Honors Discrete Mathematics: Homework 5

Due on March 12, 2022 at 11:59pm

Professor Gerandy Brito Spring 2022

You may collaborate with other students in this class, but (1) you must write up your own solutions in your own words, and (2) you must write down everyone you worked with at the top of the page, or "no collaborators" if you did it all on your own. Additionally, if you used any outside websites or textbooks besides the course text, please cite them here. You may not use question-answer sites like Chegg or MathOverflow.

Sarthak Mohanty

Exercise 0 (Not Graded)

About how many hours did you spend on this homework in total? Are there any topics that were particularly unclear that we should spend more time on?

Exercise 1

Define a relation $\leq_i$ on the Cartesian product $A_1 \times A_2 \times \cdots \times A_n$ as such: Given two sequences $(a_n) = (a_1, a_2, \ldots, a_n)$ and $(b_n) = (b_1, b_2, \ldots, b_n)$, declare that $(a_n) \leq_i (b_n)$ if and only if $a_i \leq b_i$ for all $1 \leq i \leq n$.

(a) Prove that this relation is a partial order.

(b) Let $Z = \{-3, -2, -1, 0, 1, 2, 3\}$. Draw a Hasse diagram corresponding to the poset $(Z \times Z, \leq_i)$. Briefly explain the general structure of this diagram if we were to extend the domain to all $\mathbb{Z} \times \mathbb{Z}$, and its similarity to the Cartesian plane.

(c) If we define this relation on the Cartesian product of n totally ordered sets, is the relation a total order? Either prove or provide a counterexample.

Exercise 2

(a) Draw the Hasse diagram of the poset $(D_{2470}, |)$, where D_n is the set of all positive integers that divide n, and $|$ is the divisibility relation.

(b) Draw the Hasse diagram of the poset $(\mathcal{P}(\{2, 5, 19, 13\}), \subseteq)$. Explain the similarities between the Hasse diagram you just drew and the Hasse diagram in part (a), and the underlying relationship between the two posets causing these similarities.

(c) Without explicitly drawing the Hasse diagrams, explain the similarities between the Hasse diagrams of the posets $(D_{140}, |)$ and $(D_{525}, |)$, and the underlying relationship between the two posets causing these similarities.

Hint: Consider using a similar (but not identical!) argument as in part (b).

Exercise 3

A binary tree is a structure in which each node in the tree has at most two children nodes branching off of it, as shown in Figure 1. A binary tree is *full* if all of its nodes have either zero or two children. Let B_n denote the number of full binary trees with n nodes.

For $n > 1$, give a recurrence relation for B_n (You do not have to find a closed form). Justify your answer.

Figure 1: Two examples of binary trees. The tree on the left is full, while the tree on the right is not.

Exercise 4

Use the method of your choice to solve the following recursions.

(a) $T(n) = 5T(n-1) - 8T(n-2) + 4T(n-3)$ where $T(0) = 1$, $T(1) = 0$, and $T(2) = 0$.

(b) $T(n) = 2T(n/2) + n\log(n)$ where $T(1) = c$ (here c is a constant).

Note: While you may use any of the methods discussed in class to solve these recurrences, in each part, there may be a method considered "easier" than the other.

Exercise 5

In each part, as in lecture, start with the definition and get the values of c and k.

(a) Let f, g be real functions satisfying $f = \mathcal{O}(g)$. Show that $a \cdot f = \mathcal{O}(g)$ for all $a \in \mathbb{R}^*$.

(b) Let f_1, f_2, g be real functions satisfying $f_1 = \mathcal{O}(g)$ and $f_2 = \mathcal{O}(g)$. Show that $f_1 + f_2 = \mathcal{O}(g)$.

(c) Let f_1, f_2, g_1, g_2 be real functions satisfying $f_1 = \mathcal{O}(g_1)$ and $f_2 = \mathcal{O}(g_2)$. Show that $f_1 \cdot f_2 = \mathcal{O}(g_1 \cdot g_2)$.

Exercise 6

Let A be the set of all functions $f : \mathbb{N}^+ \to \mathbb{R}^+$. Consider the relation $\sim$ on A defined as follows:

$$f \sim g \text{ if there exist constants } c_1, c_2 > 0 \text{ such that } c_1 g(n) < f(n) < c_2 g(n) \text{ for all } n \in \mathbb{N}^+.$$

(a) Show that $\sim$ is an equivalence relation.

(b) It can similarly be shown that Θ is an equivalence relation. Group the following functions by equivalence classes such that functions $f(n)$ and $g(n)$ are in the same equivalence class iff $f = \Theta(g)$ (you do not need to show your work):

- $\log_2(n) + 1$
- $\log_{16}(n!)$
- $2n + 5$
- $(\log_2(n))^{\log_2(n)}$
- $n\log_{10}(n) + 1$
- $2021n + \log_{10}(n)$
- 2^n
- $n^{1/\log_6(n)}$
- 3^n
- $\ln(n^2) + 2021$
- $\frac{8n^2+3n+9}{n^2}$
- $n^{\log_2(\log_2(n))}$

Exercise 7

(For this problem, you can use classical tools from Calculus such as $\epsilon - \delta$ definition of limits, derivatives, etc.) Let f, g be real functions with f not identically zero and $g(x) \neq 0$ for all $x \in \mathbb{R}$ satisfying

$$\lim_{x\to\infty} \left| \frac{f(x)}{g(x)} \right| = \ell.$$

(a) Show that if $0 < \ell < \infty$, then $f = \Theta(g)$.

(b) Show that, if $l = 0$, then $f = \mathcal{O}(g)$. Give an example of functions for which $g \neq \mathcal{O}(f)$.

(c) Can you deduce a relation (of the form big-$\mathcal{O}$, big-Ω, or big-Θ) between f and g if $\ell = \infty$? Prove your answer.

(d) Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ and $q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_1 x + b_0$ be two polynomials with real coefficients. Use parts (a)–(c) to deduce when $p = \mathcal{O}(q)$, $p = \Omega(q)$, or $p = \Theta(q)$ (your answer should depend on n and m).