$ s.t.

$$\|x_3 - \underbrace{x_i}_{\in Y_2}\| \geq dist(x_3, Y_2) \geq \frac{1}{2} \quad \text{for } i = 1, 2$$

If X is not finite dimensional, this process can be iterated to construct a sequence $\{x_n\} \subseteq \overline{B_1}(0)$ s.t.

$$\|x_i - x_j\| \geq \frac{1}{2} \quad \forall\, i, j \in \mathbb{N}\ i \neq j \quad (\|x_i\| = 1)$$

$\implies \{x_n\}$ has no convergent subsequence, but $\{x_n\} \subset \overline{B_1}(0)$ is bounded (since B_1 is bounded).

$\implies \overline{B_1}(0)$ is not sequentially compact $\implies \overline{B_1}(0)$ is not compact.

This is a contradiction, so the X cannot have infinite dimension.

□

Corollary:

Let X normed space.

$$\overline{B_1}(0) \text{ is compact } \iff dimX < \infty$$

Obs. This corollary is true even if, instead if $\overline{B_1}(0)$, we have a closed and bounded $E \subseteq X$

Obs. In view of this corollary, since we know that C^0 has infinite dimension. Not all the closed and bounded are compact (we need to ask equicontinuity as well).

6 Lebesgue spaces

6.1 L^p spaces

Let $(X, \mathcal{A}, \mu)$ be a measure space, $p \in [1, \infty]$,

$\mathcal{L}^p(X, \mathcal{A}, \mu) := \{f : X \to \overline{R} \text{ measurable}, \int_X |f|^p \, d\mu < \infty\}, \;\; f, g \in \mathcal{L}^p$

$$fRg \iff f = g \; a.e. \; in \; X \qquad R = \text{Equivalence relation}$$

$$L^p(X, \mathcal{A}, \mu) := \frac{\mathcal{L}^p(X, \mathcal{A}, \mu)}{R}$$

Lemma:

Let $p \in [1, \infty), \; a \geq 0, \; b \geq 0$. Then,

$$(a + b)^p \leq 2^{p-1}(a^p + b^p)$$

Proof.

$$\text{The function } x \mapsto x^p \text{ is convex in } [0, \infty) \implies \left(\frac{a+b}{2}\right)^p \leq \frac{a^p}{2} + \frac{b^p}{2} \iff (a+b)^p \leq 2^{p-1}(a^p + b^p)$$

□

Lemma:

L^p is a vector space.

Proof.

$f, g \in L^p \implies f, g$ finite a.e. in X

$\implies \lambda \in \mathbb{R} \;\; f + \lambda g$ is well defined a.e. and is measurable (sum of measurable functions)

$$\int_X |f + \lambda g|^p \, d\mu \overset{\substack{\text{previous} \\ \text{lemma}}}{\leq} 2^{p-1}\left(\int_X |f|^p \, d\mu + |\lambda|^p \int_X |g|^p \, d\mu\right) < \infty$$

$$\implies f + \lambda g \in L^p$$

□

Lemma: Young inequality

Let $p \in (1, \infty), \; a > 0, \; b > 0$. Then,

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q} \quad q = \frac{p}{p-1} \;\; (\text{from } \tfrac{1}{p} + \tfrac{1}{q} = 1)$$

Proof.

$\varphi(x) := e^x$ is convex $\implies \varphi(tx+(1-t)y) \le t\varphi(x)+(1-t)\varphi(y) \quad \forall\, x,y \in \mathbb{R} \quad \forall\, t \in [0,1]$

$$ab = e^{log\, a} e^{log\, b} = e^{\frac{1}{p} log\, a^p} e^{\frac{1}{q} log\, b^q}$$

Now we apply the definition of convexity with: $t = \dfrac{1}{p},\ 1-t = \dfrac{1}{q},\ x = log\, a^p,\ y = log\, b^q$

$$ab \le \frac{1}{p} e^{log\, a^p} + \frac{1}{q} e^{log\, b^q} = \frac{a^p}{p} + \frac{b^q}{q}$$

$\square$

Definition.

$p, q \in [1,\infty]$ are **conjugate** if

$$\begin{cases} p, q \in (1,\infty) \text{ and } \dfrac{1}{p} + \dfrac{1}{q} = 1 \\ p = 1 \text{ and } q = \infty \\ p = \infty \text{ and } q = 1 \end{cases}$$

Let $p \in [1,\infty) \quad \|f\|_p := \left(\int_X |f|^p \, d\mu \right)^{\frac{1}{p}} \quad \|f\|_\infty := \underset{X}{ess\,sup}\, |f|$

Theorem: Hölder Inequality.

Let $f, g \in \mathcal{M}(X, \mathcal{A})$ and $p, q \in [1,\infty]$ conjugate. Then,

$$\|fg\|_1 \le \|f\|_p \|g\|_q$$

Proof.

i) $p, q \in (1,\infty)$ and $\dfrac{1}{p} + \dfrac{1}{q} = 1$

The thesis is obvious if $\|f\|_p \|g\|_q = \infty$.

If $\|f\|_p \|g\|_q = 0$, then $f = 0$ a.e. or $g = 0$ a.e. $\implies fg = 0$ a.e. $\implies \|fg\|_1 = 0$.

Now, suppose that $\|f\|_p$ and $\|g\|_q$ are finite and $\neq 0$.

Let $x \in X$. Set $a := \dfrac{|f(x)|^p}{\|f\|_p^p}$, $b := \dfrac{|g(x)|^q}{\|g\|_q^q}$. Young inequality yields

$$a^{\frac{1}{p}} b^{\frac{1}{q}} = \frac{|f(x)|}{\|f\|_p} \frac{|g(x)|}{\|g\|_q} \le \frac{1}{p} \frac{|f(x)|^p}{\|f\|_p^p} + \frac{1}{q} \frac{|g(x)|^q}{\|g\|_q^q}$$

$$\Longrightarrow \quad \frac{1}{\|f\|_p\|g\|_q}\int_X |f(x)g(x)|\,d\mu \le \frac{1}{p}\underbrace{\frac{\int_X |f|^p\,d\mu}{\|f\|_p^p}}_{=1} + \frac{1}{q}\underbrace{\frac{\int_X |g|^q\,d\mu}{\|g\|_q^q}}_{=1} = \frac{1}{p}+\frac{1}{q} = 1$$

$$\|fg\|_1 = \int_X |fg|\,d\mu \le \|f\|_p\|g\|_q$$

ii) $p = 1\ ,\ q = \infty$

$$|g| \le \|g\|_\infty \ a.e.\ in\ X$$

$$\Longrightarrow\ |fg| \le |f|\|g\|_\infty \ a.e.\ in\ X$$

$$\|fg\|_1 = \int_X |fg|\,d\mu \le \|g\|_\infty \int_X |f|\,d\mu = \|f\|_1\|g\|_\infty$$

□

Theorem: Minkowski Inequality.

Let $f, g \in \mathcal{M}(X, \mathcal{A}),\ p \in [1, \infty]$. Then,

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

Proof. (Minkowski Inequality)

For $p \in (1, \infty)$

$$\|f+g\|_p^p = \int_X |f+g|^p\,d\mu = \int_X |f+g||f+g|^{p-1}\,d\mu \le \int_X |f||f+g|^{p-1}\,d\mu + \int_X |g||f+g|^{p-1}\,d\mu$$

In view of the Hölder inequality $\left(q = \dfrac{p}{p-1}\right)$:

$$\int_X |f||f+g|^{p-1}\,d\mu \le \|f\|_p \||f+g|^{p-1}\|_q$$

$$\int_X |g||f+g|^{p-1}\,d\mu \le \|g\|_p \||f+g|^{p-1}\|_q$$

$$\||f+g|^{p-1}\|_q = \left(\int_X |f+g|^{(p-1)\frac{p}{p-1}}\,d\mu\right)^{\frac{1}{q}} = \left(\int_X |f+g|^p\,d\mu\right)^{\frac{1}{q}} = \|f+g\|_p^{\frac{p}{q}}$$

$$\Longrightarrow\ \|f+g\|_p^p \le (\|f\|_p + \|g\|_p)\,\|f+g\|_p^{\frac{p}{q}}$$

$$\Longrightarrow\ \|f+g\|_p^{p-\frac{p}{q}} \le \|f\|_p + \|g\|_p \qquad \text{with } p - \frac{p}{q} = p-(p-1) = 1$$

$$\Longrightarrow\ \|f+g\|_p \le \|f\|_p + \|g\|_p$$

Now let $p = 1$

$$\|f+g\|_1 = \int_X |f+g| \, d\mu \le \int_X (|f|+|g|) \, d\mu = \|f\|_1 + \|g\|_1$$

Let $p = \infty$

$$\|f+g\|_\infty = \underset{X}{ess\,sup}\,|f+g| \le \underset{X}{ess\,sup}\,(|f|+|g|) \le \underset{X}{ess\,sup}\,|f| + \underset{X}{ess\,sup}\,|g| = \|f\|_\infty + \|g\|_\infty$$

□

Corollary:

$L^p(X, \mathcal{A}, \mu)$ is a normed space with

$$\|f\|_p := \left(\int_X |f|^p\right)^{\frac{1}{p}} \quad p \in [1,\infty) \qquad \|f\|_\infty := \underset{X}{ess\,sup}\,|f|$$

Proof.

$\|\cdot\| : L^p \to [0,\infty)$

$$\|f\|_p = 0 \iff f = 0 \text{ } a.e. \text{ } in \text{ } X \iff f = 0 \text{ } in \text{ } L^p$$

$$\forall\, \alpha \in \mathbb{R}\,,\ f \in L^p \quad \|\alpha f\|_p = |\alpha|\|f\|_p$$

Minkowski inequality $\implies \|f+g\|_p \le \|f\|_p + \|g\|_p$

□

Lezione 15 (26/10/23)

Theorem: Inclusion of L^p spaces.

Suppose $\mu(X) < \infty$. Then,

$$1 \le p < q \le \infty \implies L^q(X,\mathcal{A},\mu) \subseteq L^p(X,\mathcal{A},\mu)$$

Proof.

The thesis follows if we show that

$$\exists\, C > 0 : \quad \|f\|_p \le C\|f\|_q \quad \forall\, f \in L^q$$

Suppose $q = \infty$. Then,

$$\|f\|_p^p = \int_X |f|^p \, d\mu \le \|f\|_\infty^p \mu(X) \implies \|f\|_p \le \underbrace{[\mu(X)]^{\frac{1}{p}}}_{:=C}\|f\|_\infty$$

Now let $q < \infty$. Due to the Hölder inequality we have

$$\|f\|_p^p = \int_X |f|^p \, d\mu = \int_X 1 \cdot |f|^p \, d\mu \le \left(\int_X 1^r \, d\mu \right)^{\frac{1}{r}} \cdot \left(\int_X |f|^{ps} \, d\mu \right)^{\frac{1}{s}}$$

Now take $ps = q \implies \dfrac{1}{s} = \dfrac{p}{q} \implies \dfrac{1}{r} = 1 - \dfrac{1}{s} = \dfrac{q-p}{q}$. Therefore,

$$\|f\|_p^p \le [\mu(X)]^{\frac{q-p}{q}} \left(\int_X |f|^q \, d\mu \right)^{\frac{p}{q}}$$

$$\implies \|f\|_p \le \underbrace{[\mu(X)]^{\frac{q-p}{qp}}}_{:=C} \|f\|_q$$

$\square$

Remark:

If $\mu(X) = \infty$, in general, the preceding inclusion is false. Indeed

$$f(x) = \frac{1}{x} \ , \ x \in (1, \infty) \ , \ \lambda((1,\infty)) = \infty$$

$$f \in L^2((1,\infty)), \text{ but } f \notin L^1((1,\infty))$$

Theorem: Interpolation inequality.

Let $(X, \mathcal{A}, \mu)$ be a measure space, $1 \le p < q \le \infty$.

If $f \in L^p \cap L^q$, then $f \in L^r \ \ \forall r \in (p,q)$. Moreover,

$$\|f\|_r \le \|f\|_p^{\alpha} \|f\|_q^{1-\alpha} \ \text{ where } \alpha \in (0,1) \ s.t. \ \frac{1}{r} = \frac{\alpha}{p} + \frac{1-\alpha}{q}$$

Proof.

$$\|f\|_r^r = \int_X |f|^r \, d\mu = \int_X \underbrace{|f|^{\alpha r}}_{\varphi} \underbrace{|f|^{(1-\alpha) r}}_{\psi} \, d\mu$$

$$\varphi \in L^{\frac{p}{\alpha r}} \iff \|\varphi\|_{\frac{p}{\alpha r}} = \left(\int_X |f|^{\cancel{\alpha r} \frac{p}{\cancel{\alpha r}}} \, d\mu \right)^{\frac{\alpha r}{p}} < \infty \quad \text{since } f \in L^p$$

$$\psi \in L^{\frac{q}{(1-\alpha)r}} \iff \|\psi\|_{\frac{q}{(1-\alpha)r}} = \left(\int_X |f|^{\cancel{(1-\alpha)r} \frac{q}{\cancel{(1-\alpha)r}}} \, d\mu \right)^{\frac{(1-\alpha)r}{q}} < \infty \quad \text{since } f \in L^q$$

Define $\mathcal{P} := \dfrac{p}{\alpha r}$ and $\mathcal{Q} := \dfrac{q}{(1-\alpha)r} \implies \dfrac{1}{\mathcal{P}} + \dfrac{1}{\mathcal{Q}} = 1$. By Hölder inequality,

$$\int_X |f|^r \, d\mu = \int_X |\varphi\psi| \, d\mu \leq \left(\int_X |\varphi|^{\mathcal{P}} \, d\mu\right)^{\frac{1}{\mathcal{P}}} \left(\int_X |\psi|^{\mathcal{Q}} \, d\mu\right)^{\frac{1}{\mathcal{Q}}} = \left(\int_X |f|^p \, d\mu\right)^{\frac{\alpha r}{p}} \left(\int_X |f|^q \, d\mu\right)^{\frac{(1-\alpha)r}{q}}$$

$$\left(\int_X |f|^r \, d\mu\right)^{\frac{1}{r}} = \left(\int_X |f|^p \, d\mu\right)^{\frac{\alpha}{p}} \left(\int_X |f|^q \, d\mu\right)^{\frac{1-\alpha}{q}} \iff \|f\|_r \leq \|f\|_p^{\alpha} \|f\|_q^{1-\alpha}$$

□

6.2 Completeness of L^p spaces

Theorem.

$L^p(X, \mathcal{A}, \mu)$ is a Banach space $\forall p \in [1, \infty]$

Proof.

Let $p \in [1, \infty)$

Claim: Let $\{f_n\} \subset L^p$. If $\sum_{n=1}^{\infty} \|f_n\|_p$ converges in L^p, then $\sum_{n=1}^{\infty} f_n$ converges in L^p.

From the claim the thesis follows due to a previous result about completeness of normed spaces.

Now we prove the Claim.

$$g_k := \sum_{n=1}^{k} |f_n|$$

$$\|g_k\|_p \overset{\text{Minkowski}}{\leq} \|f_1\|_p + \ldots + \|f_k\|_p \leq M := \sum_{n=1}^{\infty} \|f_n\|_p < \infty \text{ (by hypothesis)}$$

Let $g(x) := \sum_{n=1}^{\infty} |f_n|$

$\{g_k\} \nearrow$, g_k is measurable $\implies |g_k|^p \nearrow$, measurable

By Beppo-Levi theorem,

$$\lim_{k\to\infty} \underbrace{\int_X |g_k(x)|^p \, d\mu}_{\leq M^p \quad \forall\, k \in \mathbb{N}} = \int_X \lim_{k\to\infty} |g_k(x)|^p \, d\mu = \int_X |g(x)|^p \, d\mu \leq M^p$$

$$\implies g \in L^p \implies g \text{ finite a.e. in X}$$

$$\implies \sum_{n=1}^{\infty} f_n \text{ converges absolutely a.e. in X}$$

Let $s(x) := \sum_{n=1}^{\infty} f_n(x)$ and $s_k(x) := \sum_{n=1}^{k} f_n(x)$

$$s_k \xrightarrow[k\to\infty]{} s \textit{ a.e. in } X \implies |s_k - s|^p \xrightarrow[k\to\infty]{} 0 \textit{ a.e. in } X$$

$$|s_k - s|^p = \left|\sum_{n=k+1}^{\infty} f_n\right|^p \leq \left|\underbrace{\sum_{n=k+1}^{\infty} |f_n|}_{\geq 0 \text{ and } \leq g}\right|^p \leq g^p \in L^1 \; a.e. \; in \; X$$

By DCT,

$$\lim_{k\to\infty} \int_X |s_k - s|^p \, d\mu \int_X \underbrace{\lim_{k\to\infty} |s_k - s|^p \, d\mu}_{=0} \iff \|s_k - s\|_p \xrightarrow[k\to\infty]{} 0 \iff \sum_{n=1}^{\infty} f_n \text{ converges in } L^p$$

Obs. For $p = \infty$ the proof is similar.

□

6.3 Separability in L^p

Theorem: Lusin.

Let $\Omega \in \mathcal{L}(\mathbb{R}),\ \lambda(\Omega) < \infty$.

Let $f : \mathbb{R} \to \mathbb{R}$ measurable, $f = 0$ in Ω^c. Then,

$$\forall\, \varepsilon > 0 \;\; \exists\, g \in C_c^0(\mathbb{R}) \; s.t.$$

$$\lambda(\{x \in \mathbb{R} \, : \, f(x) \neq g(x)\}) < \varepsilon \;\text{ and }\; \sup_{\mathbb{R}} |g| \leq \underset{\mathbb{R}}{ess\,sup}\,|f|$$

Definition.

$$\widetilde{\mathcal{S}}(\mathbb{R}) := \{s \in \mathcal{S}(\mathbb{R}) \;,\; \lambda(Supp\, s) < \infty\}$$

Obs. $g : \Omega \subseteq \mathbb{R}^n \to \mathbb{R} \quad Supp\, g := \overline{\{x \in \Omega \, : \, g(x) \neq 0\}}$

Remark:

$$s \in \widetilde{\mathcal{S}}(\mathbb{R}) \iff s \text{ simple and } s \in L^p(\mathbb{R}) \;\; \forall\, p \in [1, \infty)$$

Theorem.

$\widetilde{\mathcal{S}}(\mathbb{R})$ is dense in $L^p(\mathbb{R}) \;\; \forall\, p \in [1, \infty)$

Proof.

$\widetilde{\mathcal{S}}(\mathbb{R}) \subset L^p \;\; \forall\, p \in [1, \infty)$

Let $f \in L^p(\mathbb{R})$. In addition, suppose that $f \geq 0$ a.e. in $\mathbb{R}$. For the simple approximation thm, we know:

$$\exists \{s_n\} \subset \mathcal{S}(\mathbb{R}) \; s.t. \; 0 \leq s_n \leq f \;, \{s_n\} \nearrow \;, \; s_n \xrightarrow[n\to\infty]{} f \; a.e. \; in \; \mathbb{R}$$

$$\implies \{s_n\} \subset L^p \implies \{s_n\} \subset \widetilde{\mathcal{S}}(\mathbb{R})$$

<u>Claim</u>: $s_n \xrightarrow[n\to\infty]{} f$ in L^p

In fact, $|s_n - f|^p \xrightarrow[n\to\infty]{} 0$ a.e. in $\mathbb{R}$

$$|f - s_n|^p \leq (|f| + |s_n|)^p \leq (|f| + |f|)^p = 2^p |f|^p =: g \in L^1(\mathbb{R})$$

$$\overset{\text{DCT}}{\implies} \lim_{n\to\infty} \int_{\mathbb{R}} |s_n - f|^p \, d\lambda = \int_{\mathbb{R}} \underbrace{\lim_{n\to\infty} |s_n - f|^p}_{0} \, d\lambda = 0 \iff s_n \xrightarrow[n\to\infty]{} f \; in \; L^p(\mathbb{R})$$

If f is sign-changing, we argue as above for f^+ and f^-. The thesis follows, since $f = f^+ - f^-$

□

Lezione 16 (08/11/23)

Theorem.

$C_c^0(\mathbb{R})$ is dense in $L^p(\mathbb{R}) \;\; \forall \, p \in [1, \infty)$

Proof.

Let $f \in L^p(\mathbb{R})$ and $\varepsilon > 0$.

We can find $s \in \widetilde{\mathcal{S}}(\mathbb{R})$ s.t. $\|f - s\|_p < \varepsilon$

By Lusin theorem, $\exists g \in C_c^0(\mathbb{R}) \; s.t. \; \|g\|_\infty \leq \|s\|_\infty \;, \; \lambda(\{g \neq s\}) < \dfrac{\varepsilon^p}{\|s\|_\infty^p}$

$$\implies \|f - g\|_p \leq \|f - s\|_p + \|s - g\|_p < \varepsilon + \left(\int_{\mathbb{R}} |s - g|^p \, d\lambda \right)^{\frac{1}{p}} = \varepsilon + \left(\int_{\{g \neq s\}} |s - g|^p \, d\lambda \right)^{\frac{1}{p}}$$

Since $|s - g| \leq |s| + |g| \leq \|s\|_\infty + \|g\|_\infty \leq 2\|s\|_\infty$

$$\implies \|f - g\|_p \leq \varepsilon + 2\|s\|_\infty \left(\int_{\{g \neq s\}} d\lambda \right)^{\frac{1}{p}} = \varepsilon + 2\|s\|_\infty \underbrace{\lambda\left(\{g \neq s\}\right)^{\frac{1}{p}}}_{< \frac{\varepsilon}{\|s\|_\infty}} = 3\varepsilon$$

□

Theorem.

Let $\Omega \subseteq \mathbb{R}^n$ open.

$L^p(\Omega, \mathcal{L}(\Omega), \lambda)$ is separable $\;\; \forall \, p \in [1, \infty)$

Proof.

We'll demonstrate for $\Omega = \mathbb{R}$. Let $f \in L^p(\mathbb{R}),\ \varepsilon > 0$.

$\exists g \in C_c^0$ s.t. $\|f - g\|_p < \varepsilon$ by the preceding theorem.

$$\exists\, n_0 \in \mathbb{N}\ s.t.\ supp\, g \subset [-n_0, n_0]$$

Since $C^0([-n_0, n_0])$ is separable, there exists a polynomial with rational coefficients s.t.

$$\|g - \xi \chi_{(-n_0,n_0)}\|_{L^\infty(\mathbb{R})} = \|g - \xi\|_{L^\infty((-n_0,n_0))} < \varepsilon' = \frac{\varepsilon}{(2n_0)^{\frac{1}{p}}}$$

$$\implies \|f - \xi\chi_{[-n_0,n_0]}\|_{L^p(\mathbb{R})} \leq \|f - g\|_p + \|g - \xi\chi_{[-n_0,n_0]}\|_p < \varepsilon + \left(\int_{[-n_0,n_o]} |g - \xi|^p \, d\lambda\right)^{\frac{1}{p}} <$$

$$< \varepsilon + \|g - \xi\|_\infty \left(\int_{[-n_0,n_0]} d\lambda\right)^{\frac{1}{p}} = \varepsilon + \|g - \xi\|_\infty (2n_0)^{\frac{1}{p}} < \varepsilon + \frac{\varepsilon}{(2n_0)^{\frac{1}{p}}}(2n_0)^{\frac{1}{p}} = 2\varepsilon$$

The set of all such ξ is countable.

$\square$

Lemma:

Let X be a metric space.

Assume that there exists a family $\{A_i\}_{i\in I}$ s.t.

a) $\forall i \in I\ A_i$ is open

b) $A_i \cap A_j = \emptyset \quad i \neq j$

c) I is uncountable

Then X is not separable.

Proof.

Suppose by contradiction that X is separable thus $\exists \{c_n\}_{n\in\mathbb{N}} \subset X$ s.t. $\overline{\bigcup_{n\in\mathbb{N}}\{c_n\}} = X$

$$\forall\, i \in I \quad A_i \cap \bigcup_{n\in\mathbb{N}}\{c_n\} \neq \emptyset$$

$$\implies \exists\, n(i) \in \mathbb{N}\ s.t.\ c_{n(i)} \in A_i$$

$$i \in I \mapsto n(i) \in \mathbb{N} \text{ is injective}$$

In fact,

$$n(i) = n(j) \iff c_{n(i)} = c_{n(j)} \in A_i \cap A_j \quad \overset{(b)}{\implies}\ i = j$$

This is a contradiction, because if it was true, then I would be countable.

□

Theorem.

$L^\infty(\mathbb{R}, \mathcal{L}(\mathbb{R}), \lambda)$ is not separable.

Proof.

Let $\{\chi_{[-\alpha,\alpha]}\}_{\alpha>0} \subset L^\infty(\mathbb{R})$ with $\alpha \in \mathbb{R}$. Then this family is uncountable.

If $\alpha \neq \alpha'$, then $\|\chi_{[-\alpha,\alpha]} - \chi_{[-\alpha',\alpha']}\|_\infty = 1$

Define the open balls $A_\alpha := B_{\frac{1}{2}}\left(\chi_{[-\alpha,\alpha]}\right) := \{f \in L^\infty(\mathbb{R}) \ : \ \|\chi_{[-\alpha,\alpha]} - f\|_\infty < \frac{1}{2}\}$

We have seen that the 'distance' between χ with different α is 1 then $A_\alpha \cap A_{\alpha'} = \emptyset$ since the balls have 'radius' $\frac{1}{2}$. Furthermore, $\{A_\alpha\}$ is uncountable.

By the previous lemma, L^∞ is not separable.

□

6.4 l^p spaces

l^p is a Banach space $\forall p \in [1, \infty]$

$p \in [1,\infty) \quad \|x\|_p := \left(\sum_{n=1}^{\infty} |x^{(n)}|^p\right)^{\frac{1}{p}} \quad x = \{x^{(n)}\}_{n\in\mathbb{N}}$

$p = \infty \quad \|x\|_\infty := \sup_{n\in\mathbb{N}} |x^{(k)}|$

$$l^p = L^p(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu^\#) \quad f : \mathbb{N} \to \mathbb{R} \quad f \equiv \{x^{(n)}\}_{n\in\mathbb{N}} \quad x^{(1)} = f(1),\ x^{(2)} = f(2) \ldots$$

$$\|f\|_p = \left(\int_{\mathbb{N}} |f|^p \, d\mu^\#\right)^{\frac{1}{p}} = \left(\sum_{n=1}^{\infty} |x^{(n)}|^p\right)^{\frac{1}{p}} \quad p \in [1,\infty)$$

$$\|f\|_\infty = \operatorname*{ess\,sup}_{n\in\mathbb{N}} |f(n)| = \sup_{n\in\mathbb{N}} |x^{(n)}|$$

- l^p is separable $\forall p \in [1, \infty)$
- l^∞ is not separable

Inclusion in L^p and l^p

$$(X, \mathcal{A}, \mu) \ \ \mu(X) < \infty \ \ q \geq p \implies L^q \subseteq L^p$$

$$\mu^\#(\mathbb{N}) = \infty \quad q \geq p \implies l^q \supseteq l^p$$

Obs. This is because an element is in l^p when there is convergence of the series, that means the general term is convergent to zero, but by increasing q to p the general term still converges.

7 Linear operators

Let X, Y be vector spaces.

Definition.

$T : X \to Y$ is a **linear operator** if $T(\alpha v_1 + \beta v_2) = \alpha T(v_1) + \beta T(v_2) \ \ \forall\, v_1, v_2 \in X \ \ \forall\, \alpha, \beta \in \mathbb{R}$

If $Y = \mathbb{R}$, then T is called **Functional**

Remark:

$$T(0) = T(0 \cdot x) = 0 \cdot T(x) = 0 \quad (\alpha = 0)$$

Definition.

X, Y normed spaces. We say that T is **bounded** if $\exists M > 0$:

$$\|T(x)\|_Y \le M\|x\|_X \ \ \forall\, x \in X$$

Definition.

$T : X \to Y$ is **continuous** in $x_0 \in X$

$$\overset{\text{def}}{\Longleftrightarrow} \forall \{x_n\} \subset X,\ x_n \xrightarrow[n\to\infty]{} x_0 \ \text{ we have that } \ T(x_n) \xrightarrow[n\to\infty]{} T(x_0)$$

Obs. By $x_n \to x_0$ we mean $\|x_n - x_0\|_X \to 0$ and by $T(x_n) \to T(x_0)$ we mean $\|T(x_n) - T(x_0)\|_Y \to 0$

Definition.

$T : X \to Y$ is **Lipschitz** $\overset{\text{def}}{\Longleftrightarrow}$

$$\exists\, L > 0 \ : \ \|T(x) - T(y)\|_Y \le L\|x - y\|_X \ \ \forall\, x, y \in X$$

Theorem.

Let $T : X \to Y$ be a linear operator. Then the following statements are equivalent:

i) T is bounded

ii) T is Lipschitz

iii) T is continuous at $x_0 = 0$

iv) T is continuous at any $x_0 \in X$

Proof.

$i) \implies ii)$

$$\|T(x)-T(y)\|_Y = \|T(x-y)\|_Y \le M\|x-y\|_X \quad \forall\, x,y \in X$$

$ii) \implies iii)$

$$\{x_n\} \subset X,\ x_n \to 0 \Longleftrightarrow \|x_n\|_X \xrightarrow[n\to\infty]{} 0$$

$$\|T(x_n)-T(0)\|_Y = \|T(x_n)\|_Y \le M\|x_n - 0\|_X = M\|x_n\|_X \xrightarrow[n\to\infty]{} 0$$

$iii) \implies i)$

Suppose, by contradiction, that T is not bounded. Then there exists $\{x_n\} \subset X,\ x_n \neq 0$ s.t.

$$\|T(x_n)\|_Y \ge n\|x_n\|_X$$

We have that $S_n := \dfrac{x_n}{n\|x_n\|_X} \xrightarrow[n\to\infty]{} 0$, in fact $\|S_n\|_X = \dfrac{1}{n} \xrightarrow[n\to\infty]{} 0$

$$T(S_n) = \frac{1}{n\|x_n\|_X}T(x_n) \implies \|T(S_n)\|_Y = \frac{1}{n\|x_n\|_X}\|T(x_n)\|_Y \ge \frac{1}{n\|x_n\|}n\|x_n\| = 1 \quad \forall\, n \in \mathbb{N}$$

Therefore, $T(S_n) \underset{n\to\infty}{\not\longrightarrow} T(0) = 0 \quad \implies$ T is not continuous at $x_0 = 0$.

$iv) \implies iii)$ is obvious.

$iii) \implies iv)$

$$x_0 \in X,\ \{x_n\} \subset X,\ x_n \xrightarrow[n\to\infty]{} x_0 \Longleftrightarrow \|x_n - x_0\|_X \xrightarrow[n\to\infty]{} 0$$

$$\|T(x_n) - T(x_0)\|_Y \overset{iii) \implies ii)}{\le} L\|x_n - x_0\|_X \to 0 \iff T(x_n) \xrightarrow[n\to\infty]{} T(x_0)$$

$\square$

Remark:

X, Y normed spaces, $T : X \to Y$ linear and $dim X < \infty$. Then T is continuous.

7.1 Linear continuous operators

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Example:

$T : l^2 \to l^2 \quad T(x) = \left(\dfrac{x^{(1)}}{1}, \dfrac{x^{(2)}}{2}, ..., \dfrac{x^{(k)}}{k}, ...\right) \quad \forall\, x = \{x^{(k)}\} \in l^2$

T is linear.

T is continuous $\iff T$ is bounded

$$\|T(x)\|_{l^2}^2 = \sum_{k=1}^{\infty} \left(\frac{x^{(k)}}{k}\right)^2 \leq \sum_{k=1}^{\infty} (x^{(k)})^2 = \|x\|_{l^2}^2 \implies \|T(x)\|_{l^2} \leq \|x\|_{l^2} \ \ \forall\, x \in l^2$$

Let X, Y be normed spaces.

$$\mathcal{B}(X,Y) = \mathcal{L}(X,Y) := \{T : X \to Y \; s.t.\, T \text{ is linear and continuous}\}$$

If $Y = X$, then $\mathcal{L}(X) \equiv \mathcal{L}(X,X)$.

Remark:

$\mathcal{L}(X,Y)$ is a vector space. If $T \in \mathcal{L}(X,Y)$, then

$$\exists\, M > 0 \ : \ \|T(x)\|_Y \leq M \ \ \forall\, x \in X, \|x\|_X \leq 1$$

$$\implies \sup_{x \in X, \|x\|_X \leq 1} \|T(x)\|_Y \ \text{ is well defined.}$$

$(\mathcal{L}(X,Y), \|\cdot\|_{\mathcal{L}})$ is a normed space, where $\|T\|_{\mathcal{L}} := \sup_{x \in X, \|x\|_X \leq 1} \|T(x)\|_Y$ is called operator norm.

Proposition:

$$\|T\|_{\mathcal{L}} = \sup_{\|x\|_X = 1} \|T(x)\|_Y = \sup_{x \in X \setminus \{0\}} \frac{\|T(x)\|_Y}{\|x\|_X}$$

Proof.

$$\sup_{\|x\|_X \leq 1} \|T(x)\|_Y \geq \sup_{\|x\|_X = 1} \|T(x)\|_Y$$

If $\|x\|_X \leq 1$, $x \neq 0$, then

$$\|T(x)\|_Y = \|x\|_X \left\| T\left(\frac{x}{\|x\|_X}\right)\right\|_Y \leq \left\| T\left(\frac{x}{\|x\|_X}\right)\right\|_Y$$

$$\implies \sup_{\|x\|_X \leq 1} \|T(x)\| \leq \sup_{\|\xi\| = 1} \|T(\xi)\|_Y$$

This last two conditions imply the following: $\|T\|_{\mathcal{L}} = \sup_{\|x\| = 1} \|T(x)\|_Y$

Furthermore,

$$\frac{\|T(x)\|_Y}{\|x\|_X} = \left\| T\left(\frac{x}{\|x\|_X}\right)\right\|_Y$$

$$\sup_{x \in X\setminus\{0\}} \frac{\|T(x)\|_Y}{\|x\|_X} = \sup_{\|\xi\|=1} \|T(\xi)\|_Y = \|T\|_{\mathcal{L}}$$

□

Theorem.

Let X be a normed space and Y a Banach space. Then $\mathcal{L}(X,Y)$ is a Banach space.

7.2 Uniform boundedness principle (UBP) or Banach-Steinhaus theorem

X, Y Banach spaces, $\mathcal{F} \subset \mathcal{L}(X,Y)$

$$\mathcal{F} \text{ is point-wise bounded} \overset{\text{def}}{\iff} \forall x \in X \ \ \exists\, M_x > 0 \ : \ \sup_{T \in \mathcal{F}} \|T(x)\|_Y \le M_x$$

$$\mathcal{F} \text{ is uniformly bounded} \overset{\text{def}}{\iff} \exists M > 0 \ : \ \sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}} \le M$$

Obs. $\sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}} \equiv \sup_{T \in \mathcal{F}} \sup_{\|x\| \le 1} \|T(x)\|_Y$

Theorem.

Let X, Y be Banach spaces, $\mathcal{F} \subset \mathcal{L}(X,Y)$

If $\mathcal{F}$ is point-wise bounded, then $\mathcal{F}$ is also uniformly bounded.

Proof.

$\forall n \in \mathbb{N}$ Let $C_n := \{x \in X \ : \ \|T(x)\|_Y \le n \ \ \forall\, T \in \mathcal{F}\}$

i) Show that C_n is closed $\forall n \in \mathbb{N}$.

$$\text{Let } \{x_n\} \subset C_n \,,\ x_k \xrightarrow[k \to \infty]{} x_0 \in X$$

Therefore,

$$T(x_k) \xrightarrow[k \to \infty]{} T(x_0) \ \ \forall\, T \in \mathcal{F} \implies \underbrace{\|T(x_k)\|_Y}_{\le n} \xrightarrow[k \to \infty]{} \|T(x_0)\|_Y$$

$$\implies \|T(x_0)\|_Y \le n \implies x_0 \in C_N \implies C_N \text{ is closed}$$

ii) $\bigcup_{n=1}^{\infty} C_n = X$

Due to Baire's theorem there exists $n_0 \in \mathbb{N}$ s.t. $Int\, C_{n_0} \neq \emptyset$

$$\implies \quad \exists\, \overline{B}_\varepsilon(x_0) \subset C_{n_0}$$

If $\|z\|_X \leq \varepsilon \implies z + x_0 \in \overline{B}_\varepsilon(x_0) \subset C_{n_0}$

$$\implies \|T(z)\|_Y = \|T(z) + T(x_0) - T(x_0)\|_Y \leq \|T(z) + T(x_0)\|_Y + \|T(x_0)\|_y$$

$$= \underbrace{\|T(z + x_0)\|_Y}_{\leq n_0} + \underbrace{\|T(x_0)\|_Y}_{\leq n_0} \leq 2n_0 \quad \forall T \in \mathcal{F}$$

$$\forall x \in X \backslash \{0\}\,, \forall T \in \mathcal{F} \quad \|T(x)\|_Y = \frac{\|x\|_X}{\varepsilon} \left\| T\left(\frac{\varepsilon x}{\|x\|_X}\right)\right\|_Y \leq \frac{2n_0}{\varepsilon}\|x\|_X \qquad \left(z = \tfrac{\varepsilon x}{\|x\|}\right)$$

$$\implies \|T\|_{\mathcal{L}} \leq \frac{2n_0}{\varepsilon} =: M \implies \sup_{T \in \mathcal{F}} \|T\|_{\mathcal{L}} \leq M$$

$$\implies \mathcal{F} \text{ is uniformly bounded}$$

□

Corollary:

X, Y Banach spaces, $\{T_n\} \subset \mathcal{L}(X, Y)$.

Assume that $\forall x \in X \ \ \exists \lim\limits_{n\to\infty} T_n(x)$ and let

$$T : X \to Y \quad T(x) := \lim_{n\to\infty} T_n(x)$$

Then $T \in \mathcal{L}(X, Y)$

Proof.

$T : X \to Y$ is linear, we want to show that is also bounded.

$\{T_n(x)\}_{n\in\mathbb{N}}$ is bounded $\forall x \in X$ (since is convergent)

$$\iff \forall x \in X \ \ \exists\, M_x > 0 \ : \ \|T_n(x)\|_Y \leq M_x \ \ \forall\, n \in \mathbb{N}$$

$$\iff \mathcal{F} = \{T_n\}_{n\in\mathbb{N}} \text{ is pointwise bounded}$$

$$\overset{\text{UBP}}{\implies} \mathcal{F} = \{T_n\}_{n\in\mathbb{N}} \text{ is uniformly bounded}$$

$$\iff \quad \exists\, M > 0 \ : \ \sup_{n\in\mathbb{N}} \|T_n\|_{\mathcal{L}} \leq M$$

$$\implies \|T_n(x)\|_Y \leq M\|x\|_X \ \ \forall\, n \in \mathbb{N}, \ \ \forall\, x \in X$$

$$\|T(x)\|_Y = \lim_{n\to\infty} \|T_n(x)\|_Y \leq M\|x\|_X \ \ \forall\, x \in X$$

$$\implies T \text{ is bounded} \implies T \in \mathcal{L}(X, Y)$$

□

7.3 Open mapping theorem

Definition.

X, Y metric spaces, $T : X \to Y$

$$T \text{ is } \textbf{continuous} \iff T^{-1}(E) \subset X \text{ is open } \ \forall\, E \subset Y \text{ open}$$

Definition.

X, Y metric spaces, $T : X \to Y$

$$T \text{ is } \textbf{open} \overset{\text{def}}{\iff} T(A) \subset Y \text{is open } \ \forall\, A \subset X \text{ open}$$

Definition: Open mapping theorem, OMT.

Let X, Y be Banach spaces, $T \in \mathcal{L}(X, Y)$ <u>surjective</u>.

Then T is an **open** mapping.

Corollary: Inverse Bounded(continuous) Mapping, IBM

Let $T \in \mathcal{L}(X, Y)$, T bijective. Then $T^{-1} \in \mathcal{L}(Y, X)$.

Proof.

$$\left.\begin{array}{c} T \in \mathcal{L}(X, Y) \\ T \text{ bijective} \end{array}\right\} \implies \exists T^{-1} : Y \to X \text{ and } T^{-1} \text{ is linear}$$

Claim: $T^{-1} : Y \to X$ is continuous $\overset{\text{def}}{\iff} (T^{-1})^{-1}(E)$ is open $\forall E \subset X$ open.

$(T^{-1})^{-1}(E) = T(E)$ is open, in view of the OMT.

□

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Definition.

A linear operator $T : X \to Y$ is **closed** if

$$\left.\begin{array}{r} x_n \xrightarrow[n\to\infty]{} x \text{ in } X \\ T(x_n) \xrightarrow[n\to\infty]{} y \text{ in } Y \end{array}\right\} \implies T(x) = y$$

Remark:

$T \in \mathcal{L}(X,Y) \implies T$ is closed, in fact

$$x_n \to x \implies T(x_n) \to T(x) \iff y = T(x)$$

Remark:

$(X \times Y, \|x\|_X + \|y\|_Y)$ is a normed space.

Definition.

$T : X \to Y$. Then $graph\, T := \{(x, T(x)) \,:\, x \in X\} \subseteq X \times Y$

Proposition:

$T : X \to Y$ linear and closed $\iff graph\, T$ is closed $\quad \big(graph\, T \subseteq (X \times Y, \|x\|_X + \|y\|_Y)\big)$

Proof.

Let $\{(x_n, T(x_n))\} \subset graph\, T$ be s.t. $(x_n, T(x_n)) \xrightarrow[n\to\infty]{} (x,y) \in X \times Y$

This implies $x_n \to x$ and $T(x_n) \to y$.

$graph\, T$ is closed $\iff (x,y) \in graph\, T \iff y = T(x) \iff T$ is closed $\square$

Corollary: of ICM

Let $(X, \|\cdot\|_1), (X, \|\cdot\|_2)$ be Banach spaces. Suppose that $\exists M > 0$:

$$\|x\|_2 \le M\|x\|_1 \quad \forall x \in X$$

Then $\|\cdot\|_1$ and $\|\cdot\|_2$ are equivalent, i.e.

$$\exists\, m > 0 : \quad \|x\|_2 \ge m\|x\|_1 \quad \forall x \in X$$

Proof.

$I : (X, \|\cdot\|_1) \to (X, \|\cdot\|_2) \quad I(x) = x \quad \forall x \in X$

I, the identity mapping, is bijective, linear and continuous.

I continuous $\iff I$ bounded $\iff \|x\|_2 = \|I(x)\|_2 \le M\|x\|_1 \quad \forall x \in X$

By ICM, $I^{-1} : (X, \|\cdot\|_2) \to (X, \|\cdot\|_1)$ $I^{-1}(x) = x$ is linear and bounded (continuous).

Therefore, $\quad \exists\, m' > 0 : \|x\|_1 = \|I^{-1}(x)\|_1 \le m'\|x\|_2 \quad \forall x \in X$

Thus $\quad \|x\|_2 \geq \underbrace{\frac{1}{m'}}_{m} \|x\|_1 \quad \forall\, x \in X$ $\square$

THEOREM: Closed graph theorem, CGT.

Let $T : X \to Y$ be linear and closed operator. Then $T \in \mathcal{L}(X, Y)$.

PROOF.

Let $\|x\|_2 := \|x\|_X + \underbrace{\|T(x)\|_Y}_{\geq 0}$ be the graph norm.

Clearly, $(X, \|\cdot\|_2)$ is Banach and $\|x\|_X \leq \|x\|_2$

By the preceding corollary, $\exists M \geq 1$:

$$\|x\|_2 = \|x\|_X + \|T(x)\|_Y \leq M\|x\|_X \quad \forall\, x \in X$$

$$\implies \|T(x)\|_Y \leq (M-1)\|x\|_X \quad \forall\, x \in X$$

$$\implies T \text{ is bounded} \quad \implies T \in \mathcal{L}(X, Y)$$

$\square$

8 Dual spaces

DEFINITION.

X normed space. $X^* := \mathcal{L}(X, \mathbb{R})$ is called **dual space**.

X^* is a Banach space with norm $\|L\|_* := \sup\limits_{\|x\|_X = 1} |L(x)| \quad (L(x) \in \mathbb{R})$

EXAMPLE:

$X = L^p(X, \mathcal{A}, \mu)$ and q $s.t.$ $\frac{1}{p} + \frac{1}{q} = 1$

$$g \in L^q \qquad L_g : L^p \to \mathbb{R} \quad L_g(f) := \int_X fg\, d\mu$$

L_g is linear, indeed:

$$L_g(\alpha_1 f_1 + \alpha_2 f_2) = \int_X (\alpha_1 f_1 + \alpha_2 f_2) y\, d\mu = \alpha_1 \int_X f_1\, d\mu + \alpha_2 \int_X f_2\, d\mu = \alpha_1 L_g(f_1) + \alpha_2 L_g(f_2) \ \forall\, f_1, f_2 \in L^p\,,\ \alpha_1, \alpha_2 \in \mathbb{R}$$

L_g is bounded, indeed

$$|L_g(f)| = \left|\int_X fg\,d\mu\right| \leq \int_x |fg|\,d\mu \overset{\text{Hölder ineq.}}{\leq} \|f\|_p \underbrace{\|g\|_q}_{M} \quad \forall\, f \in L^p$$

$$\implies L_g \in (L^p)^*$$

We now compute $\|L_g\|_*$.

From the previous inequality,

$$\|L_g\|_* \leq \|g\|_q$$

Consider $\varphi := \dfrac{|g|^{q-2}g}{\|g\|_q^{q-1}}$

$$L_g(\varphi) = \int_X \varphi g\,d\mu = \frac{1}{\|g\|_q^{q-1}} \int_X |g|^q\,d\mu = \frac{\|g\|_q^q}{\|g\|_q^{q-1}} = \|g\|_q$$

$$\implies \|L_g\|_* = \|g\|_{L^q}$$

Example:

$(V, \langle\cdot,\cdot\rangle)\quad dimV = n$

$L : V \to \mathbb{R}$ linear ($\iff L \in \mathcal{L}(V,\mathbb{R})) \iff L \in V^* \implies \exists\,!\, y \in V : L(x) = \langle x, y\rangle \quad \forall\, x \in V$

Existence:

Consider $\mathcal{B} = \{\underline{v}_1, \underline{v}_2, ..., \underline{v}_n\}$ o.n.b. of V

$$\forall x \in V \quad x = \alpha_1\underline{v}_1 + ... + \alpha_n\underline{v}_n, \text{ where } \alpha_i = \langle x, \underline{v}_i\rangle \quad \forall\, i = 1, ..., n$$

$$L(x) = \alpha_1 L(\underline{v}_1) + ... + \alpha_n L(\underline{v}_n) = \langle x, \underline{v}_1\rangle L(\underline{v}_1) + ... + \langle x, \underline{v}_n\rangle L(\underline{v}_n) = \left\langle x, \underbrace{\underline{v}_1 L(\underline{v}_1) + ... + \underline{v}_n L(\underline{v}_n)}_{y}\right\rangle$$

Uniqueness:

By contradiction, suppose that $\exists y' \in V \;:\; L(x) = \langle x, y'\rangle \quad \forall\, x \in V$

$$0 = L(x) - L(x) = \langle x, y\rangle - \langle x, y'\rangle = \langle x, y - y'\rangle \quad \forall\, x \in V \quad \implies \quad y - y' = 0 \iff y = y'$$

Now we compute $\|L\|_*$.

$$|L(x)| = |\langle x, y\rangle| \leq \|x\| \underbrace{\|y\|}_{=:M} \quad \forall\, x \in V \implies \|L\|_* \leq \|y\|$$

$$\frac{|L(y)|}{\|y\|} = \frac{|\langle y, y\rangle|}{\|y\|} = \frac{\|y\|^2}{\|y\|} = \|y\| \quad \implies \|L\|_* = \|y\|$$

Example:

$X = \mathbb{R}^2$, Y v.s.s. of X

$\varphi \in Y^* \iff \varphi : Y \to \mathbb{R}$ is linear and continuous

We want to find $\psi \in X^* \, : \, \psi = \varphi$ in $Y \subset X \quad \|\psi\|_* = \|\varphi\|_*$

$$\varphi \in Y^* \overset{\text{preceding ex}}{\Longrightarrow} \exists\,! \eta \in Y \, : \, \varphi(x) = \langle \eta, x \rangle \quad \forall\, x \in Y$$

$$\psi : X \to \mathbb{R} \quad \psi(x) = \langle \eta, x \rangle$$

ψ is linear and bounded. Moreover, $\|\psi\|_* = \|\eta\| = \|\varphi\|_*$

8.1 Hahn-Banach theorem

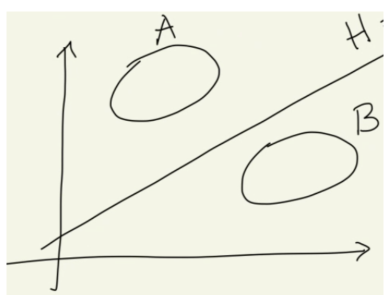
Theorem: Continuos extension.

Let X be a normed space, Y v.s.s. of X, $f \in Y^*$. Then,

$$\exists F \in X^* \, : \, F(y) = f(y) \quad \forall\, y \in Y, \quad \|F\|_{X^*} = \|f\|_{Y^*}$$

Obs. Now we define elements to state the same theorem in a geometrical way.

Let $X = \mathbb{R}^2$ and A, B disjoint convex. Then there exists a separation line $H = \{f(x) = \alpha\}$.



Definition.

X normed space, $\alpha \in \mathbb{R}$, $f \in X^*$. Then,

$$H := \{x \in X \, : \, f(x) = \alpha\} \text{ is an } \textbf{hyperplane} \text{ (closed)}$$

Definition.

We say that H **separates** $A \subseteq X$ and $B \subseteq X$ if

$$f(a) \leq \alpha \leq f(b) \;\; \forall\, a \in A,\; b \in B$$

We say that H strictly separates $A \subseteq X$ and $B \subseteq X$ if $\exists\, \varepsilon > 0$:

$$f(a) \leq \alpha - \varepsilon \;,\; f(b) \geq \alpha + \varepsilon \quad \forall\, a \in A,\; b \in B$$

Theorem: Separation form.

Let X be a normed space. If $A, B \subseteq X$, $A, B \neq \emptyset$ are disjoint convex sets and A is open.

Then there exists a closed hyperplane H which separates A and B.

Remark:

A closed, B compact. Then $\exists$ closed hyperplane H which strictly separates A and B

8.2 Consequences of the H.B. theorem

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Corollary:

Let X be a normed space and $x_0 \in X \backslash \{0\}$. Then,

$$\exists L_{x_0} \in X^* \; s.t. \; \|L_{x_0}\|_{X^*} = 1 \;,\; L_{x_0}(x_0) = \|x_0\|$$

Proof.

$Y := Span\{x_0\} = \{\lambda x_0 \;:\; \lambda \in \mathbb{R}\}$ v.s.s. of X

$$L_0 : Y \to \mathbb{R} \;\; L_0(\lambda x_0) := \lambda \|x_0\| \;\; \text{linear and bounded}$$

By the H.B. thm (continuous extension), $\exists\, \widetilde{L}_0 : X \to \mathbb{R}$, $\widetilde{L}_0 \in X^*$

$$\|\widetilde{L}_0\|_{X^*} = \|L_0\|_{Y^*} = \sup_{\lambda x_0 \in Y,\, \|\lambda x_0\| = 1} \Big| \underbrace{\lambda \|x_0\|}_{L_0(\lambda x_0)} \Big| = 1$$

$$\text{Moreover} \quad \widetilde{L}_0(\underbrace{x_0}_{\in Y}) = L_0(x_0) = \|x_0\| \qquad L_{x_0} := \widetilde{L}_0$$

$\square$

Corollary:

Let $y, z \in X$, $L(y) = L(z)$ $\forall L \in X^*$. Then $y = z$.

Proof.

Suppose, by contradiction, that $\exists y, z \in X,\ y \neq z$ s.t. $L(y) = L(z)$ $\forall L \in X^*$

$$x := y - z \neq 0 \quad L(x) = L(y - z) = L(y) - L(z) = 0 \quad \forall L \in X^*$$

By the preceding corollary,

$$\exists L_x \in X^* \ s.t.\ L_x(x) = \|x\| \neq 0 \quad \text{(contradiction)}$$

$\square$

Corollary:

Let $Y \subseteq X$ be a v.s.s. with $\overline{Y} \neq X$ and $x_0 \in X \backslash \overline{Y}$. Then,

$$\exists L \in X^* \ :\ L(x_0) \neq 0,\ L|_Y = 0$$

Proof.

$Z := \{\lambda x_0 + y \ :\ y \in Y,\ \lambda \in \mathbb{R}\} \subset X$ v.s.s.

$$L_0 : Z \to \mathbb{R} \quad L_0(\lambda x_0 + y) = \lambda$$

$$L_0(x_0) = L(\underset{=\lambda}{1} x_0 + \underset{=y}{0}) = 1 \neq 0$$

$$Ker(L_0) = \{\lambda x_0 + y \in Z \ :\ \underbrace{L_0(\lambda x_0 + y) = 0}_{\lambda = 0}\} = Y \quad L_0|_Y = 0$$

By the H.B. theorem,

$$\exists \widetilde{L}_0 \in X^* \ :\ \widetilde{L}_0 = L_0 \ in\ Z \supseteq Y$$

$$L := \widetilde{L}_0 \implies L|_Y = 0 \quad L(x_0) = L_0 = (x_0) = 1 \neq 0$$

$\square$

Theorem.

Let X be a normed space. X^* is separable $\implies$ X is separable.

8.3 Reflexive spaces

Let X be a normed space, X^* its dual. The dual of X^*, $(X^*)^* \equiv X^{**}$ is called bidual (or second dual) of X. For each $x \in X$, we define:

$$\Lambda_x : X^* \to \mathbb{R} \quad \Lambda_x(L) := L(x) \;\; \forall\, L \in X^*$$

Λ_x is linear.

$$|\Lambda_x(L)| = |L(x)| \leq \|L\|_{X^*} \underbrace{\|x\|_X}_{M} \;\; \forall\, L \in X^* \quad \Longrightarrow \Lambda_x \text{ is bounded}$$

Therefore,

$$\Lambda_x \in X^{**}, \quad \|\Lambda_x\|_{X^{**}} \leq \|x\|_X$$

$$\tau : X \to X^{**} \quad \tau(x) := \Lambda_x \;\; \forall\, x \in X$$

Definition.

τ is called **canonical map** (or evaluation map)

Theorem.

τ is linear and $\|\tau(x)\|_{X^{**}} = \|x\|_X \;\; \forall\, x \in X$

Proof.

Linearity is obvious.

$$\|\Lambda_x\|_{X^{**}} \leq \|x\|_X \;\Longrightarrow\; \|x\|_X \geq \|\tau(x)\|_{X^{**}} \;\; \forall\, x \in X$$

It remains to show that $\|x\|_X \leq \|\tau(x)\|_{X^{**}} \;\; \forall\, x \in X$

By a corollary of the H.B. theorem, for each $x \in X \backslash \{0\}$

$$\exists L \in X^* \text{ with } \|L\|_{X^*} = 1\,,\;\; L(x) = \|x\|_X$$

Therefore,

$$\|\tau(x)\|_{X^{**}} = \|\Lambda_x\|_{X^{**}} = \sup_{\|L\|_{X^*}=1} \underbrace{|\Lambda_x(L)|}_{L(x)} \geq \|x\|_X$$

$\square$

Proposition:

Let X be a Banach space. $\tau(X)$ is closed in X^{**}

Proof.

X is complete $\overset{\tau \text{ isometry}}{\Longrightarrow} \tau(X)$ is complete $\implies \tau(X)$ is closed

$\square$

Remark:

τ is injective.

Definition.

If $\tau(X) = X^{**}(\iff \tau$ is surjective), then X is said to be **reflexive**.

Remark:

X is reflexive $\iff \forall \varphi \in X^{**} \quad \varphi(L) = L(x) \quad \forall\, L \in X^* \quad$ where $x := \tau^{-1}(\varphi)$

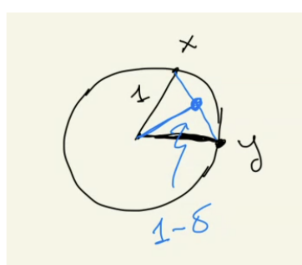
Definition.

Let X be a normed space. X is **uniformly convex** if $\forall \varepsilon > 0 \;\; \exists\, \delta > 0$ s.t.

$$\forall x, y \in X \quad \|x\| \leq 1,\; \|y\| \leq 1,\; \|x - y\| > \varepsilon \implies \left\| \frac{x + y}{2} \right\| < 1 - \delta$$

Example:

Given $(\mathbb{R}^2, \|\cdot\|_2)$ uniformly convex, take x and y on a ball with a fixed distance, then you have that the middle point is smaller then a certain distance (1 is the radius of the ball).



Theorem.

$L^p(\Omega)$ is uniformly convex $\quad \forall\, p \in (1, \infty)$

Clarkson inequalities $\implies$ uniformly convex

• I Clarkson inequality for $p \geq 2$ • II Clarkson inequality for $1 < p < 2$

Consider $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda)$ and $\Omega \in \mathcal{L}(\mathbb{R}^n)$

I) $p \geq 2$

$$\left\| \frac{f+g}{2} \right\|_p^p + \left\| \frac{f-g}{2} \right\|_p^p \leq \frac{1}{2} \left(\|f\|_p^p + \|g\|_p^p \right) \quad \forall\, f, g \in L^p(\Omega)$$

II) $1 < p < 2$

$$\left\| \frac{f+g}{2} \right\|_p^{p'} + \left\| \frac{f-g}{2} \right\|_p^{p'} \leq \frac{1}{2^{\frac{1}{p-1}}} \left(\|f\|_p^p + \|g\|_p^p \right)^{\frac{1}{p-1}} \quad \forall\, f, g \in L^p(\Omega) \;\text{ with } \tfrac{1}{p} + \tfrac{1}{p'} = 1$$

Proposition:

L^p is uniformly convex $\forall p \in (1, \infty)$

Proof.

Take any $\varepsilon > 0$, $f, g \in L^p$, $\|f\|_p \leq 1$, $\|g\|_p \leq 1$, $\|f-g\|_p > \varepsilon$

$$[p \geq 2] \text{ C. ineq. } \Longrightarrow \left\| \frac{f+g}{2} \right\|_p^p < 1 - \left(\frac{\varepsilon}{2}\right)^p \Longrightarrow \left\| \frac{f+g}{2} \right\|_p < 1 - \delta, \text{ where } \delta = 1 - \left[1 - \left(\frac{\varepsilon}{2}\right)^p\right]^{\frac{1}{p}}$$

$[1 < p < 2]$ it's similar.

□

Theorem: Milman-Pettis.

X Banach space, uniformly complex. Then X is reflexive.

Corollary:

$L^p(\Omega)$ is reflexive $\forall p \in (1, \infty)$

Remark:

$L^1(\Omega)$, $L^\infty(\Omega)$ are not reflexive.

8.4 Dual of L^p

Lezione 20 (22/11/23)

THEOREM: Riesz.

Let $(X, \mathcal{A}, \mu)$ be a measure space, $p \in (1, \infty)$. For any $\Lambda \in (L^p)^* \ \ \exists!\, g \in L^q$ with $\frac{1}{p} + \frac{1}{q} = 1$, s.t.

$$\Lambda(f) = \int_X fg \, d\mu \ \ \forall\, f \in L^p$$

Furthermore, $\|\Lambda\|_{(L^p)^*} = \|g\|_{L^q}$

Obs. We'll see the proof when we demonstrate Riesz thorem for Hilbert spaces, since it's similar.

REMARK:

The same theorem holds, when $p = 1$ and $q = \infty$. Provided that μ is σ-finite.

8.5 Dual of L^∞

$(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda), \ \ \Omega \in \mathcal{L}(\mathbb{R}^n)$

Let $g \in L^1, \ \ L_g : L^\infty \to \mathbb{R} \ \ L_g(f) := \displaystyle\int_\Omega fg \, d\lambda \ \ \forall\, f \in L^\infty$

L_g is linear.

$$|L_g(f)| \le \|f\|_\infty \underbrace{\|g\|_1}_{M} \implies \|L_g\|_{(L^\infty)^*} \le \|g\|_1$$

$$f := sgn(g) \implies |L_g(f)| = \|g\|_1 \implies \|L_g\|_{(L^\infty)^*} = \|g\|_1$$

So we can say that L^1 is kind of included in $(L^\infty)^*$ "$L^1 \subseteq (L^\infty)^*$"

But "$L^1 \supseteq (L^\infty)^*$" is not true.

REMARK:

"$L^1 \subsetneq (L^\infty)^*$"

$\exists L \in (L^\infty)^* \ : \ L$ is not of the form of L_g with $g \in L^1$

Indeed, consider $L_0 \in [C_c^0(\mathbb{R}^n)]^*$ where $(C_c^0(\mathbb{R}^n), \|\cdot\|_\infty)$ is a s.s. (supspace) of L^∞

$$L_0(f) := f(0) \ \ \forall\, f \in C_c^0(\mathbb{R}^n)$$

L^0 is linear.

$$|L_0(f)| = |f(0)| \le \|f\|_\infty \ \ \forall\, f \in C_c^0(\mathbb{R}^n) \implies L_0 \text{ is bounded}$$

By the H.B. theorem, $\exists L \in [L^\infty(\mathbb{R}^n)]^*$ which is an extension of L^0.

Claim: $\nexists\, g \in L^1(\mathbb{R}^n)$ s.t.

$$L(f) = \int_{\mathbb{R}^n} fg \, d\lambda \ \ \forall\, f \in L^\infty(\mathbb{R}^n)$$

Suppose, by contradiction, that such g exists. Then $\forall f_1 \in C_c^0(\mathbb{R}^n),\ f_1(0) = 0$

$$\int_{\mathbb{R}^n} g f_1 \, d\lambda = L(f_1) = L_0(f_1) = f_1(0) = 0 \implies g = 0 \text{ a.e. in } \mathbb{R}^n$$

$$L(f) = \int_{\mathbb{R}^n} 0 \cdot f \, d\lambda \ \ \forall f \in L^\infty(\mathbb{R}^n)$$

$$f_2 \in C_c^0(\mathbb{R}^n),\ f_2(0) \neq 0 \quad 0 = L(f_2) = L_0(f_2) = f_2(0) \neq 0$$

So, by contradiction, the claim is proven.

Remark:

$(L^\infty)^* \neq L^1$. In fact:

X^* separable $\implies$ X separable. So we have X not separable $\implies$ X^* not separable

Take $X = L^\infty$ and suppose, by contradiction, that $(L^\infty) = L^1$

Since we know that L^∞ not separable $\implies$ L^1 is not separable. But L^1 is separable.

Recap L^p spaces:

Space	Completeness	Separability	Reflective	Dual
L^p $1 < p < \infty$	Yes	Yes	Yes	$L^{p'}$ $(\frac{1}{p} + \frac{1}{p'} = 1)$
L^1	Yes	Yes	No	L^∞
L^∞	Yes	No	No	$\supsetneq L^1$

9 Weak Convergence

Definition.

Let X be a Banach space, $\{x_n\} \subset X,\ x \in X$. We say that x_n **converges weakly** to x ($x_n \rightharpoonup x$) if

$$L(x_n) \xrightarrow[n\to\infty]{} L(x) \ \ \forall L \in X^* \qquad \big(\text{where } L(x_n),\, L(x) \in \mathbb{R}\big)$$

Remark:

$$x_n \xrightarrow[n\to\infty]{} x \implies x_n \underset{n\to\infty}{\rightharpoonup} x$$

In fact,

$$\forall L \in X^* \quad |L(x_n) - L(x)| = |L(x_n - x)| \leq \|L\|_{X^*} \underbrace{\|x_n - x\|_X}_{\xrightarrow[n\to\infty]{} 0} \implies L(x_n) \to L(x) \overset{\text{def}}{\iff} x_n \rightharpoonup x$$

Obs. For $X = L^p$, in view of the representation theorem (Riesz), we can further define weakly convergence.

Definition.

$$f_n \underset{n\to\infty}{\rightharpoonup} f \text{ in } L^p(\Omega),\ p \in [1, \infty) \iff T(f_n) \xrightarrow[n\to\infty]{} T(f) \quad \forall T \in (L^p)^*$$

$$\Updownarrow \text{ Riesz theorem}$$

$$\int_\Omega f_n g \, d\lambda \xrightarrow[n\to\infty]{} \int_\Omega f g \, d\lambda \quad \forall g \in L^{p'} \qquad \left(\text{with } \tfrac{1}{p} + \tfrac{1}{p'} = 1\right)$$

Remark:

If $1 < p < \infty$, we can also define weak convergence with the following formula:

$$f_n \underset{n\to\infty}{\rightharpoonup} f \text{ in } L^p(\Omega) \iff \int_\Omega f_n \varphi \, d\lambda \xrightarrow[n\to\infty]{} \int_\Omega f \varphi \, d\lambda \quad \forall \varphi \in C_c^1$$

Obs. We can apply the Riesz theorem also for l^p spaces:

$$\Lambda \in (l^p)^*,\ p \in [1, \infty) \implies \exists !\, y \equiv y^{(k)} \in l^q \text{ with } \tfrac{1}{p} + \tfrac{1}{q} = 1 \quad s.t. \quad \Lambda(x) = \sum_{k=1}^{\infty} x^{(k)} y^{(k)} \quad \forall x \equiv x^{(k)} \in l^p$$

Definition.

$$x_n \rightharpoonup x \ in\ l^p \quad 1 \leq p < \infty \iff T(x_n) \to T(x) \quad \forall T \in (l^p)^*$$

$$\Updownarrow \text{ Riesz theorem}$$

$$\sum_{k=1}^{\infty} x_n^{(k)} y^{(k)} \xrightarrow[n\to\infty]{} \sum_{k=1}^{\infty} x^{(k)} y^{(k)} \quad \forall y \equiv y^{(k)} \in l^q \qquad \left(\text{with } \tfrac{1}{p} + \tfrac{1}{q} = 1\right)$$

Remark:

$x_n \rightharpoonup x \not\Longrightarrow x_n \to x$

$$X = l^2 \ , \ \{l_n\} \subset l^2 \ , \ l_n^{(k)} = \delta_{kn}$$

$$l_n \rightharpoonup 0 \iff \sum_{k=1}^{\infty} l_n^{(k)} y^{(k)} = y^{(n)} \xrightarrow[n\to\infty]{} 0 = \sum_{k=1}^{\infty} 0 \cdot y^{(k)} \quad \forall\, y \in l^2$$

$$\|l_n\|_2 = 1 \quad \forall\, n \in \mathbb{N} \implies l_n \not\to 0 \; in \; l^2$$

Remark:

$dimX < \infty, \quad x_n \to x \iff x_n \rightharpoonup x$

9.1 Properties of weak convergence

Proposition:

$\{x_n\}$ weakly converges $\implies$ the weak limit is unique

Proof.

Suppose, by contradiction, that $x_n \rightharpoonup x_1,\ x_n \rightharpoonup x_2$ and $x_1 \neq x_2$

$$\implies |L(x_n) - L(x_1)| \xrightarrow[n\to\infty]{} 0 \;\text{ and }\; |L(x_n) - L(x_2)| \xrightarrow[n\to\infty]{} 0 \quad \forall\, L \in X^*$$

$$\implies L(x_1) = L(x_2) \;\; \forall\, L \in X^* \implies x_1 = x_2 \;\; \text{(for corollary of H.B.)}$$

□

Proposition:

If $x_n \rightharpoonup x$, then $\{x_n\}$ is bounded. (derives from B.-S. thm)

Proposition:

$$x_n \underset{n\to\infty}{\rightharpoonup} x \implies \liminf_{n\to\infty} \|x_n\|_X \geq \|x\|_X$$

Revision:

$f : \mathbb{R} \to \mathbb{R},\ x_n \to x$

$$\liminf_{n\to\infty} f(x_n) \geq f(x) \overset{\text{def}}{\iff} f \text{ is lower semicontinuous at x}$$

$$\limsup_{n\to\infty} f(x_n) \leq f(x) \overset{\text{def}}{\iff} f \text{ is upper semicontinuous at x}$$

Obs. The proposition is equivalent to saying: $x \mapsto \|x\|$ is lower semicontinuous w.r.t. weak convergence.

Proof.

Let $x \in X \setminus \{0\}$, $L \in X^*$: $\|L\|_{X^*} = 1$, $L(x) = \|x\|$. Then,

$$0 < \|x\| = L(x) = \lim_{n\to\infty} L(x_n) = \lim_{n\to\infty} |L(x_n)|$$

On the other hand,

$$|L(x_n)| \leq \|L\|_{X^*}\|x_n\| = \|x_n\|_X$$

Therefore,

$$\|x\|_X = \lim_{n\to\infty} |L(x_n)| \leq \liminf_{n\to\infty} \|x_n\|$$

$\square$

Proposition:

$$\left.\begin{array}{r} x_n \rightharpoonup x \ in \ X \\ L_n \to L \ in \ X^* \end{array}\right\} \implies L_n(x_n) \xrightarrow[n\to\infty]{} L(x)$$

Proof.

$$L_n(x_n) - L(x) = L_n(x_n) - L(x_n) + L(x_n) - L(x)$$

$$|L_n(x_n) - L(x)| \leq \underbrace{|L_n(x_n) - L(x_n)|}_{(L_n - L)(x_n)} + |L(x_n) - L(x)| \leq \|L_n - L\|_{X^*}\|x_n\|_X + |L(x_n) - L(x)| \ \xrightarrow[n\to\infty]{} 0$$

Because $\|L_n - L\|_{X^*} \to 0$, $|L(x_n) - L(x)| \to 0$ and $\exists M > 0 : \|x_n\|_X \leq M \ \ \forall n \in \mathbb{N}$

$\square$

Proposition:

X, Y Banach spaces, $T \in \mathcal{L}(X, Y)$.

$$x_n \rightharpoonup x \ in \ X \implies T(x_n) \rightharpoonup T(x) \ in \ Y$$

Proof.

Let $L \in Y^*$ ($\iff L : Y \to \mathbb{R}$ linear and continuous)

$$\Lambda : X \to \mathbb{R} \quad \Lambda(x) = L[\underbrace{T(x)}_{\in Y}] \ \ \forall x \in X$$

$\implies \Lambda \in X^* \implies \Lambda(x_n) \to \Lambda(x)$. Therefore,

$$L[T(x_n)] \to L[T(x)] \overset{L \text{ is arbitrary}}{\iff} T(x_n) \rightharpoonup T(x)$$

□

9.2 Weak* convergence

Lezione 21 (23/11/23)

Definition.

We say that $\{L_n\} \subset X^*$ **converges weakly*** to $L \in X^*$ whenever

$$L_n(x) \to L(x) \;\; \forall\, x \in X \;\; \text{and we write } L_n \underset{n\to\infty}{\overset{*}{\rightharpoonup}} L$$

Remark:

Let $(\mathbb{R}^n, \mathcal{L}(\mathbb{R}^n), \lambda)$, $\Omega \in \mathcal{L}(\mathbb{R}^n)$, $X = L^1(\Omega)$, $X^* = L^\infty(\Omega)$, $\{f_n\} \subset L^\infty$ and $f \in L^\infty$.

Define $L_n, L : L^1(\Omega) \to \mathbb{R}$ $s.t.$ $L_n, L \in L^\infty(\Omega)$ with

$$L_n(g) := \int_\Omega f_n g \, d\lambda \;\; \forall\, g \in L^1(\Omega) \qquad L(g) = \int_\Omega f g \, d\lambda \;\; \forall\, g \in L^1$$

$$\Downarrow$$

$$L_n \overset{*}{\rightharpoonup} L \iff \int_\Omega f_n g \, d\lambda \xrightarrow[n\to\infty]{} \int_\Omega f g \, d\lambda \;\; \forall\, g \in L^1(\Omega)$$

In this case we also say that $f_n \overset{*}{\rightharpoonup} f \; in \; L^\infty(\Omega)$

Proposition:

$$L_n \underset{n\to\infty}{\rightharpoonup} L \; in \; X^* \implies L_n \overset{*}{\rightharpoonup} L$$

$$L_n \underset{n\to\infty}{\rightharpoonup} L \; in \; X^* \impliedby L_n \overset{*}{\rightharpoonup} L \quad \text{provided that X is reflexive}$$

Proof.

$(\implies)$

$$L_n \rightharpoonup L \; in \; X^* \overset{\text{def}}{\iff} \Lambda(L_n) \to \Lambda(L) \;\; \forall\, \Lambda \in X^{**} \implies \Lambda(L_n) \to \Lambda(L) \;\; \forall\, \Lambda \in \tau(X) \subseteq X^{**}$$

$$\iff L_n(x) \to L(x) \;\; \forall\, x \in X \iff L_n \overset{*}{\rightharpoonup} L$$

$(\Longleftarrow)\ \ \tau(X) = X^{**}$ $\square$

Proposition:

Let X be a Banach space.

- $\{L_n\} \subset X^*$ weakly* converges $\implies$ the limit is unique;
- $L_n \overset{*}{\rightharpoonup} L \implies \{L_n\}$ is bounded in X^*;
- $L \mapsto \|L\|_{X^*}$ is lower semic. w.r.t. weak* convergence $\iff (L_n \overset{*}{\rightharpoonup} L \implies \liminf_{n\to\infty} \|L_n\|_{X^*} \geq \|L\|_{X^*})$
- $\left.\begin{array}{l} L_n \overset{*}{\rightharpoonup} L \\ x_n \to x \end{array}\right\} \implies L_n(x_n) \to L(x)$

9.3 Banach-Alaoglu theorem

Theorem.

Let X be a separable Banach space. Then any bounded sequence $\{L_n\} \subset X^*$ admits a subsequence that weakly* converges to some $L \in X^*$.

Remark:

Any $\{f_n\}$ bounded in L^∞ posses a subsequence $\{f_{n_h}\}$ which weakly* converges to some $f \in L^\infty$.

Indeed, take $\{f_n\} \subset L^\infty$ bounded $(\exists C > 0\ :\ \|f_n\|_\infty \leq C\ \ \forall n \in \mathbb{N})$ and define $L_n(g) := \displaystyle\int_\Omega f_n g\, d\lambda\ \ \forall g \in L^1$

$$\implies |L_n(g)| \leq \|f_n\|_\infty \|g\|_1 \leq C\|g\|_1 \quad \forall n \in \mathbb{N}$$

$$\implies \|L_n\|_{(L^1)^*} \leq C \quad \forall n \in \mathbb{N} \iff \{L_n\} \text{ is bounded in } (L^1)^*$$

By B.-A. theorem, $\exists \{L_{n_h}\} \subset \{L_n\}\ :\ L_{n_h} \overset{*}{\rightharpoonup} L$ for some $L \in (L^1)^* \iff L_{n_h}(g) \xrightarrow[h\to\infty]{} L(g)\ \ \forall g \in L^1$

$$\Updownarrow$$

$\exists \{f_{n_h}\} \subset \{f_n\}$ and $\exists!\, f \in L^\infty$ *s.t.* $L(g) = \displaystyle\int_\Omega fg\, d\lambda\ \ \forall g \in L^1$ *and* $\displaystyle\int_\Omega f_{n_h} g\, d\lambda \xrightarrow[h\to\infty]{} \int_\Omega fg\, d\lambda\ \ \forall g \in L^1(\Omega)$

Remark:

Concerning weak*, what we have seen for L^∞ also holds C^∞.

Corollary:

Let X be a separable and reflexive Banach space. Then any bounded sequence $\{x_n\} \subset X$ posses a subsequence which weakly converges.

Proof.

X is separable and reflexive $\Rightarrow X^*$ separable. We also know that $\{\tau(x_n)\} \subset X^{**}$ is bounded.

So by the B.-A. theorem on X^*, $\exists\{\tau(x_{n_h})\}$ s.t. $\tau(x_{n_h}) \overset{*}{\rightharpoonup} \Lambda$ for some $\Lambda \in X^{**}$

$$\overset{\text{def}}{\iff} f(x_{n_h}) = [\tau(x_{n_h})](f) \to \Lambda(f) = f(x) \quad \forall f \in X^*$$

$$\iff x_{n_h} \underset{h\to\infty}{\rightharpoonup} x := \tau^{-1}(\Lambda)$$

□

10 Compact operators

Let X, Y be Banach spaces.

Definition.

$K : X \to Y$ linear is said to be **compact** if $\forall E \subset X$ bounded, $\overline{K(E)} \subseteq Y$ is compact.

Remark:

Equivalent definition: $\{x_n\} \subset X$ bounded $\implies$ $\{K(x_n)\}$ has a s.s. which converges in Y strongly.

Proposition:

$K : X \to Y$ linear, compact $\implies$ $K \in \mathcal{L}(X, Y)$

Proof.

$B \subset X$ closed unit ball.

$K(B)$ is relatively compact $\implies$ $K(B)$ bounded

$$\implies \exists M > 0 \; : \; \forall x \in X, \; \|x\|_X \leq 1 \; \text{ we have } \|K(x)\|_Y \leq M$$

$$\implies K \text{ is bounded.}$$

□

Definition.

$T \in \mathcal{L}(X, Y)$ has **finite rank** if $dim\,(Im\,T) < \infty$.

Remark:

$T \in \mathcal{L}(X, Y),\ rank(T)$ finite $\underset{\not\Longleftarrow}{\Longrightarrow}$ T is compact

Proposition:

$\{K_n\}$ operators of finite rank, $K_n \to K$ in $\mathcal{L}(X, Y)$ $\;(\Longleftrightarrow \|K_n - K\|_{\mathcal{L}} \xrightarrow[n\to\infty]{} 0)$

Then K is compact.

($\not\Longleftarrow$ there is a counter example by Enflo)

Lezione 22 (29/11/23)

Proposition:

Let X, Y be Banach spaces, $K \in \mathcal{L}(X, Y)$ and $dimY = \infty$.

K compact $\implies$ K is not surjective.

Proof.

Suppose, by contradiction, that K is surjective.

$$\text{open ball } B_1(0) \subset X \quad K(B_1(0)) \subset Y \text{ open (by OMT)}$$

Since $0 \in B_1(0)$, then $K(0) = 0 \in K(B_1(0))$

$$\implies \quad \exists\,\delta > 0 \ :\ B_\delta(0) \subseteq K(B_1(0)) \quad \overline{B_\delta(0)} \subset \overline{K(B_1(0))}$$

K compact $\implies$ $\overline{K(B_1(0))}$ compact $\implies$ $\overline{B_{\delta(0)}}$ compact, since it's closed.

$\implies$ $\overline{B_{\delta(0)}} \subseteq Y$ is compact, with $dimY = \infty$. This is in contradiction with Riesz theorem.

□

Definition.

$\mathcal{K}(X, Y) := \{K \in \mathcal{L}(X, Y)\ :\ K \text{ is compact}\}$. If $X = Y$, $\mathcal{K}(X, Y) = \mathcal{K}(X)$

Theorem.

i) If $K \in \mathcal{K}(X, Y)$, then

$$x_n \underset{n\to\infty}{\rightharpoonup} x \implies K(x_n) \xrightarrow[n\to\infty]{} K(x)$$

This property is called weak-strong continuity (WSC)

ii) If X is reflexive and $K \in \mathcal{L}(X, Y)$ satisfies (WSC), then $K \in \mathcal{K}(X, Y)$

10.1 Strategies for exercises

Methods to study compactness (Yes: show it's compact, No: show it isn't) :

- Definition Yes / No
- Equivalent definition with sequences Yes / No
- Approximation with finite rank operators Yes
- Surjectivity No
- WSC No / Yes if X is reflexive

Example:

With the definition: $X = C^0([a, b]) = Y \quad T : X \to Y$

$$E \subset X \text{ bounded} \quad F := T(E) \subset Y \text{ and } \overline{F} \text{ is compact } \iff T \text{ is compact}$$

So if F is equibounded and equicontinuous, then by A.-A. theorem, $\overline{F}$ is compact $\implies T$ is compact

11 Hilbert spaces

Definition.

H vector space with a scalar (inner) product is called **pre-Hilbert space** or inner product space

$$\|x\| := \sqrt{\langle x, x\rangle} \quad \forall x \in H \text{ is a norm}$$

$$(H, \langle\cdot,\cdot\rangle) \text{ pre-Hilbert} \implies (H, \|\cdot\|) \text{ normed sapce} \implies (H, d) \text{ metric space with } d(x, y) := \|y - x\|$$

$$(H, \langle\cdot,\cdot\rangle) \textbf{ Hilbert space} \implies (H, d) \text{ complete metric space}$$

Obs. This means that a Hilbert space is a vector space with an inner product, such that the norm defined with the inner product turns H into a complete metric space and if the norm defined by the norm is not complete, then H is pre-Hilbert.

Obs. An Hilbert space is always a Banach space.

Example:

i) $C^0([a,b])$ with $\langle f,g\rangle := \displaystyle\int_a^b fg\,dx$ is pre-Hilbert.

ii) $L^2(X,\mathcal{A},\mu)$ with $\langle f,g\rangle := \displaystyle\int_X fg\,d\mu$ is Hilbert.

iii) l^2 with $\langle x,y\rangle := \displaystyle\sum_{n=1}^{\infty} x^{(n)}y^{(n)}$ is Hilbert.

Example:

L^p is Hilbert $\Longleftrightarrow p=2$

Theorem.

H pre-Hilbert, then the parallelogram law holds:

$$\left\|\frac{a+b}{2}\right\|^2 + \left\|\frac{a-b}{2}\right\|^2 = \frac{1}{2}(\|a\|^2+\|b\|^2) \;\; \forall\, a,b\in H$$

Remark:

$H = L^p, \quad p\geq 2$

$$\left\|\frac{a+b}{2}\right\|_p^p + \left\|\frac{a-b}{2}\right\|_p^p \leq \frac{1}{2}(\|a\|_p^p+\|b\|_p^p) \quad \text{is the Clarkson inequality}$$

Obs. Now we see in parallel a theorem in the case of Hilbert and L^p spaces. The proof it's similar.

Theorem: Projection in H.

Let H be a Hilbert space. Let $K\subset H$ be a closed convex subset. Then,

$$(*) \quad \forall f\in H \;\; \exists!\, u\in K \;\; s.t.\; \|f-u\| = \min_{v\in K}\|f-v\| =: dist(f,K)$$

Moreover, u fulfills $(*) \Longleftrightarrow u\in K,\; \langle f-u, v-u\rangle \leq 0 \;\; \forall\, v\in K$

Theorem: Projection in L^p.

Let $1<p<\infty$. Let $K\subset L^p(X,\mathcal{A},\mu)$ be a closed convex subset. Then,

$$\forall f\in L^p \;\; \exists!\, u\in K \;\; s.t.\; \|f-u\|_p = \min_{v\in K}\|f-v\|_p =: dist(f,K)$$

Proof. (Projection in H)

Let $\{v_n\} \subset K$ be a minimizing sequence for $\min_{v\in K} \|f-v\|$. $d_n := \|f-v_n\| \to \inf_{v\in K} \|f-v\| =: d$

Claim: $\{v_n\}$ is Cauchy.

In fact, by the **parallelogram law**, applied with $a = f - v_n,\ b = f - v_m$

$$\left\| f - \frac{v_n+v_m}{2} \right\|^2 + \left\| \frac{v_n - v_m}{2} \right\|^2 = \frac{1}{2}(d_n^2 + d_m^2)$$

Since K is convex, $\dfrac{v_n+v_m}{2} \in K$. Thus $\left\| f - \dfrac{v_n+v_m}{2} \right\| \geq d$

Therefore,

$$\left\| \frac{v_n - v_m}{2} \right\|^2 \leq \frac{1}{2}(d_n^2 + d_m^2) - d^2$$

$$d_n, d_m \to d \implies \lim_{n,m\to\infty} \|v_n - v_m\| = 0 \implies \{v_n\} \text{ is Cauchy}$$

H is complete, since is Hilbert, then for the properties of Cauchy sequences

$$\exists\, u \in H \,:\, v_n \xrightarrow[n\to\infty]{} u$$

$\{v_n\} \subset K,\ K$ is closed $\implies u \in K$.

$$\implies d \leq \|f-u\| \leq \underbrace{\|f-v_n\|}_{=d_n\to d} + \underbrace{\|v_n - u\|}_{\to 0} \implies \|f-u\| = d$$

Now we prove uniqueness.

Suppose, by contradiction, that $u, \widetilde{u} \in K,\ u \neq \widetilde{u},\ d = \|f-u\| = \|f-\widetilde{u}\|$

By the parallelogram law, with $\begin{cases} a = f - u \\ b = f - \widetilde{u} \end{cases}$

$$\left\| f - \frac{u+\widetilde{u}}{2} \right\|^2 + \underbrace{\left\| \frac{u-\widetilde{u}}{2} \right\|^2}_{=\sigma} = \frac{1}{2}(d^2 + d^2) = d^2 \quad u \neq \widetilde{u} \implies \sigma > 0$$

$$\frac{u+\widetilde{u}}{2} \in K \implies d^2 \leq \|f - \frac{u+\widetilde{u}}{2}\|^2 = d^2 - \sigma < d^2$$

□

Proof. (Projection in L^p)

Let $\{v_n\} \subset K$ be a minimizing sequence for $\min_{v\in K} \|f-v\|_p$. $d_n := \|f-v_n\|_p \to \inf_{v\in K} \|f-v\|_p =: d$

Claim: $\{v_n\}$ is Cauchy.

In fact, by the **Clarckson inequality**, applied with $a = f - v_n,\ b = f - v_m$

$$\left\| f - \frac{v_n+v_m}{2} \right\|_p^p + \left\| \frac{v_n - v_m}{2} \right\|_p^p = \frac{1}{2}(d_n^p + d_m^p) \ \ p \geq 2$$

Since K is convex, $\dfrac{v_n + v_m}{2} \in K$. Thus $\left\| f - \dfrac{v_n + v_m}{2} \right\|_p^p \geq d^p$

Therefore,

$$\left\| \frac{v_n - v_m}{2} \right\|_p^p \leq \frac{1}{2}(d_n^p + d_m^p) - d^p$$

$$d_n, d_m \to d \implies \lim_{n,m\to\infty} \|v_n - v_m\|_p = 0 \implies \{v_n\} \text{ is Cauchy}$$

H is complete, then for the properties of Cauchy sequences

$$\exists\, u \in H \,:\, v_n \xrightarrow[n\to\infty]{} u$$

$\{v_n\} \subset K,\ K$ is closed $\implies u \in K$.

$$\implies d \leq \|f - u\|_p \leq \underbrace{\|f - v_n\|_p}_{=d_n \to d} + \underbrace{\|v_n - u\|_p}_{\to 0} \implies \|f - u\|_p = d$$

Now we prove uniqueness.

Suppose, by contradiction, that $u, \widetilde{u} \in K,\ u \neq \widetilde{u},\ d = \|f - u\|_p = \|f - \widetilde{u}\|_p$

By Clarckson inequality, with $\begin{cases} a = f - u \\ b = f - \widetilde{u} \end{cases}$

$$\left\| f - \frac{u + \widetilde{u}}{2} \right\|_p^p + \underbrace{\left\| \frac{u - \widetilde{u}}{2} \right\|_p^p}_{=\sigma} = \frac{1}{2}(d^p + d^p) = d^p \quad u \neq \widetilde{u} \implies \sigma > 0$$

$$\frac{u + \widetilde{u}}{2} \in K \implies d^p \leq \|f - \frac{u + \widetilde{u}}{2}\|_p^p = d^p - \sigma < d^p$$

□

Remark:

$K \subset H$ closed convex s.s. $u := Proj_K f$ is called projection.

If M is a closed vector subspace ($\implies M$ closed and convex) $u := Proj_M f$ is an orthogonal projection.

Corollary:

Assume that $M \subset H$ is a closed vector s.s. and let $f \in H$. Then,

$$u = Proj_M f \iff u \in M,\ \langle f - u, v\rangle = 0 \ \ \forall\, v \in M$$

Proof.

$(\implies)$

$$\text{By the preceding theorem, we know:} \quad \langle f - u, v - u\rangle \leq 0 \ \ \forall\, v \in M$$

$$M \text{ vector subspace } \implies tv \in M \ \ \forall\, v \in M,\, t \in \mathbb{R} \implies \langle f - u, tv - u\rangle \leq 0$$

$$\implies t\langle f-u,v\rangle \le \langle f-u,u\rangle \quad \forall t\in\mathbb{R}$$

To show the thesis, we show that $\langle f-u,v\rangle \neq 0$ is a contradiction.
Suppose $\langle f-u,v\rangle > 0$ (for <0 is similar), for $t > \dfrac{\langle f-u,u\rangle}{\langle f-u,v\rangle}$ the inequality doesn't hold.
Hence, $\langle f-u,v\rangle = 0 \ \ \forall v\in M$.
($\impliedby$) If $u\in M$ and $\langle f-u,v\rangle = 0 \ \ \forall v\in M$, then

$$\langle f-u,\xi-u\rangle = 0 \ \ \forall \xi\in M \quad (v=\xi-u\in M)$$

$$\implies \langle f-u,v-u\rangle \le 0 \ \ \forall v\in M \implies u = Proj_M f \quad \text{(seen in the previous theorem)}$$

□

Lezione 23 (30/11/23)

Obs. We want to find a property similar to orthogonal projection in L^p spaces.

Remark:

Consider $M\subset L^p$ closed vector s.s. $\quad u = Proj_M f \iff \|f-u\|_p = d = \min_{v\in M}\|f-v\|_p$
M vector s.s. $\implies u+tv\in M$ $(t\in\mathbb{R})$. Consider $t\mapsto \|f-u-tv\|_p^p$

$$\|f-u-tv\|_p^p \text{ admits a minimum in } t=0 \implies \frac{d}{dt}\|f-u-tv\|_p^p\Big|_{t=0} = 0$$

$$\frac{d}{dt}\|f-u-tv\|_p^p = \frac{d}{dt}\int_X |f-u-tv|^p\,d\mu = -\int_X p|f-u-tv|^{p-2}(f-u-tv)v\,d\mu$$

$$\frac{d}{dt}\|f-u-tv\|_p^p\Big|_{t=0} = -\int_X p|f-u|^{p-2}(f-v)v\,d\mu = 0$$

Therefore,

$$\int_X |f-u|^{p-2}(f-u)v\,d\mu = 0 \ \ \forall v\in M\subset L^p$$

Obs. This integral is the substitute of scalar product for L^p spaces.
Furthermore, $|f-u|^{p-2}(f-u)\in L^{p'}$ and $v\in M\subset L^p$. In fact, set $p' = \dfrac{p}{p-1}$ $(\Leftrightarrow \frac{1}{p}+\frac{1}{p'}=1)$

$$\int_X \left||f-u|^{p-2}(f-u)\right|^{p'}d\mu = \int_X |f-u|^{(p-1)p'}\,d\mu = \int_X |f-u|^p\,d\mu < \infty \ \text{ since } f,u\in L^P$$

11.1 Riesz theorem

Dual of Hilbert spaces

For $(V,\langle\cdot,\cdot\rangle)$, if $dimV = n$, we know that $V^* = V$.

We want to find the dual in case of $dimV = \infty$.

$$H \text{ Hilbert space, } f\in H,\ \varphi: H\to\mathbb{R} \ \ \varphi(u) := \langle f,u\rangle \quad \forall u\in H$$

$$\varphi \in H^*, \ \|\varphi\|_{\mathcal{L}} = \|f\|_H$$

So given an element in H we can always find an element in H^* with the same norm. "$H \subseteq H^*$"

Now we want to find out if "$H^* \subseteq H$". We see this with Riesz theorem.

Obs. We see the Riesz theorem in H and L^p in parallel, because they are similar.

Theorem: Riesz on H.

Let H be a Hilbert space. For any $\varphi \in H^*$ $\exists!\, f \in H$ $s.t.$ $\varphi(u) = \langle f, u \rangle \quad \forall\, u \in H$

Moreover, $\|\varphi\|_{\mathcal{L}} = \|f\|_H$

Theorem: Riesz on L^p.

Let $1 < p < \infty$. Fot any $\varphi \in [L^p(X, \mathcal{A}, \mu)]^*$ $\exists!\, f \in L^{p'}$, with $\frac{1}{p} + \frac{1}{p'} = 1$ s.t. $\varphi(u) = \displaystyle\int_X f u \, d\mu \quad \forall\, u \in L^p$

Moreover, $\|\varphi\|_{\mathcal{L}} = \|f\|_{L^{p'}}$

Proof. (Riesz on H)

Let $M := \varphi^{-1}(0)$, so M is a closed v.s.s. of H. If $M = H$, then $f = 0$.

Let $M \subsetneq H$. We claim that $\exists\, g \in H$ $s.t.$ $\|g\| = 1, \ g \in M^{\perp}$ ($\iff \langle g, v \rangle = 0 \quad \forall\, v \in M$)

In fact, let $g_0 \in H \backslash M$ and $g_1 := Proj_M g_0$

$$g := \frac{g_0 - g_1}{\|g_0 - g_1\|} \quad \|g\| = 1 \quad g \in M^{\perp}$$

For any $u \in H$, set $v := u - \lambda g$ where $\lambda = \dfrac{\varphi(u)}{\varphi(g)}$ $\left(\text{since } g_1 \in M, \ \varphi(g) = \dfrac{\varphi(g_0)}{\|g_0 - g_1\|} \neq 0\right)$

$$\varphi(v) = \varphi(u) - \lambda\varphi(y) = \varphi(u) - \frac{\varphi(u)}{\varphi(g)}\varphi(y) = 0 \quad \implies v \in M$$

Therefore,

$$g \in M^{\perp}, \ v \in M \implies 0 = \langle g, v \rangle = \langle g, u - \lambda g \rangle \implies \langle g, u \rangle = \lambda \underbrace{\|g\|^2}_{1}$$

$$\implies \varphi(u) = \varphi(g)\lambda = \varphi(g) \langle g, u \rangle = \langle f, u \rangle \quad \text{with } f = \varphi(g) g$$

For uniqueness, by contradiction, suppose that there exist $f_1, f_2 \in H$ $s.t.$ $\varphi(u) = \langle f_1, u \rangle = \langle f_2, u \rangle \quad \forall\, u \in H$

$$0 = \langle f_1 - f_2, u \rangle \quad \forall\, u \in H \implies f_1 - f_2 = 0$$

$\square$

Proof. (Riesz on L^p)

Let $M := \varphi^{-1}(0)$, so M is a closed v.s.s. of L^p. If $M = L^p$, then $f = 0$.

Let $M \subsetneq L^p$. Let $g_0 \in L^p \backslash M$ and $g_1 := Proj_M g_0$

$$g := \frac{g_0 - g_1}{\|g_0 - g_1\|_p}$$

$$\int_X |g_0 - g_1|^{p-2}(g_0 - g_1) v \, d\mu = 0 \;\; \forall\, v \in M$$

For any $u \in L^p$, set $v := u - \lambda g$ where $\lambda = \dfrac{\varphi(u)}{\varphi(g)}$ $\left(\text{since } g_1 \in M,\ \varphi(g) = \dfrac{\varphi(g_0)}{\|g_0 - g_1\|} \neq 0\right)$

$$\varphi(v) = \varphi(u) - \lambda\varphi(y) = \varphi(u) - \frac{\varphi(u)}{\varphi(g)}\varphi(y) = 0 \quad \Longrightarrow \; v \in M$$

Therefore,

$$\int_X |g_0 - g_1|^{p-2}(g_0 - g_1) v \, d\mu = 0$$

$$\Longrightarrow \int_X |g_0 - g_1|^{p-2}(g_0 - g_1) u \, d\mu = \lambda \int_X |g_0 - g_1|^{p-2} \frac{(g_0 - g_1)^2}{\|g_0 - g_1\|_p} \, d\mu = \frac{\lambda}{\|g_0 - g_1\|_p} \int_X |g_0 - g_1|^p \, d\mu =$$

$$= \lambda \|g_0 - g_1\|_p^{p-1} = \frac{\varphi(u)}{\varphi(g)} \|g_0 - g_1\|_p^{p-1}$$

$$\Longrightarrow \; \varphi(u) = \frac{\varphi(g)}{\|g_0 - g_1\|_p^{p-1}} \int_X |g_0 - g_1|^{p-2}(g_0 - g_1) u \, d\mu$$

$$\varphi(u) = \int_X f u \, d\mu \quad \text{where } f := \frac{\varphi(g)}{\|g_0 - g_1\|_p^{p-1}} |g_0 - g_1|^{p-2}(g_0 - g_1) \;\;\; f \in L^{p'}$$

For uniqueness, by contradiction, suppose exist $f_1, f_2 \in L^{p'}$ $s.t.$ $\varphi(u) = \displaystyle\int_X f_1 u \, d\mu = \int_X f_2 u \, d\mu \quad \forall\, u \in L^p$

$$\int_X u\,(f_1 - f_2)\, d\mu = 0 \;\; \forall\, u \in L^p$$

$$\text{We take } u = |f_1 - f_2|^{\frac{2-p}{p-1}}(f_1 - f_2)$$

$$u \in L^p \text{ since } \int_X |u|^p \, d\mu = \int_X |f_1 - f_2|^{\frac{2-p+p-1}{p-1} \cdot p} \, d\mu = \int_X |\underbrace{f_1 - f_2}_{\in L^{p'}}|^{p'} \, d\mu < \infty$$

Thus,

$$0 = \int_X u\,(f_1 - f_2)\, d\mu = \int_X |f_1 - f_2|^{\frac{2-p}{p-1}} (f_1 - f_2)^2 \, d\mu = \int_X |f_1 - f_2|^{\frac{2-p+2p-2}{p-1}} \, d\mu = \int_X |f_1 - f_2|^{p'} \, d\mu$$

$$\Longleftrightarrow \|f_1 - f_2\|_{p'} = 0 \Longleftrightarrow f_1 = f_2 \;\, in \;\, L^{p'}$$

$\square$

Obs. It's the same proof, but instead of the scalar product we have the integral equation.

12 Orthonormal bases

Definition.

A sequence $\{e_n\} \subset H$ (Hilbert) is said to be an **orthonormal basis** (o.n.b.) of H (or a Hilbert basis) if

i) $\langle e_n, e_m \rangle = 0 \;\; \forall\, n \neq m, \;\; \|e_n\| = 1 \;\; \forall\, n \in \mathbb{N}$

ii) $Span(\{e_n\})$ is dense in H. Where $Span(\{e_n\}) = \{$all finite linear combinations of $\{e_n\}\}$

Theorem.

Let $\{e_n\}$ be an o.n.b. of H. Then, $\forall\, u \in H$

$$u = \sum_{k=1}^{\infty} \langle u, e_k \rangle \, e_k \quad \text{where } \langle u, e_k \rangle = \text{Fourier coefficient} \quad \langle u, e_k \rangle \, e_k = \text{ Proj}\, u \text{ on } Span(e_k)$$

$$\text{We are saying that } \sum_{k=1}^{n} \langle u, e_k \rangle \, e_k \xrightarrow[n \to \infty]{} u \iff \left\| \sum_{k=1}^{n} \langle u, e_k \rangle \, e_k - u \right\| \xrightarrow[n \to \infty]{} 0$$

$$\|u\|^2 = \sum_{k=1}^{\infty} |\, \langle u, e_k \rangle \,|^2 \quad \text{called Parseval's identity}$$

Conversely, given any sequence $\{\alpha_k\} \subset l^2$, the series $\sum_{k=1}^{\infty} \alpha_k e_k$ converges to some $u \in H$ s.t.

$$\langle u, e_k \rangle = \alpha_k \quad \|u\|^2 = \sum_{k=1}^{\infty} \alpha_k^2$$

Theorem.

Every separable Hilbert space has an o.n.b.

Example:

i) In L^2 there are bases made of sin, cos or by $polynomials$

ii) In l^2 a basis is $e_n^{(k)} = \delta_{nk}$

Proposition:

Let $\{e_n\}$ be an o.n.b., then $e_n \rightharpoonup 0$, but $e_n \nrightarrow 0$

Proof.

$\forall f \in H$, Parseval's identity implies that $\|f\|^2 = \sum_{k=1}^{\infty} \underbrace{\langle f, e_n \rangle}_{\in \mathbb{R}}^2$. So the series is convergent

$$\implies \langle f, e_n \rangle \xrightarrow[n\to\infty]{} 0 \ \ \forall f \in H \overset{\text{Riesz}}{\iff} F(e_n) \xrightarrow[n\to\infty]{} F(0) = 0 \ \ \forall F \in H^* \iff e_n \rightharpoonup 0$$

On the other hand $\|e_n\| = 1 \ \ \forall n \in \mathbb{N}$. Thus $e_n \not\to 0$ □

13 Fredholm alternative

13.1 Symmetric operators

Lezione 24 (13/12/23)

Definition.

We say that $T \in \mathcal{L}(H)$ is **symmetric** whenever

$$\langle T(x), y \rangle = \langle x, T(y) \rangle \quad \forall x, y \in H$$

Proposition:

If the operator is symmetric, then $\|T\|_{\mathcal{L}} = \sup_{\|x\|=1} |\langle T(x), x \rangle|$

Theorem.

Let $T \in K(H)$, let T be symmetric. Then,

i) $dim\, Ker(I - T) < \infty$, where I is the identity

ii) $Im(I - T)$ is closed and $Im(I - T) = (Ker(I - T))^{\perp}$

iii) $Ker(I - T) = \{0\} \iff Im(I - T) = H$

Remark:

The Fredholm alternative deals with the solvability of equation $u - T(u) = f$

• Either $\forall f \in H \ \ u - T(u) = f$ has a unique solution $\leftrightsquigarrow$ (iii)

• Or $u - T(u) = 0$ admits n linearly independent solutions and in this case $u - T(u) = f$ is solvable if and only if the orthogonal condition holds: $f \in [Ker(I - T)]^{\perp}$ $\leftrightsquigarrow$ $(i)\,\&\,(ii)$

Remark:

(iii) $dimH < \infty \quad I - T$ is injective $\iff I - T$ is surjective

However, in an infinite dimensional space injectivity $\not\Longrightarrow$ $\not\Longleftarrow$ surjectivity

Obs. To show this we can respectively consider right-shift and left-shift operators in l^2

13.2 The spectrum

Let E be a Banach space and $T \in \mathcal{L}(E)$.

Definition.

The **resolvent set** is

$$\rho(T) := \{\lambda \in \mathbb{R} \ \ s.t. \ (T - \lambda I) : E \to E \text{ is bijective}\}$$

$$\sigma(T) := \mathbb{R} \backslash \rho(T) = \textbf{spectrum}$$

$\lambda \in \mathbb{R}$ is an **eigenvalue** of T, if its eigenspace:$= Ker(T - \lambda I) \neq \{0\}$ $(\iff \exists\, v \in E \backslash \{0\} \,:\, T(v) = \lambda v)$

$$EV(T) \equiv \sigma_p(T) := \{\text{all eigenvalues of } T\} = \text{point spectrum}$$

Remark:

$EV(T) \subseteq \sigma(T)$

$dim\, E < \infty \implies EV(T) \equiv \sigma(T)$

$\sigma(T) \subset \mathbb{R}$ is compact and $\sigma(T) \subseteq [-\|T\|_{\mathcal{L}}, \|T\|_{\mathcal{L}}]$

Theorem: Structure of the spectrum.

Let $T \in \mathcal{K}(E)$ with $dim\, E = \infty$. Then,

i) $0 \in \sigma(T)$

ii) $\sigma(T) \backslash \{0\} = EV(T) \backslash \{0\}$

iii) One of the following cases holds:

• $\sigma(T) = \{0\}$ • $\sigma(T) \backslash \{0\}$ is finite set • $\sigma(T) \backslash \{0\}$ is a sequence convergent to 0

Theorem.

Let H be a separable Hilbert space and let $T \in \mathcal{K}(H)$ be symmetric. Then there exists an o.n.b. of H made of eigenvectors of T.