EE23010 Assignment

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Question 12.13.3.11
How many times must a man toss a fair coin so that the probability of having at least one head is more than 90% ?
Solution:

Parameter	Value	Description
n	n	number of coin tosses
p	$\frac{1}{2}$	getting a head on a coin toss
q	$\frac{1}{2}$	getting a tail on a coin toss
$\mu = np$	$\frac{n}{2}$	mean of the distribution
$\sigma^2 = npq$	$\frac{n}{4}$	variance of the distribution
Y	≥ 1	Number of heads

1) Gaussian:
The gaussian-distribution of Y:

$$p_Y(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \qquad (x \in Y) \tag{1}$$

The Q-function from the gaussian distribution:

$$Q\left(\frac{x-\mu}{\sigma}\right) = \int_x^\infty \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(t-\mu)^2}{2\sigma^2}} dt \tag{2}$$

$$= \int_x^\infty \sqrt{\frac{2}{\pi n}} e^{-\frac{(2t-n)^2}{2n}} dt \tag{3}$$

a) Without correction

$$\Pr(Y \geq 1) = Q\left(\frac{2-n}{\sqrt{n}}\right) > 0.9 \tag{4}$$

$$\frac{2-n}{\sqrt{n}} < Q^{-1}(0.9) \tag{5}$$

$$\frac{2-n}{\sqrt{n}} < -1.28 \tag{6}$$

Squaring on both the sides

$$(n-2)^2 > (1.28\sqrt{n})^2 \tag{7}$$

$$n^2 - 5.6384n + 4 > 0 \tag{8}$$

$$n > 4.86, n < 0.8 \tag{9}$$

$$\implies n = 5 \tag{10}$$

b) With correction: 0.5 as correction term

$$\Pr(Y > 0.5) = Q\left(\frac{1-n}{\sqrt{n}}\right) > 0.9 \tag{11}$$

$$\frac{1-n}{\sqrt{n}} < Q^{-1}(0.9) \tag{12}$$

$$\frac{1-n}{\sqrt{n}} < -1.28 \tag{13}$$

Squaring on both the sides

$$(n-1)^2 > (1.28\sqrt{n})^2 \tag{14}$$

$$n^2 - 3.6384n + 1 > 0 \tag{15}$$

$$n > 3.38, n < 0.29 \tag{16}$$

$$\implies n = 4 \tag{17}$$

2) Binomial:
Let, X_i be the sequence of independent Bernoulli random varibles.

$$\implies X = \sum_{i=1}^{n} X_i \tag{18}$$

$$X_i = \begin{cases} 1, & \text{Heads} \\ 0, & \text{Tails} \end{cases} \tag{19}$$

$$X \sim \text{Bin}(n, p) \tag{20}$$

The PMF of X is given by:

$$p_X(k) = {}^nC_k (0.5)^k (0.5)^{n-k} \tag{21}$$

The CDF of X is defined as:

$$F_X(k) = \sum_{i=0}^{k} p_X(i) \tag{22}$$

$$= \sum_{i=0}^{k} {}^nC_i (0.5)^{n-i} (0.5)^i \tag{23}$$

We have

$$\Pr(X \geq 1) > 0.9 \tag{24}$$

$$1 - p_X(0) > 0.9 \tag{25}$$

$$(2)^n > 10 \tag{26}$$

$$n > \log_2(10) \tag{27}$$

$$\implies n = 4 \tag{28}$$

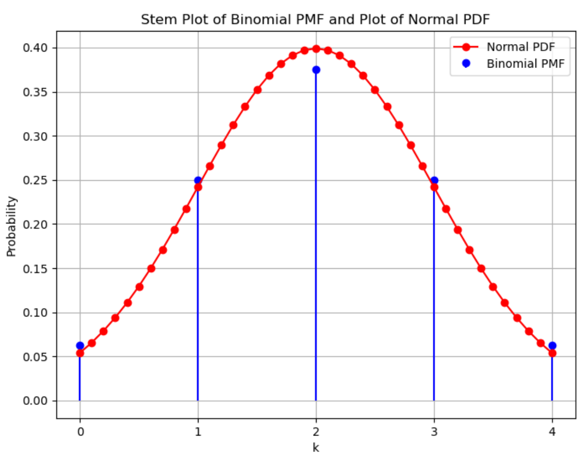

Fig. 2. Binomial PMF of X vs Normal PDF of Y