Honors Discrete Mathematics: Lecture 7 Notes

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Cardinality

There is an easy way to test if two numbers have the same cardinality, which will become useful when we deal with infinite sets.

Theorem. Two sets A and B are said to have the same cardinality if there exists a bijection (surjection?) from A to B. The proof of this theorem is outside the scope of this course.

Now consider the collection of all sets (i.e.: the power set of the universe!) and define the relation

$$A\mathcal{R}B \text{ iff } A \text{ and } B \text{ have the same cardinality.}$$

Using the above theorem, we will now show that this relation is an equivalence relation.

1. Reflexivity: There exists a bijection from A to B $f : A \to B$.
2. Symmetry: Since $a\mathcal{R}b$, we have defined a bijection from A to B. As discussed before, bijections have inverses, so we can define an inverse $f^{-1}B \to B$, so $b\mathcal{R}a$.
3. Transitivity: By definition, there exists a bijection from A to B. Furthermore, there exists a bijection from B to C. The composition $h : A \to C = g \circ f$ is a bijection as well (proof shown below), so $a\mathcal{R}c$.

Proposition. Let $f : A \to B$ and $g : B \to C$ be bijections. Then $h : A \to C$ is a bijection as well.
Proof. Let's start by proving that $g \circ f$ is one-to-one. Suppose $g \circ f(a) = g \circ f(b)$. Then $g(f(a)) = g(f(b))$. Since g is one-to-one, this implies $f(a) = f(b)$. Since f is one-to-one, $a = b$. Therefore $g \circ f$ is one-to-one.

Let's now prove that $g \circ f$ is onto. Suppose $c \in C$. Since $c \in C$ and g is onto, there exists $b \in B$ such that $g(b) = c$. Since $b \in B$ and f is onto, there exists an $a \in A$ such that $f(a) = b$. Therefore $g(f(a)) = c$, i.e., $g \circ f(a) = c$. Since $g \circ f$ is one-to-one and onto, it is a bijection.

Definition: A set is said to be *countable* if it is finite or has the same cardinality as the natural numbers.
Examples: Show the following are countable sets by defining bijections for each of the following sets.

- From $\mathbb{N}$ to the set of positive even numbers.
- From $\mathbb{N}$ to the set of negative integers.
- From $\mathbb{N}$ to the set of odd positive numbers.
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Solution:

- $f(n) = 2n$
- $f(n) = -n$
- Let A denote the set of odd natural numbers. Let $f(x) = 2x - 1$ for all $x \in \mathbb{N}$. Then $f : \mathbb{N} \to A$ is a bijection.
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Lemma. Let A, B be countable sets. Then $A \cup B$ is also countable.

Proof. Suppose both $|A|$ and $|B|$ is finite. Then the proof is trivial.
Now suppose at least one of $|A|$, $|B|$ is infinite. WLOG, suppose $|A$ is infinite and the size of b is k. Define a series of elements with the element at index i equal to the i-th element of $|A$. Then shift every element k spaces to the right, and fill the first k spaces in the series with the elements in B. Since series are functions, we have shown there exists a bijection from $|A|$ to $|B|$

Now suppose both $|A|$ and $|B|$ are infinite. Let $f(n) = 2n$ and $g(n) = 2n - 1$ Define a function $h(n)$ as

$$h(n) = \begin{cases} f(n) \text{ if } n \text{ is even} g(N) \text{ if } n \text{ is odd} \end{cases}$$

We now have another tool to show set are countable: The finite union of countable sets is also countable.

Lemma. Let A, B be countable sets. If $A \subseteq B$ and B is countable, then so is A. Proof is left as an exercise for the reader.

Lemma. Let A, B be countable sets. Then $A \times B$ is countable.

Proof. (TBC) We know $A \times B = \{a_i, b_j : a_i \in A, b_j = B\}$.
Define a function f as follows:

$$\begin{aligned}
&f(a_1, b_1) = 1, \\
&f(a_2, b_a) = 2, \quad f(a_1, b_2) = 3, \\
&f(a_3, b_1) = 4, \quad f(a_3, b_2) = 5, \quad f(a_3, b_3) = 6, \\
&f(1,4) = 7, \quad f(2,3) = 8, \quad f(3,2) = 9, \quad f(4,1) = 10,
\end{aligned}$$

and so on. Then $f : \mathbb{A} \times \mathbb{B} \to \mathbb{N}$ is a bijection.
Example. Show that $\mathbb{N}$ to $A = \{x \in \mathbb{Q} : x > 0\}$ is a bijection.
We have many options, we can use the previous lemma and define the Cartesian product as $A \times B$, where A is the set of all numerators and B is the set of all denominators.
Let A denote the set of positive rational numbers. Define a function f as follows:

$$\begin{aligned}
&f(1) = \frac{1}{1}, \\
&f(2) = \frac{1}{2}, \quad f(3) = \frac{2}{1}, \\
&f(4) = \frac{1}{3}, \quad f(5) = \frac{3}{1}, \\
&f(6) = \frac{1}{4}, \quad f(7) = \frac{2}{3}, \quad f(8) = \frac{3}{2}, \quad f(9) = \frac{4}{1},
\end{aligned}$$

Cantor's Diagonal Lemma

Let f be a function from $\mathbb{N}$ to $(0, 1)$. Prove that there exists $y \in (0, 1)$ such that y does not belong to the range of f. (in other words, prove the set of real numbers is not countable.)

Solution

We are given a function $f : \mathbb{N} \to (0, 1)$. We wish to find a number $y \in (0, 1)$ such that

$$y \notin \{f(1), f(2), f(3), f(4), \dots\}.$$

For each $n \in \mathbb{N}$ and each $k \in \mathbb{N}$, let x_{nk} be the k-th digit in the *standard* decimal expansion of $f(n)$. Then

$$\begin{aligned}
&f(1) = 0.x_{11}x_{12}x_{13}x_{14}\dots, \\
&f(2) = 0.x_{21}x_{22}x_{23}x_{24}\dots, \\
&f(3) = 0.x_{31}x_{32}x_{33}x_{34}\dots, \\
&f(4) = 0.x_{41}x_{42}x_{43}x_{44}\dots, \\
&\text{and so on.}
\end{aligned}$$

We shall define the number y by defining the digits in its decimal expansion so that they are different from the "diagonal" entries $x_{11}, x_{22}, x_{33}, x_{44}, \dots$ that are highlighted in the equations above. For each $n \in \mathbb{N}$, let

$$y_n = \begin{cases} 5 & \text{if } x_{nn} \neq 5, \\ 4 & \text{if } x_{nn} = 5. \end{cases}$$

Then for each $n \in \mathbb{N}$, $y_n \neq x_{nn}$. Now let y be the number whose standard decimal expansion is

$$y = 0.y_1y_2y_3y_4\dots.$$

Then $y \in (0,1)$. In fact, $0.444\dots \leq y \leq 0.555\dots$. To see that y is not in the range of f, note that for each $n \in \mathbb{N}$, $y \neq f(x)$ (because the numbers y and $f(n)$ differ in their n-th decimal place; in other words, $y_n \neq x_{nn}$).

Post Lecture

Question 3

Describe bijections (without justifications): Whenever the bijection is defined by a single formula, also provide its inverse.

(a) from $\mathbb{Z}$ to $\mathbb{N}$.

(b) from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$. [A suitable trigonometric function will do.]

(c) from $(0,1)$ to $\mathbb{R}$. [Compose a linear map with the map in part (b).]

Solution

(a) Let $n \in \mathbb{Z}$. Define $f(n)$ by

$$f(n) = \begin{cases} 2n+1 & \text{if } n \geq 0, \\ -2n & \text{if } n < 0. \end{cases}$$

Then $f : \mathbb{Z} \to \mathbb{N}$ is a bijection.

(b) Let $f(x) = \tan(x)$ for all $x \in (-\frac{\pi}{2}, \frac{\pi}{2})$. Let $g(y) = \tan^{-1}(y)$ for all $y \in \mathbb{R}$. Then $f^{-1} = g$ and $f : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$ is a bijection.

(c) Let $f(x) = \tan(x)$ and $g(x) = \sin^{-1}(2x-1)$. Now f is a bijection from $(-\frac{\pi}{2}, \frac{\pi}{2})$ to $\mathbb{R}$ and g is a bijection from $(0,1)$ to $(-\frac{\pi}{2}, \frac{\pi}{2})$. Then $(f \circ g)(x) = \tan(\sin^{-1}(2x-1))$ is a bijection from $(0,1)$ to $\mathbb{R}$. The inverse of $f \circ g$ is $(g^{-1} \circ f^{-1})(y) = \frac{\sin(\tan^{-1}(y))+1}{2}$ for all $y \in \mathbb{R}$.