Notes for November 8, 2021

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1 Class Updates

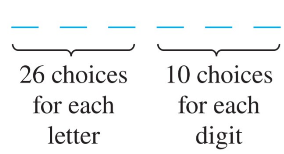
November 8 - 12:	Homework 5 due on Friday
November 15 - 19:	Quiz 4
November 22 - 26:	Thanksgiving Break
November 29 - December 3:	Exam 2 on Monday
December 6:	Last Day of Class
December 10:	Final Exam

There will be one more graded homework (on Combinatorics and Probability).

2 Product Rule

Definition: Suppose that a procedure can be broken down into a sequence of two tasks. If there are $\mathbf{n_1}$ ways to do the first task and for each of these ways of doing the first task, there are $\mathbf{n_2}$ ways to do the second task, then there are $\mathbf{n_1 n_2}$ ways to do the procedure.

Example 1: License Plates
How many license plates exist if a license plate consists of 3 letters followed by 3 digits?



$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 26^3 10^3$$

Example 2: Functions
How many functions $f : A \to B$ exist given $|A| = n$ and $|B| = m$?

$A = \{a_1, a_2, a_3, ...a_n\}$
For a_1, there are m possibilities for $f(a_1)$

For a_2, there are m possibilities for $f(a_2)$
And so on...
Total functions = $m \times m \times ... \times m$ (n times) = m^n

Example 3: One-to-one Functions
How many one-to-one functions $f : A \to B$ exist given $|A| = n$ and $|B| = m$?

Unlike example 2, possibilities reduce as we go on counting to ensure that the function is one-to-one.
$A = \{a_1, a_2, a_3, ...a_n\}$
For a_1, there are m possibilities for $f(a_1)$
For a_2, there are $m - 1$ possibilities for $f(a_2)$
For a_3, there are $m - 2$ possibilities for $f(a_3)$
For a_n, there are $m - (n - 1)$ possibilities for $f(a_n)$
If $m \leq n - 1$, there are no one-to-one functions from A to B.
If $m > n - 1$, total one-to-one functions = $m \times (m - 1) \times (m - 2) \times ... \times (m - n + 1)$

3 Sum Rule

Definition: If a task can be done either in one of $\mathbf{n_1}$ ways or in one of $\mathbf{n_2}$ ways, where none of the set of n_1 ways is the same as any of the set of n_2 ways, then there are $\mathbf{n_1} + \mathbf{n_2}$ ways to do the task.

In simpler words: If you count $\mathbf{n_1}$ elements and $\mathbf{n_2}$ elements, and you check/prove that you are not double counting and that you are counting every element, then there are $\mathbf{n_1} + \mathbf{n_2}$ total elements.

Example 4: Computer Passwords
A password has 6-8 characters (an uppercase letter or a digit) and must contain at least one digit. How many valid passwords exist?

Total passwords = No. of 6 char passwords + No. of 7 char passwords + No. of 8 char passwords
We could fix the required digit and count passwords with required digit at 1^{st} character, 2^{nd} character, and so on separately. However, this would result in double counting because we are allowed to use as many digits as we want, not just the fixed digit.

4 Subtraction Rule

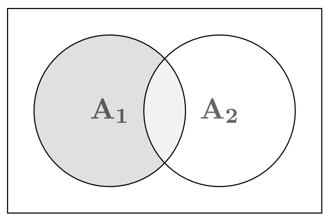
Definition: If a task can be done in either n_1 ways or n_2 ways, then the number of ways to do the task is $n_1 + n_2$ minus the number of ways to do the task that are common to the two different ways.

In simpler words: You can over-count the number of elements and then remove the elements not in your desired set.

Example 4: Computer Passwords (continued)
No. of passwords = No. of strings with letters or digits − No. of strings with only letters
No. of 6 char passwords = $36^6 - 26^6$
No. of 7 char passwords = $36^7 - 26^7$
No. of 8 char passwords = $36^8 - 26^8$
Total passwords = $(36^6 - 26^6) + (36^7 - 26^7) + (36^8 - 26^8)$

Example 5: Euler Totient Function of 1000
How many numbers $a < 1000$ are such that $g.c.d(a, 1000) = 1$?

Define $U = \{1, 2, 3, ...999, 1000\}$
Let A_1 = set of multiples of 2 and A_2 = set of multiples of 5. Then, $A_1 \cap A_2$ = set of multiples of 10 and $(A_1 \cup A_2)^c$ = set of numbers that are multiples of neither 2 nor 5.

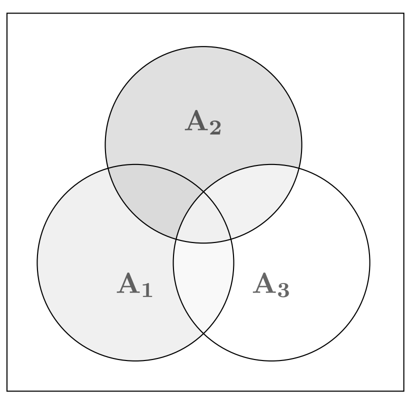


$$\begin{aligned} |(A_1 \cup A_2)^c| &= |U| - |A_1| - |A_2| + |A_1 \cap A_2| \\ &= 1000 - \tfrac{1000}{2} - \tfrac{1000}{5} + \tfrac{1000}{10} \\ &= 1000 - 500 - 200 + 100 \\ &= 400 \end{aligned}$$

From Number Theory, $\varphi(1000) = 2^{3-1} \cdot 5^{3-1} \cdot (2-1) \cdot (5-1) = 400$

5 Inclusion-Exclusion Principle

Three Sets: $$\begin{aligned} |(A_1 \cup A_2 \cup A_3)^c| &= |U| - |A_1| - |A_2| - |A_3| \\ &+ |A_1 \cap A_2| + |A_1 \cap A_3| + |A_2 \cap A_3| \\ &- |A_1 \cap A_2 \cap A_3| \end{aligned}$$



Extending to n Sets: $|(A_1 \cup A_2 \cup ... \cup A_n)^c|$

$$\begin{aligned} &= |U| - \sum_{i=1}^{n} |A_i| \\ &+ \sum_{1 \le i < j}^{n} |A_i \cap A_j| \\ &- \sum_{1 \le i < j < k}^{n} |A_i \cap A_j \cap A_k| \\ &... \\ &(-1)^n \sum |A_1 \cap A_2 \cap ... \cap A_n| \end{aligned}$$

Proof by induction with base case 1/2/3 is left as an exercise to the reader.

Example 6: Euler Totient Function is Multiplicative

Let $\varphi(n)$ be the number of a such that $0 < a < n$ and $g.c.d(a, n) = 1$. Prove $\varphi(ab) = \varphi(a)\varphi(b)$.

$\varphi(a)\varphi(b) =$ No. of $k = \hat{a}\hat{b}$ where $g.c.d(\hat{a}, a) = 1$ and $g.c.d(\hat{b}, b) = 1$

Define $U = \{1, 2, 3, ..., ab\}$

Incorrect Approach:

Let $A_1 =$ set of multiples of a and $A_2 =$ set of multiples of b. Then, $A_1 \cap A_2 =$ set of multiples of ab.

Now, $|U| - |A_1| - |A_2| + |A_1 \cap A_2|$

$= ab - \frac{ab}{a} - \frac{ab}{b} + 1$

$= ab - b - a + 1$

Problem: $(A_1 \cup A_2)^c$ includes extra numbers larger than or not relatively prime with a or b.

Correct Approach:

Let $A_1 =$ set of numbers relatively prime with a and $A_2 =$ set of numbers relatively prime with b.

To be continued in next class.