Honors Discrete Mathematics: Exam 1 Extra Credit

Due on Sunday, February 27, 2022. at 11:59pm

Professor Gerandy Brito Spring 2022.

IMPORTANT: For this assignment, you may NOT collaborate with other students, nor use resources outside the ones from class. Think of it as a take-home exam that is open book, open notes.

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Exercise 1

[0.5 points] Let n be a positive integer and $[n] = \{1, \dots, n\}$. Show that

$$\sum_{S \in \mathcal{P}([n]) \,|\, S \neq \emptyset} \prod_{i \in S} \frac{1}{i} = n.$$

Here, $\mathcal{P}([n])$ is the power set of $[n]$.

Exercise 2

In this question, we deal with *nested radicals*, an expression where radical expressions are nested within each other. The first part serves as an introduction to this concept.

(a) **[0.5 points]** Prove

$$\sqrt{1 + \sqrt{1 + \sqrt{1 + \cdots}}} = \sqrt{2 + \sqrt{2 - \sqrt{2 + \sqrt{2 - \cdots}}}}$$

(b) **[0.5 points]** Determine the validity of the following proof, and explain your answer.

CLAIM: For all $n \in \mathbb{N}$, the following sentence $P(n)$, is true.

$$\sqrt{1 + n\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{1 + (n+3)\sqrt{\dots}}}}} = n + 1.$$

BASE CASE: $P(0)$ is true, since $\sqrt{1 + \sqrt{0}\sqrt{\dots}} = 0 + 1$.
INDUCTIVE STEP: Now let $n \in \mathbb{N}$ such that $P(n)$ is true. Squaring both sides, we obtain

$$1 + n\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{1 + (n+3)\sqrt{\dots}}}} = n^2 + 2n + 1.$$

Subtracting 1 and dividing by n, we obtain

$$\sqrt{1 + (n+1)\sqrt{1 + (n+2)\sqrt{1 + (n+3)\sqrt{\dots}}}} = n + 2,$$

Hence $P(n+1)$ is true as well.
CONCLUSION: Therefore by strong induction, for all $n \geq 0$, $P(n)$ is true.

(c) **[1.5 points]** Let $\gamma \in \mathbb{N}$. Prove that for all $n \in \mathbb{N}^+$,

$$\underbrace{\sqrt{\gamma + \sqrt{\gamma + \sqrt{\gamma + \cdots + \sqrt{\gamma}}}}}_{n \text{ square roots}} < \frac{1 + \sqrt{4\gamma + 1}}{2}.$$