EE23010 Assignment

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Question 46
Let X be a random variable with cumulative distribution function

$$F_X(x) = \begin{cases} 0 & \text{if } x < -1 \\ \frac{1}{4}(x+1) & \text{if } -1 \le x < 0 \\ \frac{1}{4}(x+3) & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases} \tag{1}$$

Which one of the following statements is true?

(A)

$$\lim_{n\to\infty} \Pr\left(-\frac{1}{2} + \frac{1}{n} < X < -\frac{1}{n}\right) = \frac{5}{8} \tag{2}$$

(B)

$$\lim_{n\to\infty} \Pr\left(-\frac{1}{2} - \frac{1}{n} < X < \frac{1}{n}\right) = \frac{5}{8} \tag{3}$$

(C)

$$\lim_{n\to\infty} \Pr\left(X = \frac{1}{n}\right) = \frac{1}{2} \tag{4}$$

(D)

$$\Pr(X = 0) = \frac{1}{3} \tag{5}$$

(GATE ST 2023)

Solution:

(A)

$$\lim_{n\to\infty} \Pr\left(-\frac{1}{2} + \frac{1}{n} < X < -\frac{1}{n}\right) = \lim_{n\to\infty} F_X\left(-\frac{1}{n}\right) - \lim_{n\to\infty} F_X\left(-\frac{1}{2} + \frac{1}{n}\right) - \lim_{n\to\infty} \Pr\left(X = -\frac{1}{n}\right) \tag{6}$$

$$= \lim_{n\to\infty} F_X\left(-\frac{1}{n}\right) - \lim_{n\to\infty} F_X\left(-\frac{1}{2} + \frac{1}{n}\right) - \lim_{n\to\infty} F_X\left(-\frac{1}{n}\right) + \lim_{n\to\infty} F_X\left(-\frac{1^-}{n}\right) \tag{7}$$

$$= \lim_{n\to\infty} F_X\left(-\frac{1^-}{n}\right) - \lim_{n\to\infty} F_X\left(-\frac{1}{2} + \frac{1}{n}\right) \tag{8}$$

$$= \lim_{n\to\infty} \frac{1}{4}\left(-\frac{1^-}{n} + 1\right) - \lim_{n\to\infty} \frac{1}{4}\left(-\frac{1}{2} + \frac{1}{n} + 1\right) \tag{9}$$

$$= \frac{1}{8} \tag{10}$$

$\therefore (A)$ is not true.

(B)

$$\lim_{n\to\infty} \Pr\left(-\frac{1}{2} - \frac{1}{n} < X < \frac{1}{n}\right) = \lim_{n\to\infty} F_X\left(\frac{1}{n}\right) - \lim_{n\to\infty} F_X\left(-\frac{1}{2} - \frac{1}{n}\right) - \lim_{n\to\infty} \Pr\left(X = \frac{1}{n}\right) \tag{11}$$

$$= \lim_{n\to\infty} F_X\left(\frac{1}{n}\right) - \lim_{n\to\infty} F_X\left(-\frac{1}{2} - \frac{1}{n}\right) - \lim_{n\to\infty} F_X\left(\frac{1}{n}\right) + \lim_{n\to\infty} F_X\left(\frac{1^-}{n}\right) \tag{12}$$

$$= \lim_{n\to\infty} F_X\left(\frac{1^-}{n}\right) - \lim_{n\to\infty} F_X\left(-\frac{1}{2} - \frac{1}{n}\right) \tag{13}$$

$$= \lim_{n\to\infty} \frac{1}{4}\left(\frac{1^-}{n} + 3\right) - \lim_{n\to\infty} \frac{1}{4}\left(-\frac{1}{2} - \frac{1}{n} + 1\right) \tag{14}$$

$$= \frac{5}{8} \tag{15}$$

$\therefore (B)$ is true.

(C)

$$\lim_{n\to\infty} \Pr\left(X = \frac{1}{n}\right) = \lim_{n\to\infty} F_X\left(\frac{1}{n}\right) - \lim_{n\to\infty} F_X\left(\frac{1^-}{n}\right) \tag{16}$$

$$= \lim_{n\to\infty} \frac{1}{4}\left(\frac{1}{n} + 3\right) - \lim_{n\to\infty} \frac{1}{4}\left(\frac{1^-}{n} + 3\right) \tag{17}$$

$$= 0 \tag{18}$$

$\therefore (C)$ is not true.

(D)

$$\Pr(X=0) = F_X(0) - F_X(0^-) \tag{19}$$
$$= \frac{1}{4}(0+3) - \frac{1}{4}(0^- + 1) \tag{20}$$
$$= \frac{1}{2} \tag{21}$$

$\therefore (D)$ is not true.

Steps for the simulation of r.v X:

1) Identify the point of discontinuity (0 here).
2) Define the simulation size for the simulation data set (num_sim).
3) Define the functions of CDF and PDF of X.
4) Find $\Pr(X=0)$ from the PDF of X.
5) For this simulation, the remaining numbers in [-1,1) have probability of $1 - \Pr(X=0)$.
6) Generate random sample in $[-1,1) - \{0\}$ of the size = num_sim$\times(1 - \Pr(X=0))$.
7) Generate sample conatining zeros with size num_sim$\times \Pr(X=0)$.
8) Combine all the generated samples to make a single sample and we generate the required r.v X.

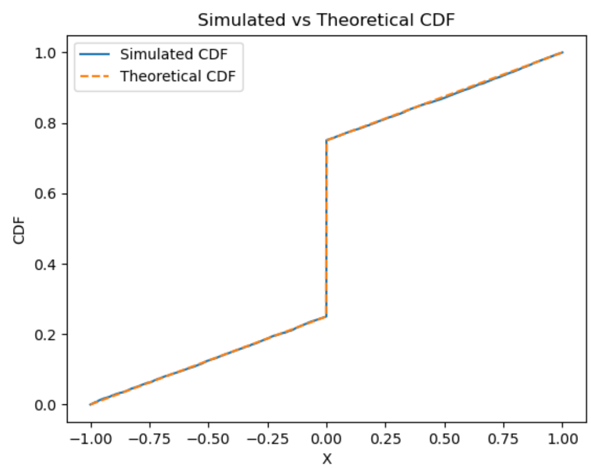

Fig. 8. CDF of X-(simulation vs actual)