Honors Discrete Mathematics: Lecture 5 Notes

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Set Theory

Definition: A *set* is an unordered collection of objects. They are the "building blocks" of modern mathematics. Many of the symbols we have used so far are sets: some examples are $\mathbb{Q}$ and $\mathbb{R}$.

Notation

- We denote sets using braces. Sets can be finite or infinite: For example, we define the set of natural numbers as $\{1, 2, 3, \dots\}$ and the set of natural numbers less than 100 as $\{1, 2, 3, \dots, 99\}$.
- We denote an element n residing in a set S by $n \in S$. If n is not in the set S, we say $n \notin S$.
- Sets can also be defined by predicates. The set of all elements x in domain D satisfying $P(x)$ is denoted with $\{x \in D : P(x)\}$ and is known as *set-builder notation*.
- A is a *subset* of B iff every element of A is also an element in B; in other words, if $(\forall a \in D)(a \in A \Rightarrow a \in B)$. We denote this by $A \subseteq B$.
- Two sets are *equal* iff they both contain exactly the same members; in other words, $X = (Y \iff X \subseteq Y) \wedge (Y \subseteq X)$.
- The *cardinality*, or size, of a set A, is denoted with $|A|$.

Operations

There are four main set operations.

- The intersection of two or more sets is the set $\{a \in D : a \in A \wedge a \in B\}$ and is denoted using $\cap$.
- The union of sets is the set $\{a \in D : a \in A \wedge a \in B\}$ and is denoted using $\cup$.
- The symmetric difference of two sets is the set $\{a \in D : a \in A \wedge a \notin B\}$ and is denoted using $\setminus$.
- The complement of a set is all elements of the domain that do not reside within the original set: in other words, the set $A^c = \{a \in D : a \notin A\}$.

Examples

- The empty set, denoted by $\{\}$ or $\emptyset$, is a set that has no elements in it. Note that nothing is an element of the empty set. That is, if x is any element, then the statement $x \in A$ must be false.

  Question. Prove the empty set is a subset of every set.

  Solution. The statement "$\emptyset$ is a subset of any set A" is equivalent to the statement "If $x \in \emptyset$, then $x \in A$." The antecedent of this statement is always false, hence the statement is always true.
- The symbol $\bigcup$ is commonly used to denote the universal set, or the set of all elements in a problem. All sets A are subsets of $\bigcup$.
- The power set is the set of all sets $\mathcal{P}(A) = \{S \subseteq D : S \subseteq A\}$.
  For example, if $A = \{0, 1\}$, then $\mathcal{P}(S) : \{\emptyset, \{0\}, \{1\}, \{0, 1\}\}$.
- An interval between two points a and b is defined as $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. In other words, a set that consists of all real numbers between two specified endpoints.
- The Cartesian product of two sets: $A \times B = \{(a, b)\, A \in A \wedge B \in B\}$. For example, the Cartesian plane is the Cartesian product of the x-axis and the y-axis.

Example: Consider the sets

$$S = \{\mathbb{N}, \mathbb{Q}, \mathbb{R}, \mathbb{Z}\} \quad \text{and} \quad \begin{aligned} A &= \{a, e, i, o, u\} \\ B &= \{a, b, c\} \\ C &= \{0, 1\}. \end{aligned}$$

Determine if the following are true or not.

(a) $\mathbb{N} \subseteq S$

(b) $\{\emptyset, \mathbb{R}\} \subseteq S$

(c) $\{a, e, o\} \in \mathcal{P}(A)$

(d) $B \times C = \begin{Bmatrix} (a,0), (a,1) \\ (b,0), (b,1) \\ (c,0), (c,1) \end{Bmatrix}$

Solution

(a) No, there are elements in $\mathbb{N}$ that are not elements in S. For example, take $x = 1$.

(b) No, $\emptyset$ is an element in $\mathbb{N}$ that is not an element in S.

(c) Yes, $\{a, e, o\}$ is a subset of A.

(d) Yes, this is the definition of Cartesian product.

Example

Show that $X \setminus Y = X \cup Y^c$ for any two sets X and Y.

Solution

For each object a, we have

$$\begin{aligned} &a \in X \setminus Y \\ &\Leftrightarrow x \in X \cap a \notin Y \\ &\Leftrightarrow x \in X \wedge x \in Y^c \\ &\Leftrightarrow x \in X \cap Y^c. \end{aligned}$$

Functions

A *function* is a mapping of inputs (x) to outputs $f(x)$.

Relations

A *relation* $\mathcal{R}$ over sets A and B is a subset of $A \times B$. $a\mathcal{R}B$ reads as "a is related to b" and it is true if (a, b) is in the subset given by $\mathbb{R}$. When $A = B$, we say that $\mathcal{R}$ is a relation on A.

Post-Lecture

Question 1

Let S be a set such that for each set A, we have $S \subseteq A$. Show that $S = \emptyset$.

Hint: Choose an appropriate A such that all other possible choices for S are eliminated.

Solution

Consider any set A, some set S, and any object x. Then $S \subseteq A$ is equivalent to the conditional sentence

$$\text{if } x \in S, \text{ then } x \in A.$$

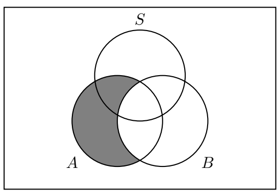
Let $A = \emptyset$. Then for any $x \notin \emptyset$, $x \notin A$. Hence the only set S for which this sentence holds is $S = \emptyset$.

Question 2

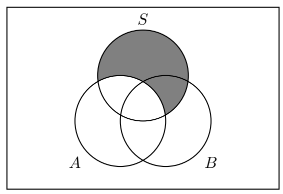
Draw a Venn Diagram to exhibit the result $S \setminus (A \setminus B) = (S \setminus A) \cup (S \cap B)$.

Solution

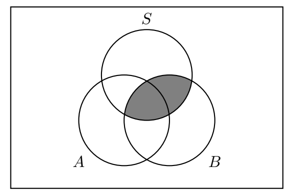
This result is represented in the following Venn Diagrams.



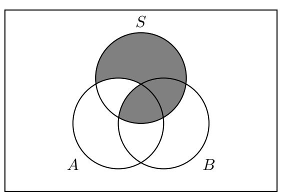
$A \setminus B$



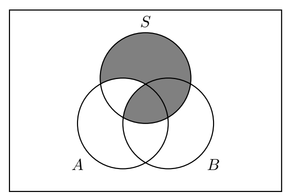
$S \setminus A$



$S \cap B$



$S \setminus (A \setminus B)$



$(S \setminus A) \cup (S \cap B)$

Question 3

Let A and B be sets and let x be any object. Prove that $x \notin A \setminus B$ iff $x \notin A$ or $x \in B$.

Solution

Using one of De Morgan's laws for propositional logic, we have

$$\begin{aligned} & x \notin A \setminus B \\ \text{iff } & \text{it is not the case that } x \in A \setminus B \\ \text{iff } & \text{it is not the case that } x \in A \text{ and } x \notin B \\ \text{iff } & x \notin A \text{ or } x \in B. \end{aligned}$$

Question 4

Let A, B, C, and D be sets. Suppose B and D are nonempty. Prove that if $A \times B = C \times D$, then $A = C$.

Solution

We are given $B \neq \emptyset$ and $D \neq \emptyset$. Suppose $A \times B = C \times D$. We wish to prove that $A = C$.

1. To prove $A \subseteq C$. Let $a \in A$. Since $B \neq \emptyset$, we pick $b_0 \in B$. Then $(a, b_0) \in A \times B$. Since $A \times B = C \times D$, we get $(a, b_0) \in C \times D$. By definition of Cartesian product, we get $a \in C$ and $b_0 \in D$. Since $a \in C$, we have proved $A \subseteq C$.
2. To prove $C \subseteq A$. Let $c \in C$. Since $D \neq \emptyset$, we pick $d_0 \in D$. Then $(c, d_0) \in C \times D$. Since $C \times D = A \times B$, we get $(c, d_0) \in A \times B$. By definition of Cartesian product, we get $c \in A$ and $d_0 \in B$. Since $c \in A$, we have proved $C \subseteq A$.

Since $A \subseteq C$ and $C \subseteq A$, we conclude that $A = C$.

Question 5 (Challenge)

This question was taken from The Ohio State University. Completing this problem, I feel, means you are adequately prepared for quiz/test questions on set theory. You may want to use the previous parts to solve the parts that appear after. Let X, A, B be sets.

(a) Prove that $X \setminus (B \setminus A) = (X \setminus B) \cup (X \cap A)$.

(b) Deduce that $X \setminus (X \setminus A) = X \cap A$.

(c) Prove that if $A \subseteq B$, then $X \setminus B \subseteq X \setminus A$.

(d) Prove that $A \subseteq X$ if and only if $A = X \setminus (X \setminus A)$.

(e) Suppose that $A \subseteq X$. Prove that if $X \setminus B \subseteq X \setminus A$, then $A \subseteq B$.

Solution

(a)
$$\begin{aligned} X \setminus (B \setminus A) &= X \cap (B \cap A^c)^c = X \cap (B^c \cup A) \\ &= (X \cap B^c) \cup (X \cap A) = (X \setminus B) \cup (X \cap A). \end{aligned}$$

(b) From part (a), we have $X \setminus (X \setminus A) = (X \setminus X) \cup (X \cap A) = X \cap A$.

(c) Suppose $A \subseteq B$. Now using the law of contraposition, we have $B^c \subseteq A^c$. Then $X \cap B^c \subseteq X \cap A^c$. Hence by definition, $X \setminus B \subseteq X \setminus A$.

(d) We have $A \subseteq X$ iff $A = A \cap X$. From part (b), we know $A \cap X = X \setminus (X \setminus A)$. Hence $A \subseteq X$ iff $A = X \setminus (X \setminus A)$.

(e) We are given $A \subseteq X$. Suppose $X \setminus B \subseteq X \setminus A$. Then it follows from part (c) that $X \setminus (X \setminus A) \subseteq X \setminus (X \setminus B)$. Since $A \subseteq X$, from part (d) we have $A = X \setminus (X \setminus A)$. Now $X \setminus (X \setminus A) \subseteq X \setminus (X \setminus B)$ and $A = X \setminus (X \setminus A)$, so $A \subseteq X \setminus (X \setminus B)$. From part (b), we know $X \setminus (X \setminus B) = B \cap X$. Since $B \cap X \subseteq B$, it follows that $X \setminus (X \setminus B) \subseteq B$. Now $A \subseteq X \setminus (X \setminus B)$ and $X \setminus (X \setminus B) \subseteq B$, so $A \subseteq B$.

Question 6 (Fun)

Find and prove the size of any power set $\mathcal{P}(A)$, where A is an arbitrary set such that $|A| = n$.

Hint: The proof follows inductively. We have not covered induction yet, so if you are not familiar, it will suffice simply to make observations and guess.

Solution

We claim by observation that the size of any power set A is 2^n, where n is the number of elements of A, and shall prove so using induction. Let $P(n)$ be the sentence

$$|\mathcal{P}(A)| = 2^n.$$

Base Case: $P(0)$ is true, since the only subset of the empty set is the empty set itself, so $|\mathcal{P}(A)| = 1 = 2^0$.
Inductive Step: Now let $n \in \mathbb{N}$ such that $P(A)$ is true. Let B be a set with $n+1$ elements, such as $B = X \cup \{b\}$. There are two types of subsets of B: those that include b and those that exclude b. The latter are exactly the subsets of A, while the former are of the form $\mathcal{P}(A) \cup \{b\}$. By the inductive hypothesis, both have 2^n elements. Therefore, the total number of subsets of B is $2^n + 2^n = 2^{n+1}$. Hence $P(n+1)$ is true as well.
Conclusion: Therefore by induction, for all $n \in \mathbb{N}$, $P(n)$ is true.