Practical works – n^{o}6–7
Filtering and Recursive filtering

1 Filtering

• **Exercice 1** *Filters response*

1.1 For Butterworth, Chebychev-I filters plot the magnitude resonse of Low-Pass, High-Pass, Band-Pass and Band-Stop of the 3^{th} order fiters.

Note: Use the same set of cutoff frequencies for all different types of filters and plot the fresponses of each filters in one figure (4 subplots).

1.2 Plot the low-pass response of the Chebychev-I filter while increasing the order of the filter $[3, 5, 10, 20]$ (ripple = 1.0). Discuss your observations.

Remarque 1 In case you are interested check the lowpass response of elliptic and chebychev-II filters.

2 Recursive Filtering

We consider the derivative filters defined by:

$$f_n(x) \quad = \quad -\text{sign}(x)^{n+1}\frac{s_j^{n+1}}{n!}x^n e^{-s|x|} \tag{1}$$

where n is the filter index and s the scale factor. Each filter is normalized in amplitude:

$$\left|\int_{-\infty}^{0} f_n(\tau)d\tau\right| = \left|\int_{0}^{+\infty} f_n(\tau)d\tau\right| = 1 \tag{2}$$

A smoothing filter is defined by integrating the corresponding derivative filter (same index n):

$$h_n(x) \quad = \quad e^{-s|x|}\sum_{i=0}^{n}\frac{s^{n-i}}{(n-i)!}|x|^{n-i} \tag{3}$$

These filters, being defined by continuous functions, have an infinite support. Filtering operations will imply a truncation of their impulse response, the width of this impulse response varying according to the scale factor. The Z-transform permits to convert these filters in a efficient recursive form. We will focus in this practical work on the second filter ($n = 1$, Canny-Deriche filter):

$$f_1(x) \quad = \quad -s^2 x e^{-s|x|} \tag{4}$$

$$h_1(x) \quad = \quad (1 + s|x|)e^{-s|x|} \tag{5}$$

The recursive form is obtained in three steps: sampling, Z transformation to design the transfer function, Z transformation^{-1} to obtain the difference (recursive) equation.

Sampling:

$$f_1[k] \quad = \quad -s^2 k T_s e^{-sT_s|k|} \tag{6}$$

$$h_1[k] \quad = \quad (1 + sT_s|k|)e^{-sT_s|k|} \tag{7}$$

Noting $\alpha = sT_s$ and $a = e^{-\alpha}$:

$$f_1[k] \quad = \quad -s\alpha k a^{|k|} \tag{8}$$

$$h_1[k] \quad = \quad (1 + \alpha|k|)a^{|k|} \tag{9}$$

Transfer function:
We now apply the unilateral Z-transform on the causal parts of these filters:

$$\begin{aligned} f_1^+[k] &= -s\alpha k a^k u[k] && (10)\\ h_1^+[k] &= (1+\alpha k)a^k u[k] && (11) \end{aligned}$$

where $u[k]$ is the Heaviside (step) function. The Z-tranform are as follows:

$$\begin{aligned} F_1^+(z) &= -s\alpha\frac{az^{-1}}{(1-az^{-1})^2} && (12)\\ H_1^+(z) &= \frac{1}{1-az^{-1}}+\alpha\frac{az^{-1}}{(1-az^{-1})^2} && (13) \end{aligned}$$

The anti-causal parts are given by:

$$\begin{aligned} F_1^-(z) &= -F_1^+(z^{-1})-f_1[0] \quad f_1 \text{ is an odd function} && (14)\\ H_1^-(z) &= H_1^+(z^{-1})-h_1[0] \quad h_1 \text{ is an even function} && (15) \end{aligned}$$

$f_1(0)=0$ and $h_1(0)=1$ are subtracted because already included in the causal parts. The causal transfer functions become:

$$\begin{aligned} F_1^+(z)=\frac{Y_F^+(z)}{X^+(z)} &= -s\alpha\frac{az^{-1}}{(1-az^{-1})^2} && (16)\\ H_1^+(z)=\frac{Y_H^+(z)}{X^+(z)} &= \frac{1+az^{-1}(\alpha-1)}{(1-az^{-1})^2} && (17) \end{aligned}$$

or :

$$\begin{aligned} (1-az^{-1})^2Y_F^+(z) &= -s\alpha a z^{-1}X^+(z) && (18)\\ (1-az^{-1})^2Y_H^+(z) &= (1+az^{-1}(\alpha-1))X^+(z) && (19) \end{aligned}$$

Concerning the anti-causal transfer functions:

$$\begin{aligned} F_1^-(z) &= s\alpha\frac{az^1}{(1-az^1)^2} && (20)\\ H_1^-(z) &= \frac{a(\alpha+1)z-a^2z^2}{(1-az^{-1})^2} && (21) \end{aligned}$$

Recursive form:
Applying now the Z-transform^{-1}, the causal recursive forms are:

$$\begin{aligned} y_k^{+F}-2ay_{k-1}^{+F}+a^2y_{k-2}^{+F} &= -s\alpha a x_{k-1} && (22)\\ y_k^{+H}-2ay_{k-1}^{+H}+a^2y_{k-2}^{+H} &= x_k+a(\alpha-1)x_{k-1} && (23) \end{aligned}$$

For the anti-causal parts:

$$\begin{aligned} y_k^{-F}-2ay_{k+1}^{-F}+a^2y_{k+2}^{-F} &= s\alpha a x_{k+1} && (24)\\ y_k^{-H}-2ay_{k+1}^{-H}+a^2y_{k+2}^{-H} &= a(\alpha+1)x_{k+1}-a^2x_{k+2} && (25) \end{aligned}$$

Finally, the expected results y of the filtering of x are given by:

$$\begin{aligned} y_k^{t,F} &= y_k^{+F}+y_k^{-F} && (26)\\ y_k^{t,H} &= y_k^{+H}+y_k^{-H} && (27)\\ & && (28) \end{aligned}$$

- **Exercice 2** – *1D filtering*

2.1 Construct a signal $x[k]_{k\in[1,40]}=\delta(k-20)$.

2.2 Apply the causal and the anti-causal parts of the smoothing filter on the signal. Analyze the result. (Keep in mind that you have to provide the value of the scaling.)

2.3 Construct a signal $x[k]_{k\in[1,40]} = u[k-10] - u[k-30]$.

2.4 Apply the causal and the anti-causal parts of the derivative filter on the signal. Analyze the result.

• **Exercice 3** – *Canny-Deriche filtering*

3.1 Load a gray image I_m and display this image.

3.2 Apply the smoothing (derivative) filter along the columns (rows) of the images to obtain the component of the gradient on the horizontal direction.

3.3 Apply the smoothing (derivative) filter along the rows (columns) of the images to obtain the component of the gradient on the vertical direction.

3.4 Write Canny edge detection function. Display the modulus and the phase of the results.