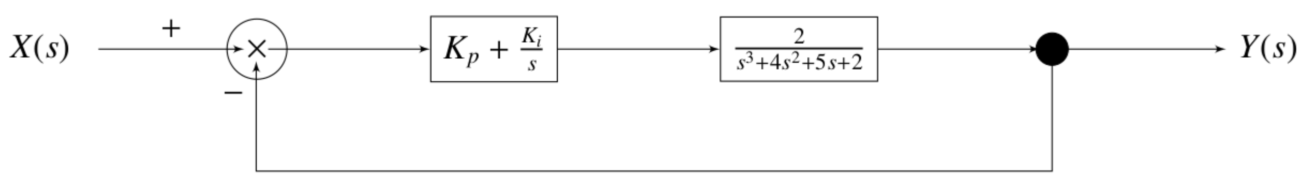
GATE:2021 - EC 48

EE23BTECH11025 - Anantha Krishnan

I. Question

A unity feedback system that uses proportional-integral (PI) control is shown in the figure. The stability



of the overall system is controlled by tuning the PI control parameters K_p and K_i. The maximum value of K_i that can be chosen so as to keep the overall system stable or, in the worst case, marginally stable (*rounded off to three decimal places*) is? (GATE EC 2021)

Solutions :

Symbols	Description	Values
$P(s)$	Plant transfer function	$\frac{2}{s^3+4s^2+5s+2}$
$C(s)$	PI controller transfer function	$K_p + \frac{K_i}{s}$
$G(s)$	Closed loop transfer function	$\frac{P(s)C(s)}{1+P(s)C(s)}$
Z	Number of zeroes with positive real part in $1 + P(s)C(s)$	?
N	Total number of anticlockwise encirclements about $-1 + 0j$ in Nyquist plot	?
P	Number of poles with positive real part in $P(s)C(s)$	?

TABLE I
Parameters, Descriptions, and Values

From table I, the characteristic equation is given as:

$$1 + \left(K_p + \frac{K_i}{s}\right)\left(\frac{2}{s^3 + 4s^2 + 5s + 2}\right) = 0 \tag{1}$$

$$s^4 + 4s^3 + 5s^2 + (2 + 2K_p)s + 2K_i = 0 \tag{2}$$

For the system to be stable, there must be no sign changes in the first column of the routh array for the above equation. From (2)

$$\begin{array}{c|ccc} s^4 & 1 & 5 & 2K_i \\ s^3 & 4 & (2+2K_p) & 0 \\ s^2 & \frac{18-2K_p}{4} & 2K_i & 0 \\ s^1 & \frac{\left(\frac{18-2K_p}{4}\right)(2+2K_p)-8K_i}{\frac{18-2K_p}{4}} & 0 & 0 \\ s^0 & 2K_i & 0 & 0 \end{array} \tag{3}$$

(4)

$$\frac{18-2K_p}{4} > 0 \tag{5}$$

$$\implies K_p < 9 \tag{6}$$

$$\frac{\left(\frac{18-2K_p}{4}\right)\left(2+2K_p\right)-8K_i}{\frac{18-2K_p}{4}} > 0 \tag{7}$$

$$K_i > 0 \tag{8}$$

For marginal stability, assuming 3 cases while maximising K_i and checking if the above inequalities hold.

1) $K_p = 9$

$$\left(\lim_{K_p\to 9^-} \frac{\left(\frac{18-2K_p}{4}\right)\left(2+2K_p\right)-8K_i}{\frac{18-2K_p}{4}} > 0\right) \cap (K_i > 0) \tag{9}$$

$$\left(\lim_{K_p\to 9^-} -8K_i > 0\right) \cap (K_i > 0) \tag{10}$$

$$\implies K_p = 9, \forall K_i \epsilon (\phi) \tag{11}$$

2) $K_i = 0$

$$\left(\left(\frac{18-2K_p}{4}\right)\left(2+2K_p\right) > 0\right) \cap \left(K_p < 9\right) \tag{12}$$

$$\implies K_i = 0, \forall K_p \epsilon (-1, 9) \tag{13}$$

3) $\frac{\left(\frac{18-2K_p}{4}\right)\left(2+2K_p\right)-8K_i}{\frac{18-2K_p}{4}} = 0$

$$\left(\frac{18-2K_p}{4}\right)\left(2+2K_p\right) = 8K_i \tag{14}$$

$$-K_p^2 + 8K_p + 9 = 8K_i \tag{15}$$

Vertex ($K_p = 4$) satisfies (6):

$$K_i = 3.125 \forall (K_p = 4, K_i > 0) \tag{16}$$

Based on the three cases for marginal stability, the maximum value of K_i is 3.125, for $K_p = 4$.

1) Verification by plotting roots of characteristic equation:
If real part of atleast 1 root is equal to zero and for the rest are less than or equal to zero, then the system is marginally stable.
2) Verification by Nyquist diagrams:
From I, if $P = 0$ and $-1 + 0j$ is neither bounded nor unbounded by the contour, then the system is marginally stable. For P:

$$s^4 + 4s^3 + 5s^2 + 2s = 0 \tag{17}$$

$$(s+1)^2(s+2) = 0 \tag{18}$$

$$\implies P = 0 \tag{19}$$

$$\implies Z = N \tag{20}$$

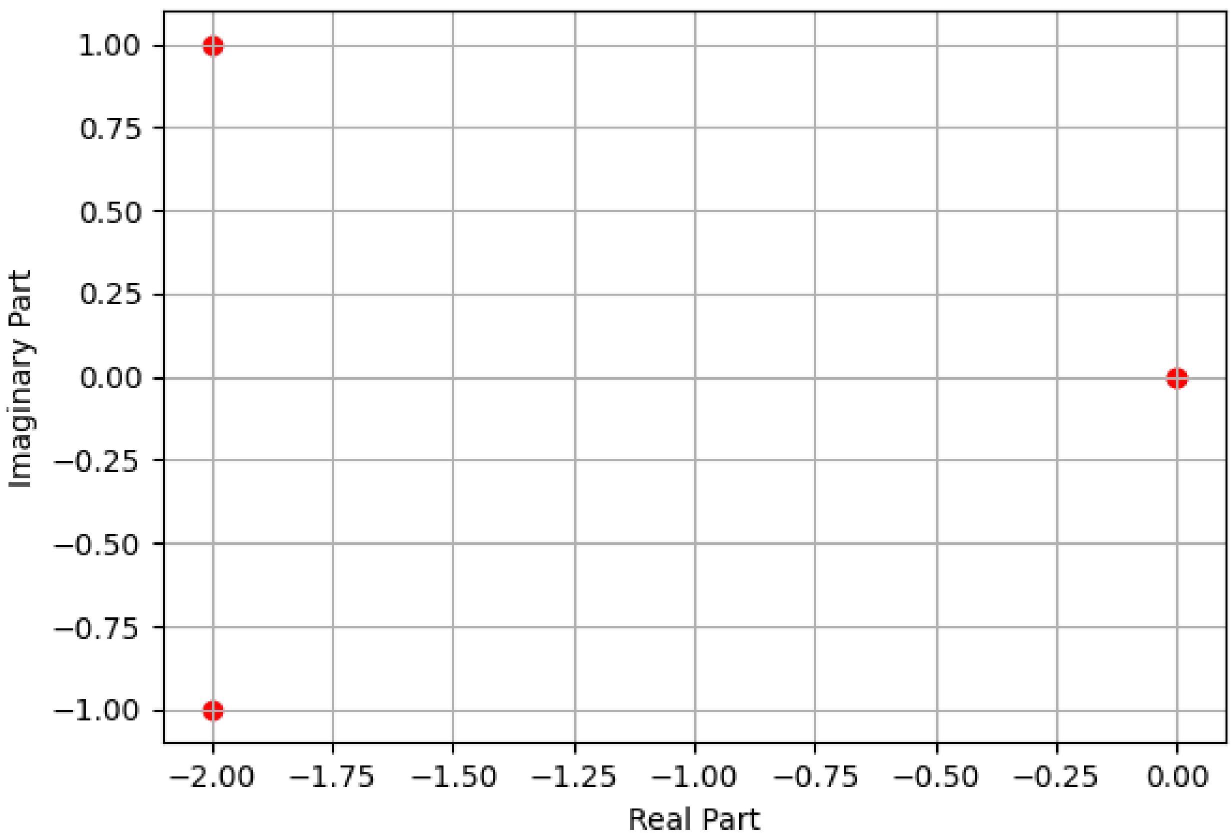


Fig. 1. Location of roots for $k_i = 0, k_p = -1$

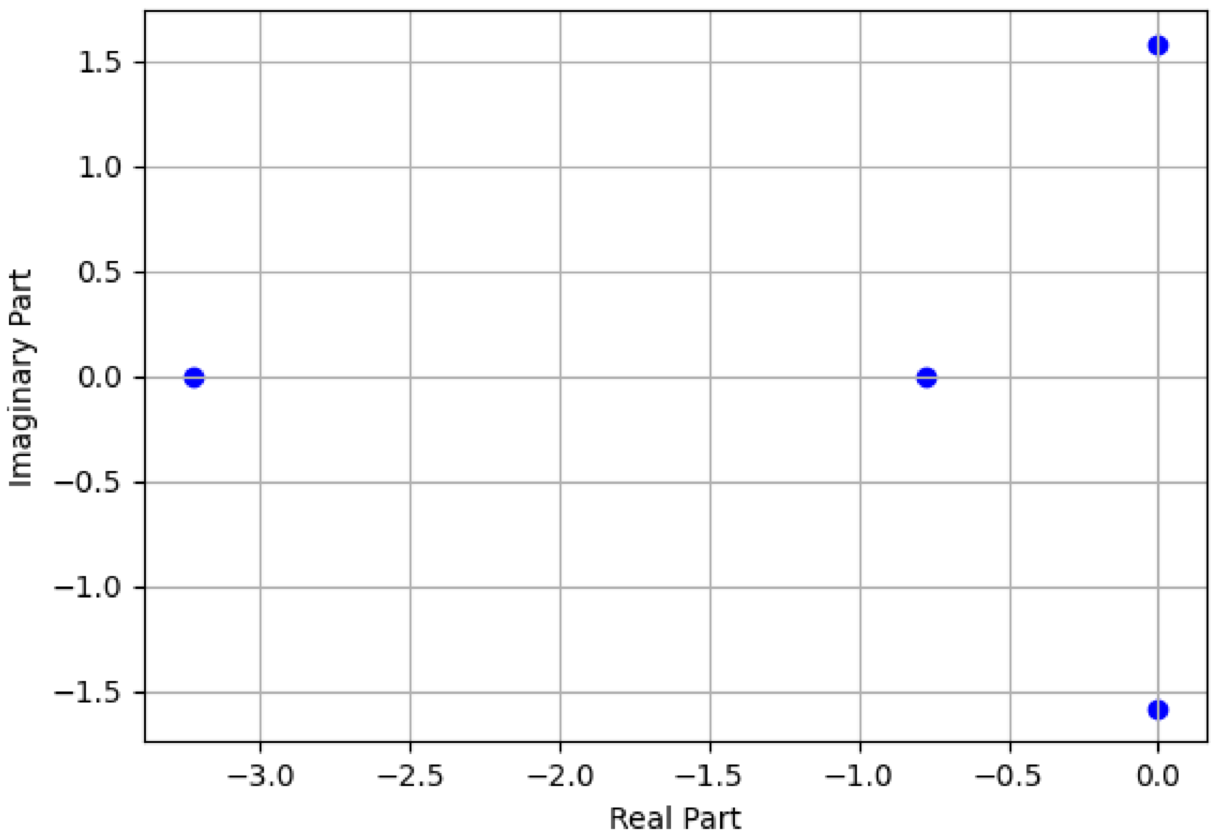


Fig. 2. Location of roots for $k_i = 0, k_p = 9$

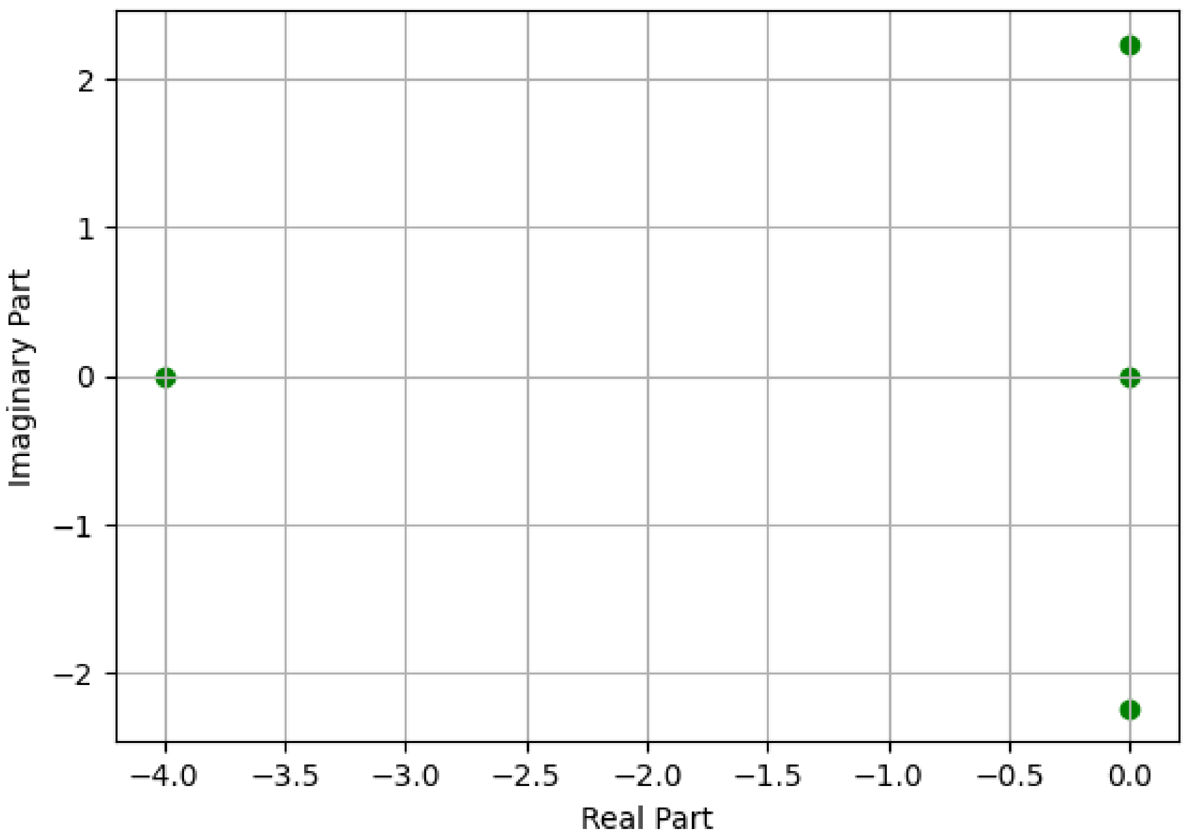


Fig. 3. Location of roots for $k_i = 3.125, k_p = 4$

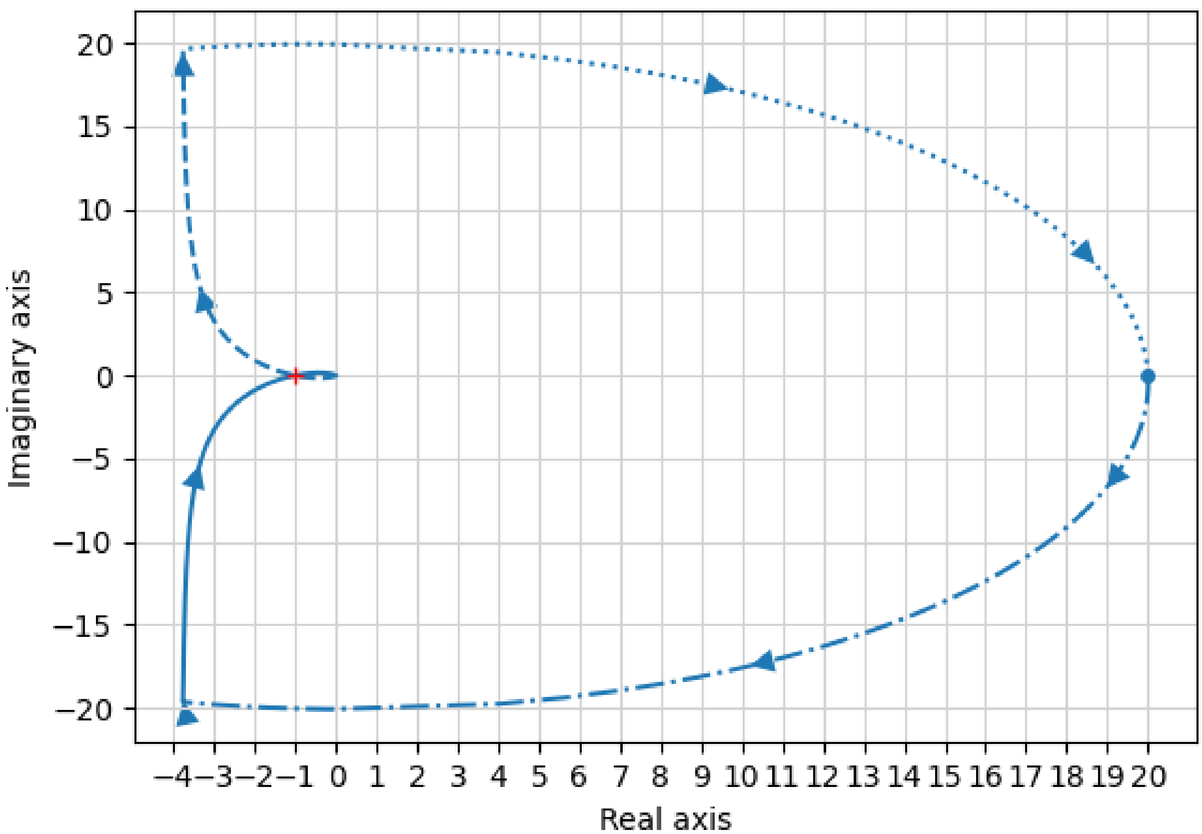


Fig. 4. Nyquist plot for $k_i = 0, k_p = -1$

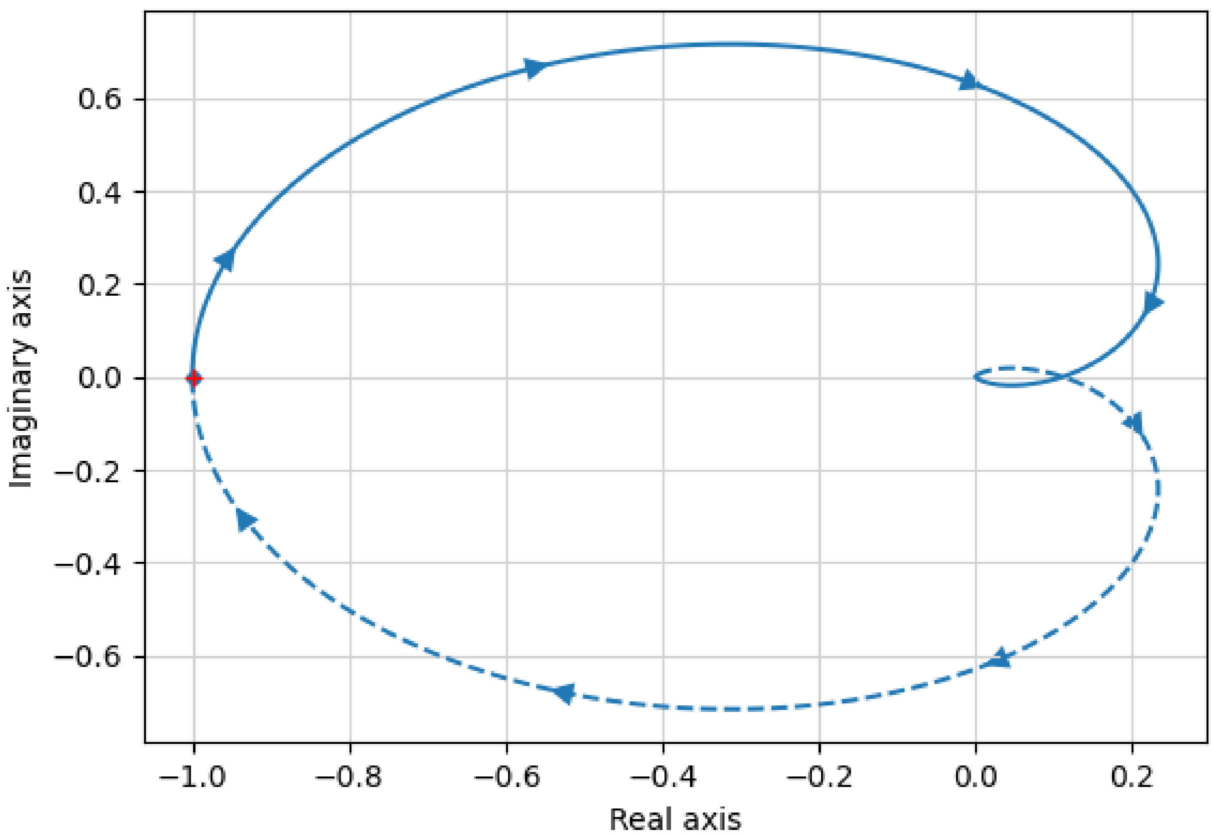


Fig. 5. Nyquist plot for $k_i = 0, k_p = 9$

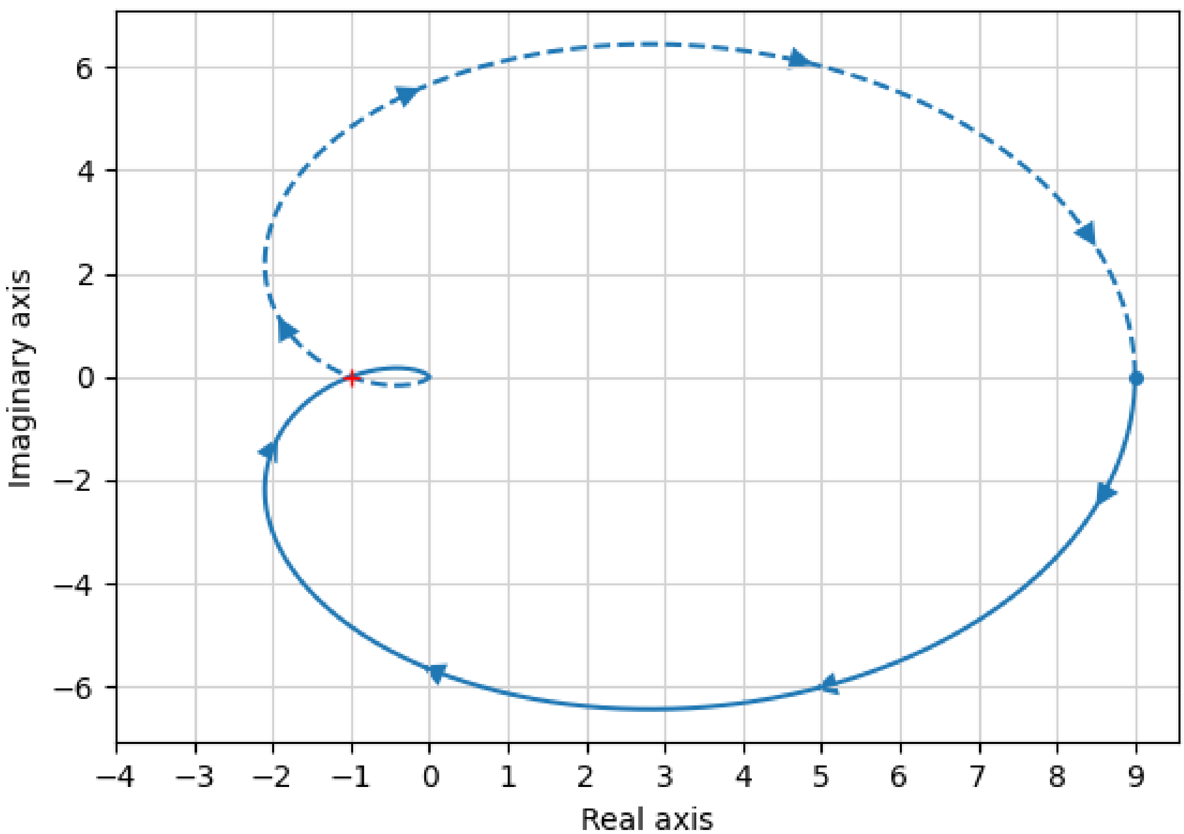


Fig. 6. Nyquist plot for for $k_i = 3.125, k_p = 4$