Honors Discrete Mathematics: Lecture 2 Notes

Professor Gerandy Brito Spring 2022

Sarthak Mohanty

Predicates & Quantifiers

Definition: A propositional function, or *predicate*, can be thought of as a function taking in one or more variables, and outputting either True or False.

- Predicates are not propositions on their own, but as soon as we assign a value to our variables, they become a proposition.
- Predicates are usually denoted using the symbols $P(x), Q(x), \dots$, but this notation is not too rigid.

As predicates are dependent upon variables, we must introduce domains upon these variables. We do so using quantifiers.

Definition: A *quantifier* is an expression that indicates the scope or domain to which it is attached.

- The most common quantifiers are the terms 'for all' and 'there exists', which are expressed using the symbols $\forall$ and $\exists$ respectively.

We can now express propositions we introduced yesterday using our new logical notation. The statement "For all real numbers x, $x+1 > x$" is logically equivalent to $(\forall x \in \mathbb{R})(P(x))$. The statement "There exists a pair of irrational numbers x, y such that x^y is rational" is logically equivalent to $(\exists x, y \in \mathbb{R} \setminus \mathbb{Q})(P(x,y))$.

Propositions of this form can also be negated. We have

$$\neg(\forall x)(P(x)) \equiv (\neg\forall x)(\neg P(x)) \equiv (\exists x)(\neg P(x))$$
$$\neg(\exists x)(P(x)) \equiv (\neg\exists x)(\neg P(x)) \equiv (\forall x)(\neg P(x)).$$

Implications

In this section, we will return to the conditional statement introduced last lecture, also known as an implication. Let us first introduce new types propositional statements branching off from implications.

- The *converse* of an implication $p \Rightarrow q$ is the statement $q \Rightarrow p$.
- The *inverse* of an implication $p \Rightarrow q$ is the statement $\neg p \Rightarrow \neg q$
- the *contrapositive* of an implication $p \Rightarrow q$ is the statement $\neg q \Rightarrow \neg p$.

The truth table of these statements is shown below.

p	q	$p \Rightarrow q$	$q \Rightarrow p$	$\neg p \Rightarrow \neg q$	$\neg q \Rightarrow \neg p$
T	T	T	T	T	T
T	F	F	T	T	F
F	T	T	F	F	T
F	F	T	T	T	T

Note that the contrapositive of an implication is logically equivalent to the implication itself. For this reason, we introduce a new type of proof, known as a *Proof by Contraposition.*

Proof by Contraposition

Example
Show that if n^2 is odd, then n is odd.

Solution
We will prove the contrapositive of this statement, in other words, "if n is even, then n^2 is even."

Suppose n is even. Then there exists an integer k such that $n = 2k$. Squaring both sides, we obtain $n^2 = 4k^2 = 2(2k^2)$, and by definition n^2 is even. Hence the contrapositive statement is true, so the original statement is true as well.

Negating Compound Propositions

Now we will show how to negate disjunctions, conjunctions, and implications.

Negating Disjunctions

Suppose your roommate made the prediction “I will get an A in CS 2051 and an A in CS 1332.” If your roommate’s prediction is incorrect, then they either did not get an A in CS 2051, or they did not get an A in CS 1332. This leads us to the first De Morgan’s law:

$$\neg(p \vee q) \equiv \neg p \wedge \neg q$$

Negating Conjunctions

Suppose your roommate made the prediction “I will get an A in CS 2051 or an A in CS 1332.” If your roommate’s prediction is incorrect, then they did not get an A in CS 2051 and did not get an A in CS 1332. This leads us to the second De Morgan’s law:

$$\neg(p \wedge q) \equiv \neg p \vee \neg q.$$

Note that De Morgan’s laws can be extended to expressions with more than one conjunction and disjunction as follows:

$$\neg(p \wedge q \wedge r) \equiv \neg p \vee \neg q \vee \neg r$$
$$\neg(p \vee q \vee r) \equiv \neg p \wedge \neg q \wedge \neg r$$

Negating Implications

One important theorem that will show up in your later CS courses is that any logical statement can be expressed using the basic operators $\vee$, $\wedge$, or $\neg$. Even implications can be expressed as such, leading us to the implication law:

$$p \Rightarrow q = \neg p \vee q.$$

Using De Morgan’s law, we can now negate this statement, leading to the implication negation law:

$$\neg(p \Rightarrow q) = p \wedge q.$$

Example
Consider the predicate “If a natural number x is greater than one, then x is even.” When would this statement be false?

Solution
Let $P(x)$ represent the statement “x is greater than one” and let $Q(x)$ represent the statement “x is even”. Then the original predicate is equivalent to $(x \in \mathbb{N})(P(x) \Rightarrow Q(x))$. Using the implication negation law, the negation of this predicate is $(x \in \mathbb{N})(P(x) \wedge \neg Q(x))$. Hence this statement is false when x is greater than one and x is odd.

Arguments

Arguments are a series of statements, known as premises, used to prove a conclusion.

Logically, this can be visualized as $(P_1 \wedge P_2 \wedge \cdots \wedge P_k) \Rightarrow t$, where $P_1, P_2, \ldots, P_k$ are propositions with a truth value of True, and t is a proposition with an unknown truth value. See the textbook for basic examples of arguments.