CS 2051: Honors Discrete Mathematics
Spring 2023 Homework 4 Supplement

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1. The purpose of this problem is to show how the power set $\mathcal{P}(S)$ of a given set S, has always a different cardinality. We have a formula for the case that S is finite, but it is less obvious for the infinite case. To fix ideas, we focus on the case $S = \mathbb{N}$.

   Show that there does not exist an onto function between $\mathbb{N}$ and its power set, $\mathcal{P}(\mathbb{N})$.

   Hint: proceed by contradiction assuming there is such function $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$. Let

   $$T = \{n \in \mathbb{N} |\ n \notin f(n)\}.$$

   Since f is onto, there exist $t \in \mathbb{N}$ such that $f(t) = T$. Argue a contradiction by looking at $t \in T$ and $t \notin T$.

   Solution:

   Proof: I proceed with a proof by contradiction. Assume $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$ is onto. Then, since $T \subseteq \mathbb{N}$ and $T \in \mathcal{P}(\mathbb{N})$, there exists $t \in \mathbb{N}$ such that $f(t) = T$.

   If $t \in T$, then by the definition of T, $t \notin f(t)$. Since $t \in T$ and $t \notin f(t)$, $T \neq f(t)$.

   If $t \notin T$, then by the definition of T, $t \in f(t)$. Since $t \notin T$ and $t \in f(t)$, $T \neq f(t)$.

   We have shown that for all $t \in \mathbb{N}$, $f(t) \neq T$. Applying De Morgan's law for quantifiers, there does not exist $t \in \mathbb{N}$ such that $f(t) = T$. This contradicts with our assumption that f is onto and that there exists such a t. Therefore, there does not exists an onto function between $\mathbb{N}$ and $\mathcal{P}(\mathbb{N})$.

2. The `Brito-Caribbean-Royal Grand Hotel` has a countable infinite number of rooms, each occupied by a guest, due to its popular demand.

   (a) How can we accommodate a new guest arriving at the fully occupied hotel without removing any of the current guests?

   Solution: First, number the rooms with positive integers, starting from 1. Then, we move every guest from their original room n to room $n + 1$. Now room 1 is empty, and the new guest can stay there.

   (b) Show that a finite group of guests arriving at the Grand Hotel can be given rooms without evicting any current guests.

   Solution: First, number the rooms with positive integers, starting from 1. Let the finite number of guests arriving at the Grand Hotel be k. Then, we move every current guest from their original room n to room $n + k$. Now the k new guests can move into the rooms numbered from 1 to k.

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(c) Brito, the owner of the hotel, decided to close all the even numbered rooms for maintenance. Show that all the guests can remain in the hotel.

Solution: First, number the rooms with positive integers, starting from 1. For all guests with room numbers larger than 1, move them from their original room n to room $2n-1$. $2n-1 = 2k+1$, where $k = n-1 \in \mathbb{Z}$, therefore $2n-1$ is odd. Since 1 is odd and $2n-1$ is odd for all $n \in \mathbb{Z}$, all even numbered rooms are empty and can be closed for maintenance.

(d) A countable infinite number of buses, each containing a countable infinite number of guests, arrive at the hotel, show that the arriving guests can be accommodated without evicting any of the current guests.

Solution: Number the rooms with positive integers, starting from 1. Also number the buses and the passengers in each bus the same way. Move all current guests from their current room n to room 2^n. Then, for the ith passenger on the jth bus, place them in room $(p_j)^i$, where p_j is the jth odd prime number. For example, the second passenger on the second bus would be in room $5^2 = 25$.

To earn full credit, carefully describe in each part how the hotel goes about accommodating the new guests.