CS 2051: Honors Discrete Mathematics
Spring 2023 Homework 4 Supplement

Nithya Jayakumar

1. The purpose of this problem is to show how the power set $\mathcal{P}(S)$ of a given set S, has always a different cardinality. We have a formula for the case that S is finite, but it is less obvious for the infinite case. To fix ideas, we focus on the case $S = \mathbb{N}$.

   Show that there does not exist an onto function between $\mathbb{N}$ and its power set, $\mathcal{P}(\mathbb{N})$.

   Hint: proceed by contradiction assuming there is such function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$. Let

   $$T = \{n \in \mathbb{N} |\ n \notin f(n)\}.$$

   Since f is onto, there exist $t \in \mathbb{N}$ such that $f(t) = T$. Argue a contradiction by looking at $t \in T$ and $t \notin T$.

2. The `Brito-Caribbean-Royal Grand Hotel` has a countable infinite number of rooms, each occupied by a guest, due to its popular demand.

   (a) How can we accommodate a new guest arriving at the fully occupied hotel without removing any of the current guests?

   (b) Show that a finite group of guests arriving at the Grand Hotel can be given rooms without evicting any current guests.

   (c) Brito, the owner of the hotel, decided to close all the even numbered rooms for maintenance. Show that all the guests can remain in the hotel.

   (d) A countable infinite number of buses, each containing a countable infinite number of guests, arrive at the hotel, show that the arriving guests can be accommodated without evicting any of the current guests.

   To earn full credit, carefully describe in each part how the hotel goes about accommodating the new guests.