Honors Discrete Mathematics: Lecture Notes

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Relations

Recall the definition of a relation:

Definition a *relation* is a subset of the cartesian product $A \times B$.

We denote relations by $\mathcal{R}$. We write $a\mathcal{R}b$ to indicate that $(a, b) \in \mathcal{R}$ (i.e.: in the subset denoted by $\mathcal{R}$). When $A = B$ we said that $\mathcal{R}$ is a relation on A.

Let $\mathcal{R}_1, \ldots, \mathcal{R}_4$ be a relation on $A = \{1, 2, 3, 4\}$.

- $\mathcal{R}_1 = \{(a, b) \mid a \leq b)\}$
- $\mathcal{R}_2 = \{(a, b) \mid a = b)\}$
- $\mathcal{R}_3 = \{(a, b) \mid a + b \leq 2022)\}$
- $\mathcal{R}_4 = \{(a, b) \mid a \text{ divides } b\}$

Create bijective (bean diagram) graph in tikz.
So why relations? They are more general and allow us to study more complex sets. Elaborate.

Properties:

- <u>Reflexive:</u> $(\forall a \in A)(a\mathcal{R}a)$
- <u>Symmetric:</u> $(\forall a, b \in A)(a\mathcal{R}b \iff b\mathcal{R}a)$
- <u>Antisymmetric:</u> $(\forall a, b \in A)(a\mathcal{R}b \wedge b\mathcal{R}a \to a = b)$
- <u>Transitive:</u> $(\forall a, b, c \in A)(a\mathcal{R}b \wedge b\mathcal{R}c \to a\mathcal{R}c)$

Create visualization graph in tikz.
We checked these properties for R_2 and R_3. $\mathcal{R}_2$ was reflexive, symmetric, and transitivity. $\mathcal{R}_3$ was symmetric and transitive?

Functions

Last lecture, we informally covered the meaning of a function. With our new knowledge of relations, we now have the tools to formally define them.

Definition A function is a relation on $A \times B$ such that

$$(\forall a \in A)(\exists! b \in B)(a\mathcal{R}b).$$

By the way, the propositional symbol $\exists!$ means "there exists only one".
Often we are presented with functions describing the unique element b for each a. In this case we use f to denote the function and write $FA \mapsto B$ or $b = f(a)$.

A function f is a way to associate each item in a set to an element in another set.

Notation:
Usually, we write functions abstractly as $F : A \mapsto B$.
A is called the domain. B is called codomain. The range of f is the set of all $b \in B$ for which there is at least one $a \in A$ satisfying $f(a) = b$.
To do: think of the representation using relations.

Classes of Functions

(a) A function is *surjective* (or onto) if every element in B is associated with at least one element in A. In other terms, the range is equal to the codomain.

(b) A function is *injective* (or one-to-one) if no two elements $b_1, b_2 \in B$ such that $b_1 \neq b_2$ but $f(b_1) = f(b_2)$. In other terms, no two distinct elements in B associate to the same $a \in A$.

(c) A function is *bijective* if it is surjective and injective.

Examples: What are the domain and codomain of the following functions. Also determine whether the following functions are surjective, injective, or both (bijective).

- f assigns to each student in the class its height in cm.
- $f(x) = x + 1$ from $\mathbb{N}$ to itself, then from $\mathbb{Z}$ to itself.
- f assigns to each bit string of length two or more its last two bits.
- f assigns to each real number the largest integer less or equal than the number

Solutions:

- Domain: Students, Codomain: $\mathbb{R}^+$. Neither injective nor surjective.
- This function is only injective when mapping from $\mathbb{N}$ to itself, as $f(x) = 1$ cannot be reached by any $x \in \mathbb{N}$. However, when mapping from $\mathbb{Z}$ to itself, this function becomes bijective.
- D: bitstrings length greater than 2, C: bitstrings length 2. Surjective
- $\mathbb{R} \to \mathbb{Z}$, Surjective

To do: discuss what it means to be one-to-one, onto, in the language of relations.
Write the definitions. Mention that, in practice, we get functions not as relations and work directly with the f description. Now give then the general strategy to prove a function is one-to-one:

$$f \text{ is one-to-one if } f(x) = f(y) \to x = y$$

and onto

$$f \text{ is onto if } (\forall b \in B)(\exists a \in A)(f(a) = b)$$

To do: give examples to prove this

To do: Briefly cover Inverses

1 Equivalence Relations

Def: A relation is said to be an equivalence relation if it is reflexive, symmetric, and transitive.

Example: Show that the relation on the real numbers defined as $a\mathcal{R}b \leftrightarrow a - b \in \mathbb{Z}$ is an equivalence relation.

Solution: Reflexive: $a = b$, then $a\mathcal{R}a$ because $a - a = 0$.
Symmetry: $a - b = -(b - a)$. Assume $a\mathcal{R}b$. Then $b\mathcal{R}a$ because $-b + a \to a - b$?
Transitivity: Assume $a\mathcal{R}b$ and $b\mathcal{R}c$, WTS $a\mathcal{R}c$ $a-b = d, d \in \mathbb{Z}$. $b-c = e, e \in \mathbb{Z}$. $d+e = (a-b)+(b-c) = a-c$).
Definition Given a set A and an equivalence relation on it, the equivalence classes are the sets

$$[a]_{\mathcal{R}} = \{x \in A \mid a\mathcal{R}x\}$$

Theorem Let A be a set and $\mathcal{R}$ an equivalence relation on it. The following are equivalent:

(i) $a\mathcal{R}b$.

(ii) $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$.

(iii) $[a]_{\mathcal{R}} \cap [b]_{\mathcal{R}} \neq \emptyset$.

Proof

- (i) to (ii): Suppose $a\mathcal{R}b$. Let $c \in [a]_{\mathcal{R}}$. Then $a\mathcal{R}c$. By the symmetry property of equivalence relations. Since $a\mathcal{R}b$ and $a\mathcal{R}c$, we have $b\mathcal{R}a$ and $a\mathcal{R}c$, so $b\mathcal{R}c$.
- (ii) to (iii): Suppose $[a]_{\mathcal{R}} = [b]_{\mathcal{R}}$. If $a = b \rightarrow = a\mathcal{R}a$ by reflexivity.
- (iii) to (i) Assume $[a]_{\mathcal{R}} \cup [b]_{\mathcal{R}} \neq \emptyset$. Then there exists some c such that $c \in [a]_{\mathcal{R}}$ and $c \in [b]_{\mathcal{R}}$. Then $a\mathcal{R}c$ and $b\mathcal{R}c$. Then by symmetry, we have $c\mathcal{R}b$. By transitivity, we conclude that $a\mathcal{R}b$, as desired.

To do: redefine **Definition**: P is a partitioning of A_i for all i iff $\bigcup_{i=1}^{k} P_i = A$, where each P_i is disjoint.

Why is every element in a class? Ans: because of the reflexive property. Why it is a partition? Because of facts (ii) and (iii) of the Theorem.

Cardinality

We defined cardinality last lecture; however, we will now modify our definition of cardinality to better suit the next few lectures.

Definition The cardinality of a set S, denoted as $|S|$ is

- the number of elements in S, if it is finite.
- ∞ otherwise.

Question: are there more integers than naturals? Answer: no. More reals than integers: Answer: Depends on definition of "more", will define next time.

Definition: Two sets A and B are said to have the same cardinality if there exists a bijection from A to B.
TA Remark: Check post-lecture to see some common examples of bijections between sets.
Now consider the collection of all sets (i.e.: the power set of the universe!) and define the relation

$$A\mathcal{R}B \text{ if } A \text{ and } B \text{ have the same cardinality.}$$

We conclude this lecture with one more definition:

Definition: A set is said to be *countable* if it is finite or has the same cardinality as the natural numbers.

Post Lecture

Question 1

Al and Bob play a game. They have the numbers $1, 2, \ldots, 9$ written on cards face up. Players alternate taking any card. The first player to have exactly 3 cards whose sum is 15 wins. Determine which player, if any, has a winning strategy.

Hint: The game 'Tic-Tac-Toe' is known to have no winning strategy for either player. Furthermore, I strongly encourage you to play this game a few times with a friend, and see what strategies work.

Solution

Either by playing this game yourself, or by doing the calculations yourself, you may have noticed there are eight possible ways to win this game. Furthermore, you can create a bijection mapping these possibilities to the magic square shown below:

4	9	2
3	5	7
8	1	6

Since Al and Bob are alternating turns, there are essentially playing Tic-Tac-Toe. As Tic-Tac-Toe is known to have no winning strategy for either player, we conclude that there is no winning strategy for Al or Bob.

Question 2 (Not finished, skip for now)

Let $f(x) = x - 1$ for all $x \in [1, \infty)$. Show that $f : [1, \infty) \to [0, \infty]$ is an injection. Then show that $\mathrm{Rng}(f) = [0, \infty)$. Conclude that f is a bijection from $[1, \infty)$ to $[0, \infty]$.

Solution

Suppose $x \in [1, \infty]$ and $y = f(x)$. Since $1 \leq x < \infty$, we have $1 - 1 \leq x - 1 < \infty - 1$, so $0 \leq x - 1 < \infty$, so $f(x) \in [0, \infty)$. Also, since $y = f(x) = x - 1$, we have $x = y + 1$. Thus $f : [1, \infty) \to [0, \infty)$, f is an injection, and for each $y \in \mathrm{Rng}(f)$.

To show that $\mathrm{Rng}(f) = [0, \infty)$, it remains only to show that each y in $[0, \infty)$ belongs to $\mathrm{Rng}(f)$. Let $y \in [0, \infty)$. We wish to show that there exists $x \in [1, \infty)$ such that $f(x) = y$. Let $x = y + 1$. Then $y = x - 1$. Hence, once we have shown that $x \in [1, \infty)$, we will have that $f(x) = x - 1 = y$. Now $0 \leq y < \infty$, so $0 + 1 \leq y + 1 < \infty + 1$, so $1 \leq y + 1 < \infty$, so $1 \leq x < \infty$, so $x \in [1, \infty)$. Thus $\mathrm{Rng}(f) = [0, \infty)$, and we can conclude that f is a bijection from $[1, \infty)$ to $[0, \infty)$.

Question 3

Describe bijections (without justifications): Whenever the bijection is defined by a single formula, also provide its inverse.

(a) From $\mathbb{N}$ to the set of odd natural numbers.

(b) from $\mathbb{Z}$ to $\mathbb{N}$.

(c) from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{N}$

(d) from $\mathbb{N}$ to $A = \{x \in \mathbb{Q} : x > 0\}$.

(e) from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$. [A suitable trigonometric function will do.]

(f) from $(0, 1)$ to $\mathbb{R}$. [Compose a linear map with the map in part e.]

Solution

(a) Let A denote the set of odd natural numbers. Let $f(x) = 2x - 1$ for all $x \in \mathbb{N}$. Let $g(y) = \frac{y+1}{2}$ for all $y \in A$. Then $f^{-1} = g$ and $f : \mathbb{N} \to A$ is a bijection.

(b) Let $n \in \mathbb{Z}$. Define $f(n)$ by

$$f(n) = \begin{cases} 2n + 1 & \text{if } n \geq 0, \\ -2n & \text{if } n < 0. \end{cases}$$

Then $f : \mathbb{Z} \to \mathbb{N}$ is a bijection.

(c) Define a function f as follows:

$$\begin{aligned}
&f(1,1) = 1, \\
&f(1,2) = 2, \quad f(2,1) = 3, \\
&f(1,3) = 4, \quad f(2,2) = 5, \quad f(3,1) = 6, \\
&f(1,4) = 7, \quad f(2,3) = 8, \quad f(3,2) = 9, \quad f(4,1) = 10,
\end{aligned}$$

and so on. Then $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is a bijection.

(d) Let A denote the set of positive rational numbers. Define a function f as follows:

$$\begin{aligned}
&f(1) = \frac{1}{1}, \\
&f(2) = \frac{1}{2}, \quad f(3) = \frac{2}{1}, \\
&f(4) = \frac{1}{3}, \quad f(5) = \frac{3}{1}, \\
&f(6) = \frac{1}{4}, \quad f(7) = \frac{2}{3}, \quad f(8) = \frac{3}{2}, \quad f(9) = \frac{4}{1},
\end{aligned}$$

and so on. Then f is a bijection from $\mathbb{N}$ to A.

(e) Let $f(x) = \tan(x)$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let $g(y) = \tan^{-1}(y)$ for all $y \in \mathbb{R}$. Then $f^{-1} = g$ and $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a bijection.

(f) Let $f(x) = \tan(x)$ and $g(x) = \sin^{-1}(2x - 1)$. Now f is a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$ and g is a bijection from $(0, 1)$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then $(f \circ g)(x) = \tan(\sin^{-1}(2x - 1))$ is a bijection from $(0, 1)$ to $\mathbb{R}$. The inverse of $f \circ g$ is $(g^{-1} \circ f^{-1})(y) = \frac{\sin(\tan^{-1}(y))+1}{2}$ for all $y \in \mathbb{R}$.