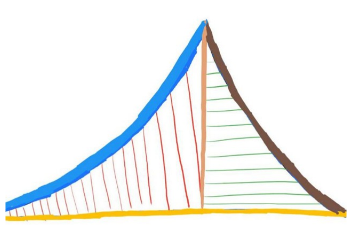
SIGNAL PROCESSING

Through GATE

EE1205-TA Group

Author: Sayyam Palrecha



Contents

Introduction

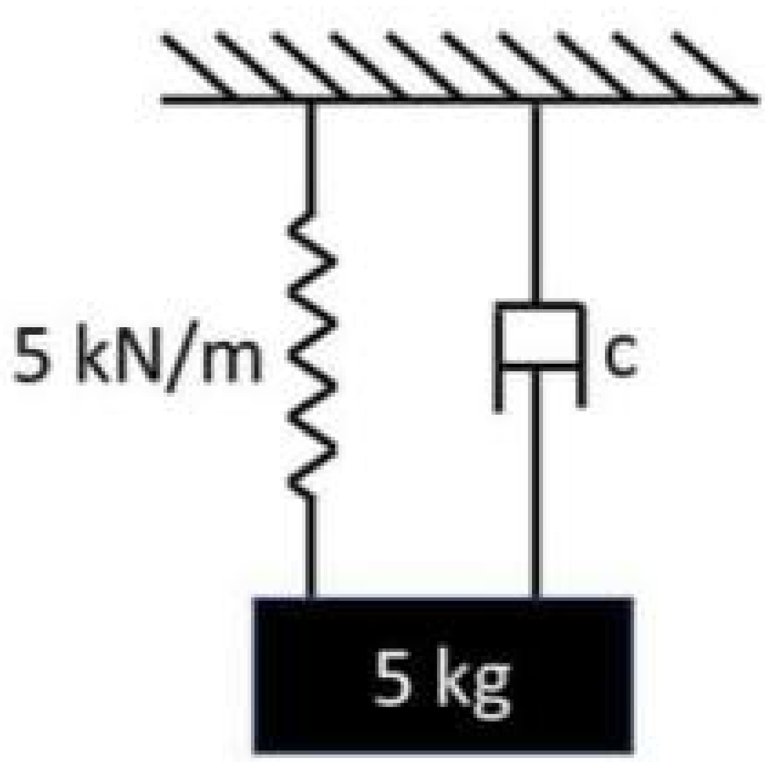
This book provides solutions to signal processing problems in GATE.

Chapter 1

Harmonics

1.1 A damper with damping coefficient, c, is attached to a mass of 5 kg and spring of stiffness 5 kN/m as shown in figure. The system undergoes under-damped oscillations. If the ratio of the 3^{rd} amplitude to the 4^{th} amplitude of oscillations is 1.5, the value of c is ?

(GATE AE-62 (2022)) **Solution:**



1.2 A uniform rigid prismatic bar of total mass m is suspended from a ceiling by two identical springs as shown in figure. Let ω_1 and ω_2 be the natural frequencies of mode I and mode II respectively ($\omega_1 < \omega_2$). The value of $\frac{\omega_2}{\omega_1}$ is ____ (rounded off to one decimal place). (GATE AE 2022 QUESTION 63)

Solution:

i: For vertical oscillations: from Fig. 1.2,

$$m\frac{d^2x\left(t\right)}{dt^2} + 2kx\left(t\right) = 0 \tag{1.1}$$

Assuming the bar is at mean position and has non-zero intitial velocity, we can

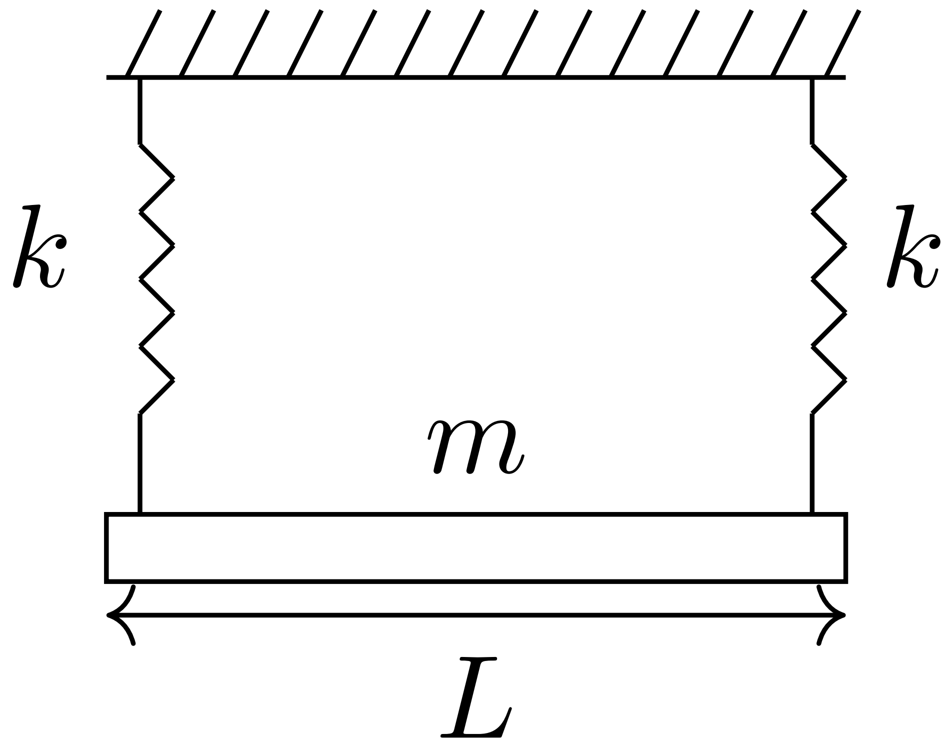


Figure 1.1: Figure given in question

write it's laplace transform as:

$$s^2mX(s) - mv(0) + 2kX(s) = 0 \tag{1.2}$$

$$\implies X(s) = \frac{v(0)}{s^2 + \frac{2k}{m}} \tag{1.3}$$

Parameter	**Description**	**Value**
$X(s)$	position in laplace domain	$X(s)$
$\Theta(s)$	angle rotated in laplace domain	$\Theta(s)$
$x(t)$	position of mass w.r.t time	$x(t)$
$\theta(t)$	angle rotated by mass w.r.t time	$\theta(t)$
$\alpha(t)$	angular acceleration of mass w.r.t time	$\alpha(t)$
k	spring constant	k
m	mass of the block	m
L	length of the mass	L
ω_o	initial angular velocity of mass	ω_o
$v(0)$	initial velocity of mass	$v(0)$

Table 1.1: input values

On taking inverse laplace transform we get,

$$x(t) = v(0)\sqrt{\frac{m}{2k}}\sin\sqrt{\frac{2k}{m}}t \tag{1.4}$$

$$\therefore \omega_1 = \sqrt{\frac{2k}{m}} \tag{1.5}$$

ii: For torsional strain from Fig. 1.3,

$$I\alpha(t) = -\frac{kL^2\theta(t)}{2} \tag{1.6}$$

Assuming it is at mean position and having non-zero angular velocity we can write it's laplace transform as:

$$s^2 I\Theta(s) - I\omega_o + \frac{kL^2\Theta(s)}{2} = 0 \tag{1.7}$$

substituting values from Table 1.1:

$$\Theta(s) = \frac{\omega_o}{s^2 + \frac{6k}{m}} \tag{1.8}$$

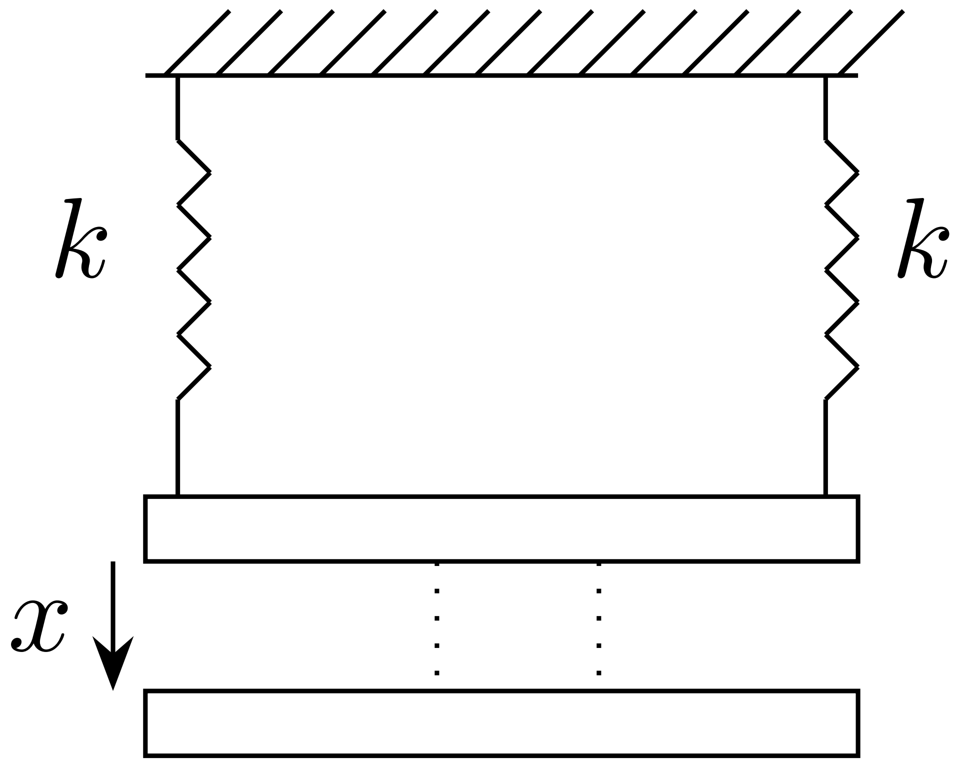


Figure 1.2: Figure for Vertical strain

On taking inverse laplace transform we get,

$$\theta(t) = \omega_o \sqrt{\frac{m}{6k}} \sin \sqrt{\frac{6k}{m}} t \tag{1.9}$$

$$\therefore \omega_2 = \sqrt{\frac{6k}{m}} \tag{1.10}$$

From (1.5) and (1.10) we see that

$$\frac{\omega_2}{\omega_1} = \sqrt{3} \tag{1.11}$$

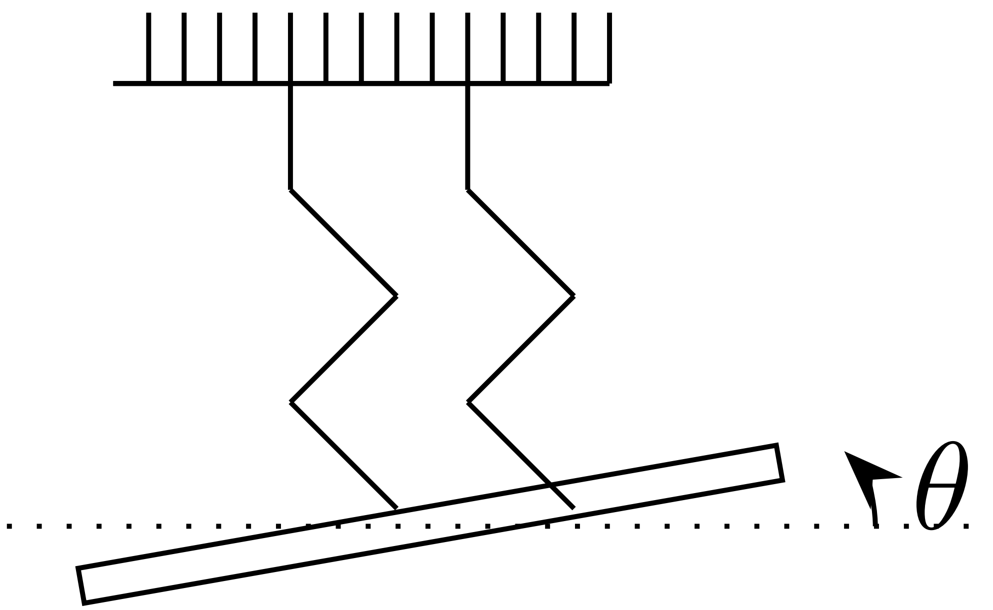


Figure 1.3: Figure for Torsional strain

Chapter 2

2.1

Chapter 3

Z-transform

3.1 Consider the following recursive iteration scheme for different values of variable P with the initial guess $x_1 = 1$:

$$x_{n+1} = \frac{1}{2}\left(x_n + \frac{P}{x_n}\right), \qquad n = 1, 2, 3, 4, 5$$

For $P = 2$, x_5 is obtained to be 1.414, rounded off to 3 decimal places. For $P = 3$, x_5 is obtained to be 1.732, rounded off to 3 decimal places.

If $P = 10$, the numerical value of x_5 is ______ . (*round off to three decimal places*)

(GATE CE 2022)

Solution:

Applying $A.M \geq G.M$ inequality,

$$\frac{x_n + \frac{P}{x_n}}{2} \geq \sqrt{P} \tag{3.1}$$

$$\implies x_{n+1} \geq \sqrt{P} \tag{3.2}$$

Solving the equation,

$$2x_{n+1}x_n - {x_n}^2 - P = 0 \tag{3.3}$$

Applying Z-transform we get,

$$X(z) * X(z) = \frac{PZ^{-1}}{(1 - z^{-1})(2 - z^{-1})} \tag{3.4}$$

$$= P\left(\frac{z^{-1}}{1 - z^{-1}} - \frac{z^{-1}}{2 - z^{-1}}\right) \tag{3.5}$$

From the transformation pairs,

$$x_{n-a} \overset{\mathcal{Z}}{\longleftrightarrow} z^{-a}X(z) \tag{3.6}$$

$$x_{n_1} \times x_{n_2} \overset{\mathcal{Z}}{\longleftrightarrow} X_1(z) * X_2(z) \tag{3.7}$$

$$\frac{u(n-1)}{a^n} \overset{\mathcal{Z}}{\longleftrightarrow} \frac{z^{-1}}{a - z^{-1}} \tag{3.8}$$

Now, applying inverse Z-tranform,

$$x_n^2 = P\left(u(n-1) - \frac{u(n-1)}{2^n}\right) \tag{3.9}$$

$$\implies x_n^2 = P\left(1 - \frac{1}{2^n}\right) \quad [\because n \geq 1] \tag{3.10}$$

Similarly,

$$x_{n+1}^2 = P\left(1 - \frac{1}{2^{n+1}}\right) \tag{3.11}$$

$$\implies \lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \lim_{n\to\infty} \sqrt{\frac{P\left(1 - \frac{1}{2^n}\right)}{P\left(1 - \frac{1}{2^{n+1}}\right)}} \tag{3.12}$$

$$= 1 \tag{3.13}$$

Hence, the system is convergent.

Now finding the limit of the sequence,

$$x^2 = \lim_{x \to \infty} P\left(1 - \frac{1}{2^n}\right) \tag{3.14}$$

$$\implies x = \pm\sqrt{P} \tag{3.15}$$

From (3.2) and (3.15),

$$x_{n+1} = \sqrt{P} \tag{3.16}$$

Therefore, for $P = 10$ the value of x_5 is,

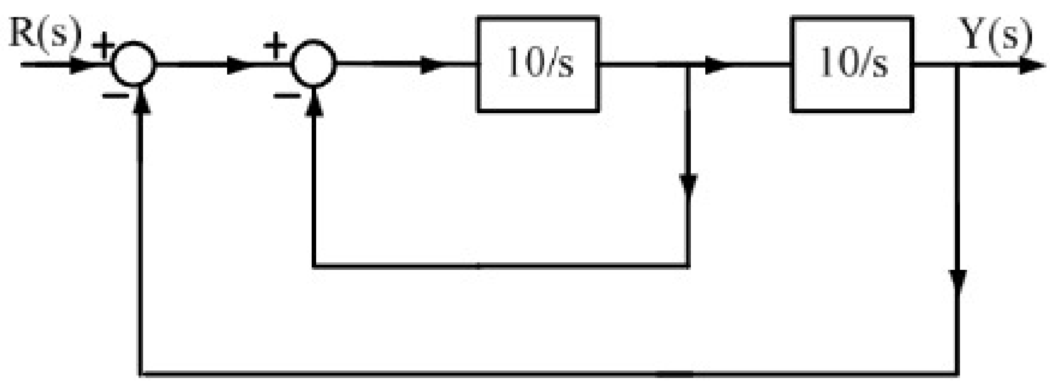
$$x_5 = \sqrt{10} \tag{3.17}$$

$$\therefore x_5 = 3.162 \tag{3.18}$$

Chapter 4

Systems

4.1 The damping ratio and undamped natural frequency of a closed loop system as shown in the figure, are denoted as ζ and ω_n, respectively. The values of ζ and ω_n are



(a) $\zeta = 0.5$ and $\omega_n = 10$ rad/s

(b) $\zeta = 0.1$ and $\omega_n = 10$ rad/s

(c) $\zeta = 0.707$ and $\omega_n = 10$ rad/s

(d) $\zeta = 0.707$ and $\omega_n = 100$ rad/s

(GATE EE 2022) **Solution:**

We will use Mason's Gain Formula to calculate the transfer function of this system.

Parameter	**Description**	**Values**
m	load of system	
k	stiffness of system	
ω_n	Natural frequency	$\sqrt{\frac{k}{m}}$
ζ	Damping ratio	$\frac{c}{2m\omega_n}$
y(t)	Output of system	
x(t)	Input to the system	
c	Damping coefficient	
T(s)	Transfer function of system	$\frac{Y(s)}{R(s)}$

Table 4.1: Parameter Table

First converting the given diagram to a signal flow graph :

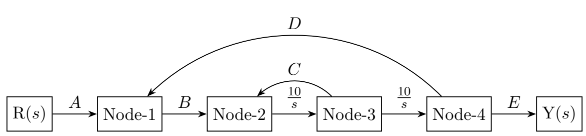


Figure 4.1: Signal Flow Diagram

Mason's Gain Formula is given by :

$$H\left(s\right) = \sum_{i=1}^{N}\left(\frac{P_i\Delta_i}{\Delta}\right) \tag{4.1}$$

This signal flow graph has only one forward path whose gain is given by :

$$P_1 = \frac{10}{s}\frac{10}{s} \tag{4.2}$$
$$= \frac{100}{s^2} \tag{4.3}$$

Parameter	Description
N	Number of forward paths
L	Number of loops
P_k	Forward path gain of k^{th} path
Δ_k	Associated path factor
Δ	Determinant of the graph

Table 4.2: Parameter Table - Mason's Gain Law

Parameter	Formula
Δ	$1 + \sum_{k=1}^{L} \left((-1)^k \text{ Product of gain of groups of k isolated loops}\right)$
Δ_k	Δ part of graph that is not touching k^{th} forward path

Table 4.3: Formula Table - Mason's Gain Law

The loop gain for loop between Node-2 and Node-3 is :

$$L_1 = \frac{10}{s}(-1) \tag{4.4}$$
$$= -\frac{10}{s} \tag{4.5}$$

The loop gain for loop between Node-1 and Node-4 is :

$$L_1 = \frac{10}{s}\frac{10}{s}(-1) \tag{4.6}$$
$$= -\frac{100}{s^2} \tag{4.7}$$

Using Table 4.3, Δ is :

$$\Delta = 1 - \left(-\frac{10}{s} - \frac{100}{s^2} \right) \tag{4.8}$$

$$= 1 + \frac{10}{s} + \frac{100}{s^2} \tag{4.9}$$

There are no two isolated loops available. Hence all further terms will b zero.

As both the loops are in contact with the only forward path,

$$\Delta_1 = 1 \tag{4.10}$$

Using equation (4.1) :

$$H(s) = \frac{\frac{100}{s^2}}{1 + \frac{10}{s} + \frac{100}{s^2}} \tag{4.11}$$

$$= \frac{100}{s^2 + 10s + 100} \tag{4.12}$$

Referring to Table 4.1, the general equation of the damping system is second order and can be written as :

$$m\ddot{y}(t) + c\dot{y}(t) + ky(t) = x(t) \tag{4.13}$$

Take the Laplace transform and solve for $\frac{Y(s)}{X(s)}$:

$$\frac{Y(s)}{X(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.14}$$

$$\implies H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \tag{4.15}$$

Comparing equations (4.12) and (4.15) ,

$$\omega_n^2 = 100 \tag{4.16}$$

$$\implies \omega_n = 10 \text{ rad/s} \tag{4.17}$$

$$2\zeta\omega_n = 10 \tag{4.18}$$

$$\implies \zeta = 0.5 \tag{4.19}$$

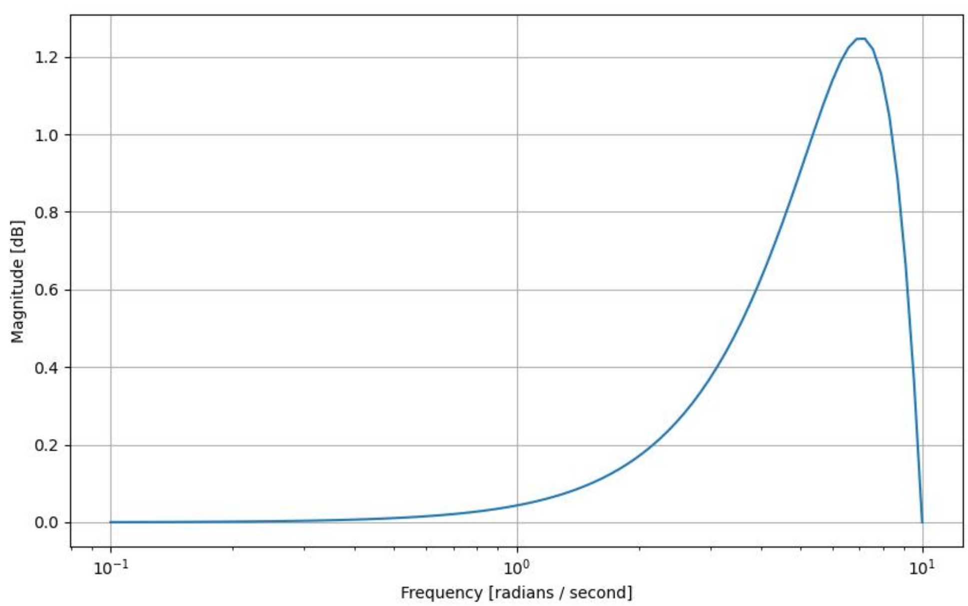


Figure 4.2: Magnitude plot

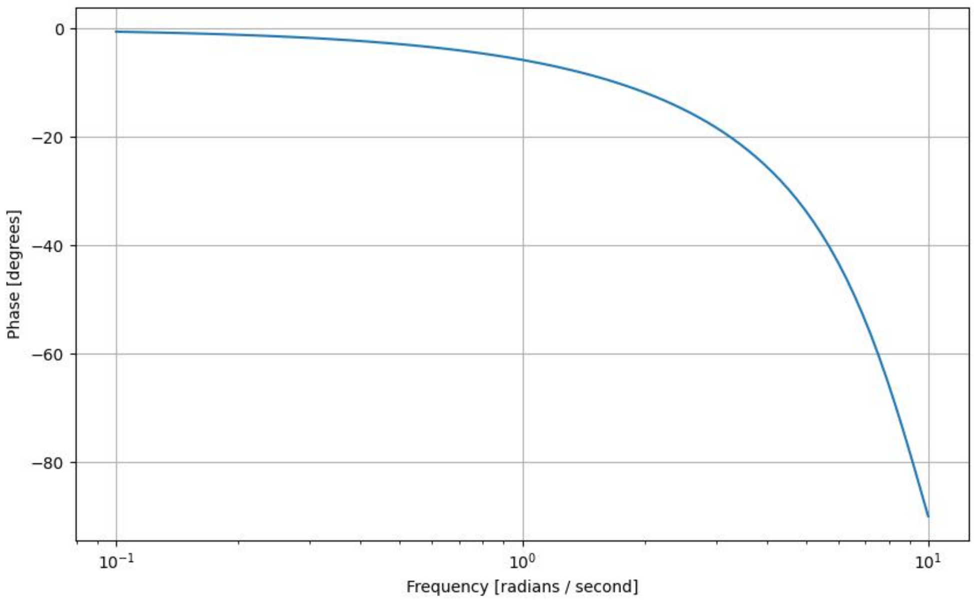
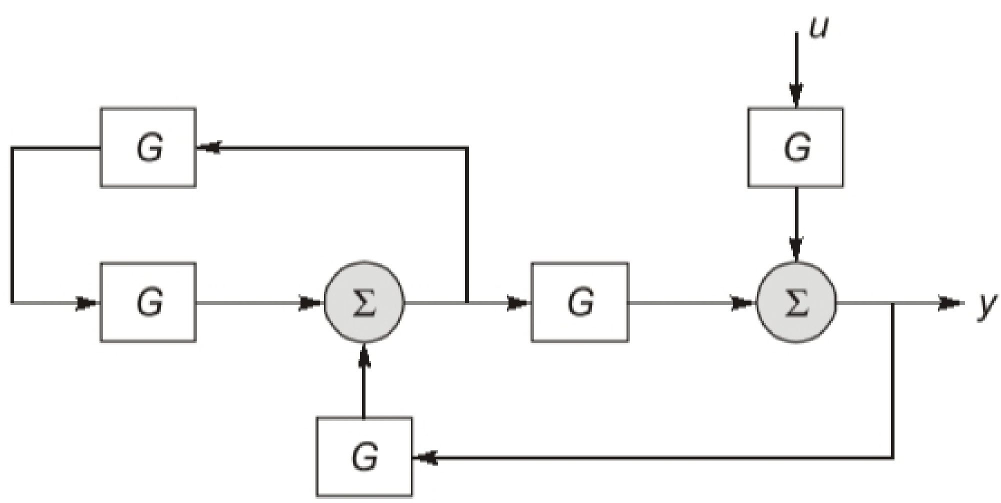


Figure 4.3: Phase plot

4.2 In the block diagram shown in the figure, the transfer function $G = \frac{K}{\tau s+1}$ with $K > 0$ and $\tau > 0$. The maximum value of K below which the system remains stable is ____(rounded off to two decimal places) (GATE CH 2022)

Solution:



Parameter	Value	Description
G	$\frac{K}{\tau s+1}$	Transfer function shown in blocks
Y		Laplace transform of y(output)
U		Laplace transform of u(input)
X,Z		Laplace transform of x and z
T	$\frac{Y}{U}$	Transfer function of complete system

Table 4.4: Parameters

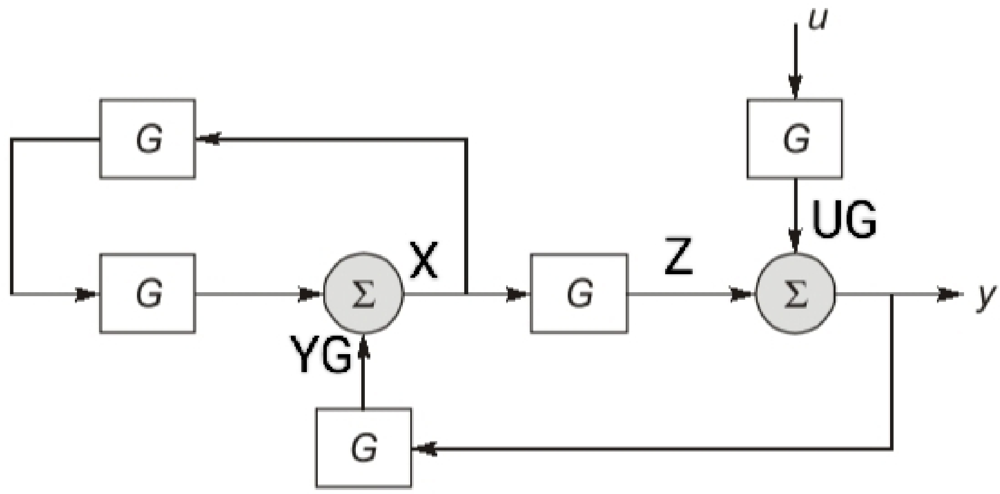


Figure 4.4: Block Diagram

$$X = XG^2 + YG \tag{4.20}$$

$$\implies X = \frac{YG}{1-G^2} \tag{4.21}$$

$$Z = XG \tag{4.22}$$

$$Y = Z + UG \tag{4.23}$$

$$Y = XG + UG \tag{4.24}$$

$$Y = \frac{YG^2}{1-G^2} + UG \tag{4.25}$$

$$\implies Y = \frac{UG(1-G^2)}{1-2G^2} \tag{4.26}$$

From Table 4.4,

$$T = \frac{G(1-G^2)}{1-2G^2} \tag{4.27}$$

$$= \frac{K\left(1-\frac{K^2}{(\tau s+1)^2}\right)}{\left(1-\frac{2K^2}{(\tau s+1)^2}\right)(\tau s+1)} \tag{4.28}$$

$$= \frac{K(\tau^2 s^2 + 2\tau s + 1 - K^2)}{\tau^3 s^3 + 3\tau^2 s^2 + (3\tau - 2K^2\tau)s + 1 - 2K^2} \tag{4.29}$$

So, Characteristic equation :

$$\tau^3 s^3 + 3\tau^2 s^2 + (3\tau - 2K^2\tau)s + 1 - 2K^2 = 0 \tag{4.30}$$

For a characteristic equation $a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + ...a_n = 0$,

s^n	a_0	a_2	a_4	...
s^{n-1}	a_1	a_3	a_5	...
s^{n-2}	$b_1 = \frac{a_1 a_2 - a_3 a_0}{a_1}$	$b_2 = \frac{a_1 a_4 - a_5 a_0}{a_1}$	...	..
s^{n-3}	$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1}$	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		
s^1	$\vdots$	$\vdots$		
s^0	a_n			

Table 4.5: Routh Array

From Table 4.5:

s^3	τ^3	$3\tau - 2K^2\tau$
s^2	$3\tau^2$	$1 - 2K^2$
s^1	$\frac{8}{3}\tau(1 - K^2)$	0
s^0	$1 - 2K^2$	

Table 4.6:

Given $\tau > 0$ and $K > 0$, for system to be stable,

$$1 - K^2 > 0 \tag{4.31}$$

$$1 - 2K^2 > 0 \tag{4.32}$$

$$\implies 0 < K < \frac{1}{\sqrt{2}} \tag{4.33}$$

$$K_{max} \approx 0.71 \tag{4.34}$$

Chapter 5

Sequences

5.1 Discrete signals $x(n)$ and $y(n)$ are shown below. The cross-correlation $r_{xy}(0)$ is:

____

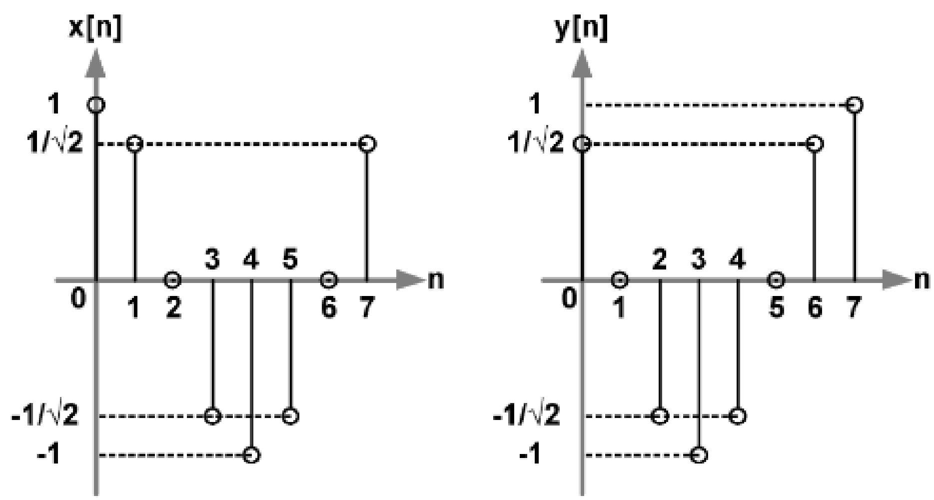


Figure 5.1: Question Figure

(GATE BM 2022)

Solution:

Parameter	Description	Value
$x\left(n\right)$	First Sequence	$x(n) = \begin{cases} 0 & ;n<0 \\ \left(1, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right) & ;0 \leq n \leq 7 \\ 0 & ;n>7 \end{cases}$
$y\left(n\right)$	Second Sequence	$y(n) = \begin{cases} 0 & ;n<0 \\ \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}, -1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\right) & ;0 \leq n \leq 7 \\ 0 & ;n>7 \end{cases}$
$r_{xy}\left(k\right)$	Cross-correlation	$\sum_{m=-\infty}^{\infty} x\left(m\right) y\left(m-k\right)$

Table 1: Parameter Table

It can be seen that :

$$y\left(n\right) = x\left(n+1\right) \tag{5.1}$$

From Table 1 :

$$r_{xy}\left(k\right) = \sum_{m=-\infty}^{\infty} x\left(m\right) y\left(m-k\right) \tag{5.2}$$

$$= x\left(k\right) * y\left(-k\right) \tag{5.3}$$

From (5.1):

$$r_{xy}\left(k\right) = x\left(k+1\right) * x\left(-k\right) \tag{5.4}$$

$$= \sum_{n=-\infty}^{\infty} x\left(n+1\right) x\left(n+k\right) \tag{5.5}$$

By definition of x(n) from Table 1:

$$r_{xy}(k) = \sum_{n=0}^{6} x(n+1)\,x(n+k) \tag{5.6}$$

$$r_{xy}(0) = \sum_{n=0}^{6} x(n+1)\,x(n) \tag{5.7}$$

Using values from Fig. 5.1:

$$r_{xy}(0) = 2\sqrt{2} \tag{5.8}$$

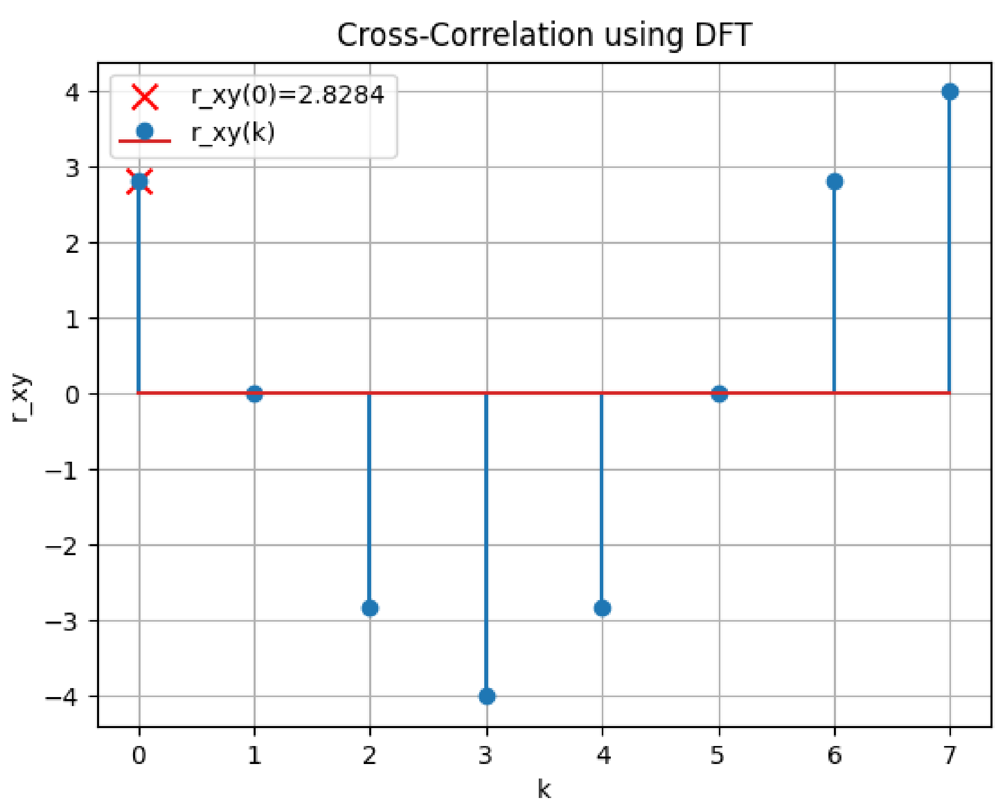


Figure 5.2: Verification of result by DFT

Chapter 6

Sampling

6.1

Chapter 7

Contour Integration

7.1 In the complex z-domain, the value of integral $\oint_C \frac{z^3-9}{3z-i}\, dz$ is

(a) $\frac{2\pi}{81} - 6i\pi$

(b) $\frac{2\pi}{81} + 6i\pi$

(c) $-\frac{2\pi}{81} + 6i\pi$

(d) $-\frac{2\pi}{81} - 6i\pi$

(GATE 2022 BM)

Solution:

Simplyfying the Contour Integral to the standard form we get,

$$\oint_C \frac{z^3-9}{3z-i}\, dz = \frac{1}{3}\oint_C \frac{z^3-9}{z-\frac{i}{3}}\, dz \tag{7.1}$$

From Cauchy's residue theorem,

$$\oint_C f(z)\, dz = 2\pi i \sum R_j \tag{7.2}$$

We can observe a non-repeated pole at $z = \frac{i}{3}$ and thus $a = \frac{i}{3}$,

$$R = \lim_{z \to a} (z - a) f(z) \tag{7.3}$$

$$\implies R = \frac{1}{3} \lim_{z \to \frac{i}{3}} \left(z - \frac{i}{3} \right) \frac{z^3 - 9}{z - \frac{i}{3}} \tag{7.4}$$

$$= \frac{-i}{81} - 3 \tag{7.5}$$

Therefore, from (7.2) and (7.5)

$$\oint_C \frac{z^3 - 9}{3z - i} \, dz = \frac{2\pi}{81} - 6i\pi \tag{7.6}$$

Chapter 8

Laplace Transform

8.1 Consider the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$. The boundary conditions are $y = 0$ and $\frac{dy}{dx} = 1$ at $x = 0$. Then the value of y at $x = \frac{1}{2}$ (GATE AE 2022)

Solution:

Parameters	**Values**	**Description**
$y(0)$	0	y at $x = 0$
$y'(0)$	1	$\frac{dy}{dx}$ at $x = 0$

Table 8.1: Parameters

$$\frac{d^2y}{dx^2} \overset{\mathcal{L}}{\longleftrightarrow} s^2Y(s) - sy(0) - y'(0) \tag{8.1}$$

$$\frac{dy}{dx} \overset{\mathcal{L}}{\longleftrightarrow} sY(s) - y(0) \tag{8.2}$$

Applying Laplace Transform, using (8.1) and (8.2),

$$s^2Y(s) - sy(0) - y'(0) - 2(sY(s) - y(0)) + Y(s) = 0 \tag{8.3}$$

From Table 8.1,

$$(s^2 - 2s + 1)Y(s) - 1 = 0 \tag{8.4}$$

$$Y(s) = \frac{1}{(s-1)^2} \tag{8.5}$$

$$t^n \overset{\mathcal{L}}{\longleftrightarrow} \frac{n!}{s^{n+1}} \tag{8.6}$$

$$e^{at}x(t) \overset{\mathcal{L}}{\longleftrightarrow} X(s-a) \tag{8.7}$$

Taking Inverse Laplace Transform for $Y(s)$, using (8.6) and (8.7),

$$y(x) = xe^x \tag{8.8}$$

$$\implies y\left(\frac{1}{2}\right) = \frac{\sqrt{e}}{2} \tag{8.9}$$

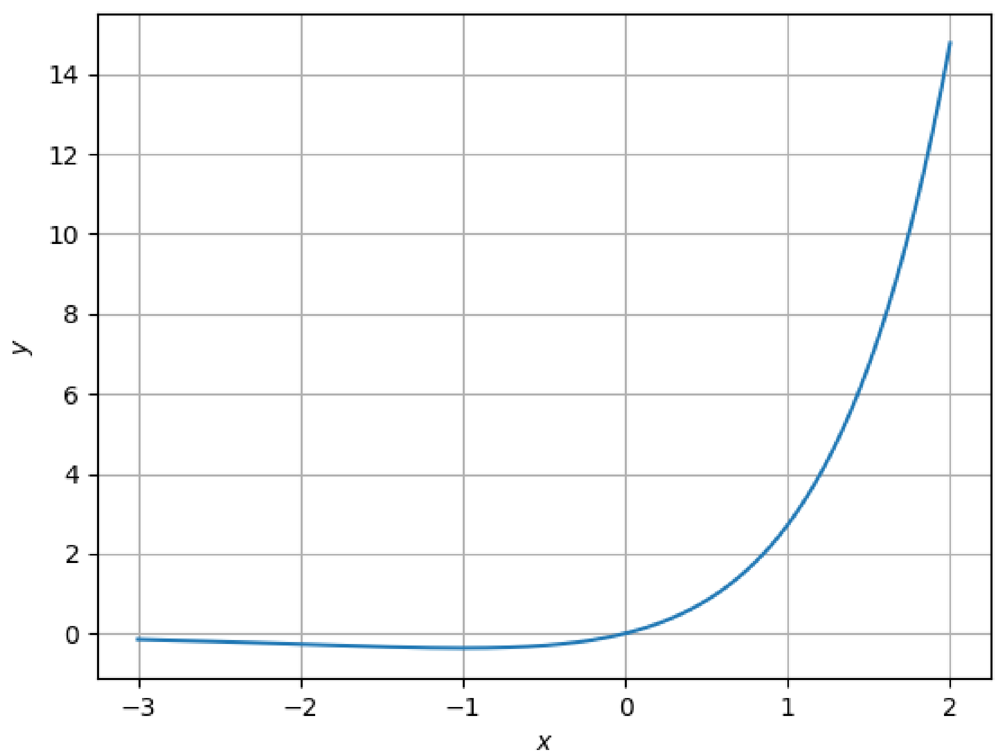


Figure 8.1: Plot of $y(x)$

8.2 A process described by the transfer function

$$G_p(s) = \frac{(10s+1)}{(5s+1)}$$

is forced by a unit step input at time $t = 0$. The output value immediately after the unit step input (at $t = 0^+$) is ? (Gate 2022 CH 34)

Solution:

Parameters	**Description**
$X(s)$	Laplace transform of $x(t)$
$Y(s)$	Laplace transform of $y(t)$
$G_p(s) = \frac{Y(s)}{X(s)}$	Transfer function
$x(t) = u(t)$	unit step function

Table 8.2: Given parameters

$$G_p(s) = \frac{Y(s)}{X(s)} = \frac{(10s+1)}{(5s+1)} \tag{8.10}$$

$$u(t) \overset{\mathcal{L}}{\longleftrightarrow} \frac{1}{s} \tag{8.11}$$

From equation (8.11):

$$Y(s) = \frac{(10s+1)}{s\,(5s+1)} \tag{8.12}$$

$$= \frac{1}{s} + \frac{5}{5s+1} \tag{8.13}$$

Taking inverse laplace transformation,

$$\frac{1}{s} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} u(t) \tag{8.14}$$

$$\frac{1}{s-c} \overset{\mathcal{L}^{-1}}{\longleftrightarrow} e^{ct}u(t) \tag{8.15}$$

$$y(t) = \left(1 + e^{\frac{-t}{5}}\right) u(t) \tag{8.16}$$

$$y(0^+) = 2 \tag{8.17}$$

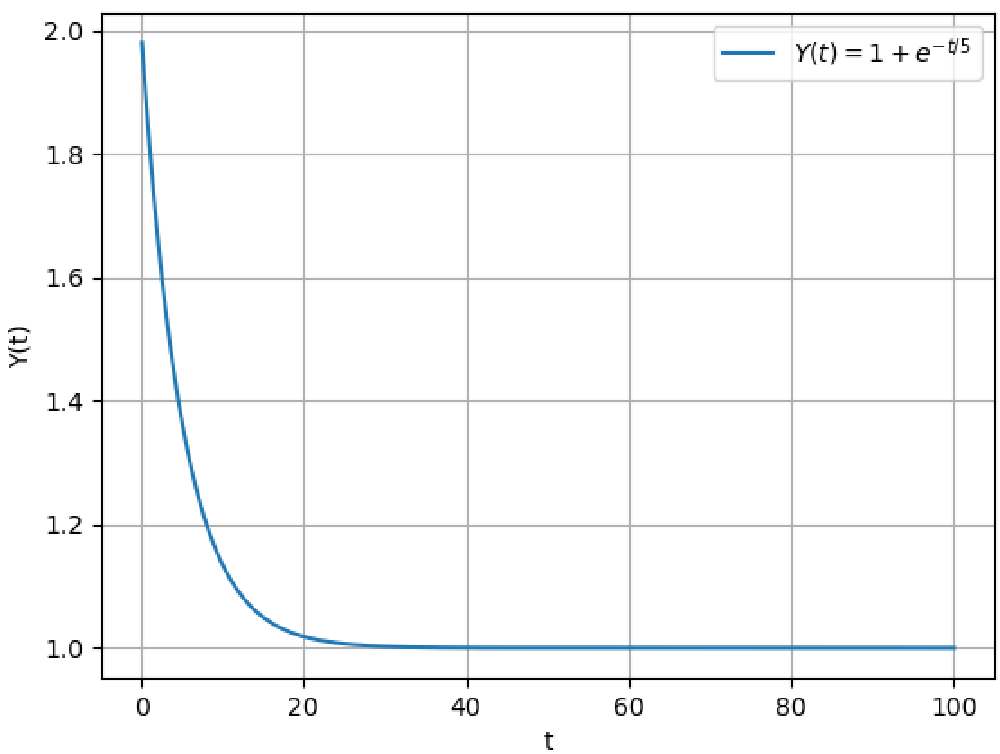


Figure 8.2: Graph of y(t)

8.3 The transfer function of a real system $H(S)$ is given as:

$$H(s) = \frac{As + B}{s^2 + Cs + D}$$

where A, B, C and D are positive constants. This system cannot operate as

(A) Low pass filter

(B) High pass filter

(C) Band pass filter

(D) An Integrator

(GATE EE 11 2022)

Solution: The transfer function $H(s)$ is given by:

$$H(s) = \frac{As + B}{s^2 + Cs + D} \tag{8.18}$$

Put $s = j\omega$ in (8.18):

$$H(j\omega) = \frac{A(j\omega) + B}{(j\omega)^2 + C(j\omega) + D} \tag{8.19}$$

$$|H(j\omega)| = \frac{\sqrt{(A\omega)^2 + B^2}}{\sqrt{(D - \omega^2)^2 + (\omega C)^2}} \tag{8.20}$$

a) Low Pass Filter:

At low frequency ($\omega = 0$):

$$|H(\omega = 0)| = \frac{B}{D} \tag{8.21}$$

$\therefore$ H(s) can operate as Low pass filter.

Parameter	Description
Low Pass Filter	The gain should be finite at low frequency
High Pass Filter	The gain should be finite at high frequency
Band Pass Filter	Finite gain over frequency band
Integrator	Transfer function should have at least one pole at origin

Table 8.3: Conditions

b) High Pass Filter:

At high frequency ($\omega = \infty$):

$$|H(\omega = \infty)| = 0 \tag{8.22}$$

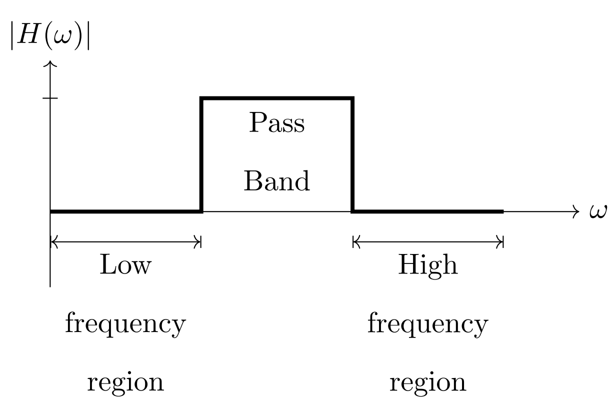
$\therefore H(s)$ cannot operate as High pass filter.

c) Band Pass Filter:

Assuming B is a very less positive valued constant as compared to others:

$$|H(j\omega)| = \frac{(A\omega)}{\sqrt{(D-\omega^2)^2 + (\omega C)^2}} \tag{8.23}$$

$$\implies |H(\omega = 0)| = 0 \text{ and } |H(\omega = \infty)| = 0 \tag{8.24}$$



$\because H(s)$ passes frequency between low and high frequencies.

$\therefore$ $H(s)$ can operate as a band pass filter.

d) Integrator:

At very high value of frequency($\omega \rightarrow \infty$):

$$H(s) \approx \frac{As}{s^2} \approx \frac{A}{s} \tag{8.25}$$

From Table 8.3:

$\therefore$ $H(s)$ can operate as an Integrator.

8.4 In a circuit, there is a series connection of an ideal resistor and an ideal capacitor. The conduction current (in Amperes) through the resistor is $2\sin\left(t+\frac{\pi}{2}\right)$. The displacement current (in Amperes) through the capacitor is ______.

(A) $2\sin(t)$

(B) $2\sin(t+\pi)$

(C) $2\sin\left(t+\frac{\pi}{2}\right)$

(D) 0

(GATE 2022 EC 24)

Solution:

Parameter	Description	Value
I_c	Conduction Current	$2\sin\left(t+\frac{\pi}{2}\right)$
A	Cross-sectional area	

Table 8.4: Parameters

Parameter	Description	Formula
Q	Charge	$\int I_c\,dt$
D	Electric Displacement	$\frac{Q}{A}$
J_D	Displacement current density	$\frac{\partial D}{\partial t}$
I_D	Displacement current	J_D x A

Table 8.5: Formulae

S Domain	Time Domain
$\frac{1}{s}$	$u(t)$
$\frac{-s}{a^2+s^2}$	$-\cos(at)$
$\frac{a}{a^2+s^2}$	$\sin(at)$
$\frac{1}{s+a}$	e^{-at}

Table 8.6: Laplace transforms

$$\mathcal{L}\left[\int f(t)\ dt\right] = \int_0^\infty \left[\int f(t)\ dt\right] e^{-st}\, dt \tag{8.26}$$

$$= \int_0^\infty u\, dv \quad \text{where} \begin{cases} u = \int f(t)\, dt \\ dv = e^{-st} dt \end{cases} \tag{8.27}$$

$$= uv - v \int du \tag{8.28}$$

$$= \frac{1}{s} \int f(t)\, dt|_0 + \frac{1}{s} \int_0^\infty f(t)\, e^{-st} dt \tag{8.29}$$

$$\implies \frac{1}{s} \int f(t)\, dt|_0 + \frac{1}{s} F(s) \tag{8.30}$$

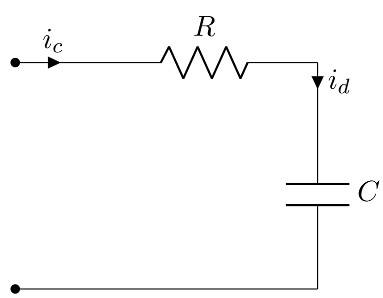


Figure 8.3: Circuit 1

From Table 8.5, Table 8.6 and eq (8.30)

$$I_c(s) = \frac{2s}{s^2+1} \tag{8.31}$$

$$Q_c(s) = \frac{2}{s(s^2+1)} \tag{8.32}$$

$$D(s) = \frac{1}{A}\left(\frac{2}{s(s^2+1)}\right) \tag{8.33}$$

$$J_D(s) = \frac{2}{A}\left(\frac{1}{s^2+1}\right) \tag{8.34}$$

$$I_D(s) = \frac{2}{s^2+1} \tag{8.35}$$

$$\implies I_D = 2\sin t \tag{8.36}$$

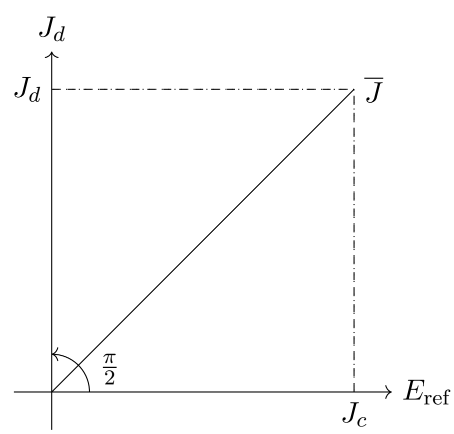


Figure 8.4: Phasor plot

From figure 8.4, phase of I_d is $\frac{\pi}{2}$

$$\therefore I_d = 2\sin\left(t+\frac{\pi}{2}\right) \tag{8.37}$$

$\therefore$ (C) is correct.

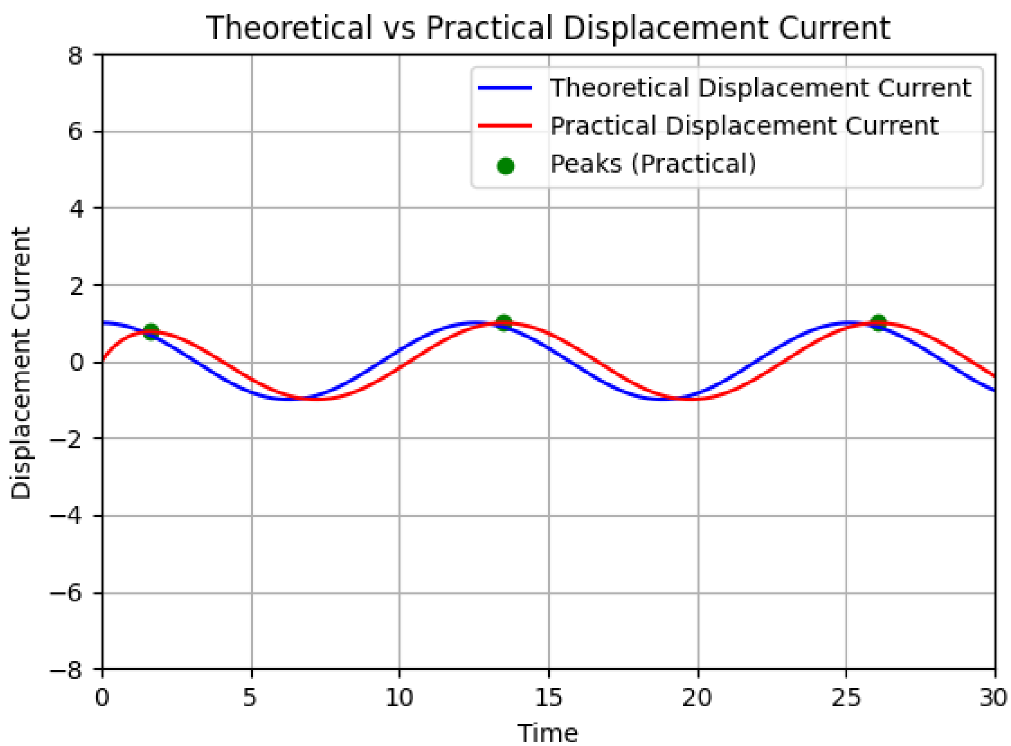


Figure 8.5: Thoritical vs Practical simulation

8.5 Given, $y = f(x)$; $\frac{d^2y}{dx2} + 4y = 0; y(0) = 0; \frac{dy}{dx}(0) = 1$. The problem is a/an

(a) initial value problem having soluition $y = x$

(b) boundary value problem having soluition $y = x$

(c) initial value problem having soluition $y = \frac{1}{2}\sin 2x$

(d) boundary value problem having soluition $y = \frac{1}{2}\sin 2x$

(GATE 2022 ES)

Solution:

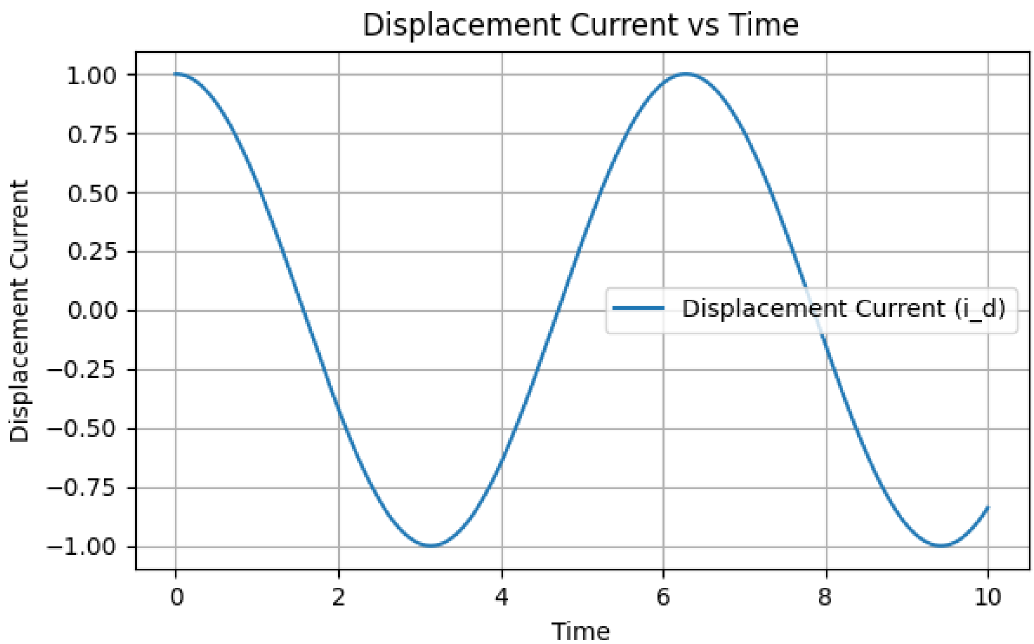


Figure 8.6: Displacement current

The above equation can be written as,

$$y''(t) + 4y(t) = 0 \tag{8.38}$$

Using the Laplace transformation pairs,

$$y''(t) \overset{\mathcal{L}}{\longleftrightarrow} s^2 Y(s) - sy(0) - y'(0) \tag{8.39}$$

$$y(t) \overset{\mathcal{L}}{\longleftrightarrow} Y(s) \tag{8.40}$$

$$\sin at \overset{\mathcal{L}}{\longleftrightarrow} \frac{a}{a^2 + s^2} \tag{8.41}$$

Applying Laplace transform for the equation we get,

$$s^2 Y(s) - 1 + 4Y(s) = 0 \tag{8.42}$$

$$\implies Y(s) = \frac{1}{4+s^2} \tag{8.43}$$

Now, applying inverse laplace transform we get,

$$y(t) = \frac{1}{2}\sin 2t \quad \text{(from (8.41))} \tag{8.44}$$

Since, the conditions at the same point(0) are mentioned, it is an initial valued problem having solution $y = \frac{1}{2}\sin 2x$.

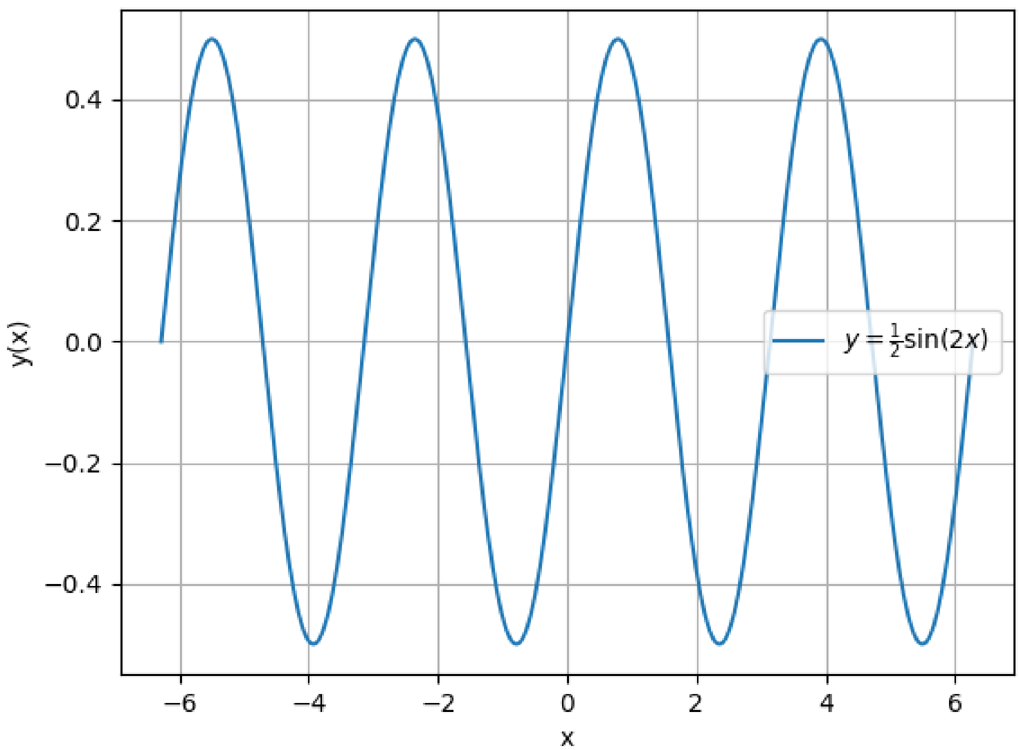


Figure 8.7: $y(x)$ vs x graph

8.6 Assuming $s > 0$; Laplace transform for $f(x) = sin(ax)$ is

(A) $\frac{a}{s^2+a^2}$

(B) $\frac{s}{s^2+a^2}$

(C) $\frac{a}{s^2-a^2}$

(D) $\frac{s}{s^2-a^2}$

(GATE ES 2022)

Solution:

$$\mathcal{L}\left(f\left(x\right)\right) = \int_{-\infty}^{\infty} e^{-sx} f\left(x\right) dx \tag{8.45}$$

$$\text{We can write} \quad \sin\left(ax\right) = \frac{e^{ax} - e^{-ax}}{2i} \tag{8.46}$$

From (8.46)

$$\mathcal{L}\left(\sin\left(ax\right)\right) = \int_{0}^{\infty} e^{-sx}\left(\frac{e^{iax} - e^{-iax}}{2i}\right) dx \tag{8.47}$$

$$= \frac{1}{2i}\int_{0}^{\infty} e^{-x(s-ia)} - e^{-x(s+ia)} dx \tag{8.48}$$

$$= \frac{1}{2i}\left(\frac{e^{-x(s-ia)}}{-\left(s-ia\right)} + \frac{e^{-x(s+ia)}}{-\left(s+ia\right)}\right)_{0}^{\infty} \tag{8.49}$$

$$= \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) \tag{8.50}$$

$$= \frac{a}{s^2 + a^2} \tag{8.51}$$

So, option (A) is correct.

Chapter 9

Fourier transform

9.1 The Fourier transform X($j\omega$) of the signal

$x(t) = \frac{t}{(1+t^2)^2}$ is ________. GATE-2022-EC-15

(A) $\frac{\pi}{2j}\omega e^{-|\omega|}$

(B) $\frac{\pi}{2}\omega e^{-|\omega|}$

(C) $\frac{\pi}{2j}e^{-|\omega|}$

(D) $\frac{\pi}{2}e^{-|\omega|}$

Solution:

Symbol	Value	Description
$x(t)$	$\dfrac{t}{(1+t^2)^2}$	Signal
$X(\omega)$	$\displaystyle\int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t} dt$	Fourier transform of $x(t)$

Table 9.1: Variable description

The Fourier transform of the form x(t)=$e^{-a|t|}$ is

$$x(t) \xleftrightarrow{\text{F.T}} X(\omega) \tag{9.1}$$

$$X(\omega) = \frac{2a}{a^2 + \omega^2} \tag{9.2}$$

Consider,

$$x(t) = e^{-|t|} \tag{9.3}$$

$$X(\omega) = \frac{2}{1 + \omega^2} \tag{9.4}$$

By using differentiation property from (A.1.5),

$$tx(t) \xleftrightarrow{\text{F.T}} j\frac{d}{d\omega}X(\omega) \tag{9.5}$$

$$tx(t) \xleftrightarrow{\text{F.T}} j\left[\frac{d}{d\omega}\left(\frac{2}{1+\omega^2}\right)\right] \tag{9.6}$$

$$te^{-|t|} \xleftrightarrow{\text{F.T}} \frac{-4j\omega}{(1+\omega^2)^2} \tag{9.7}$$

Applying duality property from (A.2.3),

$$\frac{-4jt}{(1+t^2)^2} \xleftrightarrow{\text{F.T}} 2\pi(-\omega)e^{-|-\omega|} \tag{9.8}$$

$$\frac{t}{(1+t^2)^2} \xleftrightarrow{\text{F.T}} \frac{-2\pi\omega e^{-|\omega|}}{-4j} \tag{9.9}$$

$$\frac{t}{(1+t^2)^2} \xleftrightarrow{\text{F.T}} \frac{\pi}{2j}\omega e^{-|\omega|} \tag{9.10}$$

9.2 For a vector $\bar{x} = [x[0], x[1], \ldots, x[7]]$, the 8-point discrete Fourier transform (DFT) is denoted by $\bar{X} = \text{DFT}(\bar{x}) = [X[0], X[1], \ldots, X[7]]$, where

$$X[k] = \sum_{n=0}^{7} x[n] \exp\left(-j\frac{2\pi}{8}nk\right).$$

Here $j = \sqrt{-1}$. If $\bar{x} = [1, 0, 0, 0, 2, 0, 0, 0]$ and $\bar{y} = \text{DFT}(\text{DFT}(\bar{x}))$, then the value of $y[0]$ is. GATE-2022-EC-55

Solution:

Parameter	**Description**	**Value**
$\bar{X}$	DFT($\bar{x}$)	−
$\bar{x}$	vector	$[1, 0, 0, 0, 2, 0, 0, 0]$
$\bar{y}$	DFT(DFT($\bar{x}$))	−

Table 9.2: Given Parameters

DFT of $\bar{x}$

$$X[k] = \sum_{n=0}^{7} x[n] \exp\left(-j\frac{2\pi}{8}nk\right) \tag{9.11}$$

As the only non-zero values in x are x[0] and x[4]:

$$X[k] = x[0] + x[4] \exp\left(-j\pi k\right) \tag{9.12}$$

After substituting the values of k ranging from 0 to 7,

$$\bar{X} = \text{DFT}(\bar{x}) = [X[0], X[1], \ldots, X[7]] \tag{9.13}$$

$$\bar{X} = [3, -1, 3, -1, 3, -1, 3, -1] \tag{9.14}$$

$$\bar{y} = \mathrm{DFT}(\mathrm{DFT}(\bar{x})) \tag{9.15}$$

$$\bar{y} = [3, -1, 3, -1, 3, -1, 3, -1] \tag{9.16}$$

$$y[0] = \sum_{n=0}^{7} x[n] \tag{9.17}$$

$$= x[0] + x[1] + \cdots + x[7] \tag{9.18}$$

$$= 3 - 1 + 3 - 1 + 3 - 1 + 3 - 1 = 8 \tag{9.19}$$

Appendix A

Fourier transform

A.1 The Differentiation in frequency domain is as follows

Let x(t) be a signal such that,

$$x(t) \overset{\text{F.T}}{\longleftrightarrow} X(\omega) \tag{A.1.1}$$

$$X(\omega) = \int_{t=-\infty}^{\infty} x(t)\, e^{-j\omega t} dt \tag{A.1.2}$$

$$\frac{d}{d\omega} X(\omega) = \int_{t=-\infty}^{\infty} x(t)\,(-jt)\, e^{-j\omega t} dt \tag{A.1.3}$$

$$j\frac{d}{d\omega} X(\omega) = \int_{t=-\infty}^{\infty} tx(t)\, e^{-j\omega t} dt \tag{A.1.4}$$

$$tx(t) \overset{\text{F.T}}{\longleftrightarrow} j\frac{d}{d\omega} X(\omega) \tag{A.1.5}$$

A.2 The duality property is as follows

From inverse Fourier transform we get,

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X(\omega)\, e^{j\omega t} d\omega \tag{A.2.1}$$

Replacing t by -t and multiplying 2π on both sides we get,

$$2\pi x\left(-t\right) = \int_{-\infty}^{\infty} X\left(\omega\right) e^{-j\omega t} d\omega \tag{A.2.2}$$

$$X\left(t\right) \xleftrightarrow{\text{F.T}} 2\pi x\left(-\omega\right) \tag{A.2.3}$$