

# **Honors Discrete Mathematics:**

## **Lecture 3 Notes**

*Professor Gerandy Brito Spring 2022*

**Sarthak Mohanty**

## Implications and Double Implications

Before continuing with where we left off last lecture, let us introduce one more implicative statement, the *biconditional*, or double implication. The biconditional of a proposition  $p \Rightarrow q$  is the statement  $p \Leftrightarrow q$ , or “ $p$  if and only if  $q$  ( $p$  iff  $q$ )”. These are all the major propositional statements we will discuss in this course.

**TA Remark:** The following statements are logically equivalent to the conditional statement “If  $p$ , then  $q$ ”. Make sure you understand these, as they show up many times in higher Mathematics and CS textbooks.

- $q$  if  $p$ .
- $q$  is necessary for  $p$ .
- $p$  is sufficient for  $q$ .
- $p$  only if  $q$ <sup>1</sup>

Hopefully the reason why  $p \Leftrightarrow q$  is commonly referred to as “ $p$  if and only if  $q$ ” is more clear now.

## Arguments

Another word for premises (in the context discussed in last lecture) is *rules of inference*. As said before, most of the information on arguments can be found in the textbook. In fact, this next example is taken from pg. 73 of the textbook. For the sake of completeness, we will restate it here, using the rules of inference on pg. 72. You will have access to these rules of inference on your quizzes/exams.

### Example

Show that the premises “It is not sunny this afternoon and it is colder than yesterday,” “We will go swimming only if it is sunny,” “If we do not go swimming, then we will take a canoe trip,” and “If we take a canoe trip, then we will be home by sunset” lead to the conclusion “We will be home by sunset.”

### Solution

Let  $p$  be the proposition “It is sunny this afternoon,”  $q$  the proposition “It is colder than yesterday,”  $r$  the proposition “We will go swimming,”  $s$  the proposition “We will take a canoe trip,” and  $t$  the proposition “We will be home by sunset.” Then the premises become  $\neg p \wedge q$ ,  $r \Rightarrow p$ ,  $\neg r \Rightarrow s$ , and  $s \Rightarrow t$ . The conclusion is simply  $t$ . We need to give a valid argument with premises  $\neg p \wedge q$ ,  $r \Rightarrow p$ ,  $\neg r \Rightarrow s$ , and  $s \Rightarrow t$  and conclusion  $t$ . We construct an argument to show that our premises lead to the desired conclusion as follows.

Step	Reason
1. $\neg p \wedge q$	Premise
2. $r \Rightarrow p$	Premise
3. $\neg r \Rightarrow s$	Premise
4. $s \Rightarrow t$	Premise
5. $\neg p$	Simplification using (1)
6. $\neg r$	Modus tollens using (2) and (5)
7. $s$	Modus ponens using (3) and (6)
8. $t$	Modus ponens using (4) and (7)

There are also more rules of inference using quantified statements. While there are only four of these, they are important to know, and you may be tested on these. For the sake of time, we did not cover these in class, so please refer to the textbook to cover these.

## More Proof Methods

We have already touched upon “direct proof” and “proof by contraposition”. There are a few more techniques we use to prove propositional statements. In this section, we discuss “proof by contradiction”, “proof by

<sup>1</sup>This one is especially difficult for students. I personally like [this](#) explanation.

cases”, and “proof by construction”.

## Proof by Contradiction

Proof by contradiction is best described with the statement  $p \Rightarrow (q \wedge \neg q)$ . In other words, if a statement is true, a logical fallacy occurs, meaning the statement must be false. Although not shown in lecture, a sample proof by construction is given below.

### Example

If  $S$  is the set of all real numbers, is  $S$  bounded above?

### Solution

$S$  is bounded above iff  $(\exists b \in \mathbb{R})(\forall x \in S)(x \leq b)$ . Suppose  $S$  is bounded above. Then there must exist some  $b_0 \in \mathbb{R}$  such that  $(\forall x \in S)(x \leq b_0)$ . However,  $b_0 + 1$  is a real number  $\geq b_0$ , so  $(\forall x \in S)(x \leq b_0)$  is false. This is a contradiction, therefore  $S$  is not bounded above.

## Proof by Cases

Proof by cases, also known as “proof by exhaustion”, is a subset of direct proofs where the domain of our variables is split up into disjoint cases, and the proposition we wish to prove is proven in each of the cases: in other words, we show  $p = (p_1 \vee p_2 \vee \dots \vee p_k) \Rightarrow q$ .

### Example

Show that for all real numbers  $x, y$ ,

$$|xy| = |x| \cdot |y|.$$

### Solution

We split up the domains of  $x$  and  $y$  as follows.

Case 1. Suppose  $x > 0$  and  $y > 0$ . Then  $|x| = x$  and  $|y| = y$ , so  $xy > 0$ , so  $|xy| = xy = |x| \cdot |y|$ .

Case 2. Suppose  $x < 0$  and  $y > 0$ . Then  $|x| = -x$  and  $|y| = y$ , so  $xy < 0$ , so  $|xy| = -xy = |x| \cdot |y|$ .

The remaining cases follow a similar format, and are left as an exercise for the reader.

## Proof by Construction

This method of proof is fairly tricky to pin down to one definition. For now, let’s use the Wikipedia definition: a constructive proof is one which “demonstrates the existence of a mathematical object by creating or providing a method for creating the object.”

Most of the proofs we have encountered so far as constructive as well. However, let’s introduce a *non-constructive proof*, which proves a statement without giving an exact example or method.

### Example

Show that there exists two irrational numbers  $x$  and  $y$  such that  $x^y$  is rational.

### Solution

For now, we assume  $\sqrt{2}$  is irrational (we will prove this statement in the next lecture). Now  $\sqrt{2}^2 = 2$ , by definition of the square root. If  $(\sqrt{2})^{\sqrt{2}}$  is rational, then we are done. On the other hand, if  $(\sqrt{2})^{\sqrt{2}}$  is irrational, then

$$\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2} \cdot \sqrt{2}} = \sqrt{2}^2 = 2,$$

which is rational. In either case, we have found two irrational numbers  $x, y$  such that  $x^y$  is rational. Note that this non-constructive proof also utilized the “proof by cases” method.

## Post-Lecture Practice

### Question 1

Let  $x$  be an integer. Prove that  $x(x+1)$  is even.

### Solution

Since  $x$  is an integer,  $x$  is even or  $x$  is odd.

Case 1. Suppose  $x$  is even. Since  $x$  is even, we can pick an integer  $k$  such that  $x = 2k$ . Then  $x(x+1) = 2k(2k+1) = 2(2k^2+k)$  and  $2k^2+k$  is an integer. Hence  $x(x+1)$  is even.

Case 2. Suppose  $x$  is odd. Since  $x$  is odd, we can pick an integer  $k$  such that  $x = 2k+1$ . Then  $x(x+1) = (2k+1)(2k+2) = 4k^2+6k+2 = 2(2k^2+3k+1)$  and  $2k^2+3k+1$  is an integer. Hence  $x(x+1)$  is even.

In either case,  $x(x+1)$  is even.

### Question 2

Suppose the universe of discourse is the set of people  $\{\text{Adam, Bree, Chase}\}$ . To save time, sometimes we shall write  $a$  for Adam,  $b$  for Bree, and  $c$  for Chase. Let  $P(x, y)$  be the sentence “ $x$  likes  $y$ ”. Suppose the truth value of  $P(x, y)$  depends on  $x$  and  $y$  as shown in the following table.

$P(x, y)$	$y = a$	$y = b$	$y = c$
$x = a$	F	T	F
$x = b$	T	T	F
$x = c$	F	T	T

- Is the proposition  $(\forall y)P(a, y)$  true or false? What does this proposition mean in ordinary English? Answer the same two questions about each of the two propositions  $(\forall y)P(b, y)$  and  $(\forall y)P(c, y)$ . Make a table showing how the truth value of the proposition  $(\forall y)P(x, y)$  depends on  $x$ . What does the sentence  $(\forall y)P(x, y)$  mean?
- Is the proposition  $(\exists x)(\forall y)P(x, y)$  true or false? What does this proposition mean in ordinary English?
- Make a table showing how the truth value of the proposition  $(\exists x)P(x, y)$  depends on  $y$ . What does this proposition mean in ordinary English?
- Is the proposition  $(\forall y)(\exists x)P(x, y)$  true or false? What does this proposition mean in ordinary English?
- Consider the two conditional propositions

$$(\exists x)(\forall y)P(x, y) \Rightarrow (\forall y)(\exists x)P(x, y) \text{ and } (\forall y)(\exists x)P(x, y) \Rightarrow (\exists x)(\forall y)P(x, y)$$

Which of these two propositions is true and which is false in this example?

### Solution

- The proposition  $(\forall y)P(a, y)$  means “Adam likes everybody”. This statement is false, as Adam does not like Chase nor himself. The proposition  $(\forall y)P(b, y)$  means “Bree likes everybody”. This statement is false, as Bree does not like Chase. The proposition  $(\forall y)P(c, y)$  means “Chase likes everybody”. This statement is false, as Chase doesn’t like Adam. The proposition  $(\forall y)P(x, y)$  means “ $x$  likes everyone”, and its truth value depends on  $x$  as shown in the following table.

$P(x, y)$	$\forall y$
$x = a$	F
$x = b$	F
$x = c$	F

- (b) The proposition  $(\exists x)(\forall y)P(x, y)$  means “For some  $x$ ,  $x$  likes everybody”, or in other words “Somebody likes everyone”. This statement is false.
- (c) The proposition  $(\exists x)P(x, y)$  means “Somebody likes  $y$ ”. Its truth value depends on  $y$  as shown in the following table.

$P(x, y)$	$y = a$	$y = b$	$y = c$
$\exists x$	T	T	T

- (d) The proposition  $(\forall y)(\exists x)P(x, y)$  means “For all  $y$ , somebody likes  $y$ ”, or in other words “Everybody is liked by someone”. This statement is true.
- (e) In this example, the proposition  $(\exists x)(\forall y)P(x, y) \Rightarrow (\forall y)(\exists x)P(x, y)$  is true, as the antecedent is false. On the other hand, the proposition  $(\forall y)(\exists x)P(x, y) \Rightarrow (\exists x)(\forall y)P(x, y)$  is false.

### Question 3

Let  $u$ ,  $v$ , and  $w$  be rational numbers (numbers that can be expressed in the form  $\frac{a}{b}$ , where  $a \in \mathbb{Z}$  and  $b \in \mathbb{N}$ ). Prove the following statements.

- (a)  $-v$  is a rational number.
- (b)  $u - v$  is a rational number.
- (c)  $uv$  is a rational number.
- (d) If  $w \neq 0$ , then  $1/w$  is a rational number.
- (e) If  $w \neq 0$ , then  $u/w$  is a rational number.

### Solution

- (a) Since  $v$  is rational, we can pick integers  $a$  and  $b$  such that  $b \neq 0$  and  $v = a/b$ . Then  $-v = -a/b$ . Hence  $-v$  is a rational number.
- (b) Since  $u$  is rational, we can pick integers  $a$  and  $b$  such that  $b \neq 0$  and  $u = a/b$ . Since  $v$  is rational, we can pick integers  $c$  and  $d$  such that  $d \neq 0$  and  $v = c/d$ . Then

$$u - v = \frac{a}{b} - \frac{c}{d} = \frac{ad}{bd} - \frac{bc}{bd} = \frac{ad - bc}{bd}$$

Note  $ad - bc$  and  $bd$  are integers. Also, since  $b \neq 0$  and  $d \neq 0$ ,  $bd \neq 0$ . Hence  $u - v$  is a rational number.

- (c) Since  $u$  is rational, we can pick integers  $a$  and  $b$  such that  $b \neq 0$  and  $u = a/b$ . Since  $v$  is rational, we can pick integers  $c$  and  $d$  such that  $d \neq 0$  and  $v = c/d$ . Then  $uv = ac/bd$ . Note that  $ac$  and  $bd$  are integers. Also,  $bd \neq 0$  because  $b \neq 0$  and  $d \neq 0$ . Hence  $uv$  is a rational number.
- (d) Suppose  $w \neq 0$ . Since  $w$  is rational, we can pick integers  $a$  and  $b$  such that  $b \neq 0$  and  $w = a/b$ . Then  $1/w = 1/(a/b) = b/a$ . Note that  $a \neq 0$  because  $w \neq 0$ . Hence if  $w \neq 0$ , then  $1/w$  is a rational number.
- (e) Suppose  $w \neq 0$ . We can express  $u/w$  as  $(u)(1/w)$ . From part (d), since  $w \neq 0$ ,  $1/w$  is a rational number. Now we know both  $u$  and  $1/w$  are rational numbers. Then from part (c),  $u(1/w) = u/w$  is a rational number. Hence if  $w \neq 0$ , then  $u/w$  is a rational number.

**Question 4**

Show that for each real number  $x$ ,  $\pi + x$  is irrational or  $\pi - x$  is irrational.

**Solution**

Because  $\pi$  is irrational,  $\pi \in \mathbb{R}$ . So  $\pi + x \in \mathbb{R}$  and  $\pi - x \in \mathbb{R}$ . We wish to prove that  $\pi + x \notin \mathbb{Q}$  or  $\pi - x \notin \mathbb{Q}$ , and shall do so using contradiction. Suppose it is not the case that  $\pi + x \notin \mathbb{Q}$  or  $\pi - x \notin \mathbb{Q}$ . Then by De Morgan's law,  $\pi + x \in \mathbb{Q}$  and  $\pi - x \in \mathbb{Q}$ . Then by a property of  $\mathbb{Q}$ ,  $(\pi + x) + (\pi - x) = \pi \in \mathbb{Q}$ . But as stated before,  $\pi$  is irrational. This is a contradiction, therefore  $\pi + x$  is irrational or  $\pi - x$  is irrational for each real number  $x$ .