GATE 2022 ME-32

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Question:
A rigid uniform annular disc is pivoted on a knife edge A in a uniform gravitational field as shown, such that it can execute small amplitude simple harmonic motion in the plane of the figure without slip at the pivot point. The inner radius r and outer radius R are such that $r^2 = \frac{R^2}{2}$, and the acceleration due to gravity is g. If the time period of small amplitude simple harmonic motion is given by $T = \beta\pi\sqrt{\frac{R}{g}}$, where π is the ratio of circumference to diameter of a circle, then β = (Round off to 2 decimal places) GATE 2022 ME-32

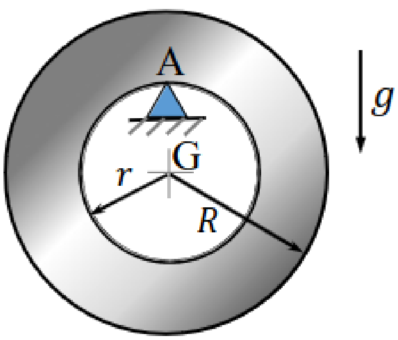

Fig. 0. Question Diagram

Parameters in expression		
Symbol	Description	Value
I	Moment of Inertia about the pivot point	$\frac{5}{4}MR^2$
$\theta(t)$	Angular displacement from vertical	?
$\theta(0)$	Value of $\theta(t)$ at $t = 0$	0
$\Theta(s)$	Laplace Transform of $\theta(t)$	?
r	Distance of center of gravity from pivot point	$\frac{R}{\sqrt{2}}$

TABLE 0
PARAMETERS

Solution:
Moment of inertia of disc about pivot point is calculated as

$$I = \frac{1}{2}\left(R^2 + \frac{R^2}{2}\right) + \frac{MR^2}{2} \tag{1}$$

$$= \frac{5}{4}MR^2 \tag{2}$$

Using D' Alambert's principle,

$$I\frac{d^2\theta(t)}{dt^2} + mgr\sin(\theta(t)) = 0 \tag{3}$$

$$\implies I\frac{d^2\theta(t)}{dt^2} + mgr\theta(t) = 0, \text{for } \theta \ll 1, \theta > 0 \tag{4}$$

Using (2) and (4), we get

$$\frac{5MR^2}{4}\frac{d^2\theta(t)}{dt^2} + Mg\frac{R}{\sqrt{2}}\theta(t) = 0 \tag{5}$$

$$\frac{d^2\theta(t)}{dt^2} + \frac{2\sqrt{2}g}{5R}\theta(t) = 0 \tag{6}$$

Taking Laplace Transform on both sides,we get

$$s^2\Theta(s) - s\theta(0) - \theta'(0) + \frac{2\sqrt{2}g}{5R}\Theta(s) = 0 \tag{7}$$

$$\Theta(s)\left(s^2 + \frac{2\sqrt{2}g}{5R}\right) = \theta'(0) \tag{8}$$

$$\Theta(s) = \frac{\theta'(0)}{\left(s^2 + \frac{2\sqrt{2}g}{5R}\right)} \tag{9}$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \tag{10}$$

$$e^{at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s-a} \tag{11}$$

$$\frac{\left(e^{jat} - e^{-jat}\right)}{2}u(t) \longleftrightarrow \frac{a}{s^2+a^2} \tag{12}$$

$$\sin(at) \longleftrightarrow \frac{a}{s^2+a^2} \tag{13}$$

From (9)

$$\Theta(s) = \frac{\theta'(0)\left(\sqrt{\frac{2\sqrt{2}g}{5R}}\right)}{\left(s^2 + \frac{2\sqrt{2}g}{5R}\right)} \frac{1}{\left(\sqrt{\frac{2\sqrt{2}g}{5R}}\right)} \tag{14}$$

Taking inverse Laplace by putting $\frac{\theta'(0)}{\left(\sqrt{\frac{2\sqrt{2}g}{5R}}\right)} = k$ and (13),

$$\theta(t) = k \sin\left(\frac{2\sqrt{2}g}{5R} t\right) \tag{15}$$

$$T = \frac{2\pi}{\frac{2\sqrt{2}g}{5R}} \tag{16}$$

$$= \sqrt{\left(5\sqrt{2}\right)}\,\pi \sqrt{\frac{R}{g}} \tag{17}$$

Thus,

$$\beta = \sqrt{\left(5\sqrt{2}\right)} \tag{18}$$

$$\beta = 2.66 \tag{19}$$

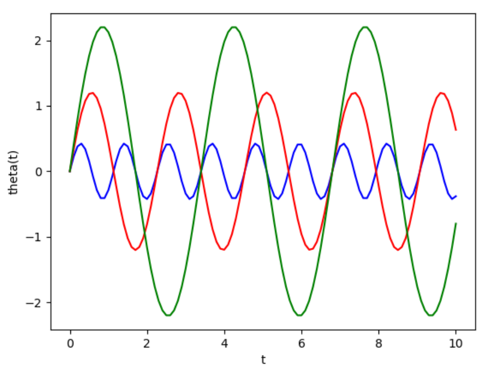

Fig. 0. Plot of $\theta(t)$ for $(\theta'(0), \mathrm{R}) \in \{(1,1)\ , (2,2)\ , (3,3)\}$