EE23010 Assignment

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Question 12.13.3.55

There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance of X.

Solution:

parameters	description
A	number on the first card
B	number on the second card

$$p_A(k) = \begin{cases} \frac{1}{5}, & 1 \le k \le 5 \\ 0, & \text{otherwise} \end{cases} \quad (1)$$

$$p_B(k) = \sum_{i=1}^{5} \Pr(B = k \mid A = i) \cdot p_A(i) \quad (2)$$

$$= \frac{1}{5} \sum_{i=1}^{5} \Pr(B = k \mid A = i) \quad (3)$$

$$\Pr(B = k \mid A = i) = \begin{cases} \frac{1}{4}, & 1 \le k \le 5, k \ne i \\ 0, & \text{otherwise} \end{cases} \quad (4)$$

$$\implies p_B(k) = \begin{cases} \frac{1}{20}, & 1 \le k \le 5, k \ne i \\ 0, & \text{otherwise} \end{cases} \quad (5)$$

X can take values ranging from 3 to 9.

$$X = A + B \quad (6)$$

$$p_X(k) = \sum_{i=1}^{5} \Pr(X = k \mid A = i) \cdot p_A(i) \quad (7)$$

Finding mean or the expectation:

$$\mathbb{E}[X] = \sum_{k=3}^{9} k \cdot p_X(k) \quad (8)$$

$$= \sum_{k=3}^{9} k \sum_{i=1}^{5} \Pr(X = k \mid A = i) \cdot p_A(i) \quad (9)$$

$$= \sum_{k=3}^{9} k \sum_{i=1}^{5} \Pr(B = k - i \mid A = i) \cdot p_A(i) \quad (10)$$

$$p_X(k) = p_X(2m + 1) + p_X(2m) \quad \forall m \in \{1, 2, 3, 4\} \quad (11)$$

$$\mathbb{E}[X] = \sum_{m=1}^{4} 2m+1 \sum_{i=1}^{5} \Pr(B = 2m + 1 - i \mid A = i) \cdot p_A(i)$$
$$+ \sum_{m=2}^{4} 2m \sum_{i=1}^{5} \Pr(B = 2m - i \mid A = i) \cdot p_A(i) \quad (12)$$

$$= \frac{1}{5} \sum_{m=1}^{4} 2m + 1 \sum_{i=1}^{5} \Pr(B = 2m + 1 - i \mid A = i)$$
$$+ \frac{1}{5} \sum_{m=2}^{4} 2m \sum_{i=1}^{5} \Pr(B = m - i \mid A = i) \quad (13)$$

$$= \frac{1}{5} \sum_{m=1}^{4} 2m + 1 \sum_{i=1}^{5} \frac{1}{4}\Big|_{1 \le 2m+1-i \le 5}$$
$$+ \frac{1}{5} \sum_{m=2}^{4} 2m \sum_{i=1}^{5} \frac{1}{4}\Big|_{1 \le 2m-i \le 5, m \ne i} \quad (14)$$

$$= \frac{1}{20} \sum_{m=1}^{4} 2m + 1 \sum_{i=1}^{5} 1\Big|_{2m-4 \le i \le 2m}$$
$$+ \frac{1}{20} \sum_{m=2}^{4} 2m \sum_{i=1}^{5} 1\Big|_{2m-5 \le i \le 2m-1, m \ne i} \quad (15)$$

$$= \frac{1}{20} \sum_{m=1}^{4} 2m+1 \left(\min(5, 2m) - \max(1, 2m - 4) + 1 \right)$$
$$+ \frac{1}{20} \sum_{m=2}^{4} 2m \left(\min(5, 2m - 1) - \max(1, 2m - 5) \right) \quad (16)$$

$$\implies \mathbb{E}[X] = \frac{1}{20}(72) + \frac{1}{20}(48) \quad (17)$$

$$= 6 \quad (18)$$

$$\text{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \qquad (19)$$

$$= \frac{1}{20} \sum_{m=1}^{4} (2m+1)^2 \left(\min(5, 2m) - \max(1, 2m - 4) + 1\right)$$

$$+ \frac{1}{20} \sum_{m=2}^{4} (2m)^2 \left(\min(5, 2m - 1) - \max(1, 2m - 5)\right) - 6^2$$

$$(20)$$

$$= \frac{1}{20}(476) + \frac{1}{20}(304) - 36 \qquad (21)$$

$$= 39 - 36 \qquad (22)$$

$$= 3 \qquad (23)$$