SIGNAL PROCESSING

Through GATE

EE1205-TA Group

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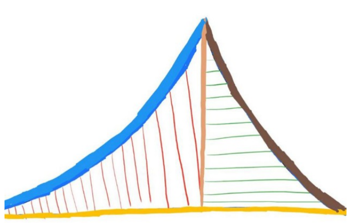


Contents

Introduction

This book provides solutions to signal processing problems in GATE.

Chapter 1

Harmonics

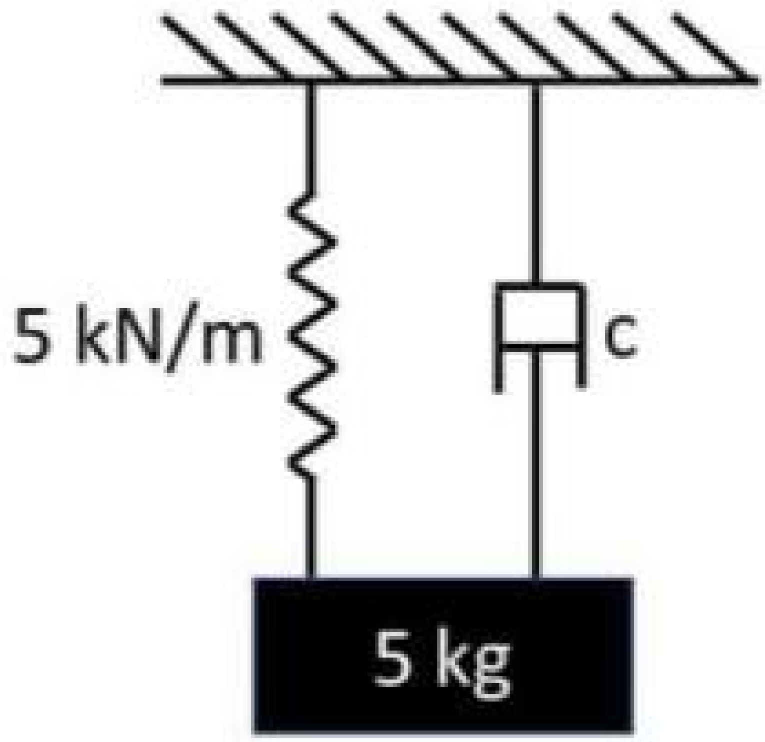
1.1. 2022

1.1 A damper with damping coefficient, c, is attached to a mass of 5 kg and spring of stiffness 5 kN/m as shown in figure. The system undergoes under-damped oscillations. If the ratio of the 3^{rd} amplitude to the 4^{th} amplitude of oscillations is 1.5, the value of c is ?

(GATE AE-62 (2022)) **Solution:**

Parameter	**Value**	**Description**
c	?	Damping Coefficient
k	5 kN/m	Stiffness
r	1.5	Ratio of 3^{rd} amplitude to 4^{th} amplitude of oscillations

Table 1.1: Parameter Table (GATE AE-62)

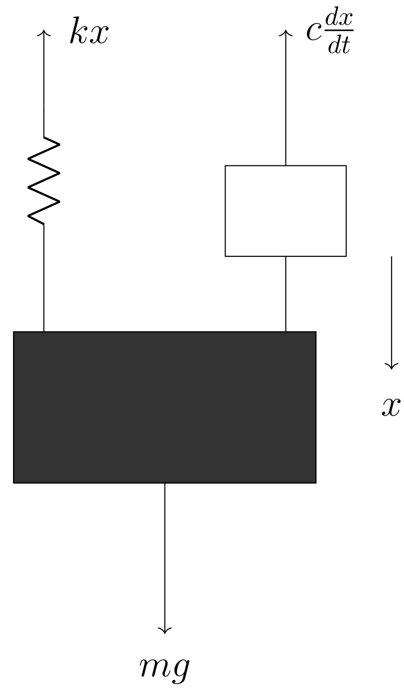


Now, as the oscillation begins, from the Fig. 1.1 we write net force on the mass as,

$$F = F_1 + F_2 + mgu(t) \tag{1.1}$$

$$\implies m\frac{d^2x(t)}{dt^2} = -kx(t) - c\frac{dx(t)}{dt} + mgu(t) \tag{1.2}$$

$$\implies \frac{d^2x(t)}{dt^2} + \left(\frac{c}{m}\right)\frac{dx(t)}{dt} + \left(\frac{k}{m}\right)x(t) = gu(t) \tag{1.3}$$



Now, taking the Laplace transform on both sides,

$$s^2X(s) + \frac{c}{m}sX(s) + \frac{k}{m}X(s) = \frac{g}{s} \tag{1.4}$$

$$\implies X(s) = \frac{g}{s\left(s^2 + \frac{c}{m}s + \frac{k}{m}\right)} \tag{1.5}$$

$$\implies X(s) = \frac{g}{s(s-s_1)(s-s_2)} \tag{1.6}$$

Where

$$s_1 = -\frac{c}{2m} + \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{1.7}$$

$$s_2 = -\frac{c}{2m} - \sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} \tag{1.8}$$

From (1.6) we get,

$$\begin{aligned}\implies X(s) = \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_1(s-s_1)} - \frac{1}{s_2(s-s_2)}\right] \\ - \frac{g}{s_1 s_2}\left(\frac{1}{s}\right)\end{aligned} \tag{1.9}$$

Now again taking the inverse Laplace transform we have,

$$x(t) = -\frac{g}{s_1 s_2}u(t) + \frac{g}{(s_1 - s_2)}\left[\frac{1}{s_1}e^{s_1 t} - \frac{1}{s_2}e^{s_2 t}\right]u(t) \tag{1.10}$$

$$\begin{aligned}\implies x(t) = -\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gc}{2mk}\right)^2}\, e^{-ct/2m}u(t) \\ \sin\left(\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}\, t + \tan^{-1}\left(\frac{2mg\sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}}{gc}\right)\right) \\ - \frac{mg}{k}u(t)\end{aligned} \tag{1.11}$$

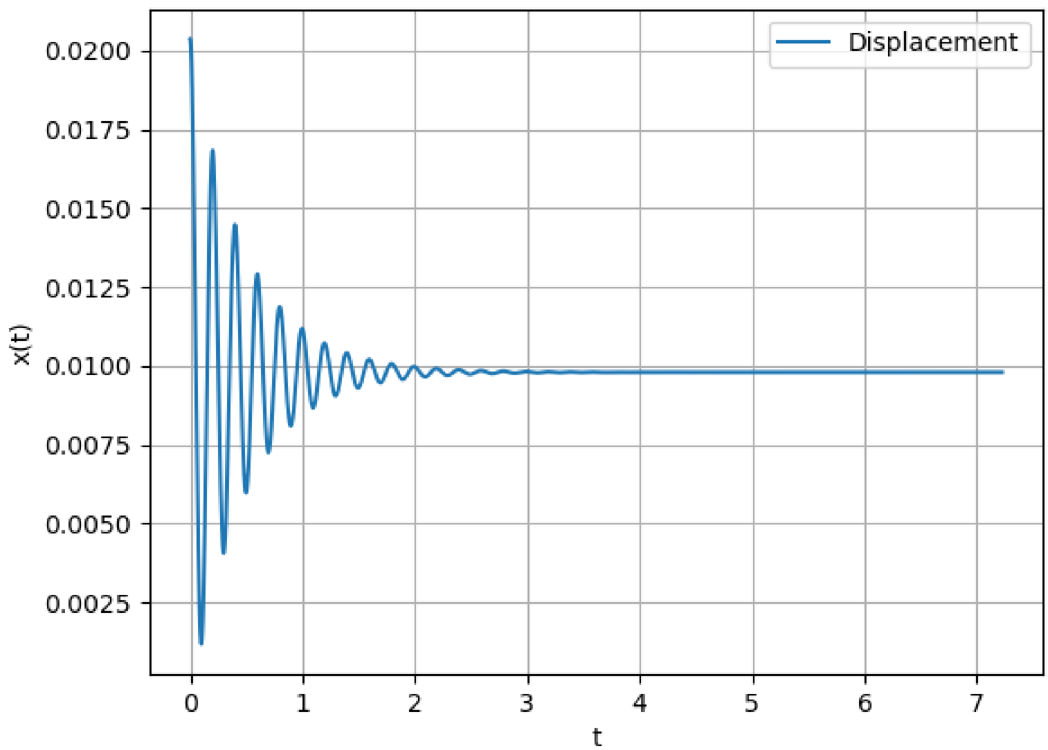
(Substituting the values of s_1 and s_2 from (1.7) and (1.8))

From (1.11), we have the ratio of 3^{rd} to 4^{th} amplitude,

$$\begin{aligned}-\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gc}{2mk}\right)^2}\, e^{-3cT/2m} = \\ -\frac{3}{2}\sqrt{\left(\frac{mg}{k}\right)^2 + \left(\frac{gc}{2mk}\right)^2}\, e^{-4cT/2m}\end{aligned} \tag{1.12}$$

$$\implies e^{\pi c/\sqrt{mk}} = \frac{3}{2} \tag{1.13}$$

$$\implies c = \frac{\sqrt{mk}\ln\frac{3}{2}}{\pi} \tag{1.14}$$



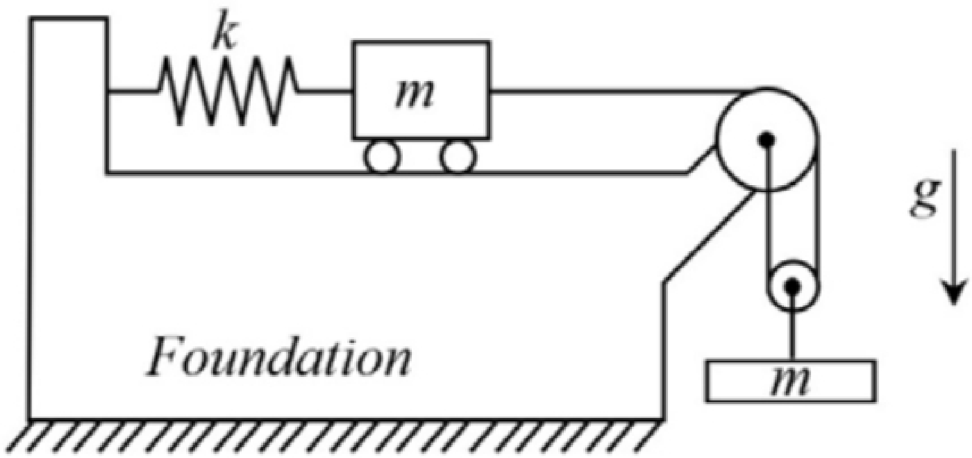
1.2 A spring-mass system having a mass m and spring constant k, placed horizontally on a foundation, is connected to a vertically hanging mass m with the help of an inextensible string. Ignore the friction in the pulleys and also the inertia of pulleys, string and spring. Gravity is acting vertically downward as shown. The natural frequency of the system in rad/s is

(A) $\sqrt{\frac{4k}{3m}}$

(B) $\sqrt{\frac{k}{2m}}$

(C) $\sqrt{\frac{k}{3m}}$

(D) $\sqrt{\frac{4k}{5m}}$



(GATE XE 2022)

Solution:

Parameters	**Description**	**Value**
$x(t)$	Displacement of mass m on foundation at time t	
$x(0)$	Displacement of mass m on foundation at time $t = 0$	0
$x'(0)$	Velocity of mass m on foundation at time $t = 0$	0

Table 1.2: Parameters

$$T - kx = m\frac{d^2x}{dt^2} \tag{1.15}$$

$$mg - 2T = m\frac{d^2\left(\frac{x}{2}\right)}{dt^2} \tag{1.16}$$

$$\implies mg - 2kx = \frac{5}{2}m\frac{d^2x}{dt^2} \tag{1.17}$$

$$\frac{d^2x}{dt^2} \xleftrightarrow{\mathcal{L}} s^2X(s) - sx(0) - x'(0) \tag{1.18}$$

$$t^n \xleftrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}} \tag{1.19}$$

From the Laplace transforms (1.18) and (1.19), we get

$$\frac{mg}{s} - 2kX(s) = \frac{5}{2}m\left(s^2X(s) - sx(0) - x'(0)\right) \tag{1.20}$$

$$\implies X(s) = \frac{\frac{2g}{5}}{s\left(s^2 + \frac{4k}{5m}\right)} \tag{1.21}$$

$$= \frac{mg}{2ks} - \frac{mgs}{2k\left(s^2 + \frac{4k}{5m}\right)} \tag{1.22}$$

$$\cos at \xleftrightarrow{\mathcal{L}} \frac{s}{s^2 + a^2} \tag{1.23}$$

From the Laplace transforms (1.19) and (1.23), we get

$$x(t) = \frac{mg}{2k}\left(1 - \cos\left(\sqrt{\frac{4k}{5m}}t\right)\right)u(t) \tag{1.24}$$

$$\implies \omega = \sqrt{\frac{4k}{5m}} \tag{1.25}$$

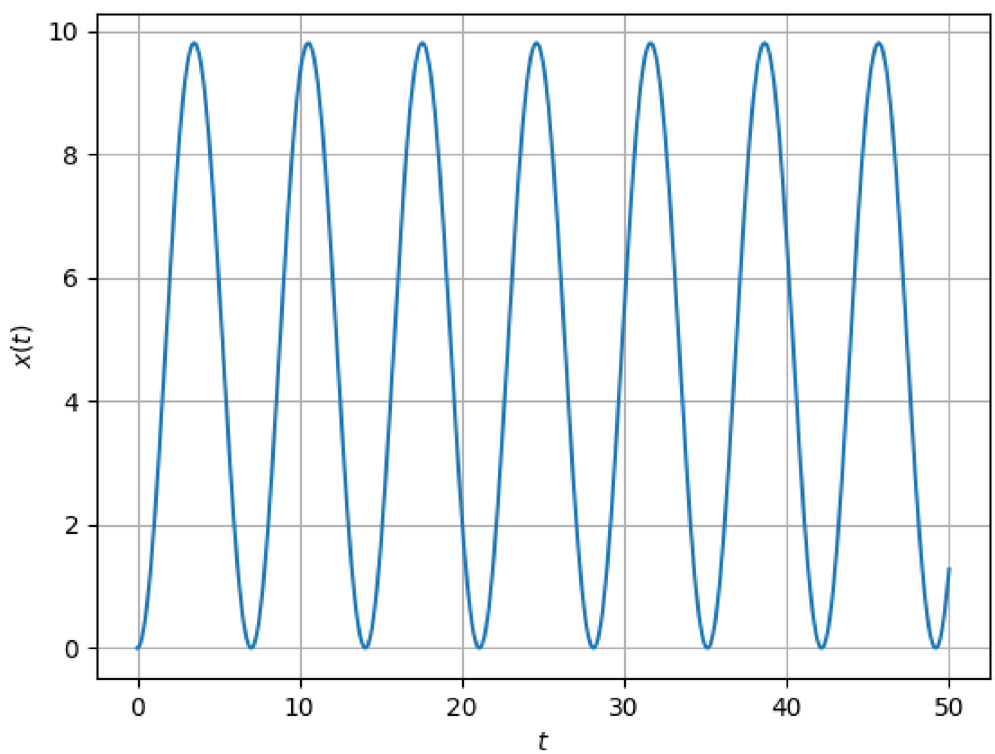


Figure 1.1: Plot of $x(t)$ for $m = 1kg$, $k = 1N/m^2$

1.3 The time delay between the peaks of the voltage signals $v_1(t) = \cos(6t + 60°)$ and $v_2(t) = -\sin(6t)$ is ______s

(A) $\frac{300\pi}{360}$

(B) $\frac{10\pi}{360}$

(C) $\frac{50\pi}{360}$

(D) $\frac{200\pi}{360}$

(GATE BM 2022 QUESTION 18)

Solution: From the values given in the Table 1.3:

Parameter	Description	Value
$v_1(t)$	Input voltage signal 1	$\cos(6t + 60°)$
$v_2(t)$	Input voltage signal 2	$-\sin(6t)$
$\Delta\phi$	Phase difference between two input signals	?
Δt	Time difference between maxima of two input signals	?
ω	angular frequency of input voltages	6

Table 1.3: input values

$$v_1(t) = \cos(6t + 60°) \tag{1.26}$$

$$v_2(t) = -\sin(6t) \tag{1.27}$$

$$\implies v_2(t) = \cos(6t + 90°) \tag{1.28}$$

From (1.27) and (1.28), phase difference between two voltage signals is 30°. From formula,

$$\Delta\phi = \frac{\Delta t}{\frac{2\pi}{\omega}} 360 \tag{1.29}$$

$$\therefore \Delta t = \frac{10\pi}{360} s \tag{1.30}$$

Hence, option B is correct.

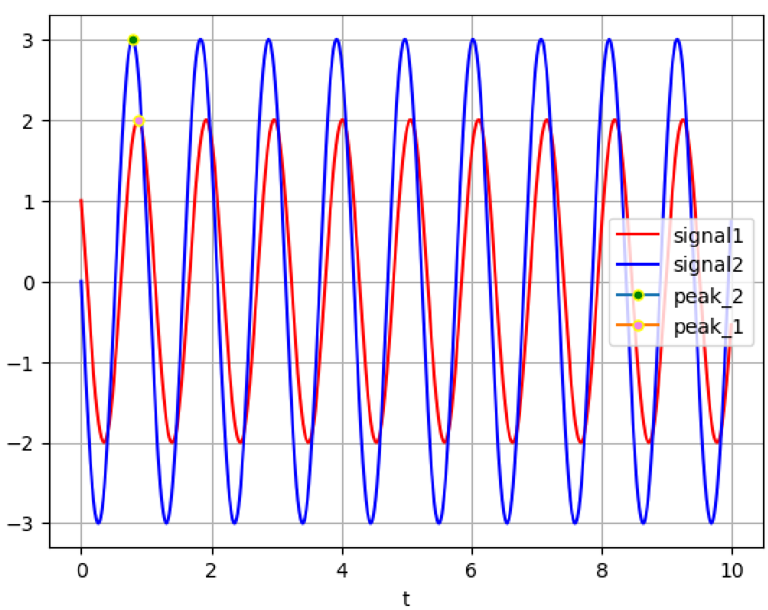


Figure 1.2: Figure of input voltage signals

1.4 A sinusoidal carrier wave with amplitude A_c and frequency f_c is amplitude modulated with a message signal $m(t)$ having frequency $0 < f_m << f_c$ to generate the modulated wave $s(t)$ given by $s(t) = A_c(1 + m(t))\, cos(2\pi f_c t)$ The message signal that can be retrieved completely using envelope detection is ____________________

(a) $m(t) = 0.5 \cos(2\pi f_m t)$

(b) $m(t) = 1.5 \sin(2\pi f_m t)$

(c) $m(t) = 2 \sin(4\pi f_m t)$

(d) $m(t) = 2 \cos(4\pi f_m t)$

(GATE IN 2022 QUESTION 16)

Solution:

Parameter	Description
$s(t)$	Amplitude Modulated Wave
$M(t)$	Message Signal
$c(t)$	Carrier Signal
f_c	Frequency of Carrier Signal
f_m	Frequency of Message Signal

Table 1.4: Variables and their descriptions

$$c\left(t\right) = A_c \cos\left(2\pi f_c t\right) \tag{1.31}$$

$$M\left(t\right) = A_m \cos\left(2\pi f_m t\right) \tag{1.32}$$

$$\begin{aligned} s\left(t\right) &= \left(A_c + M\left(t\right)\right) \cos\left(2\pi f_c t\right) && (1.33)\\ &= A_c \left(1 + \frac{A_m}{A_c} \cos\left(2\pi f_m t\right)\right) \cos 2\pi f_c t && (1.34)\\ &= A_c \left(1 + m\left(t\right)\right) \cos 2\pi f_c t && (1.35) \end{aligned}$$

Modulation Index of $s\left(t\right) = \mu = \frac{A_m}{A_c}$

- $\mu < 1$ Signal is Can be detected
- $\mu = 1$ Signal Cannot be detected
- $\mu > 1$ Over modualtion

(a) m(t) = 0.5 cos($2\pi f_m t$)

$$\frac{A_m}{A_c} = 0.5 \tag{1.36}$$

$$\mu < 1 \tag{1.37}$$

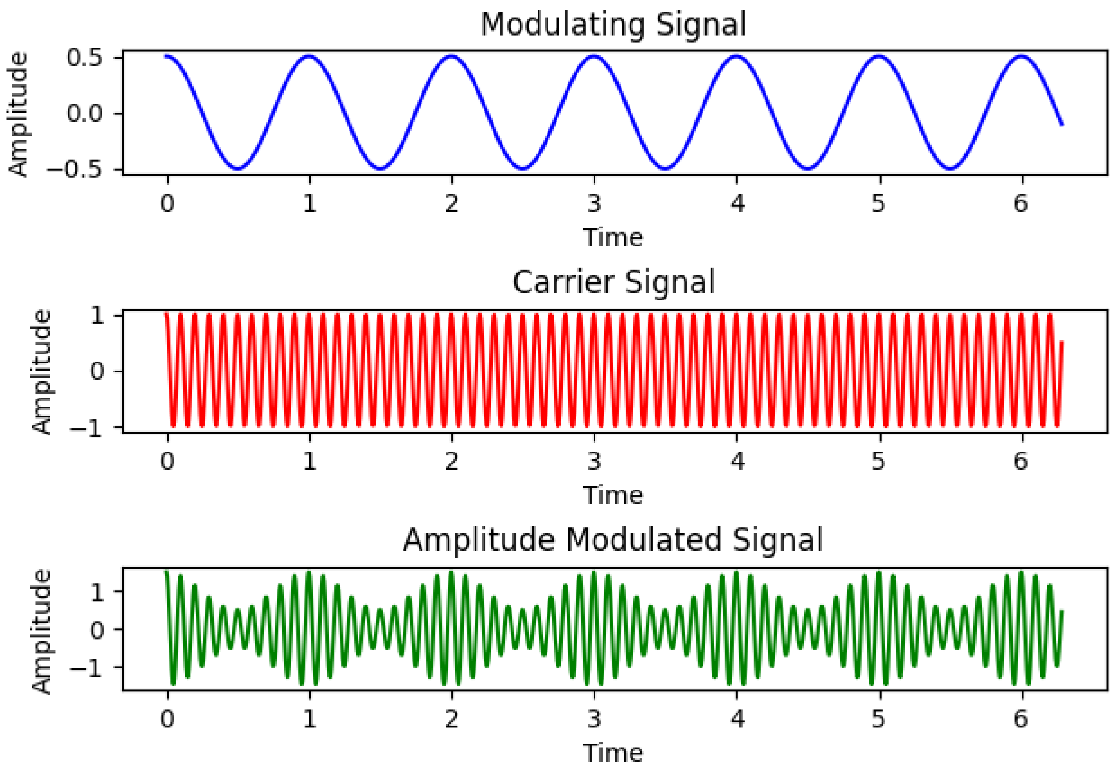
$\therefore$ Signal can be retrieved completely.

(b) m(t) = 1.5 sin($2\pi f_m t$)

$$\frac{A_m}{A_c} = 1.5 \tag{1.38}$$

$$\mu > 1 \tag{1.39}$$

$\therefore$ Signal cannot be retrieved completely.



(c) $\mathrm{m}(t) = 2\ \sin(4\pi f_m t)$

$$\frac{A_m}{A_c} = 2 \tag{1.40}$$

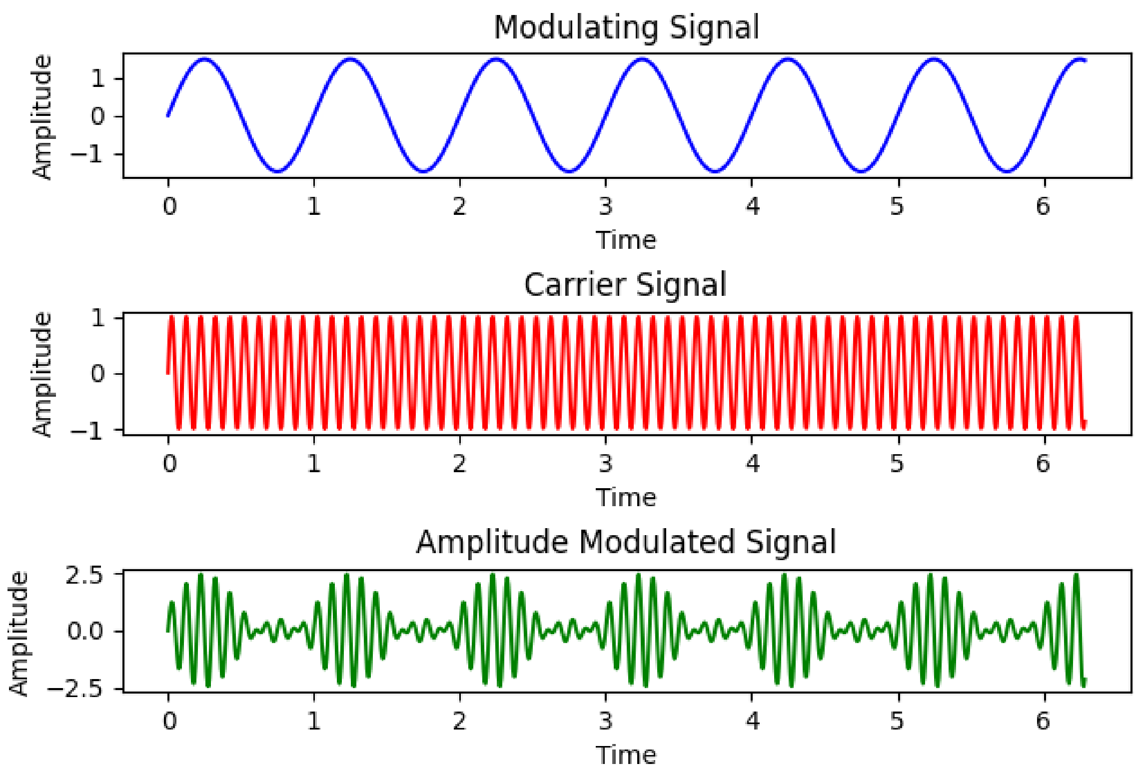
$$\mu > 1 \tag{1.41}$$

$\therefore$ Signal cannot be retrieved completely.

(d) $\mathrm{m}(t) = 2\ \cos(4\pi f_m t)$

$$\frac{A_m}{A_c} = 2 \tag{1.42}$$

$$\mu > 1 \tag{1.43}$$



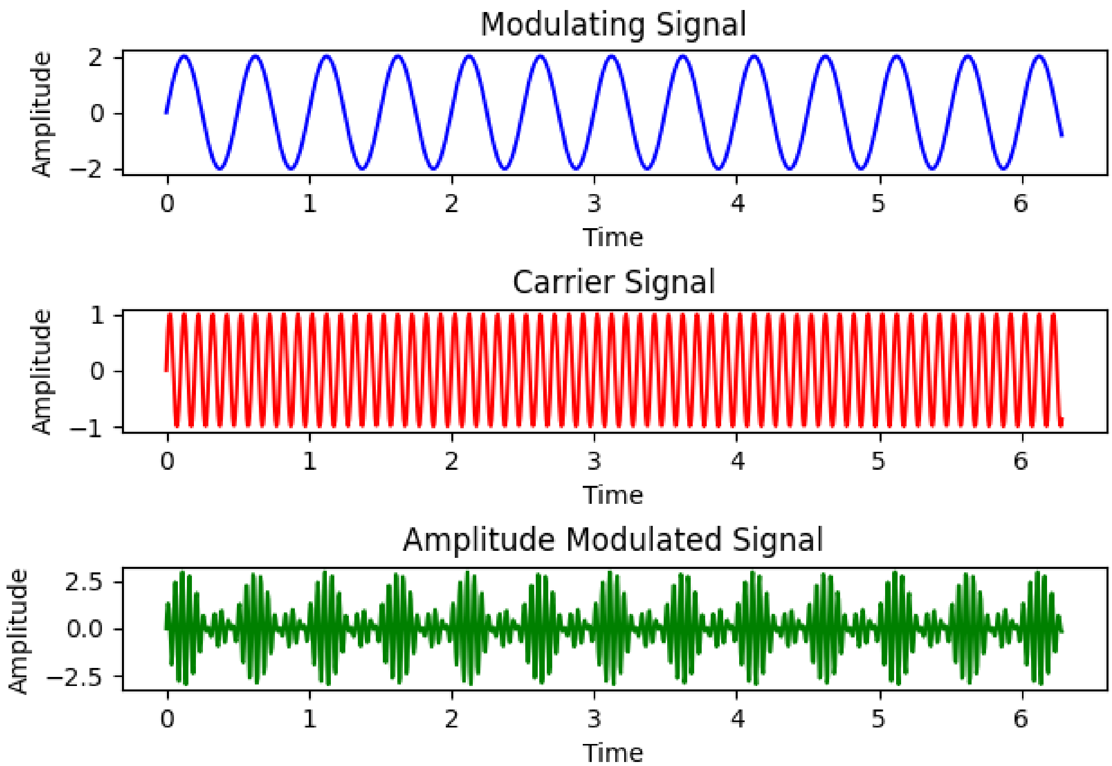
$\therefore$ Signal cannot be retrieved completely.

1.5 A uniform rigid prismatic bar of total mass m is suspended from a ceiling by two identical springs as shown in figure. Let ω_1 and ω_2 be the natural frequencies of mode I and mode II respectively ($\omega_1 < \omega_2$). The value of $\frac{\omega_2}{\omega_1}$ is ______ (rounded off to one decimal place). (GATE AE 2022 QUESTION 63)

Solution:

i: For vertical oscillations: from Fig. 1.4,

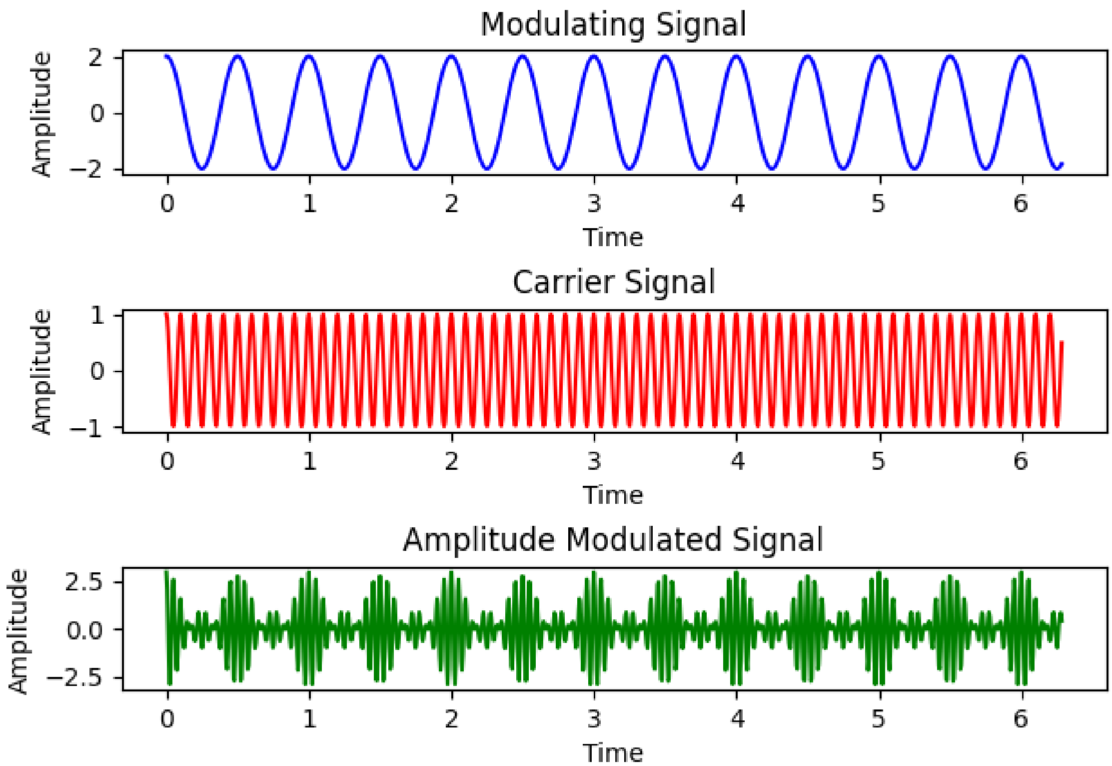
$$m\frac{d^2x\,(t)}{dt^2} + 2kx\,(t) = 0 \tag{1.44}$$



Parameter	**Description**	**Value**
$X(s)$	position in laplace domain	$X(s)$
$\Theta(s)$	angle rotated in laplace domain	$\Theta(s)$
$x(t)$	position of mass w.r.t time	$x(t)$
$\theta(t)$	angle rotated by mass w.r.t time	$\theta(t)$
$\alpha(t)$	angular acceleration of mass w.r.t time	$\alpha(t)$
k	spring constant	k
m	mass of the block	m
L	length of the mass	L
ω_o	initial angular velocity of mass	ω_o
$v(0)$	initial velocity of mass	$v(0)$

Table 1.5: input values

Assuming the bar is at mean position and has non-zero intitial velocity, we can



write it's laplace transform as:

$$s^2 mX(s) - mv(0) + 2kX(s) = 0 \tag{1.45}$$

$$\implies X(s) = \frac{v(0)}{s^2 + \frac{2k}{m}} \tag{1.46}$$

On taking inverse laplace transform we get,

$$x(t) = v(0)\sqrt{\frac{m}{2k}}\sin\sqrt{\frac{2k}{m}}t \tag{1.47}$$

$$\therefore \omega_1 = \sqrt{\frac{2k}{m}} \tag{1.48}$$

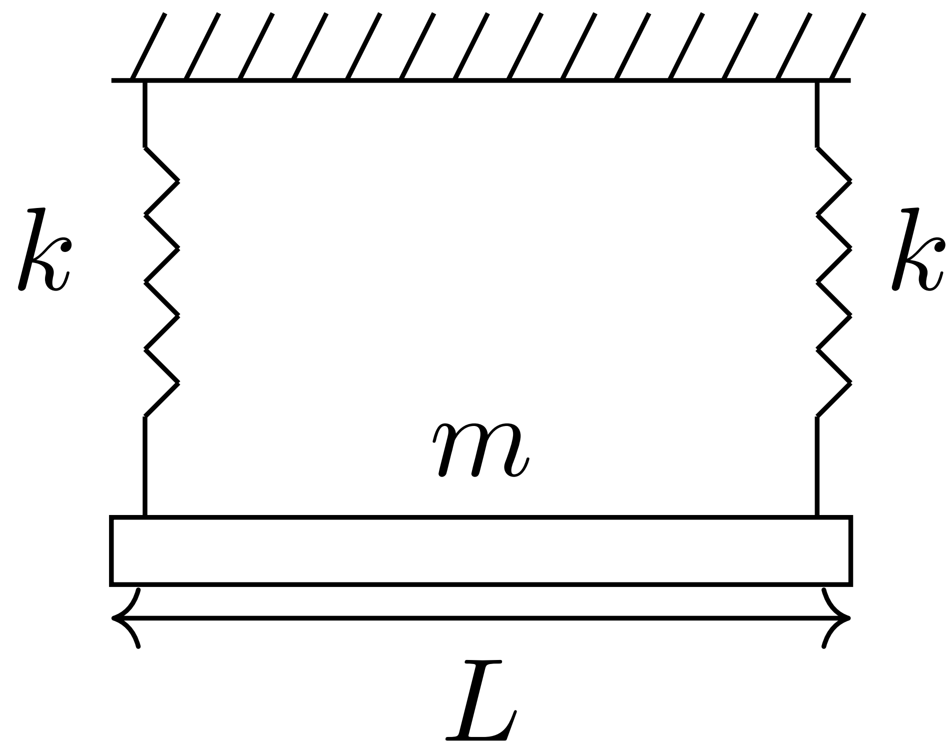


Figure 1.3: Figure given in question

ii: For torsional strain from Fig. 1.5,

$$I\alpha\left(t\right) = -\frac{kL^2\theta\left(t\right)}{2} \tag{1.49}$$

Assuming it is at mean position and having non-zero angular velocity we can write it's laplace transform as:

$$s^2 I\Theta\left(s\right) - I\omega_o + \frac{kL^2\Theta\left(s\right)}{2} = 0 \tag{1.50}$$

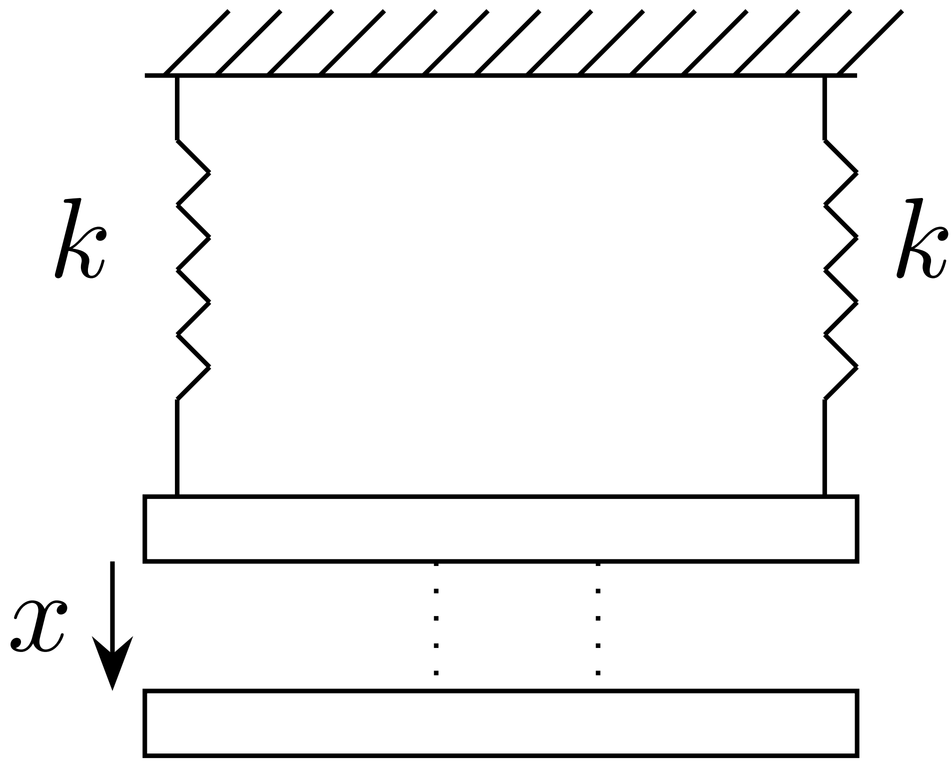


Figure 1.4: Figure for Vertical strain

substituting values from Table 1.5:

$$\Theta\left(s\right)=\frac{\omega_o}{s^2+\frac{6k}{m}} \tag{1.51}$$

On taking inverse laplace transform we get,

$$\theta\left(t\right)=\omega_o\sqrt{\frac{m}{6k}}\sin\sqrt{\frac{6k}{m}}t \tag{1.52}$$

$$\therefore\omega_2=\sqrt{\frac{6k}{m}} \tag{1.53}$$

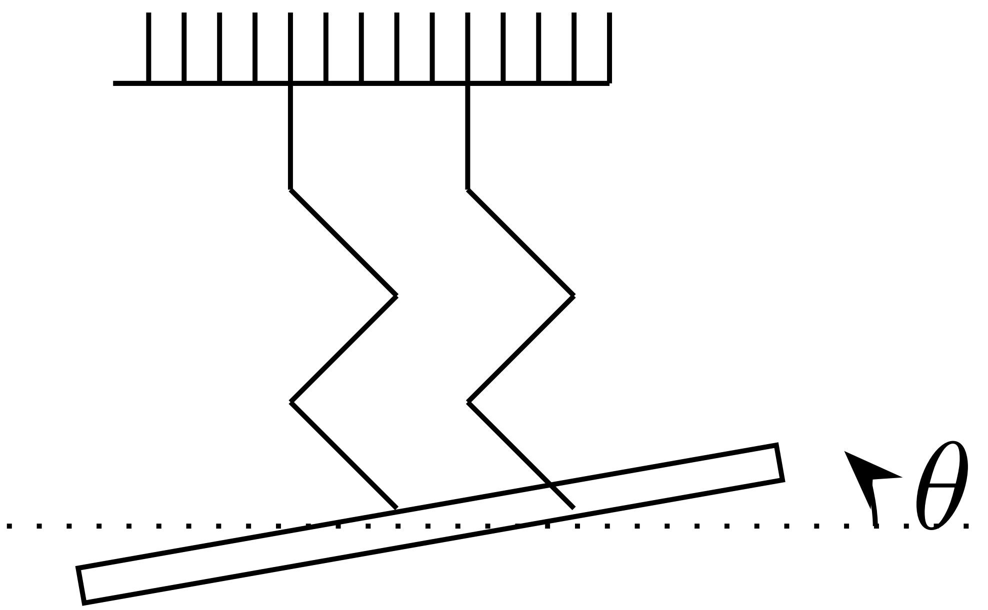


Figure 1.5: Figure for Torsional strain

1.6 A car is moving collinearly with a laser beam emitted by a transceiver. A laser pulse emitted at $t = 0\,\mathrm{s}$ is received back by the transceiver 100 ns (nanoseconds) later after reflection from the car.A second pulse emitted at $t = 0.1\,\mathrm{s}$ is received back 90 ns later. Given the speed of light is $3 \times 10^8\,\mathrm{m/s}$, the average speed of the car in this interval is __.

(A) 54 kmph, moving towards the transceiver

(B) 108 kmph,moving towards the transceiver

(C) 54 kmph, moving away from the transceiver

(D) 108 kmph,moving away from the transceiver

GATE2022-IN-39 **Solution:**

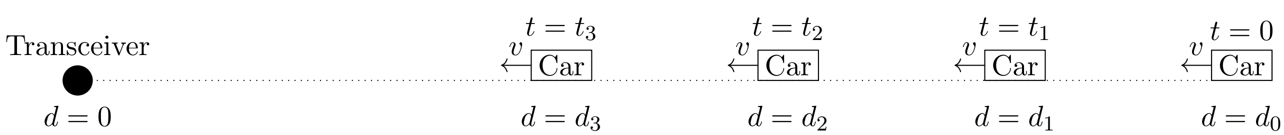


Figure 1.6: block diagram of the system

Variable	**Description**	**Value**
v_c	velocity of laser	$3 \times 10^8\,\mathrm{m/s}$
v	average speed of car	none
t_1	time at which first pulse hits car	none
t_2	time at which second pulse is emmited	0.1s
t_3	time at which second pulse hits car	none
d_0	Distance between tranceiver and car at $t = 0$	none
d_i for $i = 1, 2, 3$	Distance between tranceiver and car at $t = t_i$	none

Table 1.6: Input parameters

From Fig. 1.6

$$t_1 = \frac{d_1}{v_c} = \frac{d_0 - d_1}{v} \tag{1.55}$$

$$\implies d_1 = \frac{d_0}{\left(1 + \frac{v}{v_c}\right)} \tag{1.56}$$

Distance travelled by first pulse is given by

$$2d_1 = v_c \times 100\,\mathrm{ns} \tag{1.57}$$

$$2\frac{d_0}{\left(1 + \frac{v}{v_c}\right)} = v_c \times 100\,\mathrm{ns} \tag{1.58}$$

similarly time taken by car to move from d_2 to d_3 is given by

$$t_3 - t_2 = \frac{d_3}{v_c} = \frac{d_2 - d_3}{v} \tag{1.59}$$

$$\implies d_3 = \frac{d_2}{\left(1 + \frac{v}{v_c}\right)} \tag{1.60}$$

from Fig. 1.6

$$d_2 = d_0 - 0.1v \tag{1.61}$$

$$\therefore d_3 = \frac{d_0 - 0.1v}{\left(1 + \frac{v}{v_c}\right)} \tag{1.62}$$

Distance travelled by second pulse is given by

$$2d_3 = v_c \times 90\,\mathrm{ns} \tag{1.63}$$

$$2\frac{d_0 - 0.1v}{\left(1 + \frac{v}{v_c}\right)} = v_c \times 90\,\mathrm{ns} \tag{1.64}$$

solving (1.58) and (1.64) we get

$$v = 15\,\text{m/s} \tag{1.65}$$

$$v = 54\,\text{kmph} \tag{1.66}$$

since v is same but time taken by pulses to reach tranceiver is decreasing, v is towards tranceiver.

$\therefore$ Average speed of car is $54\,\text{kmph}$, moving towards the transceiver (option A)

1.7 A rigid uniform annular disc is pivoted on a knife edge A in a uniform gravitational field as shown, such that it can execute small amplitude simple harmonic motion in the plane of the figure without slip at the pivot point. The inner radius r and outer radius R are such that $r^2 = \frac{R^2}{2}$, and the acceleration due to gravity is g. If the time period of small amplitude simple harmonic motion is given by $T = \beta\pi\sqrt{\frac{R}{g}}$, where π is the ratio of circumference to diameter of a circle, then $\beta =$ (Round off to 2 decimal places) GATE 2022 ME-32

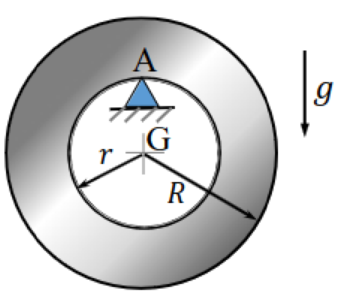


Figure 1.7: Question Diagram

Solution:

Parameters in expression		
Symbol	Description	Value
I	Moment of Inertia about the pivot point	$\frac{5}{4}MR^2$
$\theta(t)$	Angular displacement from vertical	?
$\theta(0)$	Value of $\theta(t)$ at $t = 0$	0
$\Theta(s)$	Laplace Transform of $\theta(t)$	?
r	Distance of center of gravity from pivot point	$\frac{R}{\sqrt{2}}$

Table 1.7: Parameters

Moment of inertia of disc about pivot point is calculated as

$$I = \frac{1}{2}\left(R^2 + \frac{R^2}{2}\right) + \frac{MR^2}{2} \tag{1.67}$$

$$= \frac{5}{4}MR^2 \tag{1.68}$$

Using D' Alambert's principle,

$$I\frac{d^2\theta(t)}{dt^2} + mgr\sin(\theta(t)) = 0 \tag{1.69}$$

$$\implies I\frac{d^2\theta(t)}{dt^2} + mgr\theta(t) = 0, \text{for } \theta \ll 1, \theta > 0 \tag{1.70}$$

Using (1.68) and (1.70), we get

$$\frac{5MR^2}{4}\frac{d^2\theta\left(t\right)}{dt^2}+Mg\frac{R}{\sqrt{2}}\theta\left(t\right)=0 \tag{1.71}$$

$$\frac{d^2\theta\left(t\right)}{dt^2}+\frac{2\sqrt{2}g}{5R}\theta\left(t\right)=0 \tag{1.72}$$

Taking Laplace Transform on both sides,we get

$$s^2\Theta\left(s\right)-s\theta\left(0\right)-\theta'\left(0\right)+\frac{2\sqrt{2}g}{5R}\Theta\left(s\right)=0 \tag{1.73}$$

$$\Theta\left(s\right)\left(s^2+\frac{2\sqrt{2}g}{5R}\right)=\theta'\left(0\right) \tag{1.74}$$

$$\Theta\left(s\right)=\frac{\theta'\left(0\right)}{\left(s^2+\frac{2\sqrt{2}g}{5R}\right)} \tag{1.75}$$

$$u\left(t\right)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s} \tag{1.76}$$

$$e^{at}u\left(t\right)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1}{s-a} \tag{1.77}$$

$$\frac{\left(e^{jat}-e^{-jat}\right)}{2}u\left(t\right)\longleftrightarrow\frac{a}{s^2+a^2} \tag{1.78}$$

$$\sin\left(at\right)\longleftrightarrow\frac{a}{s^2+a^2} \tag{1.79}$$

From (1.75)

$$\Theta\left(s\right)=\frac{\theta'\left(0\right)\left(\sqrt{\frac{2\sqrt{2}g}{5R}}\right)}{\left(s^2+\frac{2\sqrt{2}g}{5R}\right)}\frac{1}{\left(\sqrt{\frac{2\sqrt{2}g}{5R}}\right)} \tag{1.80}$$

Taking inverse Laplace by putting $\frac{\theta'(0)}{\left(\sqrt{\frac{2\sqrt{2}g}{5R}}\right)} = k$ and (1.79),

$$\theta(t) = k \sin\left(\frac{2\sqrt{2}g}{5R} t\right) \tag{1.81}$$

$$T = \frac{2\pi}{\frac{2\sqrt{2}g}{5R}} \tag{1.82}$$

$$= \sqrt{\left(5\sqrt{2}\right)}\, \pi \sqrt{\frac{R}{g}} \tag{1.83}$$

Thus,

$$\beta = \sqrt{\left(5\sqrt{2}\right)} \tag{1.84}$$

$$\beta = 2.66 \tag{1.85}$$

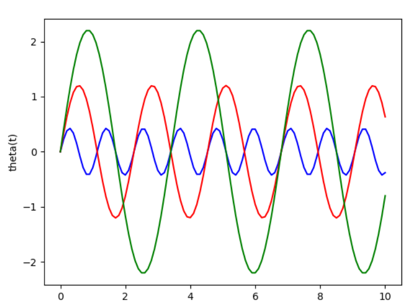


Figure 1.8: Plot of $\theta(t)$ for $(\theta'(0), \mathrm{R}) \in \{(1{,}1)\ , (2{,}2)\ , (3{,}3)\}$

1.2. 2021

1.1 A two degree of freedom spring-mass system undergoing free vibration with generalized coordinates x_1 and x_2 has natural frequencies $\omega_1 = 233.9$ rad/s and $\omega_2 = 324.5$ rad/s , respectively. The corresponding mode shapes $\phi_1 = \begin{bmatrix} 1 \\ -3.16 \end{bmatrix}$ and $\phi_2 = \begin{bmatrix} 1 \\ 3.16 \end{bmatrix}$. If the system is disturbed with certain deflections and zero initial velocities, then which of the following statement(s) is/are true?

(A) An initial deflection of $x_1(0) = 6.32$cm and $x_2(0) = -3.16$cm would make the system oscillate with only the second natural frequency.

(B) An initial deflection of $x_1(0) = 2$cm and $x_2(0) = -6.32$cm would make the system oscillate with only the first natural frequency.

(C) An initial deflection of $x_1(0) = 2$cm and $x_2(0) = -2$cm would make the system oscillate with linear combination of first and second natural frequency.

(D) An initial deflection of $x_1(0) = 1$cm and $x_2(0) = -6.32$cm would make the system oscillate with only the first natural frequency.

(GATE AE 2021 QUESTION 32)

Solution:

The F.B.D for above system is written as:

$$m_1\frac{d^2x_1}{dt^2} - k_c(x_2 - x_1) + k_1x_1 = 0 \tag{1.86}$$

$$m_2\frac{d^2x_2}{dt^2} + k_c(x_2 - x_1) + k_2x_2 = 0 \tag{1.87}$$

Parameter	Description	Value
m_1, m_2	mass of block attached to springs	m_1
k_1, k_c, k_2	spring constants of springs	k_1, k_c, k_2
$x_1(0)$	Initial vibration of first spring from mean position	?
$x_2(0)$	Initial vibration of second spring from mean position	?
$x_1(t)$	Vibration of first spring from the respective mean position	?
$x_2(t)$	Vibration of second spring from the respective mean position	?
A_{11}, A_{12}	Amplitudes of block 1 under natural conditions	?
A_{21}, A_{22}	Amplitudes of block 2 under natural conditions	?
ω_1	First natural frequency of the system	233.9 rad/s
ω_2	Second natural frequency of the system	324.5 rad/s
λ	Phase angle of wave motion exhibited by masses	$\frac{\pi}{2}$rad
$\phi_{\mathbf{1}}$	mode shape for first natural frequency	$\begin{bmatrix} 1 \\ -3.16 \end{bmatrix}$
$\phi_{\mathbf{2}}$	mode shape for second natural frequency	$\begin{bmatrix} 1 \\ 3.16 \end{bmatrix}$

Table 1.8: input values

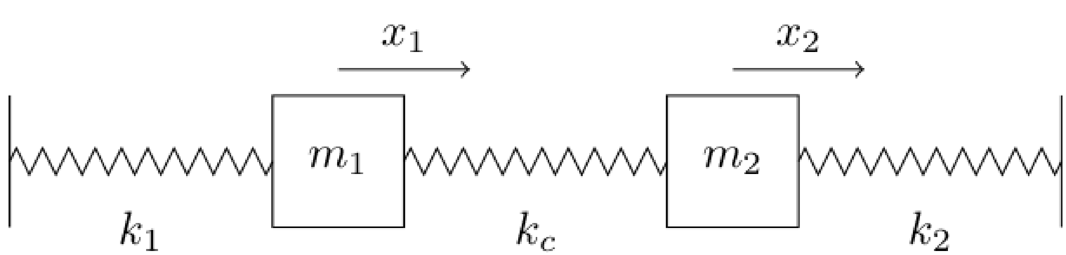


Figure 1.9: System with D.O.F =2

Which can be written in the form of matrices as:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \frac{d^2x_1}{dt^2} \\ \frac{d^2x_2}{dt^2} \end{pmatrix} = - \begin{bmatrix} k_1 + k_c & -k_c \\ -k_c & k_2 + k_c \end{bmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} \tag{1.88}$$

Taking laplace transform:

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} s^2 X_1(s) - s x_1(0) \\ s^2 X_2(s) - s x_2(0) \end{pmatrix} = - \begin{bmatrix} k_1 + k_c & -k_c \\ -k_c & k_2 + k_c \end{bmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} \tag{1.89}$$

$$\implies \begin{pmatrix} s^2X_1(s) - sx_1(0) \\ s^2X_2(s) - sx_2(0) \end{pmatrix} = -\begin{bmatrix} \frac{1}{m_1} & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix} \begin{bmatrix} k_1 + k_c & -k_c \\ -k_c & k_2 + k_c \end{bmatrix} \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} \tag{1.90}$$

$$\implies \begin{pmatrix} X_1(s) \\ X_2(s) \end{pmatrix} = \frac{\begin{bmatrix} s^2 + \frac{k_1+k_c}{m_1} & \frac{-k_c}{m_1} \\ \frac{-k_c}{m_2} & s^2 + \frac{k_2+k_c}{m_2} \end{bmatrix} \begin{pmatrix} sx_1(0) \\ sx_2(0) \end{pmatrix}}{\left(s^2 + \frac{k_1+k_c}{m_1}\right)\left(s^2 + \frac{k_c+k_2}{m_2}\right) - \frac{k_c^2}{m_1 m_2}} \tag{1.91}$$

Assuming the solutions to the equations are:

$$x_1(t) = A_1 \sin(\omega t + \lambda) \tag{1.92}$$

$$x_2(t) = A_2 \sin(\omega t + \lambda) \tag{1.93}$$

Substituting (1.92) and (1.93) in (1.88), we get:

$$\begin{bmatrix} k_1 + k_c - m_1\omega^2 & -k_c \\ -k_c & k_2 + k_c - m_2\omega^2 \end{bmatrix} \begin{Bmatrix} A_1 \\ A_2 \end{Bmatrix} \sin(\omega t + \lambda) = 0 \tag{1.94}$$

$$\implies \det\left(\begin{bmatrix} k_1 + k_c - m_1\omega^2 & -k_c \\ -k_c & k_2 + k_c - m_2\omega^2 \end{bmatrix}\right) = 0 \tag{1.95}$$

Let the roots of this equation be ω_1 and ω_2. Which are the two modes of the system. Substituting ω_1 in (1.88):

we obtain

$$\phi_{\mathbf{1}} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}_1 \tag{1.96}$$

and

Substituting ω_2 in (1.88):

we obtain

$$\phi_{\mathbf{2}} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix}_2 \tag{1.97}$$

These are called mode shapes.

Since the initial velocities of both the masses are zero:

$$\lambda = \frac{\pi}{2} \text{ rad} \tag{1.98}$$

So any oscillation can be represented as:

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \left\{\phi\right\}_1 \cos\left(\omega_1 t\right) + \left\{\phi\right\}_2 \cos\left(\omega_2 t\right) \tag{1.99}$$

From Table 1.8:

$$\implies x_1(t) = A_{11}\cos(\omega_1 t) + A_{12}\cos(\omega_1 t) \tag{1.100}$$

$$\implies x_2(t) = A_{21}\cos(\omega_2 t) + A_{22}\cos(\omega_2 t) \tag{1.101}$$

$$\therefore x_1(0) = A_{11} + A_{12} \tag{1.102}$$

$$\therefore x_2(0) = A_{21} + A_{22} \tag{1.103}$$

(a) For first natural frequency:

$$\frac{x_1(0)}{x_2(0)} = \frac{A_{11}}{A_{21}} \tag{1.104}$$

$$\implies \frac{x_1(0)}{x_2(0)} = \frac{1}{-3.16} \tag{1.105}$$

(b) For second natural frequency:

$$\frac{x_1(0)}{x_2(0)} = \frac{A_{12}}{A_{22}} \tag{1.106}$$

$$\implies \frac{x_1(0)}{x_2(0)} = \frac{1}{3.16} \tag{1.107}$$

So, option (B) is correct.

(c) For linear combination of first and second natural frequencies:

$$x_1(0) = A_{11} + A_{12}x_2(0) \qquad = A_{21} + A_{22} \tag{1.108}$$

i. If $\boldsymbol{\phi_1} \neq \boldsymbol{\phi_2}$ solution always exists

ii. If $\boldsymbol{\phi_1} = \boldsymbol{\phi_2}$ solution exists only if $x_1(0) = x_2(0)$

So, option (C) is also correct.

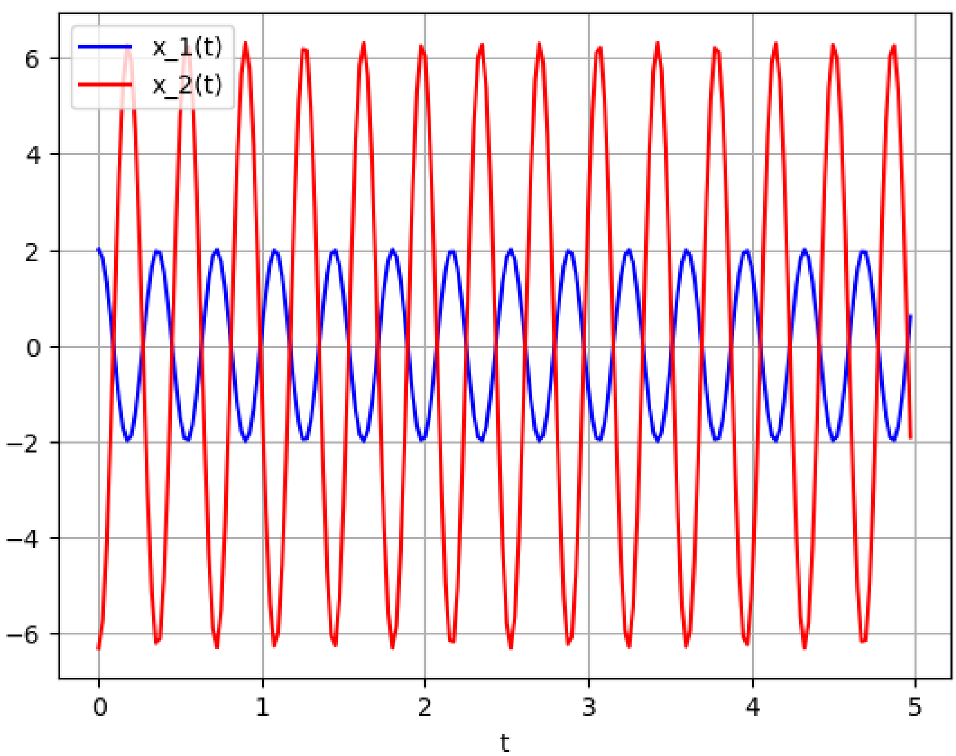


Figure 1.10: $x_1\left(t\right), x_2\left(t\right)$for option B

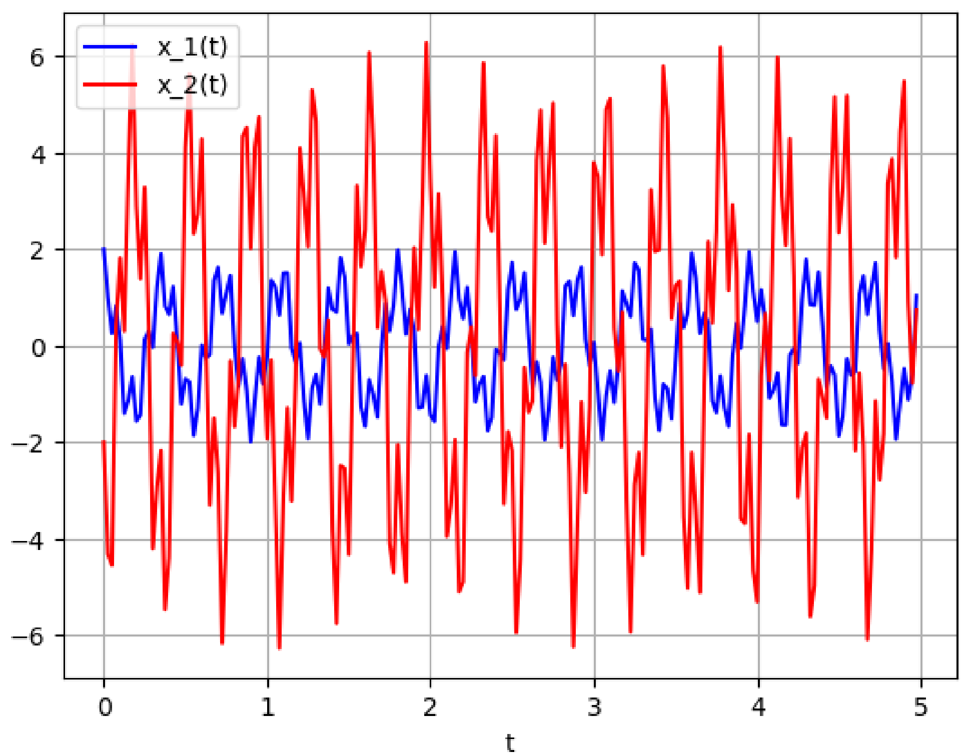


Figure 1.11: $x_1(t), x_2(t)$for option C

Chapter 2

Filters

2.1. 2022

2.1 The network shown below has a resonant frequency of 150 kHz and bandwidth of 600 Hz. The Q-factor of the network is ______ (rounded off to one decimal place).
(GATE 2022 EC)

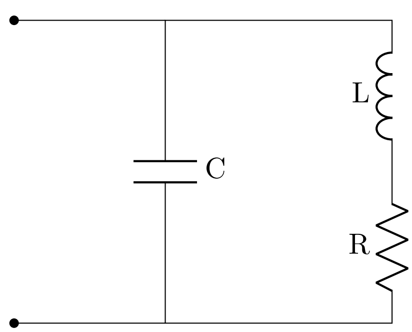


Figure 2.1: Circuit 1

Solution:

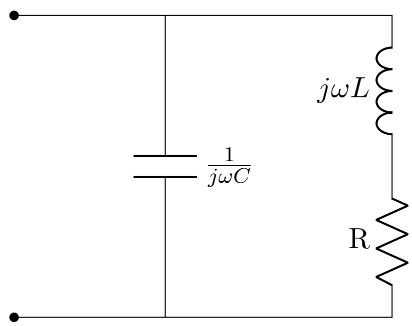


Figure 2.2: Circuit 2

Parameter	Description	Value
f_0	Resonant frequency	150 kHz
B	Bandwidth	600 Hz

Table 2.1: Parameters

At Resonance,

$$X_L = X_C \tag{2.1}$$

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{2.2}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{2.3}$$

$$2\pi f_0 = \frac{1}{\sqrt{LC}} \tag{2.4}$$

$$\implies f_0 = \frac{1}{2\pi\sqrt{LC}} \tag{2.5}$$

Parameter	Description	Formula
Q	Quality factor	$\frac{X_L}{R}$
B	Bandwidth	$\frac{R}{2\pi L}$
ω_0	Radial resonant frequency	$2\pi f_0$
X_L	Inductive reactance	ωL
X_C	Capacitive reactance	$\frac{1}{\omega C}$

Table 2.2: Formulae

Using Table 2.2,

$$\begin{aligned} Q &= \frac{X_L}{R} && (2.6) \\ &= \frac{\omega_0 L}{R} && (2.7) \\ &= \left(\frac{1}{\sqrt{LC}}\right)\frac{L}{R} && (2.8) \\ \implies Q &= \frac{1}{R}\sqrt{\frac{L}{C}} && (2.9) \end{aligned}$$

From eq (2.5) and Table 2.2

$$\begin{aligned} \frac{f_0}{B} &= \left(\frac{1}{2\pi\sqrt{LC}}\right)\frac{2\pi L}{R} && (2.10) \\ &= \left(\frac{1}{\sqrt{LC}}\right)\frac{L}{R} && (2.11) \\ \implies \frac{f_0}{B} &= \frac{1}{R}\sqrt{\frac{L}{C}} && (2.12) \end{aligned}$$

From Table 2.1, eq (2.9) and eq (2.12),

$$Q = \frac{f_0}{B} \tag{2.13}$$
$$= \frac{150 \text{ x } 10^3}{600} \tag{2.14}$$
$$= 250 \tag{2.15}$$

$\therefore$ Q-factor is 250

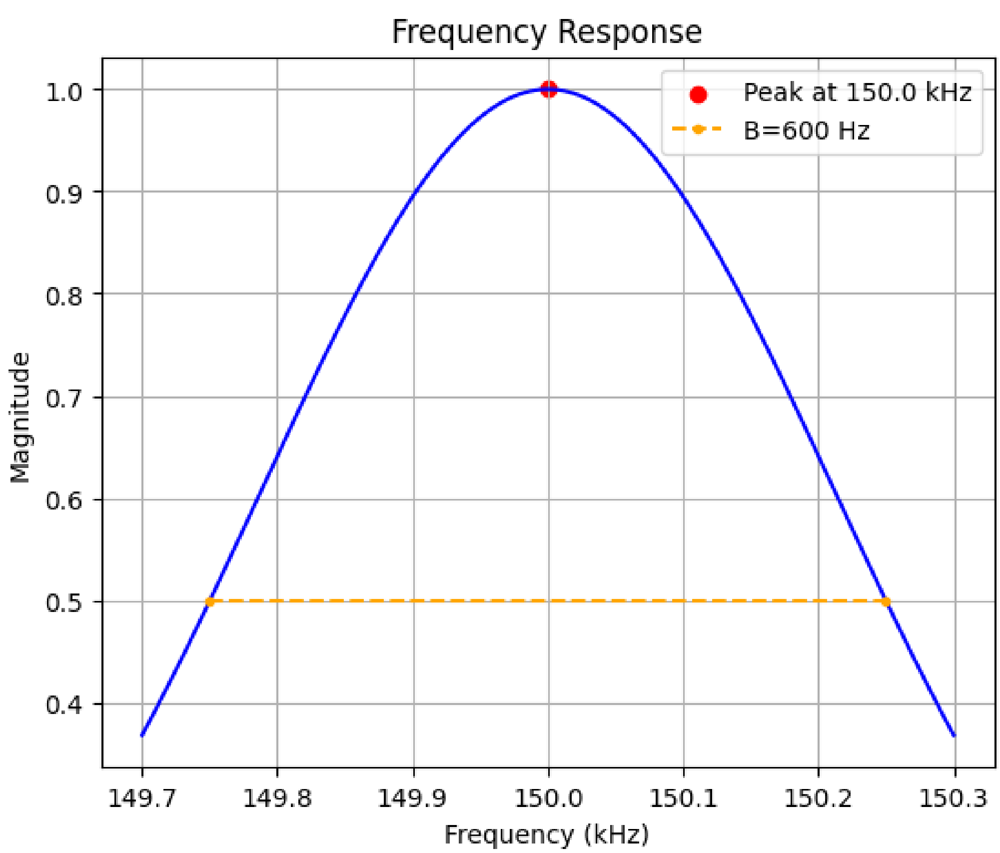
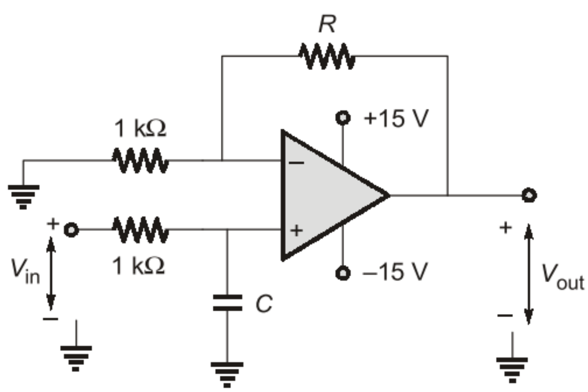
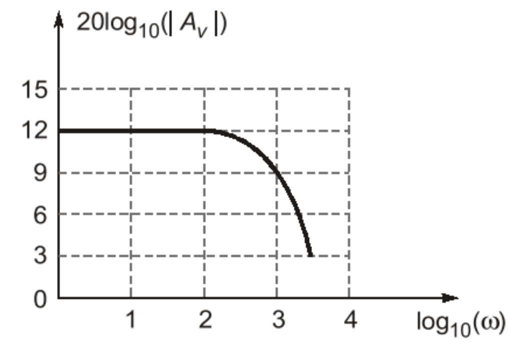


Figure 2.3: Plot of Q-factor

2.2 A circuit with an ideal OPAMP is shown. The Bode plot for the magnitude (in dB) of the gain transfer function $(A(j\omega)) = \dfrac{V_{out}(j\omega)}{V_{in}(j\omega)}$ of the circuit is also provided (here, ω is the angular frequency in rad/s). The values of R and C are





(A) $R = 3k\Omega$, $C = 1\mu F$

(B) $R = 1k\Omega$, $C = 3\mu F$

(C) $R = 4k\Omega$, $C = 1\mu F$

(D) $R = 3k\Omega$, $C = 2\mu F$

(GATE 2022 EC)

Solution:

Parameter	Description	Value
R	Resistance	?
C	Capacitance	?
R_1	Resistance	1000
ω_{dB}	Cut-off frequency	1000
A_V	Gain Transfer	$\frac{V_{out}}{V_{in}}$

Table 2.3: Given Parameters

On applying KVL,

$$sR_1 i_1(s) + \frac{i_1(s)}{C} = V_{IN}(s)\, s \tag{2.16}$$

$$\frac{i_1(s)}{C} - sRi_2(s) = sV_0(s) \tag{2.17}$$

From (2.16) and (2.17),

$$\frac{sV_{IN}(s)}{sR_1C+1} - sRi_2(s) = sV_0(s) \tag{2.18}$$

$$-\frac{i_1(s)}{C} = sR_1 i_2(s) \tag{2.19}$$

From (2.16), (2.17) and (2.19) ,

$$V_{OUT}(s) = \frac{1+10^{-3}R}{1+sC10^3} V_{IN}(s) \tag{2.20}$$

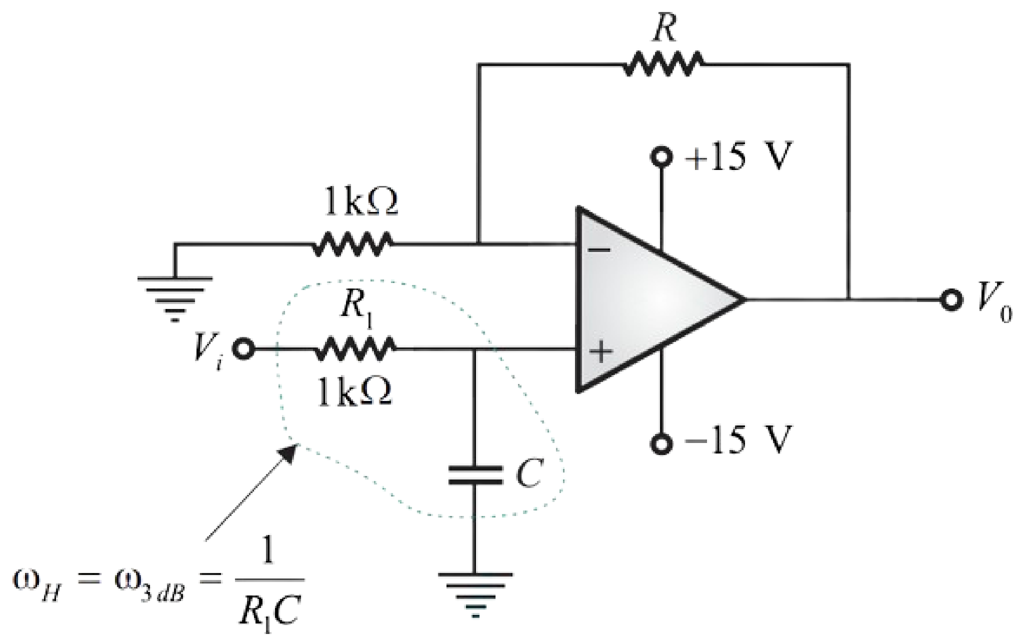


Figure 2.4: Active Low Pass Filter

The 3-dB frequency from bode magnitude plot,

$$\implies \omega_{3dB} = 1000\, rad/sec \tag{2.21}$$

$$\omega_{3dB} = \frac{1}{R_1C} \tag{2.22}$$

$$\implies C = 1\mu F \tag{2.23}$$

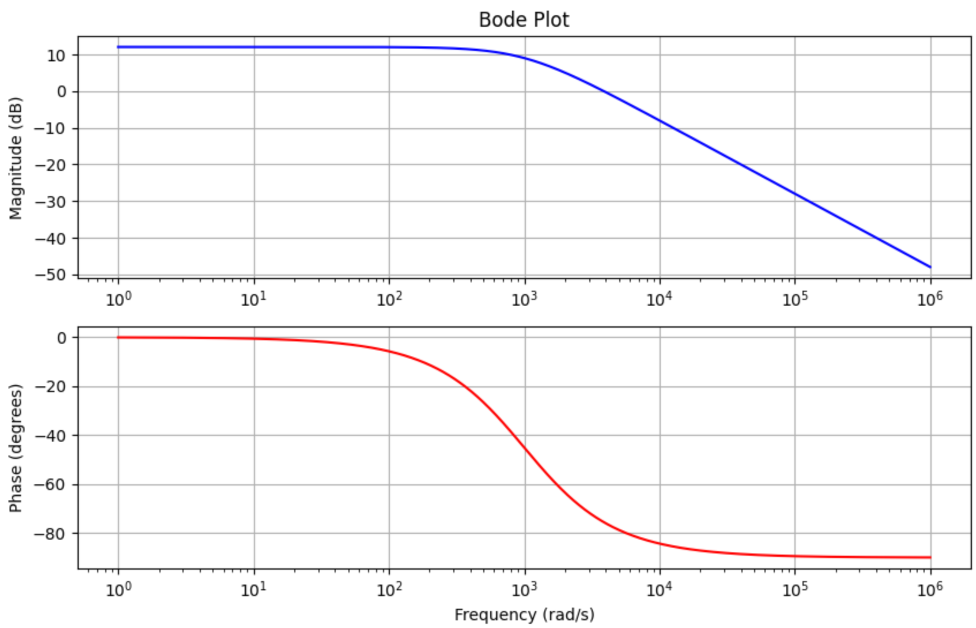


Figure 2.5: bode plot

$$\implies A(s) = \frac{V_{OUT}(s)}{V_{IN}(s)} \tag{2.24}$$

$$= \frac{1 + 10^{-3}R}{1 + sC10^3} \tag{2.25}$$

$$|A(s)| = \frac{1 + 10^{-3}R}{\sqrt{1 + \omega^2 10^{-6}}} \tag{2.26}$$

$$\tag{2.27}$$

A_V at low frequency,

$$|A_V| = 1 + 10^{-3}R \tag{2.28}$$

$$1 + 10^{-3}R = 10^{\frac{3}{5}} \tag{2.29}$$

$$R = 3k\Omega \tag{2.30}$$

Hence, The correct option is (A).

2.3 In the circuit shown, the load is driven by a sinusoidal A.C. voltage source $V_1 = 100\angle 0° V$ at $50Hz$. Given $R_1 = 20\Omega$, $C_1 = \left(\frac{1000}{\pi}\right)\mu F$, $L_1 = \left(\frac{20}{\pi}\right) mH$ and $R_2 = 4\Omega$, the power factor is ___ (round off to one decimal place)

(GATE 2022 IN Q52) **Solution:**

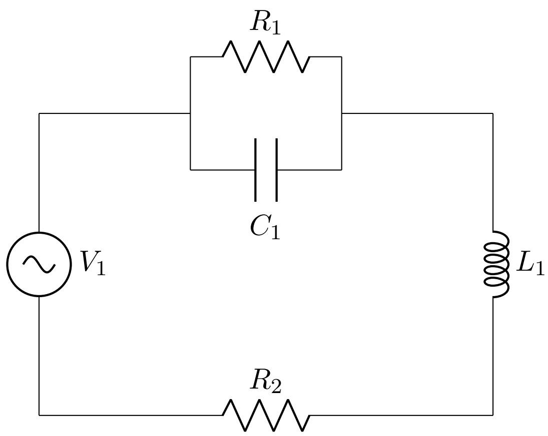


Figure 2.6:

Symbol	**Value**	**Description**
V_1	$100\angle 0° V$	Input Voltage
f	$50Hz$	Frequency
ω	$2\pi f$	Angular Frequency
R_1	20Ω	Resistance
R_2	4Ω	Resistance
C_1	$\left(\frac{1000}{\pi}\right)\mu F$	Capacitance
L_1	$\left(\frac{20}{\pi}\right) mH$	Inductance
Z_{eff}		Impedance
$\cos(\phi)$	$\frac{\text{Re}(Z_{\text{eff}})}{\lvert Z_{\text{eff}}\rvert}$	Power Factor

Table 2.4: Given Parameters

$$Z_{\text{eff}} = R_2 + j\omega L_1 + \frac{\frac{R_1}{j\omega C_1}}{R_1 + \frac{1}{j\omega C_1}} \tag{2.31}$$

$$= 4 + 2j + \frac{-200j}{20 - 10j} \tag{2.32}$$

$$= 8 - 6j \tag{2.33}$$

$\therefore$ Power Factor:

$$\cos(\phi) = \frac{\text{Re}(Z_{\text{eff}})}{|Z_{\text{eff}}|} \tag{2.34}$$

$$= \frac{8}{\sqrt{8^2 + 6^2}} \tag{2.35}$$

$$= 0.8 \tag{2.36}$$

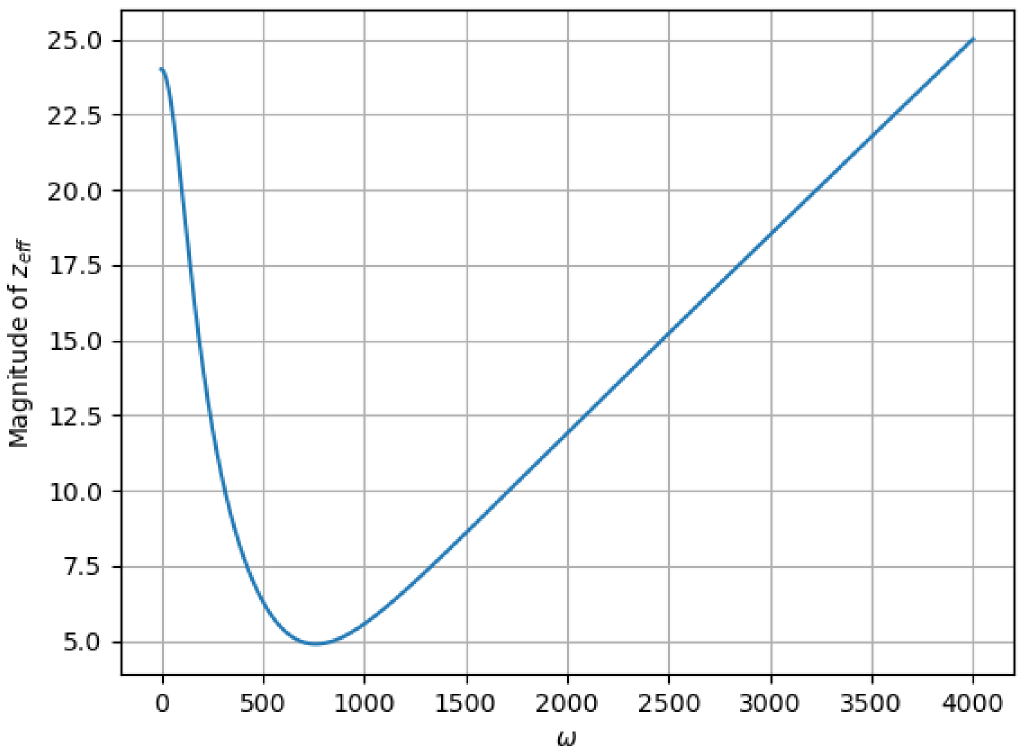


Figure 2.7: Plot of Z_{eff} vs ω

2.4 For the circuit shown, the locus of the impedance $Z(j\omega)$ is plotted as ω increases from zero to infinity. The values of R_1 and R_2 are:

(A) $R_1 = 2\ \text{k}\Omega, R_2 = 3\ \text{k}\Omega$

(B) $R_1 = 5\ \text{k}\Omega, R_2 = 2\ \text{k}\Omega$

(C) $R_1 = 5\ \text{k}\Omega, R_2 = 2.5\ \text{k}\Omega$

(D) $R_1 = 2\ \text{k}\Omega, R_2 = 5\ \text{k}\Omega$

(GATE ECE 2022 QUESTION 38)

Solution:

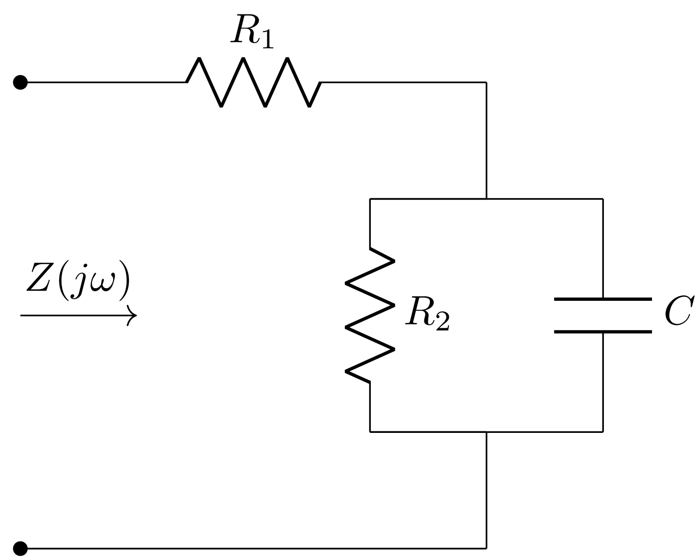


Figure 2.8: Figure of circuit

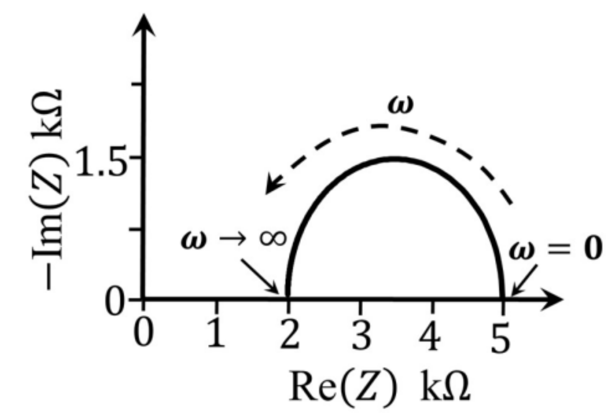


Figure 2.9:

In ω domain (i.e. after Laplace transform) Fig. 2.8 can be represented as Fig. 2.10 So, the impedance for the circuit in ω domain is:

$$Z\left(j\omega\right) = R_1 + \frac{1}{\frac{1}{R_2} + j\omega C} \tag{2.37}$$

Parameter	Description	Value
$Z(j\omega)$	Impedance of circuit	?
R_1	Resistor 1	?
R_1	Resistor 2	?
C	Capacitor	?
ω	angular frequency of input voltage	ω

Table 2.5: input values

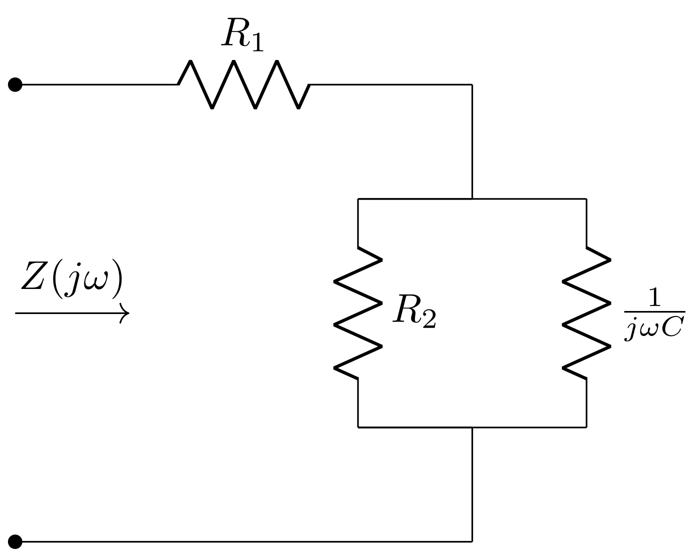


Figure 2.10:

From Fig. 2.9, $Z(j\omega) = 2$ as $\omega \to \infty$ and $Z(j\omega) = 5$ as $\omega \to 0$

$$2 = R_1 + \lim_{\omega\to\infty} \frac{1}{\frac{1}{R_2} + j\omega C} \tag{2.38}$$

$$\implies 2 = R_1 + \lim_{\omega\to\infty} \frac{\frac{1}{R_2} - j\omega C}{\left(\frac{1}{R_2}\right)^2 + (\omega C)^2} \tag{2.39}$$

$$\implies 2 = R_1 + \lim_{\omega\to\infty} \frac{\frac{1}{R_2\omega^2} - j\frac{C}{\omega}}{\left(\frac{1}{R_2\omega}\right)^2 + C^2} \tag{2.40}$$

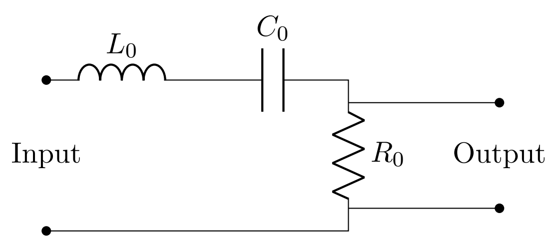
$$\therefore 2\Omega = R_1 \tag{2.41}$$

$$5 = R_1 + \frac{1}{\frac{1}{R_2} + j(0)} \tag{2.42}$$

$$\implies 5 = R_1 + R_2 \tag{2.43}$$

Hence, option (A) is correct.

2.5 In the bandpass filter circuit shown, $R_0 = 50\Omega$, $L_0 = 1mH$, $C_0 = 10nF$. The q factor of the filter is



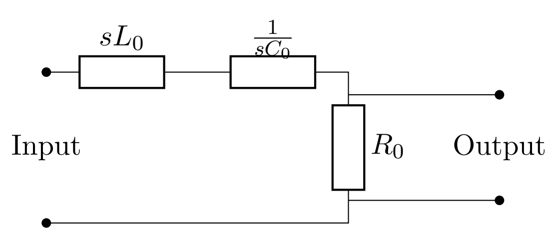
(GATE 2022 IN Q33)

Solution:

Variable	Description	Value
R_0	Resistance	50Ω
L_0	Inductance	$1mH$
C_0	Capacitance	$10nF$
ω_0	Resonant Angular Frequency	$\frac{1}{\sqrt{L_0C_0}}$

Table 1: Variables and their description

The corresponding Laplace domain circuit is



Input $X(s)$ can be written as

$$X(s) = I(s)\left(sL_0 + \frac{1}{sC_0} + R_0\right) \tag{2.45}$$

Output $Y(s)$ can be written as

$$Y(s) = I(s) R_0 \tag{2.46}$$

Transfer function $H(s)$ can be written as

$$H(s) = \frac{Y(s)}{X(s)} \tag{2.47}$$
$$= \frac{sC_0R_0}{s^2C_0L_0 + C_0R_0s + 1} \tag{2.48}$$

substituting $s = j\omega$

$$H(j, \omega) = \frac{j\omega C_0R_0}{-\omega^2C_0L_0 + jC_0R_0\omega + 1} \tag{2.49}$$
$$\implies |H(j, \omega)| = \frac{\omega C_0R_0}{\sqrt{(1-\omega^2C_0L_0)^2 + (C_0R_0\omega)^2}} \tag{2.50}$$

Differentiating w.r.t ω and equating to 0, we get

$$\frac{d|H(j,\omega)|}{d\omega} = \frac{C_0R_0}{\sqrt{(1-\omega^2C_0L_0)^2 + (C_0R_0\omega)^2}} + \frac{\omega C_0R_0}{2\left((1-\omega^2C_0L_0)^2 + (C_0R_0\omega)^2\right)^{\frac{3}{2}}} \left(2\omega (C_0R_0)^2 - 2\left(1-\omega^2C_0L_0\right)2\omega\right) = 0 \tag{2.51}$$
$$\implies \omega_0 = \frac{1}{\sqrt{L_0C_0}} \tag{2.52}$$

from Table 1,

$$\omega_0 = 316227.76 \tag{2.53}$$

$Q - factor$ defined with reference to inductor

$$Q = \left|\frac{V_L}{V_R}\right|_{\omega_0} \tag{2.54}$$
$$= \frac{L_0\omega_0}{R_0} \tag{2.55}$$
$$= \frac{1}{R_0}\sqrt{\frac{L_0}{C_0}} \quad \text{(from (2.52))} \tag{2.56}$$

$Q - factor$ defined with reference to capacitor

$$Q = \left|\frac{V_C}{V_R}\right|_{\omega_0} \tag{2.57}$$
$$= \frac{1}{C_0\omega_0 R_0} \tag{2.58}$$
$$= \frac{1}{R_0}\sqrt{\frac{L_0}{C_0}} \quad \text{(from (2.52))} \tag{2.59}$$

Substituting the values from Table 1, we get

$$Q = 200 \tag{2.60}$$

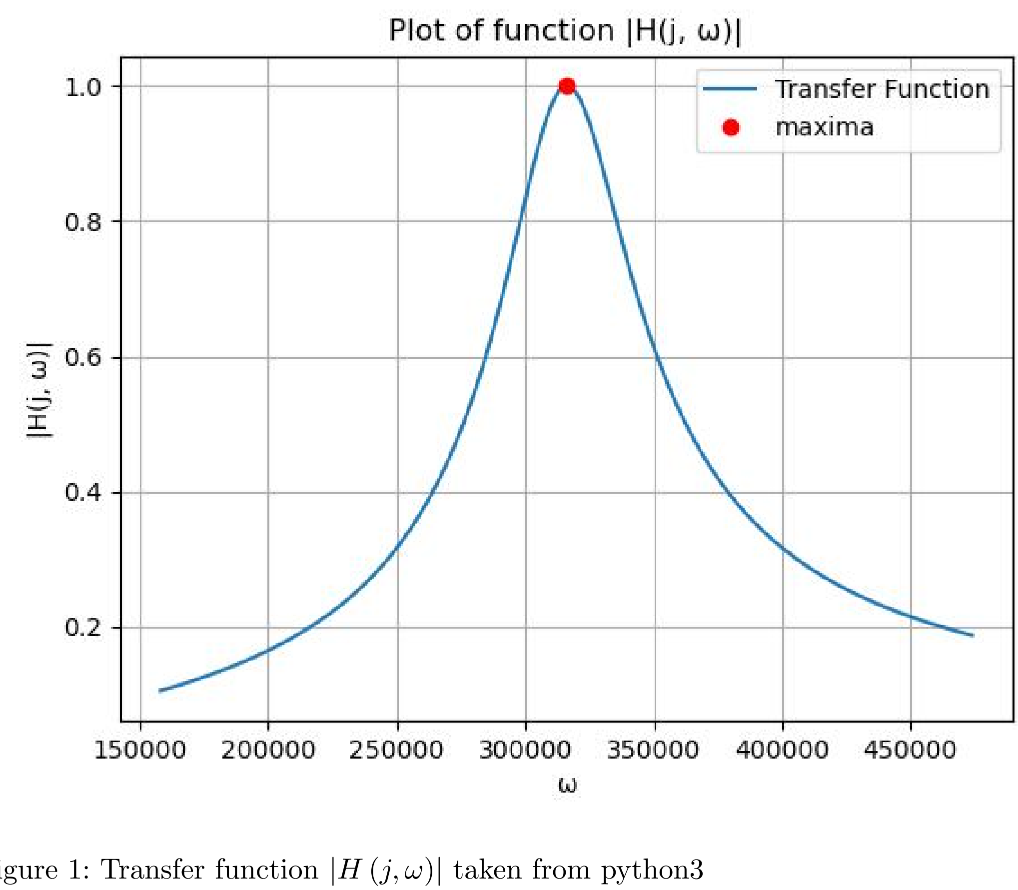


Figure 1: Transfer function $|H\,(j,\omega)|$ taken from python3

2.6 In the circuit shown below, the switch S is closed at $t = 0$. The magnitude of the steady state voltage, in volts, across the 6Ω resistor is ______.(*round off to two decimal places*)

(GATE 2022 EE Q31)

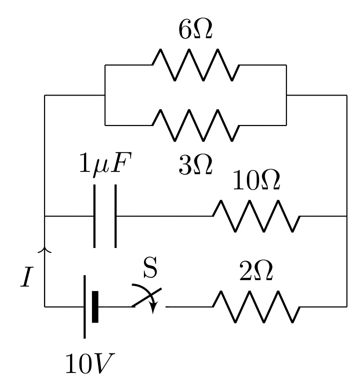


Figure 2.12:

Solution: Consider a sinusoidal input source of angular frequency ω.

Symbol	**Value**	**Description**
ω	0 for D.C.	Angular Frequency
C	$1\mu F$	Capacitance
$V_{in}(t)$	$10\cos(\omega t)$	Input Voltage
$V_{out}(t)$		Output Voltage across 6Ω
$V_{out}(j\omega)$	$H(j\omega)V_{in}(j\omega)$	Output in Frequency Domain
$H(j\omega)$		Transfer Function
$I(j\omega)$		Total Current
Z_{eff}		Overall Impedance

Table 2.7: Given Parameters

Using KCL and KVL, we can calculate:

$$Z_{\text{eff}} = \frac{2\left(10 + \frac{1}{j\omega C}\right)}{12 + \frac{1}{j\omega C}} + 2 \tag{2.61}$$

$$\implies I(j\omega) = \frac{V_{in}}{\left(\frac{2\left(10+\frac{1}{j\omega C}\right)}{12+\frac{1}{j\omega C}} + 2\right)} \tag{2.62}$$

$$\implies V_{out}(j\omega) = 2\left[\left(\frac{10 + \frac{1}{j\omega C}}{12 + \frac{1}{j\omega C}}\right) I(j\omega)\right] \tag{2.63}$$

$$= 2\left[\left(\frac{10 + \frac{1}{j\omega C}}{12 + \frac{1}{j\omega C}}\right) \frac{V_{in}(j\omega)}{\left(\frac{2\left(10+\frac{1}{j\omega C}\right)}{12+\frac{1}{j\omega C}} + 2\right)}\right] \tag{2.64}$$

$$\implies H(j\omega) = \frac{1 + 10j\omega C}{2(1 + 11j\omega C)} \tag{2.65}$$

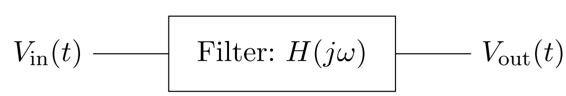


Figure 2.13: Filter Equivalent of Circuit

$$H(j\omega) = \left(\frac{\sqrt{1 + 100\omega^2C^2}}{2\sqrt{1 + 121\omega^2C^2}}\right) e^{j(\tan^{-1}(10\omega C) - \tan^{-1}(11\omega C))} \tag{2.66}$$

$$= \left(\frac{\sqrt{1 + 100\omega^2C^2}}{2\sqrt{1 + 121\omega^2C^2}}\right) e^{j\tan^{-1}\left(\frac{-\omega C}{1+110\omega^2C^2}\right)} \tag{2.67}$$

$$\therefore V_{out}(t) = 10\,|H(j\omega)|\cos(\omega t + \angle H(j\omega)) \tag{2.68}$$

$$= \frac{5\sqrt{1 + 100\omega^2C^2}}{\sqrt{1 + 121\omega^2C^2}} \cos\left(\omega t - \tan^{-1}\left(\frac{\omega C}{1 + 110\omega^2C^2}\right)\right) \tag{2.69}$$

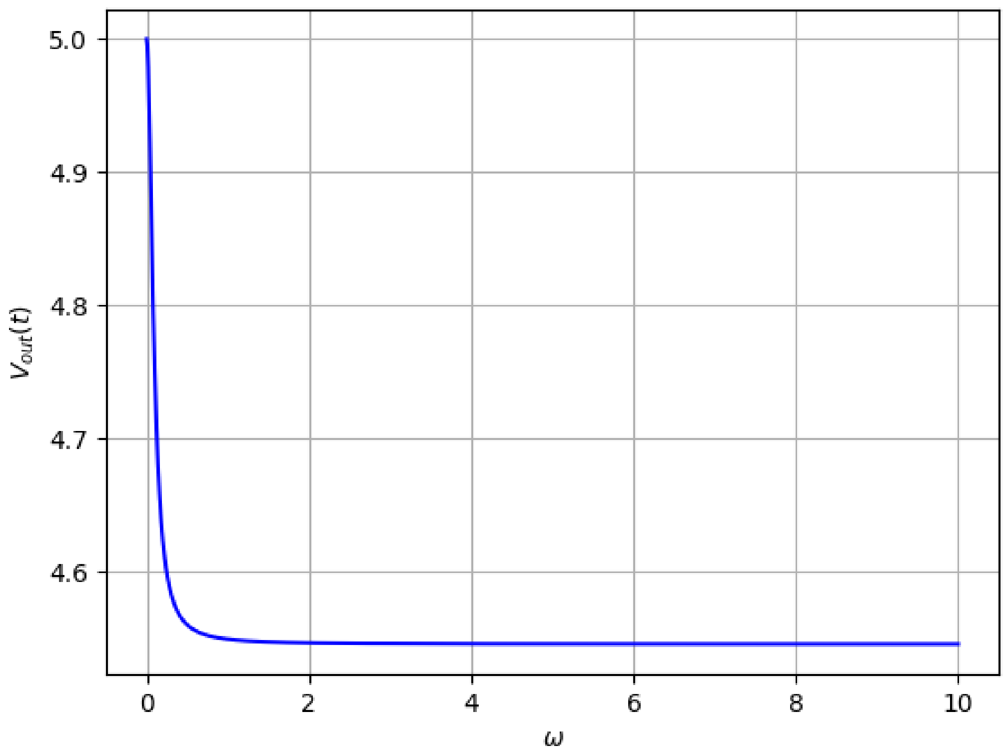


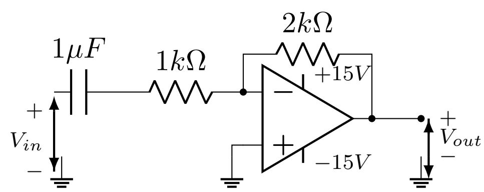
Figure 2.14: Plot of $V_{out}(t)$ at $t = 0$ w.r.t ω

As $\omega \to 0$, $V_{in}(t)$ approaches being a D.C. input source ($10V$).

$\therefore$ substituting $\omega = 0$, we get:

$$V_{out}(t) = 5V \tag{2.70}$$

2.7 An ideal OPAMP circuit with a sinusoidal input is shown in the figure. The 3dB frequency is the frequency at which the magnitude of the voltage gain decreases by 3 dB from the maximum value. Which of the options is/are correct?



(A) The circuit is a low pass filter.

(B) The circuit is a high pass filter.

(C) The 3 dB frequency is 1000rad/s.

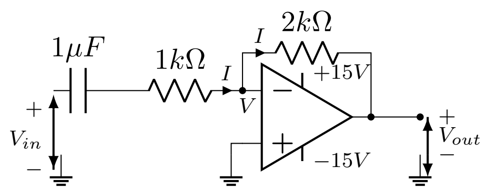
(D) The 3 dB frequency is $\frac{1000}{3}$rad/s.

(GATE EC 2022)

Solution:

Parameter	Description	Value
V_{in}	Input Voltage	–
V_{out}	Output Voltage	–
C	Capacitor	$1\mu F$
R_1	Resistance	$1k\Omega$
R_2	Feedback Resistance	$2k\Omega$
V	Voltage at Negative terminal	–
V^+	Voltage at positive terminal	0

Table 2.8: Input Parameters



$$\frac{V_{in} - V}{\frac{1}{sC} + R_1} = \frac{V - V_{out}}{R_2} \tag{2.71}$$

As Op-Amp is ideal

$$V = V^{+} = 0V \tag{2.72}$$

$$\left|\frac{V_{out}}{V_{in}}\right| = \frac{sCR_2}{1 + sCR_1} \tag{2.73}$$

$$H(s) = \frac{sCR_2}{1 + sCR_1} \tag{2.74}$$

Keeping $s = j\omega$

For determining nature of Filter

Put $j\omega = 0$

$$H(j\omega) = 0 \tag{2.75}$$

Put $j\omega \to \infty$

$$H(j\omega) = \frac{R_2}{R_1} = 2 \quad \text{(Finite)} \tag{2.76}$$

$\therefore$ It is high pass filter.

On simplifying (2.74) further

$$H(j\omega) = \frac{R_2}{R_1}\left(\frac{j\omega}{j\omega + \frac{1}{CR_1}}\right) \tag{2.77}$$

$$|H(j\omega)|_{max} = \frac{R_2}{R_1} \tag{2.78}$$

$$|H(j\omega)|_{\omega=\omega_c} = \frac{R_2}{R_1}\left|\frac{j\omega_c}{j\omega_c + \frac{1}{CR_1}}\right| \tag{2.79}$$

Given:

$$20\log(|H(j\omega)|_{max}) - 20\log(|H(j\omega)|_{\omega=\omega_c}) = 3dB \tag{2.80}$$

$$\frac{|H(j\omega)|_{max}}{|H(j\omega)|_{\omega=\omega_c}} = \sqrt{2} \tag{2.81}$$

From (2.78) and (2.79)

$$\frac{R_2}{R_1}\left|\frac{j\omega_c}{j\omega_c + \frac{1}{CR_1}}\right| = \frac{1}{\sqrt{2}}\frac{R_2}{R_1} \tag{2.82}$$

$$\left|\frac{j\omega_c}{j\omega_c + \frac{1}{CR_1}}\right| = \frac{1}{\sqrt{2}} \tag{2.83}$$

$$\implies \omega_c = \frac{1}{CR_1} \tag{2.84}$$

From Table 2.8

$$\omega_c = 1000\text{rad/s} \tag{2.85}$$

Where ω_c is 3 dB frequency.

Finally, Correct options are (B) and (C).

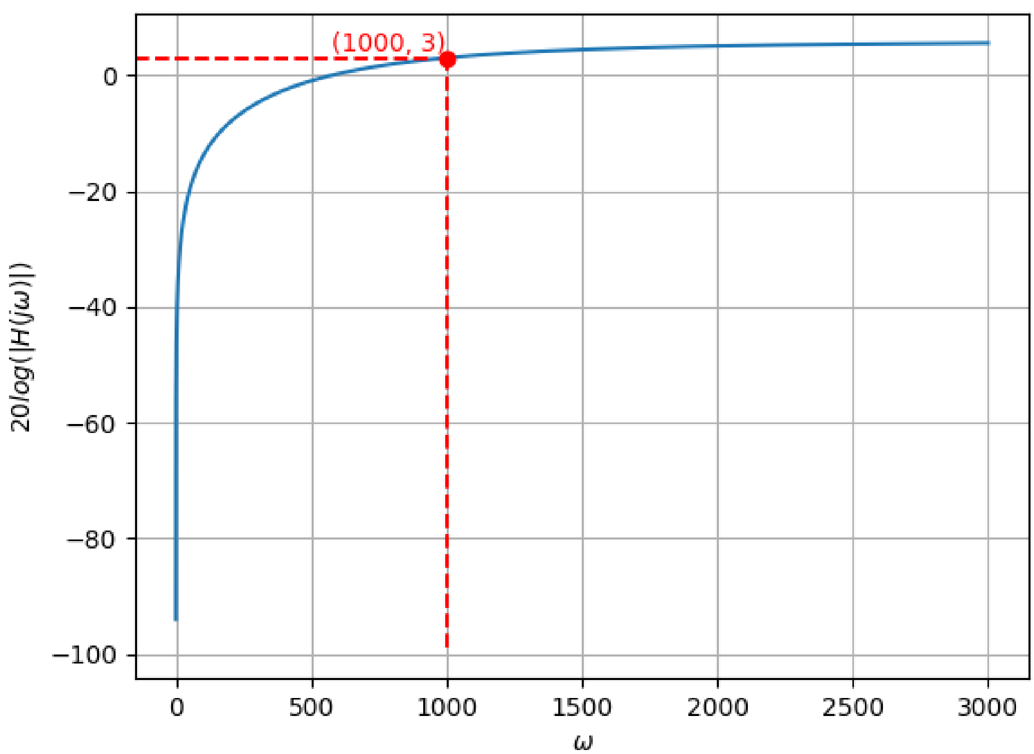


Figure 2.15: Frequency response plot

2.8 A series RLC circuit with $R = 10\Omega$, $L = 50mH$ and $C = 100\mu F$ connected to 200 V, 50 Hz supply consumes power P. The value of L is changed such that this circuit consumes same power P but operates with lagging power factor. The new value of L is ________ mH (rounded off to two decimal places). (GATE 33 BM 2022)

Solution:

Parameter	**Description**	**Value**
R	Resistance	10Ω
C	Capacitance	$100\mu F$
L_{old}	Inductor	$50mH$
L_{new}	New Inductor	
Z_{old}	Old Impedance	
Z^*	New Impedance	

Table 2.9:

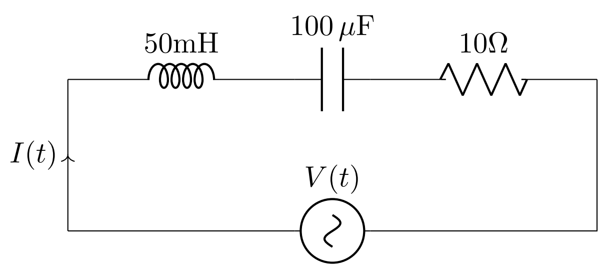
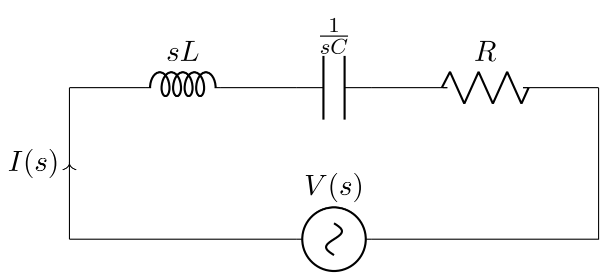


Figure 2.16:

From Fig. 2.16

In s - domain,



$$Z = R + sL_{old} + \frac{1}{sC} \tag{2.86}$$

As the circuit consumes same power P but operates with lagging power factor :

The new impedance(Z^*) will be :

$$Z^* = R + sL_{new} + \frac{1}{sC} \tag{2.87}$$

Comparing the imaginary parts of the impedances:

$$sL_{old} + \frac{1}{sC} = -\left(sL_{new} + \frac{1}{sC}\right) \tag{2.88}$$

Taking $s = j2\pi f$:

$$j\left(2\pi f L_{old} - \frac{1}{2\pi f C}\right) = -j\left(2\pi f L_{new} - \frac{1}{2\pi f C}\right) \tag{2.89}$$

From Table 2.9:

$$L_{new} \approx 152.7\text{mH} \tag{2.90}$$

2.9 A single-phase full-bridge diode rectifier feeds a resistive load of 50Ω from a 200 V, 50 Hz single phase AC supply. If the diodes are ideal, then the active power, in watts, drawn by the load is ____ (round off to nearest integer).

(GATE EE 32)

Solution:

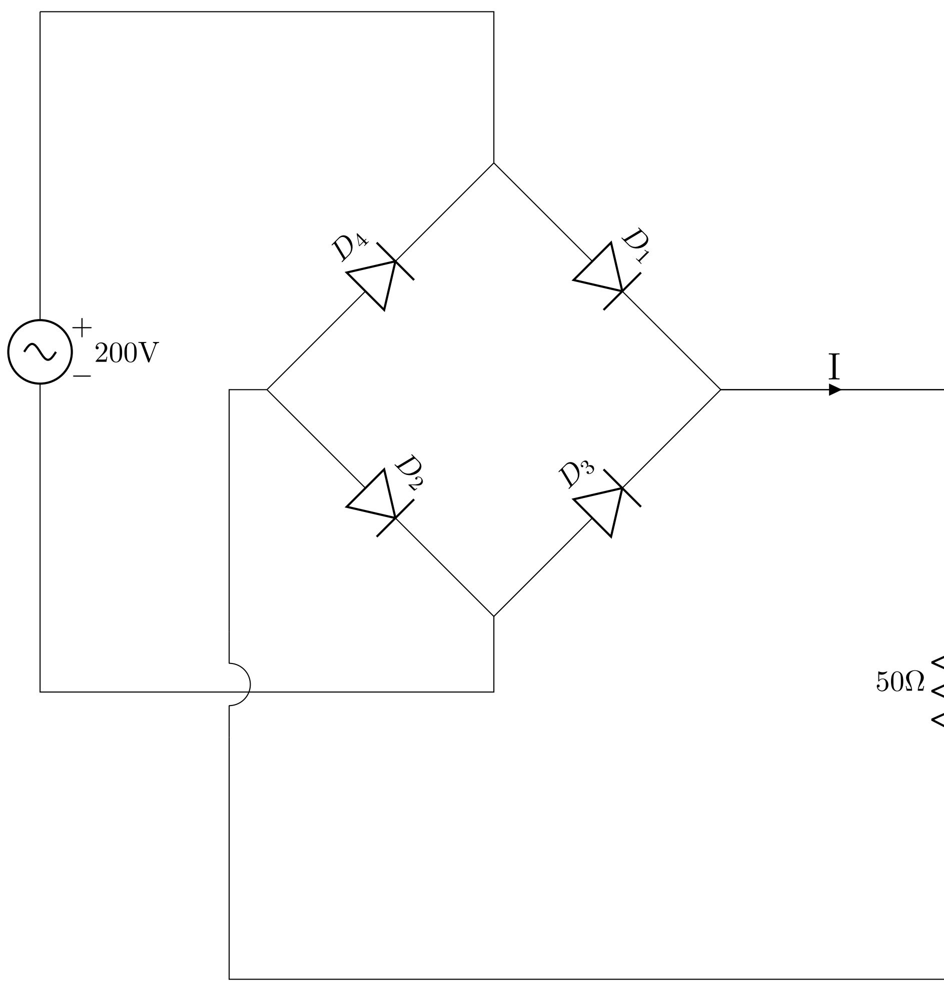


Figure 2.17: Circuit-1

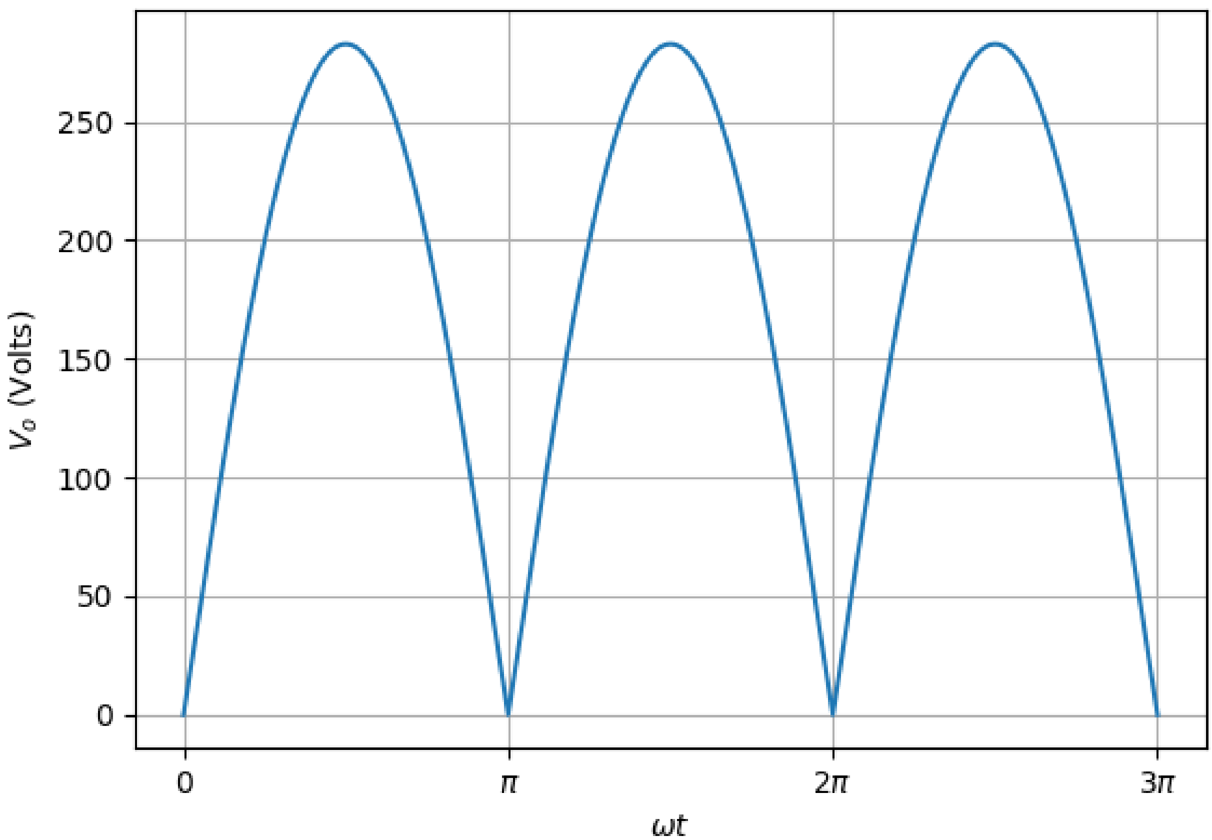


Figure 2.18: Output voltage waveform of single-phase full bridge rectifier

$$V_{rms} = 200 \tag{2.91}$$

$$P = \frac{(V_{rms})^2}{R} \tag{2.92}$$

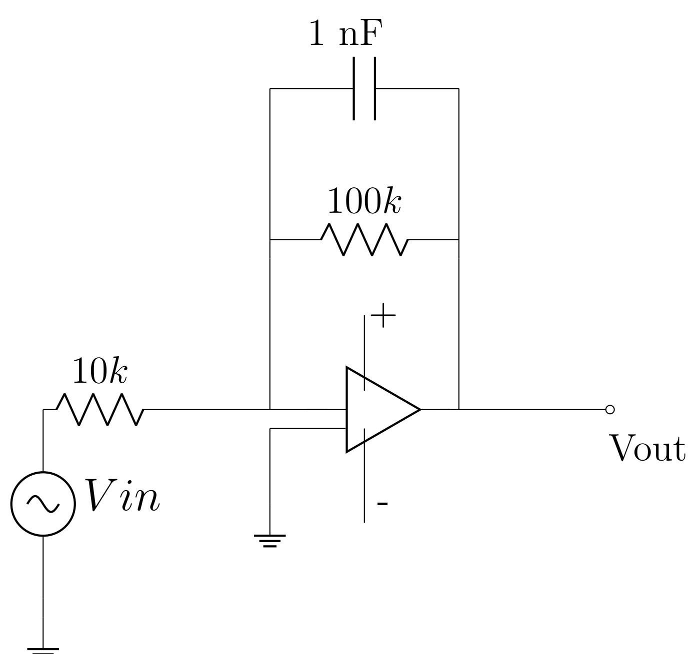
$$P = \frac{(200)^2}{50} W \tag{2.93}$$

$$P = 800W \tag{2.94}$$

Table 2.10: Input Parameters

Symbol	Description	value
R	Load Resistance	50Ω
V_{rms}	RMS Voltage	200V
f	Frequency	50Hz

2.10 The circuit shown is driven by a sinusoidal input voltage, V_{in}, resulting in the output voltage V_{out}. The frequency (in kilohertz) at which the voltage gain is 0 dB is (rounded off to two decimal places). (GATE IN 2022)



Solution:

This circuit is an inverting OP-AMP. The transfer function of an inverting OP-AMP is given by

Parameter	Value	Description
$20\log\left(\frac{V_{\text{out}}}{V_{\text{in}}}\right)$	0	Voltage gain
Sinusoidal input voltage	V_{in}	Input voltage applied to the circuit
Output voltage	V_{out}	Voltage across the output of the circuit
R_1	10 kΩ	Resistor connected to the inverting input of the OP-AMP
R_2	100 kΩ	Feedback resistor connected from the output to the inverting input of the OP-AMP
C	1 nF	Capacitor connected in parallel with R_2
Z_1	?	Impedance of resistor R_1
Z_2	?	Impedance of capacitor C in series with resistor R_2

Table 2.11: Parameters

$$Z_1 = R_1 \tag{2.95}$$

$$Z_2 = \frac{R_2}{1 + j\omega R_2 C} \tag{2.96}$$

$$\frac{1}{Z_2} = \frac{1}{R_2} + j\omega C \tag{2.97}$$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = -\frac{Z_2}{Z_1} \tag{2.98}$$

$$\frac{|V_{\text{out}}|}{|V_{\text{in}}|} = \frac{|Z_2|}{|Z_1|} \tag{2.99}$$

$$20\log\left(\frac{V_{\text{out}}}{V_{\text{in}}}\right) = 0 \tag{2.100}$$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = 1 \tag{2.101}$$

$$\frac{|V_{\text{out}}|}{|V_{\text{in}}|} = \frac{|R_2|}{|(1 + j\omega R_2 C)R_1|} = 1 \tag{2.102}$$

$$\frac{R_2}{R_1} = \sqrt{1 + (R_2 \omega C)^2} \tag{2.103}$$

$$10 = \sqrt{1 + (10^5 \cdot \omega 10{-}9)^2} \tag{2.104}$$

$$99 = \omega^2 \times 10^{-8} \tag{2.105}$$

$$\omega = \sqrt{99} \times 10^4 \tag{2.106}$$

$$2\pi f = 99.49 \times 10^3 \tag{2.107}$$

$$f = 15.84\,\text{kHz} \tag{2.108}$$

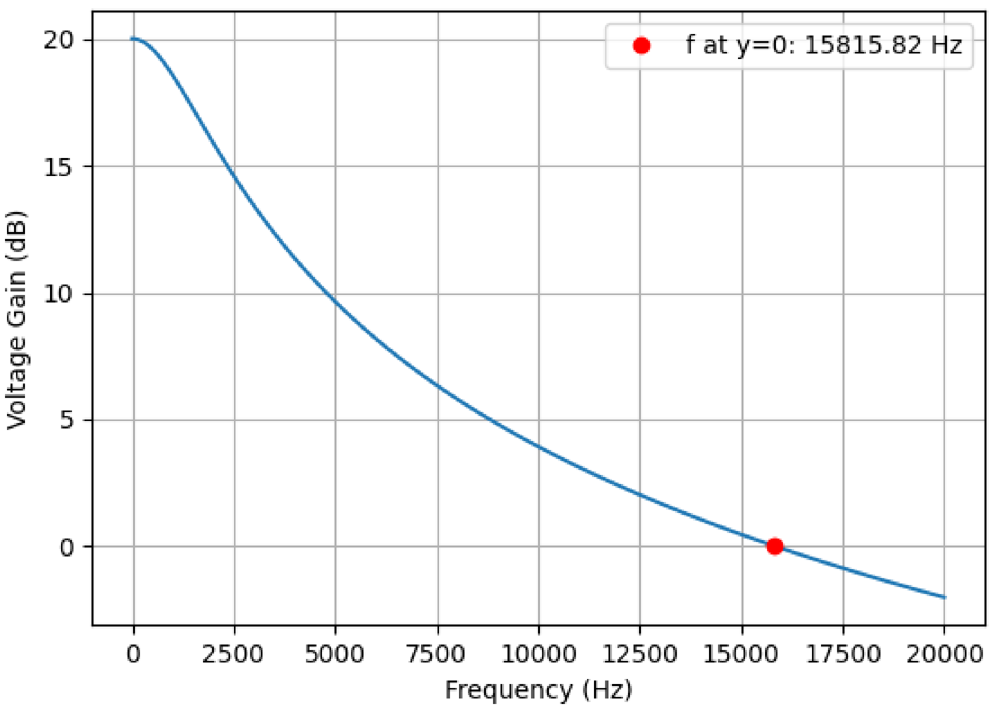
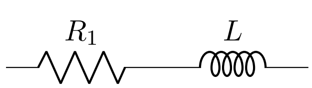


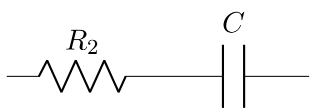
Figure 2.19: Frequency vs Voltage gain

2.11 An inductor having a Q-factor of 60 is connected in series with a capacitor having a Q-factor of 240. The overall Q-factor of the circuit is __________. (Round off to the nearest integer)

Gate 2022 EE Question 27

Solution: 

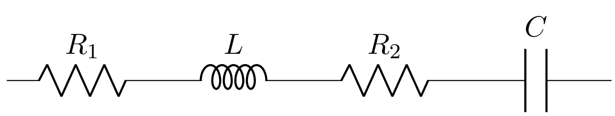
$$Q_1 = \frac{\omega_0 L}{R_1} \tag{2.109}$$



$$Q_2 = \frac{1}{\omega_0 C R_2} \tag{2.110}$$

at resonance as $\omega_0 L = \frac{1}{\omega_0 C}$ hence

$$Q_2 = \frac{\omega_0 L}{R_2} \tag{2.111}$$



$$Q = \frac{\omega_0 L}{R_1 + R_2} \tag{2.112}$$

$$Q = \frac{1}{\frac{R_1}{\omega_0 L} + \frac{R_2}{\omega_0 L}} \tag{2.113}$$

$$Q = \frac{Q_1 Q_2}{Q_1 + Q_2} \tag{2.114}$$

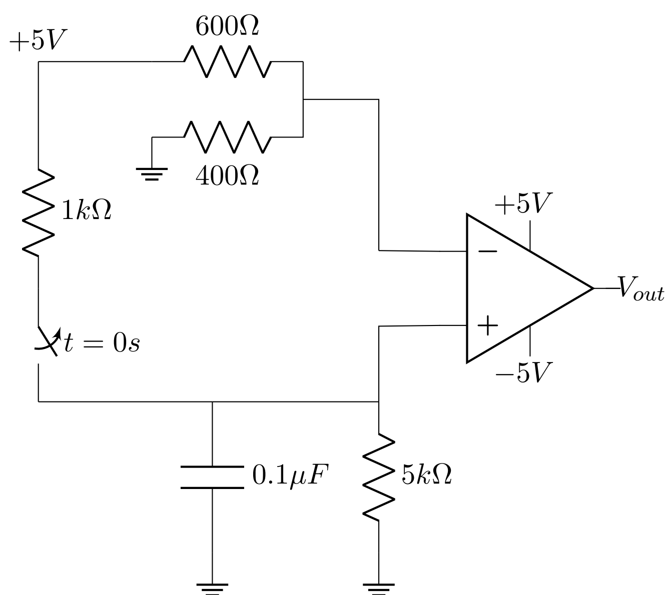
then from (2.114)

$$Q = \frac{60 \times 240}{60 + 240} \tag{2.115}$$

$$Q = 48 \tag{2.116}$$

2.12 In the circuit shown, the switch is initially closed. It is opened at t= 0 s and remains open thereafter. The time (in milliseconds) at which the output voltage V_{out} becomes LOW is (round off to three decimal places) (GATE IN 2022)

Solution:



At t= 0^-, when the switch is closed,

The voltage across the capacitor is:

$$V_c\left(0^-\right) = 5 \times \frac{5}{5+1} \tag{2.117}$$

$$= \frac{25}{6} V \tag{2.118}$$

$V_c\left(0^-\right)$ is also the non inverting voltage of the OP-AMP

At $t = 0^+$, when the switch is open,

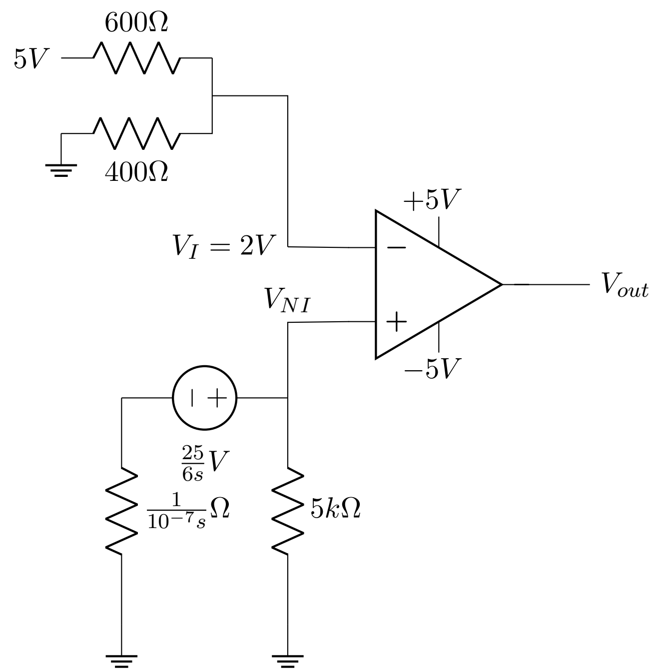


Figure 2.20: circuit diagram in laplace domain at $t = 0^+$

The voltage across inverting terminal is:

$$V_I = 5 \times \frac{600}{600 + 400} \tag{2.119}$$

$$= 2V \tag{2.120}$$

Analysing the circuit at t=0^+ in laplace domain:

Using voltage divider rule,

$$V_{NI}(s) = V \times \left[\frac{R}{R + \frac{1}{sC}}\right] \tag{2.121}$$

$$= \frac{25}{6s} \times \left[\frac{s}{s + \frac{1}{RC}}\right] \tag{2.122}$$

$$= \frac{25}{6} \times \left[\frac{1}{s + \frac{1}{RC}}\right] \tag{2.123}$$

Applying inverse laplace:

$$V_{NI}(t) = \frac{25}{6} e^{\frac{-t}{RC}} \tag{2.124}$$

$$\implies 2 = \frac{25}{6} \times e^{\frac{-t}{RC}} \tag{2.125}$$

$$\implies t = RC \ln\left(\frac{25}{12}\right) \tag{2.126}$$

$$= 0.1 \times 10^{-6} \times 5 \times 10^{3} \ln\left(\frac{25}{12}\right) \tag{2.127}$$

$$= 0.367ms \tag{2.128}$$

2.13 The steady state output v_{out} of the circuit shown below, will

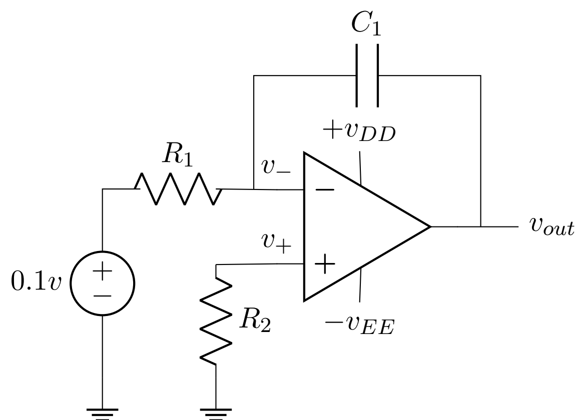


Figure 2.21: Circuit

(a) saturate to $+V_{DD}$

(b) saturate to $-V_{EE}$

(c) become equal to $0.1V$

(d) become equal to $-0.1V$

Solution:

Parameters	Description
v_{out}	Steady State Output Voltage
V_{out}	Laplace Transform of v_{out}

Table 2.12: Parameter description

for an ideal OP amp,

$$V_+ = 0V \tag{2.129}$$

$$V_- = 0V \tag{2.130}$$

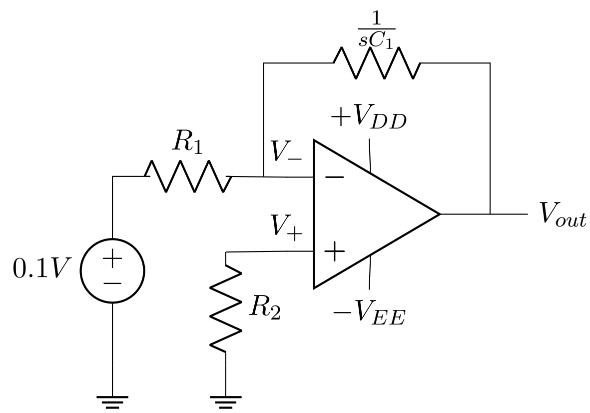


Figure 2.22: s-domain circuit

using KVL,

$$0 = \frac{V_- - 0.1}{R_1} + C_1 s \left(V_- - V_{out}\right) \tag{2.131}$$

$$V_{out} = \frac{V_- - 0.1}{R_1 C_1 s} + V_- \tag{2.132}$$

$$= -\frac{0.1}{R_1 C_1 s} \qquad \text{using (2.130)} \tag{2.133}$$

$$V_{out} \xleftrightarrow{\mathcal{L}^-} v_{out} \tag{2.134}$$

$$v_{out} = -\frac{0.1}{R_1 C_1} t \tag{2.135}$$

$$v_{out} = max\left\{-v_{EE}, -\frac{1}{R_1 C_1} t\right\} \tag{2.136}$$

Hence, v_{out} saturates to $-v_{EE}$

2.14 For the circuit shown below with ideal diodes, the output will be :

(A) $V_{\text{out}} = V_{\text{in}}$ for $V_{\text{in}} > 0$

(B) $V_{\text{out}} = V_{\text{in}}$ for $V_{\text{in}} < 0$

(C) $V_{\text{out}} = -V_{\text{in}}$ for $V_{\text{in}} > 0$

(D) $V_{\text{out}} = -V_{\text{in}}$ for $V_{\text{in}} < 0$

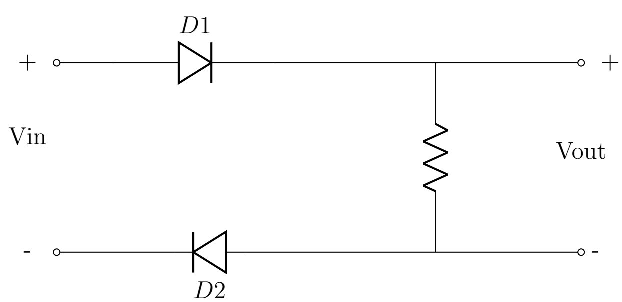


Figure 2.23: Gate EE 25 fig-1

Solution:

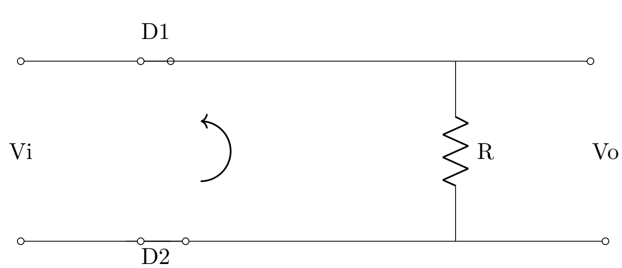


Figure 2.24: Gate EE fig-2

Postive Half Cycle- D_1 and D_2 will be ON

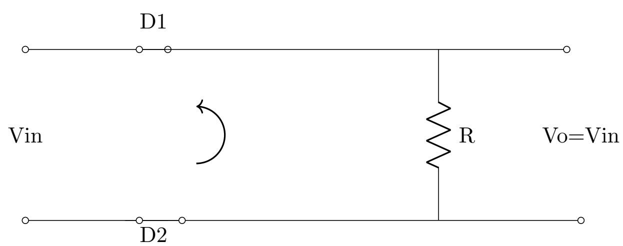


Figure 2.25: Gate EE fig-3

Negative Half Cycle- D_1 and D_2 will be OFF, Vo=0 at$Vin < 0$

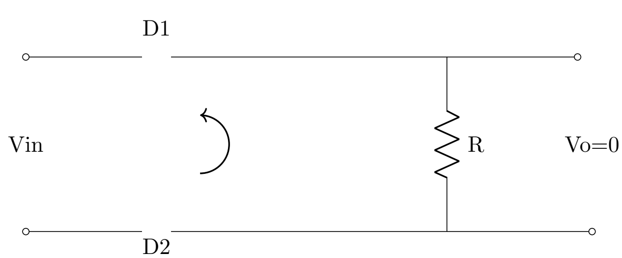


Figure 2.26: Gate EE fig-4

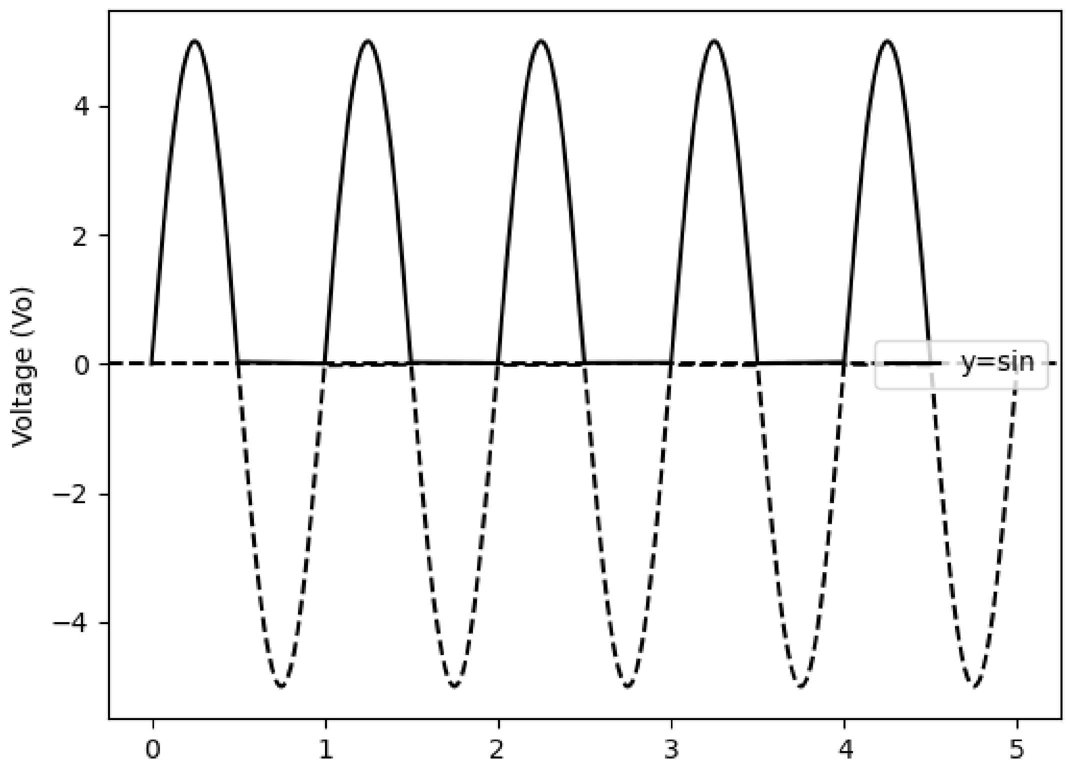


Figure 2.27: Output Waveform

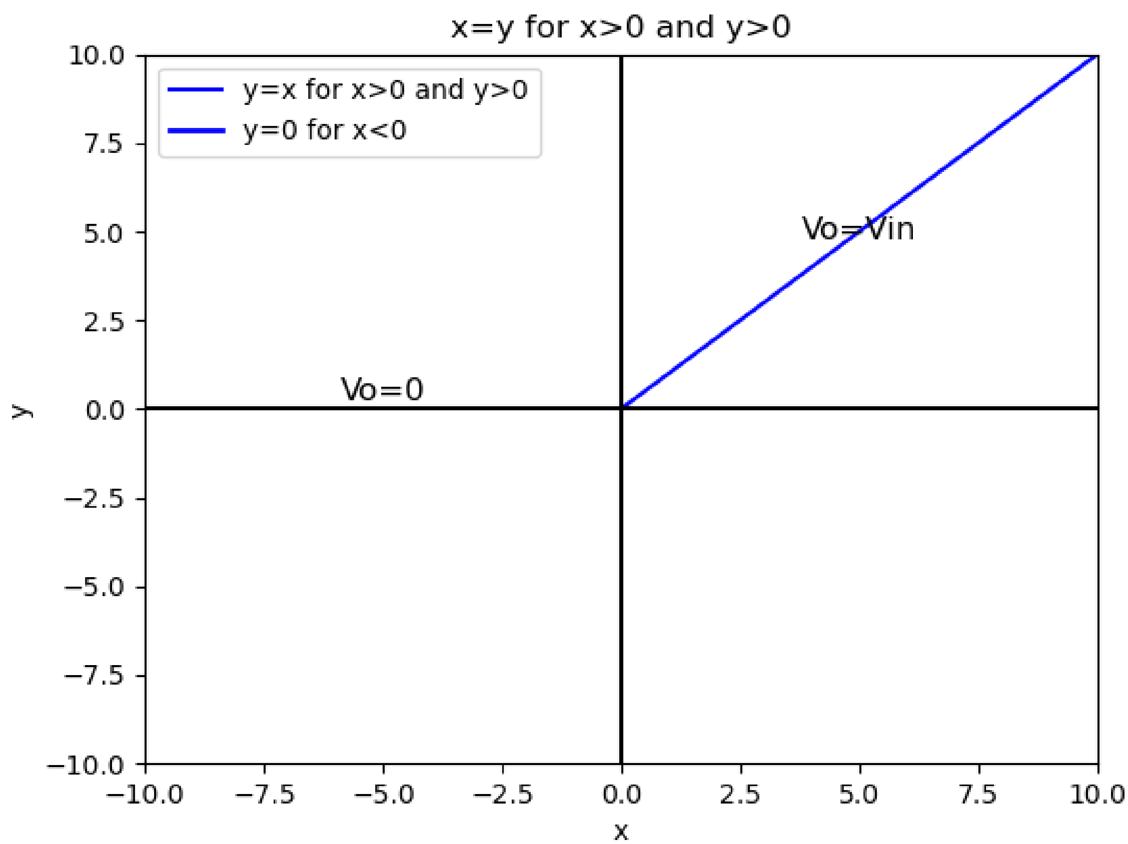


Figure 2.28: charecterstic graph

The solution is Option A

2.2. 2021

2.1 A speech signal, band limited to 4 kHz, is sampled at 1.25 times the Nyquist rate. The speech samples, assumed to be statistically independent and uniformly distributed in the range -5 V to +5 V, are subsequently quantized in an 8-bit uniform quantizer and then over a voice-grade AWGN telephone channel. If the ratio of transmitted signal power to channel noise power is 26 dB, the minimum channel bandwidth required to ensure reliable transmission of the signal with arbitrarily small probability of transmission error (*rounded off to one decimal place*) is ______ kHz. (GATE EC 2021)

Solution:

Parameter	Value	Description
B_0	4 kHz	Bandwidth of signal
R_N	$2B_0$	Nyquist Rate
f_s	$1.25R_N$	Sampling Frequency
R	nf_s	Data Rate
C	$B\log_2\left(1+\frac{P}{N}\right)$	Capacity of AWGN Channel with bandwidth B
$10\log_{10}\frac{P}{N}$	26 dB	Signal to Noise Ratio

Table 2.13: Input Parameters

The signal is band limited to 4 kHz.

$$B_0 = 4\text{kHz} \tag{2.137}$$

$$\implies R_N = 8\text{kHz} \tag{2.138}$$

$$\implies f_s = 10\text{kHz} \tag{2.139}$$

$$R = (8)\,(10\text{kHz}) \tag{2.140}$$

$$\implies R = (8)\left(10^4\right)\ \text{bits/second} \tag{2.141}$$

Channel capacity for an Additive White Gaussian Noise channel is

$$C = B\log_2\left(1 + \frac{P}{N}\right) \text{ bits/second} \tag{2.142}$$

where P is the maximum channel power and N is the noise power and B is the channel bandwidth.

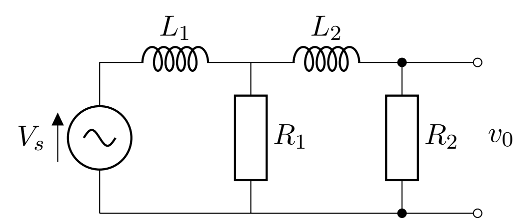
$$\begin{aligned} 10\log_{10}\frac{P}{N} &= 26\text{dB} && (2.143)\\ \implies \frac{P}{N} &= 10^{2.6} && (2.144)\\ &\approx 398.107 && (2.145) \end{aligned}$$

For reliable transmission:

$$\begin{aligned} R &\leq C && (2.146)\\ 8\left(10^4\right) &\leq B\log_2 399.107 && (2.147)\\ B &\geq \frac{8\left(10^4\right)}{\log_2 399.107} && (2.148)\\ \implies B &\geq 9258.58\text{Hz} && (2.149) \end{aligned}$$

$\therefore$ the minimum channel bandwidth required to ensure reliable transmission of the signal is ≈ 9.26 kHz.

2.2 In the circuit shown below, $R_1 = 2\Omega$,$R_2 = 1\Omega$,$L_1 = 2$ h,and $L_2 = 0.5$ H. Which of the following describe(s) the correct characteristics of the circuit ?



(a) Second order high pass filter

(b) Second order low pass filter

(c) Under damped system

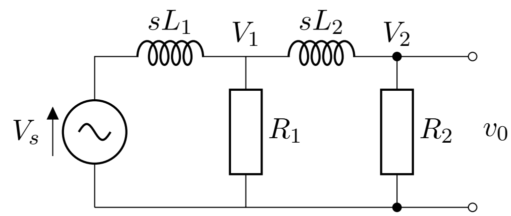
(d) Overdamped system

GATE BM 2021

Solution:

Converting above circuit to frequency domain using laplace transform

let V_1 and V_2 be voltages at shown positions



Variable	**Value**
R_1	2Ω
R_2	1Ω
L_1	2 H
L_2	0.5 H

Table 2.14: input parameters

$$V_0 = V_1 \left(\frac{R_2}{R_2 + sL_2} \right) \tag{2.150}$$

$$V_1 = V_s \left(\frac{R_1 \left(\frac{sL_2 + R_2}{R_1 + R_2 + SL_2} \right)}{sL_1 + R_1 \left(\frac{sL_2 + R_2}{R_1 + R_2 + SL_2} \right)} \right) \tag{2.151}$$

$$V_1 = V_s \left(\frac{2 + s}{(2 + s) + s(6 + s)} \right) \tag{2.152}$$

$$V_0 = V_s \left(\frac{2}{s^2 + 7s + 2} \right) \tag{2.153}$$

$$\text{let } s = \jmath\omega \tag{2.154}$$

$$V_0 = V_s \left(\frac{2}{-\omega^2 + 7\jmath\omega + 2} \right) \tag{2.155}$$

$$= V_s \left(\frac{4 - 2\omega^2 - 7\jmath\omega}{\omega^4 + 45\omega^2 + 4} \right) \tag{2.156}$$

For lower frequency V_0 is finite and for higher frequency V_0 is zero

$\therefore$ Second order low pass filter

From 2.154

$$s^2 + 7s + 2 = 0 \tag{2.157}$$

$$\text{for } as^2 + bs + c = 0 \tag{2.158}$$

$$\text{(Damping Factor)}\zeta = \frac{b}{2\sqrt{ac}} \tag{2.159}$$

$$\text{By comparing } \zeta = \frac{7}{2\sqrt{2}} \tag{2.160}$$

$$\implies \zeta > 1 \tag{2.161}$$

$\therefore$ Over-damped System

$\therefore$ B,D are correct options

Chapter 3

Z-transform

3.1. 2022

3.1 Consider the following recursive iteration scheme for different values of variable P with the initial guess $x_1 = 1$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{P}{x_n}\right), \qquad n = 1, 2, 3, 4, 5$$

For $P = 2$, x_5 is obtained to be 1.414, rounded off to 3 decimal places. For $P = 3$, x_5 is obtained to be 1.732, rounded off to 3 decimal places.

If $P = 10$, the numerical value of x_5 is ______ . (*round off to three decimal places*) (GATE CE 2022)

Solution:

Applying $A.M \geq G.M$ inequality,

$$\frac{x_n + \frac{P}{x_n}}{2} \geq \sqrt{P} \tag{3.1}$$

$$\implies x_{n+1} \geq \sqrt{P} \tag{3.2}$$

Solving the equation,

$$2x_{n+1}x_n - {x_n}^2 - P = 0 \tag{3.3}$$

Applying Z-transform we get,

$$X\left(z\right) * X\left(z\right) = \frac{PZ^{-1}}{\left(1-z^{-1}\right)\left(2-z^{-1}\right)} \tag{3.4}$$

$$= P\left(\frac{z^{-1}}{1-z^{-1}} - \frac{z^{-1}}{2-z^{-1}}\right) \tag{3.5}$$

From the transformation pairs,

$$x_{n-a} \overset{\mathcal{Z}}{\longleftrightarrow} z^{-a}X\left(z\right) \tag{3.6}$$

$$x_{n_1} \times x_{n_2} \overset{\mathcal{Z}}{\longleftrightarrow} X_1\left(z\right) * X_2\left(z\right) \tag{3.7}$$

$$\frac{u\left(n-1\right)}{a^n} \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{a-z^{-1}} \tag{3.8}$$

Now, applying inverse Z-tranform,

$$x_n^2 = P\left(u\left(n-1\right) - \frac{u\left(n-1\right)}{2^n}\right) \tag{3.9}$$

$$\implies x_n^2 = P\left(1-\frac{1}{2^n}\right) \quad \left[\because n \geq 1\right] \tag{3.10}$$

Similarly,

$$x_{n+1}^2 = P\left(1-\frac{1}{2^{n+1}}\right) \tag{3.11}$$

$$\implies \lim_{n\to\infty}\frac{x_{n+1}}{x_n} = \lim_{n\to\infty}\sqrt{\frac{P\left(1-\frac{1}{2^n}\right)}{P\left(1-\frac{1}{2^{n+1}}\right)}} \tag{3.12}$$

$$= 1 \tag{3.13}$$

Hence, the system is convergent.

Now finding the limit of the sequence,

$$x^2 = \lim_{x \to \infty} P\left(1 - \frac{1}{2^n}\right) \tag{3.14}$$

$$\implies x = \pm\sqrt{P} \tag{3.15}$$

From (3.2) and (3.15),

$$x_{n+1} = \sqrt{P} \tag{3.16}$$

Therefore, for $P = 10$ the value of x_5 is,

$$x_5 = \sqrt{10} \tag{3.17}$$

$$\therefore x_5 = 3.162 \tag{3.18}$$

3.2 The block diagram of a two-tap high-pass FIR filter is shown below. The filter transfer function is given by $H(z) = Y(z)/X(z)$.

If the ratio of maximum to minimum value of $H(z)$ is 2 and $|H(z)|_{max} = 1$, the coefficients β_0 and β_1 are ______________ and ______________, respectively.

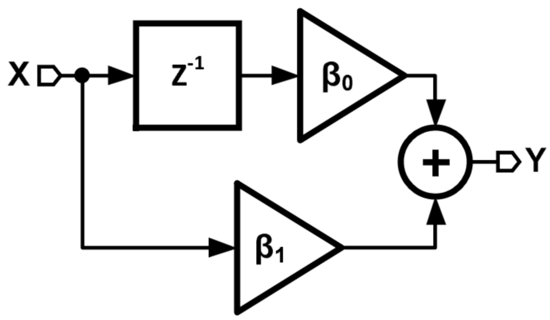


Figure 3.1: Block diagram

(A) 0.75, -0.25

(B) 0.67, 0.33

(C) 0.60, -0.40

(D) -0.64, 0.36

GATE BM 2022

Solution:

Results and Proofs:

Time Shift Property:

$$x(n) \overset{\mathcal{Z}}{\longleftrightarrow} X(z) \tag{3.19}$$

$$x(n-n_0) \overset{\mathcal{Z}}{\longleftrightarrow} z^{-n_0}X(z) \tag{3.20}$$

Proof:

Let

$$y(n) = x(n - n_0) \tag{3.21}$$

Taking z-transform

$$\mathcal{Z}\left(y(n)\right) = \mathcal{Z}\left(x(n - n_0)\right) \tag{3.22}$$

$$\tag{3.23}$$

Simplifying LHS

$$Y(z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n} \tag{3.24}$$

From (3.21)

$$Y(z) = \sum_{n=-\infty}^{\infty} x(n - n_0) z^{-n} \tag{3.25}$$

Let

$$n - n_0 = s \tag{3.26}$$

$$\implies n = s + n_0 \tag{3.27}$$

From (3.25) and (3.27)

$$Y(z) = \sum_{s=-\infty}^{\infty} x(s) z^{-(s+n_0)} \tag{3.28}$$

$$= z^{-n_0} \sum_{s=-\infty}^{\infty} x(s) z^{-s} \tag{3.29}$$

As variable in Z-transform is dummy, on replacing it, we get

$$Y(z) = z^{-n_0} \sum_{n=-\infty}^{\infty} x(n) z^{-n} \tag{3.30}$$

$$= z^{-n_0} X(z) \tag{3.31}$$

From (3.22) and (3.31)

$$\mathcal{Z}\left(x(n-n_0)\right) = z^{-n_0} X(z) \tag{3.32}$$

Hence proved

<u>Result:</u>

$$z^{-n_0} X(z) \overset{\mathcal{Z}^-}{\longleftrightarrow} x(n-n_0) \tag{3.33}$$

Sol:

Variable	**Description**	**Value**
$H(z)$	Transfer Function	$\beta_0 z^{-1} + \beta_1$
$\lvert H(z)\rvert_{max}$	Maximum value of Transfer Function	1
$\lvert H(z)\rvert_{min}$	Minimum value of Transfer Function	$\frac{1}{2}$

Table 3.1: input parameters

In (3.33), put

$$n_0 = 1, \quad x(n) = \delta(n)$$

Since

$$1 \overset{\mathcal{Z}^-}{\longleftrightarrow} \delta(n)$$

$$z^{-1} \overset{\mathcal{Z}^{-}}{\longleftrightarrow} \delta(n-1) \tag{3.34}$$

This is a unit delay in discrete time and represents unit amplitude sinosoidal signal. So,

$$z^{-1} = e^{-jw} \tag{3.35}$$

$$\implies \left|z^{-1}\right| = 1 \tag{3.36}$$

Since $H(z)$ is complex, on using Triangle Inequality, we get

$$|x+y| \leq |x| + |y| \tag{3.37}$$

And its corollary

$$||x| - |y|| \leq |x+y| \tag{3.38}$$

where x and y are complex numbers.

$$\left|\left|z^{-1}\beta_0\right| - |\beta_1|\right| \leq \left|z^{-1}\beta_0 + \beta_1\right| \leq \left|z^{-1}\beta_0\right| + |\beta_1| \tag{3.39}$$

From Table 3.1

$$\left|\left|z^{-1}\beta_0\right| - |\beta_1|\right| \leq |H(z)| \leq \left|z^{-1}\beta_0\right| + |\beta_1| \tag{3.40}$$

From (3.36)

$$||\beta_0| - |\beta_1|| \leq |H(z)| \leq |\beta_0| + |\beta_1| \tag{3.41}$$

So, we can conclude that

$$|H(z)|_{max} = |\beta_0| + |\beta_1| \tag{3.42}$$

Now from Table 3.1

$$1 = |\beta_0| + |\beta_1| \tag{3.43}$$

Similarly,

$$\frac{1}{2} = ||\beta_0| - |\beta_1|| \tag{3.44}$$

On solving (3.43) and (3.44), we get

$$|\beta_0| = 0.75, |\beta_1| = 0.25 \tag{3.45}$$

OR

$$|\beta_0| = 0.25, |\beta_1| = 0.75 \tag{3.46}$$

Hence the correct answer is option (A)

3.2. 2021

3.1 The causal signal with Z transform $z^2(z-a)^{-2}$ is ($u(n)$ is unit step signal)

(a) $a^{2n}u(n)$

(b) $(n+1)a^n u(n)$

(c) $n^{-1}a^n u(n)$

(d) $n^2 a^n u(n)$

(GATE 31 EE 2021)

Solution:

Z-transform of a causal signal is,

$$X(z) = z^2(z-a)^{-2} = \frac{1}{(1-az^{-1})^2}; |z| > |a| \tag{3.47}$$

The Z transform pair for $a^n u(n)$ signal is given by :

$$a^n u(n) \longleftrightarrow \frac{1}{1-az^{-1}} \tag{3.48}$$

Using differentiation in z-domain property:

$$na^n u(n) \longleftrightarrow -z\frac{d}{dz}\left(\frac{1}{1-az^{-1}}\right) \tag{3.49}$$

$$\implies na^n u(n) \longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2} \tag{3.50}$$

Using time-shifting property:

$$(n+1)a^{n+1}u(n+1) \longleftrightarrow \frac{az^{-1}}{(1-az^{-1})^2}z \tag{3.51}$$

$$(n+1)a^{n}u(n+1) \longleftrightarrow \frac{1}{(1-az^{-1})^2} \tag{3.52}$$

From (3.47) and (3.52), Inverse Z transform is :

$$x(n) = (n+1)a^n u(n+1) \tag{3.53}$$

Sequence $u(n+1)$ exist for$-1 \leq n < \infty$, but the factor $(n+1)$ is zero for $n = -1$, so $x(n)$ may be expressed as a causal sequence.

$$x(n) = (n+1)a^n u(n) \tag{3.54}$$

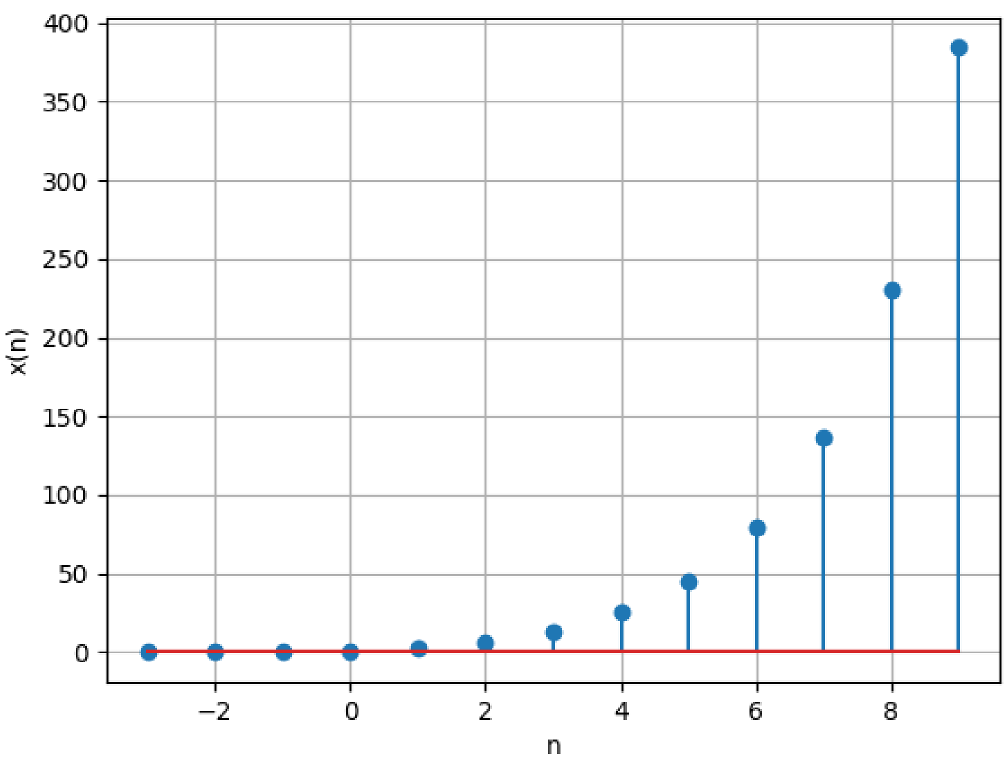


Figure 3.2: $x(n) vs n$ using $a = 1.5$

3.2 The sum of the infinite geometric series

$$1+\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+...$$

(rounded off to one decimal place) is ______. (GATE 2021 BT Q20)

Solution:

Symbol	Value	Description
$x(n)$		$(n+1)^{th}$ term of series
$x(0)$	1	1^{st} term of series
r	$\frac{1}{3}$	Common ratio
$y(n)$		Sum of $(n+1)$ terms

Table 3.2: Given Parameters

General term:

$$x(n) = x(0)r^n u(n) \tag{3.55}$$

$$\implies X(z) = \frac{1}{1-rz^{-1}} \tag{3.56}$$

$$y(n) = x(n) * u(n) \tag{3.57}$$

$$\implies Y(z) = X(z)U(z) \tag{3.58}$$

$$= \frac{1}{(1-rz^{-1})(1-z^{-1})} \tag{3.59}$$

$$= \frac{1}{r-1}\left(\frac{r}{1-rz^{-1}} - \frac{1}{1-z^{-1}}\right) \tag{3.60}$$

(3.61)

Taking inverse Z-transform:

$$y(n) = \frac{1}{r-1}\left(r(r^n u(n)) - u(n)\right) \tag{3.62}$$

$$= \left(\frac{r^{n+1}-1}{r-1}\right)u(n) \tag{3.63}$$

$$= \left(\frac{1-r^{n+1}}{1-r}\right)u(n) \tag{3.64}$$

For infinite terms:

$$y(\infty) = \lim_{n\to\infty}\left(\frac{1-r^{n+1}}{1-r}\right)u(n) \tag{3.65}$$

$$= \frac{1}{1-r} \tag{3.66}$$

$$= \frac{3}{2} \tag{3.67}$$

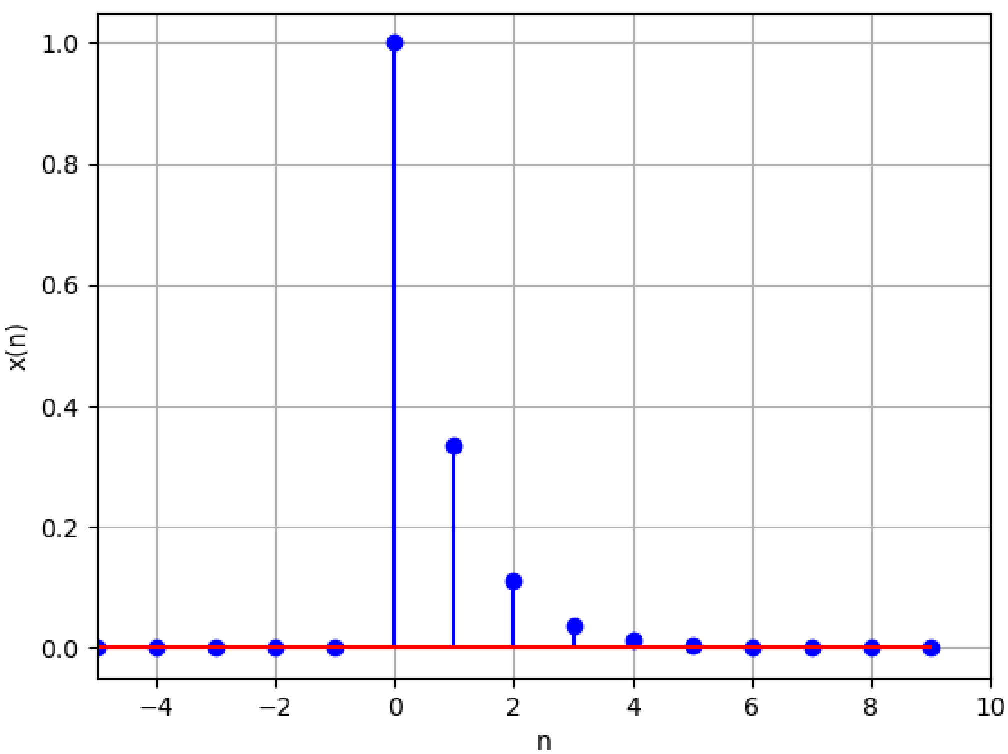


Figure 3.3: Plot of $x(n)$ vs n

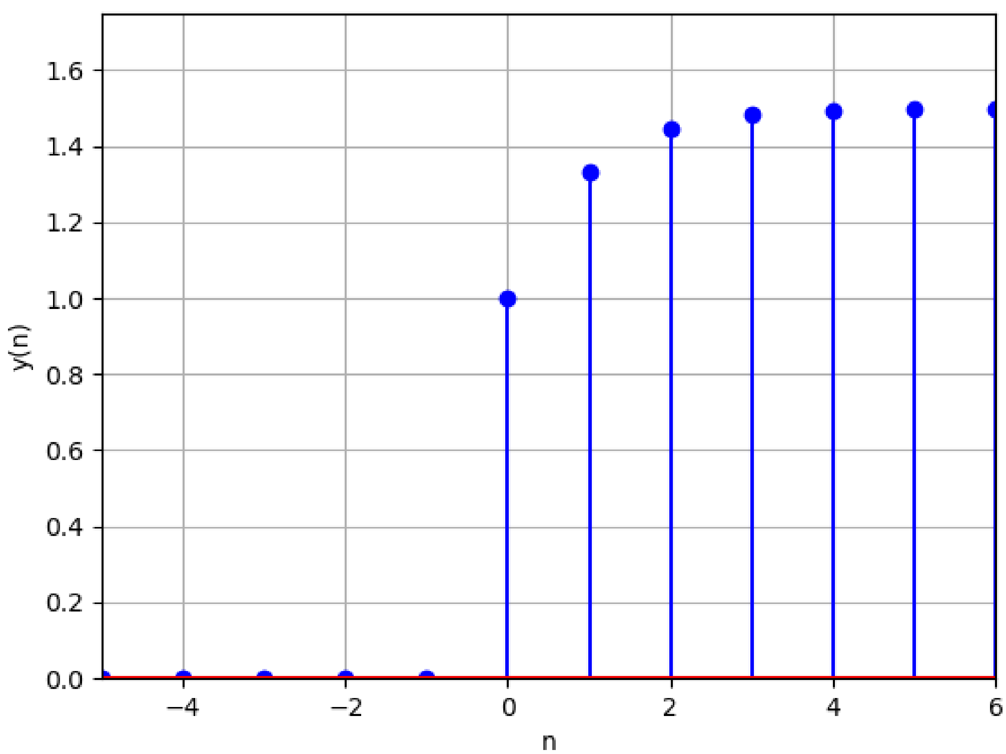


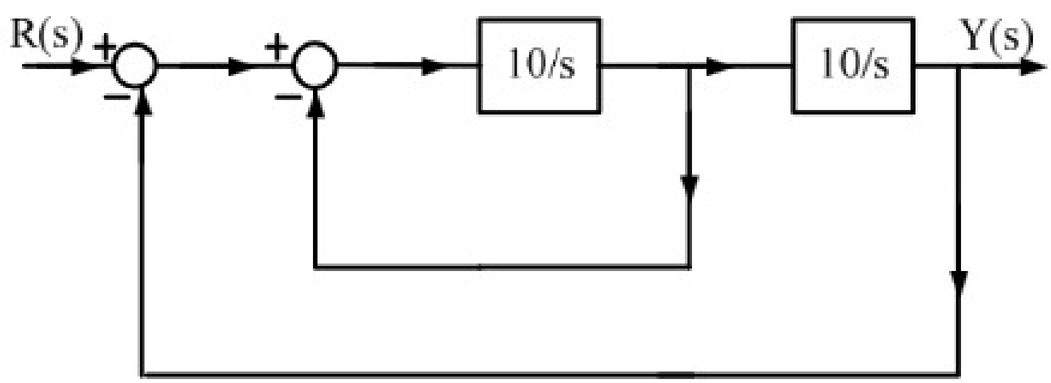
Figure 3.4: Plot of $y(n)$ vs n

Chapter 4

Systems

4.1. 2022

4.1 The damping ratio and undamped natural frequency of a closed loop system as shown in the figure, are denoted as ζ and ω_n, respectively. The values of ζ and ω_n are



(a) $\zeta = 0.5$ and $\omega_n = 10$ rad/s

(b) $\zeta = 0.1$ and $\omega_n = 10$ rad/s

(c) $\zeta = 0.707$ and $\omega_n = 10$ rad/s

(d) $\zeta = 0.707$ and $\omega_n = 100$ rad/s

(GATE EE 2022) **Solution:**

We will use Mason's Gain Formula to calculate the transfer function of this system.

Parameter	**Description**	**Values**
m	load of system	
k	stiffness of system	
ω_n	Natural frequency	$\sqrt{\frac{k}{m}}$
ζ	Damping ratio	$\frac{c}{2m\omega_n}$
y(t)	Output of system	
x(t)	Input to the system	
c	Damping coefficient	
T(s)	Transfer function of system	$\frac{Y(s)}{R(s)}$

Table 4.1: Parameter Table

First converting the given diagram to a signal flow graph :

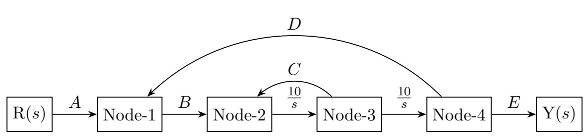


Figure 4.1: Signal Flow Diagram

Mason's Gain Formula is given by :

$$H\left(s\right) = \sum_{i=1}^{N}\left(\frac{P_i\Delta_i}{\Delta}\right) \tag{4.1}$$

This signal flow graph has only one forward path whose gain is given by :

Parameter	Description
N	Number of forward paths
L	Number of loops
P_k	Forward path gain of k^{th} path
Δ_k	Associated path factor
Δ	Determinant of the graph

Table 4.2: Parameter Table - Mason's Gain Law

Parameter	Formula
Δ	$1 + \sum_{k=1}^{L} \left((-1)^k \text{ Product of gain of groups of k isolated loops} \right)$
Δ_k	Δ part of graph that is not touching k^{th} forward path

Table 4.3: Formula Table - Mason's Gain Law

$$P_1 = \frac{10}{s}\frac{10}{s} \tag{4.2}$$

$$= \frac{100}{s^2} \tag{4.3}$$

The loop gain for loop between Node-2 and Node-3 is :

$$L_1 = \frac{10}{s}(-1) \tag{4.4}$$

$$= -\frac{10}{s} \tag{4.5}$$

The loop gain for loop between Node-1 and Node-4 is :

$$L_1 = \frac{10}{s}\frac{10}{s}(-1) \tag{4.6}$$
$$= -\frac{100}{s^2} \tag{4.7}$$

Using Table 4.3, Δ is :

$$\Delta = 1 - \left(-\frac{10}{s} - \frac{100}{s^2}\right) \tag{4.8}$$
$$= 1 + \frac{10}{s} + \frac{100}{s^2} \tag{4.9}$$

There are no two isolated loops available. Hence all further terms will b zero.

As both the loops are in contact with the only forward path,

$$\Delta_1 = 1 \tag{4.10}$$

Using equation (4.1) :

$$H(s) = \frac{\frac{100}{s^2}}{1 + \frac{10}{s} + \frac{100}{s^2}} \tag{4.11}$$
$$= \frac{100}{s^2 + 10s + 100} \tag{4.12}$$

Referring to Table 4.1, the general equation of the damping system is second order and can be written as :

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = x(t) \tag{4.13}$$

Take the Laplace transform and solve for $\frac{Y(s)}{X(s)}$:

$$\frac{Y(s)}{X(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.14}$$

$$\implies H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.15}$$

Comparing equations (4.12) and (4.15) ,

$$\omega_n^2 = 100 \tag{4.16}$$

$$\implies \omega_n = 10 \text{ rad/s} \tag{4.17}$$

$$2\zeta\omega_n = 10 \tag{4.18}$$

$$\implies \zeta = 0.5 \tag{4.19}$$

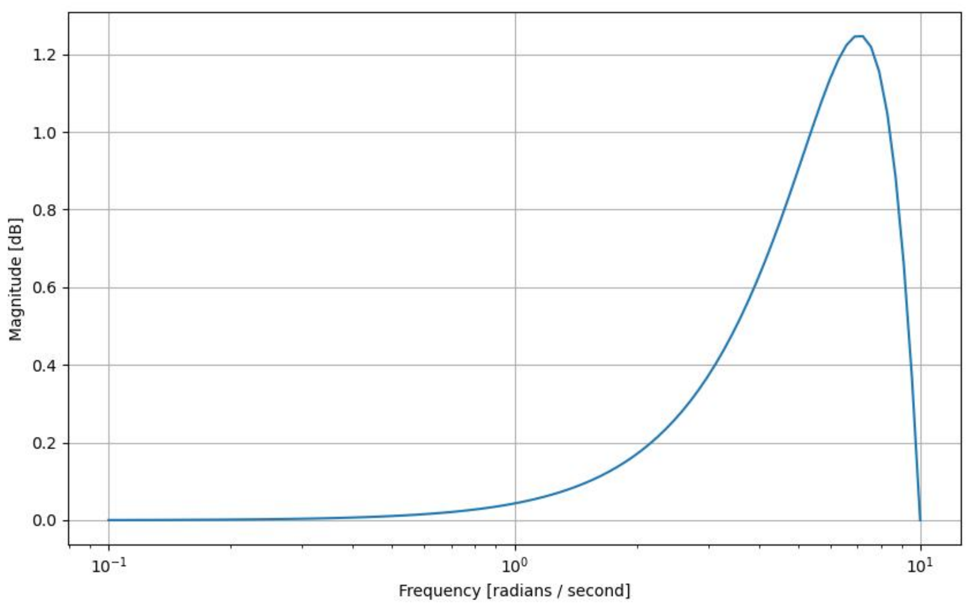


Figure 4.2: Magnitude plot

4.2 In the block diagram shown in the figure, the transfer function $G = \frac{K}{\tau s+1}$ with $K > 0$ and $\tau > 0$. The maximum value of K below which the system remains stable is _____(rounded off to two decimal places) (GATE CH 2022)

Solution:

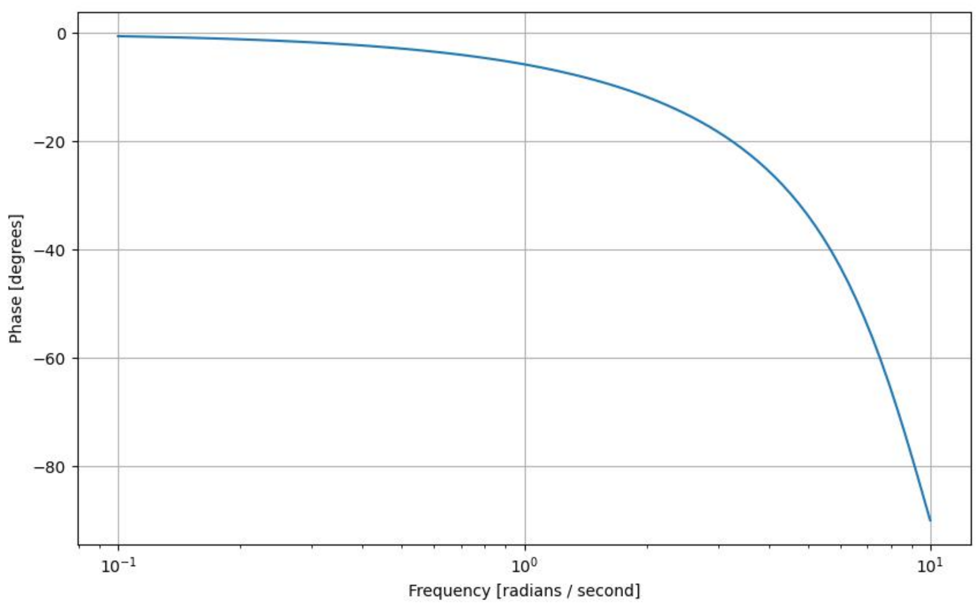


Figure 4.3: Phase plot

$$X = XG^2 + YG \tag{4.20}$$

$$\implies X = \frac{YG}{1 - G^2} \tag{4.21}$$

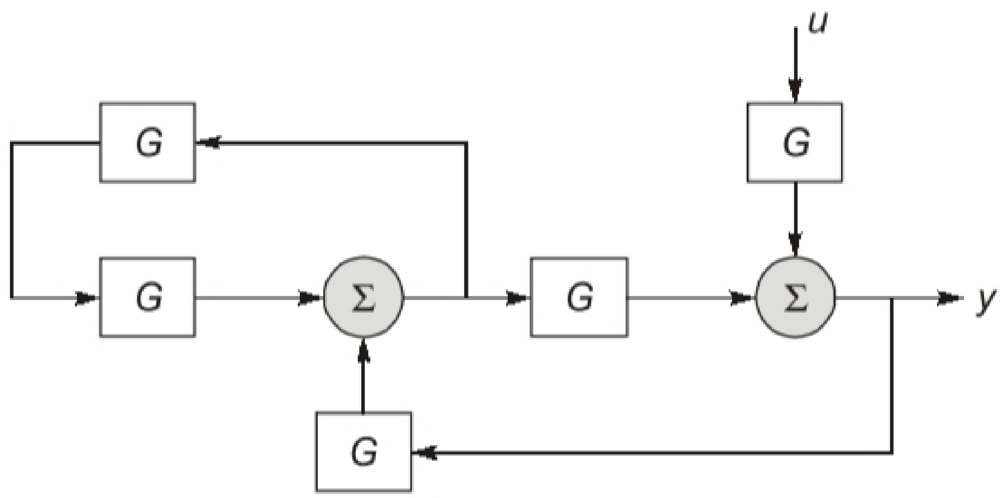
$$Z = XG \tag{4.22}$$

$$Y = Z + UG \tag{4.23}$$

$$Y = XG + UG \tag{4.24}$$

$$Y = \frac{YG^2}{1 - G^2} + UG \tag{4.25}$$

$$\implies Y = \frac{UG(1 - G^2)}{1 - 2G^2} \tag{4.26}$$



Parameter	Value	Description
G	$\frac{K}{\tau s+1}$	Transfer function shown in blocks
Y		Laplace transform of y(output)
U		Laplace transform of u(input)
X,Z		Laplace transform of x and z
T	$\frac{Y}{U}$	Transfer function of complete system

Table 4.4: Parameters

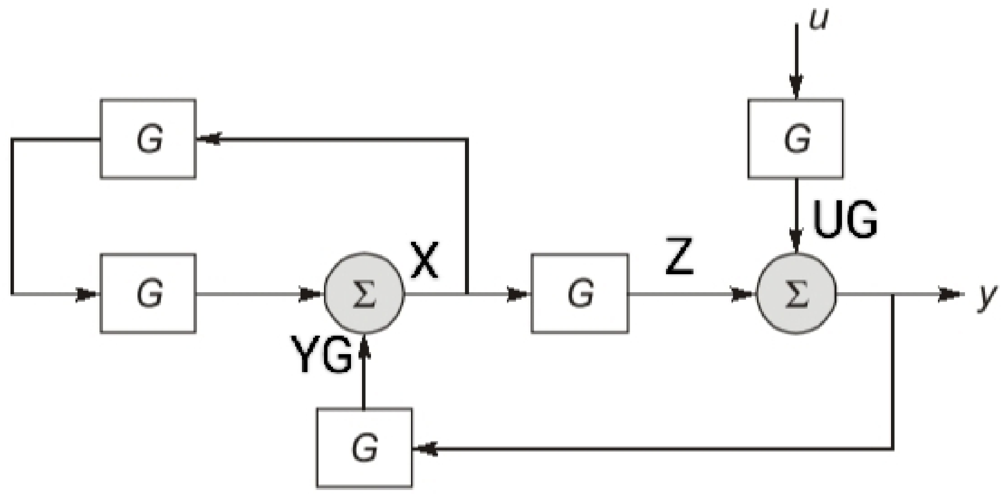


Figure 4.4: Block Diagram

From Table 4.4,

$$T = \frac{G(1-G^2)}{1-2G^2} \tag{4.27}$$

$$= \frac{K\left(1-\frac{K^2}{(\tau s+1)^2}\right)}{\left(1-\frac{2K^2}{(\tau s+1)^2}\right)(\tau s+1)} \tag{4.28}$$

$$= \frac{K(\tau^2 s^2 + 2\tau s + 1 - K^2)}{\tau^3 s^3 + 3\tau^2 s^2 + (3\tau - 2K^2\tau)s + 1 - 2K^2} \tag{4.29}$$

So, Characteristic equation :

$$\tau^3 s^3 + 3\tau^2 s^2 + (3\tau - 2K^2\tau)s + 1 - 2K^2 = 0 \tag{4.30}$$

For a characteristic equation $a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...a_n = 0$,

From Table 4.5:

s^n	a_0	a_2	a_4	...
s^{n-1}	a_1	a_3	a_5	...
s^{n-2}	$b_1 = \dfrac{a_1a_2 - a_3a_0}{a_1}$	$b_2 = \dfrac{a_1a_4 - a_5a_0}{a_1}$	...	..
s^{n-3}	$c_1 = \dfrac{b_1a_3 - b_2a_1}{b_1}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		
s^1	$\vdots$	$\vdots$		
s^0	a_n			

Table 4.5: Routh Array

s^3	τ^3	$3\tau - 2K^2\tau$
s^2	$3\tau^2$	$1 - 2K^2$
s^1	$\frac{8}{3}\tau(1 - K^2)$	0
s^0	$1 - 2K^2$	

Table 4.6:

Given $\tau > 0$ and $K > 0$, for system to be stable,

$$1 - K^2 > 0 \tag{4.31}$$

$$1 - 2K^2 > 0 \tag{4.32}$$

$$\implies 0 < K < \frac{1}{\sqrt{2}} \tag{4.33}$$

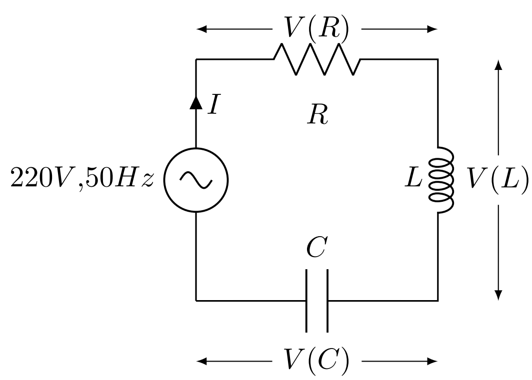
$$K_{max} \approx 0.71 \tag{4.34}$$

4.3 A series RLC circuit is connected to 220 V, 50 Hz supply. For a fixed a value of R and C, the inductor L is varied to deliver the maximum current. This value 0.4A and the corresponding potential drop across the capacitor is 330 V. The value of the inductor L is ? (Rounded off to two decimal places). (GATE BM 2022)

Solution:

Symbols	Description	Values
V_s	Input voltage	220 V and 50Hz
χ_L	Impedance across inductor	$j\omega L$
χ_C	Impedance across capacitor	$\frac{-j}{\omega C}$
Z	Impedance across the entire circuit	$\frac{1}{R+j\omega L+\frac{-j}{\omega C}}$

Table 4.7: Parameters, Descriptions, and Values



During maximum current $|Z|$ is minimum .

$$I = \frac{V_s}{Z} \tag{4.35}$$

$$= \frac{V_s}{R + \chi_L + \chi_C} \tag{4.36}$$

$$= \frac{V_s}{R + j\omega L + \frac{1}{j\omega C}} \tag{4.37}$$

$$|I| = \frac{|V_s|}{\sqrt{R + \left(\omega L - \frac{1}{\omega C}^2\right)}} \tag{4.38}$$

Varying L for maximum value of I :

$$\omega L = \frac{1}{\omega C} \tag{4.39}$$

Putting in (4.37):

$$I_{max} = \frac{V_s}{R} \tag{4.40}$$

I_{max} has same phase as V_s (Assume $\angle\phi$). For impedance across the capacitor :

$$V_C|_{I=I_{max}} = I_{max}\chi_C \tag{4.41}$$

$$-330\angle(90+\phi) = (0.4\angle\phi)\,\chi_C \tag{4.42}$$

$$-330\angle 90 = 0.4\chi_C \tag{4.43}$$

$$\implies \chi_C = -825j\Omega \tag{4.44}$$

For value of Capacitor and inductor, using (4.39) :

$$L = \frac{825}{100\pi}H \tag{4.45}$$

$$\approx 2.63\text{H} \tag{4.46}$$

$$C = 3.858 * 10^{-6}\text{F} \tag{4.47}$$

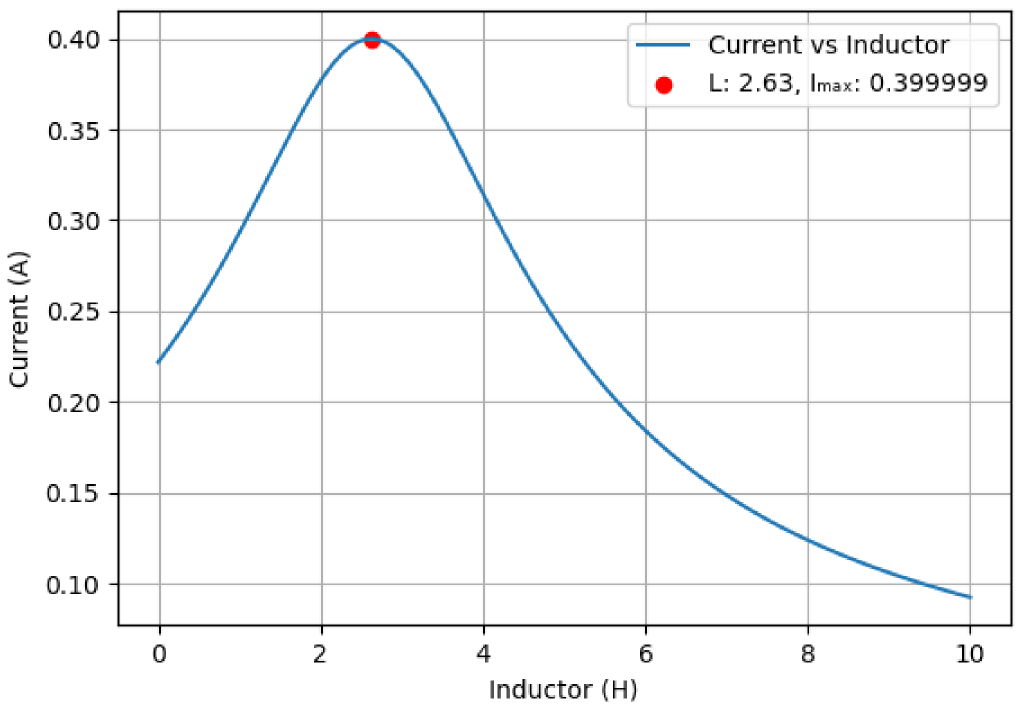


Figure 4.5: I vs L

4.4 The open loop transfer function of a unity gain negative feedback system is given by $G\left(s\right)=\frac{k}{s^2+4s-5}$. The range of k for which the system is stable,is (GATE EE 2022)

Solution:

Variable	**Description**	**value**
$G\left(s\right)$	Open loop transfer function	$\frac{k}{s^2+4s-5}$
1+G(s)	Characteristic equation	0

Table 4.8: A Table with input parameters

from Table4.8

Characteristic equation:

$$1 + G(s) = 0 \tag{4.48}$$

$$\implies 1 + \frac{k}{s^2 + 4s - 5} = 0 \tag{4.49}$$

$$\implies s^2 + 4s + (k - 5) = 0 \tag{4.50}$$

By routh table analysis, for a stable system:

$$\begin{array}{c|cc} s^2 & 1 & k-5 \\ s^1 & 4 & 0 \\ s^0 & \frac{4(k-5)-0}{4} & 0 \end{array}$$

$$\frac{4(k-5) - 0}{4} > 0 \tag{4.51}$$

$$k - 5 > 0 \tag{4.52}$$

$$\implies k > 5 \tag{4.53}$$

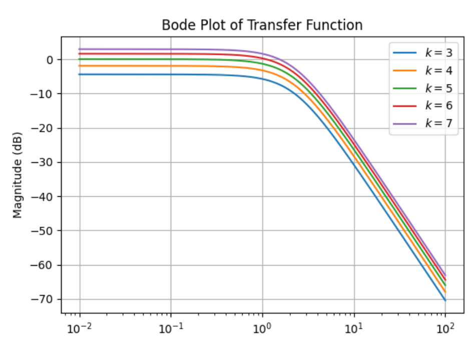


Figure 4.6: Graph showing $k < 5, k = 5, k > 5$

For an open transfer function to be stable, its magnitude in the bode plot should be positive for some positive frequency.

In the below graph we can observe that the above condition satisfies for k>5.

4.5 The signal flow graph of a system is shown. The expression for $\frac{Y(s)}{X(s)}$ is

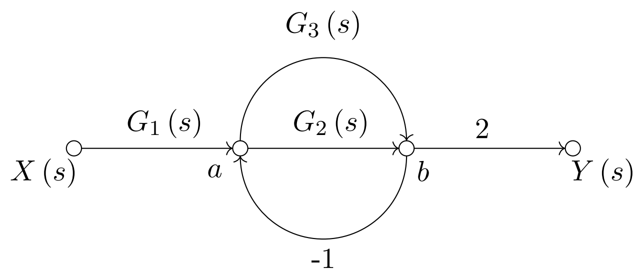


Figure 4.7: Signal Flow Graph of the System

(a) $\frac{2G_1(s)G_2(s)+2G_1(s)G_3(s)}{1+G_2(s)+G_3(s)}$

(b) $2 + G_1(s) + G_3(s) + \frac{G_2(s)}{1+G_2(s)}$

(c) $G_1(s) + G_3(s) - \frac{G_2(s)}{2+G_2(s)}$

(d) $\frac{2G_1(s)G_2(s)+2G_1(s)G_3(s)-G_1(s)}{1+G_2(s)+G_3(s)}$

(GATE 2022 IN Question 37)

Solution:

Parameter	Description	Value
$Y(s)$	Output node variable	
$X(s)$	Input node variable	
$\frac{Y(s)}{R(s)}$	Transfer function	?
P_1	Forward Path Gain a-b through $G_2(s)$	$2G_1(s)G_2(s)$
P_2	Forward Path Gain a-b through $G_3(s)$	$2G_1(s)G_3(s)$
Δ_1	Determinant of Forward Path a-b through $G_2(s)$	1
Δ_2	Determinant of Forward Path a-b through $G_3(s)$	1
L_1	Gain of Loop a-b through $G_2(s)$ and back	$-G_2(s)$
L_2	Gain of Loop a-b through $G_3(s)$ and back	$-G_3(s)$
Δ	Determinant of System	$1 + G_2(s) + G_3(s)$
n	Number of forward paths	2

Table 4.9: Variables Used

$$P_1 = (G_1(s))(G_2(s))(2) = 2G_1(s)G_2(s) \tag{4.54}$$

$$P_2 = (G_1(s))(G_3(s))(2) = 2G_1(s)G_3(s) \tag{4.55}$$

$$\Delta_1 = 1 - (0) = 1 \tag{4.56}$$

$$\Delta_2 = 1 - (0) = 1 \tag{4.57}$$

$$L_1 = -G_2(s) \tag{4.58}$$

$$L_2 = -G_3(s) \tag{4.59}$$

$$\Delta = 1 - (L_1 + L_2) = 1 + G_1(s) + G_2(s) \tag{4.60}$$

From 4.7, Using Mason's Gain Formula,

$$\frac{Y(s)}{X(s)} = \frac{\sum_{i=1}^{n} P_i \Delta_i}{\Delta} \tag{4.61}$$

$$= \frac{P_1\Delta_1 + P_2\Delta_2}{\Delta} \tag{4.62}$$

$$= \frac{2G_1(s)G_2(s)(1) + 2G_1(s)G_3(s)(1)}{1 + G_2(s) + G_3(s)} \tag{4.63}$$

$$\implies \frac{Y(s)}{X(s)} = \frac{2G_1(s)G_2(s) + 2G_1(s)G_3(s)}{1 + G_2(s) + G_3(s)} \tag{4.64}$$

4.6 The output of the system y(t) is related to its input x(t) according to the relation $y(t) = x(t)\, sin(2\pi t)$.This system is

(A) Linear and time-variant

(B) Non-Linear and time-invariant

(C) Linear and time-invariant

(D) Non-linear and time-variant

(GATE 2022 IN Question 14) **Solution:**

Symbol	**Value**	**Description**
$x(t)$		input signal
$y(t)$	$x(t)\, sin(2\pi t)$	output signal
τ		Time delay

Table 1: input parameters

From Table 1

$$y_1(t) \leftrightarrow x_1(t) \tag{4.65}$$

$$y_2(t) \leftrightarrow x_2(t) \tag{4.66}$$

$$ay_1(t) + by_2(t) \leftrightarrow ax_1(t) + bx_2(t) \tag{4.67}$$

$$ay_1(t) + by_2(t) = (ax_1(t) + bx_2(t))\, sin(2\pi t) \tag{4.68}$$

$\therefore$ satisfies principle of superposition

$$ky\left(t\right) \leftrightarrow kx\left(t\right) \tag{4.69}$$

$$ky\left(t\right) = k\left(x\left(t\right) sin\left(2\pi t\right)\right) \tag{4.70}$$

$\therefore$ satisfies principle of homogenity

$\therefore$ it is linear

Delay in input x(t):

$$y_1\left(t\right) = x\left(t-\tau\right) sin\left(2\pi t\right) \tag{4.71}$$

Delay in output y(t):

$$y\left(t-\tau\right) = x\left(t-\tau\right) sin\left(2\pi\left(t-\tau\right)\right) \tag{4.72}$$

$$y_2\left(t\right) = x\left(t-\tau\right) sin\left(2\pi\left(t-\tau\right)\right) \tag{4.73}$$

$$y_1\left(t\right) \neq y_2\left(t\right) \tag{4.74}$$

$\therefore$ it is time variant

$\therefore$ (A) linear and time variant

4.7 Two linear time-invariant systems with transfer functions

$$G_1(s) = \frac{10}{s^2 + s + 1}$$

and

$$G_2(s) = \frac{10}{s^2 + s\sqrt{10} + 10}$$

have unit step responses $y_1(t)$ and $y_2(t)$, respectively. Which of the following statements is/are true?

(a) $y_1(t)$ and $y_2(t)$ have the same percentage peak overshoot.

(b) $y_1(t)$ and $y_2(t)$ have the same steady state values.

(c) $y_1(t)$ and $y_2(t)$ have the same damped frequency of oscillation.

(d) $y_1(t)$ and $y_2(t)$ have the same 2% settling time.

(GATE 2022 EC Q50)

Solution: The general second-order transfer function is given by:

$$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.75}$$

After comparing the coefficients of $G_1(s)$ and $G_2(s)$, as $\zeta = \frac{1}{2}$ is less than 1, the

Parameter	Description	value
$X_1(s)$	input	$\frac{1}{s}$
$X_2(s)$	input	$\frac{1}{s}$
$G_1(s)$	transfer function	$\frac{10}{s^2+s+1}$
$G_2(s)$	transfer function	$\frac{10}{s^2+s\sqrt{10}+10}$
$y_1(t)$	unit step response	$-$
$y_2(t)$	unit step response	$-$
ω_n	natural frequency	$-$
ζ	damping ratio	$-$

Table 4.11: Given Parameters

Tranfer function	ω_n	ζ
$G_1(s)$	1	$\frac{1}{2}$
$G_1(s)$	$\sqrt{10}$	$\frac{1}{2}$

Table 4.12: Given Parameters

system is underdamped.

$$Y(s) = X(s) G(s) \tag{4.76}$$
$$= \frac{1}{s}\left(\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}\right) \tag{4.77}$$

Applying inverse laplace transform,

$$y(t) = 1 - \frac{e^{-\zeta\omega_n t}}{1-\zeta^2}\sin(\omega_d t+\phi) \tag{4.78}$$

where ω_d is the damped frequency of oscillation.

$$\omega_d = \omega_n\sqrt{1-\zeta^2} \tag{4.79}$$

The percentage peak overshoot (PO):

$$PO = \left(\frac{y_{\text{max}} - y_{\text{ss}}}{y_{\text{ss}}} \right) \times 100\% \tag{4.80}$$

y_{max} is obtained by differentiating (4.78) with respect to time and equating it to zero, substituting the value in (4.78),

$$y_{\text{max}} = 1 + \frac{1}{\sqrt{1-\zeta^2}} \tag{4.81}$$

y_{ss} is obtained by final value theorem,

$$y_{\text{ss}} = \lim_{s \to 0} sY(s) \tag{4.82}$$

$$= \lim_{s \to 0} s \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \frac{1}{s} \tag{4.83}$$

$$= 1 \tag{4.84}$$

Substituting the values of y_{max} and y_{ss} in (4.80),

$$PO = \frac{1}{\sqrt{1-\zeta^2}} \times 100\% \tag{4.85}$$

$y_1(t)$ and $y_2(t)$ have same ζ, they have same percentage peak overshoot.So, option (1) is correct.

The steady state value of $y(t)$ is given by final value theorem:

$$y_{1ss} = \lim_{s\to 0} sY_1(s) \tag{4.86}$$

$$= \lim_{s\to 0} s\frac{10}{s^2+s+1}\frac{1}{s} \tag{4.87}$$

$$= 10 \tag{4.88}$$

$$y_{2ss} = \lim_{s\to 0} sY_2(s) \tag{4.89}$$

$$= \lim_{s\to 0} s\frac{10}{s^2+s\sqrt{10}+10}\frac{1}{s} \tag{4.90}$$

$$= 1 \tag{4.91}$$

as both the unit step responses have different steady state values, option (2) is incorrect.

From (4.80), as ω_n is different for $y_1(t)$ and $y_2(t)$, they have different damped frequency of oscillation. Hence option (3) is incorrect.

Settling time T_s:

$$T_s = \frac{4}{\zeta\omega_n} \tag{4.92}$$

As, ω_n is different for $y_1(t)$ and $y_2(t)$, they have different 2% settling time, Hence option (4) is incorrect.

So, only option (1) is correct.

4.8 Consider a single-input-single-output (SISO) system with the transfer function

$$G_p(s) = \frac{2\,(s+1)}{\left(\frac{1}{2}s+1\right)\left(\frac{1}{4}s+1\right)}$$

where the time constants are in minutes. The system is forced by a unit step input at time $t = 0$. The time at which the output response reaches the maximum is ______ minutes (rounded off to two decimal places). (GATE CH 2022)

Solution:

Parameters	**Description**	**Value**
$y(t)$	Output response	
$G_p(s)$	Transfer function	$\frac{2(s+1)}{\left(\frac{1}{2}s+1\right)\left(\frac{1}{4}s+1\right)}$
$x(t)$	Input	$u(t)$
$X(s)$	Laplace transform of x(t)	$\frac{1}{s}$
$y'(t)$	$\frac{dy}{dt}$	

Table 4.13: Parameters

$$Y(s) = G_p(s)X(s) \tag{4.93}$$

$$= \frac{16\,(s+1)}{s\,(s+2)\,(s+4)} \tag{4.94}$$

$$= \frac{2}{s} + \frac{4}{s+2} - \frac{6}{s+4} \tag{4.95}$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \tag{4.96}$$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \tag{4.97}$$

From Laplace transforms (4.96) and (4.97), we get

$$y(t) = \left(2 + 4e^{-2t} - 6e^{-4t}\right) u(t) \tag{4.98}$$

For maximum value of $y(t)$,

$$y'(t) = 0 \tag{4.99}$$

$$\implies -8e^{-2t} + 24e^{-4t} = 0 \tag{4.100}$$

$$e^{2t} = 3 \tag{4.101}$$

$$\implies t = \frac{\ln 3}{2} \tag{4.102}$$

$$\approx 0.55 \tag{4.103}$$

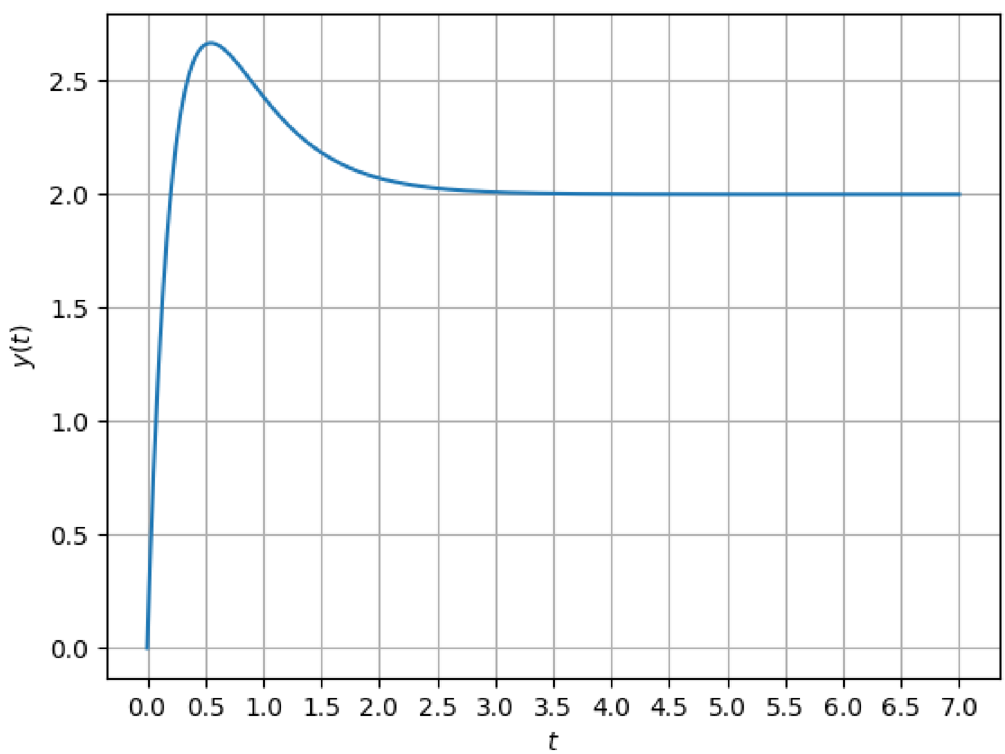
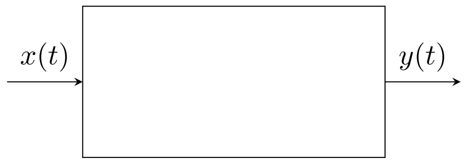


Figure 4.8: Plot of $y(t)$

4.9 Consider the system as shown below:



The system is described by the equation

$$y(t) = x(e^{-t}).$$

The system is:

(A) non-linear and causal.

(B) linear and non-causal.

(C) non-linear and non-causal.

(D) linear and causal.

(GATE EE 2022)

Solution:

Homogeneity Test:

For input $x_1(e^{-t})$, the output will be $y_1(t)$.

$$y_1(t) = x_1(e^{-t}) \tag{4.104}$$

Multiplying both sides by a scalar quantity 'a'

$$ay_1(t) = ax_1(e^{-t}) \tag{4.105}$$

For input $x_2(e^{-t})$, the output will be $y_2(t)$.

$$y_2(t) = x_2(e^{-t}) \tag{4.106}$$

Multiplying both sides by a scalar quantity 'b'

$$by_2(t) = bx_2(e^{-t}) \tag{4.107}$$

Adding the above equations we get:

$$ay_1(t) + by_2(t) = ax_1(e^{-t}) + bx_2(e^{-t}) \tag{4.108}$$

Let us assume that, for input $ax_1(e^{-t}) + bx_2(e^{-t})$, the output will be $y_3(t)$.

$$y_3(t) = ax_1(e^{-t}) + bx_2(e^{-t}) \tag{4.109}$$

But, $ay_1(t) + by_2(t) = ax_1(e^{-t}) + bx_2(e^{-t})$

Therefore;

$$y_3(t) = ay_1(t) + by_2(t) \tag{4.110}$$

The system satisfies homogeneity, as scaling the input scales the output.

Additivity Test:

From the given system;

$$y(t) = x(e^{-t}) \tag{4.111}$$

$$y(0) = x(e^0) \tag{4.112}$$

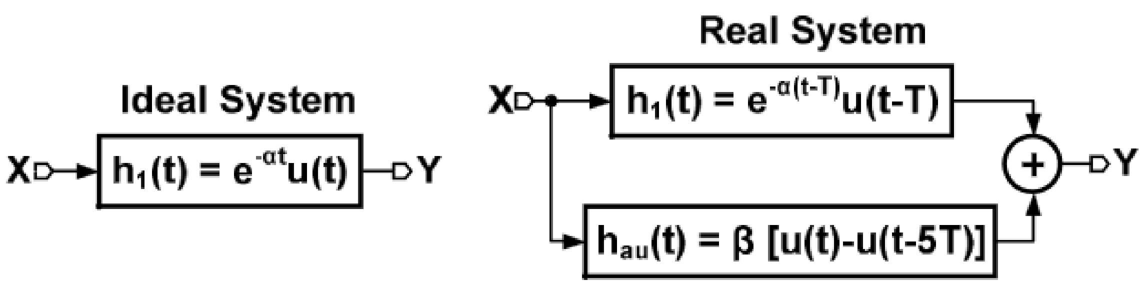
$$y(1) = x(e) = x(2.71) \tag{4.113}$$

So, the present value of output depends on the future value of input, indicating non-causality.

Therefore, the correct answer is: **(B) linear and non-causal**

4.10 The block diagrams of an ideal system and a real system with their impulse responses are shown below. An auxiliary path is added to the delayed impulse response in the real system.

For a unit impulse input $(x(t)) = \delta(t)$ to both systems, gain β is chosen such that $y(4T)$ is same for both systems. The value of β is:



$(A)\, e^{-3\alpha T}\left(1 - e^{-2\alpha T}\right)$

$(B) - e^{-\alpha T}\left(1 - e^{-3\alpha T}\right)$

$(C) - e^{-3\alpha T}\left(1 - e^{-\alpha T}\right)$

$(D)\, e^{-2\alpha T}\left(1 - e^{-2\alpha T}\right)$

(GATE BM 2022)

Solution: For both signals to be equal at t = 4T:

No.	Output	Function
1	y_I	$e^{-\alpha t}u\left(t\right)$
2	y_R	$\beta\left(u\left(t\right)-u\left(t-5T\right)\right)+e^{-\alpha(t-T)}u\left(t-T\right)$

Table 4.14: Values

$$e^{-\alpha 4T}u\left(4T\right)=\left[\beta\left(u\left(4T\right)-u\left(-T\right)\right)+e^{-\alpha(3T)}u\left(3T\right)\right] \tag{4.114}$$

$$e^{-\alpha 4T}=\beta+e^{-\alpha 3T} \tag{4.115}$$

$$\implies \beta=-e^{-3\alpha T}\left(1-e^{-\alpha T}\right) \tag{4.116}$$

4.11 In a unity-gain feedback control system, the plant $P(s) = \frac{0.001}{s(2s+1)(0.01s+1)}$ is controlled by a lag compensator $C(s) = \frac{s+10}{s+0.1}$ The slope (in dB/decade) of the asymptotic Bode magnitude plot of the loop gain at $\omega = 3$rad/s is ________ (in integer) (GATE 2022 IN)

Solution:

Parameter	Description	Value
$P(s)$	Plant Transfer Function	$\frac{0.001}{s\left(\frac{s}{0.5}+1\right)\left(\frac{s}{100}+1\right)}$
$C(s)$	Lag Compensator	$\frac{100\left(\frac{s}{10}+1\right)}{\frac{s}{0.1}+1}$
$T(s)$	Loop gain	$P(s)C(s)$
ω	Angular Frequency	3rad/s

Table 4.15: Given Parameters list

$$|T(s)| = \frac{0.1\left(\frac{s}{10}+1\right)}{s\left(\frac{s}{0.5}+1\right)\left(\frac{s}{100}+1\right)\left(\frac{s}{0.1}+1\right)} \tag{4.117}$$

Here, $10, 0.5, 100, 0.1$ are corner frequencies of loop gain L(s)

Corner Frequency	**Description**	**Change in slope**
10	Zero	$20dB/dec$
0.1	Pole	$-20dB/dec$
0.5	Pole	$-20dB/dec$
100	Pole	$-20dB/dec$

Table 4.16: Caption

$$Gain(K) = \lim_{s\to 0} sT(s) \tag{4.118}$$

$$K = 0.1 \tag{4.119}$$

$$|T(s)| = 20\log_{10} K \tag{4.120}$$

$$= -20dB \tag{4.121}$$

$$T(\omega) = \begin{cases} -20log_{10}(w) & \omega < 0.1 \\ -20.0\left(2log_{10}(w) - 0.1\right) & 0.1 \leq \omega < 0.5 \\ -20.0\left(3log_{10}\omega - 0.1 + log_{10}0.5\right) & 0.5 \leq \omega < 10 \\ -20.0\left(2log_{10}\omega + 0.9 + log_{10}0.5\right) & 10 \leq \omega < 100 \\ -20.0\left(3log_{10}\omega - 1.9 + log_{10}0.5\right) & \omega \geq 100 \end{cases}$$

Slope of Bode magnitude plot (at $\omega = 3$) = -60 dB/decade

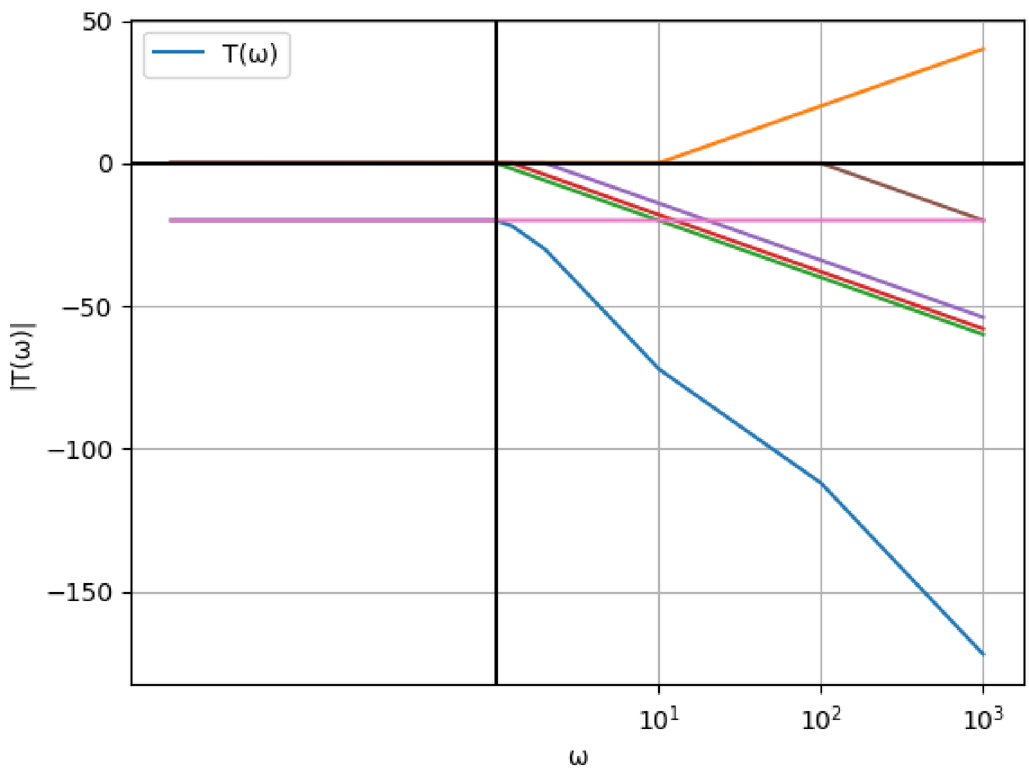


Figure 4.9: Pink Line = Bode magnitude plot of loop gain

4.12 The voltage at the input of an AC-DC rectifier is given by $v(t) = 230\sqrt{2}\sin\omega t$, where $\omega = 2\pi \times 50\text{rad/s}$. The input current drawn by the rectifier is given by

$$i(t) = 10\sin\left(\omega t - \frac{\pi}{3}\right) + 4\sin\left(3\omega t - \frac{\pi}{6}\right) + 3\sin\left(5\omega t - \frac{\pi}{3}\right)$$

The power input, (rounded off to two decimal places), is________lag. (Gate 2022 EE 33Q)

Solution:

$$P_{avg} = \frac{1}{T} \int_0^T V(t)\, I(t)\, dt \tag{4.122}$$

For current sources of the form $I(t)=I_0+I_1(t)+...+I_n(t)$

$$P_{avg} = \frac{1}{T} \sum_1^n \int_0^T V(t)\, I_n(t)\, dt \tag{4.123}$$

$$P_{avg} = \frac{1}{T} \sum_1^n \int_0^T V_{pk} sin(\omega t)\, I_{pk(n)} \sin(\omega t + \varphi)\, dt \tag{4.124}$$

$$P_{avg} = \sum_0^n \frac{v_{pk} I_{(n)pk}}{2} \cos\varphi \tag{4.125}$$

For a sine wave signal $V_{pk} = V_{rms}\sqrt{2}$

$$P_{avg} = \sum_0^n (v_{rms}) \left(I_{(n)rms}\right) \cos\varphi \tag{4.126}$$

$$\text{Power Factor} = \frac{\sum_0^n I_{(n)rms} \cos\varphi}{I_{rms}} \tag{4.127}$$

$$I_{rms} = \sqrt{\left(\frac{10}{\sqrt{2}}\right)^2 + \left(\frac{4}{\sqrt{2}}^2\right) + \left(\frac{3}{\sqrt{2}}\right)^2} = 7.905A \tag{4.128}$$

The rms value of fundamental value of current

$$\left(I_{(1)rms}\right) = \sqrt{\left(\frac{10}{\sqrt{2}}\right)^2} \tag{4.129}$$

$$\varphi = 30^{\circ} \tag{4.130}$$

$$\text{Power Factor} = \frac{\frac{10}{\sqrt{2}}\cos 30}{7.905} \tag{4.131}$$

$$\implies 0.4473 \tag{4.132}$$

4.13 The transfer function of a system is:

$$\frac{(s+1)(s+3)}{(s+5)(s+7)(s+9)}$$

In the state-space representation of the system, the minimum number of state variables (in integer) necessary is_____.

(GATE IN 2022)

Solution:

From Table 4.18

$$H(s) = \frac{(s+1)(s+3)}{(s+5)(s+7)(s+9)} \tag{4.133}$$

$$H(s) = \frac{P}{s+5} + \frac{Q}{s+7} + \frac{R}{s+9} \tag{4.134}$$

$$(s+1)(s+3) = P(s+7)(s+9) + Q(s+5)(s+9) + R(s+5)(s+7) \tag{4.135}$$

By solving equation (4.135) , we get

P = 1

Q = -6

R = 6

$$\implies H(s) = \frac{1}{s+5} - \frac{6}{s+7} + \frac{6}{s+9} \tag{4.136}$$

(4.137)

The state-space representation of the system is given by:

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + Bu(t) \tag{4.138}$$

$$\mathbf{y}(t) = C\mathbf{x}(t) + Du(t) \tag{4.139}$$

$$H(s) = \frac{Y(s)}{U(s)} = C\left(sI - A\right)^{-1} B + D \tag{4.140}$$

Comparing the coefficients:

$$A = \text{coefficient of } s \text{ in } (sI - A)^{-1} \tag{4.141}$$

$$B = \text{coefficient of } U(s) \tag{4.142}$$

$$C = \text{coefficient of } Y(s) \tag{4.143}$$

$$D = \text{constant term} \tag{4.144}$$

The denominator $(s+5)(s+7)(s+9)$ suggests that the system has three poles. Thus, we'll have a third-order state-space model, and A will be a 3×3matrix.

$$(s+5)(s+7)(s+9) = s^3 + 21s^2 + 143s + 315 \tag{4.145}$$

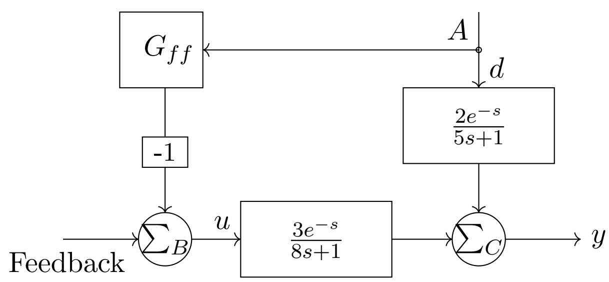
$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -21 & -143 & -315 \end{pmatrix} \tag{4.146}$$

$$\tag{4.147}$$

A is a 3×3 matrix, then the characteristic polynomial will have a degree equal to the size of A, which is 3.

Therefore, the system order, and hence the minimum number of state variables, will be 3.

4.14 The appropriate feedforward compensator, G_{ff}, in the shown block diagram is



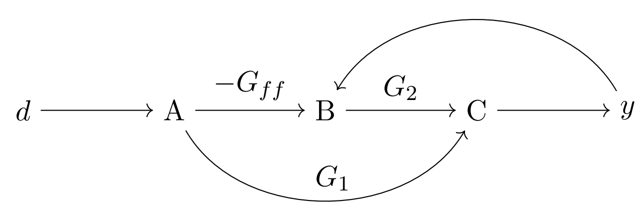
(GATE CH 22)

Solution:

In an ideal system, the output y must be independent of the disturbance d. This means, the transfer function

$$\frac{Y(s)}{D(s)} = 0$$

The signal flow graph for this system is given by



(a) **Theory:** For such a system, Mason's Gain formula can be used. From Table 4.19

$$H = \sum_i \frac{\Delta_i P_i}{\Delta} \tag{4.148}$$

If L_i denotes the loop gain of ith Loop

$$\Delta = 1 - \sum_i L_i + \sum_i \sum_j L_i L_j - ... \tag{4.149}$$

Δ_i is the value of Δ without the nodes contained by the ith path.

(b) Here, there are 2 forward paths

$$d \longrightarrow \mathrm{A} \xrightarrow{-G_{ff}} \mathrm{B} \xrightarrow{G_2} \mathrm{C} \longrightarrow y$$

$$d \longrightarrow \mathrm{A} \xrightarrow{G_1} \mathrm{C} \longrightarrow y$$

For these paths,

$$P_1 = -G_2 G_{ff} \tag{4.150}$$

$$P_2 = G_1 \tag{4.151}$$

$$\Delta_1 = \Delta_2 = 1 - (0) \tag{4.152}$$

$$\Delta = 1 - (0) = 1 \tag{4.153}$$

From (4.148) and Table 4.19

$$H = G_1 - G_2 G_{ff} \tag{4.154}$$

Since $H = 0$,

$$G_1 - G_2 G_{ff} = 0 \tag{4.155}$$

$$\Longrightarrow G_{ff} = \frac{G_1}{G_2} = \frac{2}{3}\frac{8s+1}{5s+1} \tag{4.156}$$

4.2. 2021

4.1 A sinusoidal message signal having root mean square value of 4V and frequency of 1 kHz fed to a phase modulator with phase deviation constant 2 rad/volt. If the carrier signal is $c(t) = 2\cos\left(2\pi 10^6 t\right)$, the maximum instantaneous frequency of the phase modulated signal (rounded off to one decimal place) is ____ Hz. (GATE 2021 EC)

Solution:

Phase Modulation Signal Proof:

let e_m and e_c be message and carrier signals, $2\pi f_m$ and $2\pi f_c$ be radial frequencies and A_m and A_c be their amplitudes respectively. Then,

$$e_m = A_m \cos(2\pi f_m t) \tag{4.157}$$

$$e_c = A_c \sin(2\pi f_c t) \tag{4.158}$$

On rewriting the equation 4.158

$$e = E_c \sin(\theta) \tag{4.159}$$

$$\theta = 2\pi f_c t + k_p e_m \tag{4.160}$$

$$= 2\pi f_c t + k_p E_m \cos(2\pi f_m t) \tag{4.161}$$

$$m_p = k_p E_m \tag{4.162}$$

$$\theta = 2\pi f_c t + m_p \cos(2\pi f_m t) \tag{4.163}$$

$$\implies s(t) = E_c \sin\left(2\pi f_c t + \underbrace{m_p \cos(2\pi f_m t)}_{\theta_i(t)}\right) \tag{4.164}$$

$$m(t)_{rms} = 4V \tag{4.165}$$

$$A_m = 4\sqrt{2} \tag{4.166}$$

From Table 4.20, eq (4.165) and eq (4.166)

$$m(t) = 4\sqrt{2}\sin\left(2\pi 10^3 t\right) \tag{4.167}$$

$$\tag{4.168}$$

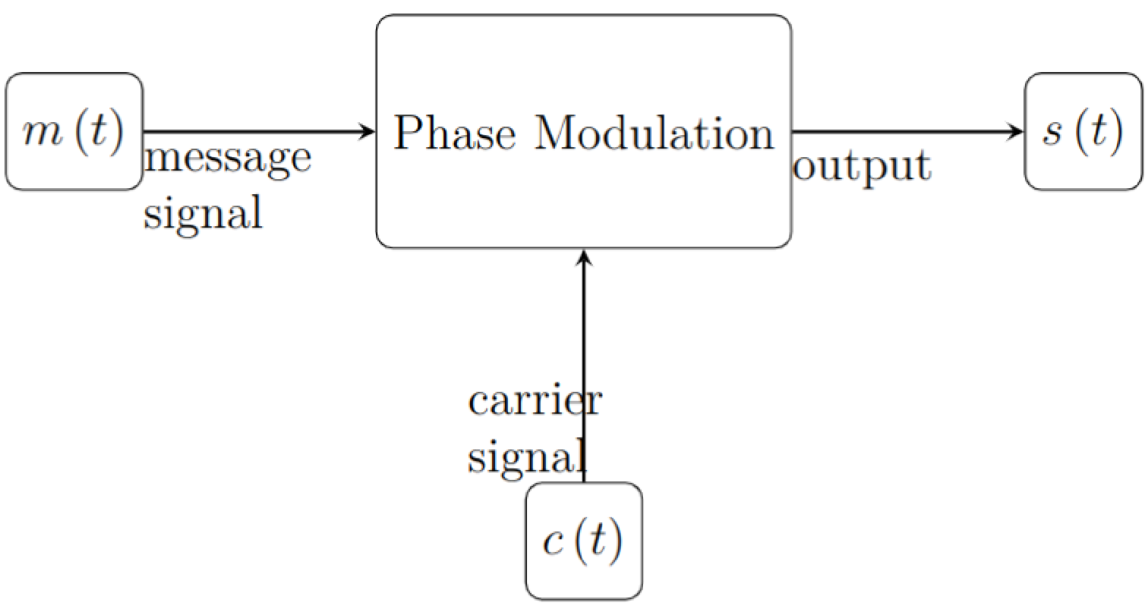


Figure 4.10: Block diagram of phase modulation

From Table 4.20, 4.21 and using eq (4.164) instantaneous frequency is given as,

$$\begin{aligned}
f_i(t) &= f_c + \frac{1}{2\pi}\frac{d}{dt}\theta_i(t) && (4.169)\\
&= f_c + \frac{1}{2\pi}\frac{d}{dt}\left[k_p m(t)\right] && (4.170)\\
&= f_c + \frac{1}{2\pi}\frac{d}{dt}\left(4\sqrt{2}\sin\left(2\pi 10^3 t\right)\right) && (4.171)\\
&= f_c + \frac{2}{2\pi}4\sqrt{2}\left(2\pi 10^3\right)\left(\cos\left(2\pi 10^3 t\right)\right) && (4.172)\\
&= 1000 + 8\sqrt{2}\text{ x }10^3\cos\left(2\pi 10^3 t\right) && (4.173)
\end{aligned}$$

Thus,

$$\implies f_{i_{max}} = 1011313.7\,Hz \tag{4.174}$$

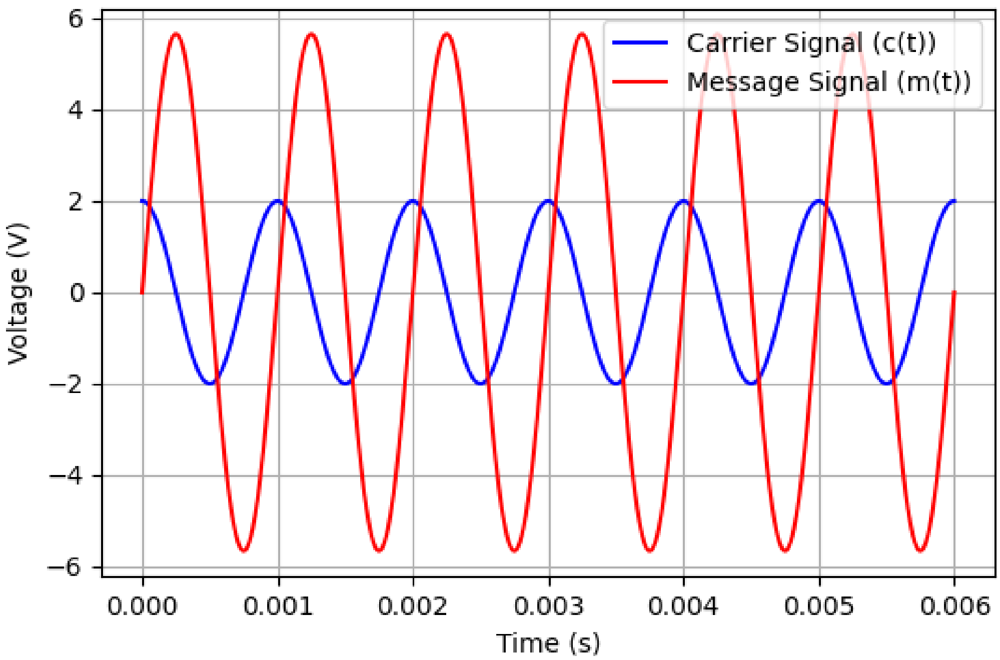


Figure 4.11: plot of $m(t)$ and $c(t)$

4.2 Two discrete-time linear time-invarient systems with impulse responses $h_1[n] = \delta[n-1] + \delta[n+1]$ and $h_2[n] = \delta[n] + \delta[n-1]$ are connected in cascade, where $\delta[n]$ is the Kronecker delta. The impulse response of the cascaded system is

(a) $\delta[n-2] + \delta[n+1]$

(b) $\delta[n-1]\delta[n] + \delta[n+1]\delta[n-1]$

(c) $\delta[n-2] + \delta[n-1] + \delta[n] + \delta[n+1]$

(d) $\delta[n]\delta[n-1] + \delta[n-2]\delta[n+1]$

(GATE 2021 EE)

Solution:

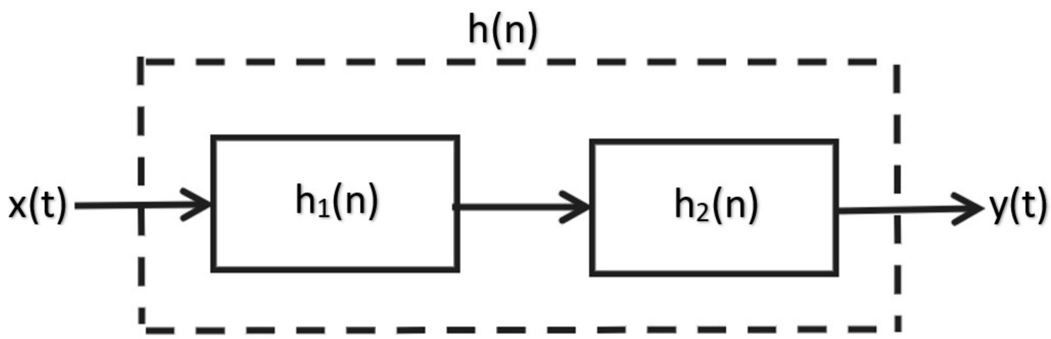


Figure 4.12: Block Diagram

From the Z-transformation pairs,

$$\delta[n] \xleftrightarrow{\mathcal{Z}} 1 \tag{4.175}$$

$$x\left(n-k\right) \xleftrightarrow{\mathcal{Z}} z^{-k} X\left(z\right) \tag{4.176}$$

$$x_1\left(n\right) * x_2\left(n\right) \xleftrightarrow{\mathcal{Z}} X_1\left(z\right) X_2\left(z\right) \tag{4.177}$$

If $h_1\left(n\right)$ and $h_2\left(n\right)$ are cascade connected then the resultant impulse can be given by:

$$h\left(n\right) = h_1\left(n\right) * h_2\left(n\right) \tag{4.178}$$

$$\implies H\left(z\right) = H_1\left(z\right) H_2\left(z\right) \tag{4.179}$$

$$H\left(z\right) = \left(z^{-1} + z\right)\left(1 + z^{-1}\right) \tag{4.180}$$

$$= \left(z^{-1} + z^{-2} + z + 1\right), \quad |z| \neq 0 \tag{4.181}$$

Using the Z-transformation pairs to find the the inverse Z-transform,

$$h\left(n\right) = \delta[n-2] + \delta[n-1] + \delta[n] + \delta[n+1] \tag{4.182}$$

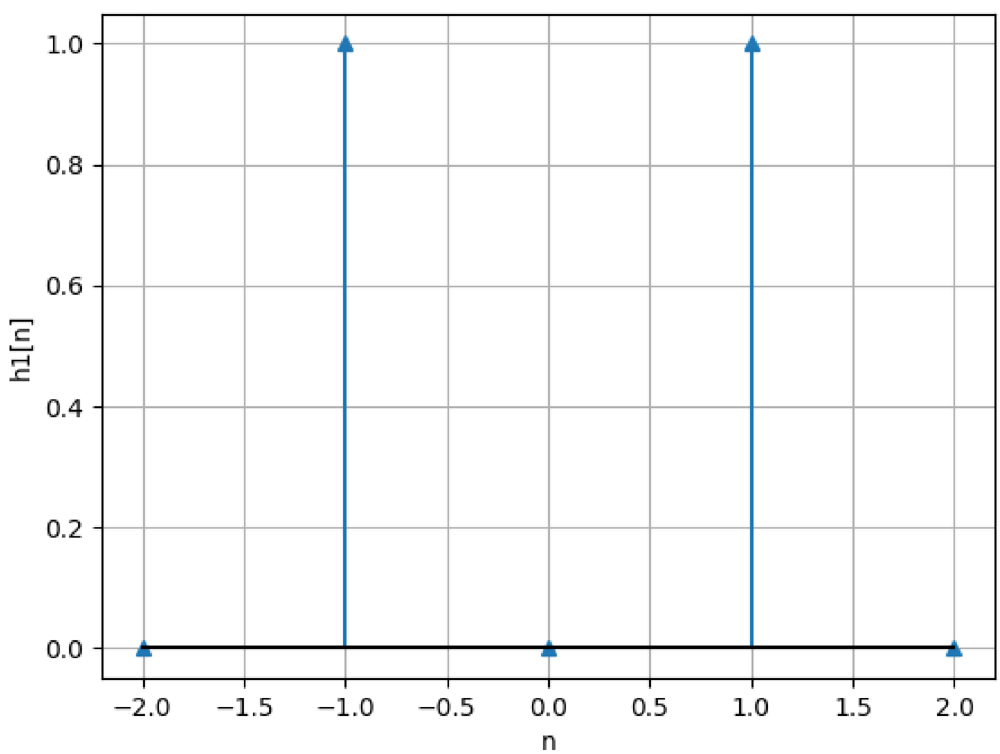


Figure 4.13: $h_1\left(n\right)$ vs n graph

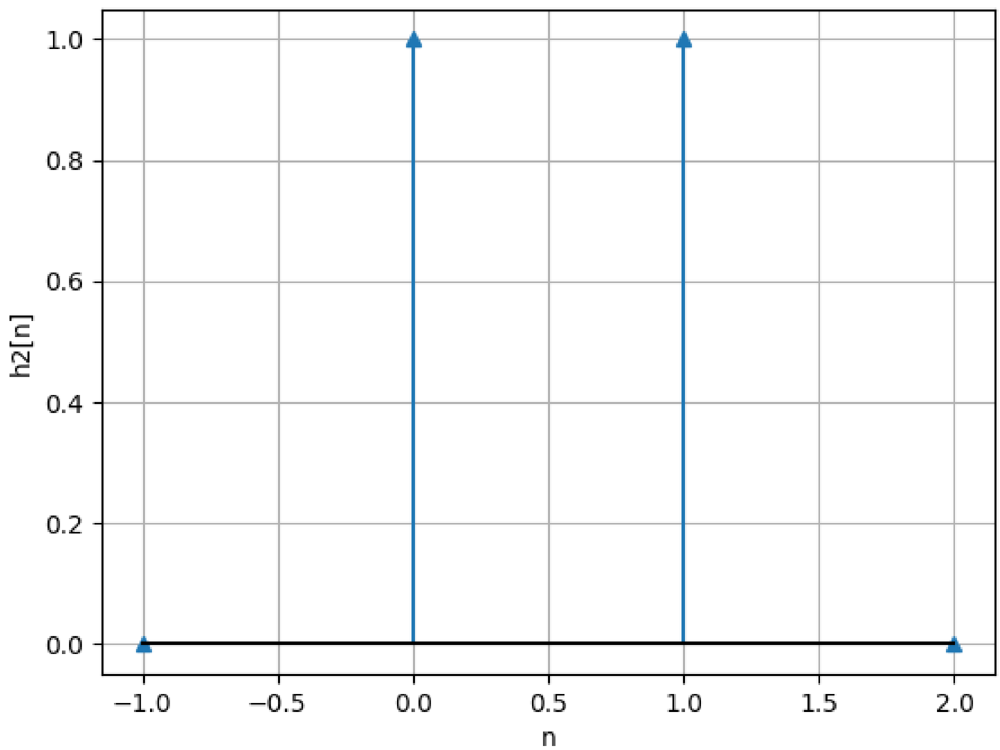


Figure 4.14: $h_2(n)$ vs n graph

4.3 Consider a superheterodyne receiver tuned to 600 kHz. If the local oscillator feeds a 1000 kHz signal to the mixer, the image frequency (in integer) is ______ kHz. (GATE EC 2021)

Solution: Let f_x be the intermediate frequency given by $|f_l - f_r|$.

$$f_x = |1000 - 600| = 400\,\text{kHz} \tag{4.183}$$

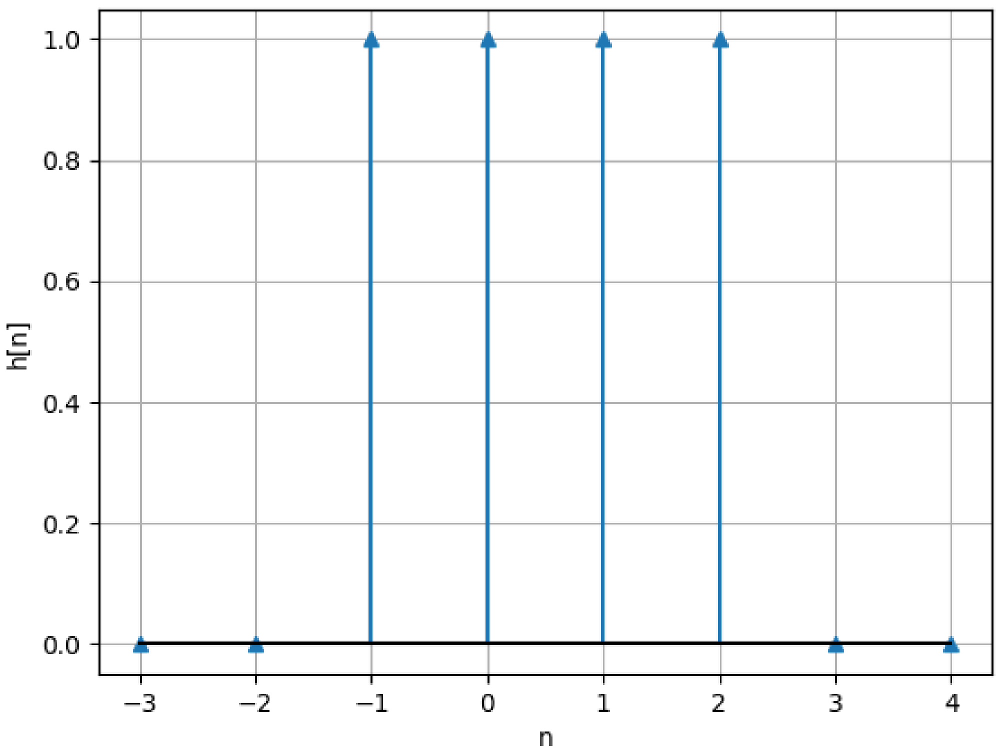


Figure 4.15: $h\left(n\right)$ vs n graph

From the above diagram, we can observe that:

$$f_i = f_r + 2(f_x) = 600 + 2(400) = 1400\,\text{kHz} \tag{4.184}$$

Therefore the Image frequency is **1400 kHz**

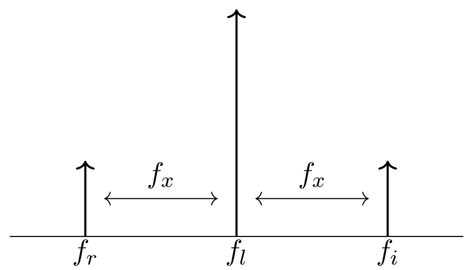


Figure 4.16: Diagram

4.4 Consider a unity feedback system with closed loop transfer function

$$\frac{C(s)}{R(s)} = \frac{s+90}{s^2+10s+90}$$

The steady state error with respect to a unit ramp input is ______ . (GATE 2021 BM)

Solution:

$$\frac{C(s)}{R(s)} = \frac{s+90}{s^2+10s+90} \tag{4.185}$$

where $C(s)$ is the output and $R(s)$ is the input. Given that input is unit ramp function:

$$r(t) = tu(t) \tag{4.186}$$

$$\implies R(s) = \frac{1}{s^2} \tag{4.187}$$

$$\implies C(s) = \frac{s+90}{s^2(s^2+10s+90)} \tag{4.188}$$

$$E(s) = R(s) - C(s) \tag{4.189}$$

$$= \frac{s^2+9s}{s^2(s^2+10s+90)} \tag{4.190}$$

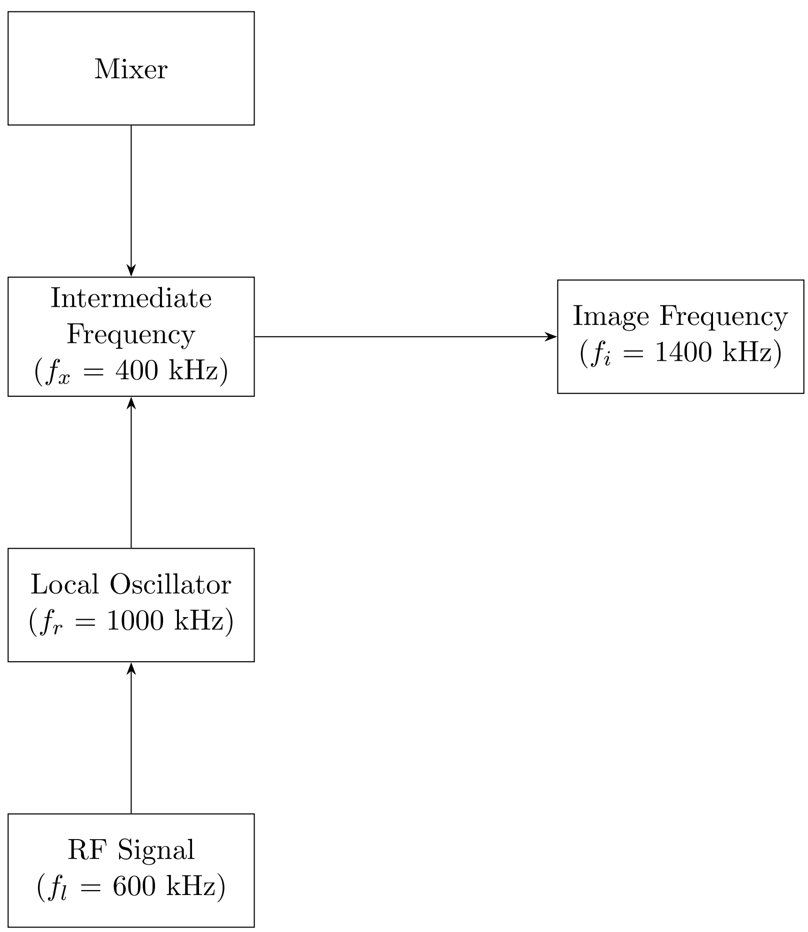


Figure 4.17: Superheterodyne Receiver Block Diagram

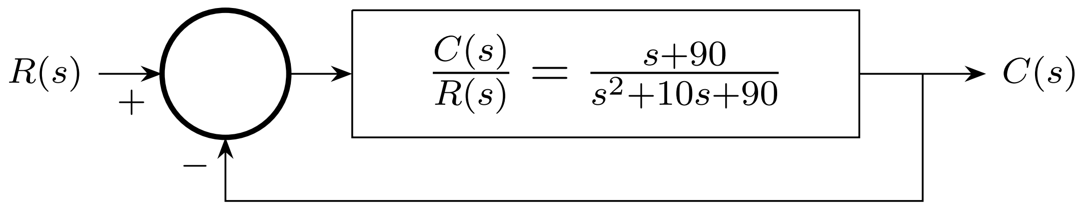


Figure 4.18: Block Diagram of the System

Steady state error is:

$$\lim_{s\to 0} sE(s) = \frac{s+9}{s^2+10s+90} \tag{4.191}$$
$$= \frac{1}{10} \tag{4.192}$$

$\therefore$ steady state error for unit ramp input is 0.1.

$$C(s) = \frac{s+90}{s^2(s^2+10s+90)} \tag{4.193}$$
$$= -\frac{1}{10s} + \frac{1}{s^2} + \frac{s}{10(s^2+10s+90)} \tag{4.194}$$
$$= -\frac{1}{10s} + \frac{1}{s^2} + \frac{s+5}{(s+5)^2+65} - \frac{1}{2}\left(\frac{1}{(s+5)^2+65}\right) \tag{4.195}$$

$$c(t) = u(t)\left(-\frac{1}{10} + t + \frac{e^{-5t}}{10}\cos\left(\sqrt{65}t\right) - \frac{e^{-5t}}{2\sqrt{65}}\sin\left(\sqrt{65}t\right)\right) \tag{4.196}$$

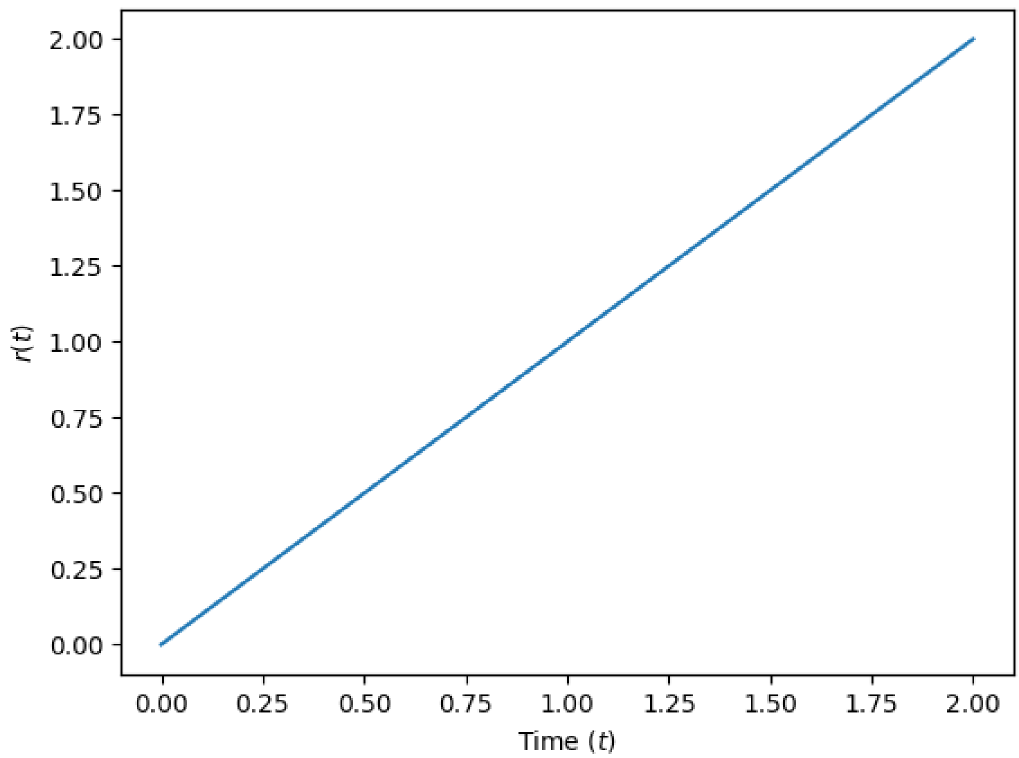


Figure 4.19: Plot of $r\left(t\right)$ vs t

$$
\begin{aligned}
E\left(s\right) &= R\left(s\right) - C\left(s\right) && (4.197)\\
\implies e\left(t\right) &= r\left(t\right) - c\left(t\right) && (4.198)\\
&= u\left(t\right)\left(\frac{1}{10} - \frac{e^{-5t}}{10}\cos\left(\sqrt{65}t\right) + \frac{e^{-5t}}{2\sqrt{65}}\sin\left(\sqrt{65}t\right)\right) && (4.199)
\end{aligned}
$$

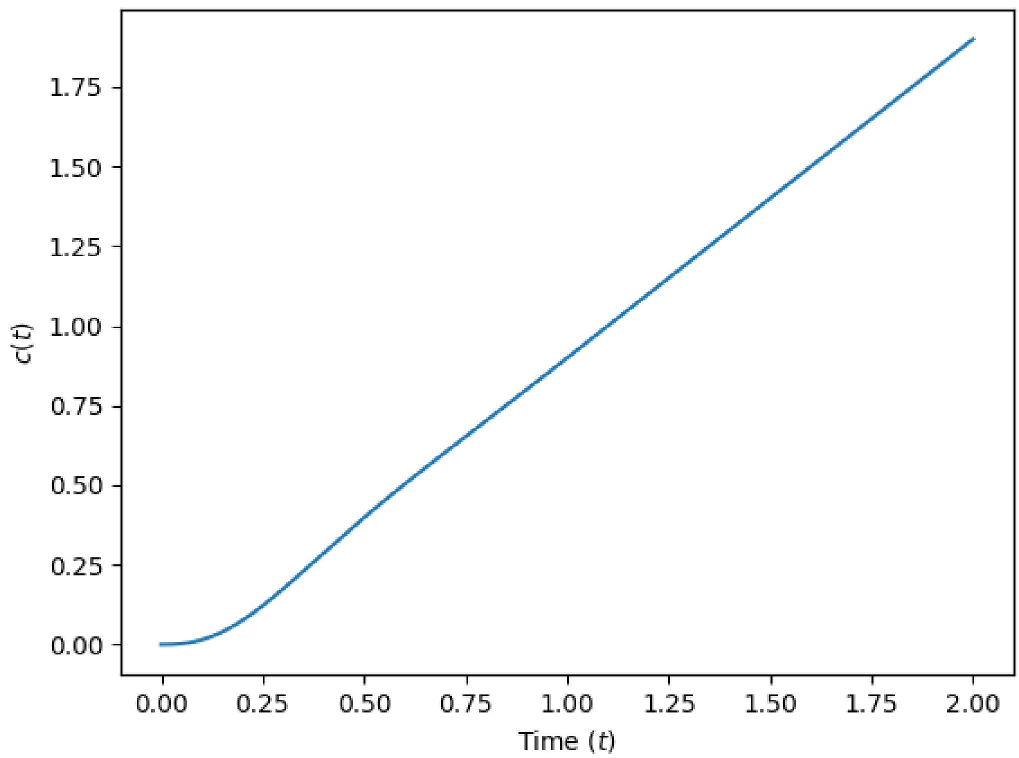


Figure 4.20: Plot of $c(t)$ vs t

$$\text{Feedback Gain } = \frac{\frac{C(s)}{R(s)}}{1+\frac{C(s)}{R(s)}} \tag{4.200}$$

$$= \frac{s+90}{s^2+11s+180} \tag{4.201}$$

4.5 A unit step input is applied to a system with impulse response $\text{H}(s) = \frac{1-\frac{s}{\omega_z}}{1+\frac{s}{\omega_p}}$ at t=0. The output of the system $y(t)$ at t=0^+ is:

a) 1

b) $-\frac{\omega_z}{\omega_p}$

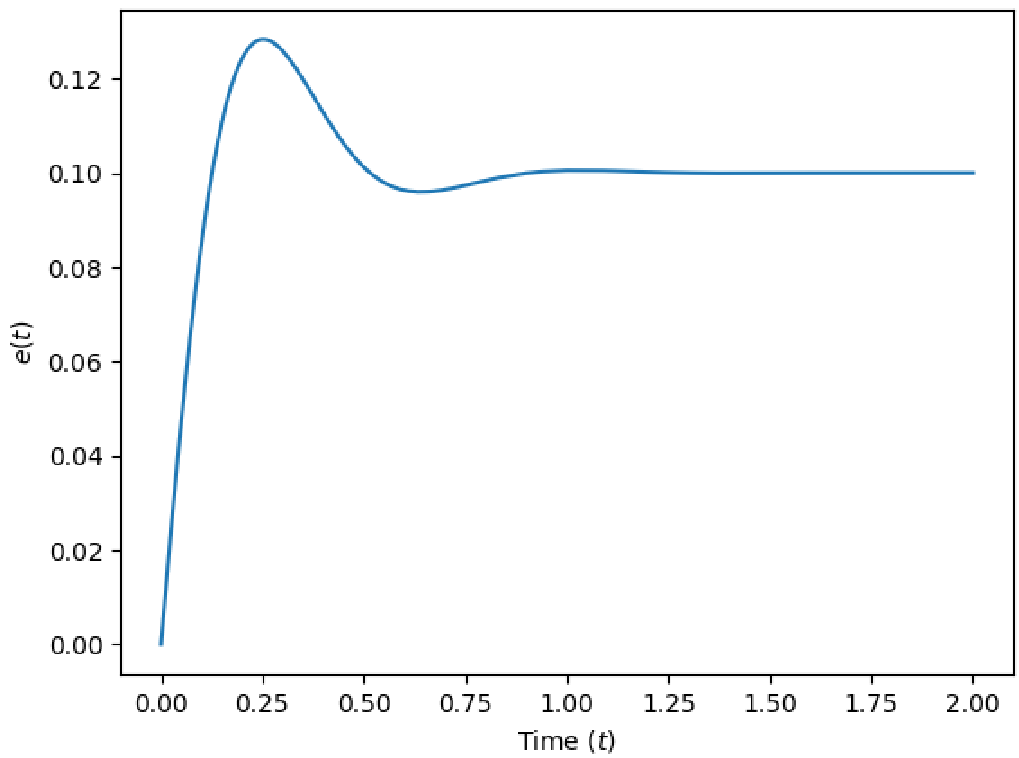


Figure 4.21: Plot of $e(t)$ vs t

c) $-\frac{\omega_p}{\omega_z}$

d) 0

(GATE 2021 BM)

Solution: Given, input signal

$$x(t) = u(t)$$

$$Y(s) = H(s).X(s) \tag{4.202}$$

$$= \frac{1}{s}.\frac{1-\frac{s}{\omega_z}}{1+\frac{s}{\omega_p}} \tag{4.203}$$

By initial value theorem,

$$y\left(t=0^+\right) = \lim_{s\to\infty} sY(s) \tag{4.204}$$

$$= \lim_{s\to\infty} \frac{1-\frac{s}{\omega_z}}{1+\frac{s}{\omega_p}} \tag{4.205}$$

$$= \lim_{s\to\infty} \frac{\frac{1}{s}-\frac{1}{\omega_z}}{\frac{1}{s}+\frac{1}{\omega_p}} \tag{4.206}$$

$$= -\frac{\omega_p}{\omega_z} \tag{4.207}$$

Hence, option (c) is correct

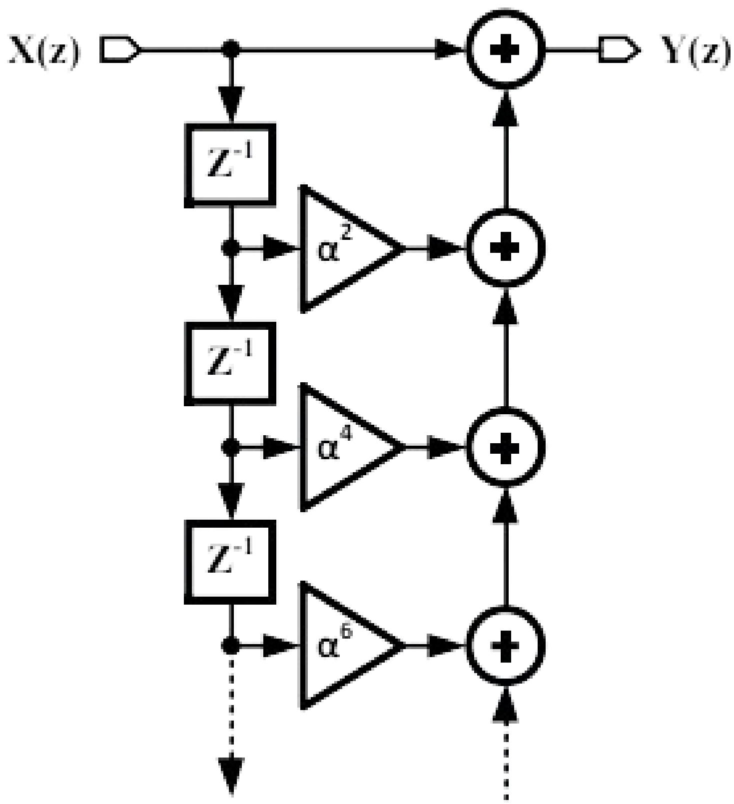
4.6 In the block diagram shown below, an infinite tap FIR filter with transfer function $H(z) = \frac{Y(z)}{X(z)}$ is realized. If $H(z) = \frac{1}{1-0.5z^{-1}}$.

the value of α is (GATE 2021 BM)

Solution:

From diagram we have:

$$Y(z) = X(z)\left(\sum_{n=0}^{\infty}\left(z^{-1}\alpha^2\right)^n\right) \tag{4.208}$$



Dividing by $X(z)$ in both sides:

$$\frac{Y(z)}{X(z)} = \sum_{n=0}^{\infty} \left(z^{-1}\alpha^2\right)^n \tag{4.209}$$

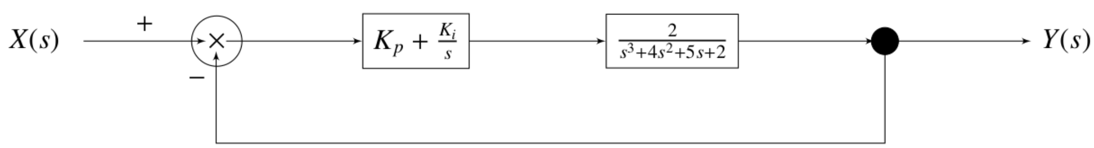
$$\implies H(z) = \sum_{n=0}^{\infty} \left(z^{-1}\alpha^2\right)^n \tag{4.210}$$

$$\frac{1}{1-0.5z^{-1}} = \sum_{n=0}^{\infty} \left(z^{-1}\alpha^2\right)^n \tag{4.211}$$

$$\frac{1}{1-0.5z^{-1}} = \frac{1}{1-z^{-1}\alpha^2} \tag{4.212}$$

$$\implies \alpha = \frac{1}{\sqrt{2}} \tag{4.213}$$

4.7 A unity feedback system that uses proportional-integral (PI) control is shown in the figure. The stability of the overall system is controlled by tuning the PI control pa-



rameters K_p and K_i. The maximum value of K_i that can be chosen so as to keep the overall system stable or, in the worst case, marginally stable (*rounded off to three decimal places*) is? (GATE EC 2021)

Solution:

From table 4.24, the characteristic equation is given as:

$$1 + \left(K_p + \frac{K_i}{s}\right)\left(\frac{2}{s^3 + 4s^2 + 5s + 2}\right) = 0 \tag{4.214}$$

$$s^4 + 4s^3 + 5s^2 + (2 + 2K_p)s + 2K_i = 0 \tag{4.215}$$

For the system to be stable, there must be no sign changes in the first column of the routh array for the above equation. From (4.215)

$$\begin{array}{c|ccc} s^4 & 1 & 5 & 2K_i \\ s^3 & 4 & (2+2K_p) & 0 \\ s^2 & \frac{18-2K_p}{4} & 2K_i & 0 \\ s^1 & \frac{\left(\frac{18-2K_p}{4}\right)(2+2K_p)-8K_i}{\frac{18-2K_p}{4}} & 0 & 0 \\ s^0 & 2K_i & 0 & 0 \end{array} \tag{4.216}$$

(4.217)

$$\frac{18 - 2K_p}{4} > 0 \tag{4.218}$$

$$\implies K_p < 9 \tag{4.219}$$

$$\frac{\left(\frac{18-2K_p}{4}\right)(2 + 2K_p) - 8K_i}{\frac{18-2K_p}{4}} > 0 \tag{4.220}$$

$$K_i > 0 \tag{4.221}$$

For marginal stability, assuming 3 cases while maximising K_i and checking if the above inequalities hold.

(a) $K_p = 9$

$$\left(\lim_{K_p \to 9^-} \frac{\left(\frac{18-2K_p}{4}\right)(2 + 2K_p) - 8K_i}{\frac{18-2K_p}{4}} > 0\right) \cap (K_i > 0) \tag{4.222}$$

$$\left(\lim_{K_p \to 9^-} -8K_i > 0\right) \cap (K_i > 0) \tag{4.223}$$

$$\implies K_p = 9, \forall K_i \epsilon(\phi) \tag{4.224}$$

(b) $K_i = 0$

$$\left(\left(\frac{18 - 2K_p}{4}\right)(2 + 2K_p) > 0\right) \cap (K_p < 9) \tag{4.225}$$

$$\implies K_i = 0, \forall K_p \epsilon(-1, 9) \tag{4.226}$$

(c) $\frac{\left(\frac{18-2K_p}{4}\right)(2+2K_p)-8K_i}{\frac{18-2K_p}{4}} = 0$

$$\left(\frac{18-2K_p}{4}\right)(2+2K_p) = 8K_i \tag{4.227}$$

$$-K_p^2 + 8K_p + 9 = 8K_i \tag{4.228}$$

Vertex ($K_p = 4$) satisfies (4.219):

$$K_i = 3.125 \forall (K_p = 4, K_i > 0) \tag{4.229}$$

Based on the three cases for marginal stability, the maximum value of K_i is 3.125, for $K_p = 4$.

(a) Verification by plotting roots of characteristic equation:

If real part of atleast 1 root is equal to zero and for the rest are less than or equal to zero, then the system is marginally stable.

(b) Verification by Nyquist diagrams:

From 4.24, if $P = 0$ and $-1 + 0j$ is neither bounded nor unbounded by the contour, then the system is marginally stable. For P:

$$s^4 + 4s^3 + 5s^2 + 2s = 0 \tag{4.230}$$

$$(s+1)^2(s+2) = 0 \tag{4.231}$$

$$\implies P = 0 \tag{4.232}$$

$$\implies Z = N \tag{4.233}$$

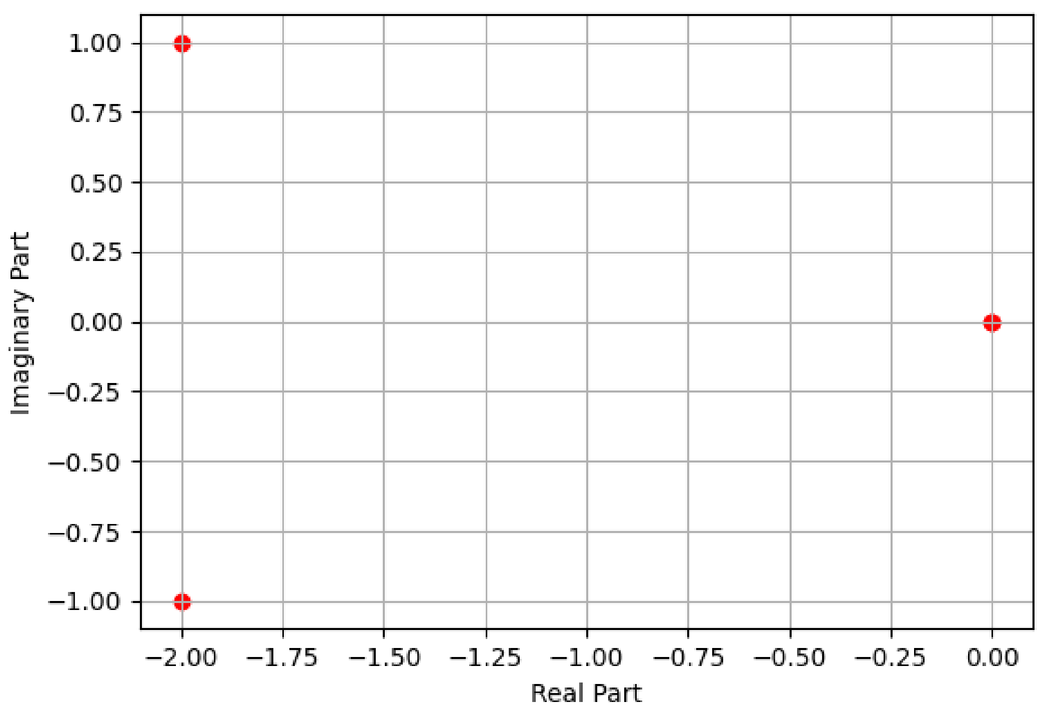


Figure 4.22: Location of roots for $k_i = 0$,$k_p = -1$

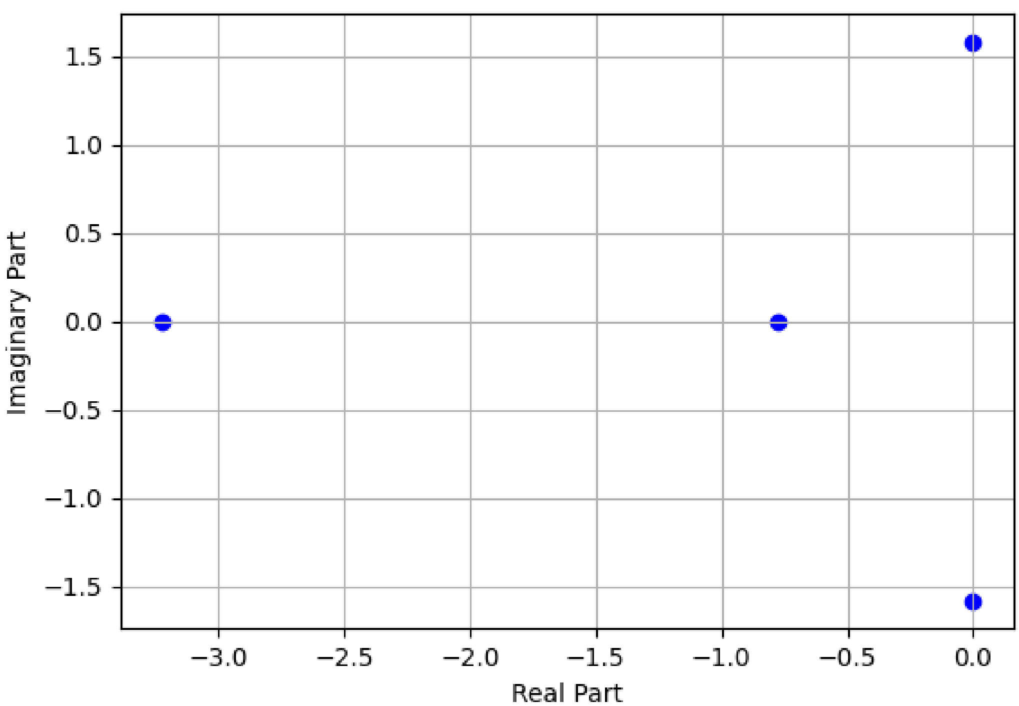


Figure 4.23: Location of roots for $k_i = 0$,$k_p = 9$

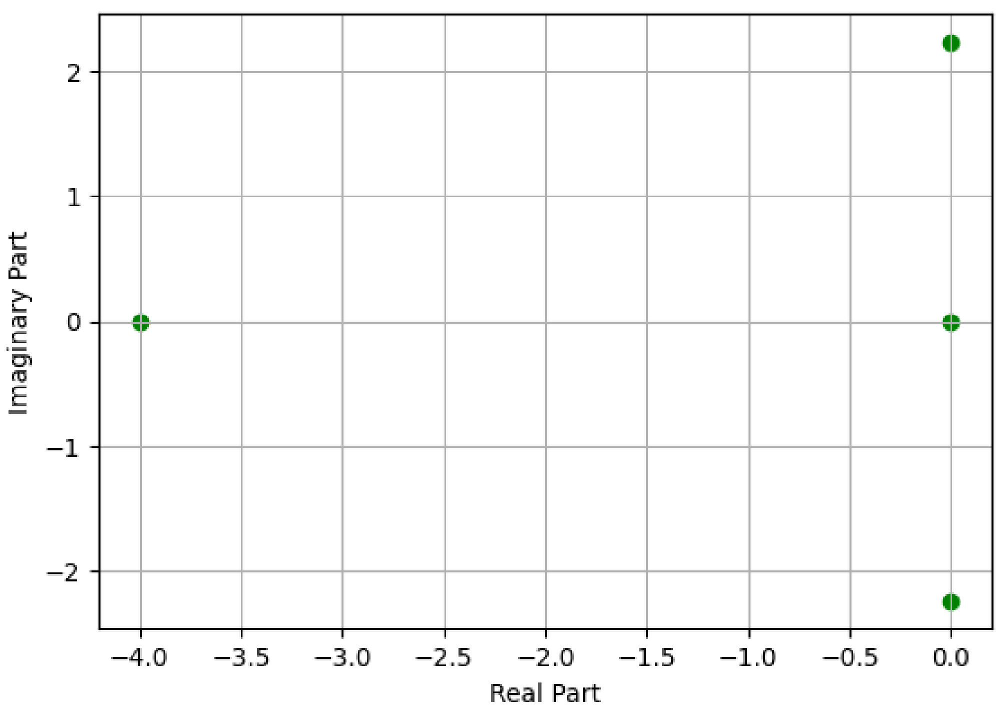


Figure 4.24: Location of roots for $k_i = 3.125$,$k_p = 4$

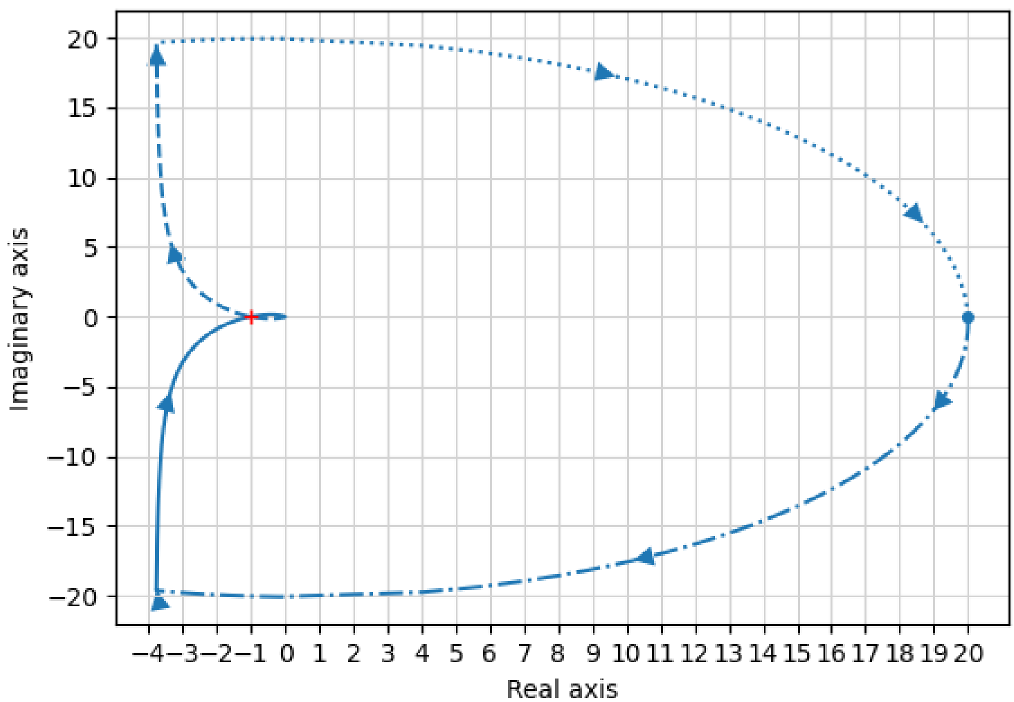


Figure 4.25: Nyquist plot for $k_i = 0$,$k_p = -1$

4.8 In the given figure, plant $G_p(s) = \frac{2.2}{(1+0.1s)(1+0.4s)(1+1.2s)}$ and compensator $G_c(s) = K\left(\frac{1+T_1 s}{1+T_2 s}\right)$. The external disturbance input is D(s). It is desired that when the disturbance is a unit step, the steady-state error should not exceed 0.1 unit. The minimum value of K is (GATE EE 2021)

Solution:

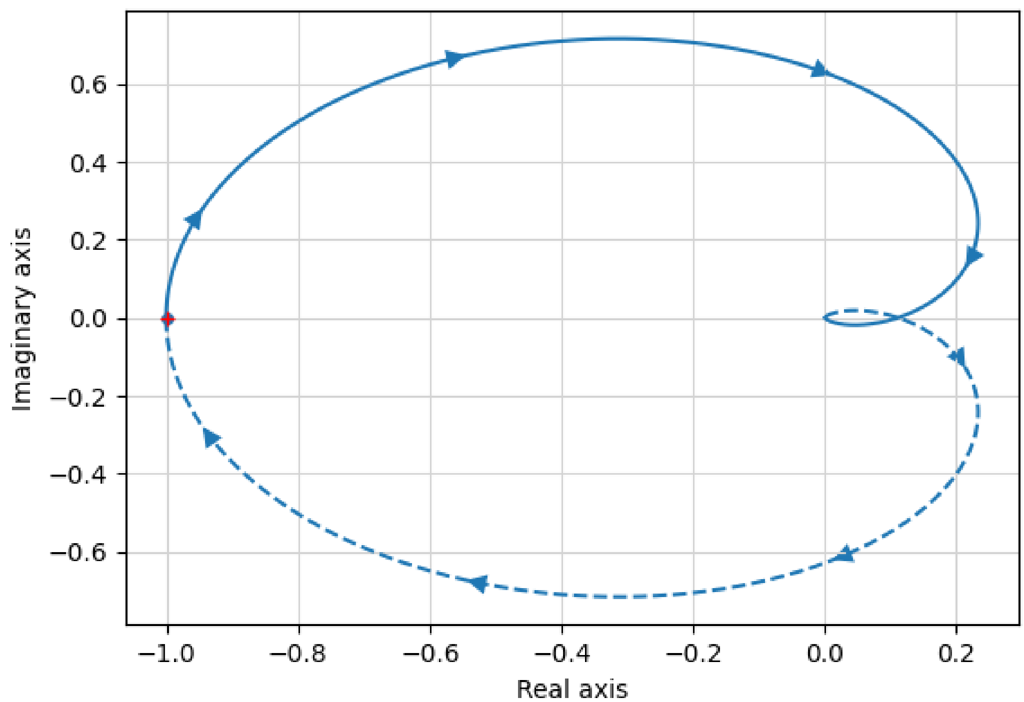


Figure 4.26: Nyquist plot for $k_i = 0$,$k_p = 9$

From Fig. 4.28

$$E(s) = R(s) - C(s) \tag{4.234}$$

Assume R(s)=0

$$E(s) = -C(s) \tag{4.235}$$

$$C(s) = (E(s)G_c(s) + D(s))\,G_p(s) \tag{4.236}$$

$$-E(s) = (E(s)G_c(s) + D(s))\,G_p(s) \tag{4.237}$$

$$E(s) = \frac{-D(s)G_p(s)}{1 + G_c(s)G_p(s)} \tag{4.238}$$

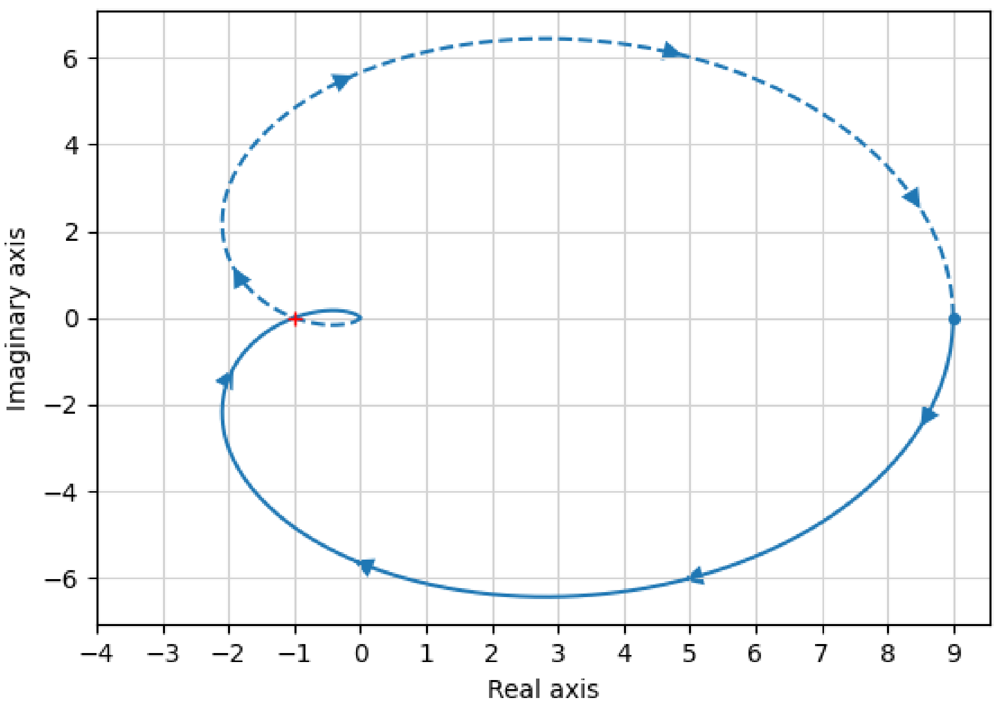


Figure 4.27: Nyquist plot for for $k_i = 3.125$,$k_p = 4$

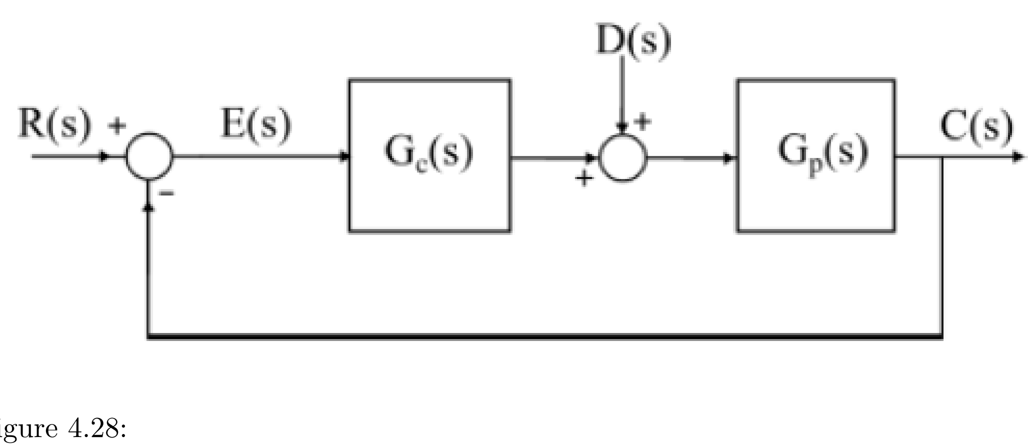


Figure 4.28:

Using final value theorem

$$e_{ss} = \lim_{t\to\infty} e(t) = \lim_{s\to 0} sE(s) \tag{4.239}$$

$$\text{Where} \quad \mathcal{L}\{e(t)\} = E(s)$$

$$e_{ss} = \lim_{s\to 0} sE(s) \tag{4.240}$$

$$= \lim_{s\to 0} \left(\frac{-sD(s)G_p(s)}{1 + G_c(s)G_p(s)} \right) \tag{4.241}$$

$$\tag{4.242}$$

$$D(s) = \mathcal{L}\{u(t)\}$$

$$= \frac{1}{s} \tag{4.243}$$

$$e_{ss} = \lim_{s\to 0} \left(\frac{-s\frac{1}{s}G_p(s)}{1 + G_c(s)G_p(s)} \right) \tag{4.244}$$

$$= \lim_{s\to 0} \frac{\frac{-2.2}{(1+0.1s)(1+0.4s)(1+1.2s)}}{1 + K\left(\frac{1+T_1 s}{1+T_2 s}\right)\frac{2.2}{(1+0.1s)(1+0.4s)(1+1.2s)}} \tag{4.245}$$

$$= \lim_{s\to 0} \frac{-2.2(1+T_2 s)}{(1+0.1s)(1+0.4s)(1+1.2s)(1+T_2 s) + 2.2K(1+T_1 s)} \tag{4.246}$$

$$|e_{ss}| = \frac{2.2}{1+2.2K} \tag{4.247}$$

given

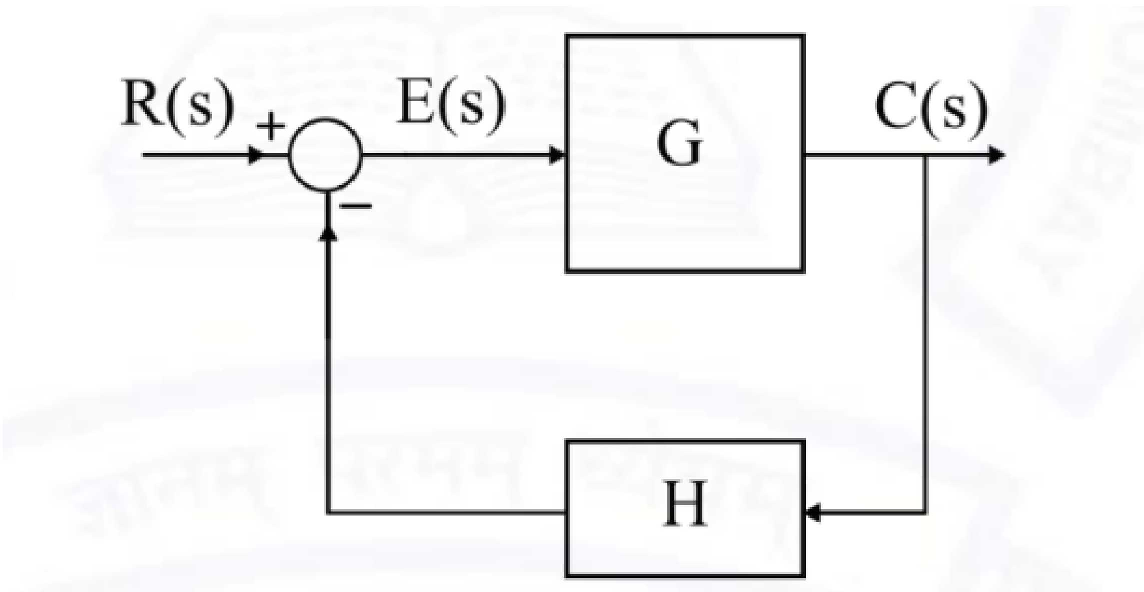
$$|e_{ss}| \leq 0.1 \tag{4.248}$$

$$\frac{2.2}{1+2.2K} \leq 0.1 \tag{4.249}$$

$$K \geq 9.54 \tag{4.250}$$

$$K_{\min} = 9.54 \tag{4.251}$$

4.9 For the closed loop system shown , the transfer function $\frac{E(s)}{R(s)}$ is



(a) $\frac{G}{1+GH}$

(b) $\frac{GH}{1+GH}$

(c) $\frac{1}{1+GH}$

(d) $\frac{1}{1+G}$

(GATE EE 2021)

Given,

$$C(s) = G \times E(s) \tag{4.252}$$

$$\text{Feedback signal} = H \times C(s) \tag{4.253}$$

$$E(s) = R(s) - H \times C(s) \tag{4.254}$$

from eq (1),

$$E(s) = R(s) - H \times G \times E(s) \tag{4.255}$$

$$E(s) + H \times G \times E(s) = R(s) \tag{4.256}$$

$$\therefore \frac{E(s)}{R(s)} = \frac{1}{1 + GH} \tag{4.257}$$

Symbol	Value	Description
$v(t)$	$230\sqrt{2}\sin\omega t$	input voltage
ω	$100\pi rad/s$	Angular velocity
Power Factor	$\dfrac{P_{avg}}{V_{rms}I_{rms}}$	$-$
$\cos\varphi$	$\dfrac{1}{2}$	Fundamental displacement factor
φ	$\dfrac{\pi}{3}$	angle between v(t) and I_n
I_n	$10\sin\left(\omega t - \dfrac{\pi}{3}\right)$	fundamental component of current

Table 4.17: Variable description

variable	value	description
U(s)	-	input function of the system
Y(s)	-	output function of the system
H(s)	$\frac{(s+1)(s+3)}{(s+5)(s+7)(s+9)}$	transfer function of the system.
I	-	identity matrix
$\dot{\mathbf{x}}(t)$	$A\mathbf{x}(t) + Bu(t)$	derivative of State function of $\mathbf{x}(t)$

Table 4.18: Table: Input Parameters

Symbol	**Value**	**Description**
y	-	Signal
d	-	Disturbance
H	?	Transfer function of the system
G_1	$\frac{2e^{-s}}{5s+1}$	Gains Given
G_2	$\frac{3e^{-s}}{8s+1}$	
P_1	$-G_2G_{ff}$	Gain of the 1st forward path
P_2	G_1	Gain of the 2nd forward path
Δ	1	Determinant of the graph
Δ_1	1	Determinant of the graph removing the 1st forward path
Δ_2	1	Determinant of the graph removing the 2nd forward path

Table 4.19: Input Parameters

Parameter	Description	Value
f_m	Message signal frequency	1 kHz
$c(t)$	Carrier signal	$2\cos\left(2\pi 10^6 t\right)$
k_p	Phase sensitivity factor	2 rad V^{-1}
$m(t)$	message signal	$A_m \sin 2\pi f_m t$
f_c	Carrier signal frequency	1 kHz
A_c	Amplitude of carrier signal	2
A_m	Amplitude of message signal	

Table 4.20: Input Parameters

Parameter	Description	Formula
$m(t)_{rms}$	rms value of $m(t)$	$\frac{A_m}{\sqrt{2}}$
$s(t)$	Phase modulation	$A_c \sin\left[2\pi f_c t + \theta_i(t)\right]$
$\theta_i(t)$	phase	$k_p m(t)$

Table 4.21: Formulae

Parameter	**Symbol**	**Value**
Receiver Frequency	f_r	600 kHz
Local Oscillator Frequency	f_l	1000 kHz
Image Frequency	f_i	___________ kHz

Table 4.22: Given Parameters with Symbols

Parameter	**Definition**
$H(z)$	$\frac{1}{1-0.5z^{-1}}$

Table 4.23: Parameter Table

Symbols	Description	Value
$P(s)$	Plant transfer function	$\frac{2}{s^3+4s^2+5\ldots}$
$C(s)$	PI controller transfer function	$K_p + \ldots$
$G(s)$	Closed loop transfer function	$\frac{P(s)C(\ldots}{1+P(s)C\ldots}$
Z	Number of zeroes with positive real part in $1 + P(s)C(s)$	?
N	Total number of anticlockwise encirclements about $-1 + 0j$ in Nyquist plot	?
P	Number of poles with positive real part in $P(s)C(s)$	?

Table 4.24: Parameters, Descriptions, and Values

Symbol	**Value**
$G_p(s)$	$\frac{2.2}{(1+0.1s)(1+0.4s)(1+1.2s)}$
$G_c(s)$	$K\left(\frac{1+T_1 s}{1+T_2 s}\right)$
$\lvert e_{ss}\rvert$	≤ 0.1
K_{min}	??

Table 4.25: Input Parameters

symbol	**description**
G	Forward path gain
H	Feedback path gain
$R(s)$	Input signal
$C(s)$	Output signal
$E(s)$	Error signal

Table 4.26: Parameters

Chapter 5

Sequences

5.1. 2022

5.1 Discrete signals $x(n)$ and $y(n)$ are shown below. The cross-correlation $r_{xy}(0)$ is:

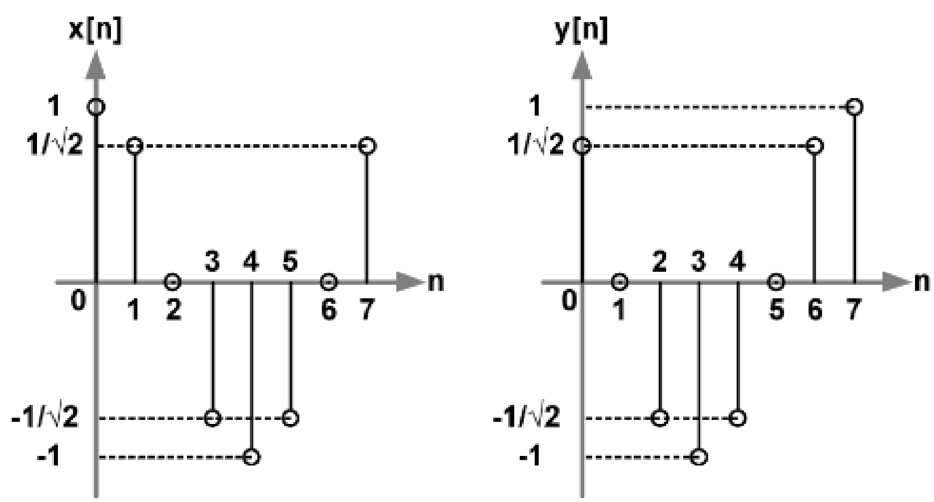


Figure 5.1: Question Figure

(GATE BM 2022)

Solution:

Parameter	Description	Value
$x\left(n\right)$	First Sequence	$x(n) = \begin{cases} 0 & ;n<0 \\ \left(1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) & ;0\leq n\leq 7 \\ 0 & ;n>7 \end{cases}$
$y\left(n\right)$	Second Sequence	$y(n) = \begin{cases} 0 & ;n<0 \\ \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\right) & ;0\leq n\leq 7 \\ 0 & ;n>7 \end{cases}$
$r_{xy}\left(k\right)$	Cross-correlation	$\sum_{m=-\infty}^{\infty} x\left(m\right) y\left(m-k\right)$

Table 1: Parameter Table

It can be seen that :

$$y\left(n\right) = x\left(n+1\right) \tag{5.1}$$

From Table 1 :

$$r_{xy}\left(k\right) = \sum_{m=-\infty}^{\infty} x\left(m\right) y\left(m-k\right) \tag{5.2}$$

$$= x\left(k\right) * y\left(-k\right) \tag{5.3}$$

From (5.1):

$$r_{xy}\left(k\right) = x\left(k+1\right) * x\left(-k\right) \tag{5.4}$$

$$= \sum_{n=-\infty}^{\infty} x\left(n+1\right) x\left(n+k\right) \tag{5.5}$$

By definition of x(n) from Table 1:

$$r_{xy}(k) = \sum_{n=0}^{6} x(n+1)\, x(n+k) \tag{5.6}$$

$$r_{xy}(0) = \sum_{n=0}^{6} x(n+1)\, x(n) \tag{5.7}$$

Using values from Fig. 5.1:

$$r_{xy}(0) = 2\sqrt{2} \tag{5.8}$$

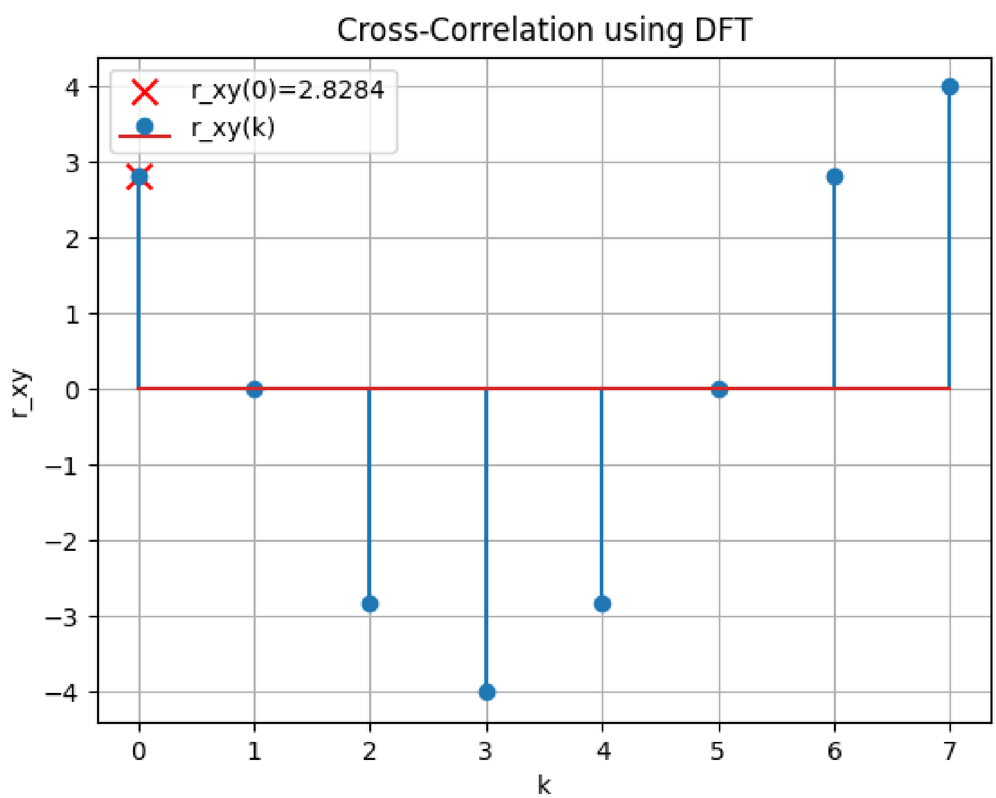


Figure 5.2: Verification of result by DFT

5.2 Which one of the following is the closed form for the generating function of the sequence $\left\{a\right\}_{n\geq 0}$ defined below?

$$a_n = \begin{cases} n+1 & \text{, n is odd} \\ 1 & \text{otherwise} \end{cases} \tag{5.9}$$

(A) $\frac{x(1+x)^2}{(1-x^2)^2} + \frac{1}{1-x}$

(B) $\frac{x\left(3-x^2\right)}{(1-x^2)^2} + \frac{1}{1-x}$

(C) $\frac{2x}{(1-x^2)^2} + \frac{1}{1-x}$

(D) $\frac{x}{(1-x^2)^2} + \frac{1}{1-x}$

(GATE CS 2022 QUESTION 36)

Solution: For the given sequence:

Parameter	Description	Value
$X(z)$	Generating function for a sequence $\left\{a_n\right\}$	?
a_n	n^{th} term of the sequence	$(n+1)\, u(n)$ (when odd)
		$u(n)$ (when even)

Table 5.2: input values

$$X(z) = \sum_{k=-\infty}^{\infty} u(2k)\, z^{-2k} + \sum_{k=-\infty}^{\infty} \left((2k+2)\, u(2k+1)\right) z^{-(2k+1)} \tag{5.10}$$

$$\implies X(z) = \left(1 + z^{-2} + z^{-4} + \dots\right) + \left(2z^{-1} + 4z^{-3} + 6z^{-5} + \dots\right) \tag{5.11}$$

$$\implies X(z) = \frac{1}{1-z^{-2}} + \left(2z^{-1} + 4z^{-3} + 6z^{-5} \dots\right) \quad |z| > 1 \tag{5.12}$$

$$\implies X(z) = \frac{1}{1-z^{-2}} + 2z^{-1}\left(\frac{1}{1-z^{-2}} + \frac{z^{-2}}{(1-z^{-2})^2}\right) \quad |z| > 1 \tag{5.13}$$

$$\therefore X(z) = \frac{1}{1 - z^{-1}} + \frac{z^{-1}\left(1 + z^{-2}\right)}{\left(1 - z^{-2}\right)^2} \quad |z| > 1 \tag{5.14}$$

(5.14) is the closed form of generating function required in the question.

Hence, option (A) is correct.

$$X(z) = X_1(z) + X_2(z) \tag{5.15}$$

$$X_1(z) = \frac{1}{1 - z^{-1}} \quad |z| > 1 \tag{5.16}$$

$$\implies x_1(n) = u(n) \tag{5.17}$$

$$\implies a_n = x_1(n) + x_2(n) \tag{5.18}$$

To find inverse z-transform of $X_2(z)$ we use contour integration technique:

$$x_2(n) = \frac{1}{2\pi j} \oint_C X_2(z)\ z^{n-1}\, dz \tag{5.19}$$

$$= \frac{1}{2\pi j} \oint_C \frac{z^n\left(z^2 + 1\right)}{\left(z^2 - 1\right)^2}\, dz \tag{5.20}$$

We can observe that we have two poles at

$z = 1, -1$. And poles are repeated twice, thus by applying residue theorem two times for poles 1 and -1:

$$x_2(n) = \frac{1}{(1)!} \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 X_2(z) \right) + \frac{1}{(1)!} \lim_{z \to -1} \frac{d}{dz} \left((z+1)^2 X_2(z) \right) \quad (5.21)$$

$$\implies x_2(n) = \lim_{z \to 1} \frac{d}{dz} \left((z-1)^2 \frac{z^n \left(z^2+1\right)}{\left(z^2-1\right)^2} \right) + \lim_{z \to -1} \frac{d}{dz} \left((z+1)^2 \frac{z^n \left(z^2+1\right)}{\left(z^2-1\right)^2} \right) \quad (5.22)$$

$$\implies x_2(n) = \lim_{z \to 1} \frac{(z+1)^2 \left(nz^{n-1} + (n+2) z^{n+1}\right) - 2z^n \left(1+z^2\right)(z+1)}{(z+1)^4} + \lim_{z \to -1} \frac{(z-1)^2 \left(nz^{n-1} + (n+2) z^{n+1}\right) - 2z^n \left(1+z^2\right)(z-1)}{(z-1)^4} \quad (5.23)$$

on simplification, we get

$$x_2(n) = \frac{n + n(-1)^{n-1}}{2} \quad (5.24)$$

$$\therefore a_n = u(n) + \frac{n + n(-1)^{n-1}}{2} u(n) \quad (5.25)$$

Which is the sequence given in the Question.

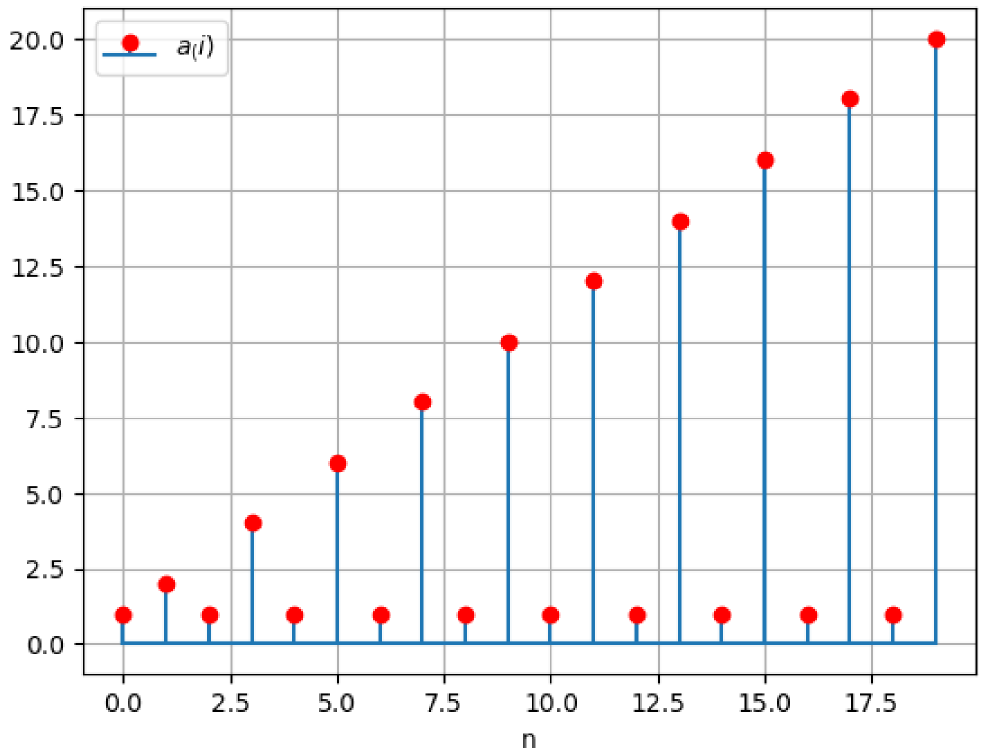


Figure 5.3: Terms of the sequence given

5.3 **Question:** Consider the following recurrence:

$$f(1) = 1; \tag{5.26}$$

$$f(2n) = 2f(n) - 1, \text{ for } n \geq 1; \tag{5.27}$$

$$f(2n+1) = 2f(n) + 1, \text{ for } n \geq 1. \tag{5.28}$$

Then, which of thefollowing is/are **TRUE?**

(A) $f(2^n - 1) = 2^n - 1$

(B) $f(2^n) = 1$

(C) $f(5 \cdot 2^n) \ = \ 2^{n+1} + 1$

(D) $f(2^n + 1) \ = \ 2^n + 1$

[GATE-CS.51 2022]

Solution: (A)

let $x(2^k - 1) = 2^k - 1$ for any $k \geq 1$,

$$x(2^{k+1} - 1) = x(2(2^k - 1) + 1) \tag{5.29}$$

From (5.28),

$$= 2x(2^k - 1) + 1 \tag{5.30}$$

$$= 2(2^k - 1) + 1 \tag{5.31}$$

$$= 2^{k+1} - 1 \tag{5.32}$$

From (5.26),(5.32)

$$x(2 - 1) = 2 - 1 \ (k = 0) \tag{5.33}$$

$$= 1 \tag{5.34}$$

Hence $x(2^n - 1) = 2^n - 1$ for $n \geq 1$

So statement A is TRUE

(B)

Let $x(2^k) = 1$ for any $k \geq 0$

$$x(2^{k+1}) = x(2 \cdot 2^k) \tag{5.35}$$

From (5.27)

$$= 2x(2^k) - 1 \tag{5.36}$$

$$= 1 \tag{5.37}$$

From (5.36),(5.27)

$$x(2) = 2x(1) - 1 \tag{5.38}$$

$$= 1 \tag{5.39}$$

Hence $x(2^n) = 1$ for every $n \geq 0$ value.

So statement B is TRUE.

(C)

Let,$x(5 \cdot 2^k) = 2^{k+1} + 1$ be true for any $k \geq 0$,

$$x(5 \cdot 2^{k+1}) = x(2(5 \cdot 2^k)) \tag{5.40}$$

From (5.27)

$$= 2x(5 \cdot 2^k) - 1 \tag{5.41}$$

$$= 2^{k+2} + 1 \tag{5.42}$$

$k = -1$,From (5.42)

$$x(5) = 2^1 + 1 \tag{5.43}$$

$$= 3 \tag{5.44}$$

Proof:-

$$x(5) = x(2 \cdot 2 + 1) \tag{5.45}$$

From (5.28),(5.39)

$$= 2x(2) + 1 \tag{5.46}$$

$$= 3 \tag{5.47}$$

Hence $x(5.2^n) = 2^{n+1} + 1$ for $n \geq 1$

So statement C is TRUE.

(D)

$$x(2^n + 1) = x(2 \cdot 2^{n-1} + 1) \tag{5.48}$$

From (5.28),(5.39)

$$= 2x(2^{n-1}) + 1 \tag{5.49}$$

$$= 3 \tag{5.50}$$

Hence statement D is FALSE.

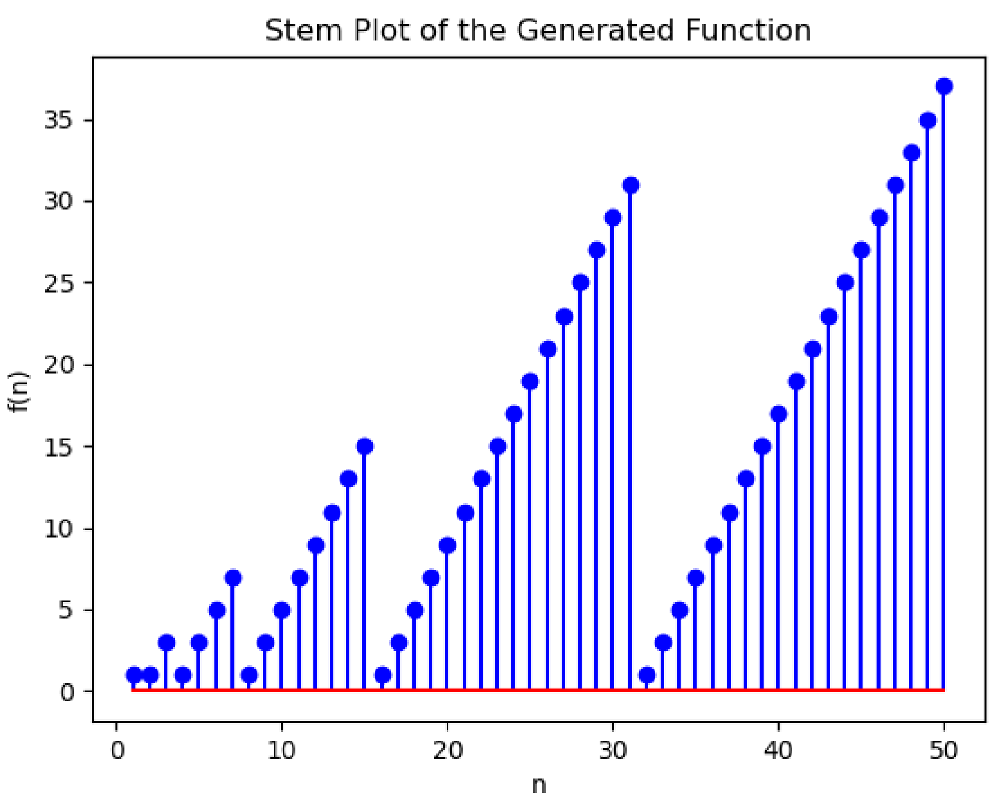


Figure 5.4: plot of $x(n)$

5.2. 2021

5.1 **Solution:**

Chapter 6

Sampling

6.1. 2022

6.1 Consider the transfer function

$$H_c(s) = \frac{1}{(s+1)(s+3)}$$

Bilinear transformation with a sampling period of $0.1s$ is employed to obtain the discrete-time transfer function $H_d(z)$. Then $H_d(z)$ is

(A) $\frac{(1+z^{-1})^2}{(19-21z^{-1})(23-17z^{-1})}$

(B) $\frac{(1-z^{-1})^2}{(21-19z^{-1})(17-23z^{-1})}$

(C) $\frac{(1+z^{-1})^2}{(21-19z^{-1})(23-17z^{-1})}$

(D) $\frac{(1+z^{-1})^2}{(21-19z^{-1})(17-23z^{-1})}$

(GATE IN 2022)

Solution:

Parameters	**Value**	**Description**
$H_c(s)$	$\dfrac{1}{(s+1)(s+3)}$	Transfer function in s domain
T_s	$0.1s$	Sampling period
$H_d(z)$		Transfer function of sampled signal

Table 6.1: Input Parameters

$$H_c(s) \xleftrightarrow{BilinearTransform} H_d(z) \tag{6.1}$$

To get $H_d(z)$, substitute s with

$$s = \frac{2}{T_s}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{6.2}$$

where T_s is the sampling period. Then,

$$H_d(z) = \frac{1}{\left(\frac{2}{0.1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+1\right)\left(\frac{2}{0.1}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+3\right)} \tag{6.3}$$

$$= \frac{(1+z^{-1})^2}{(21-19z^{-1})(23-17z^{-1})} \tag{6.4}$$

ROC : $|z| > \frac{19}{21}$

Using partial fractions,

$$H_d(z) = \frac{1}{323} + \frac{340}{323}\left(\frac{1}{21-19z^{-1}}\right) - \frac{380}{323}\left(\frac{1}{23-17z^{-1}}\right) \tag{6.5}$$

By applying inverse z-transform,

$$\delta(n) \overset{\mathcal{Z}}{\longleftrightarrow} 1 \tag{6.6}$$

$$x(0)r^n u(n) \overset{\mathcal{Z}}{\longleftrightarrow} \frac{x(0)}{1 - rz^{-1}} \tag{6.7}$$

$$H_d(n) = \frac{1}{323}\delta(n) + \frac{340}{6783}\left(\frac{19}{21}\right)^n - \frac{380}{7429}\left(\frac{17}{23}\right)^n \tag{6.8}$$

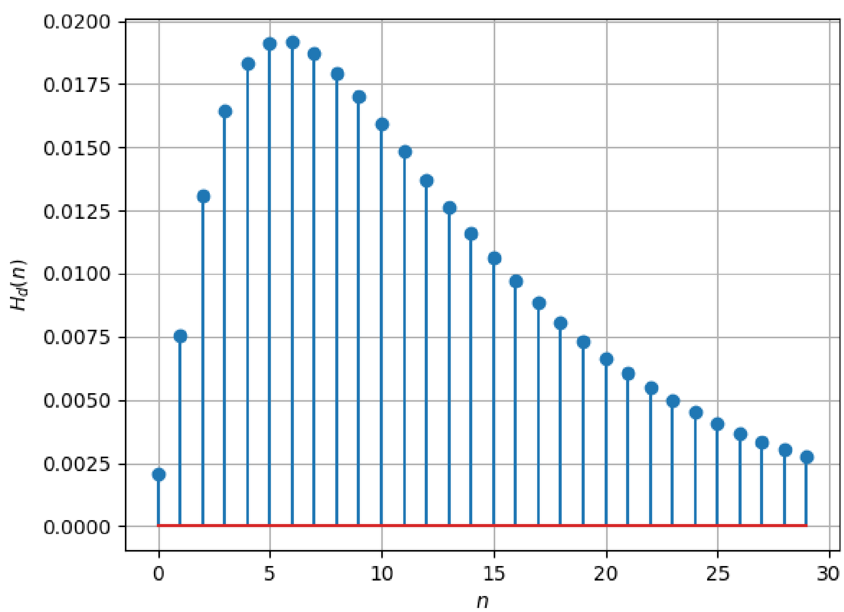


Figure 6.1: stem plot of $H_d(n)$

6.2 $x(t)$ is a real continuous-time signal whose magnitude frequency response $|X(j\Omega)|$ is shown below. After sampling $x(t)$ at 100 $rad.s^{-1}$, the spectral point P is down-converted to ______ $rad.s^{-1}$ in the spectrum of the sampled signal. (GATE 2022 BM 14 Q) **Solution:**

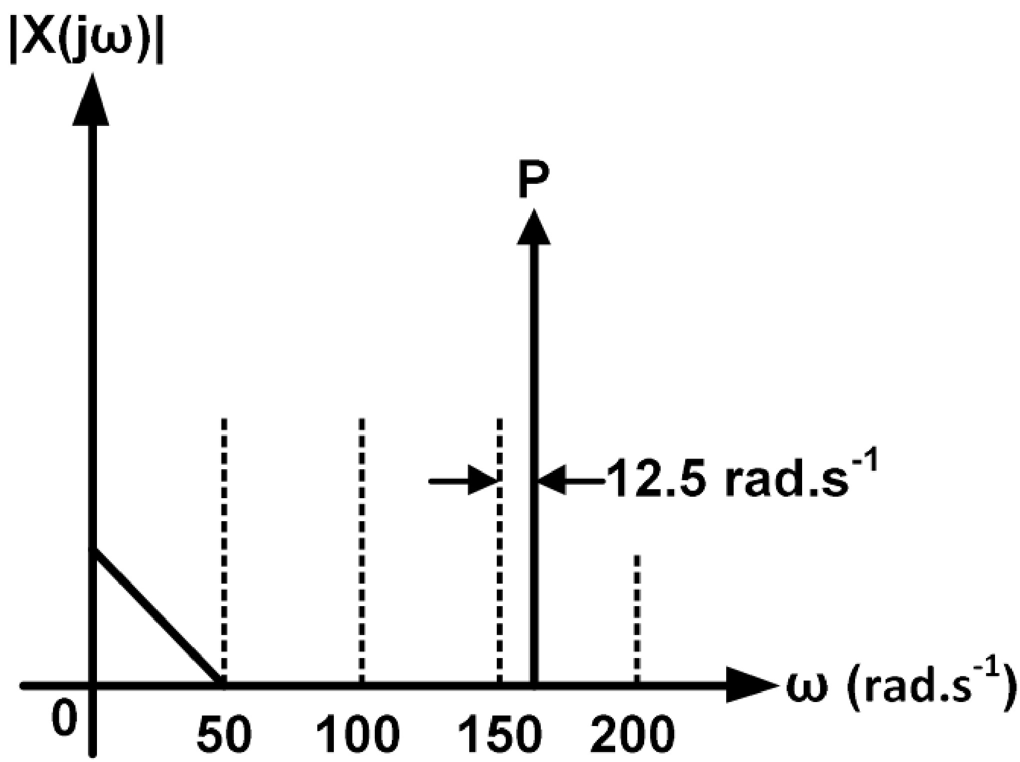


Figure 6.2: Plot of $|X(j\omega)|$

The sampling function is:

$$w(t) = \sum_{k=-\infty}^{\infty} \delta\left(t - \frac{2\pi k}{100}\right) \tag{6.9}$$

$$W(j\omega) = 100 \sum_{k=-\infty}^{\infty} \delta\left(j\left(\omega - 100k\right)\right) \tag{6.10}$$

Parameter	Description
$w(t)$	Sampling Function
$W(j\omega)$	Fourier Transform of $w(t)$
$x(t)$	Input Signal
$X(j\omega)$	Input Signal Frequency Spectrum
$x_s(t)$	Sampled Input Signal
$X_s(j\omega)$	Sampled Signal Frequency Spectrum

Table 1: Table of parameters

then the sampled function:

$$x_s(t) = x(t)\, w(t) \tag{6.11}$$

$$X_s(j\omega) = X(j\omega) * W(j\omega) \tag{6.12}$$

$$X_s(j\omega) = \int_{-\infty}^{\infty} X(j\theta)\, W(j(\omega - \theta))\, d\theta \tag{6.13}$$

$$X_s(j\omega) = 100 \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} X(j\theta)\, \delta(j(\omega - 100k - \theta))\, d\theta \tag{6.14}$$

$$X_s(j\omega) = 100 \sum_{k=-\infty}^{\infty} X(j(\omega - 100k)) \tag{6.15}$$

Thus, The down sampled point is at:

$$\omega = |162.5 - 100k| \tag{6.16}$$

where k is the nearest integer to $\frac{162.5}{100}$, which is 2

Thus,

$$\omega = 37.5\, rad\, s^{-1} \tag{6.17}$$

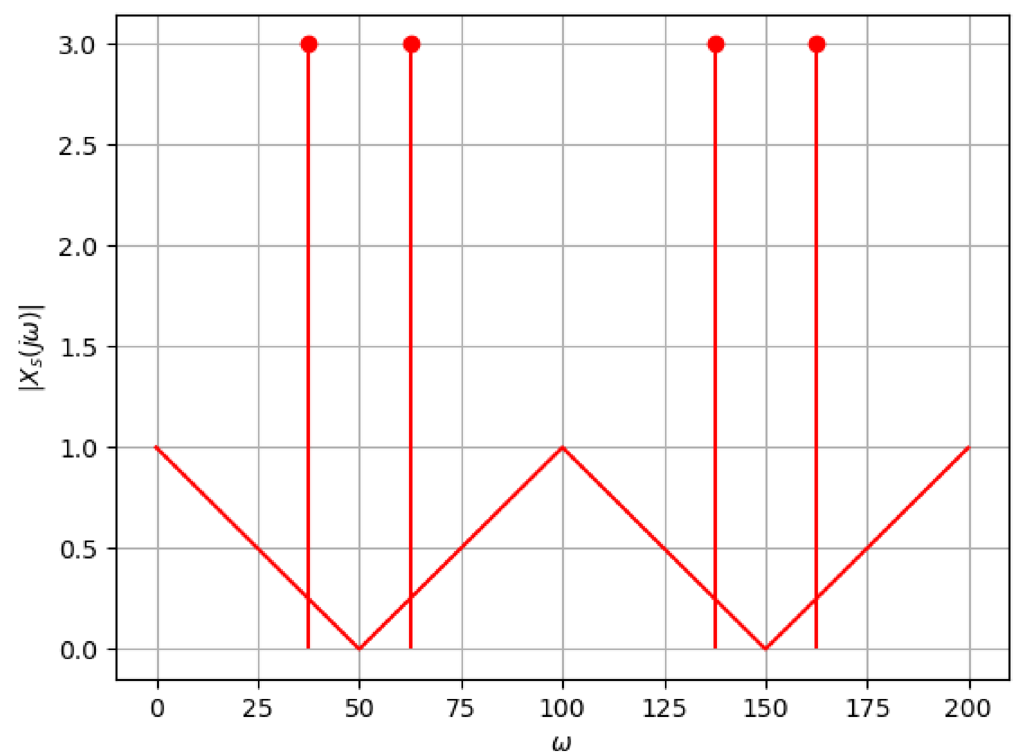


Figure 2: Plot of $|X_s\,(j\omega)|$

6.2. 2021

6.1 An analog signal is sampled at 100 MHz to generate 1024 samples. Only these samples are used to evaluate 1024-point FFT. The separation between adjacent frequency points (ΔF) in FFT is ____ kHz.

(GATE BM 2021)

Solution:

Table 6.3: Input Parameters

Symbol	Description	value
f_s	Sampling frequency	100 MHz
N	No of samples	1024

$$\Delta F = \frac{f_s}{N} \tag{6.18}$$

$$\Delta F = \frac{100}{1024} MHz \tag{6.19}$$

$$\Delta F = \frac{10^5}{1024} kHz \tag{6.20}$$

$$\Delta F = 97.66 kHz \tag{6.21}$$

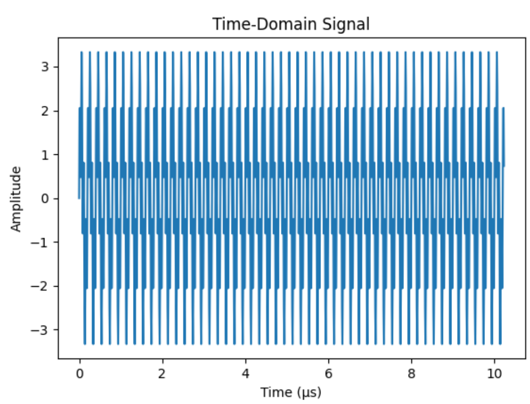


Figure 6.4: Time Domain Signal

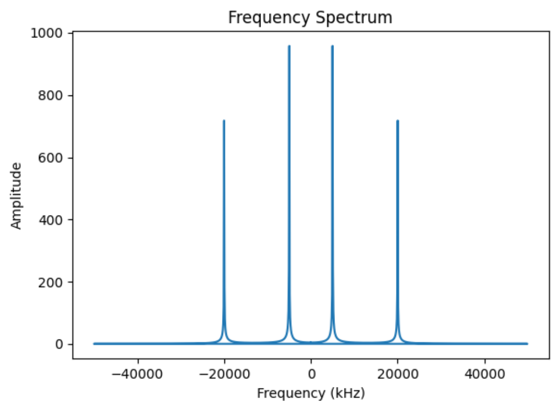


Figure 6.5: Frequency Spectrum

6.2 Consider a real-valued base-band signal $x(t)$, band limited to $10kHz$. The Nyquist rate for the signal

$y(t) = x(t)x(1 + \frac{t}{2})$ is

(A) $15kHz$

(B) $30kHz$

(C) $60kHz$

(D) $20kHz$

(GATE EC 2021)

Solution:

Parameter	Value	Description
$x(t)$		base-band signal
f	$10kHz$	Maximum frequency of $X(f)$
$y(t)$	$x(t)x(1+\frac{t}{2})$	new signal
f_{max}		Maximum frequency of $Y(f)$

Table 6.4: Input Parameters

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \tag{6.22}$$

$$x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a}X(j\omega) \tag{6.23}$$

$$x(t-t_o) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_o}X(j\omega) \tag{6.24}$$

$$x(1+\frac{t}{2}) \overset{\mathcal{F}}{\longleftrightarrow} 2e^{j\omega}X(j2\omega) \tag{6.25}$$

$$y(t) = x(t)x(1+\frac{t}{2}) \tag{6.26}$$

$$x_1(t)x_2(t) \overset{\mathcal{F}}{\longleftrightarrow} X_1(f) * X_2(f) \tag{6.27}$$

$$Y(f) = X(f) * 2e^{j2\pi f}X(2f) \tag{6.28}$$

Nyquist rate is $2f_{max} = 2(15kHz)$ which is $30kHz$

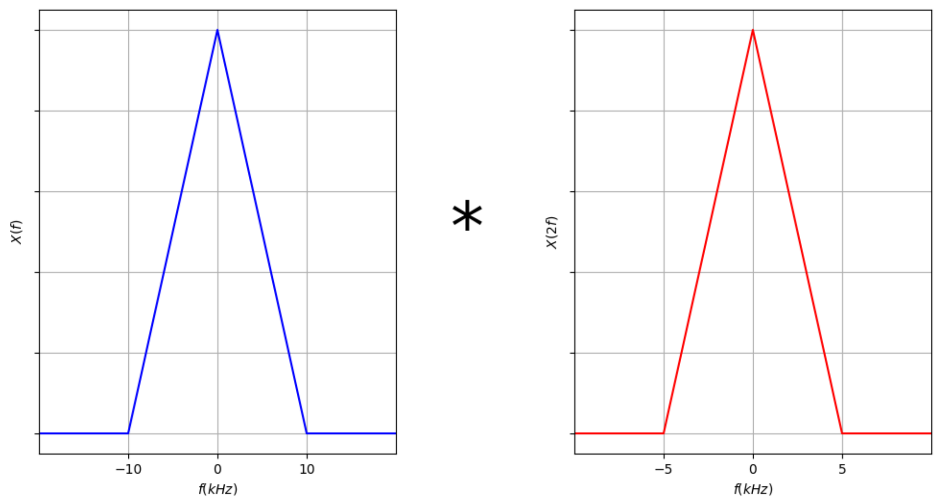


Figure 6.6: Plot of $X(f)$ and $X(2f)$

6.3 A continous time transfer function, $\mathrm{H(s)} = \frac{1+\frac{\mathrm{s}}{10^6}}{\mathrm{s}}$ is coverted to a discrete time transfer function, H(z) using a bi-linear transformation at 100 MHz sampling rate. The pole of H(z) is located at z = ? (GATE BM 2021)

Solution:

Variable	Condition
$\mathrm{F_s} = 100$ MHz	sampling rate
$\mathrm{T_s} = \frac{1}{\mathrm{F_s}}$	sampling period
$\mathrm{s_0}$	pole of H(z)

Table of Parameters

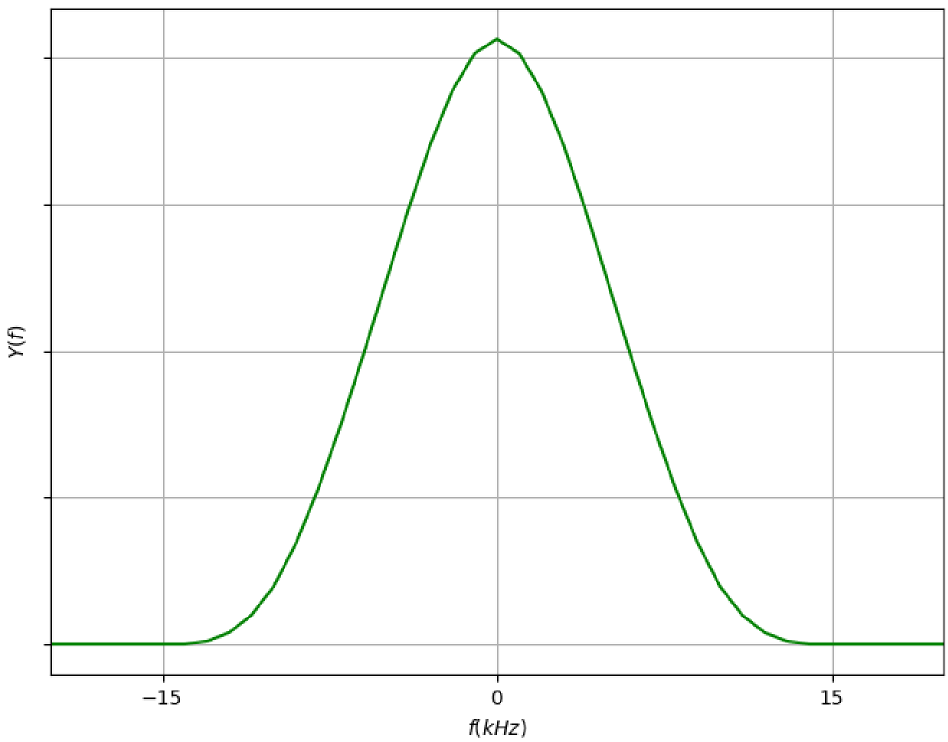
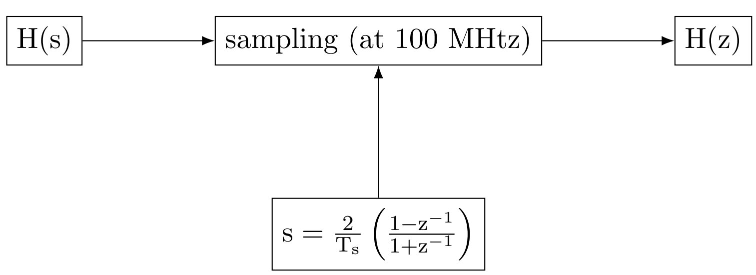


Figure 6.7: Plot of $Y(f)$



From above,

$$\mathrm{H(z)} = \mathrm{H}\left(2\mathrm{F_s}\left(\frac{1-\mathrm{z}^{-1}}{1+\mathrm{z}^{-1}}\right)\right) \tag{6.29}$$

So, from (6.29)

$$\mathrm{H(z)} = \frac{1 + \frac{2\mathrm{F_s}}{10^6}\left(\frac{1-\mathrm{z}^{-1}}{1+\mathrm{z}^{-1}}\right)}{2\mathrm{F_s}\left(\frac{1-\mathrm{z}^{-1}}{1+\mathrm{z}^{-1}}\right)} \tag{6.30}$$

$$\mathrm{H(z)} = \frac{1}{200 \times 10^6}\left(\frac{1+\mathrm{z}^{-1}+200\left(1-z^{-1}\right)}{1-z^{-1}}\right) \tag{6.31}$$

$$\mathrm{H(z)} = 5 \times 10^{-9}\left(\frac{201-199\mathrm{z}^{-1}}{1-z^{-1}}\right) \tag{6.32}$$

So, $\mathrm{s_0}$ is at z=1

6.4 A 4 kHz sinusoidal message signal having amplitude 4 V is fed to a delta modulator (DM) operating at a sampling rate of 32 kHz. The minimum step size required to avoid slope overload noise in the DM is? (GATE EC 2021) **Solution:**

Parameter	**Value**	**Description**
δ	-	Step size
f_s	32 kHz	Sampling rate
A_{max}	4 V	Maximum amplitude of message signal
f_m	4 kHz	Frequency of message signal

Table 6.5: Input Table

To avoid slope overload distortion,

$$\delta f_s \geq 2\pi A_{max} f_m \tag{6.33}$$

The minimum slope can be obtained when,

$$\delta_{min} f_s = 2\pi A_{max} f_m \tag{6.34}$$

$$\delta_{min}\left(32\right) = 2\pi\left(4\right)\left(4\right) \tag{6.35}$$

$$\delta_{min} = \pi \tag{6.36}$$

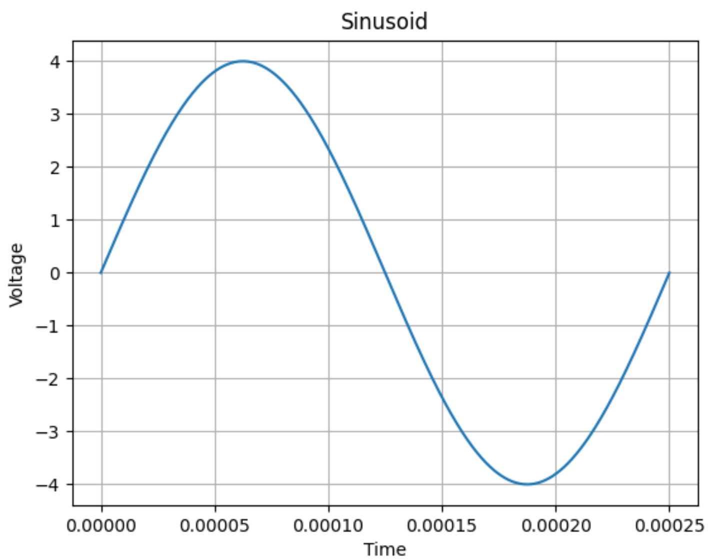


Figure 6.8: (a) Plot of the sinusoid

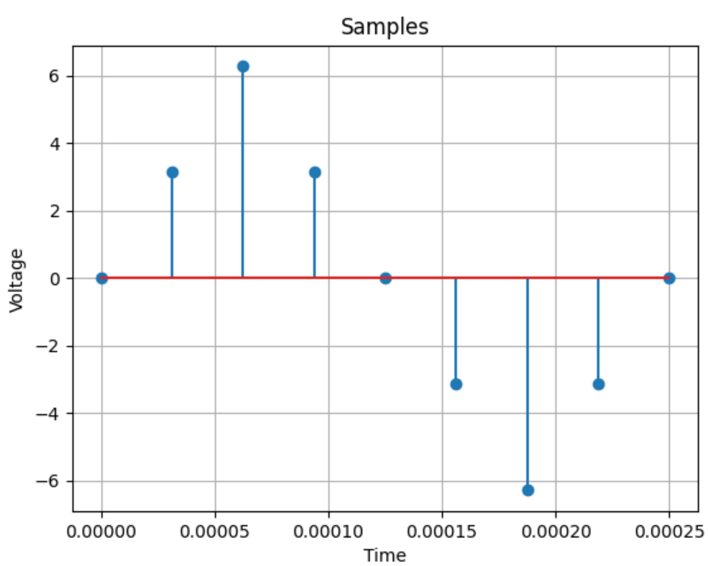


Figure 6.9: (b) Plot of the samples with δ_{min}

Chapter 7

Contour Integration

7.1. 2022

7.1 In the complex z-domain, the value of integral $\oint_C \frac{z^3-9}{3z-i}\, dz$ is

(a) $\frac{2\pi}{81} - 6i\pi$

(b) $\frac{2\pi}{81} + 6i\pi$

(c) $-\frac{2\pi}{81} + 6i\pi$

(d) $-\frac{2\pi}{81} - 6i\pi$

(GATE 2022 BM)

Solution:

Simplyfying the Contour Integral to the standard form we get,

$$\oint_C \frac{z^3-9}{3z-i}\, dz = \frac{1}{3}\oint_C \frac{z^3-9}{z-\frac{i}{3}}\, dz \tag{7.1}$$

From Cauchy's residue theorem,

$$\oint_C f(z)\, dz = 2\pi i \sum R_j \tag{7.2}$$

We can observe a non-repeated pole at $z = \frac{i}{3}$ and thus $a = \frac{i}{3}$,

$$R = \lim_{z \to a} (z - a) f(z) \tag{7.3}$$

$$\implies R = \frac{1}{3} \lim_{z \to \frac{i}{3}} \left(z - \frac{i}{3} \right) \frac{z^3 - 9}{z - \frac{i}{3}} \tag{7.4}$$

$$= \frac{-i}{81} - 3 \tag{7.5}$$

Therefore, from (7.2) and (7.5)

$$\oint_C \frac{z^3 - 9}{3z - i} \, dz = \frac{2\pi}{81} - 6i\pi \tag{7.6}$$

7.2 Consider the function

$$f(z) = \frac{1}{(z+1)(z+2)(z+3)}$$

The residue of $f(z)$ at $z = -1$, is ______ (GATE 2022 IN)

Solution: Residue of a function $f(z)$ at a simple pole c is

$$\operatorname{Res}(f, c) = \lim_{z \to c} (z - c) f(z) \tag{7.7}$$

$$\implies \operatorname{Res}(f, -1) = \lim_{z \to -1} \frac{z+1}{(z+1)(z+2)(z+3)} \tag{7.8}$$

$$= \frac{1}{2} \tag{7.9}$$

$\therefore$ residue of $f(z)$ at $z = -1$ is $\frac{1}{2}$.

7.3 The value of the integral

$$\int_C \frac{z^{100}}{z^{101}+1}\,dz$$

where C is the circle of radius 2 centred at the origin taken in the anti-clockwise direction is

(A) $-2\pi i$

(B) 2π

(C) 0

(D) $2\pi i$

(GATE 2022 MA)

Solution:

Using B.1 and B.2 Solving the integral,

$$f(z) = \int_C \frac{z^{100}}{z^{101}+1}\,dz \tag{7.10}$$

Since the pole $z = -1$ is inside the circle, Using eq (B.2.2)

$$Res\, f(-1) = \lim_{z\to -1}\left(\frac{z^{100}}{z^{101}+1}\right)\left(z^{101}+1\right) \tag{7.11}$$

$$= 1 \tag{7.12}$$

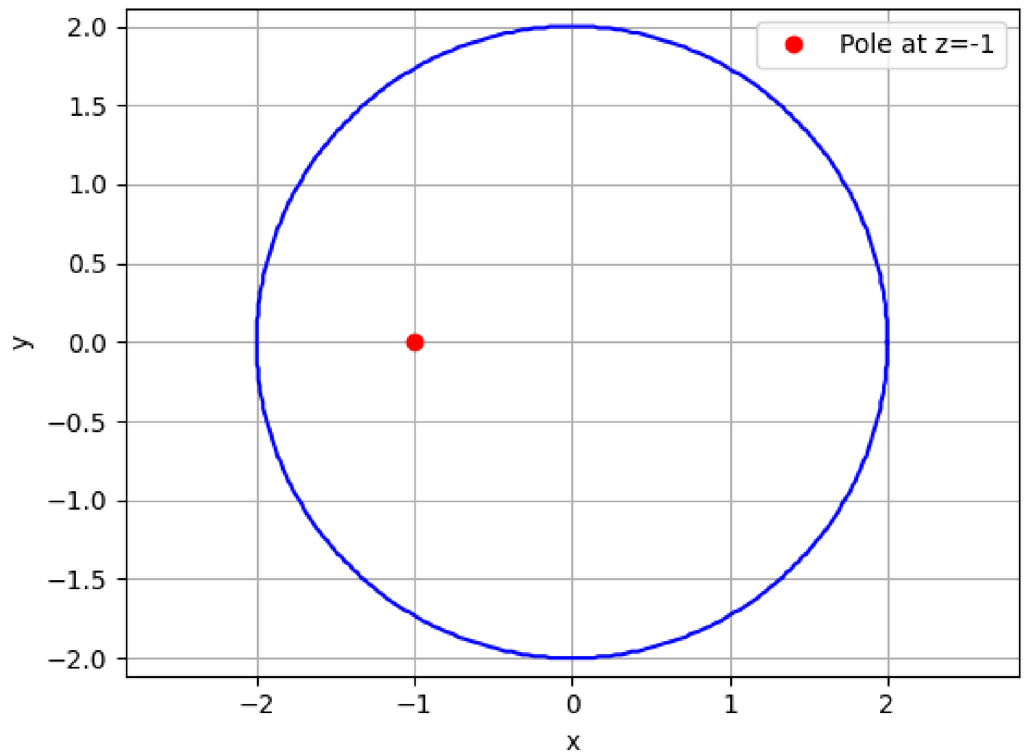


Figure 7.1: plot of C with it's pole

From eq (B.2.1), and eq (7.12)

$$\int f(z)\, dz = 2\pi i\,(1) \tag{7.13}$$

$$\implies \int_C \frac{z^{100}}{z^{101}+1}\, dz = 2\pi i \tag{7.14}$$

$\therefore$ option (D) is correct.

7.2. 2021

7.1 **Solution:**

Chapter 8

Laplace Transform

8.1. 2022

8.1 Consider the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$. The boundary conditions are $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$. Then the value of y at $x = \frac{1}{2}$ (GATE AE 2022)

Solution:

Parameters	**Values**	**Description**
$y(0)$	0	y at $x = 0$
$y'(0)$	1	$\frac{dy}{dx}$ at $x = 0$

Table 8.1: Parameters

$$\frac{d^2y}{dx^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0) - y'(0) \tag{8.1}$$

$$\frac{dy}{dx} \overset{\mathcal{L}}{\longleftrightarrow} sY(s) - y(0) \tag{8.2}$$

Applying Laplace Transform, using (8.1) and (8.2),

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + Y(s) = 0 \tag{8.3}$$

From Table 8.1,

$$(s^2 - 2s + 1)Y(s) - 1 = 0 \tag{8.4}$$

$$Y(s) = \frac{1}{(s-1)^2} \tag{8.5}$$

$$t^n \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{n!}{s^{n+1}} \tag{8.6}$$

$$e^{at}x(t) \stackrel{\mathcal{L}}{\longleftrightarrow} X(s-a) \tag{8.7}$$

Taking Inverse Laplace Transform for $Y(s)$, using (8.6) and (8.7),

$$y(x) = xe^x \tag{8.8}$$

$$\implies y\left(\frac{1}{2}\right) = \frac{\sqrt{e}}{2} \tag{8.9}$$

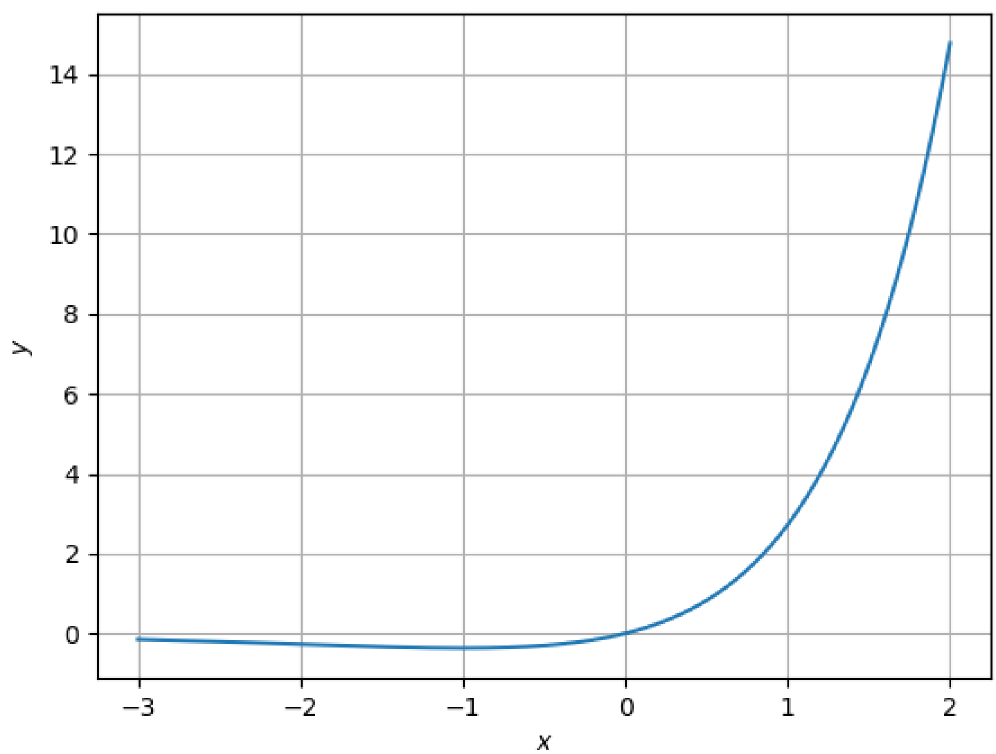


Figure 8.1: Plot of $y(x)$

8.2 A process described by the transfer function

$$G_p(s) = \frac{(10s+1)}{(5s+1)}$$

is forced by a unit step input at time $t = 0$. The output value immediately after the unit step input (at $t = 0^+$) is ? (Gate 2022 CH 34)

Solution:

Parameters	**Description**
$X(s)$	Laplace transform of $x(t)$
$Y(s)$	Laplace transform of $y(t)$
$G_p(s) = \frac{Y(s)}{X(s)}$	Transfer function
$x(t) = u(t)$	unit step function

Table 8.2: Given parameters

$$G_p(s) = \frac{Y(s)}{X(s)} = \frac{(10s+1)}{(5s+1)} \tag{8.10}$$

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \tag{8.11}$$

From equation (8.11):

$$Y(s) = \frac{(10s+1)}{s\,(5s+1)} \tag{8.12}$$

$$= \frac{1}{s} + \frac{5}{5s+1} \tag{8.13}$$

Taking inverse laplace transformation,

$$\frac{1}{s} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} u(t) \tag{8.14}$$

$$\frac{1}{s-c} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} e^{ct}u(t) \tag{8.15}$$

$$y(t) = \left(1 + e^{\frac{-t}{5}}\right) u(t) \tag{8.16}$$

$$y(0^+) = 2 \tag{8.17}$$

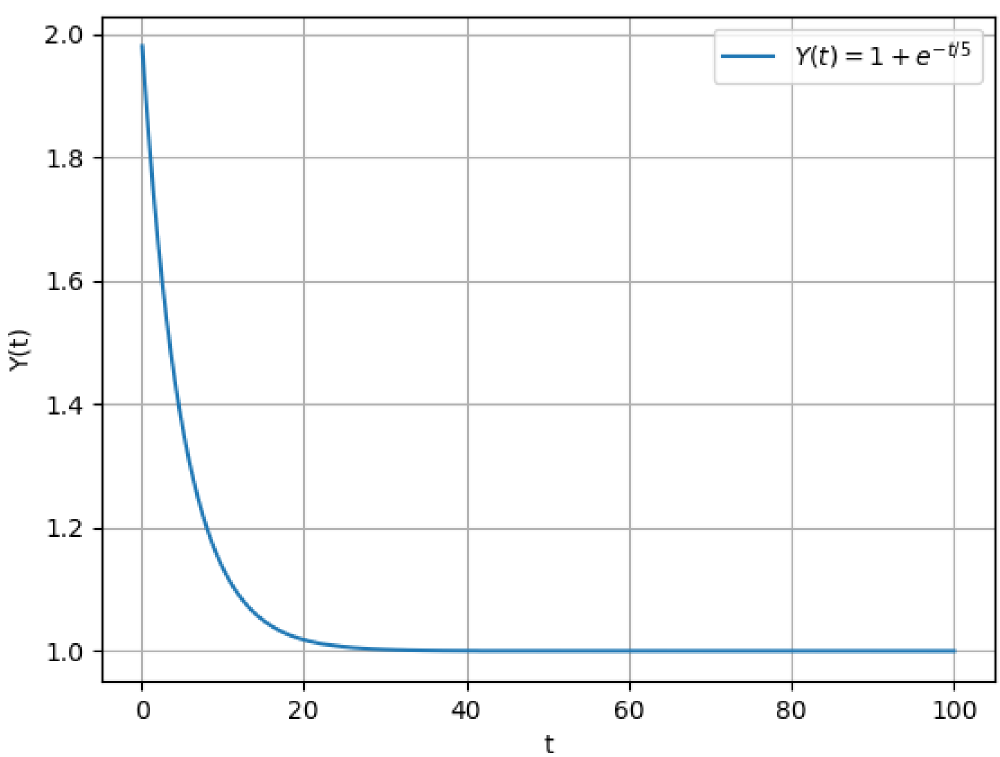


Figure 8.2: Graph of y(t)

8.3 The transfer function of a real system $H(S)$ is given as:

$$H(s) = \frac{As + B}{s^2 + Cs + D}$$

where A, B, C and D are positive constants. This system cannot operate as

(A) Low pass filter

(B) High pass filter

(C) Band pass filter

(D) An Integrator

(GATE EE 11 2022)

Solution: The transfer function $H(s)$ is given by:

$$H(s) = \frac{As + B}{s^2 + Cs + D} \tag{8.18}$$

Put $s = j\omega$ in (8.18):

$$H(j\omega) = \frac{A(j\omega) + B}{(j\omega)^2 + C(j\omega) + D} \tag{8.19}$$

$$|H(j\omega)| = \frac{\sqrt{(A\omega)^2 + B^2}}{\sqrt{(D - \omega^2)^2 + (\omega C)^2}} \tag{8.20}$$

a) Low Pass Filter:

At low frequency ($\omega = 0$):

$$|H(\omega = 0)| = \frac{B}{D} \tag{8.21}$$

$\therefore$ H(s) can operate as Low pass filter.

Parameter	Description
Low Pass Filter	The gain should be finite at low frequency
High Pass Filter	The gain should be finite at high frequency
Band Pass Filter	Finite gain over frequency band
Integrator	Transfer function should have at least one pole at origin

Table 8.3: Conditions

b) High Pass Filter:

At high frequency ($\omega = \infty$):

$$|H(\omega = \infty)| = 0 \tag{8.22}$$

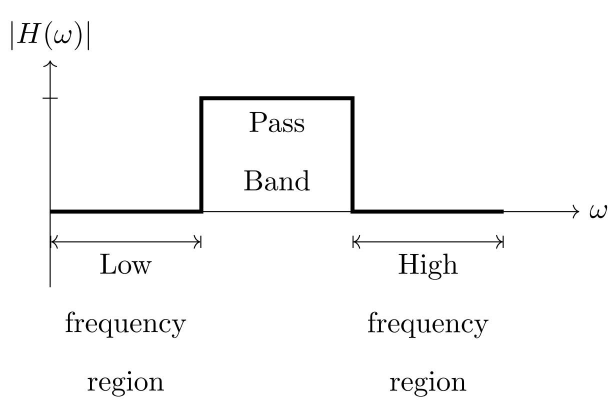
$\therefore H(s)$ cannot operate as High pass filter.

c) Band Pass Filter:

Assuming B is a very less positive valued constant as compared to others:

$$|H(j\omega)| = \frac{(A\omega)}{\sqrt{(D-\omega^2)^2 + (\omega C)^2}} \tag{8.23}$$

$$\implies |H(\omega = 0)| = 0 \text{ and } |H(\omega = \infty)| = 0 \tag{8.24}$$



$\because H(s)$ passes frequency between low and high frequencies.

$\therefore H(s)$ can operate as a band pass filter.

d) Integrator:

At very high value of frequency($\omega \rightarrow \infty$):

$$H(s) \approx \frac{As}{s^2} \approx \frac{A}{s} \tag{8.25}$$

From Table 8.3:

$\therefore H(s)$ can operate as an Integrator.

8.4 In a circuit, there is a series connection of an ideal resistor and an ideal capacitor. The conduction current (in Amperes) through the resistor is $2\sin\left(t+\frac{\pi}{2}\right)$. The displacement current (in Amperes) through the capacitor is ____.

(A) $2\sin(t)$

(B) $2\sin(t+\pi)$

(C) $2\sin\left(t+\frac{\pi}{2}\right)$

(D) 0

(GATE 2022 EC 24)

Solution:

Parameter	Description	Value
I_c	Conduction Current	$2\sin\left(t+\frac{\pi}{2}\right)$
A	Cross-sectional area	

Table 8.4: Parameters

Parameter	Description	Formula
Q	Charge	$\int I_c\, dt$
D	Electric Displacement	$\frac{Q}{A}$
J_D	Displacement current density	$\frac{\partial D}{\partial t}$
I_D	Displacement current	J_D x A

Table 8.5: Formulae

S Domain	Time Domain
$\frac{1}{s}$	$u\left(t\right)$
$\frac{-s}{a^2+s^2}$	$-\cos\left(at\right)$
$\frac{a}{a^2+s^2}$	$\sin\left(at\right)$
$\frac{1}{s+a}$	e^{-at}

Table 8.6: Laplace transforms

$$\mathcal{L}\left[\int f\left(t\right)\,dt\right] = \int_0^{\infty}\left[\int f\left(t\right)\,dt\right]e^{-st}\,dt \tag{8.26}$$

$$= \int_0^{\infty} u\,dv \quad \text{where} \begin{cases} u = \int f\left(t\right)dt \\ dv = e^{-st}dt \end{cases} \tag{8.27}$$

$$= uv - v\int du \tag{8.28}$$

$$= \frac{1}{s}\int f\left(t\right)dt|_0 + \frac{1}{s}\int_0^{\infty} f\left(t\right)e^{-st}dt \tag{8.29}$$

$$\implies \frac{1}{s}\int f\left(t\right)dt|_0 + \frac{1}{s}F\left(s\right) \tag{8.30}$$

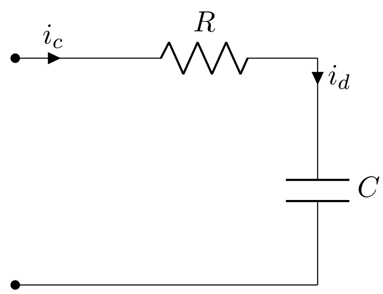


Figure 8.3: Circuit 1

From Table 8.5, Table 8.6 and eq (8.30)

$$I_c\,(s) = \frac{2s}{s^2+1} \tag{8.31}$$

$$Q_c\,(s) = \frac{2}{s\,(s^2+1)} \tag{8.32}$$

$$D\,(s) = \frac{1}{A}\left(\frac{2}{s\,(s^2+1)}\right) \tag{8.33}$$

$$J_D\,(s) = \frac{2}{A}\left(\frac{1}{s^2+1}\right) \tag{8.34}$$

$$I_D\,(s) = \frac{2}{s^2+1} \tag{8.35}$$

$$\implies I_D = 2\sin t \tag{8.36}$$

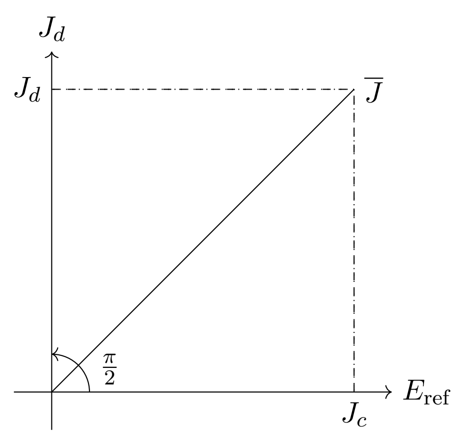


Figure 8.4: Phasor plot

From figure 8.4, phase of I_d is $\frac{\pi}{2}$

$$\therefore I_d = 2\sin\left(t + \frac{\pi}{2}\right) \tag{8.37}$$

$\therefore$ (C) is correct.

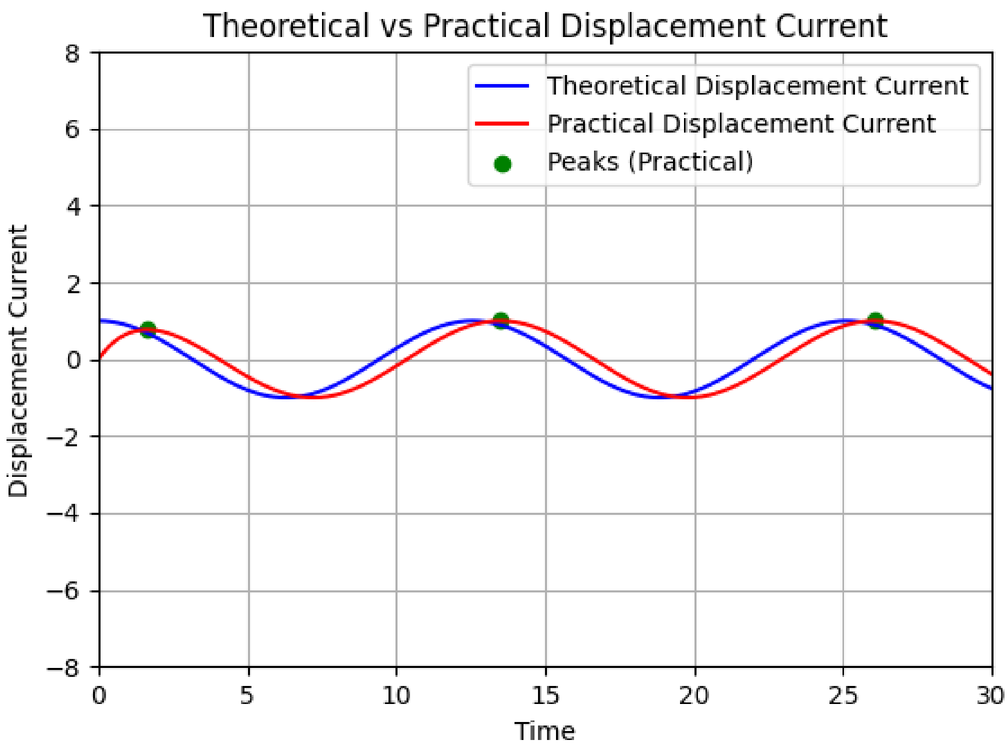


Figure 8.5: Thoritical vs Practical simulation

8.5 Given, $y = f(x)$; $\frac{d^2y}{dx2} + 4y = 0; y(0) = 0; \frac{dy}{dx}(0) = 1$. The problem is a/an

(a) initial value problem having soluition $y = x$

(b) boundary value problem having soluition $y = x$

(c) initial value problem having soluition $y = \frac{1}{2}\sin 2x$

(d) boundary value problem having soluition $y = \frac{1}{2}\sin 2x$

(GATE 2022 ES)

Solution:

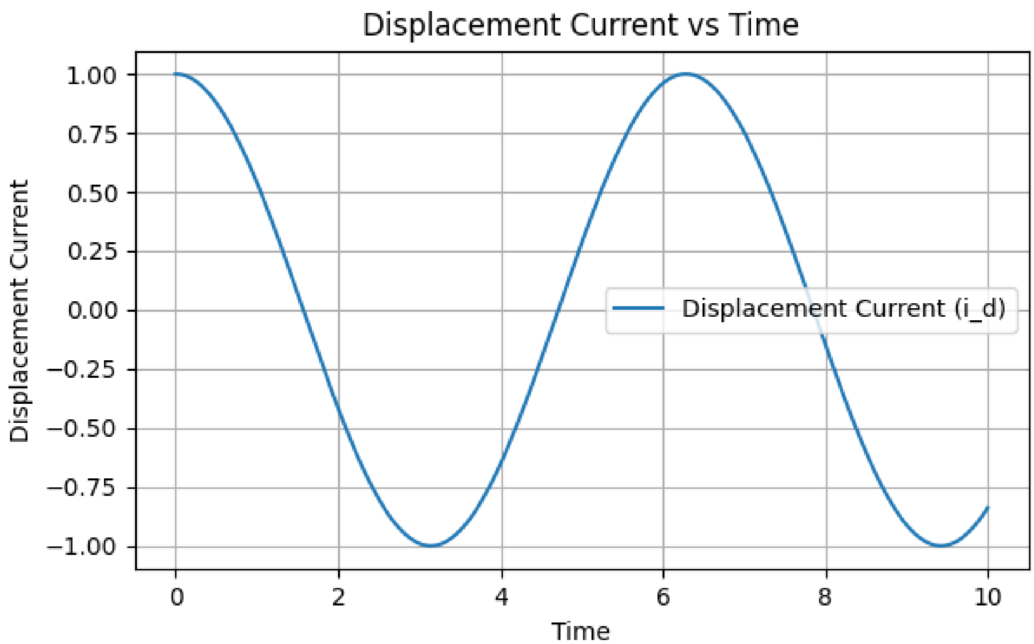


Figure 8.6: Displacement current

The above equation can be written as,

$$y''(t) + 4y(t) = 0 \tag{8.38}$$

Using the Laplace transformation pairs,

$$y''(t) \overset{\mathcal{L}}{\longleftrightarrow} s^2 Y(s) - sy(0) - y'(0) \tag{8.39}$$

$$y(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) \tag{8.40}$$

$$\sin at \overset{\mathcal{L}}{\longleftrightarrow} \frac{a}{a^2 + s^2} \tag{8.41}$$

Applying Laplace transform for the equation we get,

$$s^2Y(s) - 1 + 4Y(s) = 0 \tag{8.42}$$

$$\implies Y(s) = \frac{1}{4+s^2} \tag{8.43}$$

Now, applying inverse laplace transform we get,

$$y(t) = \frac{1}{2}\sin 2t \quad \text{(from (8.41))} \tag{8.44}$$

Since, the conditions at the same point(0) are mentioned, it is an initial valued problem having solution $y = \frac{1}{2}\sin 2x$.

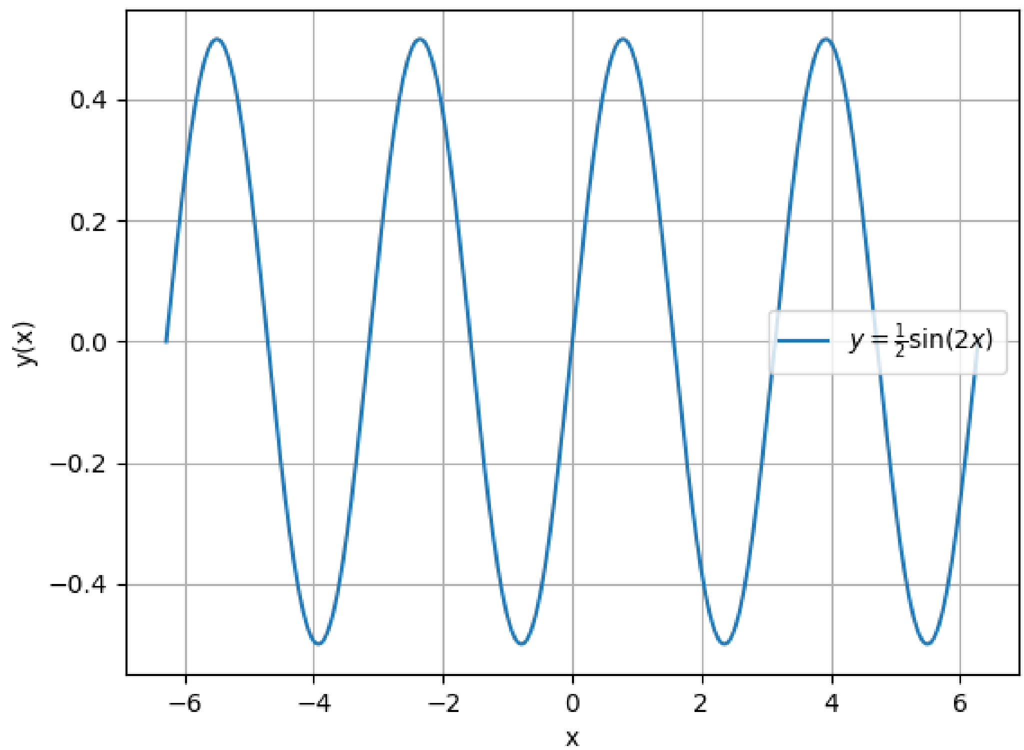


Figure 8.7: $y(x)$ vs x graph

8.6 Let a causal LTI system be governed by the following differential equation,

$$y(t) + \frac{1}{4}\frac{dy}{dt} = 2x(t) \tag{8.45}$$

where $x(t)$ and $y(t)$ are the input and output respectively. It's impulse response is (GATE EE-2022)

Solution: Solution:

From (8.45), corresponding Laplace transform,

$$Y(s) + \frac{1}{4}(sY(s) - y(0)) = 2X(s) \tag{8.46}$$

Since it is causal LTI system,

$$y(0) = 0 \tag{8.47}$$

$$\implies Y(s) + \frac{1}{4}sY(s) = 2X(s) \tag{8.48}$$

$$\implies Y(s) = X(s)\frac{8}{4+s} \tag{8.49}$$

$$\implies H(s) = \frac{8}{4+s} \quad ROC : Re(s) > -4 \tag{8.50}$$

Taking inverse laplace transform and applying causality conditions

$$h(t) = 8e^{-4t}u(t) \tag{8.51}$$

8.7 Assuming $s > 0$; Laplace transform for $f(x) = sin(ax)$ is

(A) $\frac{a}{s^2+a^2}$

(B) $\frac{s}{s^2+a^2}$

(C) $\frac{a}{s^2-a^2}$

(D) $\frac{s}{s^2-a^2}$

(GATE 2022 ES)

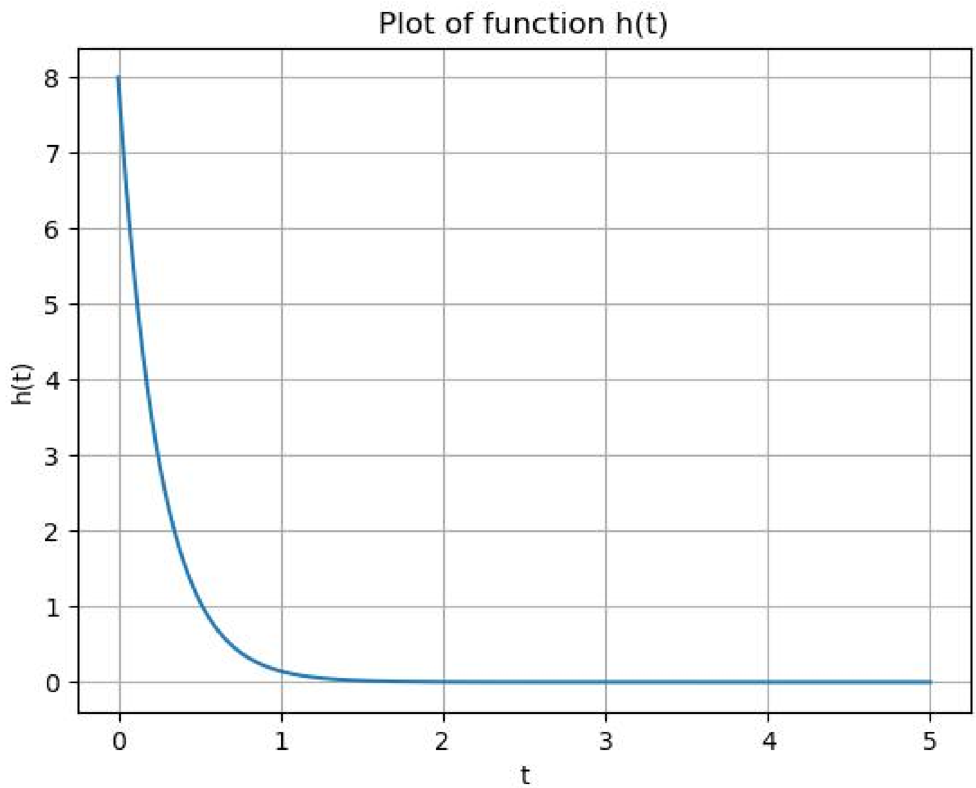


Figure 1: Plot of $h(n)$, taken from python3

Solution:

$$\mathcal{L}(f(x)) = \int_{-\infty}^{\infty} e^{-sx} f(x)\, dx \tag{8.52}$$

$$\text{We can write} \quad \sin(ax) = \frac{e^{ax} - e^{-ax}}{2i} \tag{8.53}$$

From (8.53)

$$\mathcal{L}\left(\sin\left(ax\right)\right) = \int_0^\infty e^{-sx}\left(\frac{e^{iax} - e^{-iax}}{2i}\right)dx \tag{8.54}$$

$$= \frac{1}{2i}\int_0^\infty e^{-x(s-ia)} - e^{-x(s+ia)}dx \tag{8.55}$$

$$= \frac{1}{2i}\left(\frac{e^{-x(s-ia)}}{-\left(s-ia\right)} + \frac{e^{-x(s+ia)}}{-\left(s+ia\right)}\right)_0^\infty \tag{8.56}$$

$$= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) \tag{8.57}$$

$$= \frac{a}{s^2+a^2} \tag{8.58}$$

So, option (A) is correct.

8.8 The input $x(t)$ to a system is related to its output $y(t)$ as

$$\frac{dy(t)}{dt} + y(t) = 3x(t-3)u(t-3)$$

Here $u(t)$ represents a unit-step function.

The transfer function of this system is

(A) $\frac{e^{-3s}}{s+3}$

(B) $\frac{3e^{-3s}}{s+1}$

(C) $\frac{3e^{-(s/3)}}{s+1}$

(D) $\frac{e^{-(s/3)}}{s+3}$

(GATE IN 2022)

Solution:

$$\frac{dy(t)}{dt} + y(t) = 3x(t-3)u(t-3) \tag{8.59}$$

By applying Laplace Transform on both sides

$$x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s) \tag{8.60}$$

$$x(t-t_o) \overset{\mathcal{L}}{\longleftrightarrow} X(s)e^{-st_o} \tag{8.61}$$

$$sY(s) + Y(s) = 3X(s)e^{-3s} \tag{8.62}$$

$$Y(s)\,(s+1) = 3X(s)e^{-3s} \tag{8.63}$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{3e^{-3s}}{s+1} \quad (Re(s) > 0) \tag{8.64}$$

$$H(j\omega) = \frac{3e^{-3j\omega}}{1+j\omega} \tag{8.65}$$

$$= \frac{3\,(cos3\omega - jsin3\omega)}{1+j\omega} \tag{8.66}$$

$$|H(j\omega)| = \frac{3}{\sqrt{1+\omega^2}} \tag{8.67}$$

$$phase = \tan^{-1}\left(\frac{\omega\cos(3\omega) + \sin(3\omega)}{\omega\sin(3\omega) - \cos(3\omega)}\right) \tag{8.68}$$

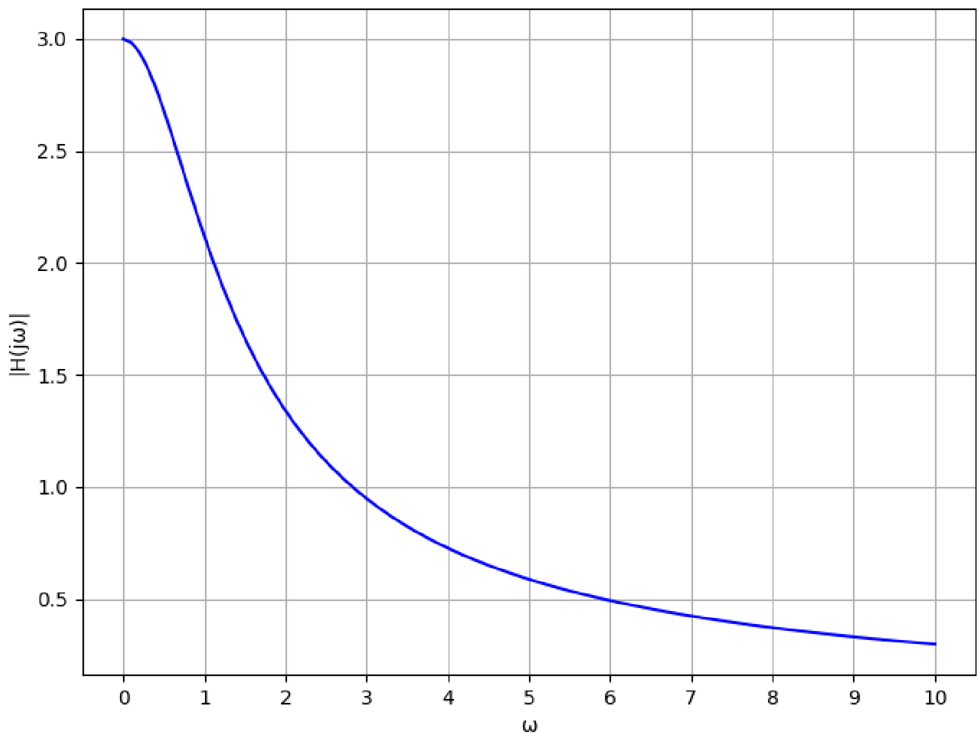


Figure 8.9: Plot for magnitude of transfer function

8.9 Let $x_1(t) = e^{-t}u(t)$ and $x_2(t) = u(t) - u(t-2)$, where $u(.)$ denotes the unit step function. If $y(t)$ denotes the convolution of $x_1(t)$ and $x_2(t)$,then $\lim_{t \to \infty} y(t) =$ ______.
(Rounded off to one decimal place)

(GATE EC 2022)

Solution:

$$y(t) = x_1(t) * x_2(t) \tag{8.69}$$

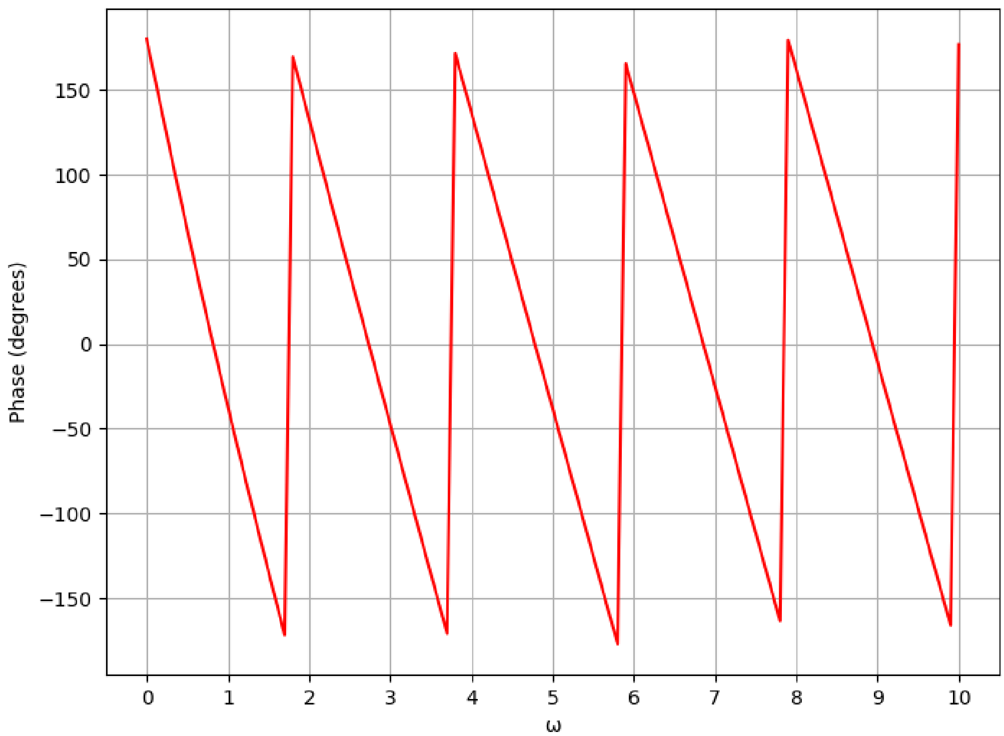


Figure 8.10: Plot for phase of transfer function

variable	value	description
$x_1(t)$	$e^{-t}u(t)$	given function 1
$x_2(t)$	$u(t)-u(t-2)$	given function 2
$y(t)$	-	convolution of $x_1(t)$ and $x_2(t)$

Table 8.7: Table: Input Parameters

from Table 8.7

$$y(t) = e^{-t}u(t) * (u(t) - u(t-2)) \tag{8.70}$$

By applying Laplace transform

$$Y(s) = X_1(s) . X_2(s) \tag{8.71}$$

$$e^{-t}u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{1+s}, \quad \mathrm{Re}(s) > -1 \tag{8.72}$$

$$u(t) - u(t-2) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1-e^{-2s}}{s}, \quad \mathrm{Re}(s) > 0 \tag{8.73}$$

$$Y(s) = \left(\frac{1}{1+s}\right)\left(\frac{1-e^{-2s}}{s}\right), \quad \mathrm{Re}(s) > 0 \tag{8.74}$$

$$= \frac{1-e^{-2s}}{s(s+1)} \tag{8.75}$$

Final value theorem

$$\lim_{t\to\infty} x(t) = \lim_{s\to 0} sX(s) \tag{8.76}$$

$$\tag{8.77}$$

Proof:

$$\mathcal{L}\left[x\left(t\right)\right] = X\left(s\right) = \int_{0}^{\infty} x\left(t\right) e^{-st}\, dt \tag{8.78}$$

$$\mathcal{L}\left[\frac{dx\left(t\right)}{dt}\right] = \int_{0}^{\infty} \frac{d}{dt}\left(x\left(t\right) e^{-st}\right)\, dt \tag{8.79}$$

$$= sX\left(s\right) - x\left(0^{-}\right) \tag{8.80}$$

$$\lim_{s\to 0}\left[\int_{0}^{\infty} \frac{d}{dt}\left(x\left(t\right) e^{-st}\right)\, dt\right] = \lim_{s\to 0}\left[sX\left(s\right) - x\left(0^{-}\right)\right] \tag{8.81}$$

$$\int_{0}^{\infty} \frac{dx\left(t\right)}{dt}\, dt = \lim_{s\to 0}\left[sX\left(s\right) - x\left(0^{-}\right)\right] \tag{8.82}$$

$$\left[x\left(t\right)\right]_{0}^{\infty} = \lim_{s\to 0}\left[sX\left(s\right) - x\left(0^{-}\right)\right] \tag{8.83}$$

$$x\left(\infty\right) - x\left(0^{-}\right) = \lim_{s\to 0}\left[sX\left(s\right) - x\left(0^{-}\right)\right] \tag{8.84}$$

$$\implies x\left(\infty\right) = \lim_{s\to 0} sX\left(s\right) \tag{8.85}$$

$$\lim_{t\to\infty} x\left(t\right) = \lim_{s\to 0} sX\left(s\right) \tag{8.86}$$

By applying Final value theorem

$$\lim_{t\to\infty} y\left(t\right) = \lim_{s\to 0} sY\left(s\right) \tag{8.87}$$

$$= \lim_{s\to 0} s\left(\frac{1-e^{-2s}}{s\left(s+1\right)}\right) \tag{8.88}$$

$$= \lim_{s\to 0}\left(\frac{1-e^{-2s}}{\left(s+1\right)}\right) \tag{8.89}$$

$$= \left(\frac{1-e^{0}}{0+1}\right) \tag{8.90}$$

$$\lim_{t\to\infty} y\left(t\right) = 0 \tag{8.91}$$

8.10 A unity-gain negative-feedback control system has a loop-gain $L(s)$ given by

$$L(s) = \frac{6}{s(s-5)} \tag{8.92}$$

The closed loop system is ____

(a) Causal and stable

(b) Causal and unstable

(c) Non-causal and stable

(d) Non-causal and unstable

(GATE IN 2022)

Solution: From Table 8.8, the transfer function of the system is given by,

Parameter	**Description**	**Value**
$L(s)$	Forward loop transfer function	$\frac{6}{s(s-5)}$
$H(s)$	Feedback path transferfunction	1
$T(s)$	Transfer function	$\frac{L(s)}{1+L(s)H(s)}$

Table 8.8: Parameter Table

$$T(s) = \frac{\frac{6}{s(s-5)}}{1 + 1\frac{6}{s(s-5)}} \tag{8.93}$$

$$= \frac{6}{s^2 - 5s + 6} \tag{8.94}$$

The poles of the system are given by the roots of the denominator of transfer function,

$$s^2 - 5s + 6 = 0 \tag{8.95}$$

$\therefore$ The poles of the system are $s = 2$ and $s = 3$.

As the poles are positive, the output will increase without bound, causing the system to be unstable.

The transfer function of the system is ,

$$T(s) = \frac{6}{(s-2)(s-3)} \tag{8.96}$$

Clearly, it is dependent only on the past values. Hence, the system is causal.

Thus the correct option is B. The system is causal and unstable.

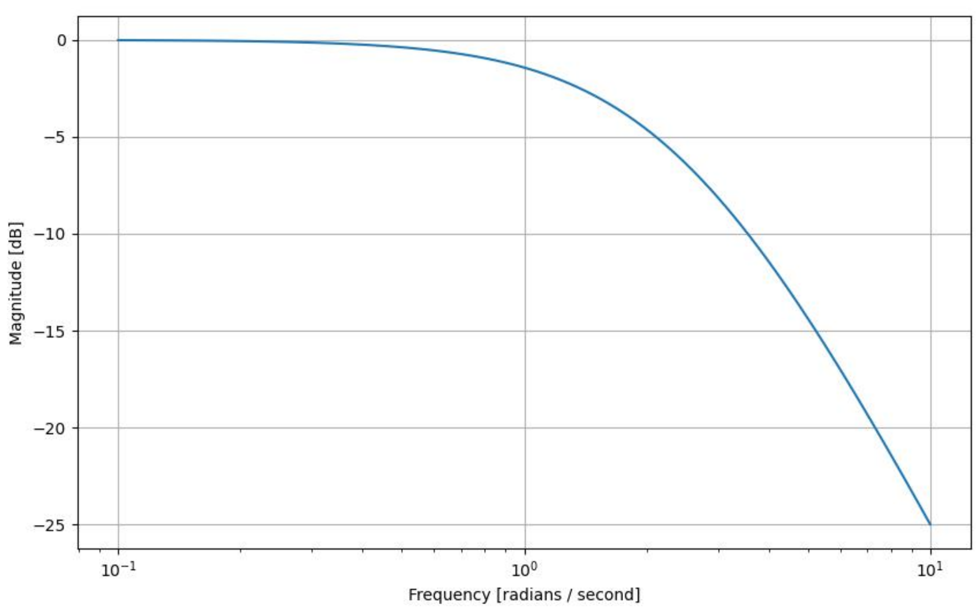


Figure 8.11: Magnitude plot for the transfer function

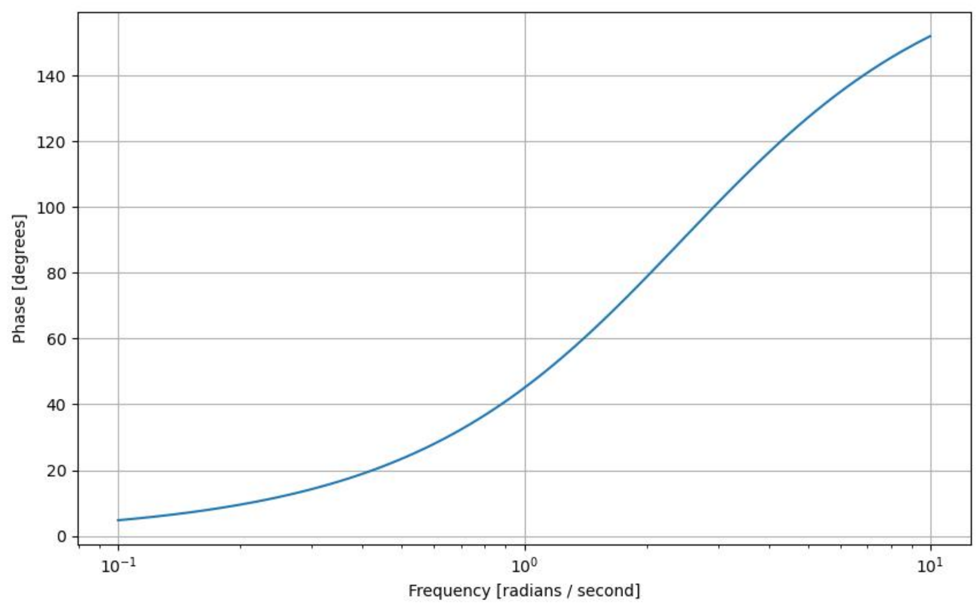


Figure 8.12: Phase plot for the transfer function

8.11 An input $x(t)$ is applied to a system with a frequency transfer function given by $H(j\omega)$ as shown below. The magnitude and phase response of the transfer function are shown below. If $y(t_d) = 0$ for $x(t) = u(t)$, the time $t_d(> 0)$ is.

(Gate 2022 BM.38) **Solution:**

Parameter	Description
$x(t) = u(t)$	Input signal
$y(t)$	Output signal
$X(j\omega)$	Fourier Transform of $x(t)$
$Y(j\omega)$	Fourier Transform of $y(t)$
$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$	Transfer function

Table 8.9: Input Parameters Table

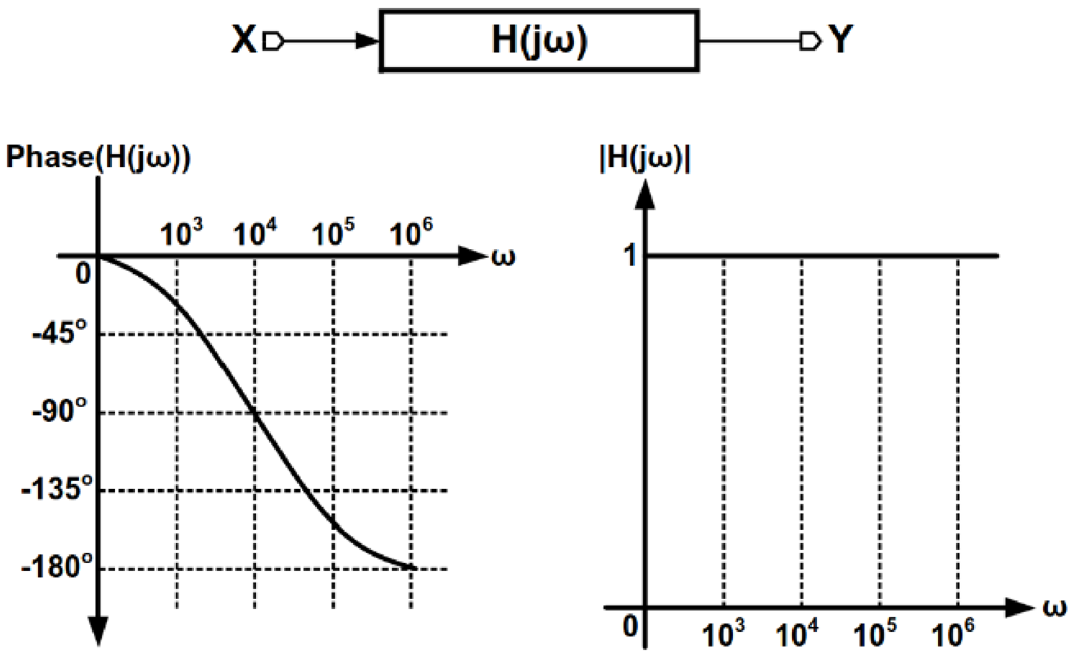


Figure 8.13: Graph of y(t)

from graph 8.14

$$\angle H(j\omega) = -2\tan^{-1}\left(\frac{\omega}{a}\right) \tag{8.97}$$

At $\omega = 10^4, \angle H(j\omega) = -\frac{\pi}{2}$

$$\implies a = 10^4 \tag{8.98}$$

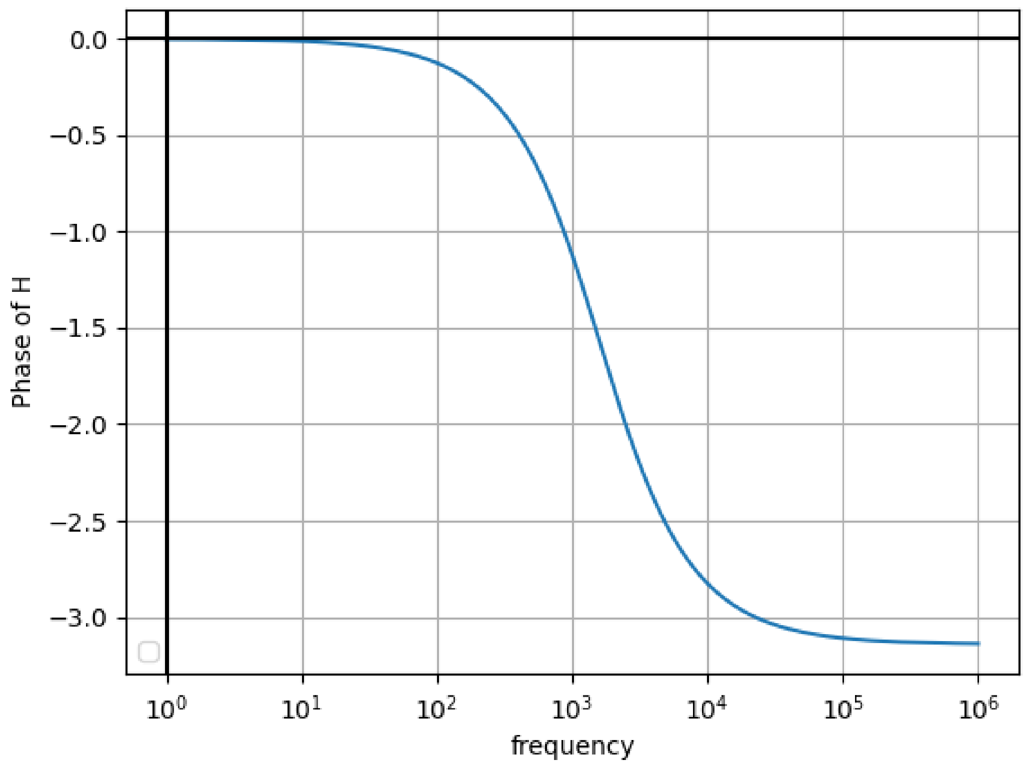


Figure 8.14: Phase of H(f)

$$\angle H(j\omega) = \tan^{-1}\left(\frac{-\omega}{a}\right) - \tan^{-1}\left(\frac{\omega}{a}\right) \tag{8.99}$$

$$H(j\omega) = \frac{e^{j\tan^{-1}\left(\frac{-\omega}{a}\right)}}{e^{j\tan^{-1}\left(\frac{\omega}{a}\right)}} \tag{8.100}$$

$$= \frac{\frac{a-j\omega}{\sqrt{a^2+\omega^2}}}{\frac{a+j\omega}{\sqrt{a^2+\omega^2}}} \tag{8.101}$$

$$= \frac{a-j\omega}{a+j\omega} \tag{8.102}$$

Substitute $s = j\omega$

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \tag{8.103}$$

$$Y(s) = \frac{1}{s}\frac{a-s}{a+s} \tag{8.104}$$

$$= \frac{1}{s} - \frac{2}{a+s} \tag{8.105}$$

$$\frac{1}{s} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} u(t) \tag{8.106}$$

$$\frac{1}{a+s} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} e^{-at}u(t) \tag{8.107}$$

$$y(t) = (1 - 2e^{-at})u(t) \tag{8.108}$$

Verification of laplace transform:

$$\because y(t_d) = 0 \tag{8.109}$$

$$t_d = 100 \ln 2\mu s \tag{8.110}$$

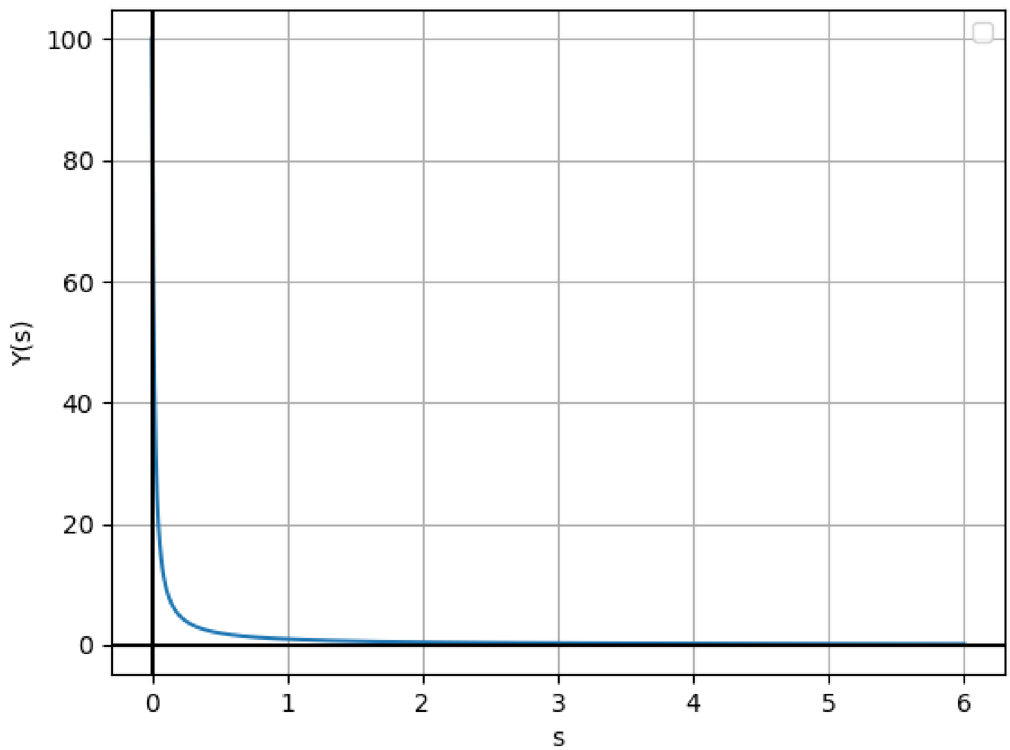


Figure 8.15: Laplace Transform of $u(t)$

8.12 Consider the circuit shown in the figure with input V(t) in volts.The sinusoidal steady state current I(t) flowing through the circuit is shown graphically(where t is in seconds). The circuit element Z can be________. GATE 2022 EC 39

(a) a capacitor of 1 F

(b) an inductor of 1 H

(c) a capacitor of $\sqrt{3}$ H

(d) an inductor of $\sqrt{3}$ H

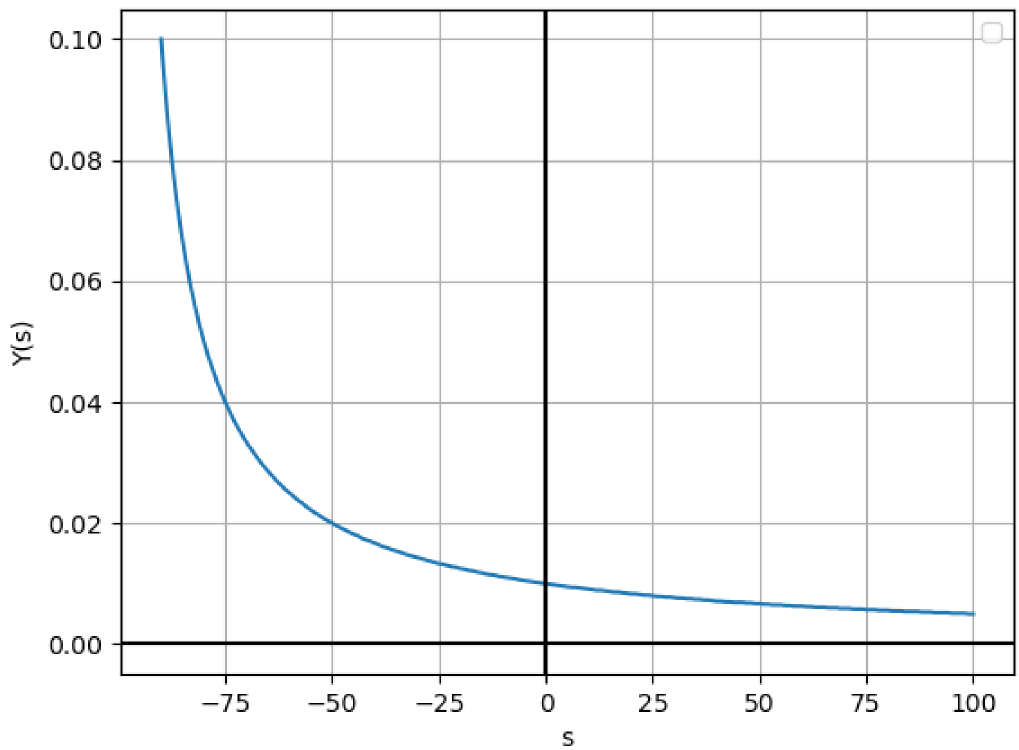
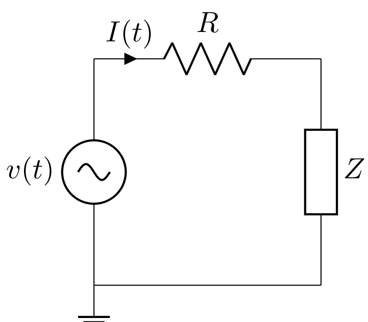
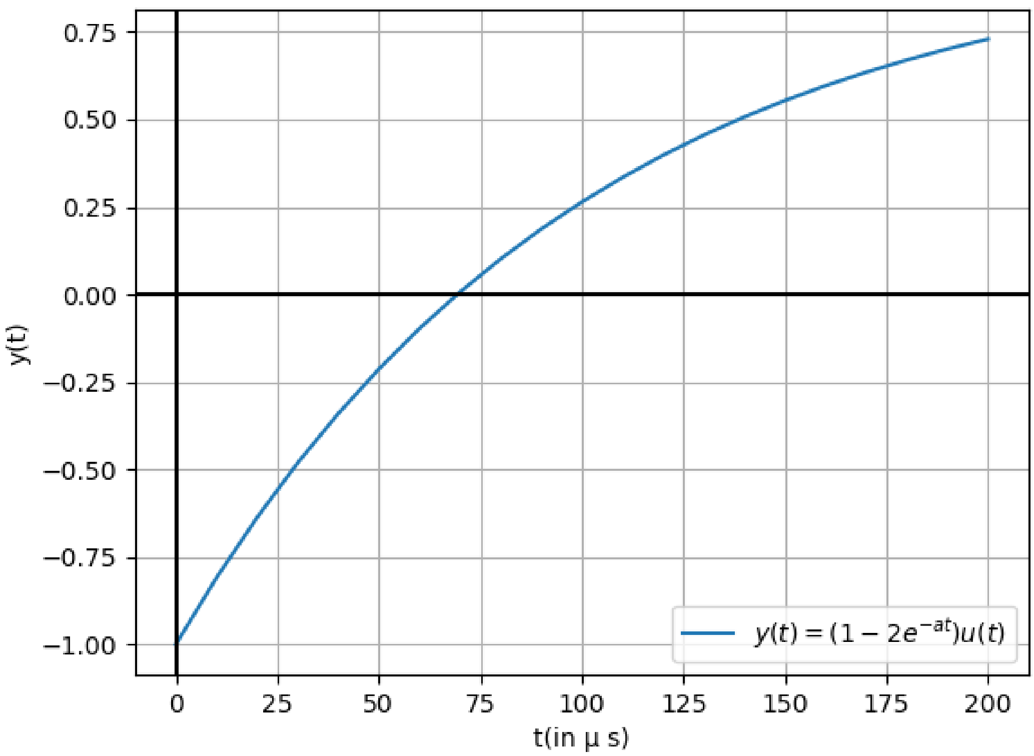


Figure 8.16: Laplace Transform $of e^{-at}u(t)$



Solution:



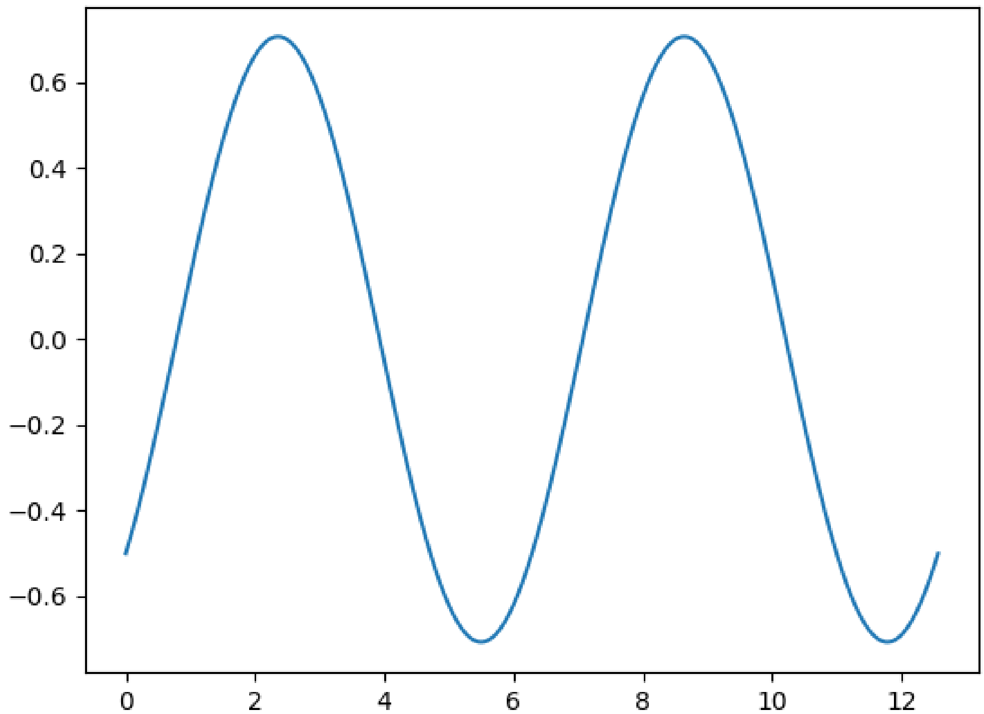
The current through the circuit can be expressed as

$$I(t) = \sin\left(t - \frac{\pi}{4}\right) \tag{8.111}$$

Since, the voltage seems to be leading the current the circuit element z is an inductor with inductance L.

Applying KVL in the circuit,

$$R.I\left(t\right) + L\frac{dI\left(t\right)}{dt} = sin\left(t\right) \tag{8.112}$$



Applying Fourier transform to the differential equation,

$$R.I\left(s\right)+sL.I\left(s\right)-\frac{1}{s^{2}+1}=0 \tag{8.113}$$

$$I\left(s\right)\left(R+sL\right)=\frac{1}{s^{2}+1} \tag{8.114}$$

$$\sin\left(at+b\right)\overset{\mathcal{L}}{\longleftrightarrow}\frac{a\cos\left(b\right)+s\sin\left(b\right)}{a^{2}+s^{2}} \tag{8.115}$$

$$\sin\left(t-\frac{\pi}{4}\right)\overset{\mathcal{L}}{\longleftrightarrow}\frac{1-s}{2\left(s^{2}+1\right)} \tag{8.116}$$

$$\frac{1-s}{2\left(s^{2}+1\right)}\left(R+sL\right)=\frac{1}{s^{2}+1} \tag{8.117}$$

Symbol	Value	Description
$V(t)$	$\sin t$	Time varying voltage source
$I(t)$	$\sin t - \frac{\pi}{4}$	Current flowing in the circuit
R	1Ω	Resistor in series to Z
Z	Z	Circuit element

Table 8.10: Variable description

Upon plugging in R=1Ω,

$$L = \frac{1}{s} \tag{8.118}$$

Applying inverse Laplace,

$$L = 1H \tag{8.119}$$

8.13 Consider the differential equation $\frac{dy}{dx} = 4(x+2) - y$ For the initial condition $y = 3$ at $x = 1$, the value of y at $x = 1.4$ obtained using Euler's method with a step-size of 0.2 is ? (*round off to one decimal place*) (GATE CE 2022)

Solution:

Symbols	Description	Values
R	Residue Formula	$\frac{1}{(m-1)!} \lim_{s \to a} \frac{d^{m-1}}{ds^{m-1}} \left((s-a)^m f(s) e^{st} \right)$
$\phi(x)$	Transformation of $y(x)$	$y(x+1)$
$g(x)$	Euler's Approximated function of f(x)	$g_{(n-1)}(x) + hf'(x_{n-1}, y_{n-1})$
h	Step-size	0.2

Table 8.11: Parameters, Descriptions, and Values

(a) Solution of the differential:

Applying the transformation from table 8.11 and laplace transform

$$s\mathscr{L}(\phi(x)) - \phi(0) = 4\left(\frac{1}{s^2} + \frac{3}{s}\right) - \mathscr{L}(\phi(x)) \tag{8.120}$$

$$\mathscr{L}(\phi(x)) = \frac{3}{s+1} + 4\left(\frac{1}{s^2(s+1)} + \frac{3}{s(s+1)}\right) \tag{8.121}$$

$$= \frac{-5}{s+1} + \frac{8}{s} + \frac{4}{s^2} \tag{8.122}$$

(b) Inverse Laplace Transform:

Using Bromwich integrals and extension of Jordans lemma :

$$\phi(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \mathscr{L}(\phi(x)) e^{st} \, dt, c > 0 \tag{8.123}$$

$$= \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \left(\frac{-5}{s+1} + \frac{8}{s} + \frac{4}{s^2}\right) e^{st} \, dt \tag{8.124}$$

Here, the poles $s = -1$ (non repeated, $m = 1$) and $s = 0$ (repeated, $m = 2$) lie inside a semicircle for some $c > 0$. Using method of residues from 8.11:

$$R_1 = \lim_{s \to -1} \left((s+1) \left(\frac{-5}{s+1} \right) e^{st} \right) \tag{8.125}$$

$$= -5e^{-t} \tag{8.126}$$

$$R_2 = \lim_{s \to 0} \left((s) \left(\frac{8}{s} \right) e^{st} \right) \tag{8.127}$$

$$= 8 \tag{8.128}$$

$$R_3 = \frac{1}{(1)!} \lim_{s \to 0} \frac{d}{dz} \left((s)^2 \left(\frac{4}{s^2} \right) e^{st} \right) \tag{8.129}$$

$$= 4t \tag{8.130}$$

$$\phi(t) = R_1 + R_2 + R_3 \tag{8.131}$$

$$= -5e^{-t} + 8 + 4t \tag{8.132}$$

Reverting to $y(x)$, we get :

$$y(x) = -5e^{-x+1} + x + 4x \tag{8.133}$$

Now, approaching to $y(1.4)$ using euler's approximation from 8.11

$$g_{(1.2)} = 3 + (0.2)\, f'\,(1, 3) \tag{8.134}$$

$$= 4.8 \tag{8.135}$$

$$g_{(1.4)} = 4.8 + (0.2)\, f'\,(1.2, 4.8) \tag{8.136}$$

$$= 6.4 \tag{8.137}$$

$$\implies y(1.4) \approx 6.4 \tag{8.138}$$

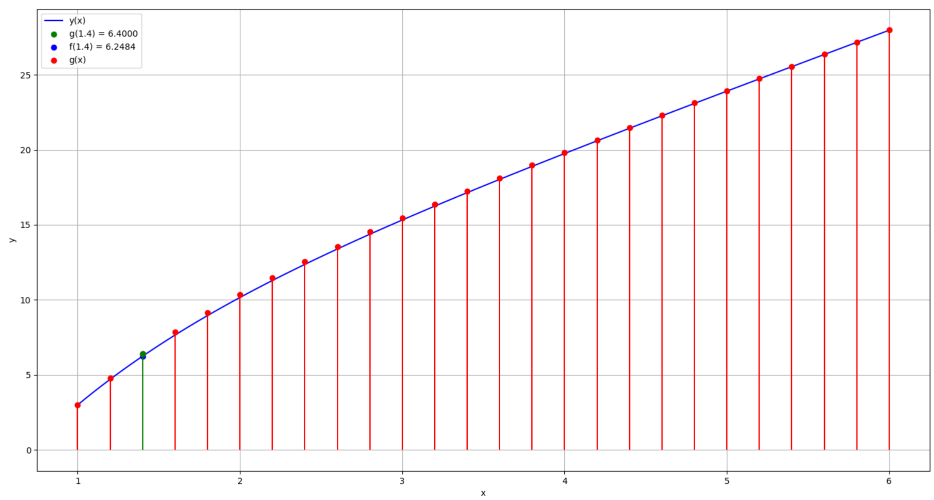


Figure 8.17: Euler's approximated function vs Original function

8.14 A proportional-integral-derivative (PID) controller is employed to stably control a plant with transfer function

$$P(s) = \frac{1}{(s+1)(s+2)} \tag{8.139}$$

Now, the proportional gain is increased by a factor of 2, the integral gain is increased by a factor of 3, and the derivative gain is left unchanged. Given that the closed-loop system continues to remain stable with the new gains, the steady-state error in tracking a ramp reference signal (GATE IN 2022)

Solution:

The transfer function of PID controller,

Parameter	**Description**
K_P	Proportional Gain
K_I	Integral Gain
K_D	Derivative Gain
$r(t)$	Reference Input
$G_c(t)$	Controller Output
$L(t)$	Plant Output
$e(t)$	Error Input

Table 8.12:

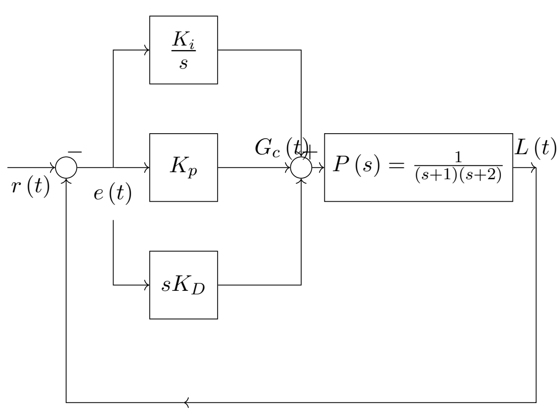


Figure 8.18: Block Diagram of System

$$G_c(s) = K_P + \frac{K_I}{s} + sK_D \tag{8.140}$$

$$= \frac{s^2K_D + sK_P + K_I}{s} \tag{8.141}$$

Overall loop-transfer function,

$$L(s) = G_c(s) \cdot P(s) \tag{8.142}$$

$$L(s) = \frac{s^2K_D + sK_P + K_I}{s(s+1)(s+2)} \tag{8.143}$$

Steady-state error due to ramp signal,

$$e_{ss} = \frac{1}{K_v} \tag{8.144}$$

where,

$$K_v = \lim_{s \to 0} sL(s) \tag{8.145}$$

$$K_v = \frac{K_I}{2} \tag{8.146}$$

$$e_{ss} = \frac{2}{K_I} \tag{8.147}$$

Now, the proportional gain is increased by a factor of 2, the integral gain is increased by a factor of 3, and the derivative gain is left unchanged.

$$K_P' = 2K_P, K_I' = 3K_I \text{ and } K_D' = K_D \tag{8.148}$$

$$K'_v = \lim_{s\to 0} sL'(s) \tag{8.149}$$

$$K'_v = \frac{3K_I}{2} \tag{8.150}$$

$$e'_{ss} = \frac{1}{K'_V} = \frac{2}{3K_I} \tag{8.151}$$

8.15 Let $y(x)$ be the solution of the differential equation

$$y^{''} - 4y^{'} - 12y = 3e^{5x} \tag{8.152}$$

satisfying $y(0) = \frac{18}{7}$ and $y^{'}(0) = \frac{-1}{7}$.

Then $y(1)$ is ____________ (rounded off to nearest integer).

GATE NM 2022

Solution:

Parameter	Description	Value
$y^{''} - 4y^{'} - 12y = 3e^{5x}$	Differential equation	none
$y(x)$	Solution of differential equation	$y(0) = \frac{18}{7}$
$y^{'}(x)$	First order derivative of solution of differential equation	$y^{'}(0) = \frac{-1}{7}$

Table 8.13: Input Parameters

$$y^{''}(t) \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0) - y^{'}(0) \tag{8.153}$$

$$y^{'}(t) \overset{\mathcal{L}}{\longleftrightarrow} sY(s) - y(0) \tag{8.154}$$

$$y(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) \tag{8.155}$$

$$e^{at} \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s-a} \tag{8.156}$$

Applying Laplace transform on both sides of (8.152),

$$\mathcal{L}\left(y^{''}(t) - 4y^{'}(t) - 12y(t)\right) = \mathcal{L}\left(3e^{5x}\right) \tag{8.157}$$

From (8.153), (8.154), (8.155), (8.156)

$$Y(s)\left(s^2-4s-12\right)-y(0)\,(s-4)-y^{'}(0)=\frac{3}{s-5} \tag{8.158}$$

$$Y(s)\left(s^2-4s-12\right)-\frac{(18s-73)}{7}=\frac{3}{(s-5)} \tag{8.159}$$

$$Y(s)=\frac{3}{(s-5)\,(s^2-4s-12)}+\frac{(18s-73)}{7\,(s^2-4s-12)} \tag{8.160}$$

$$\Longrightarrow Y(s)=\frac{1}{(s-6)}-\frac{3}{7\,(s-5)}+\frac{1}{(s+2)} \tag{8.161}$$

$$\frac{1}{s-a}\overset{\mathcal{L}^{-1}}{\longleftrightarrow}e^{at} \tag{8.162}$$

Now finding Inverse Laplace Transform on both sides of (8.161) ,

From (8.162)

$$\Longrightarrow y(t)=\left(e^{6t}-\frac{3}{7}e^{5t}+2e^{-2t}\right)u(t) \tag{8.163}$$

$$\Longrightarrow y(1)=e^{6}-\frac{3}{7}e^{5}+2e^{-2} \tag{8.164}$$

$$\therefore y(1)=340 \tag{8.165}$$

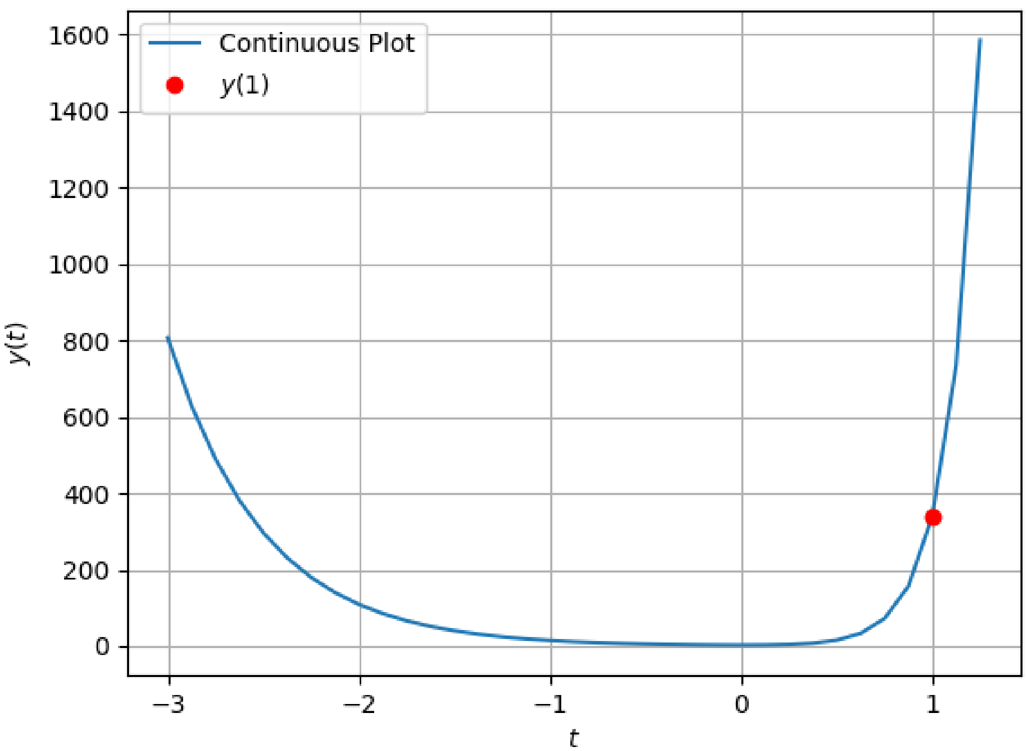


Figure 8.19:

8.16 The signal $x(t) = (t-1)^2 u(t-1)$,where u(t) is unit-step function,has the Laplace transform X(s).The Value of X(1) is

(a) $\frac{1}{e}$

(b) $\frac{2}{e}$

(c) $2e$

(d) e^2

(GATE 2022 IN 40)

Solution:

PARAMETER	VALUE	DESCRIPTION
$x(t)$	$x(t) = (t{-}1)^2u(t{-}1)$	Function

Table 8.14: INPUT PARAMETER TABLE

$$x(t) = (t-1)^2u(t-1) \tag{8.166}$$

Taking Laplace-Transform:

$$t^nu(t) \leftrightarrow \frac{n!}{s^{n+1}} \tag{8.167}$$

if $X(s)$ is Laplace transform of $x(t)$ then,

$$x(t-t_0) = e^{-st_0}X(s) \tag{8.168}$$

using 8.167 and 8.168

$$(t-1)^2u(t-1) \leftrightarrow \frac{2e^{-s}}{s^3} \tag{8.169}$$

$$X(s) = \frac{2e^{-s}}{s^3} \tag{8.170}$$

$$X(1) = \frac{2}{e} \tag{8.171}$$

$\therefore$ 2 is Correct.

8.17 The bridge shown is balanced when $R_1 = 100\Omega, R_2 = 210\Omega, C_2 = 2.9\mu F, R_4 = 50\Omega$. The 2kHz sine-wave generator supplies a voltage of $10V_{p-p}$. The value of L_3(in mH) is?

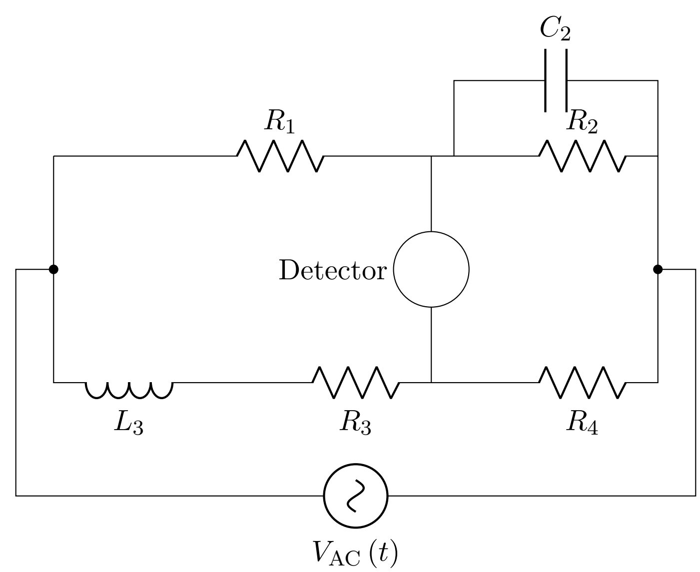


Figure 1: Circuit in T domain

(GATE 2022 IN 63) **Solution:**

Variables	Description	value
R_1	Resistor 1	100Ω
R_2	Resistor 2	210Ω
R_3	Resistor 3	?
R_4	Resistor 4	50Ω
C_2	Capacitor 2	$2.9\mu F$
L_3	Inductor 3	?

Table 1: Caption

The circuit in S domain is

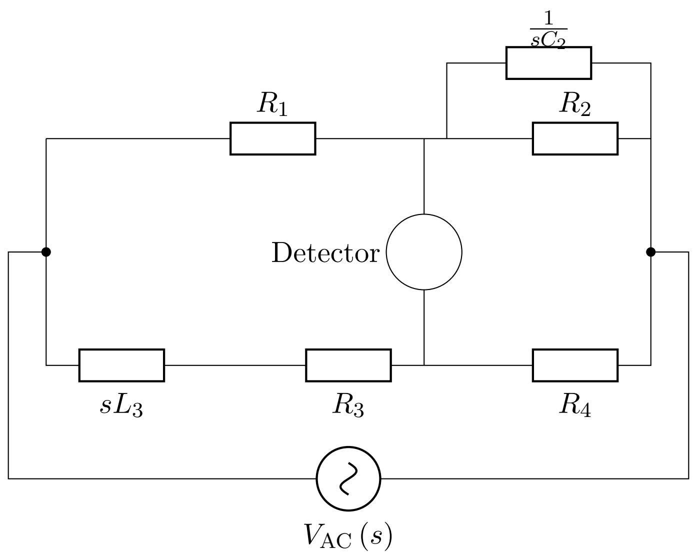


Figure 2: Circuit in S domain

Applying Wheatstone bridge condition,

$$\frac{R_1}{sL_3 + R_3} = \frac{\frac{1}{\frac{1}{R_2} + sC_2}}{R_4} \tag{8.172}$$

$$\implies \frac{R_1}{sL_3 + R_3} = \frac{1}{\left(\frac{1}{R_2} + sC_2\right) R_4} \tag{8.173}$$

$$\implies \frac{sL_3 + R_3}{R_1} = \left(\frac{1}{R_2} + sC_2\right) R_4 \tag{8.174}$$

$$\implies \frac{sL_3}{R_1} + \frac{R_3}{R_1} = \frac{R_4}{R_2} + sC_2 R_4 \tag{8.175}$$

Comparing coefficients, we get

$$\frac{L_3}{R_1} = C_2 R_4 \tag{8.176}$$

$$\implies L_3 = R_1 C_2 R_4 \tag{8.177}$$

$$\frac{R_3}{R_1} = \frac{R_4}{R_2} \tag{8.178}$$

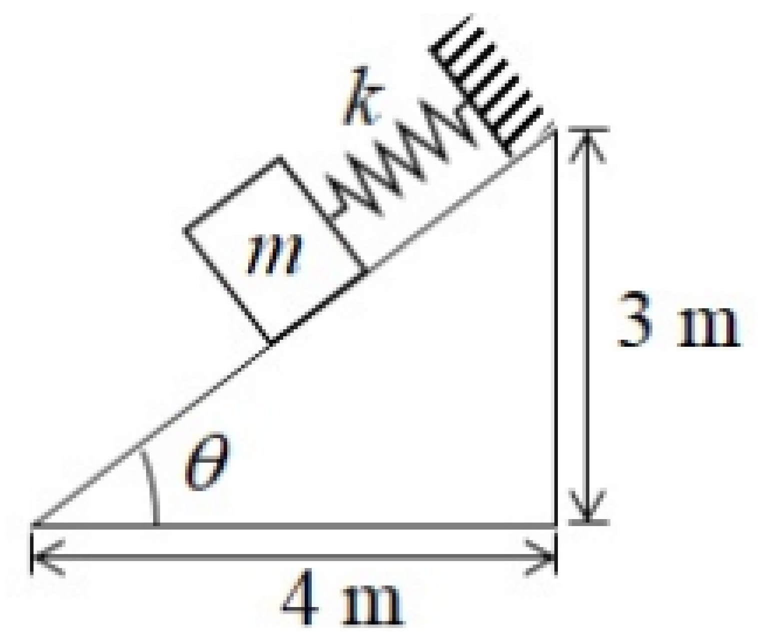
$$\implies R_3 = \frac{R_1 R_4}{R_2} \tag{8.179}$$

From Table 1, substituting in (8.177) and (8.179), we get

$$L_3 = 14.5mH \tag{8.180}$$

$$R_3 = 23.80\Omega \tag{8.181}$$

8.18 A mass m = 10 kg is attached to a spring as shown in the figure. The coefficient of friction between the mass and the inclined plane is 0.25. Assume that the acceleration due to gravity is 10 m/s^2 and that static and kinematic friction coefficients are the same. Equilibrium of the mass is impossible if the spring force is



(a) 30 N

(b) 45 N

(c) 60 N

(d) 75 N

(GATE XE 2022)

Solution: If the spring force is imum, frictional force is downwards and block is just about to move upwards and is at rest and equilibrium currently.

Parameter	Description	Value
m	Mass of object	10 Kg
μ	Frictional coefficient (*static*)	0.25
x(t)	Displacement of block	
$x\,(0)$	Initial displacement	0 (*assumed*)
g	Gravitational acceleration	10 m/s^2
F_s	Spring force	
f	frictional force	μ N
N	Normal Force	mg $cos\,(\theta)$

Table 8.16: Parameter Table

From Fig. 8.22 and Table 8.16, the force equation for the object is

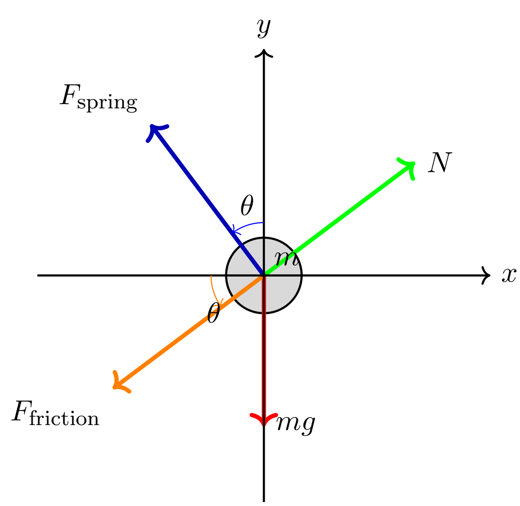


Figure 8.22: Maximum spring force FBD

$$\mu mg\cos\theta + mg\sin\theta - F_s = m\frac{d^2x}{dt^2} \tag{8.182}$$

the Laplace transform of terms is

$$k \overset{\mathcal{L}}{\longleftrightarrow} \frac{k}{s} \tag{8.183}$$

$$\frac{d^2x}{dt^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2 X(s) - sx(0) - \dot{x}(0) \tag{8.184}$$

Applying Laplace transform to equation (8.182),

$$\frac{\mu mg\cos\theta + mg\sin\theta - F_s}{s} = s^2 X(s) - sx(0) - \dot{x}(0) \tag{8.185}$$

$$\implies \frac{\mu mg\cos\theta + mg\sin\theta - F_s}{s^3} = X(s) \tag{8.186}$$

The inverse Laplace transform is

$$\frac{k}{s^3} \overset{\mathcal{L}^-}{\longleftrightarrow} \frac{k}{2}t^2 \tag{8.187}$$

The inverse Laplace of (8.186) is

$$(\mu mg\cos\theta + mg\sin\theta - F_s)\, t^2 = x(t) \tag{8.188}$$

As it is always at equilibrium, $\frac{dx}{dt}$ is 0

$$2t\,(\mu mg\cos\theta + mg\sin\theta - F_s) = 0 \tag{8.189}$$

$$\implies \mu mg\cos\theta + mg\sin\theta - F_s = 0 \tag{8.190}$$

$$\implies \mu mg\cos\theta + mg\sin\theta = F_s \tag{8.191}$$

Using Table 8.16 , the maximum value for equilibrium is

$$F_s = 80N \tag{8.192}$$

Now consider the other case. F_s is minimum possible for equilibrium. The block is about to move downwards.

From Fig. 8.23 and Table 8.16, the force equation for the object is

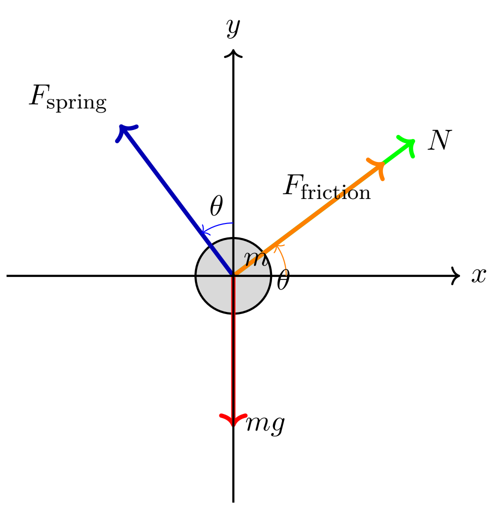


Figure 8.23: Minimum spring force FBD

$$mg\sin\theta - \mu mg\cos\theta - F_s = m\frac{d^2x}{dt^2} \tag{8.193}$$

Laplace transform is

$$\frac{mg\sin\theta - \mu mg\cos\theta - F_s}{s^3} = X\left(s\right) \tag{8.194}$$

The inverse Laplace of (8.186) is

$$(mg\sin\theta - \mu mg\cos\theta - F_s)\, t^2 = x\,(t) \tag{8.195}$$

Hence the minimum force for equilibrium is

$$F_s = mg\sin\theta - \mu mg\cos\theta \tag{8.196}$$

$$= 40N \tag{8.197}$$

Hence , block is in equilibrium for F_s between 40 and 80N. At 30 N, it is not at equilibrium.

8.19 The Bode magnitude plot of a first order stable system is constant with frequency. The asymptotic value of the high frequency phase, for the system, is $-180°$. This system has

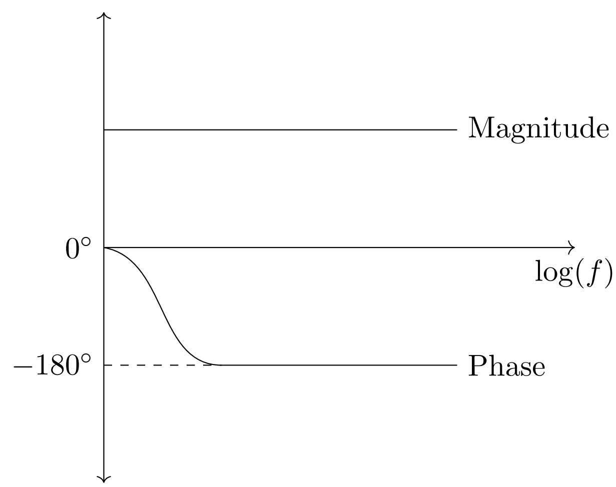


Figure 8.24:

(A) one LHP pole and one RHP zero at the same frequency.

(B) one LHP pole and one LHP zero at the same frequency.

(C) two LHP poles and one RHP zero.

(D) two RHP poles and one LHP zero.

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Solution: Flat constant magnitude response for all frequency of system shows that it is an all pass system.

In all pass system, poles and zeros are symmetrical about $j\omega$ axis.

Possible transfer functions are

$$T_1(s) = \frac{s-a}{s+a} \quad a > 0 \tag{8.198}$$

$$T_2(s) = \frac{a-s}{a+s} \quad a > 0 \tag{8.199}$$

$$s = j\omega \tag{8.200}$$

From the phase plot as $\omega \longleftrightarrow \infty$ shows $\phi = -180°$.

(a) For $T_1(s)$:

Using equation (8.200)

$$T_1(j\omega) = \frac{j\omega - a}{j\omega + a} \quad a > 0 \tag{8.201}$$

$$\angle T_1(j\omega) = 180° - \tan^{-1}\left(\frac{\omega}{a}\right) - \tan^{-1}\left(\frac{\omega}{a}\right) \tag{8.202}$$

$$= 180° - 2\tan^{-1}\left(\frac{\omega}{a}\right) \tag{8.203}$$

At $\omega = \infty$,

$$\angle T_1(j\omega) = 0° \tag{8.204}$$

(b) For $T_2(s)$:

Using equation (8.200)

$$T_2(j\omega) = \frac{a - j\omega}{a + j\omega} \quad a > 0 \tag{8.205}$$

$$\angle T_2(j\omega) = -\tan^{-1}\left(\frac{\omega}{a}\right) - \tan^{-1}\left(\frac{\omega}{a}\right) \tag{8.206}$$

$$= -2\tan^{-1}\left(\frac{\omega}{a}\right) \tag{8.207}$$

At $\omega = \infty$,

$$\angle T_2(j\omega) = -180^\circ \tag{8.208}$$

Hence, the transfer function of given all pass filter.

$$T(s) = \frac{a - s}{a + s} \quad a > 0 \tag{8.209}$$

Hence, the system has one LHP pole and one RHP zero at the same frequency.

8.20 If

$$g(t) = \frac{df(t)}{dt} \tag{8.210}$$

$$F(s) = \frac{1+s}{s^2+12s+32} \tag{8.211}$$

where $F(s)$ is the Laplace transform of the function $f(t)$, then what is the value of $g(t)$ at $t = 0$?

(GATE BM 2022)

Solution:

Value	Parameter	Description
$g(t)$	$\frac{df(t)}{dt}$	Derivative of $f(t)$ with respect to t
$F(s)$	$\frac{1+s}{s^2+12s+32}$	Laplace transform of the function $f(t)$

Table 8.17: Given Parameters

Using Initial value Theorem

$$f(0) = \lim_{s\to\infty} sF(s) \tag{8.212}$$

$$= \lim_{s\to\infty} \frac{s(s+1)}{s^2+12s+32} \tag{8.213}$$

$$= 1 \tag{8.214}$$

$$G(s) = sF(s) - f(0) \tag{8.215}$$

$$= \frac{s(s+1)}{s^2+12s+32} - 1 \tag{8.216}$$

$$= \frac{s^2+s-(s^2+12s+32)}{s^2+12s+32} \tag{8.217}$$

$$G(s) = \frac{-11s-32}{s^2+12s+32} \tag{8.218}$$

Using Partial fraction decomposition

$$G(s) = \frac{A}{s+4} + \frac{B}{s+8} \tag{8.219}$$

$$-11s - 32 = A(s+8) + B(s+4) \tag{8.220}$$

$$-11s - 32 = (A+B)s + (8A+4B) \tag{8.221}$$

Equating coefficients:

$$-11 = A + B \tag{8.222}$$

$$-32 = 8A + 4B \tag{8.223}$$

By solving these equations , we get

$$A = 3 \tag{8.224}$$

$$B = -14 \tag{8.225}$$

$$G(s) = \frac{3}{s+4} - \frac{14}{s+8}; \quad Re(s) > -4 \tag{8.226}$$

Inverse Laplace transform of $G(s)$

$$g(t) = \mathcal{L}^{-1}\left(\frac{3}{s+4}\right) - \mathcal{L}^{-1}\left(\frac{14}{s+8}\right) \tag{8.227}$$

$$= 3e^{-4t} - 14e^{-8t} \tag{8.228}$$

$$g(0) = 3e^{-4\cdot 0} - 14e^{-8\cdot 0} \tag{8.229}$$

$$= 3 - 14 \tag{8.230}$$

$$= -11 \tag{8.231}$$

Verifying g(0) by Initial value theorem

$$g(0) = \lim_{s\to\infty} sG(s) \tag{8.232}$$
$$= \lim_{s\to\infty} s \cdot \frac{-11s - 32}{s^2 + 12s + 32} \tag{8.233}$$
$$= \lim_{s\to\infty} \frac{-11 - \frac{32}{s}}{1 + \frac{12}{s} + \frac{32}{s^2}} \tag{8.234}$$
$$= \lim_{s\to\infty} \frac{-11}{1} \tag{8.235}$$
$$= -11 \tag{8.236}$$

The value of $g(t)$ at $t = 0$ is -11.

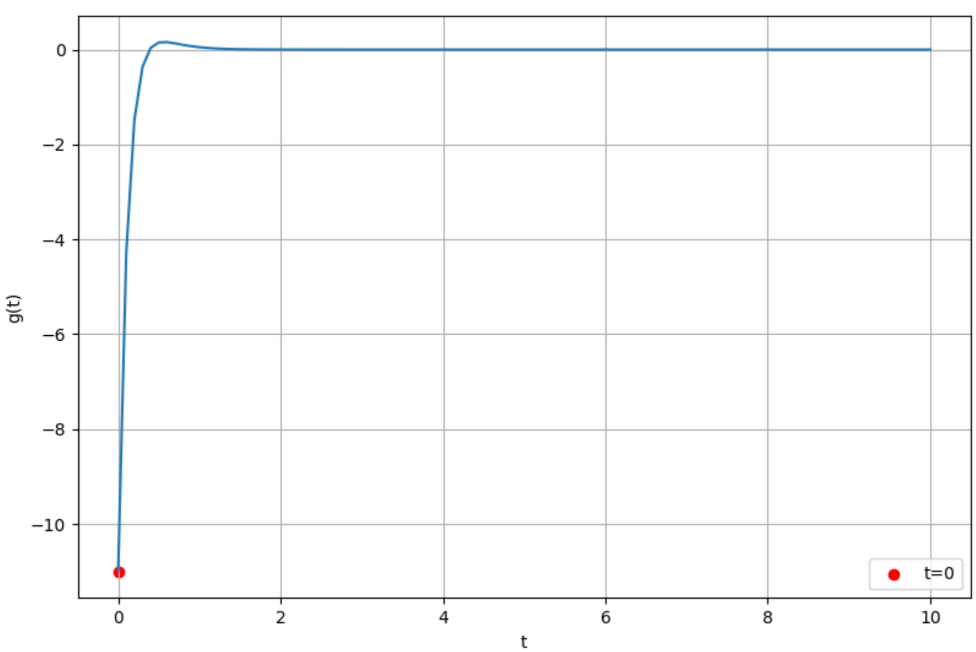


Figure 8.25: Plot $g(t)$ vs t

8.21 Solution of the differential equation $\frac{dy}{dx} - y = cos\left(x\right)$ is

$$(A) \quad y = \frac{sin\left(x\right) - cos\left(x\right)}{2} + ce^{x} \tag{8.237}$$

$$(B) \quad y = \frac{sin\left(x\right) + cos\left(x\right)}{2} + ce^{x} \tag{8.238}$$

$$(C) \quad y = \frac{sin\left(x\right) + cos\left(x\right)}{2} + ce^{-x} \tag{8.239}$$

$$(D) \quad y = \frac{sin\left(x\right) - cos\left(x\right)}{2} + ce^{-x} \tag{8.240}$$

(GATE BM 2022)

Solution:

$$\frac{dy}{dx} - y = cos\left(x\right) \tag{8.241}$$

Apply laplace transform

$$\mathcal{L}\left(\frac{dy}{dx}\right) - \mathcal{L}\left(y\right) = \mathcal{L}\left(cos\left(x\right)\right) \tag{8.242}$$

parameter	**laplace transform**
$\frac{dy}{dx}$	$sY\left(s\right) - y\left(0\right)$
y	$Y\left(s\right)$
$cos\left(x\right)$	$\frac{s}{s^2+1}$
$sin\left(x\right)$	$\frac{1}{s^2+1}$
e^{x}	$\frac{1}{s-1}$

Table 8.18: transformation

$$\implies sY(s) - y(0) - Y(s) = \frac{s}{s^2+1} \tag{8.243}$$

$$Y(s)(s-1) = y(0) + \frac{s}{s^2+1} \tag{8.244}$$

$$Y(s) = \frac{s + y(0)\left(s^2+1\right)}{(s-1)(s^2+1)} \tag{8.245}$$

$$Y(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+1} \tag{8.246}$$

$$A = y(0), B = \frac{-1}{2}, c = \frac{1}{2} \tag{8.247}$$

$$\implies Y(s) = \frac{y(0)}{s-1} + \frac{-s+1}{2(s^2+1)} \tag{8.248}$$

apply inverse laplace transform

$$\mathcal{L}^{-}(Y(s)) = \mathcal{L}^{-}\left(\frac{y(0)}{s-1}\right) + \mathcal{L}^{-}\left(\frac{-s+1}{2(s^2+1)}\right) \tag{8.249}$$

$$y(x) = y(0)e^x + \frac{sin(x) - cos(x)}{2} \tag{8.250}$$

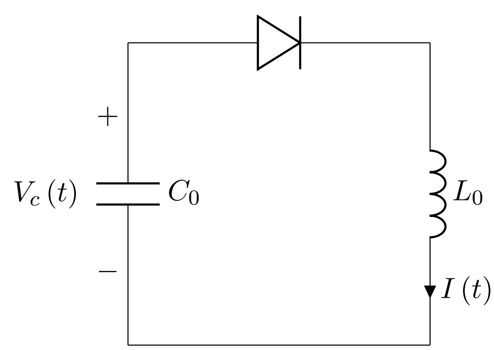
$$y(0) = c \tag{8.251}$$

As both are constants

$$\implies y = \frac{sin(x) - cos(x)}{2} + ce^x \tag{8.252}$$

Option A is correct

8.22 In the circuit shown, the capacitance $C_0 = 10\mu F$ and inductance $L_0 = 1mH$ and the diode is ideal. The capacitor is initially charged to 10V and the current in the inductor is initially zero. If the switch is closed at t=0, the voltage $V_c(t)$(in volts) across the capacitor at t=0.5s is? (round off to one decimal place)



(GATE IN 2022)

Solution:

$$V_c\left(0^-\right) = 10V \tag{8.253}$$

$$t > 0 \tag{8.254}$$

convert circuit into laplace form

parameter	**laplace transform**
C_0	$\frac{1}{sC_0} - V\left(0^-\right)C_0$
L_0	sL_0
$i\left(t\right)$	$I\left(s\right)$
$10cos\left(10^{-4}t\right)V$	$\frac{10V}{10^{-8}s^2-1}$
$sin\left(10^4t\right)A$	$\frac{10^{-7}sA}{10^{-8}s^2-1}$
$v\left(t\right)$	$V\left(s\right)$

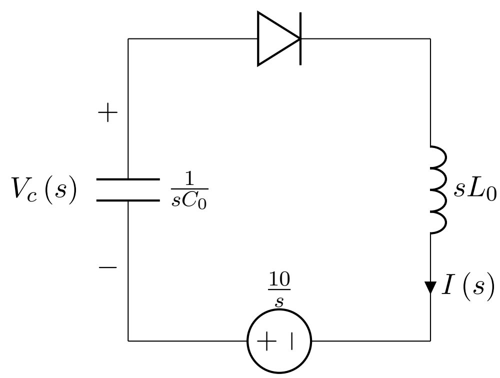


Figure 8.26: s domain circuit

apply KVL,

$$-V_c(s) + I\left(s\right)sL_0 - \frac{10}{s} = 0 \tag{8.255}$$

$$\frac{10}{s} - I\left(s\right)sL_0 = -V_c(s) \tag{8.256}$$

$$\frac{10}{s} - V_c(s)sC_0sL_0 = -V_c(s) \tag{8.257}$$

$$\frac{10V}{10^{-8}s^2-1} = V_c(s) \tag{8.258}$$

apply inverse laplace transform

$$V_c\left(t\right) = 10cos\left(10^{-4}t\right)V \tag{8.259}$$

for the inductor

$$V_L(s) = \frac{I_L(s)}{sL_0C_0} \tag{8.260}$$

$$I_L(s) = \frac{10^{-7}sA}{10^{-8}s^2 - 1} \tag{8.261}$$

apply inverse laplace transform

$$i_L(t) = sin\left(10^4 t\right) A \tag{8.262}$$

At,

$$10^4 t = \pi \tag{8.263}$$

$$i_L(\pi) = 0 \tag{8.264}$$

$$V_c(\pi) = 10cos(\pi) = -10V \tag{8.265}$$

So, this time capacitor plates are changed opposite to its initial, so after

$$10^4 t = \pi, \tag{8.266}$$

$$t = \frac{\pi}{10^4} \tag{8.267}$$

$$t = 10^{-4}\pi sec \tag{8.268}$$

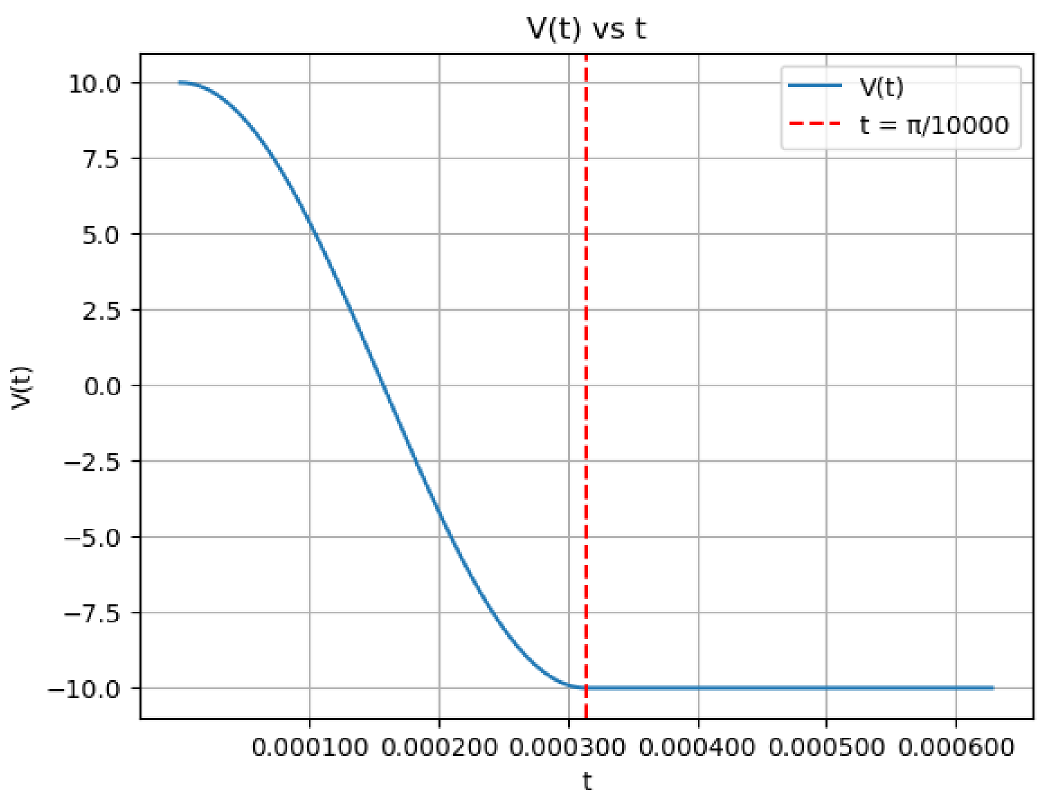
capacitor voltage is always

$$-10V \tag{8.269}$$

as,

$$0.5s > 10^{-4}\pi \tag{8.270}$$

$$\implies V_c\,(0.5) = -10V \tag{8.271}$$



8.2. 2021

8.1 For the ordinary differential equation

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 1,$$

with initial conditions $y(0) = y'(0) = y''(0) = y'''(0) = 0$, the value of

$$\lim_{t \to \infty} y(t) = ?$$

(round off to 3 decimal places). (GATE CH 2021)

Solution:

Parameter	Value	Description
$y(0)$	0	Initial displacement
$y'(0)$	0	First derivative at $t = 0$
$y''(0)$	0	Second derivative at $t = 0$
$y'''(0)$	0	Third derivative at $t = 0$

Table 8.19: Parameters

$$\frac{d^3y}{dt^3} + 6\frac{d^2y}{dt^2} + 11\frac{dy}{dt} + 6y = 1 \tag{8.272}$$

Applying the Laplace transform to both sides:

$$s^3Y(s) + 6s^2Y(s) + 11sY(s) + 6Y(s) = \frac{1}{s} \tag{8.273}$$

$$Y(s)\left(s^3 + 6s^2 + 11s + 6\right) = \frac{1}{s} \tag{8.274}$$

$$\implies Y(s) = \frac{1}{s\left(s+1\right)\left(s+2\right)\left(s+3\right)}; \quad Re(s) > 0 \tag{8.275}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2} + \frac{D}{s+3} \tag{8.276}$$

$$1 = A(s+1)(s+2)(s+3) + Bs(s+2)(s+3) + Cs(s+1)(s+3) + Ds(s+1)(s+2) \tag{8.277}$$

$$1 = A\left(s^3 + 6s^2 + 11s + 6\right) + Bs\left(s^2 + 5s + 6\right) + Cs\left(s^2 + 4s + 3\right) + Ds\left(s^2 + 3s + 2\right) \tag{8.278}$$

$$\tag{8.279}$$

Comparing the coefficients on both sides

$$A + B + C + D = 0 \tag{8.280}$$

$$6A + 5B + 4C + 3D = 0 \tag{8.281}$$

$$11A + 6B + 3C + 2D = 0 \tag{8.282}$$

$$6A = 1 \tag{8.283}$$

$$A = 1/6, B = -11/26, C = 5/26, D = 5/78 \tag{8.284}$$

Substitute these values

$$Y(s) = \frac{6}{s} - \frac{11}{26(s+1)} + \frac{5}{26(s+2)} + \frac{5}{78(s+3)} \tag{8.285}$$

Apply Inverse Laplace Transform

$$y(t) = 6\mathcal{L}^{-1}\left(\frac{1}{s}\right) - 11\mathcal{L}^{-1}\left(\frac{1}{26\,(s+1)}\right) + 5\mathcal{L}^{-1}\left(\frac{1}{26\,(s+2)}\right) + 5\mathcal{L}^{-1}\left(\frac{1}{78\,(s+3)}\right) \tag{8.286}$$

$$y(t) = 6 - \frac{11}{26}e^{-t} + \frac{5}{26}e^{-2t} + \frac{5}{78}e^{-3t} \tag{8.287}$$

Consider

$$\lim_{t\to\infty} y(t) = \lim_{t\to\infty}\left(6 - \frac{11}{26}e^{-t} + \frac{5}{26}e^{-2t} + \frac{5}{78}e^{-3t}\right) \tag{8.288}$$

$$= 6 \tag{8.289}$$

Verifying using Final Value Theorem :

$$\lim_{t\to\infty} y(t) = \lim_{s\to 0} sY(s) \tag{8.290}$$

$$\lim_{s\to 0} sY(s) = \lim_{s\to 0} s\left(\frac{6}{s} - \frac{11}{26\,(s+1)} + \frac{5}{26\,(s+2)} + \frac{5}{78\,(s+3)}\right) \tag{8.291}$$

$$= \lim_{s\to 0}\left(6 - \frac{11s}{26\,(s+1)} + \frac{5s}{26\,(s+2)} + \frac{5s}{78\,(s+3)}\right) \tag{8.292}$$

$$= 6 \tag{8.293}$$

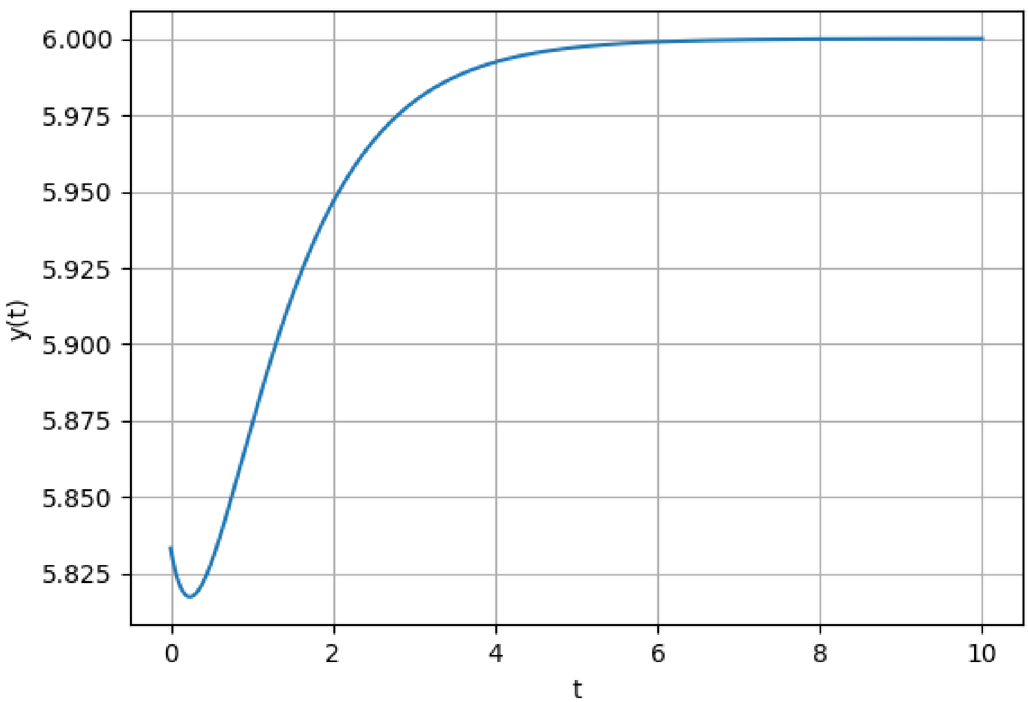


Figure 8.27: Plot $y(t)$ vs t

8.2 **Question:** Consider the differential equation

$\frac{d^2y}{dx^2}+8\frac{dy}{dx}+16y=0$ and the boundary conditions $y(0)=1$ and $\frac{dy}{dx}(0)=0$. The solution to equation is:

(GATE.AE-1.2021)

Solution:

From C.6.5

$$\frac{d^2y}{dx^2}+8\frac{dy}{dx}+16y \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s)-sy(0)-y'(0)+8sY(s)-8y(0)+16Y(s) \tag{8.294}$$

Symbol	**Values**	**Description**
$Y(s)$	-	y in s domain
$y(x)$	-	y in x domain
$y(0)$	1	y at $x = 0$
$y'(0)$	0	$y'(x)$ at $x = 0$
$u(x)$	$= \begin{cases} 1 & \text{if } x > 0 \\ 0 & o.w \end{cases}$	unit step function

Table 8.20: Parameters

$$Y(s)(s^2 + 8s + 16) = s + 8 \tag{8.295}$$

$$\Rightarrow Y(s) = \frac{s+8}{s^2 + 8s + 16} \tag{8.296}$$

$$= \frac{1}{s+4} + \frac{4}{(s+4)^2} \tag{8.297}$$

For inversion of $Y(s)$ in partial fractions-

From C.5.1

$$\frac{b}{(s+a)^n} \xleftrightarrow{\mathcal{L}^{-1}} \frac{b}{(n-1)!} \cdot x^{n-1} e^{-ax} \cdot u(x) \tag{8.298}$$

Applying Laplace inverse-

From (8.297),(8.298)

$$y(x) = \frac{1}{0!} e^{-4x} \cdot u(x) + \frac{4}{1!} x \cdot e^{-4x} \cdot u(x) \tag{8.299}$$

$$= (1 + 4x) e^{-4x} u(x) \tag{8.300}$$

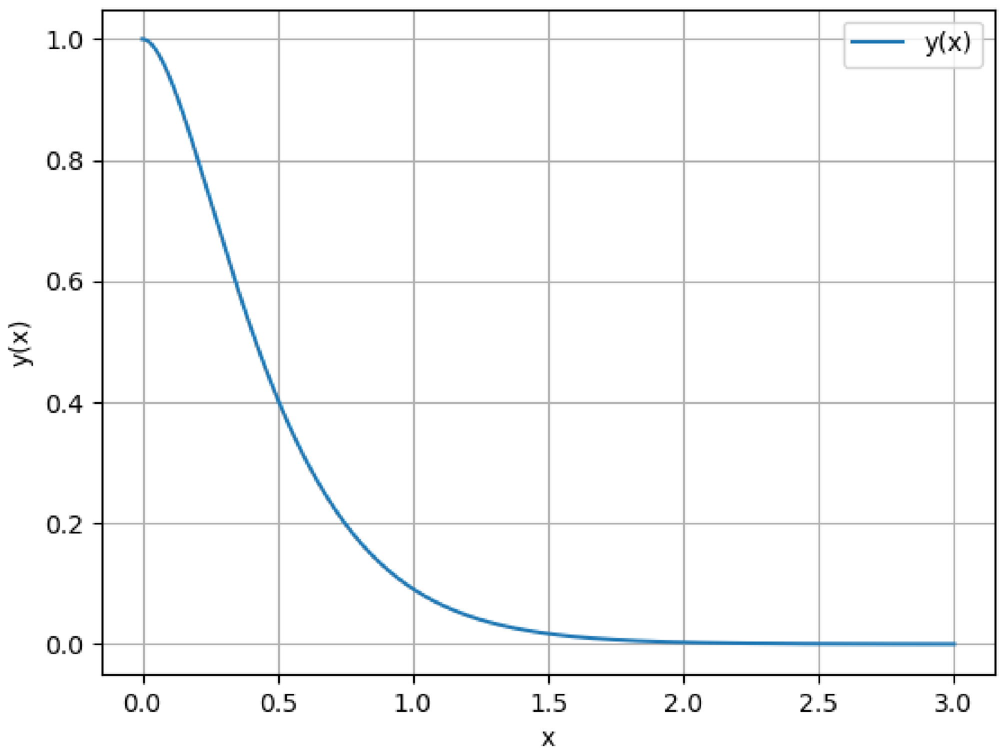


Figure 8.28: Plot of y(x)

8.3 **Question:** The solution of second-order differential equation $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ with boundary conditions $y(0) = 1$ and $y(1) = 3$.

(GATE 2021 CE.26)

Solution:

Symbol	**Values**	**Description**
$Y(s)$	-	y in s domain
$y(x)$	-	y in x domain
$y(0)$	1	y at $x = 0$
$y(1)$	3	$y(x)$ at $x = 1$
$u(x)$	$= \begin{cases} 1 & \text{if } x > 0 \\ 0 & o.w \end{cases}$	unit step function

Table 8.21: Parameters

Applying Laplace transform

From C.6.5

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0) - y'(0) + 2sY(s) - 2y(0) + Y(s) \tag{8.301}$$

$$Y(s)(s^2 + 2s + 1) = s - 2 - y'(0) \tag{8.302}$$

$$\Rightarrow Y(s) = \frac{s - 2 - y'(0)}{s^2 + 2s + 1} \tag{8.303}$$

$$= \frac{1}{s+1} - \frac{2 + y'(0)}{(s+1)^2} \tag{8.304}$$

For inversion of $Y(s)$ in partial fractions-

From C.5.1

$$\frac{b}{(s+a)^n} \stackrel{\mathcal{L}}{\longleftrightarrow} \frac{b}{(n-1)!} \cdot x^{n-1} e^{-ax} \cdot u(x) \tag{8.305}$$

Applying Laplace inverse-

From (8.304),(8.305)

$$y(x) = \frac{1}{0!} e^{-x} \cdot u(x) - \frac{3+y'(0)}{1!} x \cdot e^{-x} \cdot u(x) \tag{8.306}$$

$$= (1-(3+y'(0))x)e^{-x}u(x) \tag{8.307}$$

From (8.307),

$$y(1) = (1-3-y'(0))e^{-1} \tag{8.308}$$

$$3 = (1-3-y'(0))e^{-1} \tag{8.309}$$

$$\Rightarrow y'(0) = -(2+3e) \tag{8.310}$$

From(8.310),(8.307)

$$y(x) = (e^x + (3e-1)xe^{-x})u(x) \tag{8.311}$$

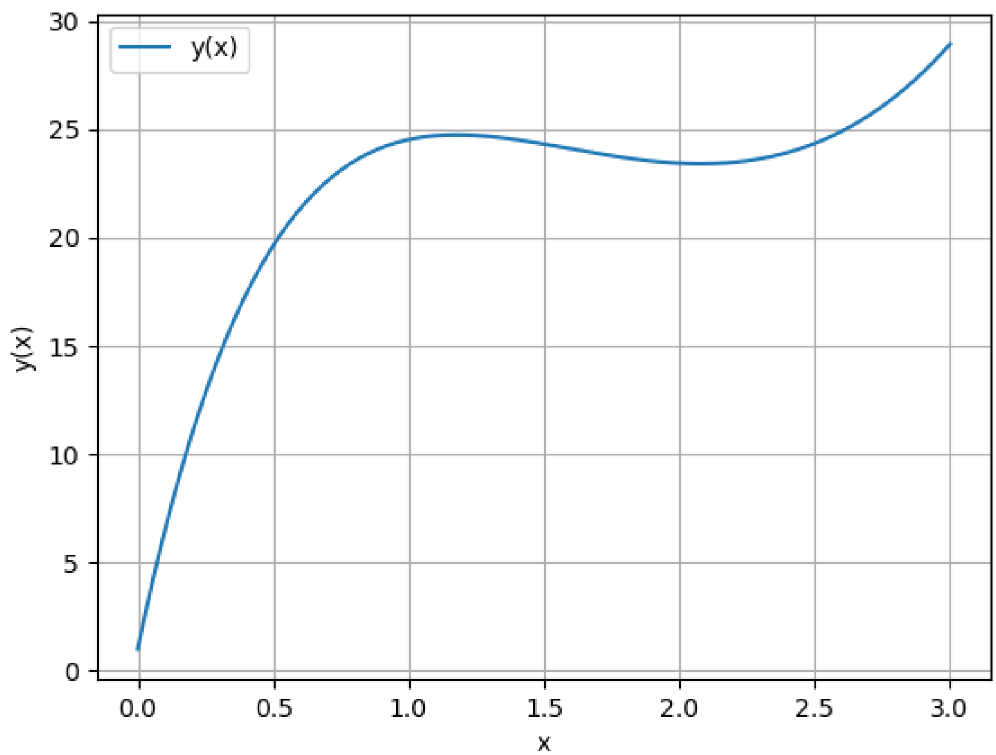


Figure 8.29: Plot of y(x)

8.4 A system has a transfer function

$$G(s) = \frac{3e^{-4s}}{12s+1}$$

When a step-change of magnitude M is given to the system input, the final value of the system output is measured to be 120. The value of M is ______. (GATE 2021 CH Q52)

Solution:

Symbol	**Value**	**Description**
$x(t)$	$Mu(t)$	Input Signal
$X(s)$	$\frac{M}{s}$	s-domain Input Signal
$y(t)$		Output Signal
$Y(s)$		s-domain Output Signal
$G(s)$	$\frac{3e^{-4s}}{12s+1}$	Transfer Function

Table 8.22: Given Parameters

Given, input step-change:

$$x(t) = Mu(t) \tag{8.312}$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \tag{8.313}$$

$$\implies X(s) = \frac{M}{s} \tag{8.314}$$

Transfer Function:

$$G(s) = \frac{Y(s)}{X(s)} = \frac{3e^{-4s}}{12s+1} \tag{8.315}$$

$$\implies Y(s) = \frac{3e^{-4s}}{12s+1}\frac{M}{s} \tag{8.316}$$

$\therefore$ system output

$$\lim_{s\to 0} sY(s) = 120 \tag{8.317}$$

$$\implies \lim_{s\to 0}\left(\frac{3e^{-4s}}{12s+1}M\right) = 120 \tag{8.318}$$

$$\implies 3M = 120 \tag{8.319}$$

$$\implies M = 40 \tag{8.320}$$

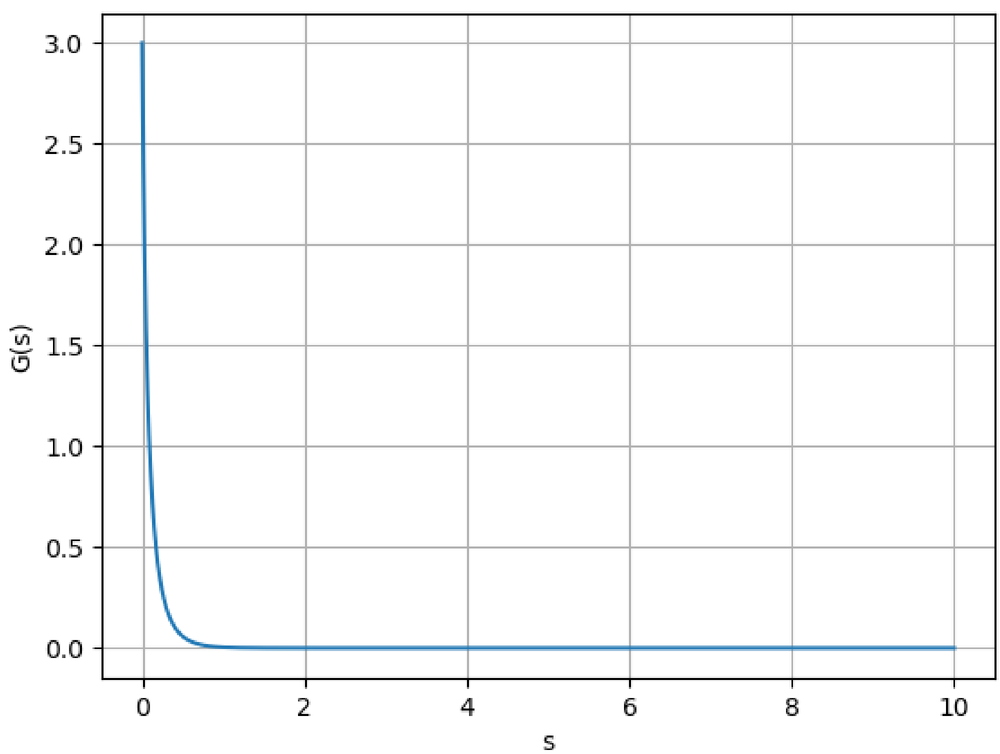
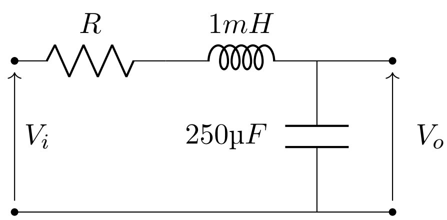
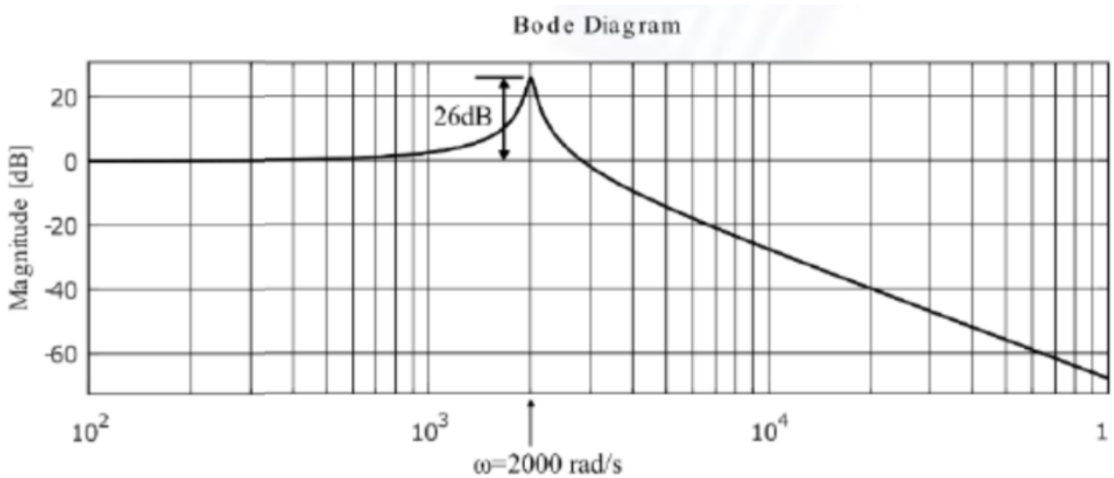


Figure 8.30: Plot of $G(s)$ vs s

8.5 The Bode magnitude plot for the transfer function $\frac{V_o(s)}{V_i(s)}$ of the circuit is as shown. The value of R is ____Ω. (GATE 2021 EE Q20) **Solution:** Applying KVL,





Parameter	Description	Value
C	Capacitance	$250\mu F$
L	Inductor	$1mH$
I	Current	
$I(0)$	Initial Current	0A
V_o	Voltage across capacitor	
V_i	Input Voltage	
$T(s)$	Transfer Function	$\frac{V_o(s)}{V_i(s)}$

Table 8.23: Given Parameters table

$$V_i - RI - L\frac{dI}{dt} - \frac{\int I dt}{C} = 0 \tag{8.321}$$

Taking Laplace Transform ,

$$V_i(s) - RI(s) - LsI(s) + LI(0^+) - \frac{I(s)}{sC} = 0 \tag{8.322}$$

$$I(s) = \frac{V_i(s) + LI(0)}{R + sL + \frac{1}{sC}} \tag{8.323}$$

$$V_o(s) = \frac{V_i(s) + LI(0)}{RsC + s^2LC + 1} \tag{8.324}$$

Substituting $I(0) = 0$ and $s = j\omega$,

$$\frac{V_o(j\omega)}{V_i(j\omega)} = \frac{1}{\omega RCj - \omega^2 LC + 1} \tag{8.325}$$

$\because$ Magnitude in bode plot $= 20\log|T(s)|$

From given graph,At $\omega = 2000$

$$26 = 20\log\left|\frac{V_o}{V_i}\right| \tag{8.326}$$

$$\left|\frac{V_o}{V_i}\right| = 20 \tag{8.327}$$

$$\implies 20 = \left|\frac{1}{\omega RCj - \omega^2 LC + 1}\right| \tag{8.328}$$

$$R = 0.1\Omega \tag{8.329}$$

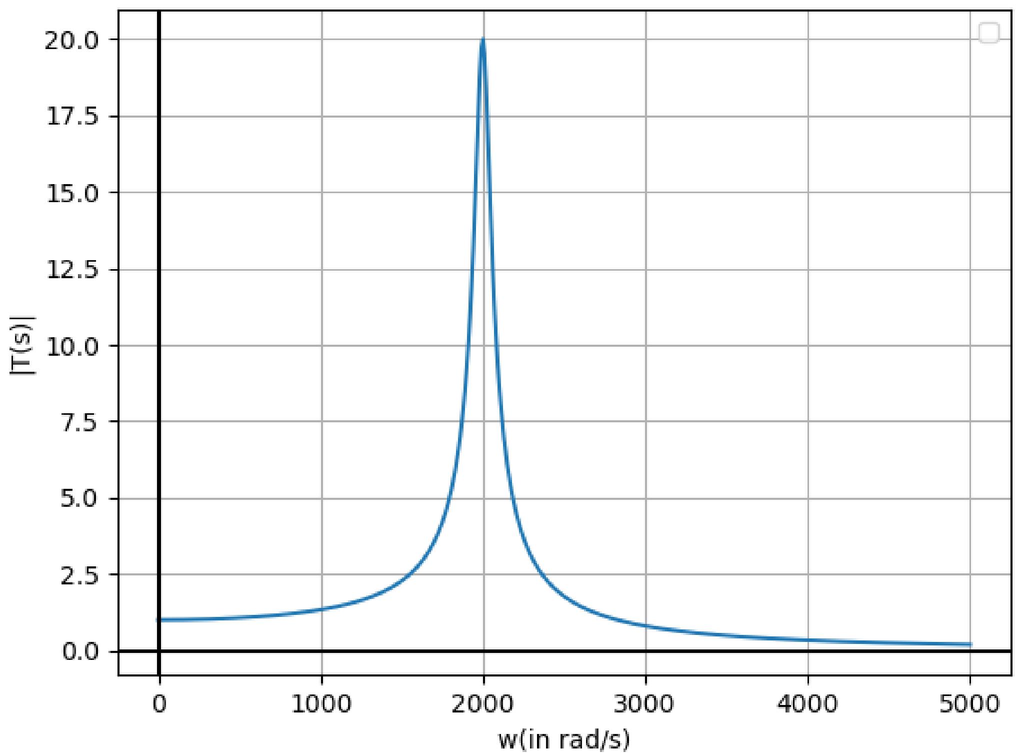


Figure 8.31: Frequency response of V_o

8.6 Consider a system with transfer-function $G(s) = \frac{2}{s+1}$. A unit-step function $\mu(t)$ is applied to the system, which results in an output y(t).

If $e(t) = y(t) - \mu(t)$ then $\lim_{t\to\infty} e(t)$ is________. **Solution:**

Symbol	Value	Description
$G(s)$	$\frac{2}{s+1}$	Transfer function
$e(t)$	$y(t) - \mu(t)$	Function of y(t) and $\mu(t)$
$Y(s)$	$G(s) \times U(s)$	Convolution in t domain is multiplication in s domain.
$\mu(t)$	$\begin{cases} 0 & \text{if t<0} \\ 1 & \text{if t>0} \end{cases}$	Unit step function

Table 8.24: Variable description

Applying Laplace transform on $\mu(t)$

$$\mu(t) \overset{\mathcal{L}}{\longleftrightarrow} U(s) \tag{8.330}$$

$$U(s) = \frac{1}{s} \tag{8.331}$$

$$Y(s) = \left(\frac{2}{s+1}\right)\left(\frac{1}{s}\right) \tag{8.332}$$

$$Y(s) = \frac{2}{s} - \frac{2}{s+1} \tag{8.333}$$

The inverse Laplace transform of $\frac{a}{s+b}$ is $ae^{-bt}\mu\left(t\right)$

$$y\left(t\right) = 2\mu\left(t\right) - 2e^{-t}\mu\left(t\right) \tag{8.334}$$

$$e\left(t\right) = \mu\left(t\right)\left(1 - 2e^{-t}\right) \tag{8.335}$$

$$\lim_{t\to\infty} e\left(t\right) = \lim_{t\to\infty} \mu\left(t\right)\left(1 - 2e^{-t}\right) \tag{8.336}$$

$$\lim_{t\to\infty} e\left(t\right) = 1 \tag{8.337}$$

8.7 Solution of differential equation $y'' + y' + 0.25y = 0$ with initial values $y(0) = 3$ and $y'(0) = -3.5$ is

(A) $y = (3 - 2x)e^{0.5x}$

(B) $y = (3 - 2x)e^{-0.25x}$

(C) $y = (3 - 2x)e^{-0.5x}$

(D) $y = (2 - 3x)e^{-0.5x}$

(GATE AG 2021)

Solution:

Parameter	Description	Value
$y(t)$	y in time domain	?
$y(0)$	y at $t = 0$	3
$y'(0)$	y' at $t = 0$	-3.5

Table 8.25: Given parameters

By applying laplace transform to the differential equation,

$$y'' + y' + 0.25y \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0) - y'(0) + sY(s) - y(0) + 0.25Y(s) \tag{8.338}$$

$$Y(s)(s^2 + s + 0.25) = 3s - 0.5 \tag{8.339}$$

$$\implies Y(s) = \frac{3s - 0.5}{s^2 + s + 0.25} \tag{8.340}$$

$$= \frac{3}{s + 0.5} - \frac{2}{(s + 0.5)^2}; Re(s) > -0.5 \tag{8.341}$$

As we know,

$$\frac{b}{(s+a)^n} \stackrel{\mathcal{L}^{-1}}{\longleftrightarrow} \frac{b}{(n-1)!} t^{n-1} e^{-at} u(t) \tag{8.342}$$

By taking inverse laplace of (8.341), we get

$$y(t) = \frac{3}{0!} e^{-0.5t} u(t) - \frac{2}{1!} t e^{-0.5t} u(t) \tag{8.343}$$

$$\implies y(t) = \left[(3-2x)e^{-0.5t}\right] u(t) \tag{8.344}$$

Hence the correct answer is option (C)

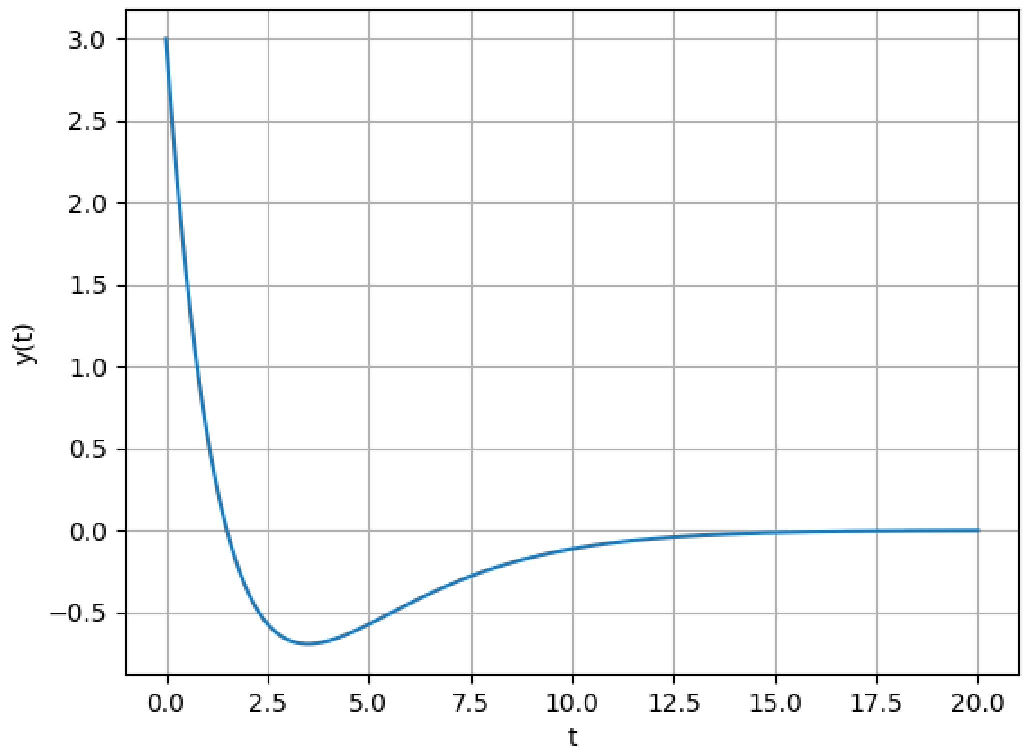


Figure 8.32: $y(t) = \left[(3-2t)e^{-0.5t}\right]u(t)$

8.8 Consider the following first order partial differential equation, also known as the transport equation

$$\frac{\partial y\,(x,t)}{\partial t} + 5\frac{\partial y\,(x,t)}{\partial x} = 0$$

with initial conditions given by $y(x,0) = \sin x, -\infty < x < \infty$. The value of $y(x,t)$ at $x = \pi$ and $t = \frac{\pi}{6}$ is ______.

(A) 1

(B) 2

(C) 0

(D) 0.5

(GATE 2021 BM Q28)

Solution:

Parameters	**Description**	**Value**
$Y(x,s)$	Laplace transform of $y(x,t)$ in t considering x as a parameter	
$y(x,0)$	$y(x,t)$ at $t=0$	$\sin x$

Table 8.26: Parameters

From Laplace transforms (C.1.5) and (C.1.7), we get

$$sY(x,s) - y(x,0) + 5\frac{dY(x,s)}{dx} = 0 \tag{8.345}$$

$$\implies \frac{dY(x,s)}{dx} + \frac{s}{5}Y(x,s) = \frac{\sin x}{5} \tag{8.346}$$

$$e^{\frac{s}{5}x}Y(x,s) = \frac{1}{5}\int e^{\frac{s}{5}x}\sin x dx \tag{8.347}$$

$$= \frac{1}{s^2+25}e^{\frac{s}{5}x}(s\sin x - 5\cos x) + c \tag{8.348}$$

$$Y(x,s) = \frac{1}{s^2+25}(s\sin x - 5\cos x) + ce^{-\frac{s}{5}x} \tag{8.349}$$

$$\cos at \xleftrightarrow{\mathcal{L}} \frac{s}{s^2+a^2} \tag{8.350}$$

$$\sin at \xleftrightarrow{\mathcal{L}} \frac{a}{s^2+a^2} \tag{8.351}$$

From Laplace transforms (8.350) and (8.351), we get

$$y(x,t) = ((\sin x \cos 5t - \cos x \sin 5t))\, u(t) + ce^{-\frac{s}{5}x}\delta(t) \tag{8.352}$$

$$= (\sin(x-5t))\, u(t) + ce^{-\frac{s}{5}x}\delta(t) \tag{8.353}$$

$$y(x,0) = \sin x + ce^{-\frac{s}{5}x}\delta(0) \tag{8.354}$$

$$\implies c = 0 \tag{8.355}$$

$$\therefore y(x,t) = (\sin(x-5t))\, u(t) \tag{8.356}$$

$$\implies y\left(\pi, \frac{\pi}{6}\right) = 0.5 \tag{8.357}$$

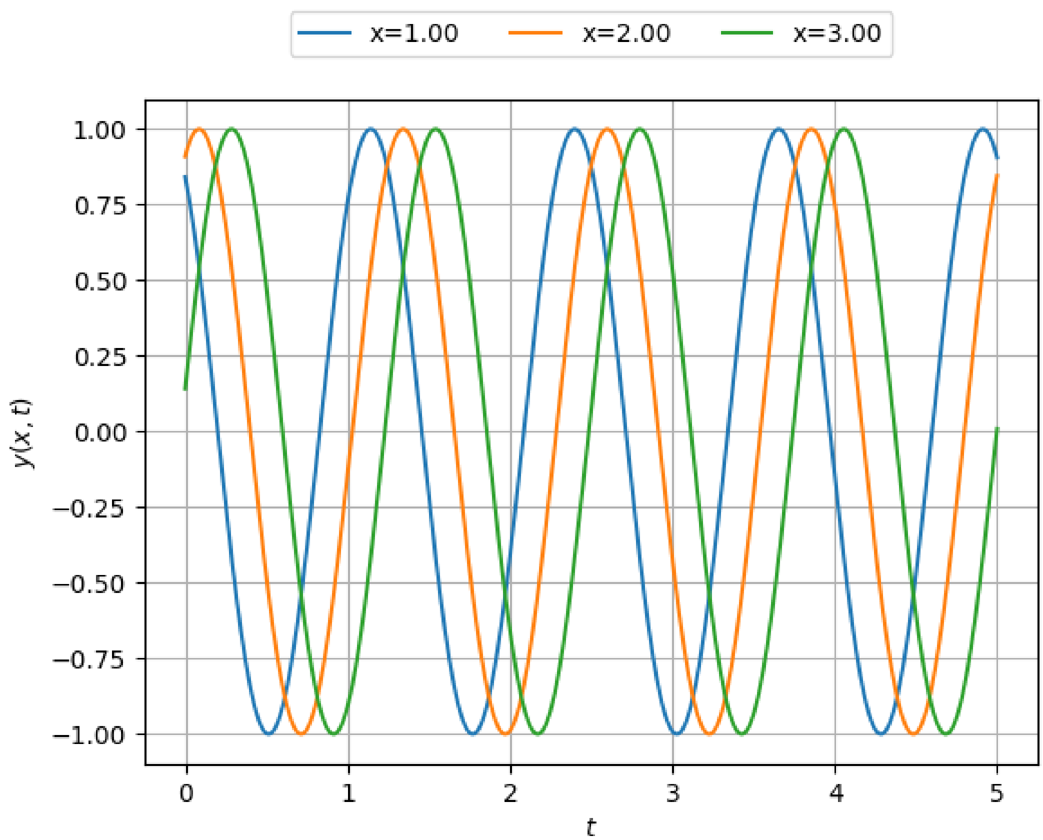


Figure 8.33: Plot of $y(x,t)$

Chapter 9

Fourier transform

9.1. 2022

9.1 The outputs of four systems (S_1, S_2, S_3, S_4) corresponding to the input signal $\sin(t)$, for all time t , are shown in the figure. Based on the given information, which of the four systems is/are definately NOT LTI(linear and time-invariant)?

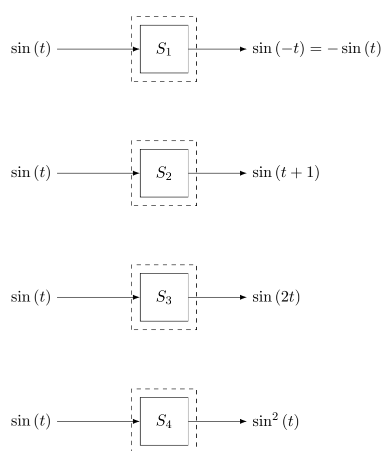


Figure 9.1: Block Diagram of Systems

(GATE22 EC Q46)

Solution:

Parameter	**Description**
(S_1, S_2, S_3, S_4)	Systems Given
$\sin(t)$	Input
$H(\omega)$	Transfer Function
$X(\omega)$	Fourier-Transform of input
$Y(\omega)$	Fourier-Transform of output
$\Phi(\omega)$	Phase of Transfer Function

Table 1: Parameter Table

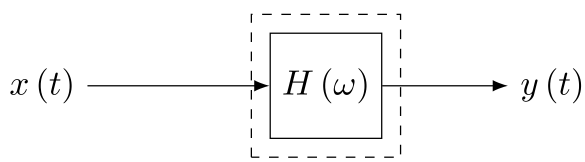


Figure 9.2: Block Diagram of LTI System

For an LTI system :

$$y(t) = h(t) * x(t) \tag{9.1}$$

$$Y(\omega) = H(\omega) X(\omega) \tag{9.2}$$

$H(\omega)$ is a complex exponential :

$$H(j\omega) = |H(j\omega)| e^{j\Phi(\omega)} \tag{9.3}$$

$x(t) = \sin(t)$, and $w_o = 1 rad/sec$

$$X(\omega) = j\pi \left(\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right) \tag{9.4}$$

Now,

$$Y(\omega) = (\delta(\omega+\omega_0) - \delta(\omega-\omega_0))\, \pi \, |H(\omega)|\, e^{j\Phi(\omega)} \tag{9.5}$$

$$x(t)\, \delta(t - t_o) = x(t_0)\, \delta(t - t_o) \tag{9.6}$$

Using property (9.6) in (9.5) :

$$\begin{aligned} Y(\omega) =& j\pi\, |H(-\omega_0)|\, e^{j\Phi(-\omega_0)} \delta(\omega+\omega_0) \\ & - j\pi\, |H(\omega_0)|\, e^{j\Phi(j\omega_0)} \delta(\omega-\omega_0) \end{aligned} \tag{9.7}$$

By definition of the Fourier transform,

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t}\, dt \tag{9.8}$$

$$X^*(\omega) = \int_{-\infty}^{\infty} x^*(t) e^{j\omega t}\, dt \tag{9.9}$$

$$X^*(-\omega) = \int_{-\infty}^{\infty} x^*(t) e^{-j\omega t}\, dt \tag{9.10}$$

For real-time domain signal :

$$x(t) = x^*(t) \tag{9.11}$$

Therefore , from (9.10):

$$X(\omega) = X^*(-\omega) \tag{9.12}$$

By (9.12) , Given $h(t)$ a real-time domain signal, $H(\omega)$ is conjugate symmetric.

$$|H(\omega)| = |H(-\omega)| \tag{9.13}$$

$$\Phi(-\omega) = -\Phi(\omega) \tag{9.14}$$

Therefore using (9.13) and (9.14) in (9.7),

$$Y(\omega) = j\pi\, |H(\omega_0)| \left(e^{-j\Phi(\omega_0)}\delta(\omega+\omega_0) - e^{j\Phi(\omega_0)}\delta(\omega-\omega_0)\right) \tag{9.15}$$

Taking Inverse Fourier Transform,

$$\delta(\omega-\omega_0) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}e^{j\omega_0 t} \tag{9.16}$$

$$\delta(\omega+\omega_0) \xleftrightarrow{\mathcal{F}} \frac{1}{2\pi}e^{-j\omega_0 t} \tag{9.17}$$

$$\implies y(t) = j\,|H(\omega_0)|\,\frac{1}{2}\left(e^{-j(\omega_0 t+\Phi(\omega_0))} - e^{j(\omega_0 t+\Phi(\omega_0))}\right) \tag{9.18}$$

$$\implies y(t) = |H(\omega_0)|\sin(\omega_0 t + \Phi(\omega_0)) \tag{9.19}$$

$w_0 = 1$ rad/sec :

$$y(t) = |H(1)|\sin(t + \Phi(1)) \tag{9.20}$$

From (9.20) we can see output cant have scaled frequency nor a squared output. But can have a shifted output or amplitude-scaled output.

So, S_3 and S_4 cannot be LTI system.

9.2 The Fourier transform X($j\omega$) of the signal

$x(t) = \frac{t}{(1+t^2)^2}$ is ________. GATE-2022-EC-15

(A) $\frac{\pi}{2j}\omega e^{-|\omega|}$

(B) $\frac{\pi}{2}\omega e^{-|\omega|}$

(C) $\frac{\pi}{2j}e^{-|\omega|}$

(D) $\frac{\pi}{2}e^{-|\omega|}$

Solution:

Symbol	Value	Description
$x(t)$	$\frac{t}{(1+t^2)^2}$	Signal
$X(\omega)$	$\int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$	Fourier transform of $x(t)$

Table 9.2: Variable description

The Fourier transform of the form x(t)=$e^{-a|t|}$ is

$$x(t) \xleftrightarrow{\text{F.T}} X(\omega) \tag{9.21}$$

$$X(\omega) = \frac{2a}{a^2+\omega^2} \tag{9.22}$$

Consider,

$$x(t) = e^{-|t|} \tag{9.23}$$

$$X(\omega) = \frac{2}{1+\omega^2} \tag{9.24}$$

By using differentiation property from (A.1.5),

$$tx(t) \xleftrightarrow{\text{F.T}} j\frac{d}{d\omega}X(\omega) \tag{9.25}$$

$$tx(t) \xleftrightarrow{\text{F.T}} j\left[\frac{d}{d\omega}\left(\frac{2}{1+\omega^2}\right)\right] \tag{9.26}$$

$$te^{-|t|} \xleftrightarrow{\text{F.T}} \frac{-4j\omega}{(1+\omega^2)^2} \tag{9.27}$$

Applying duality property from (A.2.3),

$$\frac{-4jt}{(1+t^2)^2} \xleftrightarrow{\text{F.T}} 2\pi(-\omega)e^{-|-\omega|} \tag{9.28}$$

$$\frac{t}{(1+t^2)^2} \xleftrightarrow{\text{F.T}} \frac{-2\pi\omega e^{-|\omega|}}{-4j} \tag{9.29}$$

$$\frac{t}{(1+t^2)^2} \xleftrightarrow{\text{F.T}} \frac{\pi}{2j}\omega e^{-|\omega|} \tag{9.30}$$

9.3 For a vector $\bar{x} = [x[0], x[1], \ldots, x[7]]$, the 8-point discrete Fourier transform (DFT) is denoted by $\bar{X} = \text{DFT}(\bar{x}) = [X[0], X[1], \ldots, X[7]]$, where

$$X[k] = \sum_{n=0}^{7} x[n] \exp\left(-j\frac{2\pi}{8}nk\right).$$

Here $j = \sqrt{-1}$. If $\bar{x} = [1, 0, 0, 0, 2, 0, 0, 0]$ and $\bar{y} = \text{DFT}(\text{DFT}(\bar{x}))$, then the value of $y[0]$ is. GATE-2022-EC-55

Solution:

Parameter	Description	Value
$\bar{X}$	DFT($\bar{x}$)	–
$\bar{x}$	vector	$[1, 0, 0, 0, 2, 0, 0, 0]$
$\bar{y}$	DFT(DFT($\bar{x}$))	–

Table 9.3: Given Parameters

DFT of $\bar{x}$

$$X[k] = \sum_{n=0}^{7} x[n] \exp\left(-j\frac{2\pi}{8}nk\right) \tag{9.31}$$

As the only non-zero values in x are x[0] and x[4]:

$$X[k] = x[0] + x[4] \exp(-j\pi k) \tag{9.32}$$

After substituting the values of k ranging from 0 to 7,

$$\bar{X} = \text{DFT}(\bar{x}) = [X[0], X[1], \ldots, X[7]] \tag{9.33}$$

$$\bar{X} = [3, -1, 3, -1, 3, -1, 3, -1] \tag{9.34}$$

$$\bar{y} = \mathrm{DFT}(\mathrm{DFT}(\bar{x})) \tag{9.35}$$

$$\bar{y} = [3, -1, 3, -1, 3, -1, 3, -1] \tag{9.36}$$

$$y[0] = \sum_{n=0}^{7} x[n] \tag{9.37}$$

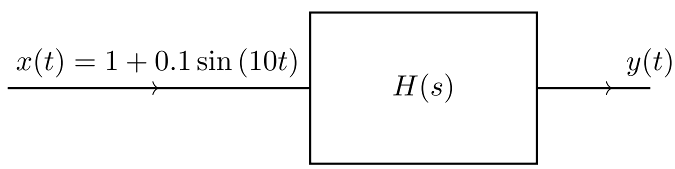
$$= x[0] + x[1] + \cdots + x[7] \tag{9.38}$$

$$= 3 - 1 + 3 - 1 + 3 - 1 + 3 - 1 = 8 \tag{9.39}$$

9.4 **Question:** An LTI system is shown in the figure where

$$H(s) = \frac{100}{s^2 + 0.1s + 10}$$

The steady state output of the system for an input $x(t)$ is given by $y(t) = a + b\sin(10t + \theta)$. The values of $'a'$ and $'b'$ are



Solution:

Symbol	Value	Description
$x(t)$	$1 + 0.1\sin(10t)$	Input Signal
$y(t)$	?	Output of the system
$H(s)$	$\frac{100}{s^2+0.1s+10}$	Impulse Response

Table 9.4: Given Information

(a) **Theory:** If a sinusoidal input is given to a system, whose transfer function is known, the output can be calculated as follows

$$y(t) = h(t) * x(t) \tag{9.40}$$

$$Y(s) = H(s)X(s) \tag{9.41}$$

Let $s = j\omega$

$$Y(j\omega) = H(j\omega)X(j\omega) \tag{9.42}$$

If Φ is the phase of $H(j\omega)$,

$$H(j\omega) = |H(j\omega)|\, e^{j\Phi(\omega)} \tag{9.43}$$

If $x(t) = \cos(\omega_0 t)$,

$$X(j\omega) = \pi\left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right) \tag{9.44}$$

Now,

$$Y(j\omega) = \left(\delta(\omega - \omega_0) + \delta(\omega + \omega_0)\right)|H(j\omega)|\, e^{j\Phi(\omega)} \tag{9.45}$$

$$\tag{9.46}$$

Since $|H(j\omega)|\, \delta(\omega - \omega_0)$ is zero everywhere except at ω_0

$$Y(j\omega) = |H(j\omega_0)|\, e^{j\Phi(\omega_0)}\delta(\omega - \omega_0) \tag{9.47}$$

$$+ |H(-j\omega_0)|\, e^{j\Phi(-j\omega_0)}\delta(\omega + \omega_0) \tag{9.48}$$

As $h(t)$ is real,

$$H(\omega) = H^*(-\omega)$$

$$\Phi(-\omega_0) = -\Phi(\omega_0)$$

Hence

$$Y(\omega) = |H(\omega_0)|\left(e^{j\Phi(\omega_0)}\delta(\omega - \omega_0) + e^{-j\Phi(\omega_0)}\delta(\omega + \omega_0)\right) \tag{9.49}$$

Taking Inverse Fourier Transform,

$$\delta(\omega - \omega_0) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}e^{j\omega_0 t} \tag{9.50}$$

$$\implies y(t) = |H(\omega_0)| \frac{1}{2}\left(e^{j(\omega_0 t + \Phi(\omega_0))} + e^{-j(\omega_0 t + \Phi(\omega_0))}\right) \tag{9.51}$$

$$\implies y(t) = |H(\omega_0)| \cos\left(\omega_0 t + \Phi(\omega_0)\right) \tag{9.52}$$

(b) The given input can be assumed to be a superposition of $u(t)$ and $0.1\sin(\omega_0 t)u(t)$.

$$\omega_0 = 0 \text{ and } \omega_0 = 10$$

for the constant input and the sinusoidal input respectively.

$$y(t) = |H(0)| + |H(10)| \sin\left(10t + \Phi(10)\right) \tag{9.53}$$

Here

$$H(\omega) = \frac{100}{(j\omega)^2 + 0.1(j\omega) + 10} \tag{9.54}$$

$$\implies H(\omega) = \frac{100}{10 - \omega^2 + j(0.1\omega)} \tag{9.55}$$

$$\implies |H(\omega)| = \frac{100}{\sqrt{(10-\omega^2)^2 + (0.1\omega)^2}} \tag{9.56}$$

$$\therefore |H(0)| = 10 \text{ and } |H(10)| \approx 1 \tag{9.57}$$

The phase $\Phi(\omega)$ is given by

$$\Phi(\omega) = \tan^{-1}\frac{0.1\omega}{\omega^2 - 10} \tag{9.58}$$

$$\implies \Phi(10) = \tan^{-1}\frac{1}{90} \tag{9.59}$$

Hence the output of the system

$$y(t) = 10 + \sin(10t + \tan^{-1}\frac{1}{90}) \tag{9.60}$$

Hence $a = 10$ and $b = 1$

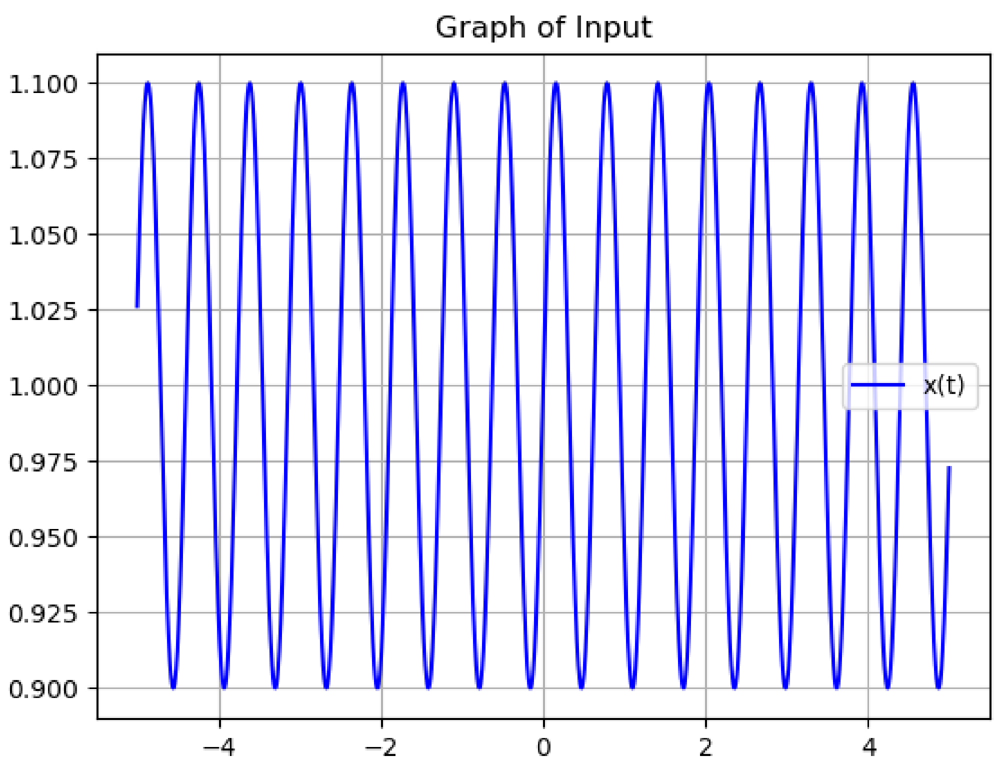


Figure 9.3: Input of the system, $x(t)$

9.5 A periodic function $f(x)$, with period 2, is defined as

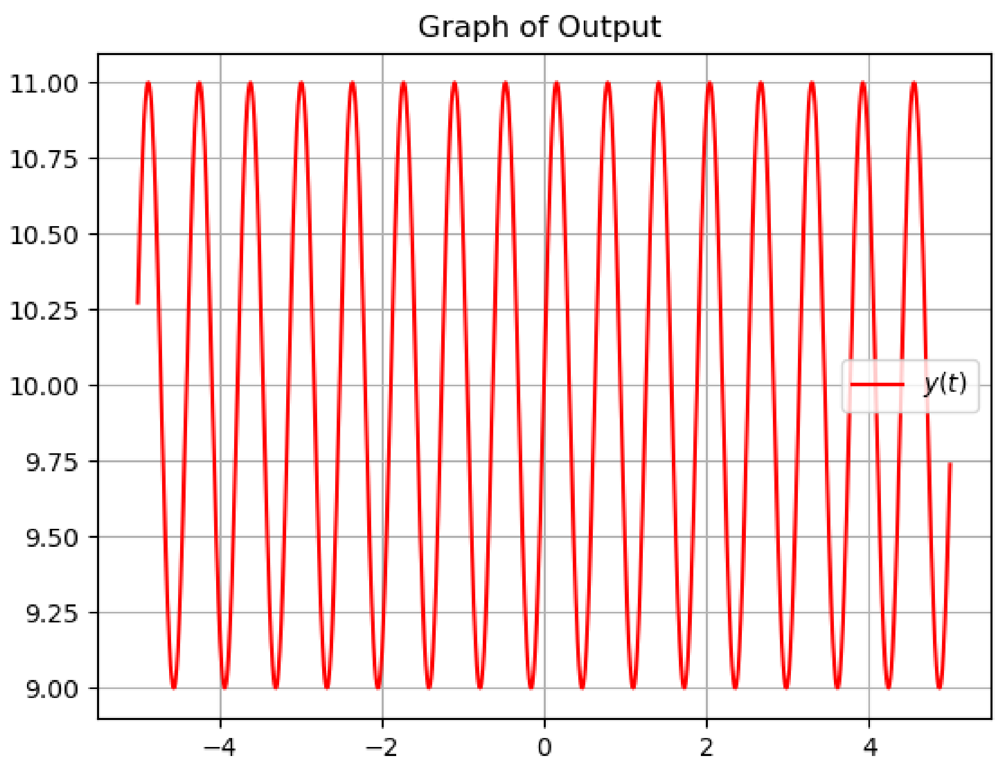


Figure 9.4: Output of the system, $y(t)$

$$f(x) = \begin{cases} -1 - x & -1 \leq x < 0 \\ 1 - x & 0 < x \leq 1 \end{cases} \tag{9.61}$$

The Fourier series of this function contains

A. Both $\cos(n\pi x)$ and $sin(n\pi x)$ where n=1,2,3...

B. Only $\sin(n\pi x)$ where n=1,2,3...

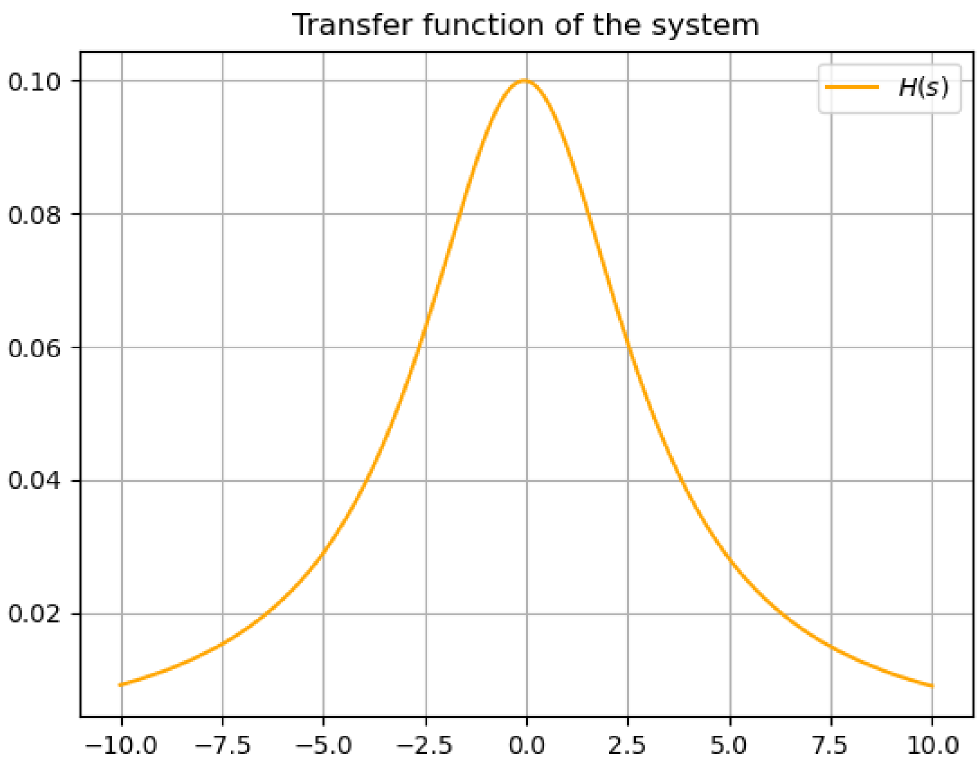


Figure 9.5: Transfer function of the system, $H(s)$

C. Only $\cos(n\pi x)$ where n=1,2,3...

D. Only $\cos(2n\pi x)$ where n=1,2,3... GATE IN 2022

Solution:

Parameter	Description
$f(x)$	Polynomial function
$2L$	Period of the Polynomial function
$c(n)$	Complex Fourier Coefficients
$a(0)$, $a(n)$, $b(n)$	Trigonometric Fourier Coefficients

Table 9.5: Input Parameters

The complex exponential Fourier Series of $f\left(x\right)$ is,

$$f\left(x\right)=\sum_{n=-\infty}^{\infty}c(n)e^{jn\omega x} \tag{9.62}$$

$$\Longrightarrow\ c(n)=\frac{1}{2L}\int_{-L}^{L}f\left(x\right)e^{-jn\omega x}\,dx \tag{9.63}$$

For $n\neq 0$;

$$c(n)=\frac{1}{2}\int_{-1}^{1}f(x)e^{-jn\omega x}\,dx \tag{9.64}$$

$$=\frac{1}{2}\left(\int_{-1}^{0}\left(-1-x\right)e^{-jn\omega x}\,dx+\int_{0}^{1}\left(+1-x\right)e^{-jn\omega x}\,dx\right) \tag{9.65}$$

$$=\frac{1}{2}\left(-\int_{-1}^{0}e^{-jn\omega x}\,dx-\int_{-1}^{1}xe^{-jn\omega x}\,dx+\int_{0}^{1}e^{-jn\omega x}\,dx\right) \tag{9.66}$$

$$=\frac{1}{2}\left[\frac{-1}{jn\omega}\left[-\left(1-e^{+jn\omega}\right)+\left(e^{-jn\omega}-1\right)\right]-\int_{-1}^{1}xe^{-jn\omega x}\,dx\right] \tag{9.67}$$

$$=\frac{1}{2}\left[\frac{-1}{jn\omega}\left[-2+e^{+jn\omega}+e^{-jn\omega}\right]+\left(\frac{e^{-jn\omega x}}{jn\omega}\left[x+\frac{1}{jn\omega}\right]\right)_{-1}^{1}\right] \tag{9.68}$$

$$=\frac{-1}{jn\omega}\left[-1+\cos(n\omega)\right]+\frac{1}{2(jn\omega)^2}\left[\left(e^{-jn\omega}\right)\left(1+jn\omega\right)-\left(e^{jn\omega}\right)\left(-jn\omega+1\right)\right] \tag{9.69}$$

$$\Longrightarrow\ c\left(n\right)=\frac{-1}{(jn\omega)^2}\left[-jn\omega+j\sin(n\omega)\right] \tag{9.70}$$

For $n = 0$;

$$c(0) = \frac{1}{2}\int_{-1}^{1} f(x)\,dx \tag{9.71}$$
$$= \frac{1}{2}\left[\int_{-1}^{0} (-1-x)\ dx + \int_{0}^{1} (1-x)\ dx\right] \tag{9.72}$$
$$= \frac{1}{2}\left[\left(-x - \frac{x^2}{2}\right)_{-1}^{0} + \left(x - \frac{x^2}{2}\right)_{0}^{1}\right] \tag{9.73}$$
$$= \frac{1}{2}\left[0 - 1 + \frac{1}{2} + 1 - \frac{1}{2} - 0\right] \tag{9.74}$$
$$\implies c(0) = 0 \tag{9.75}$$

The trigonometric Fourier Series of $f\,(x)$ is,

$$f\,(x) = a(0) + \sum_{n=1}^{\infty}\{a(n)\cos\,(n\omega x) + b(n)\sin\,(n\omega x)\} \tag{9.76}$$

Finding the Fourier Coefficient a_0,

$$a(0) = c(0) \tag{9.77}$$
$$\implies a(0) = 0 \tag{9.78}$$

Finding the Fourier Coefficients $a(n)$,

$$a(n) = \frac{1}{L}\int_{-L}^{L} f(x)\cos\,(n\omega x)\ \ dx, n \geq 0 \tag{9.79}$$
$$= \frac{1}{L}\int_{-L}^{L} f(x)\left(e^{-jn\omega x} + e^{jn\omega x}\right)\ dx \tag{9.80}$$
$$\implies a(n) = c(n) + c(-n) \tag{9.81}$$
$$\implies a(n) = 0 \tag{9.82}$$

Finding the Fourier Coefficients $b(n)$,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin(n\omega x)\ dx, n \geq 0 \tag{9.83}$$

$$= \frac{1}{L}\int_{-L}^{L} f(x)j\left(e^{-jn\omega x} - e^{jn\omega x}\right)\ dx \tag{9.84}$$

$$\implies b(n) = j\left(c(n) - c(-n)\right) \tag{9.85}$$

$$\implies b(n) = \frac{-2}{(n\omega)^2}\left[-n\omega + \sin(n\omega)\right] \tag{9.86}$$

$\implies$ The trigonometric Fourier Series of $f(x)$ is,

$$f(x) = \sum_{n=1}^{\infty}\left\{0 + 0 + b(n)\sin(n\omega x)\right\} \tag{9.87}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{-2}{(n\omega)^2}\left[-n\omega + \sin(n\omega)\right]\sin(n\omega x)\right\} \tag{9.88}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{-2}{(n\pi)^2}\left[-n\pi + \sin(n\pi)\right]\sin(n\pi x)\right\} \tag{9.89}$$

$$f(x) = \sum_{n=1}^{\infty}\left\{\frac{2}{n\pi}\sin(n\pi x)\right\} \tag{9.90}$$

$\therefore$ The Fourier series of this function contains only $\sin(n\pi x)$ where n=1,2,3...

9.6 A Simple closed path C in the Complex Plane is shown in the figure.

$$\oint_C \frac{2^z}{z^2-1}dz = -\jmath\pi A$$

Where $\jmath = \sqrt{-1}$, Then find the value of A is ______(Rounded of to two decimals)

(GATE 2022 EC)

Solution: Let

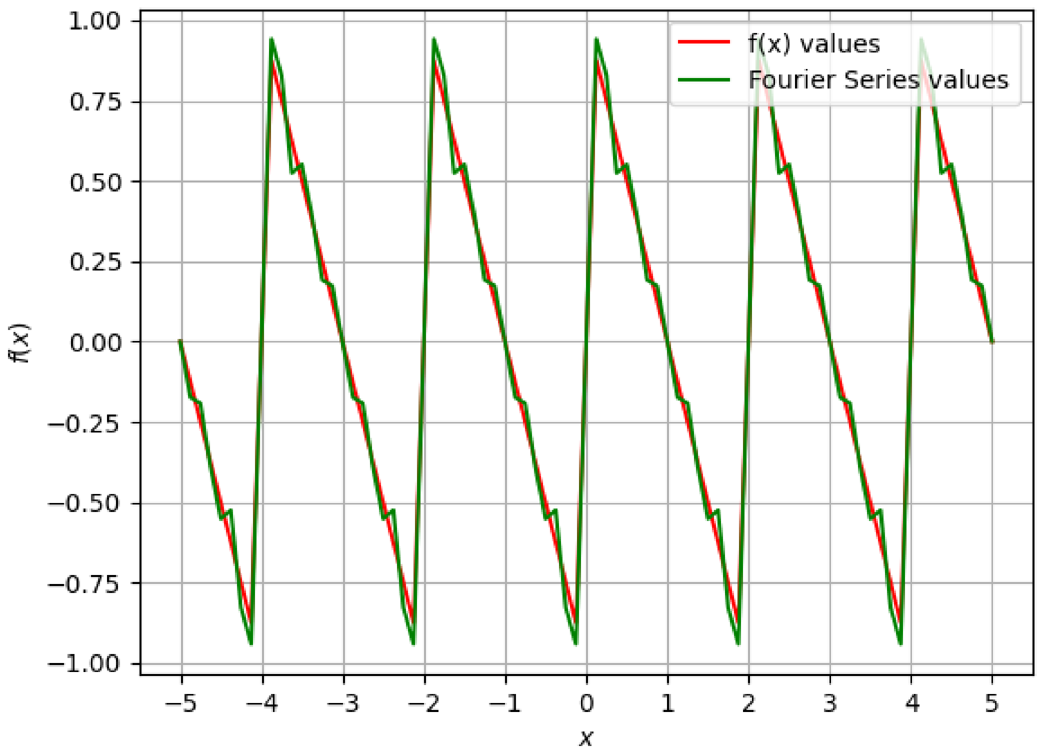
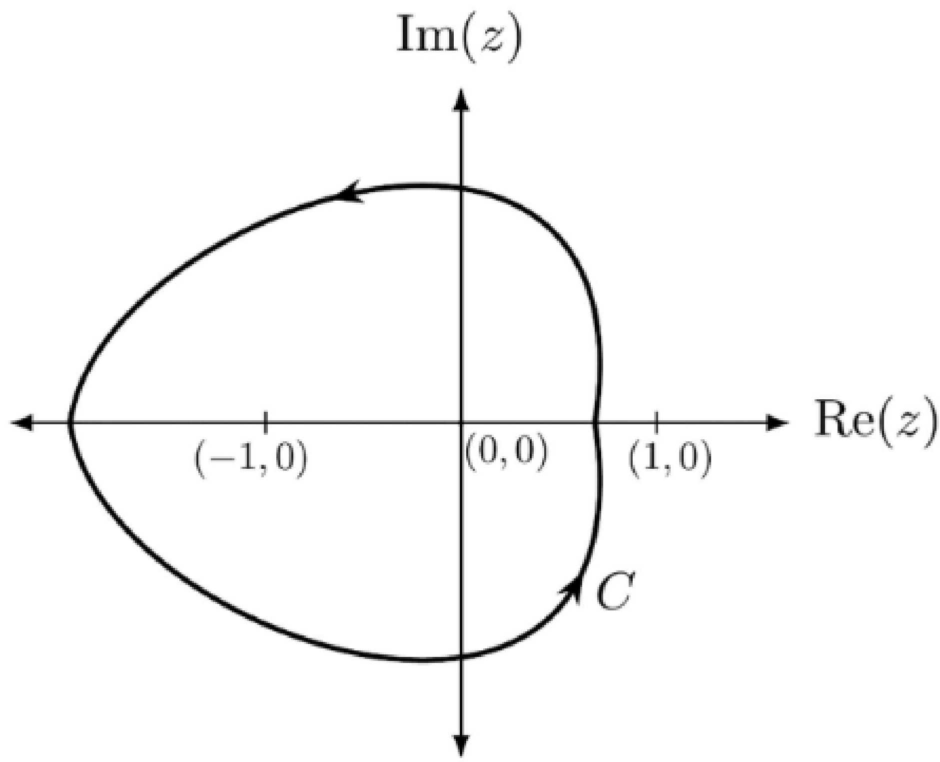


Figure 9.6:

$$f(z) = \frac{2^z}{z^2 - 1}$$



For poles

$$z^2 - 1 = 0 \tag{9.91}$$

$$\implies z = \pm 1 \tag{9.92}$$

As $Z = -1$ lies inside the C and $z = 1$ lies outside C

$$\oint_C f(z)dz = \oint_C \frac{\frac{2^z}{z-1}}{z+1}dz \tag{9.93}$$

$$= 2\pi\jmath\left(\frac{2^z}{z-1}\right) \text{ At } z = -1 \tag{9.94}$$

(By Cauchy's integral formula)

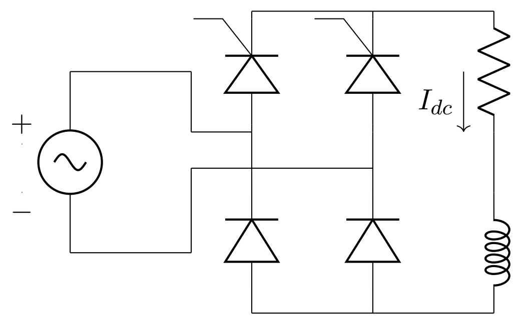
$$= 2\pi \jmath \left(\frac{-1}{4} \right) \tag{9.95}$$

$$= -\pi \jmath \left(\frac{1}{2} \right) \tag{9.96}$$

By comparing

$$A = \frac{1}{2} = 0.50 \tag{9.97}$$

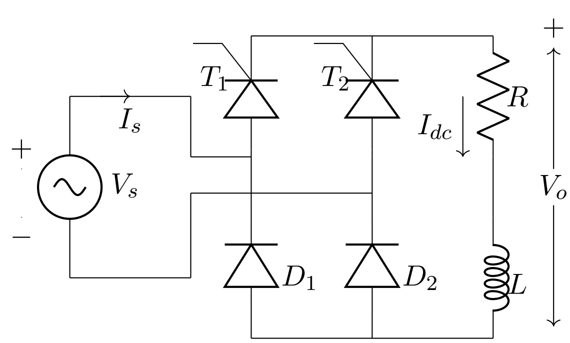
9.7 For the ideal AC-DC rectifier circuit shown in the figure below, the load current magnitude is I_{dc} = 15 A and is ripple free. The thyristors are fired with a delay angle of 45°. The amplitude of the fundamental component of the source current, in amperes, is ________(Round off to 2 decimal places). (GATE 59 EE 2022) **Solution:**



Parameter	Description	Value
I_{dc}	Load current	15A
α	Firing angle	45°

Table 9.6:

A symmetrical single phase semi converter is shown below,



The Fourier series representation of supply current is given by:

$$i_s(t) = a_o + \sum_{n=1}^{\infty} C_n \sin(n\omega t + \theta_n) \tag{9.98}$$

where,

$$a_o = \frac{1}{2\pi}\int_0^{2\pi} i_s(t)d\omega t \tag{9.99}$$

$$C_n = \sqrt{a_n^2 + b_n^2} \tag{9.100}$$

$$\theta_n = \tan^{-1}\left(\frac{a_n}{b_n}\right) \tag{9.101}$$

$$\implies a_o = \frac{1}{2\pi}\int_\alpha^{\pi} I_o d\omega t - \int_{\pi+\alpha}^{2\pi} I_o d\omega t = 0 \tag{9.102}$$

$$\implies a_n = \frac{1}{\pi}\int_\alpha^{\pi} I_o \cos n\omega t d\omega t - \int_{\pi+\alpha}^{2\pi} I_o \cos n\omega t d\omega t \tag{9.103}$$

$$a_n = \begin{cases} \frac{-2I_o}{n\pi}\sin n\alpha & \text{for } n = 1,3,5... \\ 0 & \text{for } n = 2,4..... \end{cases} \tag{9.104}$$

$$\implies b_n = \frac{1}{\pi}\int_\alpha^{\pi} I_o \sin n\omega t d\omega t - \int_{\pi+\alpha}^{2\pi} I_o \sin n\omega t d\omega t \tag{9.105}$$

$$b_n = \begin{cases} \frac{2I_o}{n\pi}\left(1+\cos n\alpha\right) & \text{for } n = 1,3,5... \\ 0 & \text{for } n = 2,4.... \end{cases} \tag{9.106}$$

From (9.100):

$$\therefore C_n = \frac{2\sqrt{2}I_o}{n\pi}\left(\sqrt{1+\cos n\alpha}\right) \tag{9.107}$$

$$\implies C_n = \frac{4I_o}{n\pi}\cos\frac{n\alpha}{2} \tag{9.108}$$

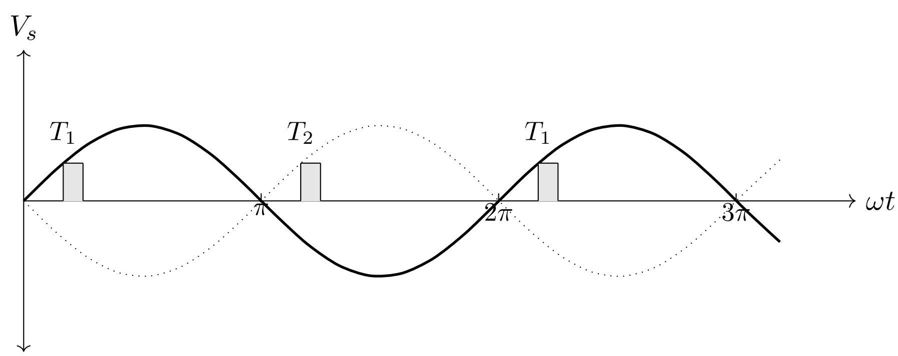
From (9.101):

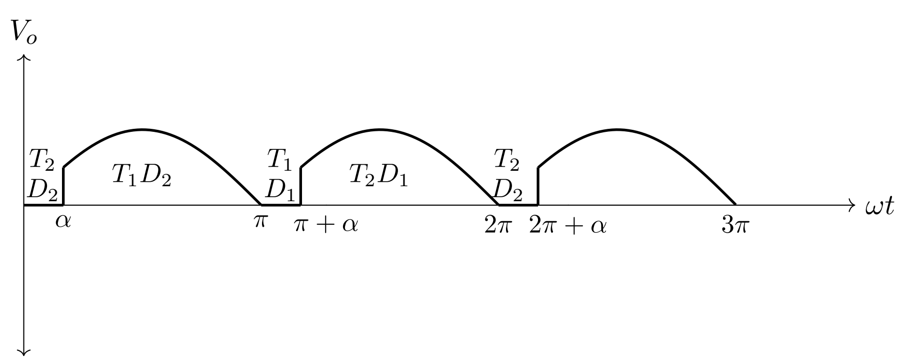
$$\theta_n = \tan^{-1}\left(\frac{-\sin n\alpha}{1+\cos n\alpha}\right) \tag{9.109}$$

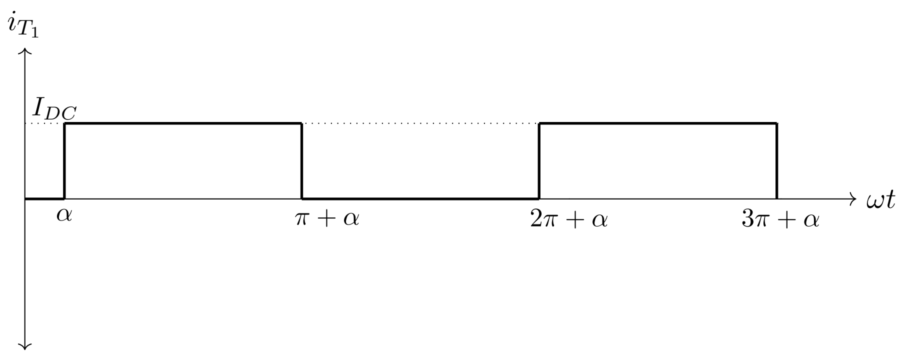
$$\implies \theta_n = \frac{-n\alpha}{2} \tag{9.110}$$

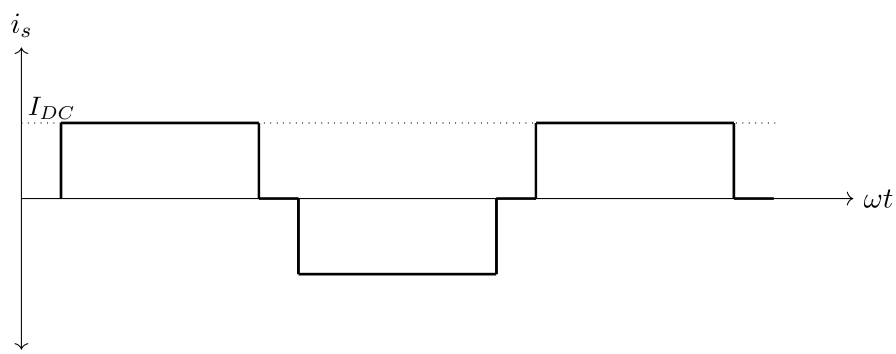
From (9.98),(9.108) and (9.110):

$$I_s(t) = \sum_{n=1,3,5\ldots}^{\infty} \frac{4I_o}{n\pi}\cos\frac{n\alpha}{2}\sin\left(n\omega t - \frac{n\alpha}{2}\right) \tag{9.111}$$









From Table 9.6:

$$(I_{s_1})_{peak} = \frac{4I_{dc}}{\pi}\cos\left(\frac{\alpha}{2}\right) \tag{9.112}$$

$$= \frac{4\times 15}{\pi}\times\cos\frac{45^\circ}{2} \tag{9.113}$$

$$= 17.64A \tag{9.114}$$

9.8 If

$$\frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(nx)$$

is the Fourier cosine series of the function

$$f(x) = \sin(x), 0 < x < \pi$$

then which of the following are TRUE?

(a) $a_0 + a_1 = \frac{4}{\pi}$

(b) $a_0 = \frac{4}{\pi}$

(c) $a_0 + a_1 = \frac{2}{\pi}$

(d) $a_1 = \frac{2}{\pi}$

(GATE 2022 NM Q24)

Solution:

Symbol	**Value**	**Description**
a_0, a_n, b_n		Fourier Series Coefficients
T	π	Time Period
n		Positive Integer

Table 9.7: Input Parameters

Fourier series of a function $f(x)$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\omega x) + \sum_{n=1}^{\infty} b_n \sin(n\omega x) \tag{9.115}$$

where,

$$a_0 = \frac{1}{T}\int_T f(x)dx \tag{9.116}$$
$$a_n = \frac{2}{T}\int_T f(x)\cos(n\omega x)dx \tag{9.117}$$
$$b_n = \frac{2}{T}\int_T f(x)\sin(n\omega x)dx \tag{9.118}$$

Calculating for given function:

$$\frac{a_0}{2} = \frac{1}{\pi}\int_0^{\pi} \sin(x)dx \tag{9.119}$$
$$\implies a_0 = \frac{4}{\pi} \tag{9.120}$$
$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)dx \tag{9.121}$$
$$\implies a_1 = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(x)dx \tag{9.122}$$
$$= 0 \tag{9.123}$$

Calculating general a_n:

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin(x)\cos(nx)dx \tag{9.124}$$
$$= \frac{1}{\pi}\int_0^{\pi} (\sin(x+nx)+\sin(x-nx))dx \tag{9.125}$$
$$= \frac{1}{\pi}\left[\frac{-\cos((n+1)x)}{n+1} + \frac{-\cos((1-n)x)}{1-n}\right]_0^{\pi} \tag{9.126}$$
$$= \frac{2(1+\cos(n\pi))}{\pi(1-n^2)} \tag{9.127}$$

From (9.120) and (9.123),

$$a_0 + a_1 = \frac{4}{\pi} \tag{9.128}$$

$$a_0 = 0 \tag{9.129}$$

$\therefore$ correct options are (a) and (b).

9.9 The fourier series expansion of x^3 in the interval $-1 \leq x \leq 1$with periodic continuation has

(a) only sine terms

(b) only cosine terms

(c) both sine and cosine terms

(d) only sine terms and a non-zero constant

(GATE 2022 ME)

Solution:

Fourier series expansion of the function $x\left(t\right)$ in the interval $[-L, L]$ can be given by:

$$x\left(t\right) = a_0 + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi t}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \tag{9.130}$$

where,

$$a_0 = \frac{1}{2L}\int\limits_{-L}^{L} f\left(t\right)\, dt \tag{9.131}$$

$$a_n = \frac{1}{2L}\int\limits_{-L}^{L} f\left(t\right)\cos\left(\frac{n\pi t}{L}\right)\, dt \tag{9.132}$$

$$b_n = \frac{1}{2L}\int\limits_{-L}^{L} f\left(t\right)\sin\left(\frac{n\pi t}{L}\right)\, dt \tag{9.133}$$

(9.134)

Therefore, the expansion can be given by:

$$t^3 = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\pi t) + \sum_{n=1}^{\infty} b_n \sin(n\pi t) \tag{9.135}$$

Since t^3 is an odd function,

$$a_0 = a_n = 0 \tag{9.136}$$

$$b_n = \frac{1}{2} \int_{-1}^{1} t^3 \sin(n\pi t)\ dt \tag{9.137}$$

$$= (-1)^{n+1} \left(\frac{2}{n\pi} - \frac{12}{(n\pi)^3} \right) \tag{9.138}$$

$$\implies t^3 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi t}{L}\right) \tag{9.139}$$

$\therefore$It contains only sine terms.

9.10 The discrete time Fourier series representation of a signal $x[n]$ with period N is written as $x[n] = \sum_{k=0}^{N-1} a_k e^{j(2kn\pi/N)}$. A discrete time periodic signal with period $N = 3$, has the non-zero Fourier series coefficients: $a_{-3} = 2$ and $a_4 = 1$. The signal is

(A) $2 + 2e^{-\left(j\frac{2\pi}{6}n\right)} \cos\left(\frac{2\pi}{6}n\right)$

(B) $1 + 2e^{\left(j\frac{2\pi}{6}n\right)} \cos\left(\frac{2\pi}{6}n\right)$

(C) $1 + 2e^{\left(j\frac{2\pi}{3}n\right)} \cos\left(\frac{2\pi}{6}n\right)$

(D) $2 + 2e^{\left(j\frac{2\pi}{6}n\right)} \cos\left(\frac{2\pi}{6}n\right)$

(GATE EE 2022)

Solution:

Parameters	**Description**	**Value**
$x[n]$	Signal	
N	Period	3
a_k	Fourier series coefficient	
a_{-3}	a_k at $k = -3$	2
a_4	a_k at $k = 4$	1

Table 9.8: Parameters

$$x[n] = \sum_{k=0}^{N-1} a_k e^{j(2kn\pi/N)} \tag{9.140}$$

$$= a_0 + a_1 e^{j\frac{2\pi}{3}n} \tag{9.141}$$

From (A.5.10),

$$a_0 = a_{-3} \tag{9.142}$$

$$a_1 = a_4 \tag{9.143}$$

$$x[n] = 2 + e^{j\frac{2\pi}{3}n} \tag{9.144}$$

$$= 1 + 1 + e^{j\frac{2\pi}{3}n} \tag{9.145}$$

$$= 1 + e^{j\frac{2\pi}{6}n} e^{-j\frac{2\pi}{6}n} + e^{j\frac{2\pi}{6}n} e^{j\frac{2\pi}{6}n} \tag{9.146}$$

$$= 1 + 2e^{j\frac{2\pi}{6}n} \left(\frac{e^{j\frac{2\pi}{6}n} + e^{-j\frac{2\pi}{6}n}}{2} \right) \tag{9.147}$$

$$= 1 + 2e^{j\frac{2\pi}{6}n} \cos\left(\frac{2\pi}{6}n\right) \tag{9.148}$$

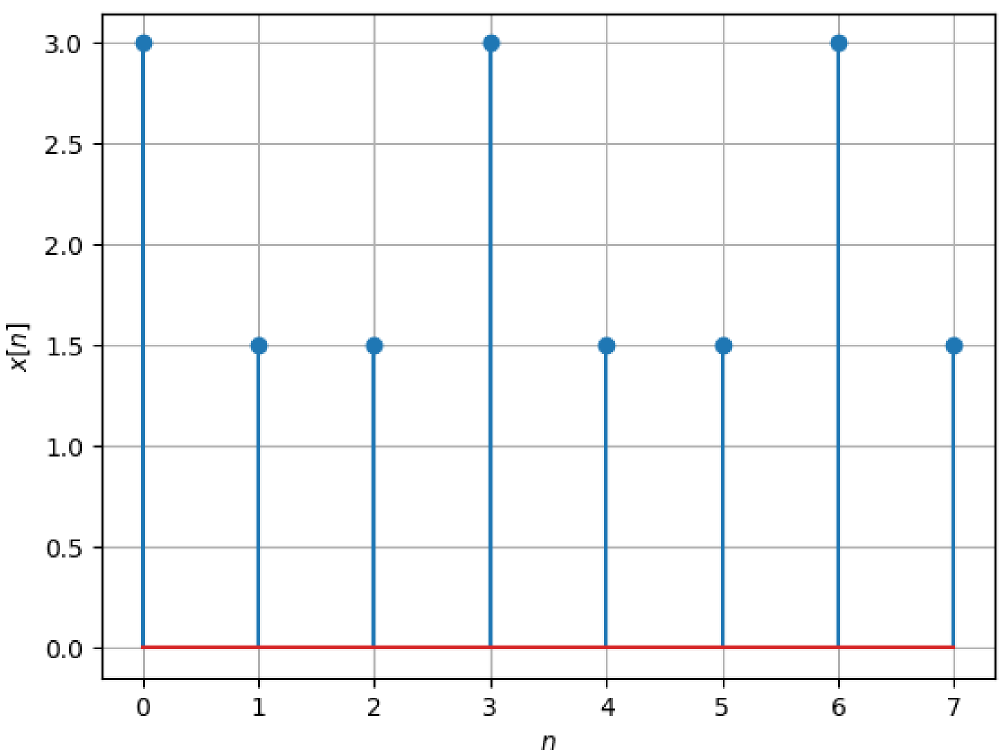


Figure 9.7: Stem Plot of $x[n]$

9.11 Consider the wave elevation spectrum $S_{\eta\eta}(\omega)$ as shown in the figure. Then, the significant wave height is ________________ m.

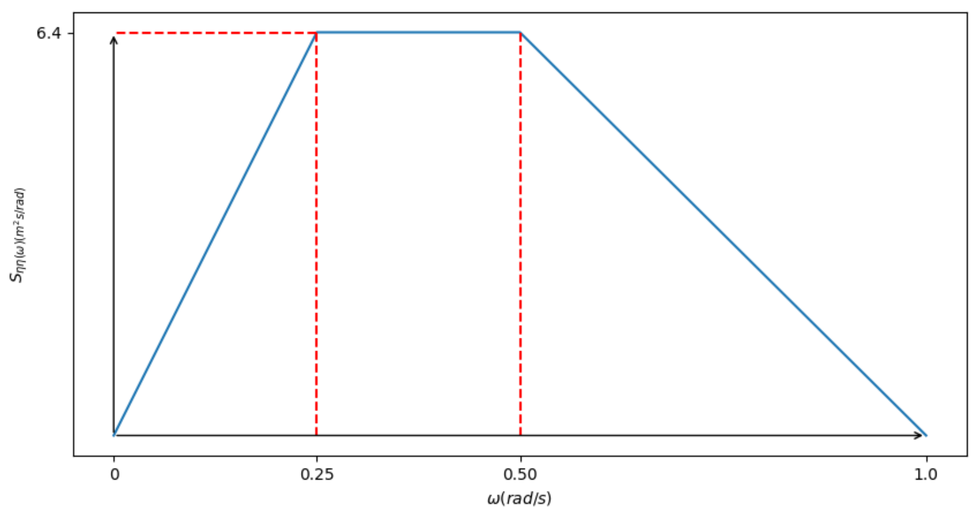


Figure 9.8: Wave Elevation Spectrum

(A) 2

(B) 4

(C) 6

(D) 8

GATE NM 2022

Solution:

Given:

$$S_{\eta\eta}(\omega)(m^2s/rad) = \begin{cases} 25.6\omega & \text{if } \omega \in [0, 0.25] \\ 6.4 & \text{if } \omega \in (0.25, 0.50] \\ 12.8\omega - 12.8 & \text{if } \omega \in (0.50, 1.0] \\ 0 & o.w \end{cases} \tag{9.149}$$

In terms of f:

$$S_{\eta\eta}(f)(m^2s) = \begin{cases} 51.2\pi f & \text{if } f \in [0, \frac{\pi}{2}] \\ 6.4 & \text{if } f \in (\frac{\pi}{2}, \pi] \\ 25.6\pi f - 12.8 & \text{if } f \in (\pi, 2\pi] \\ 0 & o.w \end{cases} \tag{9.150}$$

Significant Wave Height:

$$H_s = 4\sqrt{\int_0^\infty S(f)df} \tag{9.151}$$

From (9.150)

$$H_s = 4\sqrt{\int_0^{\frac{\pi}{2}} 51.2\pi f df + \int_{\frac{\pi}{2}}^{\pi} 6.4 df + \int_{\pi}^{2\pi} (25.6\pi f - 12.8) df} \tag{9.152}$$

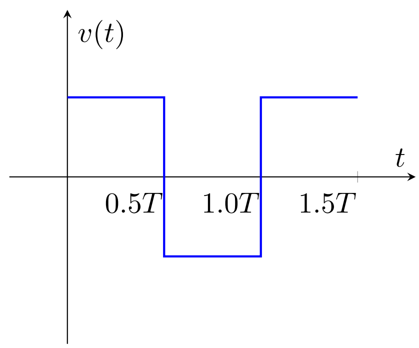
$$= 4\sqrt{0.8 + 1.6 + 1.6} \tag{9.153}$$

$$\therefore H_s = 8 \tag{9.154}$$

Hence the answer is option (D).

9.12 Consider an ideal full-bridge single-phase DC-AC inverter with a DC bus voltage magnitude of $1000V$. The inverter output voltage $v(t)$ shown below is obtained when diagonal switches of the inverter are switched with 50% duty cycle. The inverter feeds a load with a sinusoidal current given by $i(t) = 10\sin(\omega t - \frac{\pi}{3})$ A, where $\omega = \frac{2\pi}{T}$. The active power, in watts, delivered to the load is __.[Gate2022-EE-58]

Solution:



Variable	Description	Value
V_s	input DC voltage	$1000V$
$i(t)$	output current	$\sin(\omega t - \frac{\pi}{3})$
$v(t)$	Output voltage	given
ω	Frequency	$\frac{2\pi}{T}$
v_0^{rms}	RMS output voltage at the fundamental frequency	none
i_{rms}	RMS output current at the fundamental frequency	none
$v_0(t)$	output voltage at the fundamental frequency	none
ϕ	phase difference between $v_0(t)$ and $i(t)$	none
i_0	amplitude of output current	1
P	active power delivered	none

Table 9.9: Input parameters

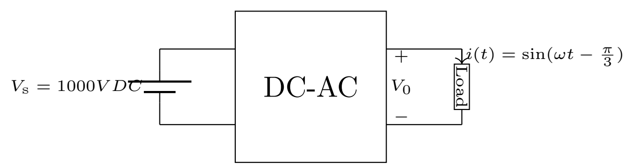


Figure 9.9: circuit diagram of the system

The Fourier series expansion of the given voltage $v(t)$ is,

$$v(t) = \sum_{n=1,3,5,\ldots}^{\infty} \frac{4V_{\text{dc}}}{n\pi} \sin(n\omega t) \tag{9.155}$$

$$v_0(t) = \frac{4V_{\text{dc}}}{\pi} \sin(\omega t) \tag{9.156}$$

$$\therefore \quad \phi = \frac{\pi}{3} \tag{9.157}$$

$$v_0^{\text{rms}} = \frac{4V_{\text{dc}}}{\pi\sqrt{2}} \tag{9.158}$$

$$= \frac{4000}{\pi\sqrt{2}} \tag{9.159}$$

$$i_{\text{rms}} = \frac{i_0}{\sqrt{2}} = \frac{1}{\sqrt{2}} \tag{9.160}$$

Active power delivered to load in Watts is given by,

$$P = v_0^{\text{rms}} \times i_{\text{rms}} \times \cos\phi \tag{9.161}$$

$$= \frac{4000}{\pi\sqrt{2}} \times \frac{1}{\sqrt{2}} \times \cos\frac{\pi}{3} \tag{9.162}$$

$$\approx 3183 \tag{9.163}$$

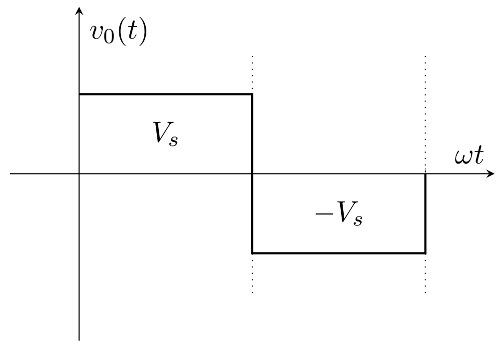


Figure 9.10: output voltage and current of the system

9.13 If $G(f)$ is the Fourier Transform of $f(x)$, then which of the following are true?

(a) $G(-f) = +G^*(f)$ implies $f(x)$ is real.

(b) $G(-f) = -G^*(f)$ implies $f(x)$ is purely imaginary.

(c) $G(-f) = +G^*(f)$ implies $f(x)$ is purely imaginary.

(d) $G(-f) = -G^*(f)$ implies $f(x)$ is real.

(GATE 2022 PH Question 26)

Solution:

Symbol	**Description**
$f(x)$	Function
$G(f)$	Fourier Transform of the function $f(x)$
$f^*(x)$	Complex Conjugate of $f(x)$
$G^*(f)$	Complex Conjugate of $G(f)$
$\mathrm{Im}(G(f))$	Imaginary Part of $G(f)$

Table 9.10: Given Information

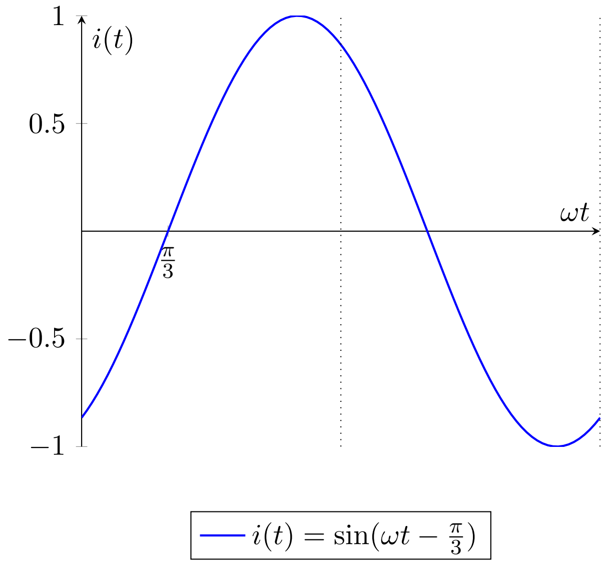


Figure 9.11: output voltage and current of the system

$$f(x) \xleftrightarrow{\mathcal{F}} G(f) \tag{9.164}$$

$$G(f) = \int_{-\infty}^{\infty} f(x)\, e^{-j2\pi f x}\, dx \tag{9.165}$$

$$\implies G(-f) = \int_{-\infty}^{\infty} f(x)\, e^{j2\pi f x}\, dx \tag{9.166}$$

$$\implies G^*(f) = \int_{-\infty}^{\infty} f^*(x)\, e^{j2\pi f x}\, dx \tag{9.167}$$

If $G(-f) = +G^*(f)$, from (9.166) and (9.167),

$$f(x) = f^*(x) \tag{9.168}$$

Hence, $f(x)$ is real.

Consider, $G(f) = \frac{j}{2}\left(\delta\left(f + f_0\right) - \delta\left(f - f_0\right)\right)$,

$$G(-f) = -\frac{j}{2}\left(\delta\left(f + f_0\right) - \delta\left(f - f_0\right)\right) \tag{9.169}$$

$$G^*(f) = -\frac{j}{2}\left(\delta\left(f + f_0\right) - \delta\left(f - f_0\right)\right) \tag{9.170}$$

$$\implies G(-f) = +G^*(f) \tag{9.171}$$

Here, $f(x) = \sin\left(2\pi f_0 x\right)$, is real.

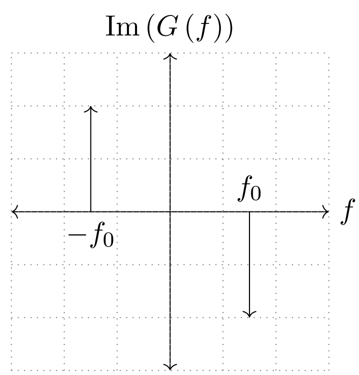


Figure 9.12: Plot of Im$(G(f))$ vs f

If $G(-f) = -G^*(f)$, from (9.166) and (9.167),

$$f(x) = -f^*(x) \tag{9.172}$$

Hence, $f(x)$ is purely imaginary.

Consider, $G(f) = \frac{j}{2}(\delta(f - f_0) + \delta(f + f_0))$,

$$G(-f) = \frac{j}{2}(\delta(f + f_0) + \delta(f - f_0)) \tag{9.173}$$

$$G^*(f) = -\frac{j}{2}(\delta(f - f_0) + \delta(f + f_0)) \tag{9.174}$$

$$\implies G(-f) = -G^*(f) \tag{9.175}$$

Here, $f(x) = j\cos(2\pi f_0 x)$, is purely imaginary.

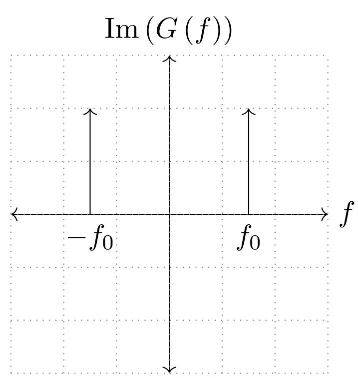


Figure 9.13: Plot of Im($G(f)$) vs f

Therefore, (a) and (b) are true.

9.14 The value of Integral

$$\oint \left(\frac{6z}{2z^4-3z^3+7z^2-3z+5}\right)dz$$

evaluated over a counter-clockwise circular contour in the complex plane enclosing only the pole $z=\jmath$, where $\jmath$ is the imaginary unit,is

(a) $(-1+\jmath)\pi$

(b) $(1+\jmath)\pi$

(c) $2(1-\jmath)\pi$

(d) $(2+\jmath)\pi$

(GATE 2022 ME)

Solution:

Given $z=\jmath$ is only enclosing pole

$$\oint \left(\frac{6z}{2z^4-3z^3+7z^2-3z+5}\right)dz = \oint \left(\frac{\frac{6z}{2z^3+(2\jmath-3)z^2+(5-3\jmath)z+5\jmath}}{z-\jmath}\right)dz \tag{9.176}$$

$$= 2\pi\jmath\left(\frac{6z}{2z^3+(2\jmath-3)z^2+(5-3\jmath)z+5\jmath}\right) \text{ At } z=\jmath \text{ (Cau} \tag{9.177}$$

$$= 2\pi\jmath\left(\frac{6\jmath}{2\jmath^3+(2\jmath-3)\jmath^2+(5-3\jmath)\jmath+5\jmath}\right) \tag{9.178}$$

$$= 2\pi\jmath\left(\frac{\jmath}{\jmath+1}\right) \tag{9.179}$$

$$= -2\pi\frac{\jmath-1}{\jmath^2-1} \tag{9.180}$$

$$= (-1+\jmath)\pi \tag{9.181}$$

9.2. 2021

9.1 Consider the signals $x(n)$=$2^{n-1}u(-n+2)$ and $y(n)$=$2^{-n+2}u(n+1)$, where $u(n)$ is the unit step sequence. Let $X\left(e^{j\omega}\right)$ and $Y\left(e^{j\omega}\right)$ be the discrete-time Fourier of $x(n)$ and $y(n)$,respectively. The value of the integral $\frac{1}{2\pi}\int_0^{2\pi} X\left(e^{j\omega}\right)Y\left(e^{-j\omega}\right)d\omega$ (rounded off to one decimal place) is ___________________

(GATE EC 41 2021)

Solution:

Parameter	Description
$u(n)$	unit step function
$z(n)$	$x(n)*y(-n)$
$Z\left(e^{j\omega}\right)$	$X\left(e^{j\omega}\right)Y\left(e^{-j\omega}\right)$

Table 9.11: Variables and their descriptions

$$V = \frac{1}{2\pi}\int_0^{2\pi} X\left(e^{j\omega}\right) Y\left(e^{-j\omega}\right) d\omega \tag{9.182}$$

$$Z\left(e^{j\omega}\right) = X\left(e^{j\omega}\right) Y\left(e^{-j\omega}\right) \tag{9.183}$$

$$z(n) \xrightarrow{\mathcal{F}} Z(e^{j\omega}) \tag{9.184}$$

$$z\left(n\right) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Z\left(e^{j\omega}\right) e^{j\omega n} d\omega \tag{9.185}$$

$$z\left(0\right) = \frac{1}{2\pi}\int_0^{2\pi} Z\left(e^{j\omega}\right) d\omega \tag{9.186}$$

$$z\left(n\right) = x\left(n\right) * y\left(-n\right) \tag{9.187}$$

$$= \sum_{k=-\infty}^{\infty} 2^{k-1} u\left(-k+2\right) 2^{n-k+2} u\left(-n+k+1\right) \tag{9.188}$$

$$= \sum_{k=-\infty}^{2} 2^{n+1} u\left(k-n+1\right) \tag{9.189}$$

$$z(0) = \sum_{k=-\infty}^{2} 2u\left(k+1\right) \tag{9.190}$$

$$= 2\sum_{k=-1}^{2} u\left(k+1\right) \tag{9.191}$$

$$\therefore z\left(0\right) = 8 \tag{9.192}$$

$\therefore \frac{1}{2\pi}\int_0^{2\pi} X\left(e^{j\omega}\right) Y\left(e^{-j\omega}\right) d\omega = 8$

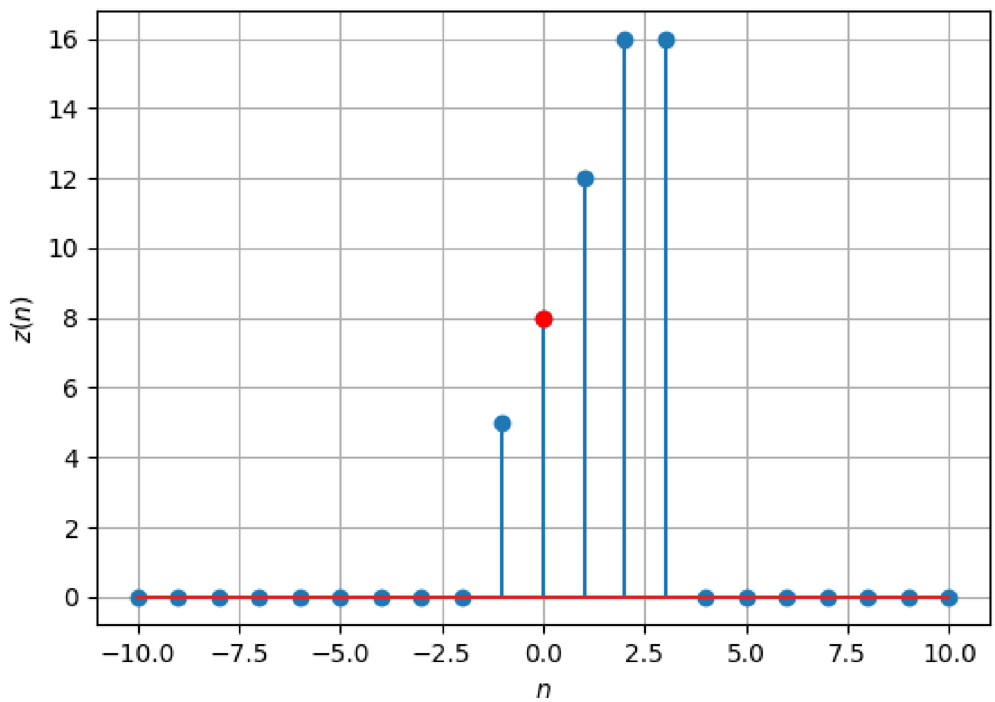


Figure 9.1: Stem Plot of $z(n)$

9.2 Given that $\mathcal{S}$ is the unit circle in the counter clock-wise direction with its centre at origin, the integral $\oint \left(\frac{z^3}{4z-\jmath}\right) dz =$ ______ (round off to theree decimal places) (GATE 2022 AE)

Solution:

For pole

$$4z - \jmath = 0 \tag{9.193}$$

$$z = \frac{\jmath}{4} \text{ order of pole is 1} \tag{9.194}$$

Pole inside unit circle is $\frac{\jmath}{4}$

$$\oint \left(\frac{z^3}{4z-\jmath}\right) dz = \oint \left(\frac{\frac{z^3}{4}}{z-\frac{\jmath}{4}}\right) dz \tag{9.195}$$

$$= 2\pi\jmath \left(\frac{z^3}{4}\right) \text{ at } z = \frac{\jmath}{4} \text{ (using Cauchy integral)} \tag{9.196}$$

$$= 2\pi\jmath \left(\frac{-\jmath}{256}\right) \tag{9.197}$$

$$= \frac{\pi}{128} \tag{9.198}$$

$$= 0.02 \tag{9.199}$$

9.3 Consider the signals $x[n] = 2^{n-1}u[-n+2]$ and $y[n] = 2^{-n+2}u[n+1]$, where $u[n]$ is the unit step sequence. Let $X(e^{j\omega})$ and $Y(e^{j\omega})$ be the discrete-time Fourier transform of $x[n]$ and $y[n]$, respectively. The value of the integral

$$\frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})d\omega$$

(rounded off to one decimal place) is.

GATE 2021 EC 41 Q **Solution:**

Symbol	Value	description
$x[n]$	$2^{n-1}u[-n+2]$	Discrete time signal
$y[n]$	$2^{-n+2}u[n+1]$	Discrete time signal

Table 9.3:

$$x[n] * y[n] \quad \xleftrightarrow[transform]{Fourier} \quad X(e^{j\omega})Y(e^{j\omega}) \tag{9.200}$$

$$x[n] \quad \xleftrightarrow[transform]{Fourier} \quad X(e^{j\omega}) \tag{9.201}$$

$$y[n] \quad \xleftrightarrow[transform]{Fourier} \quad Y(e^{j\omega}) \tag{9.202}$$

The

$$y(n) \quad \xleftrightarrow[transform]{Fourier} \quad y(e^{j\omega}) \tag{9.203}$$

By using the time reversal property:

$$y[-n] \quad \xleftrightarrow[transform]{Fourier} \quad y(e^{-j\omega}) \tag{9.204}$$

Let assume

$$z[n] = x[n] * y[-n] \tag{9.205}$$

$$Z\left(e^{j\omega}\right) = X(e^{j\omega})Y(e^{-j\omega}) \tag{9.206}$$

$$z[n] = \frac{1}{2\pi}\int_0^{2\pi} Z(e^{j\omega})e^{j\omega n}d\omega \tag{9.207}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})e^{j\omega n}d\omega. \tag{9.208}$$

putting n=0, we get

$$z[0] = \frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})d\omega \tag{9.209}$$

$$
\begin{aligned}
z[n] &= x[n] * y[-n] && (9.210)\\
&= \sum_{k=-\infty}^{\infty} 2^{k-1}u[-k+2] \cdot 2^{n-k+2}u[-n+k+1] && (9.211)\\
&= \sum_{k=-\infty}^{2} 2^{k-1} \cdot 2^{n-k+2}u[-n+k+1] && (9.212)\\
&= \sum_{k=-\infty}^{2} 2^{k-1+n-k+2}u[-n+k+1] && (9.213)\\
&= \sum_{k=-\infty}^{2} 2^{n+1}u[-n+k+1] && (9.214)
\end{aligned}
$$

Putting $n = 0$, we get:

$$
\begin{aligned}
\frac{1}{2\pi}\int_0^{2\pi} X(e^{j\omega})Y(e^{-j\omega})d\omega &= z[0] && (9.215)\\
&= \sum_{k=-\infty}^{2} 2 \cdot u[k+1] && (9.216)\\
&= \sum_{k=-1}^{2} 2(1) = 2 \times 4 && (9.217)\\
&= 8 && (9.218)
\end{aligned}
$$

9.4 Consider a continuous-time signal $x(t)$ defined by $x(t) = 0$ for $|t| > 1$, and $x(t) = 1 - |t|$ for $|t| \leq 1$. Let the Fourier transform of $x(t)$ be defined as $X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$. The maximum magnitude of $X(\omega)$ is ______. (GATE 2021 EE 43)

Solution:

$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt \tag{9.219}$$

$$X(f) = \int_{-1}^{1} (1 - |t|)\, e^{-j2\pi ft} dt \tag{9.220}$$

$$X(f) = \int_{-1}^{1} e^{-j2\pi ft} dt - \int_{-1}^{1} |t|\, e^{-j2\pi ft} dt \tag{9.221}$$

$$X(f) = 2\int_{0}^{1} \cos(2\pi ft)\, dt - 2\int_{0}^{1} t\cos(2\pi ft)\, dt \tag{9.222}$$

$$X(f) = 2\frac{\sin(2\pi f)}{2\pi f} - 2\left[\frac{\sin(2\pi f)}{2\pi f} + \frac{\cos(2\pi f)}{(2\pi f)^2} - \frac{1}{(2\pi f)^2}\right] \tag{9.223}$$

$$X(f) = 2\frac{1 - \cos(2\pi f)}{(2\pi f)^2} \tag{9.224}$$

$$X(f) = 2\frac{2\sin^2\left(\frac{2\pi f}{2}\right)}{(2\pi f)^2} \tag{9.225}$$

$$X(f) = \frac{\sin^2(\pi f)}{(\pi f)^2} \tag{9.226}$$

$$f \to 0 \implies X(f) \to 1 \tag{9.227}$$

Maximum of magnitude of $X(f) = 1$

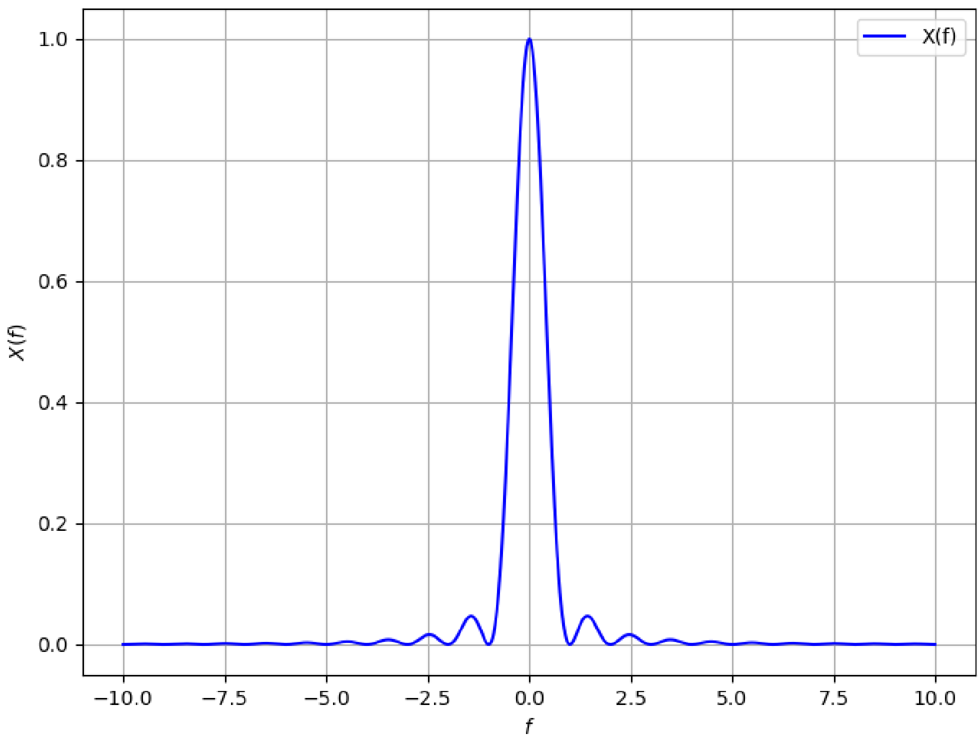


Figure 9.4: plot of X(f)

9.5 Let $f(t)$ be an even function, i.e.$f(-t) = f(t)$ for all t.Let the Fourier transform of $f(t)$ be defined as $F(\omega) = \int_{-\infty}^{\infty} f(t)e^{-j\omega t}\, dt$. Suppose $\frac{dF(\omega)}{d\omega} = -\omega F(\omega)$ for all ω , and $F(0) = 1$. Then

(A) $f(0) < 1$

(B) $f(0) > 1$

(C) $f(0) = 1$

(D) $f(0) = 0$

(GATE EE 2021)

Solution: Given,

$$\frac{dF(\omega)}{d\omega} = -\omega F(\omega) \tag{9.228}$$

$$\frac{dF(\omega)}{d\omega} + \omega F(\omega) = 0 \tag{9.229}$$

$$ln|F(\omega)| = -\frac{\omega^2}{2} + c \tag{9.230}$$

$$F(\omega) = Ke^{-\frac{\omega^2}{2}} \tag{9.231}$$

Put $\omega = 0$,

$$F(0) = K \tag{9.232}$$

$$K = 1 \tag{9.233}$$

$$\therefore F(\omega) = e^{-\frac{\omega^2}{2}} \tag{9.234}$$

$$f(t) \longleftrightarrow F(\omega)$$

$$e^{-at^2} \longleftrightarrow \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}} \; ; \; a > 0 \tag{9.235}$$

$$At \; a = \frac{1}{2}, \; e^{-\frac{t^2}{2}} \longleftrightarrow \sqrt{2\pi} e^{-\frac{\omega^2}{2}} \tag{9.236}$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \longleftrightarrow e^{-\frac{\omega^2}{2}} = F(\omega) \tag{9.237}$$

$$\text{Thus , } f(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \tag{9.238}$$

At $t = 0$

$$f(0) = \frac{1}{\sqrt{2\pi}} < 1 \tag{9.239}$$

Hence , option (a) is correct.

9.6 The exponential Fourier series representation of a continous-time periodic signal x(t) is defined as

$$x\left(t\right) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

where ω_0 is the fundamental angular frequency of x(t) and the coefficients of the series are a_k.The following information is given about x(t) and a_k

I. x(t) is real and even,having a fundamental period of 6

II. The average value of x(t) is 2.

III.

$$a_k = \begin{cases} k, & 1 \leq k \leq 3 \\ 0, & k > 3 \end{cases} \tag{9.240}$$

$$\tag{9.241}$$

The average power of the signal x(t) (rounded off to one decimal place) is _____.

(GATE EC 2021)

Solution:

x(t) is even and real so, a_k=a_{-k}

Parswal's Power Theorem

variable	value	description
T_0	6	Fundamental time period
P_{avg}	-	average power of the signal
x(t)	$\sum\limits_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$	Input signal
a_k	$\begin{cases} k, & 1 \leq k \leq 3 \\ 0, & k > 3 \end{cases}$	coefficients of the series
a_0	2	average of x(t)

Table 9.6: Table: Input Parameters

Proof

$$P = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} \left| x\left(t\right)^2 \right| dt, \quad \left| x\left(t\right)^2 \right| = x\left(t\right) x^*\left(t\right) \tag{9.242}$$

$$P = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x\left(t\right) x^*\left(t\right) dt \tag{9.243}$$

$$x\left(t\right) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \tag{9.244}$$

$$P = \frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} \left(\sum_{k=-\infty}^{\infty} C_n e^{jn\omega_0 t} \right) x^*\left(t\right) dt \tag{9.245}$$

$$P = \sum_{n=-\infty}^{\infty} C_n \left(\frac{1}{T} \int_{\frac{-T}{2}}^{\frac{T}{2}} x^*\left(t\right) e^{jn\omega_0 t} dt \right) \tag{9.246}$$

$$= \sum_{n=-\infty}^{\infty} C_n C_n^* \tag{9.247}$$

$$\implies P = \sum_{n=-\infty}^{\infty} \left| C_n \right|^2 \tag{9.248}$$

By using Parsval's Power Theorem

$$\frac{1}{T}\int_{0}^{T}\left|x\left(t\right)^{2}\right|dt = \sum_{k=-\infty}^{\infty}\left|a_k\right|^2 \tag{9.249}$$

$$P_{avg} = \sum_{k=-\infty}^{\infty}\left|a_k\right|^2 \tag{9.250}$$

$$= 2\sum_{k=1}^{\infty}\left|a_k\right|^2 + \left|a_0\right|^2 \tag{9.251}$$

$$= 2\sum_{k=1}^{3}\left|a_k\right|^2 + \left|a_0\right|^2 \tag{9.252}$$

$$= 2\left(1^2 + 2^2 + 3^2\right) + 2^2 \tag{9.253}$$

$$= 32 \tag{9.254}$$

$$x\left(n\right) = 2\text{Re}\left(e^{\frac{j\pi t}{3}} + 2e^{\frac{2j\pi t}{3}} + 3e^{j\pi t}\right) + 2 \tag{9.255}$$

$$\implies x\left(n\right) = 2\left(cos\left(\frac{\pi t}{3}\right) + 2cos\left(\frac{2\pi t}{3}\right) + 3cos\left(\pi t\right)\right) + 2 \tag{9.256}$$

The average power of the signal x(t) (rounded off to one decimal place) is 32

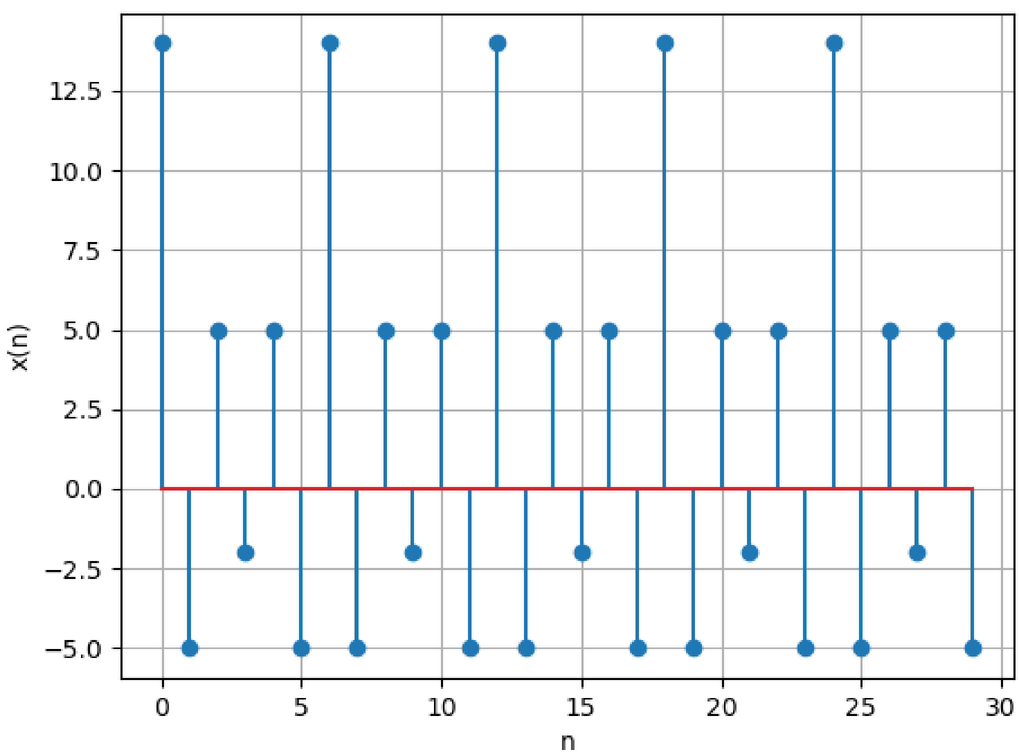


Figure 9.6: STEM PLOT OF $y\,(n)$

Chapter 10

Numerical Methods

10.1. 2022

10.1 **Solution:**

10.2. 2021

10.1 An ordinary differential equation (ODE) $\frac{dy}{dx} = 2y$ with an initial condition $y(0) = 1$ has the analytical solution $y = e^{2x}$ Using Runge-Kutta second order method, numerically integrate the ODE to calculate y at $x = 0.5$ using a step size of $h = 0.5$. If the relative percentage error is defined as

$$\epsilon = \left| \frac{y_{analytical} - y_{numerical}}{y_{analytical}} \right| \times 100$$

then the value of ϵ at $x = 0.5$ is

(GATE 1 CH 2021) **Solution:**

Constant	**Description**
$y(0.5)$	$y_{analytical}$
y_1	$y_{numerical}$
$\frac{dy}{dx} = f(x, y)$	Function representing the ODE
h	Step size
K_1	First slope estimate in the Runge-Kutta
K_2	Second slope estimate in the Runge-Kutta
K	Weighted average of K_1 and K_2

Table 10.1:

Analytical solution is given by:

$$y = e^{2x} \tag{10.1}$$

At $x = 0.5$, analytical solution is

$$y(0.5) = e^{2\times 0.5} = 2.718 \tag{10.2}$$

According to question:

$$f(x, y) = \frac{dy}{dx} = 2y \tag{10.3}$$

By Runge-kutta 2^{nd} order method,

$$K_1 = hf(x_o, y_o) = h(2y_o) \tag{10.4}$$

$$K_1 = 0.5(2 \times 1) = 1 \tag{10.5}$$

$$K_2 = h[f(x_o + h, y_o + K_1)] \tag{10.6}$$

$$K_2 = h[2(1 + 1)] \tag{10.7}$$

$$K_2 = 0.5 \times 4 = 2 \tag{10.8}$$

$$K = \frac{K_1 + K_2}{2} = \frac{1 + 2}{2} = 1.5 \tag{10.9}$$

$$y_1 = y_{numerical} = y_o + K = 1 + 1.5 = 2.5 \tag{10.10}$$

Hence,

$$\epsilon = \left| \frac{2.718 - 2.5}{2.718} \right| \times 100 \tag{10.11}$$

$$\epsilon = \frac{0.218}{2.718} \times 100 = 8\% \tag{10.12}$$

Appendix A

Fourier transform

A.1 The Differentiation in frequency domain is as follows

Let x(t) be a signal such that,

$$x(t) \xleftrightarrow{\text{F.T}} X(\omega) \tag{A.1.1}$$

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t) e^{-j\omega t} dt \tag{A.1.2}$$

$$\frac{d}{d\omega} X(\omega) = \int_{t=-\infty}^{\infty} x(t)(-jt) e^{-j\omega t} dt \tag{A.1.3}$$

$$j\frac{d}{d\omega} X(\omega) = \int_{t=-\infty}^{\infty} tx(t) e^{-j\omega t} dt \tag{A.1.4}$$

$$tx(t) \xleftrightarrow{\text{F.T}} j\frac{d}{d\omega} X(\omega) \tag{A.1.5}$$

A.2 The duality property is as follows

From inverse Fourier transform we get,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega) e^{j\omega t} d\omega \tag{A.2.1}$$

Replacing t by -t and multiplying 2π on both sides we get,

$$2\pi x\left(-t\right) = \int_{-\infty}^{\infty} X\left(\omega\right) e^{-j\omega t} d\omega \tag{A.2.2}$$

$$X\left(t\right) \xleftrightarrow{\text{F.T}} 2\pi x\left(-\omega\right) \tag{A.2.3}$$

A.3 Complex Fourier Series

$$x(t) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi nft} \tag{A.3.1}$$

where c_n is the exponential fourier coefficient.

$$c_n = \frac{1}{T} \int_0^T x(t) e^{-j2\pi nft}\, dt \tag{A.3.2}$$

where T is the time period of function $x(t)$.

A.4 Trigonometric Fourier Series

$$e^{j2\pi nft} = cos(2\pi nft) + jsin(2\pi nft) \tag{A.4.1}$$

Substituting (A.4.1) in (A.3.1)

$$x(t) = \sum_{n=-\infty}^{\infty} c_n \left(\cos(2\pi nft) + j\sin(2\pi nft)\right) \tag{A.4.2}$$

$$= a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(2\pi nft)\right) + \left(b_n \sin(2\pi nft)\right) \tag{A.4.3}$$

where a_0, a_n and b_n are trigonometric fourier series.

$$a_0 = c_0 \tag{A.4.4}$$

$$= \frac{1}{T}\int_0^T x(t)\,dt \tag{A.4.5}$$

$$a_n = 2Re(c_n) \tag{A.4.6}$$

$$= \frac{2}{T}\int_0^T x(t)\cos(2\pi nft)\,dt \tag{A.4.7}$$

$$b_n = -2Im(c_n) \tag{A.4.8}$$

$$= \frac{2}{T}\int_0^T x(t)\sin(2\pi nft)\,dt \tag{A.4.9}$$

A.5 Periodicity of Fourier coefficients

For a signal $x[n] = \sum_{k=0}^{N-1} a_k e^{j\left(\frac{2kn\pi}{N}\right)}$, defining $X[k] = Na_k$.

$$x[n] = \frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\left(\frac{2kn\pi}{N}\right)} \tag{A.5.1}$$

$$\sum_{n=0}^{N-1} x[n]e^{-j\left(\frac{2rn\pi}{N}\right)} = \sum_{n=0}^{N-1}\frac{1}{N}\sum_{k=0}^{N-1} X[k]e^{j\left(\frac{2kn\pi}{N}\right)}e^{-j\left(\frac{2rn\pi}{N}\right)} \tag{A.5.2}$$

$$= \sum_{k=0}^{N-1} X[k]\sum_{n=0}^{N-1}\frac{1}{N}e^{j\left(\frac{2(k-r)n\pi}{N}\right)} \tag{A.5.3}$$

$$\sum_{n=0}^{N-1}\frac{1}{N}e^{j\left(\frac{2(k-r)n\pi}{N}\right)} = \begin{cases} 1 & k-r = mN, m \in \mathbb{Z} \\ 0 & \text{otherwise} \end{cases} \tag{A.5.4}$$

$$\implies X[k] = \sum_{n=0}^{N-1} x[n]e^{-j\left(\frac{2kn\pi}{N}\right)} \tag{A.5.5}$$

If $x[n] = x[n + rN], r \in \mathbb{Z}$, from (A.5.5),

$$X[k + rN] = \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2(k+rN)n\pi}{N}\right)} \tag{A.5.6}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2kn\pi}{N}\right)} e^{-j(2rn\pi)} \tag{A.5.7}$$

$$= \sum_{n=0}^{N-1} x[n] e^{-j\left(\frac{2kn\pi}{N}\right)} \tag{A.5.8}$$

$$= X[k] \tag{A.5.9}$$

$$\implies a_k = a_{k+rN}, r \in \mathbb{Z} \tag{A.5.10}$$

Appendix B

Contour Integration

B.1 Cauchy's Theorem:

From Figure B.1.1

$$\int_C f(z)\, dz = \int_{Cr} f(z)\, dz \tag{B.1.1}$$

Since $g(z)$ is continuous we know that $|g(z)|$ is bounded inside C_r. Say, $|g(z)| < M$.

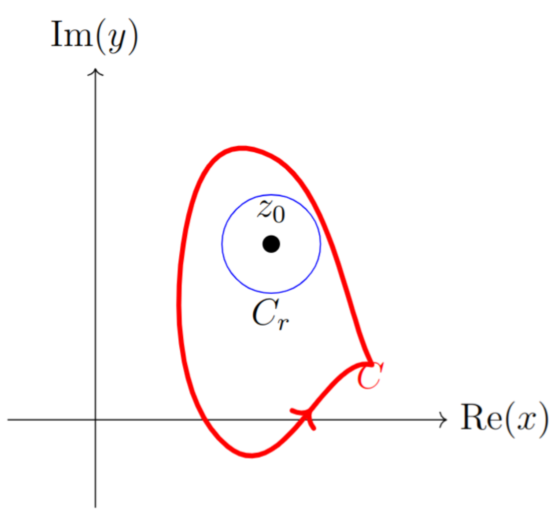


Figure B.1.1: Figure1

The corollary to the triangle inequality says that

$$\left|\int_{C_r} f\left(z\right)\, dz\right| \leq M2\pi r. \tag{B.1.2}$$

Since r can be as small as we want, this implies that

$$\int_{Cr} f\left(z\right) dz = 0 \tag{B.1.3}$$

let

$$g\left(z\right) = \frac{f\left(z\right) - f\left(z_0\right)}{z - z_0} \tag{B.1.4}$$

$$\lim_{z \to z_0} g(z) = f'(z_0) \tag{B.1.5}$$

$$\int_C g(z)\, dz = 0, \implies \int_C \frac{f(z) - f(z_0)}{z - z_0}\, dz = 0 \tag{B.1.6}$$

Thus,

$$\int_C \frac{f(z)}{z - z_0}\, dz = \int_C \frac{f(z_0)}{z - z_0}\, dz = 2\pi i f(z_0) \tag{B.1.7}$$

Using Cauchy's Theorem,

$$\int_C f\left(z\right) = 0 \tag{B.1.8}$$

$$f\left(a\right) = \frac{1}{2\pi i} \oint_C \frac{f\left(z\right)}{z - a}\, dz \tag{B.1.9}$$

$$f^n\left(a\right) = \frac{n!}{2\pi i} \oint_C \frac{f\left(z\right)}{\left(z - a\right)^{n+1}}\, dz \tag{B.1.10}$$

B.2 Residue Theorem:

From eq (B.1.7)

$$\int_C f(z)\ dz = 2\pi i \sum Res\, f(a) \tag{B.2.1}$$

where, for n repeated poles,

$$Res\, f(a) = \lim_{z\to a} \frac{1}{(n-1)!}\left(\frac{d^{n-1}}{dz^{n-1}}\left[(z-a)^n\ f(z)\right]\right) \tag{B.2.2}$$

Appendix C

Laplace Transform

C.1 Laplace Transform of Partial Differentials

Let a function $y\left(x,t\right)$ be defined for all $t>0$ and assumed to be bounded. Appling Laplace transform in t considering x as a parameter,

$$\mathcal{L}\left(y\left(x,t\right)\right)=\int_{0}^{\infty}e^{-st}y\left(x,t\right)dt \tag{C.1.1}$$

$$=Y\left(x,s\right) \tag{C.1.2}$$

Let $\dfrac{\partial y\left(x,t\right)}{\partial t}$ be $y_t\left(x,t\right)$ and $\dfrac{\partial y\left(x,t\right)}{\partial x}$ be $y_x\left(x,t\right)$, then

$$\mathcal{L}\left(y_t\left(x,t\right)\right)=\int_{0}^{\infty}e^{-st}y_t\left(x,t\right)dt \tag{C.1.3}$$

$$=e^{-st}y\left(x,t\right)\Big|_0^{\infty}+s\int_{0}^{\infty}e^{-st}y\left(x,t\right)dt \tag{C.1.4}$$

$$=sY\left(x,s\right)-y\left(x,0\right) \tag{C.1.5}$$

$$\mathcal{L}\left(y_x\left(x,t\right)\right)=\int_{0}^{\infty}e^{-st}y_x\left(x,t\right)dt \tag{C.1.6}$$

$$=\frac{d}{dx}\int_{0}^{\infty}e^{-st}y\left(x,t\right)dt \tag{C.1.7}$$

$$=\frac{dY\left(x,s\right)}{dx} \tag{C.1.8}$$

C.2 Laplace transform of f(t):

$$f(t)u(t) \xleftrightarrow{\mathcal{L}} \int_0^\infty f(t)e^{-st}\,dt \tag{C.2.1}$$

$$= F(s) \tag{C.2.2}$$

C.3 Laplace transform of powers of t

Let $f(t) = t^n u(t)$

From (C.2.2),and considering $h = st$

$$F(s) = \frac{1}{s^{n+1}} \int_0^\infty h^n e^{-h}\,dh \tag{C.3.1}$$

$$(n-1)! = \int_0^\infty e^{-t} t^{n-1}\,dt \text{ (Gamma function)} \tag{C.3.2}$$

From (C.3.1),(C.3.2)

$$F(s) = \frac{n!}{s^{n+1}} \tag{C.3.3}$$

$$t^n u(t) \xleftrightarrow{\mathcal{L}} \frac{n!}{s^{n+1}} \tag{C.3.4}$$

C.4 Frequency shift property:

Let $f(t) = y(t)e^{-at}u(t)$

From(C.2.2),

$$F(s) = \int_0^\infty y(t)e^{-(s+a)t}\,dt \tag{C.4.1}$$

$$y(t)e^{-at}u(t) \xleftrightarrow{\mathcal{L}} Y(s+a) \tag{C.4.2}$$

C.5 Inverse Laplace for partial fractions

From (C.3.4),(C.4.2) we get

$$\frac{b}{(s+a)^n} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} \frac{b}{(n-1)!} \cdot t^{n-1}e^{-at} \cdot u(t) \tag{C.5.1}$$

C.6 Laplace transform of derivatives:

Let $f(t) = y'(t)u(t)$

From (C.2.2), integration by parts,recursion

$$F(s) = \int_0^\infty e^{-st}\, dy \tag{C.6.1}$$
$$= [y(t)e^{-st}]_0^\infty + s\int_0^\infty y(t)e^{-st}dt \tag{C.6.2}$$
$$= -y(0) + sY(s) \tag{C.6.3}$$

From(C.6.3),recursion

$$y'(t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} sY(s) - \int y'(t)\, dt|_{t=0} \tag{C.6.4}$$
$$y^{(n)}(t)u(t) \overset{\mathcal{L}}{\longleftrightarrow} s^nY(s) - \sum_{k=0}^{n-1} s^{(n-1-k)}y^{(k)}(0) \tag{C.6.5}$$

C.7 Laplace transform of integrals:

Let the function defined as $y(t) = \int_0^t f(u)du$ for all $t > 0$

Laplace transform of y(t) in t

$$\mathcal{L}\left(y(t)\right) = \int_0^\infty e^{-st} y(t)dt \tag{C.7.1}$$

$$= \int_0^\infty e^{-st} \int_0^t f(u)dudt \tag{C.7.2}$$

$$= \int_0^t f(u)du \left[-\frac{e^{-st}}{s}\right]_0^\infty + \int_0^\infty \frac{e^{-st}}{s} f(t)dt \tag{C.7.3}$$

$$= \frac{F(s)}{s} \tag{C.7.4}$$

Appendix D

Filters

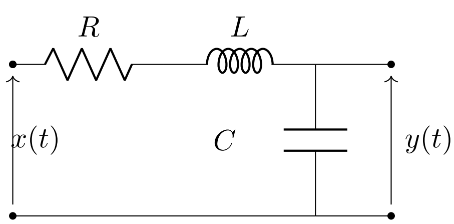


Figure D.1.1: RLC Low pass filter

Parameter	Description
Z_C	Reactance of Capacitor
Z_R	Reactance of Resistor
Z_L	Recatance of Inductor
$x(t) = u(t)$	Input Response
$y(t)$	Output across capacitor
ω_0	Angular resonant frequency

Table D.1.1: Input Parameters

D.1

$$Y(s) = I(s)Z_C \tag{D.1.1}$$
$$= \frac{X(s)}{Z_L + Z_R + Z_C} Z_C \tag{D.1.2}$$
$$H(s) = \frac{Y(s)}{X(s)} \tag{D.1.3}$$
$$= \frac{Z_C}{Z_L + Z_R + Z_C} \tag{D.1.4}$$
$$= \frac{\frac{1}{sC}}{sL + R + \frac{1}{sC}} \tag{D.1.5}$$
$$= \frac{1}{s^2LC + sRC + 1} \tag{D.1.6}$$
$$\implies H(s) = \omega_0^2 \frac{1}{(s - p_1)(s - p_2)} \tag{D.1.7}$$

where,

$$\omega_0 = \frac{1}{\sqrt{LC}} \tag{D.1.8}$$
$$p_{1,2} = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}} \tag{D.1.9}$$
$$= -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \tag{D.1.10}$$

where

$$\alpha = \frac{R}{2L} \tag{D.1.11}$$

Damping Factor is given by,

$$\zeta = \frac{\alpha}{\omega_0} \tag{D.1.12}$$

$$= \frac{R}{2}\sqrt{\frac{C}{L}} \tag{D.1.13}$$

ζ	**Pole Location**	**Referred to as**	**Condition**
$\zeta > 1$	Different locations on the negative real axis	Overdamped	$R > 2\sqrt{\frac{L}{C}}$
$\zeta = 1$	Coincide on the negative real axis	Critically Damped	$R = 2\sqrt{\frac{L}{C}}$
$\zeta < 1$	Complex Conjugate poles in the left half of s-plane	Underdamped	$R < 2\sqrt{\frac{L}{C}}$

Table D.1.2: Effect of Damping Coefficient ζ on system behaviour

(a) Overdamped Response

$$
\begin{aligned}
Y(s) &= X(s)H(s) && \text{(D.1.14)}\\
&= \omega_0^2 \frac{1}{s(s-p_1)(s-p_2)} && \text{(D.1.15)}\\
&= \frac{c_0}{s} + \frac{c_1}{s-p_1} + \frac{c_2}{s-p_2} && \text{(D.1.16)}
\end{aligned}
$$

where,

$$
\begin{aligned}
c_0 &= 1 && \text{(D.1.17)}\\
c_1 &= \frac{p_2}{p_1-p_2} && \text{(D.1.18)}\\
c_2 &= \frac{p_1}{p_2-p_1} && \text{(D.1.19)}
\end{aligned}
$$

Taking inverse Laplace,

$$
\begin{aligned}
y(t) &= c_0 + c_1 e^{p_1 t} + c_2 e^{p_2 t} && \text{(D.1.20)}\\
&= \left(1 + \frac{p_2}{p_1-p_2} e^{p_1 t} + \frac{p_1}{p_2-p_1} e^{p_2 t}\right) u(t) && \text{(D.1.21)}
\end{aligned}
$$

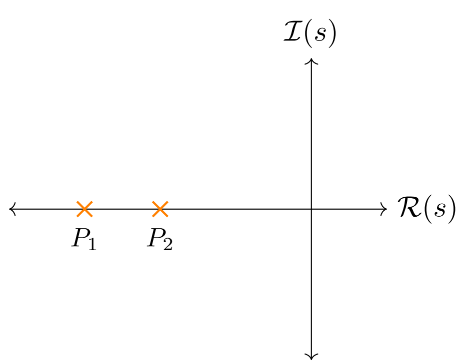


Figure D.1.2: s-Plane for Overdamped case

(b) Critically Damped Response

$$Y(s) = X(s)H(s) \tag{D.1.22}$$
$$= \omega_0^2 \frac{1}{s(s-p)^2} \tag{D.1.23}$$
$$= \frac{c_0}{s} + \frac{c_1}{(s-p)^2} + \frac{c_2}{s-p} \tag{D.1.24}$$

where,

$$c_0 = 1 \tag{D.1.25}$$
$$c_1 = p \tag{D.1.26}$$
$$c_2 = -1 \tag{D.1.27}$$

Taking Inverse Laplace,

$$y(t) = c_0 + (c_1 t + c_2)e^{pt} \tag{D.1.28}$$
$$= \left(1 + (pt-1)e^{pt}\right) u(t) \tag{D.1.29}$$

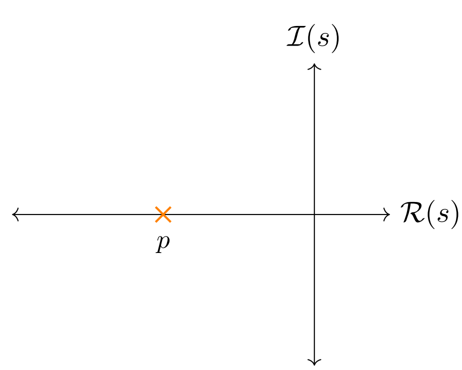


Figure D.1.3: s-Plane for Critically damped case

(c) Underdamped Response

$$
\begin{aligned}
Y(s) &= X(s)H(s) && \text{(D.1.30)}\\
&= \omega_0^2 \frac{1}{s(s-p)(s-p^*)} && \text{(D.1.31)}\\
&= \frac{c_0}{s} + \frac{c_1}{s-p} + \frac{c_2}{s-p^*} && \text{(D.1.32)}
\end{aligned}
$$

where,

$$
\begin{aligned}
c_0 &= 1 && \text{(D.1.33)}\\
c_1 &= \frac{p^*}{p-p^*} && \text{(D.1.34)}\\
c_2 &= \frac{p}{p^*-p} && \text{(D.1.35)}
\end{aligned}
$$

Taking Inverse Laplace,

$$
\begin{aligned}
y(t) &= c_0 + c_1 e^{pt} + c_2 e^{p^* t} && \text{(D.1.36)}\\
&= 1 + \frac{|p|}{\omega_d}\ e^{-\sigma t} \frac{e^{j(\omega_d t+\varphi)} + e^{-j(\omega_d t+\varphi)}}{2} && \text{(D.1.37)}\\
&= \left(1 + \frac{|p|}{\omega_d} e^{-\sigma t} \cos(\omega_d t + \varphi)\right) u(t) && \text{(D.1.38)}
\end{aligned}
$$

where,

$$
\begin{aligned}
|p| &= \sqrt{{\omega_d}^2 + \sigma^2} && \text{(D.1.39)}\\
\omega_d &= \omega_0 \sqrt{1-\zeta^2} && \text{(D.1.40)}\\
\sigma &= \omega_0 \zeta && \text{(D.1.41)}\\
\varphi &= \pi - \tan^{-1} \frac{\sigma}{\omega_d} && \text{(D.1.42)}
\end{aligned}
$$

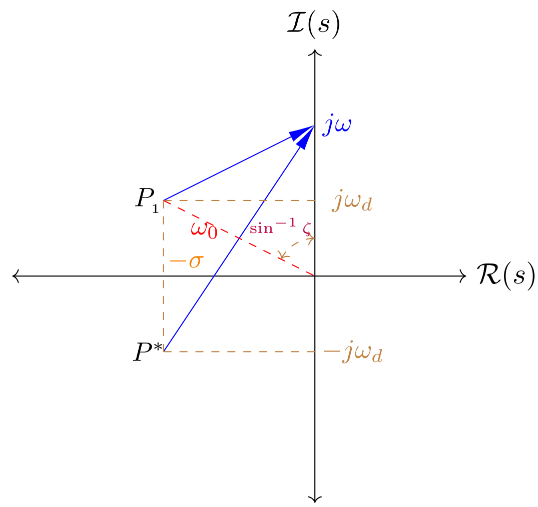


Figure D.1.4: s-Plane for Under damped case

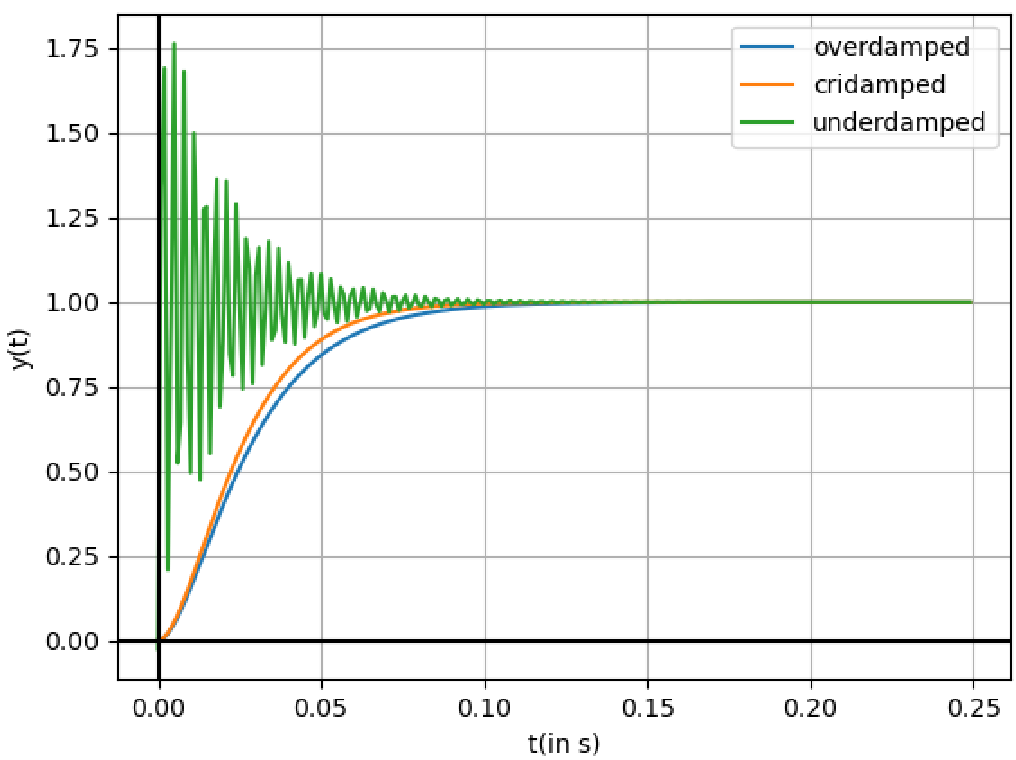


Figure D.1.5: Step response in all three cases

Appendix E

Analogous Systems

Analogous systems of electrical-mechanical systems are like a common language that bridges the gap between electrical and mechanical engineering. They allow us to understand and predict how different components interact, whether they're electrical circuits or mechanical machines. By recognizing similarities between electrical and mechanical elements, engineers can solve problems more efficiently and design better systems. This approach encourages collaboration between specialists from different fields and helps us develop innovative solutions that seamlessly integrate both electrical and mechanical aspects.

ELECTRICAL ANALOGIES OF MECHANICAL SYSTEMS:

Two systems are said to be analogous if :

1. The two systems are physically different.
2. Differential equation modelling of these two systems are same.

There are two types of electrical analogies of translational mechanical systems:

1. Force-Voltage analogy
2. Force-Current analogy

FORCE-VOLTAGE ANALOGY:

In this, the mathematical equations of translational mechanical system are compared with mesh equations of the electrical system.

Translational Mechanical System	**Electrical System**
Force(F)	Voltage(V)
Mass(M)	Inductance (L)
Frictional coefficient(B)	Resistance(R)
Spring constant(K)	Reciprocal of Capacitance(1/C)
Displacement(x)	Charge(q)
Velocity(v)	Current(i)

Table 2.1: Input Parameters

Example:

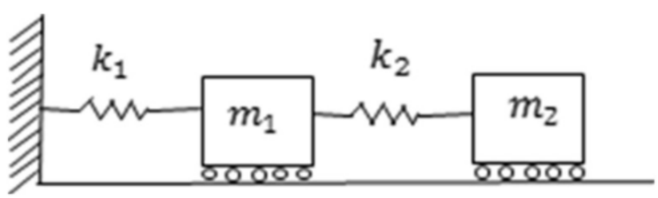


Figure 2.1:

Equations of translational mechanical system:

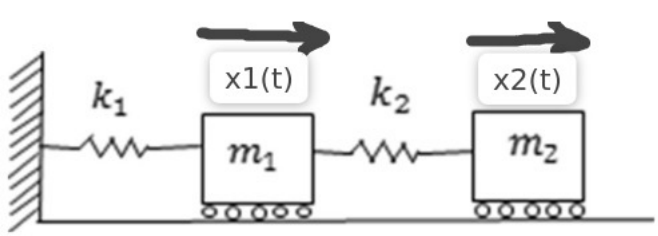


Figure 2.2:

$$m_1\ddot{x}_1(t) - k_2\left(x_2(t) - x_1(t)\right) + k_1x_1(t) = 0 \tag{2.1}$$

$$m_2\ddot{x}_2(t) + k_2\left(x_2(t) - x_1(t)\right) = 0 \tag{2.2}$$

Mesh equations of electrical system:

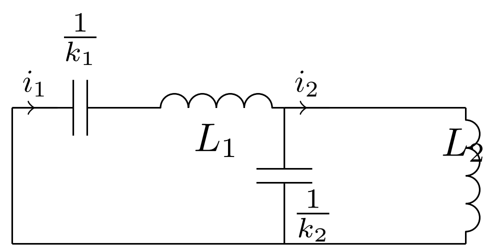


Figure 2.3:

$$k_1\int i_1\,dt + L_1\frac{di_1}{dt} + k_2\int\left(i_1 - i_2\right)\,dt = 0 \tag{2.3}$$

$$L_2\frac{di_2}{dt} - k_2\int\left(i_1 - i_2\right)\,dt = 0 \tag{2.4}$$

but we know, $i = \frac{dq}{dt}$

$$\implies L_1\ddot{q}_1 - k_2\left(q_2 - q_1\right) + k_1q_1 = 0 \tag{2.5}$$

$$\implies L_2\ddot{q}_2 + k_2\left(q_2 - q_1\right)\,dt = 0 \tag{2.6}$$

FORCE-CURRENT ANALOGY:

In this , the mathematical equations of the translational mechanical system are compared with the nodal equations of the electrical system.

Translational Mechanical System	**Electrical System**
Force(F)	Current(i)
Mass(M)	Capacitance(C)
Frictional coefficient(B)	Reciprocal of Resistance(1/R)
Spring constant(K)	Reciprocal of Inductance(1/L)
Displacement(x)	Magnetic Flux(ψ)
Velocity(v)	Voltage(V)