EE23010 Assignment

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Question 12.13.3.55
There are 5 cards numbered 1 to 5, one number on one card. Two cards are drawn at random without replacement. Let X denote the sum of the numbers on two cards drawn. Find the mean and variance of X.

Solution:

parameters	description
A	number on the first card
B	number on the second card

$$p_A(k) = \begin{cases} \frac{1}{5}, & 1 \le k \le 5 \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

$$p_B(k) = \sum_{i=1}^{5} \Pr\left(B = k \mid A = i\right) \cdot p_A(i) \tag{2}$$

$$= \frac{1}{5} \sum_{i=1}^{5} \Pr\left(B = k \mid A = i\right) \tag{3}$$

$$\Pr\left(B = k \mid A = i\right) = \begin{cases} \frac{1}{4}, & 1 \le k \le 5, k \ne i \\ 0, & \text{otherwise} \end{cases} \tag{4}$$

$$\implies p_B(k) = \begin{cases} \frac{1}{20}, & 1 \le k \le 5, k \ne i \\ 0, & \text{otherwise} \end{cases} \tag{5}$$

X can take values ranging from 3 to 9.

$$X = A + B \tag{6}$$

$$p_X(k) = \sum_{i=1}^{5} \Pr\left(X = k \mid A = i\right) \cdot p_A(i) \tag{7}$$

Finding mean or the expectation:

$$\mathbb{E}[X] = \sum_{k=3}^{9} k \cdot p_X(k) \tag{8}$$

$$= \sum_{k=3}^{9} k \sum_{i=1}^{5} \Pr\left(X = k \mid A = i\right) \cdot p_A(i) \tag{9}$$

$$= \sum_{k=3}^{9} k \sum_{i=1}^{5} \Pr\left(B = k - i \mid A = i\right) \cdot p_A(i) \tag{10}$$

$$p_X(k) = p_X(2m+1) + p_X(2m) \quad \forall m \in \{1, 2, 3, 4\} \tag{11}$$

Using (11), we get:

$$\mathbb{E}[X] = \sum_{m=1}^{4} (2m+1) \sum_{i=1}^{5} \Pr\left(B = 2m + 1 - i \mid A = i\right) \cdot p_A(i) + \sum_{m=2}^{4} 2m \sum_{i=1}^{5} \Pr\left(B = 2m - i \mid A = i\right) \cdot p_A(i) \tag{12}$$

$$= \frac{1}{5} \sum_{m=1}^{4} (2m+1) \sum_{i=1}^{5} \Pr\left(B = 2m + 1 - i \mid A = i\right) + \frac{1}{5} \sum_{m=2}^{4} 2m \sum_{i=1}^{5} \Pr\left(B = m - i \mid A = i\right) \tag{13}$$

$$= \frac{1}{5} \sum_{m=1}^{4} (2m+1) \sum_{i=1}^{5} \left.\frac{1}{4}\right|_{1 \le 2m+1-i \le 5} + \frac{1}{5} \sum_{m=2}^{4} 2m \sum_{i=1}^{5} \left.\frac{1}{4}\right|_{1 \le 2m-i \le 5, m \ne i} \tag{14}$$

$$= \frac{1}{20} \sum_{m=1}^{4} (2m+1) \sum_{i=1}^{5} \left.1\right|_{2m-4 \le i \le 2m} + \frac{1}{20} \sum_{m=2}^{4} 2m \sum_{i=1}^{5} \left.1\right|_{2m-5 \le i \le 2m-1, m \ne i} \tag{15}$$

$$= \frac{1}{20} \sum_{m=1}^{4} (2m+1) \left(\min(5, 2m) - \max(1, 2m-4) + 1\right) + \frac{1}{20} \sum_{m=2}^{4} 2m \left(\min(5, 2m-1) - \max(1, 2m-5)\right) \tag{16}$$

$$\implies \mathbb{E}[X] = \frac{1}{20}(72) + \frac{1}{20}(48) \tag{17}$$

$$= 6 \tag{18}$$

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 \tag{19}$$

$$= \frac{1}{20}\sum_{m=1}^{4}(2m+1)^2\,(\min(5,2m) - \max(1,2m-4) + 1) + \frac{1}{20}\sum_{m=2}^{4}(2m)^2\,(\min(5,2m-1) - \max(1,2m-5)) - 6^2 \tag{20}$$

$$= \frac{1}{20}(476) + \frac{1}{20}(304) - 36 \tag{21}$$

$$= 39 - 36 \tag{22}$$

$$= 3 \tag{23}$$