CS 2051: Honors Discrete Mathematics
Spring 2023 Homework 8 Supplement

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Question 1

(Warm-up) The concept of induction is often compared to the domino analogy. It starts with the assumption that the first domino falls, and from there, we can infer that if the nth domino falls, then the $(n+1)$th domino will also fall. This leads to the eventual falling of all the dominoes. But what if the dominoes fell in the opposite direction?

Suppose we have proven the following facts with respect to some predicate $P(n)$:

$$P(1) \tag{1}$$
$$\forall n \in \mathbb{N}^+, P(n) \implies P(n-1) \tag{2}$$
$$\forall n \in \mathbb{Z}, P(n) \implies P(kn) \tag{3}$$

Let k be an integer such that $|k| > 1$. Prove that $(\forall n \in \mathbb{N})(P(n))$.

I proceed by weak induction to prove that $P(n)$ is true for all $n \in \mathbb{N}$.

Base Case: $P(1)$ is true because it is given. $P(0)$ is true because using Fact (2), $P(1) \to P(0)$.

Inductive Hypothesis: Now let $m \geq 1, m \in \mathbb{N}$ such that $P(m)$ is true.

Inductive Step: We now show that $P(m+1)$ is true.
Since $P(m)$ is true by the Inductive Hypothesis, using Fact (3), $P(km)$ is also true. Using Fact (3) again, $P(k^2m)$ is true. We can rewrite this as $P(m+(k^2-1)m)$.

Since $m \geq 1$ and $(k^2-1) > 0$, we know that $m+(k^2-1)m \in \mathbb{N}^+$. Therefore, we can apply Fact (2), so $P(m+(k^2-1)m-1)$ is true. Using Fact (2) again, we know that $P(m+(k^2-1)m-2)$ is also true, and so on. We continuously apply Fact (2) until we reach the fact that $P(m+(k^2-1)m-((k^2-1)m-1))$ is true, which simplifies to $P(m+1)$. We have shown that $P(m+1)$ is true if $P(m)$ is true. This completes the inductive step.

Conclusion: Therefore by induction, for all $n \in \mathbb{N}$, $P(n)$ is true.

Question 2

(Brito) Let $S = \{s_1, s_2, \ldots, s_{2n-1}\}$ natural numbers, where $n = 2^k > 1$. Show that one can choose a subset $S' \subset S$ of cardinality $|S'| = n$ such that the sum of the elements in S' is a multiple of n.

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Let $P(k)$ be the statement

Any set S of $2n-1$ natural numbers has a subset of n elements whose sum is a multiple of n

where $n = 2^k > 1$. I proceed by weak induction to prove that $P(k)$ is true for all $k \in \mathbb{N}^+$.

Base Case: We first prove $P(1)$ is true.
In this case, $|S| = 2^{1+1} - 1 = 3$. For any set with 3 natural numbers, we can pick two with the same parity. Their sum will be a multiple of 2. Therefore, $P(1)$ is true. This completes the base case.

Inductive Hypothesis: Now let $j \in \mathbb{N}^+$ such that $P(j)$ is true.

Inductive Step: We now show that $P(j+1)$ is true.

First, we split the set with $2^{j+2} - 1$ elements into two sets with $2^{j+1} - 1$ elements and one set with 1 element. By the Inductive Hypothesis, for each of the sets with $2^{j+1} - 1$ elements, we can choose a subset of 2^j elements whose sum is a multiple of 2^j. The number of elements that are not in either of the two chosen subsets is $2^{j+2} - 1 - 2^j - 2^j = 2^{j+1} - 1$. Again, by the Inductive Hypothesis, from these remaining elements, we can choose a subset of 2^j elements whose sum is a multiple of 2^j.

A multiple of 2^j must be congruent to either 0 or 2^j modulo 2^{j+1}. We have three subsets of 2^j elements whose sum is a multiple of 2^j. This means there must be at least two subsets whose sums are congruent modulo 2^{j+1}. As stated before, these two subsets have congruent sums that are also either congruent to 0 or 2^j modulo 2^{j+1}. Therefore, the union of these two subsets is a subset with 2^{j+1} elements whose sum is a multiple of 2^{j+1}, and we choose this as S'. We have shown that $P(j+1)$ is true if $P(j)$ is true. This completes the inductive step.

Conclusion: Therefore by induction, for all $k \in \mathbb{N}^+$, $P(k)$ is true.