Homework 3

CS 2051, Spring 2022

Georgia Tech

Due February 12

Reminder: You may collaborate with other students in this class, but (1) you must write up your own solutions in your own words, and (2) you must write down everyone you worked with at the top of the page, or "no collaborators" if you did it all on your own. Additionally, if you used any outside websites or textbooks besides the course text, please cite them here. You may not use question-answer sites like Chegg or MathOverflow.

Question 0

About how many hours did you spend on this homework in total? Are there any topics that were particularly unclear that we should spend more time on? (not graded)

Question 1

Let $A, B \subseteq U$ where U is the universe.

1. Prove that $(A \cup B)^C = A^C \cap B^c$.
2. Prove that $(A \cap B)^C = A^C \cup B^C$.
3. Prove that

$$(X \cap Y \cap Z) \cup (X \cup Y \cup Z)^C$$

and

$$(X \cup Y^C) \cap (X \cup Z^C) \cap (Y \cup X^C) \cap (Y \cup Z^C) \cap (Z \cup X^C) \cap (Z \cup Y^C)$$

are equal.

Question 2

For each of the following parts, determine if each of the following are true: $A \subseteq B$, $B \subseteq A$, $A = B$. Prove your answers.

1. $A = \{2n : n \in \mathbb{Z}\}, B = \{4n : n \in \mathbb{Z}\}$
2. $A = \mathbb{Q}, B = \{7n : n \in \mathbb{Q}\}$
3. $A = \{2n : n \in \mathbb{Z}\}$, $B = \{n = 2k + 1 \text{ for some k in } \mathbb{Z} : n \in \mathbb{R}\}^C$

Question 3

Out of the following four functions, one is a surjection only, one is an injection only, one is neither, and one is a bijection. State which is which and demonstrate your answers by (1) proving that it is an injection or naming two distinct elements of the domain that are associated to the same element of the target, and (2) proving that it is a surjection or naming an element of the target for which there is no corresponding element in the domain.

1. $f : \mathbb{Z}^2 \mapsto \mathbb{Z}$, $f(m, n) = m^2 + n^2$
2. $f : \mathbb{Z}^2 \mapsto \mathbb{Z}$, $f(m, n) = m$
3. $f : \mathbb{R}^2 \mapsto \mathbb{C}$, $f(m, n) = m - 2ni$
4. $f : \mathbb{Z}^2 \mapsto \mathbb{Z}$, $f(m, n) = 2^m 3^n$

Question 4

Determine whether the relation $\mathcal{R}$ on the set of all people is reflexive, symmetric, antisymmetric, and/or transitive. You do not need to justify your answers.

$(a, b) \in \mathcal{R}$ if and only if

1. a is taller than b.
   ◯ `Reflexive` ◯ `Symmetric` ◯ `Antisymmetric` ◯ `Transitive`
2. a and b were born on the same day.
   ◯ `Reflexive` ◯ `Symmetric` ◯ `Antisymmetric` ◯ `Transitive`
3. a has the same first name as b.
   ◯ `Reflexive` ◯ `Symmetric` ◯ `Antisymmetric` ◯ `Transitive`
4. a and b have a common grandparent.
   ◯ `Reflexive` ◯ `Symmetric` ◯ `Antisymmetric` ◯ `Transitive`
5. a and b have met.
   ◯ `Reflexive` ◯ `Symmetric` ◯ `Antisymmetric` ◯ `Transitive`

Question 5

The set of rational numbers can be seen as a set of the equivalence classes of a certain relation on $\mathbb{Z} \times \mathbb{Z}$ defined by $(a, b)R(c, d)$ if and only if $ad = bc$. Show that R is an equivalence relation.

Question 6 *(Hilbert's Grand Hotel)*

The Grand Hotel has countable many rooms, numbered 1, 2, 3, . . . , and it is fully occupied.

1. Suppose that k guests arrive at the hotel. Show how you can accommodate all new guests without removing any of the current guests.
2. Suppose that a countable infinite number of guests arrive. Show how you can accommodate all new guests without removing any of the current guests.
3. Suppose Hilbert (the owner of the Grand Hotel) expands the property to a second building, also with infinite rooms numbered $1, 2, 3, \ldots$. Show how you can spread the current guests to fill every room in both buildings.

Question 7 (*Cantor's theorem*)

1. Prove that there does not exist an onto function between $\mathbb{N}$ and its power set, $\mathcal{P}(\mathbb{N})$.

   Hint: proceed by contradiction assuming there is such function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$. Let

   $$T = \{n \in \mathbb{N} |\ n \notin f(n)\}.$$

   Since f is onto, there exist $t \in \mathbb{N}$ such that $f(t) = T$. Argue a contradiction by looking at $t \in T$ and $t \notin T$.
2. Show that the set of all countable sets is uncountable