Milstein Approximation for Free Stochastic Differential Equations

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Abstract

This paper derives a new numerical method for approximating Free Stochastic Differential Equations with strong convergence order one. Previously, the authors derived a free variant of the Euler-Maruyama method, which obeys strong convergence order of 0.5. In this paper these results are extended using multiple operator integrals and Taylor expansion of Operator Functions. The new method can be viewed as the free variant of the Milstein-Method for Stochastic Differential Equations. In addition, we generalize the results of the free Euler-Maruyama method to $L_p(\varphi), 1 \leq p \leq \infty$.

Keywords free stochastic differential equations, free probability theory, Euler-Maruyama method, random matrix theory, stochastic differential equations, weak convergence, strong convergence **AMS Codes** 46L53, 46L54, 60H10, 65C30

1 Introduction

Free stochastic differential equations (fSDE) emerged up after D. Voiculescu had developed the concept of free probability in the beginning 80's ([44]). In the sequel several researchers, as P. Biane and R. Speicher ([10], [9]), M. Anshelevich ([5]) showed that in principle the Doeblin-Itô-calculus can be transferred to theses non-commutative differential equations in an appropriate way. Nevertheless, there are certain differences, which indicate that a merely literal translation of the classical stochastic calculus hits its limits. While the classical stochastic differential equations are driven by at most vector-valued stochastic processes (e.g. Brownian motion or Lévy processes), here, the state space is an abstract von Neumann algebra with unital, normal, faithful trace and the driven process is a so called free Brownian motion with values in an abstract finite von Neumann algebra (see section 2). Thus, we have to encounter the non-commutativity. To get a good idea, one should consider the von Neumann algebra of

 $n \times n$ -matrices $M_n(\mathbb{C})$. P. Biane and R. Speicher showed in [10] that asymptotically the GUE random matrices converge to a so called free Brownian motion (see section 2)). In addition, it was I. Nourdin and M. Taqqu [30] who presented a non-commutative version of the central limit theorem, both results showing that the free stochastic equations are a good approximation and helpful modelling tool for the wide-spread used random matrices.

A Picard-Lindelöf-type existence result was first gained by V. Kargin ([20]). Similar to the classical case, it is understandable that solutions of free stochastic equations may not be found explicitly. Hence, numerical methods have come into play to obtain approximation solutions to the underlying fSDE. Looking at the classical counterpart we have the Euler-Maruyama as well as the general Milstein scheme at hand ([21], [25]). One of the major questions concerns the speed of convergence of the numerical iterations and here especially in the strong sense. In [34] the authors developed a free analog of the Euler-Maruyama scheme (fEMM) to converging with order $\frac{1}{2}$ in the strong sense and order one in weak sense. In this paper, we complete the results on the Euler-Mayurama scheme to general $L_p(\varphi)$ -spaces $(1 \le p \le \infty)$ and develop on a free analog of a Milstein scheme in addition.

As known in the classical case the major ingredient in developing higher order methods is a Taylor-like expansion of the underlying functions. So, we need an appropriate tool in the non-commutative case of a von Neumann algebra. Here, the deep result of N. Azamov, A. Carey, P. Dodds and F. Sukochev ([6]) on multiple operator integrals hits the scene (see section 2) and [36]). The Taylor approximation and the representation of the derivative in the operator sense enables us to formulate a free analog of the Milstein scheme (fMM) avoiding any derivative explicitly, as done in the commutative case ([25]). Terms of higher order are represented using multiple operator integrals, which suits well to estimate them properly to gain the speed of convergence order $\gamma=1$ in the strong sense, similar to the commutative classical case of stochastic differential equations. We will show, that the iterated free stochastic integrals in the terms of higher order in the stochastic Taylor expansion can be converted into a product by help of the Itô-formalims, developed by [9]. To do so, it is necessary to commute factors. We will show, that the error due to commuting factors is small enough, such that the convergence order $\gamma=1$ is retained.

At start the convergence rate of the free Milstein method is given only for a single, self-adjoint diffusion term. The extension to the general case is easily possible.

The result in [34] is extended into all L_p -norms. We give an analogous proof since the Milstein approximation is built upon it and to keep the paper more self-contained. It should be mentioned that using different methods quite recently Y.-L. Niu, J.-X. Wei, Z. Yin and D. Weng extended the result on Euler-Mayurama approximation for free stochastic differential equations to stochastic theta methods (see Niu et al. [23]).

Finally we give numerical examples for different cases which show the difference in convergence orders. Just as in the commutative case, for simple diffusion terms the theory and numerical examples show the fEMM has strong convergence order of one in special situations.

The paper is organised as follows. Section 2 and section 3 contains some preliminaries on free stochastic differential equations. Section 4 presents the technique on multiple operator integrals, which in fact serves for the Taylor-like expansion. It also contains an alternative derivation of the free Itô formula. Section 5 contains the definition of fMM based on an iterated free Itô expansion. Section 6 shows the complete result of the strong convergence of the Euler-Mayurama scheme in all non-commutative L_p -spaces $(1 \le p \le \infty)$. Finally in section 7 we intensively use the multiple operators

technique to show the strong convergence of the Milstein approximations based on fMM with convergence rate of $\gamma = 1$. Section 8 is devoted to several examples to show the desired convergence rates of fEMM and fMM numerically.

2 Preliminaries - Free Stochastic Calculus

Consider a classical probability space $(\Omega, \mathcal{F}, \mu)$ and random variables as measurable functions $X:\Omega\to\mathbb{R}$. By taking an algebraic viewpoint, μ -integrable random variables X form an (commutative) algebra, where it is possible to assign an expectation $\mathbb{E}(X)$. This change of viewpoint allows to consider cases, where the random variables are non-commutative. The space $\mathcal{M}_N(\mathbb{C}) = L^\infty\left(\Omega, \mu, \operatorname{Mat}_N(\mathbb{C})\right)$ builds up a star-algebra with the unit matrix as identity and $\varphi(M) = \frac{1}{N}\mathbb{E}(\operatorname{tr}(M))$ as a trace. $\operatorname{Mat}_N(\mathbb{C})$ is the space of $N\times N$ -matrices with complex entries. By the help of non-commutative algebras it is possible to develop non-commutative probability theory. The limits $N\to\infty$ can be handled properly in algebraic structures and lead to fruitful concepts. It turns out that non-commutative probability theory is realized by using operator algebras such as von Neumann algebras. We refer to [43], [10], [4] for setting up non-commutative probability theory and relations to random matrices. To be complete, we give the following general definition (see e.g. [45]).

Definition 2.1. A non-commutative probability space is a pair (A, φ) , where A denotes a von Neumann operator algebra and $\varphi : A \to \mathbb{C}$ a faithful unital normal trace.

We refer to [29, Chapter 8.1] for a short introduction. In a non-commutative probability space, the self-adjoint elements of the algebra are called non-commutative random variables. Since we consider von Neumann algebras with a unital, faithful and normal trace $\varphi: \mathcal{A} \to \mathbb{C}$, we can introduce for $1 \leq p < \infty$ a norm on \mathcal{A} by $\|X\|_p = \varphi(|X|^p)^{\frac{1}{p}}$. The Banach space completion is denoted by $L_p(\varphi)$ (see e.g. [33]). Since the trace is finite we may consider \mathcal{A} as a subset of the predual $L_1(\varphi)$ of the von Neumann algebra $\mathcal{A} = L_{\infty}(\varphi)$. By $\|\cdot\|$ we denote the usual operator norm in \mathcal{A} .

An important property of a non-commutative probability space is stated in [29, Proposition 8.1]. Simply speaking, for a non-commutative random variable $X \in \mathcal{A}^{sa}$, there is a unique probability measure on \mathbb{R} with compact support having the same moments as X.

The notion of independence of classical random variables is extended to the non-commutative setting by the concept of freeness of subalgebras of \mathcal{A} . Let $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be a family of $n \in \mathbb{N}$ subalgebras of \mathcal{A} . They are called freely independent (or simply free) in the sense of Voiculescu, if $\varphi(X_1 X_2 \ldots X_m) = 0$ whenever the following conditions

1.
$$X_j \in \mathcal{A}_{i(j)}$$
, where $i(1) \neq i(2), i(2) \neq i(3), \dots, i(n-1) \neq i(n), j = 1, \dots, m$
2. $\varphi(X_i) = 0$ for all $i = 1, \dots, n$

hold ([26, Definition 11]). If $X \in \mathcal{A}$ is a self adjoint element, then there is a unique spectral measure μ on \mathbb{R} so that the moments of X are the same as the moments of the probability measure μ defined by $\varphi(X^k) = \int_{\mathbb{R}} x^k d\mu(x)$, see [26, pp. 51]. An important role in the subsequent plays the Cauchy transform G_X of μ defined by $G_X(z) = \int_{\mathbb{R}} \frac{d\mu(x)}{x-z}$, which is an analytic function defined on \mathbb{C}^+ with values in \mathbb{C}^+ . The Cauchy transform G_X is the expectation of the resolvent of X, i.e. $G_X(z) = \varphi\left((X-z)^{-1}\right)$. The Cauchy transform carries all the properties of the spectral probability distribution of the self-adjoint operator X. In [10] and [20] it is shown how fSDEs can be handled by

a corresponding deterministic partial differential equations of the Cauchy transform G_X . We will strongly depend on these results since it allows us to check the numerical results.

2.1 Free Brownian Motion

Motivated by the concept of classical Brownian motion the definition within non-commutative probability is as follows. Consider a von Neumann algebra \mathcal{A} with a faithful normal trace $\varphi: \mathcal{A} \to \mathbb{C}$. A filtration $\mathbb{F} = (\mathcal{A}_t)_{t \geq 0}$ is a family of subalgebras \mathcal{A}_t of \mathcal{A} with $\mathcal{A}_s \subset \mathcal{A}_t$ for $s \leq t$. A free stochastic process is a family of elements $(X_t)_{t \geq 0}$ for which the increments $X_t - X_s$ are free with respect to the subalgebra \mathcal{A}_s . A process $(X_t)_{t \geq 0}$ is called adapted to the filtration \mathbb{F} if $X_t \in \mathcal{A}_t$ for all $t \geq 0$.

Definition 2.2. A free Brownian motion is a family $(W_t)_{t\geq 0}$ of self-adjoint elements in A, which admits the properties

- 1. $W_0 = 0$.
- 2. The increments $W_t W_s$ are free of W_s for all $0 \le s < t$. The subalgebra W_s is the smallest von Neumann algebra containing W_τ with $0 < \tau < s$.
- 3. The increment $W_t W_s$ has a semicircle distribution with mean 0 and variance t-s for all $0 \le s < t$.

We define the filtration $\mathbb{F} = (\mathcal{W}_t)_{t \geq 0}$ where \mathcal{W}_t is generated by all elements $W_s, s \leq t$.

Remark 2.3. Free Brownian motion $(W_t)_{t\geq 0}$ can be viewed as the limit $N\to\infty$ of $N\times N$ hermitian random matrices having classical independent Brownian motion entries $b_{ij}(t)$ (see [5], [10]). Considering the symmetric N-dimensional quadratic random matrix $W_t^N:=\frac{1}{\sqrt{N}}\left(b_{ij}(t)\right)_{N\times N}$, the limit $\lim_{N\to\infty}W_t^N$ defines an element W_t in a von Neumann algebra $\mathcal A$ with trace $\varphi(\cdot)=\lim_{N\to\infty}\mathbb E(\frac1N tr(\cdot))$. Free Brownian motion as a random variable in a von Neumann algebra $\mathcal A$ is uniformly bounded in operator norm. The corresponding measure, the semicircle distribution, has compact support ([29, chapter 8.1]). This is a major difference to the commutative case and has strong impact on the proofs of strong convergence properties of fEMM and fMM.

2.2 Stochastic Integration with Respect to Free Brownian Motion

Let $(W_t)_{t\geq 0}$ be a free Brownian motion. Let $a,b:[0,T]\to \mathcal{A}$ be mappings such that $\|a(\cdot)\|\|b(\cdot)\|\in L_2([0,T])$ and $a(t),b(t)\in \mathcal{W}_t$. We shorten the notation $a(t)=a_t,b(t)=b_t$ in the following, if there is no danger of confusion. Under these assumptions it is possible to define an Itô-style free stochastic integration with respect to free Brownian motion (see [10] and [5]). We follow [20]. Let $t_0=0\leq t_1\leq \cdots \leq t_n=t$ a decomposition of [0,t] with $a_i=a_{\tau_i},b_i=b_{\tau_i}$ for $0\leq \tau_i\leq t_i$. The decomposition by t_i and t_i is simply denoted by t_i . Let t_i 0 sup t_i 1 sup t_i 2 is t_i 3. Then the operator norm limit

$$\int_0^t a_s dW_s b_s = \lim_{d(\Delta) \to 0} \sum_{i=0}^{n-1} \Delta a_{\tau_i} \Delta W_i b_{\tau_i}$$

defines the free stochastic integral. For details of the definition and conditions for the existence and properties we refer to [20, subsection 2.2], [5, chapter 3], [10]. The free stochastic integral fulfills a free analog of Burkholder-Gundy martingale inequality (Section 3.2. in [10]), i.e.

$$\left\| \int_0^t a_s dW_s b_s \right\| \le 2\sqrt{2} \left(\int_0^t \|a_s\|^2 \|b_s\|^2 ds \right)^{\frac{1}{2}}.$$
 (2.1)

Hence, the free Burkholder-Gundy inequality implies $\left\| \int_{\tau}^{t} a_{s} dW_{s} b_{s} \right\| = O(\sqrt{t-\tau})$.

2.3 Free Itô Formula and - Process

An important ingredient in the development of numerical methods for fSDEs and their convergence properties is a free analog of the Itô-formula (see e.g., [10, Section 4], [22], [5], [20]). In terms of stochastic integrals the stochastic product rule is given in [10, Theorem 4.1.2] and can simply be written in differential form as (see [20])

$$a_t dW_t b_t \cdot c_t dW_t d_t = \varphi \left(b_t c_t \right) a_t d_t dt. \tag{2.2}$$

In the important case $a_t = c_t = d_t = 1$ this yields the formal rules $dW_t b(X_t) dW_t = \varphi\left(b(X_t)\right) dt$ and $dW_t dW_t = dt$. As in the classical case we can apply the relations $dt^2 = 0$, $dW_t dt = 0$.

In the following we restrict ourselves to self-adjoint elements $a_t, b_t, c_t, d_t \in \mathcal{A}$ and denote the set of self-adjoint elements of \mathcal{A} by \mathcal{A}^{sa} .

Definition 2.4. Let $(W_t)_{t\geq 0}$ be a free Brownian motion and \mathbb{F} it's natural filtration. An adapted mapping $X_t: [0,T] \to \mathcal{A}^{sa}$ is called a free Itô-process, if there are operator valued functions $a_i, b_i, c_i: [0,T] \to \mathcal{A}^{sa}$ and an element $X_0 \in \mathcal{A}^{sa}_0$ so that

$$X_t = X_0 + \int_0^t a(s)ds + \sum_{i=1}^d \int_0^t b^i(s)dW_s c^i(s).$$
 (2.3)

Remark 2.5. If X_0 is a self-adjoint element, for X_t to be self-adjoint, it is required that a(t) and the sum $S = \sum_{i=1}^d \int_0^t b^i(s) dW_s c^i(s)$ is self-adjoint for each $t \in [0,T]$.

A simple calculation shows, that free Itô Formula (2.2) (in integral form see [10, Theorem 4.1.2]) implies the following $L_2(\varphi)$ isometry ($\tau < t$),

$$\left\| \int_{\tau}^{t} b_{s} dW_{s} c_{s} \right\|_{2}^{2} = \int_{\tau}^{t} \|c_{s}\|_{2}^{2} \|b_{s}\|_{2}^{2} ds. \tag{2.4}$$

Note that this equality implies that $\left\| \int_{\tau}^{t} b_{s} dW_{s} c_{s} \right\|_{2} = O(\sqrt{t-\tau})$.

3 Free Stochastic Differential Equations (fSDEs)

Definition 3.1. Let X_0 be a self-adjoint element in \mathcal{A}^{sa} and $a, b^i, c^i : \mathcal{A} \to \mathcal{A}$ continuous functions in the operator norm (resp. in the $L_p(\varphi)$ -norm) such that $a(\mathcal{A}^{sa}) \subset \mathcal{A}^{sa}$. We call

$$dX_{t} = a(X_{t})dt + \sum_{i=1}^{d} b^{i}(X_{t})dW_{t}c^{i}(X_{t})$$
(3.1)

a (formal) free Stochastic Differential Equation (fSDE). A solution to (3.1) with initial condition $X(0) = X_0$ is a process $(X_t)_{t \ge 0}$ with the following properties:

- 1. $X(0) = X_0$ is a self-adjoint element in \mathcal{A}_0^{sa}
- 2. $X_t \in \mathcal{A}_t^{sa}$ for all $t \geq 0$
- 3. The equation

$$X_{t} = X_{0} + \int_{0}^{t} a(X_{s})ds + \sum_{i=1}^{d} \int_{0}^{t} b^{i}(X_{s})dW_{s}c^{i}(X_{s})$$
(3.2)

is fulfilled for all $t \geq 0$.

Remark 3.2. Due to the continuity of a, b^i, c^i the integrals in (2.3) are well defined. It should be noted that these functions can be taken from more general spaces (see [10]), but for our purposes the continuity requirement is sufficient. We only consider the autonomous case, where a, b^i, c^i do not explicitly depend on t.

In order to fullfil the self-adjoint condition, the function b^i, c^i cannot be chosen arbitrarily. Either $b^i = c^i$, or for each term with $b^i \neq c^i$, we need a symmetric equivalent, i.e. for each $b^i dW_s c^i$ there must be a term $c^i dW_s b^i$. We therefore rewrite (3.2) as

$$X_{t} = X_{0} + \int_{0}^{t} a(X_{s})ds + \sum_{i=1}^{d_{1}} \int_{0}^{t} b^{i}(X_{s})dW_{s}b^{i}(X_{s}) +$$

$$+ \sum_{j=1}^{d_{2}} \left(\int_{0}^{t} b^{j}(X_{s})dW_{s}c^{j}(X_{s}) + \int_{0}^{t} c^{j}(X_{s})dW_{s}b^{j}(X_{s}) \right)$$
(3.3)

Remark 3.3. An existence and uniqueness theorem for fSDEs and several examples are given in [20]. These results rely on locally operator-Lipschitz functions a, b^i, c^i . The existence proofs in [20], originally formulated in operator norm, can easily be formulated in $L_2(\varphi)$ by applying (2.4) instead of the free Burkholder-Gundy inequality. The solution X_t is therefore uniformly bounded in $\|\cdot\|_2$ and operator norm, which is a significant difference to commutative SDEs, for which boundedness in operator norm is not necessarily given. The boundedness property of X_t will play a major role in the proofs of strong convergence properties in the following.

As an initial example consider the free analog of the Ornstein-Uhlenbeck process (see [20]) defined by the fSDE

$$dX_t = \theta X_t dt + \sigma dW_t, \ t \ge 0, \ \theta, \sigma \in \mathbb{R}. \tag{3.4}$$

Spectral information about the solution can be obtained by taking the Cauchy transform G of the self-adjoint element X_t . G fulfills a deterministic partial differential equation ([20, Proposition 3.7]).

Applying the Stieltjes inversion formula (see [20]) to its solution, it is possible to recover the spectral distribution of X_t . In the case $\theta < 0$ it turns out that the density of X_t is a semicircle distribution with radius

$$R(t) = \sqrt{\frac{2\sigma^2}{|\theta|}(1 - e^{-2|\theta|t})}.$$

For $t \to \infty$ the probability distribution function (PDF) converges to a semicircle with radius $\sigma \sqrt{\frac{2}{|\theta|}}$. The case $\theta \ge 0$ is treated in the same way. For more examples we refer to [20].

4 Operator Integrals, Free Itô-Formula in Functional Form

For functions with certain properties, which will be defined below, it is possible to give the formentioned Taylor approximation with appropriate remainder term [36, Chapter 5.4] and derive [10, Proposition 4.3.4] directly from a Taylor expansions of operator functions. Let $W_n(\mathbb{R})$ be the set of functions $f \in C^n(\mathbb{R})$, such that the k-th derivative $f^{(k)}$, $k = 0, \ldots, n$ is the Fourier transform of a finite measure m_f on \mathbb{R} (see [6, pp. 243]). At this point we use the results in [6, Corollary 5.8], which allow to apply Taylor's formula to $f \in W_n(\mathbb{R})$. Note that f can be taken from more general spaces (see also [28], [27], [11]), but for our purpose $W_n(\mathbb{R})$ is sufficient. Consider $[r,t] \subseteq \mathbb{R}, r \geq 0$, divided into n intervals. Write

$$f(X_t) - f(X_r) = \sum_{i=0}^{n-1} f(X_{i+1}) - f(X_i)$$
(4.1)

Applying the Taylor series expansion [6, Corollary 5.8], for $f \in W_3(\mathbb{R})$ we obtain

$$f(X_{i+1}) - f(X_i) = T_{f_1^{[1]}}^{X_i, X_i}(\Delta X) + T_{f_2^{[2]}}^{X_i, X_i, X_i}(\Delta X, \Delta X) + O(\|\Delta X\|_2^3)$$
(4.2)

by setting $\Delta X = X_{i+1} - X_i$. The definition of multiple operator integrals $T_{f^{[1]}}, T_{f^{[2]}}$ is given in [6, Definition 4.1] and [6, Lemma 4.5]. To repeat, for $X, Y \in \mathcal{A}$,

$$\begin{split} T_{f^{[1]}}^{X,X}(Y) &= \int_{\Pi^{(2)}} e^{i(s_0-s_1)X} Y e^{is_1X} d\nu_f^{(2)}(s_0,s_1), \\ T_{f^{[2]}}^{X,X,X}(Y,Y) &= \int_{\Pi^{(3)}} e^{i(s_0-s_1)X} Y e^{i(s_1-s_2)X} Y e^{is_2X} d\nu_f^{(3)}(s_0,s_1,s_2). \end{split}$$

The definition of the set $\Pi^{(n)}$ and the measure $\nu_f^{(n)}$ can be found in [6, Lemma 2.1].

Theorem 4.1 (Free Itô Formula in Integral Form). Suppose a,b,c are continuous functions $\mathcal{A} \to \mathcal{A}$ in the operator norm such that $a(\mathcal{A}^{sa}) \subset \mathcal{A}^{sa}, b(\mathcal{A}^{sa}) \subset \mathcal{A}^{sa}, c(\mathcal{A}^{sa}) \subset \mathcal{A}^{sa}$. Furthermore b,c are so that the product $b(X_t)dW_tc(X_t)$ is self-adjoint (resp. the sum for d > 1, see Definition 3.1). Let $(X_t)_{t\geq 0}$ be a free Itô-process and $X_0 \in \mathcal{A}_0^{sa}$ be a self-adjoint element. Then for functions $f \in W_3(\mathbb{R})$, it follows that

$$f(X_t) = f(X_0) + \int_0^t L^0 [f(X_s)] ds + L^1 [f(X_s)]_0^t$$
(4.3)

where the operators $L^0, L^1: \mathcal{A}^{sa} \to \mathcal{A}^{sa}$ are introduced as an abbreviation for the expressions

$$L^{0}[f(X_{s})] = T_{f^{[1]}}^{X_{0},X_{0}}(a_{s}) + \sum_{(i,j)\in\{1,\dots,d\}^{2}} T_{f^{[2]}}^{X_{0},X_{0},X_{0}}(b_{s}^{i}dW_{s}c_{s}^{i},b_{s}^{j}dW_{s}c_{s}^{j})$$
(4.4)

and

$$L^{1}[f(X_{s})]_{0}^{t} = \sum_{j=1}^{d} \int_{0}^{t} T_{f^{[1]}}^{X_{0}, X_{0}} (b_{s}^{j} dW_{s} c_{s}^{j}).$$

$$(4.5)$$

We end this section by an extension of [36, Theorem 5.1.4], which will be central in defining the free analog of the Milstein method to be developed in 5. Let I be an interval on the real line. Define $Lip(I) = \{f: I \to \mathbb{C}, \sup_{t,s \in I, t \neq s} \frac{|f(t) - f(s)|}{|t-s|} < \infty\}$, the set of all complex-valued Lipschitz functions on an interval I.

Lemma 4.2. Let $A, B, U, X \in \mathcal{A}$ self-adjoint with $\sigma(A) \cup \sigma(B) \subseteq [a, b]$. If $f \in Lip([a, b])$, then

$$Uf(A)X - XUf(B) = T_{f^{[1]}}^{A,B}(UAX - XUB).$$

Proof. By the spectral theorem it follows that

$$Uf(A)X = \int_{\sigma(A)} Uf(\lambda)d(E_{\lambda})X = \int_{\sigma(A)} Uf(\lambda)d(E_{\lambda})(X)$$
$$XUf(B) = X \int_{\sigma(B)} Uf(\mu)d(F_{\mu}) = \int_{\sigma(B)} Uf(\mu)d(F_{\mu})(X)$$

Then the proof of [36, Theorem 5.1.4] readily carries over.

5 Definition of the free analog of the Euler-Maruyama (fEMM) and Milstein Method (fMM)

In this section we give a motivation for fEMM and fMM out of a free stochastic Taylor expansion of X_t . Since we extend the results in [34] to $L_p(\varphi)$ for $p \in [1, \infty]$, we partially repeat the Taylor expansion in the following. The expansion is also necessary to derive fMM and for the proof of strong convergence. In order to define a numerical approximation with higher strong convergence order than fEMM, we extend [34] and perform one addition iteration step in the stochastic Taylor expansion. As in the commutative case, we then discretize suitable terms in this free stochastic expansion in order to obtain strong convergence order of 1. Due to the non-commutativity, the free Itô formula and the multiple operator integrals the derivation is different.

The chapter is organized as follows. First we give the iterated stochastic Taylor expansion. As a next step we repeat [34] to define fEMM. We continue the expansion in order to define a free variant of the Milstein method (fMM). We give a discretization of operator integrals based on the Taylor formula [6, Corollary 5.8].

We will show that only in the case d=1 (resp. $b_t=c_t$), it is possible to resolve the iterated free stochastic integrals into a non-iterated product. As a consequence, fMM is first derived for the case d=1. The convergence proofs are given in section 6 and section 7. The case d>1 is then addressed in section 5.2, where we will show how to handle the general case by fMM based on the method derived for d=1.

Now consider the free Itô process (2.3) over the time interval of length Δt ,

$$X_{t+\Delta t} = X_t + \int_t^{t+\Delta t} a(X_s) ds + \sum_{i=1}^d \int_t^{t+\Delta t} b^i(X_s) dW_s c^i(X_s).$$
 (5.1)

Assuming $a, b^i, c^i \in W_3(\mathbb{R})$ we can apply the free Itô formula (4.3) for $f = a, b^i, c^i$ in (5.1). This yields an iterated free Itô formula which allows to motivate and define a free analog of the Euler-Maruyama method. Using the abbreviations $a(X_t) = a_t$

(similar notation for b^i, c^i) and $t_1 = t + \Delta t$ we obtain

$$X_{t_{1}} - X_{t} = \int_{t}^{t_{1}} a_{t} ds + \int_{t}^{t_{1}} \int_{t}^{s} L^{0}[a_{u}] du ds + \int_{t}^{t_{1}} L^{1}[a_{u}]_{t}^{s} ds + \sum_{i=1}^{d} \int_{t}^{t_{1}} \left\{ \left(b_{t}^{i} + \int_{t}^{s} L^{0}[b_{u}^{i}] du + L^{1}[b_{u}^{i}]_{t}^{s} \right) \right) dW_{s} \left(c_{t}^{i} + \int_{t}^{s} L^{0}[c_{u}^{i}] du + L^{1}[c_{u}^{i}]_{t}^{s} \right) \right\}$$

$$(5.2)$$

Since a_t, b_t^i, c_t^i do not depend on the integration variable s, we rewrite (5.2) as

$$X_{t_1} - X_t = a_t \Delta t + \sum_{i=1}^d b_t^i (W_{t_1} - W_t) c_t^i + \sum_{i=1}^d M_t^i(t_1) + \sum_{i=1}^d R_t^i(t_1),$$
 (5.3)

where

$$M_t^i(t_1) = \int_t^{t_1} b_t^i dW_s \left(L^1[c_u^i]_t^s \right) + \int_t^{t_1} \left(L^1[b_u^i]_t^s \right) dW_s c_t^i$$
 (5.4)

and

$$\begin{split} R_t^i(t_1) &= \int_t^{t_1} \int_t^s L^0[a_u^i] du \, ds + \int_t^{t_1} L^1[a_u^i]_t^s ds + \\ &+ \int_t^{t_1} b_t^i dW_s \left(\int_t^s L^0[c_u^i] du \right) + \int_t^{t_1} \left(\int_t^s L^0[b_u^i] du \right) dW_s c_t^i + \\ &+ \int_t^{t_1} \left(\int_t^s L^0[b_u^i] du \right) dW_s \left(\int_t^s L^0[c_u^i] du \right) + \\ &+ \int_t^{t_1} \left(\int_t^s L^0[b_u^i] du \right) dW_s \left(L^1[c_u^i]_t^s \right) + \int_t^{t_1} \left(L^1[b_u^i]_t^s \right) dW_s \left(\int_t^s L^0[c_u^i] du \right) + \\ &+ \int_t^{t_1} \left(L^1[b_u^i]_t^s \right) dW_s \left(L^1[c_u^i]_t^s \right). \quad (5.5) \end{split}$$

By the boundedness and continuity of the involved functions a,b^i,c^i the above integrals are well defined.

Since $\|\int_t^{t_1} b_s dW_s c_s\|_2 = \mathcal{O}(\sqrt{\Delta t})$, we have $\|M_t^i(t_1)\|_2^2 = \mathcal{O}((t_1-t)^2)$ (see Section 7) and $\|R_t^i(t_1)\|_2^2 = \mathcal{O}((t_1-t)^3)$. The free Euler-Maruyama method can now be motivated from (5.3) by simply skipping the terms M_t^i and R_t^i . The free Milstein method fMM will be motivated by skipping R_t^i and modifying the terms M_t^i such the iterated free stochastic integrals can be resolved by the Itô formula in product form (see section 5.1).

Definition 5.1 (fEMM). Given T > 0, consider a partition of [0,T] into $L \in \mathbb{N}$ intervals $[t_k, t_{k+1}], k = 0, \ldots, L-1$ with constant step size $\Delta t = \frac{T}{L}$. Define the one-step free Euler-Maruyama approximation (fEMM) \overline{X}_k of the solution $X_{t_k} = X_k$ of (3.1) at $t = t_k \in [0, \Delta T]$ by

$$\overline{X}_{k+1} = \overline{X}_k + a(\overline{X}_k)\Delta t + \sum_{i=1}^d b^i(\overline{X}_k)\Delta W_k c^i(\overline{X}_k), \quad k = 0, 1, \dots, L - 1$$
 (5.6)

with start value $X_0 = \overline{X}_0 \in \mathcal{A}^{sa}$ and $\Delta W_k = W_{k+1} - W_k$. \overline{X}_k denotes the numerical approximation to X_t at timepoint t_k .

To be able to define a free variant of the Milstein method we need to apply (4.3) to (5.4) and (5.5) once more, but we only take the terms b_t^i and c_t^i , which are the functions b, c evaluated at timepoint t. This yields

$$X_{t_1} = X_t + a_t \Delta t + b_t \Delta W_t c_t + \sum_{i=1}^d m_t^i(t_1) + \sum_{i=1}^d \rho_t^i(t_1),$$
 (5.7)

where

$$m_t^i(t_1) = \int_t^{t_1} b_t^i dW_s \left(T_{c^{i,[1]}}^{X_t, X_t} \left(\sum_{j=1}^d b_t^j \int_t^s dW_u c_t^j \right) \right) + \int_t^{t_1} \left(T_{b^{i,[1]}}^{X_t, X_t} \left(\sum_{j=1}^d b_t^j \int_t^s dW_u c_t^j \right) \right) dW_s c_t^i.$$
 (5.8)

Note that the functions b_t^j , c_t^j in the operator integrals in m_t^i in (5.8) do not depend on the integration variable u, as in the case for M_t^i (resp. R_t^i).

As mentioned in the introduction of this section, we will now derive the Milstein method fMM in the case d=1. The general case will be discussed in section 5.2.

5.1 The case d = 1, $b_t = c_t$

We develop a numerical method for the fSDE

$$dX_t = a_t dt + b_t dW_t b_t.$$

In Chapter 7 we prove strong convergence order of 1.

In the commutative case, the development of stochastic Taylor methods ([21]) of higher order requires to deal with iterated stochastic integrals. As we can see in (5.9), in the case of non-commutativity we end up in such iterated integrals, too. We are looking for a derivative free method, which, comparable to the Milstein method in the commutative setting, does not require to calculate iterated integrals. This is also due to the fact, that derivatives are expressed by operator integrals. Our strategy is finding cases, where the sum of iterated stochastic integrals can be converted into a product by applying the free Itô rule in product form ([11, Theorem 4.1.2]). This will be shown in the following.

The case d=1 allows a simplification of (5.8) in the form

$$m_{t}(t_{1}) = \int_{t}^{t_{1}} b_{t}dW_{s} \left(T_{b^{[1]}}^{X_{t},X_{t}} \left(b_{t} \int_{t}^{s} dW_{u}b_{t} \right) \right) +$$

$$+ \int_{t}^{t_{1}} \left(T_{b^{[1]}}^{X_{t},X_{t}} \left(b_{t} \int_{t}^{s} dW_{u}b_{t} \right) \right) dW_{s}b_{t} =$$

$$= b_{t} \int_{t}^{t_{1}} dW_{s} \left(T_{b^{[1]}}^{X_{t},X_{t}} \left(b_{t} \int_{t}^{s} dW_{u} \right) \right) b_{t} +$$

$$+ b_{t} \int_{t}^{t_{1}} \left(T_{b^{[1]}}^{X_{t},X_{t}} \left(\int_{t}^{s} dW_{u}b_{t} \right) \right) dW_{s}b_{t}. \quad (5.9)$$

The goal is to modify (5.9) in a way, to get rid of the iterated stochastic integrals by applying the Itô formula in product form [11, Theorem 4.1.2]. This theorem allows to

convert a sum of iterated free stochastic integrals into a product. This requests that we push the outer stochastic integral into the double operator integral $T_{b^{[1]}}^{X_t, X_t}$. Due to non-commutativity, this is not possible in general. The strategy is, based on [36, Theorem 5.1.4] and [36, Theorem 5.1.5], to transform (5.9) into

$$m_t(t_1) = b_t \left(T_{b^{[1]}}^{X_t, X_t} \left(\int_t^{t_1} dW_s \int_t^s b_t dW_u + \int_t^{t_1} \int_t^s dW_u b_t dW_s \right) \right) b_t + \tilde{R}. \quad (5.10)$$

The term \tilde{R} counts for pushing dW_s into the operator integral $T_{b^{[1]}}^{X_t,X_t}(\cdot)$, which requires to commute dW_s with terms of the form $e^{i\sigma X}$, where $\sigma \in \Pi$ and $X \in \mathcal{A}$. This will be shown in Section 5.3, resp. Proposition 5.5.

We will show that \tilde{R} either consists of terms $\mathcal{O}(\Delta t || X_t - \overline{X}_t ||_2^2)$ or $\mathcal{O}(\Delta t^3)$, which allows to enable the Gronwall argumentation also for the proof of fMM.

Now starting from (5.10), the free Itô-formula in product form (see [10, Theorem 4.1.2], [20, Formula (9)] allows to convert the sum of the iterated stochastic integrals in (5.10) to a product of stochastic integrals over the interval $[t, t_1]$. The Itô-formula in product form states

$$\int_{t}^{t_{1}} dW_{s} \left(\int_{t}^{s} b_{t} dW_{u} \right) + \int_{t}^{t_{1}} \left(\int_{t}^{s} dW_{u} b_{t} \right) dW_{s} =$$

$$= \int_{t}^{t_{1}} dW_{s} b_{t} \int_{t}^{t_{1}} dW_{s} - \int_{t}^{t_{1}} \varphi(b_{t}) dt. \quad (5.11)$$

Combining (5.10) and (5.11) we obtain

$$m_t(t_1) = b_t \left(T_{b_1^{[1]}}^{X_t, X_t} \left(\int_t^{t_1} dW_s b_t \int_t^{t_1} dW_s - \int_t^{t_1} \varphi(b_t) dt \right) \right) b_t + \tilde{R}.$$
 (5.12)

Since we are seeking a derivative free method, we make use of Taylor's formula applied to b, see [6, Corollary 5.8] and [36, Theorem 5.4.5] to obtain

$$T_{b^{[1]}}^{X_n, X_n}(V) = b(X_n + V) - b(X_n) + \mathcal{O}(\|V\|_4^2).$$
 (5.13)

where $V = \int_t^{t_1} dW_s b_t \int_t^{t_1} dW_s - \int_t^{t_1} \varphi(b_t) dt$. By applying (5.13) to (5.14) we summarize

$$m_t(t_1) = b_t \left(b \left(X_t + \int_t^{t_1} dW_s b_t \int_t^{t_1} dW_s - \int_t^{t_1} \varphi(b_t) \right) - b(X_t) \right) b_t + R, \quad (5.14)$$

where $R = \tilde{R} + \overline{R}$. The term \overline{R} is the remainder in (5.13). Since $||dW||_4^2 = \mathcal{O}(dt^2)$ we have the nice property that $||\overline{R}||_2^2 = \mathcal{O}(\Delta t^4)$. The $L_2(\varphi)$ - norm of \tilde{R} will be handeled in section 5.3, see Lemma 5.4.

By simply skipping R, we are ready to motivate a free analog of the Milstein method for a fSDE in case d=1. In Section 7 we will show that in this way we obtain a method with strong order of 1.

Definition 5.2 (fMM). Given T > 0 and the fSDE (3.1) with d = 1 and $b_t = c_t$, i.e.

$$dX_t = a(X_t)dt + b(X_t)dW_tb(X_t).$$

Consider a partition of [0,T] into $L \in \mathbb{N}$ intervals $[t_k,t_{k+1}], k=0,\ldots,L-1$ with constant step size $\Delta t = \frac{T}{L}$. Define the one-step free Milstein approximation (fMM)

 \overline{X}_k to the solution X_t on [0,T] by

$$\overline{X}_{k+1} = \overline{X}_k + a(\overline{X}_k)\Delta t + b(\overline{X}_k)\Delta W_k b(\overline{X}_k) + b(\overline{X}_k) \left(b(\hat{X}_{k+1}) - b(\overline{X}_k)\right) b(\overline{X}_k),$$

$$k = 0, 1, \dots, L - 1, \quad (5.15)$$

where $\hat{X}_{k+1} = \overline{X}_k + \Delta W_k b(\overline{X}_k) \Delta W_k - \varphi(b(\overline{X}_k)) \Delta t$. \overline{X}_k denotes the numerical approximation to X_t at timepoint t_k either by (5.6) or (5.15).

Before we go into details to the convergence properties of fMM, we extend the case d=1 to d>1.

5.2 The case d > 1

First, consider a fSDE of the form

$$dX_{t} = a(X_{t})dt + b(X_{t})dW_{t}c(X_{t}) + c(X_{t})dW_{t}b(X_{t}).$$
(5.16)

To handle this case we write (i.e. $b_t = b(X_t)$, etc.)

$$b_t dW_t c_t + c_t dW_t b_t = (b_t + c_t) dW_t (b_t + c_t) - b_t dW_t b_t - c_t dW_t c_t.$$
 (5.17)

We define

$$\sigma_0(x) = b(x)$$

$$\sigma_1(x) = b(x) + c(x)$$

$$\sigma_2(x) = c(x).$$

Then

$$dX_{t} = a(X_{t})dt + \sum_{i=0}^{3} (-1)^{i+1} \sigma_{i}(X_{t}) dW_{t} \sigma_{i}(X_{t})$$
$$= a(X_{t})dt + \sum_{i=0}^{3} \sigma_{i}(X_{t})((-1)^{i+1} dW_{t}) \sigma_{i}(X_{t})$$

Applying fMM on each term $\sigma_i(X_t)((-1)^{i+1}dW_t)\sigma_i(X_t)$ we obtain the formular

$$X_{k+1} = X_k + a(X_k)\Delta t + \sum_{i=0}^{3} (-1)^{i+1} \sigma_i(X_k) \Delta W_k \sigma_i(X_k)$$
$$+ \sum_{i=0}^{3} \sigma_i(X_k) \Big(\sigma_i(\hat{X}_{k+1}^{(i)}) - \sigma_i(X_k) \Big) \sigma_i(X_k), k = 0, \dots, L - 1$$
(5.18)

where $\hat{X}_{k+1}^{(i)} = X_k + \Delta W_k \sigma_i(X_k) \Delta W_k - \varphi(\sigma_i(X_k)) \Delta t$. X_k denotes the numerical approximation to X_t at timepoint t_k either by fMM or (5.2).

5.3 Commuting Milstein-Terms

In this chapter we now discuss the details how to transform (5.9) into (5.10). We further describe the remainder R in (5.14) and estimate the remainder in order to be applicable in the Gronwall argument in the proof of Proposition 7.1. At this stage, we require the numerical solution to be uniformly bounded in $L_2(\varphi)$. This is indeed the case as shown in Lemma 7.2.

We start with the first term in (5.9) and apply the definition of the free stochastic integral, see Section 2.2. To shorten the notation we will make use of the commutator bracket [A, B] = AB - BA for $A, B \in \mathcal{A}$.

$$\int_{t}^{t_{1}} dW_{s} \left(T_{b^{[1]}}^{X_{t}, X_{t}} \left(b_{t} \int_{t}^{t_{i}} dW_{u} \right) \right) =
= \lim_{d(\Delta) \to 0} \sum_{i=0}^{n-1} \Delta W_{i} T_{b^{[1]}}^{X_{t}, X_{t}} \left(b_{t} (W_{i} - W_{t}) \right) =
= \lim_{d(\Delta) \to 0} \sum_{i=0}^{n-1} \Delta W_{i} \iint_{\Pi} e^{i(s_{0} - s_{1})X_{t}} \left(b_{t} (W_{i} - W_{t}) e^{i(s_{0} - s_{1})X_{t}} \right) d\nu_{b} \quad (5.19)$$

Now ΔW_i does not commute with $e^{i(s_0-s_1)X_t}$. We therefore seek to obtain $\Delta W_i e^{i(s_0-s_1)X_t} = e^{i(s_0-s_1)X_t} \Delta W_i + A$ such that A can be estimated to be able to handle it in the Gronwall argument in the proof of Proposition 7.1. Direct application of Theorem [36, 3.3.6] to $\Delta W_i e^{i(s_0-s_1)X_t}$ ends up in a term which has $\mathcal{O}(\Delta t^2)$ in $L_2(\varphi)$ (the norm to the power of 2).

Now consider X_t , the solution of an fSDE 3.1 and the numerical solution \overline{X}_t obtained by fMM. At this stage, we simply write t as a superscript for both X and \overline{X} . As a first step, apply [6, Corollary 5.8] to $\epsilon(\cdot) = e^{i\sigma(\cdot)}$ at $X_t - \overline{X}_t$ ($\sigma = s_0 - s_1$) and obtain

$$\Delta W_i e^{i\sigma X_t} = \Delta W_i e^{i\sigma(X_t - \overline{X}_t)} + \Delta W_i \sum_{j=1}^N T_{\epsilon^{[j]}}^{X_t - \overline{X}_t, \dots, X_t - \overline{X}_t} \left(\overline{X}_t \right) + + R_N(\overline{X}_t). \quad (5.20)$$

with the remainder (see (5.4.7) in [36]). Due to Theorem [36, Theorem 5.4.4], the boundedness of all the derivatives and the bound of the numerical solution it follows that $||R_N(\overline{X}_t)||_2^2 \le c_\epsilon \frac{(i\sigma)^N}{(N-1)!}$. Let

$$N = \min_{M \in \mathbb{N}} \left\{ \left| \frac{c_{\epsilon}(i\sigma)^{M}}{(M-1)!} \right| \le \Delta t \right\}.$$
 (5.21)

Consider the first two summands in (5.20). The first is handeled by Lemma 4.2 with U=id

$$\Delta W_i e^{i\sigma(X_t - \overline{X}_t)} = e^{i\sigma(X_t - \overline{X}_t)} \Delta W_i + T_i.$$

where $T_i = T_{\epsilon}^{X_t - \overline{X}_t, X_t - \overline{X}_t} \left([\Delta W_i, X_t - \overline{X}_t] \right)$. It follows that $||T_i||_2^2 \le c_{\epsilon} \Delta t ||X_t - \overline{X}_t||_2^2$. The second summand in (5.20) can be treated by Lemma 5.3 below, therefore (5.20) results in

$$\Delta W_i e^{i\sigma X_t} = \left(e^{i\sigma(X_t - \overline{X}_t)} + \sum_{j=1}^N T_{\epsilon^{[j]}}^{X_t - \overline{X}_t, \dots, X_t - \overline{X}_t} \left(\overline{X}_t \right) \right) \Delta W_i +$$

$$+ T_i + \sum_{j=1}^{N-1} V_j + \Delta W_i R_N =$$

$$= e^{i\sigma X_t} \Delta W_i + T_i + \sum_{j=1}^{N-1} V_j + [R_N, \Delta W_i].$$

Therefore (5.19) turns into

$$\begin{split} \int_{t}^{t_{1}} dW_{s} \left(T_{b^{[1]}}^{X_{t}, X_{t}} \left(b_{t} \int_{t}^{t_{i}} dW_{u} \right) \right) &= T_{b^{[1]}}^{X_{t}, X_{t}} \left(\int_{t}^{t_{1}} dW_{s} \left(b_{t} \int_{t}^{t_{i}} dW_{u} \right) \right) + \\ &+ \lim_{d(\Delta) \to 0} \sum_{i=0}^{n-1} \iint_{\Pi} \left(\Theta_{1} + \Theta_{2} + \Theta_{3} \right) \left(b_{t} (W_{i} - W_{t}) e^{i(s_{0} - s_{1})X_{t}} \right) d\nu_{b} = \\ &= T_{b^{[1]}}^{X_{t}, X_{t}} \left(\int_{t}^{t_{1}} dW_{s} \left(b_{t} \int_{t}^{t_{i}} dW_{u} \right) \right) + R_{1}^{1} + R_{2}^{1} + R_{3}^{1}. \end{split} (5.22)$$

As a next step we need to estimate the remainders R_1^j , j=1,2,3. We assume, at this point, that the numerical solution \overline{X}_k is bounded in $L_2(\varphi)$ independent of Δt , The bound $\overline{M} > 0$ only depends on the solution X_k .

Lemma 5.3. Consider X_t , the solution to (3.1) and the numerical solution \overline{X}_t . Let $\epsilon : \mathbb{R} \to \mathbb{C}$ defined by $\epsilon(X) = e^{i\sigma X}$, where $\sigma = s_0 - s_1$ (see (5.20)). Then for every $j \in \mathbb{N}$ we have the relation

$$\Delta W T_{\epsilon^{[j]}}^{X_t - \overline{X}_t, \dots, X_t - \overline{X}_t} \left(\overline{X}_t \right) = T_{\epsilon^{[j]}}^{X_t - \overline{X}_t, \dots, X_t - \overline{X}_t} \left(\overline{X}_t \right) \Delta W + V_j,$$

where $||V_j||_2^2 \le C\Delta t ||\overline{X}_t||_2^{2j} ||X_t - \overline{X}_t||_2^2$ and C is independent of j.

Proof. The definition of the multiple operator integrals states that

$$\Delta W T_{\epsilon^{[j]}}^{X_t - \overline{X}_t, \dots, X_t - \overline{X}_t} \left(\overline{X}_t \right) =$$

$$= \iint_{\Pi} \Delta W \underbrace{e^{i(u_0 - u_1)(X_t - \overline{X}_t)} \overline{X}_t e^{i(u_1 - u_2)(X_t - \overline{X}_t)} \dots \overline{X}_t}_{=U} e^{iu_j(X_t - \overline{X}_t)} d\nu_{\epsilon}(u_0, \dots, u_j)$$

Then $||U||_2^2 \leq ||\overline{X}_t||_2^{2j}$. Let $g(x) = e^{ix}$. Using Lemma 4.2, we see that

$$\Delta W T_{\epsilon^{[j]}}^{X_t - \overline{X}_t, \dots, X_t - \overline{X}_t} \left(\overline{X}_t \right) = T_{\epsilon^{[j]}}^{X_t - \overline{X}_t, \dots, X_t - \overline{X}_t} \left(\overline{X}_t \right) \Delta W + V_j$$

where V_i counts for the error due to commuting ΔW ,

$$V_{j} = \iint_{\Pi} T_{g^{[1]}}^{X_{t} - \overline{X}_{t}, X_{t} - \overline{X}_{t}} \left(\Delta W U u_{j} (X_{t} - \overline{X}_{t}) - U u_{j} (X_{t} - \overline{X}_{t}) \Delta W \right) d\nu_{\epsilon}(u_{0}, \dots, u_{j}).$$

Using freeness of ΔW to $U \cdot (X_t - \overline{X}_t)$ and Cauchy-Schwarz we see that

$$||V_{j}||_{2}^{2} \leq \iint_{\Pi} ||T_{g^{[1]}}^{(X_{t} - \overline{X}_{t}),(X_{t} - \overline{X}_{t})}(\Delta W \cdot U \cdot u_{j}(X_{t} - \overline{X}_{t}) - \dots \cdots - U \cdot u_{j} \cdot (X_{t} - \overline{X}_{t})\Delta W||_{2}^{2} d\nu_{\epsilon}(u_{0},\dots,u_{j})$$

$$\leq \iint_{\Pi} C_{2}|u_{j}|^{2}||U||_{2}^{2}||(X_{t} - \overline{X}_{t})||_{2}^{2}||\Delta W||_{2}^{2} d\nu_{\epsilon}(u_{0},\dots,u_{j})$$

$$\leq C\Delta t||\overline{X}_{t}||_{2}^{2j}||X_{t} - \overline{X}_{t}||_{2}^{2},$$

where we used, that the operators integrals $T_{\epsilon^{[j]}}^{X_t - \overline{X}_t, \dots, X_t - \overline{X}_t}$ and $T_{g^{[j]}}^{X_t - \overline{X}_t, \dots, X_t - \overline{X}_t}$ are uniformly bounded according to [36].

We are then ready to estimate the remainders in (5.22).

Lemma 5.4. Assume the numerical solution to be uniformly bounded, $\|\overline{X}_t\| \leq \overline{M}$, independent of Δt . Then for Δt small enough there exist constants $a_i \geq 0$, independent of N resp. Δt such that

$$||R_1^1||_2^2 \leq a_1 \Delta t^2 ||X_t - \overline{X}_t||_2^2$$

$$||R_2^1||_2^2 \leq a_2 \Delta t ||X_t - \overline{X}_t||_2^2$$

$$||R_3^1||_2^2 \leq a_3 \Delta t^3.$$

Proof. The estimation of R_1^1 follows from the fact that $\|\Delta W\|_2^2 = \mathcal{O}(\Delta t)$, $\|\Delta W\|_4^2 = \mathcal{O}(\Delta t^2)$, the Cauchy-Schwarz inequality and the bounds on the multiple operator integrals, e.g. Theorem [36, 5.3.5].

For the estimation of R_2^1 we need to handle $\sum\limits_{j=0}^{N-1}V_j$ first. Lemma 5.3 and the triangle inequality yield

$$\|\sum_{j=0}^{N-1} V_j\|_2^2 \leq \sum_{j=0}^{N-1} C \|\overline{X}_t\|_2^{2j} \Delta t \|X_t - \overline{X}_t\|_2^2 \leq C \|X_t - \overline{X}_t\|_2^2 \Delta t \frac{\overline{M}^{2N+1} - 1}{\overline{M} - 1}.$$

Considering in (5.21), that N depends on Δt , it is possible to choose Δt small enough, such that

$$\Delta t \frac{\overline{M}^{2N+1} - 1}{\overline{M} - 1} \le 1.$$

Due to the bounds of the multiple operator integrals and the factor $W_i - W_t$ in (5.22) the assertion follows.

The estimation of R_3^1 follows directly from the application of Cauchy-Schwarz inequality, freeness arguments and the bounds on the multiple operator integrals.

Now we formulate the main statement regarding (5.14). The remainder term R counts for the error, which is made by commuting the factors, necessary to formulation fMM.

Proposition 5.5. Consider (5.14). Then for Δt small enough, the remainder term R can be estimated by

$$||R||_2^2 \le C_1 \Delta t ||X_t - \overline{X}_t||_2^2 + C_2 \Delta t^3.$$
 (5.23)

Proof. In (5.14) the remainder R is defined as $R = \tilde{R} + \overline{R}$. As already mentioned, the term \overline{R} gives a Δt^4 . The term \tilde{R} in (5.10) consists of the summands R_j^1 in (5.22) estimated by Lemma (5.4) and the similar estimates from the second summand in (5.9). Combining the estimates gives the statement of the proposition.

6 Strong Convergence of fEMM in $\|\cdot\|_p$, $1 \le p \le$

 ∞

In [34] the authors proved strong convergence order of $\gamma=\frac{1}{2}$ of fEMM (5.6) in L_2 -norm under certain assumptions. In addition, we present the proof, since essential techniques are used later in Chapter 7 for the final demonstration of the fMM speed. This section is devoted to the strong convergence of fEMM (5.6) with at least speed of $O(\sqrt{\Delta t})$ in all norms $1 \leq p \leq \infty$ provided the coefficient functions are local Lipschitz in the operator norm. The results will be numerically verified in section 8. We start with the definition of the speed of strong convergence in $L_p(\varphi)$.

Definition 6.1. The numerical approximation fEMM resp. fMM defined by (5.6) is said to converge strongly to the solution X_t of (3.1) in L_p -norm $(1 \le p \le \infty)$ with order $\gamma > 0$, if there is a constant C > 0 independent of Δt , so that

$$\|\overline{X}_k - X_k\|_p \le C(\Delta t)^{\gamma} \tag{6.1}$$

for all k = 0, ..., L and $L \in \mathbb{N}$. X_k denotes the solution (i.e. the process) X_t evaluated at $t_k = k\Delta t \in [0, T]$ and \overline{X}_k the numerical approximation calculated by fEMM or fMM.

Before we start to consider strong convergence properties, we introduce the following notation. We call a function $f: \mathbb{R} \to \mathbb{R}$ locally operator Lipschitz, if it is a locally bounded, measurable function such that for all A > 0, there is a constant $L_f(A) > 0$ such that

$$||f(X) - f(Y)|| \le L_f(A)||X - Y||,$$
 (6.2)

for elements $X,Y\in\mathcal{A}^{sa}$ and $\|X\|,\|Y\|< A$. Examples of operator Lipschitz functions are functions of type $f(x)=\int_{\mathbb{R}}e^{ixy}d\mu(y)$ where μ is a bounded complex measure with certain properties (e.g. [11], [31], [36]). It turns out that C^2 functions are locally operator Lipschitz, but C^1 is not sufficient.

As mentioned in Remark 3.3, the exact solution X_t of the fSDE (3.2) is uniformly bounded in the operator-norm under the assumption of locally operator Lipschitz functions a, b, c of the fSDE. We will prove in the following Proposition 6.2 that under locally operator Lipschitz functions the order of strong convergence of fEMM is $\frac{1}{2}$. In Proposition 6.2 we make the assumption that all numerical solutions \overline{X}_k are uniformly bounded in the operator-norm. This will be shown in Theorem 6.4. The proofs of Proposition 6.2, Theorem 6.4, Lemma 6.5 and Lemma 6.6 are taken from [34] and extended to the operator-norm. We will refer to this proposition in the following.

Proposition 6.2. Consider the fSDE (3.1). Let $a: A \to A$ be an operator function with $a(A^{sa}) \subset A^{sa}$. Additionally let the function a be locally operator Lipschitz with constant $L_a > 0$. The functions b, c are operator functions with the same properties as a. Let X_t be a solution to the fSDE on [0,T]. Furthermore assume that there is a constant $\overline{M} > 0$ such that $\|\overline{X}_k\| < \overline{M}$ for $k = 0, \ldots, L$ and $L \in \mathbb{N}$, i.e. \overline{M} is

independent of the discretization. Then the approximation (5.1) has strong convergence order of $\gamma = \frac{1}{2}$, i.e.

$$\|\overline{X}_k - X_k\| \le C(\Delta t)^{\frac{1}{2}} \tag{6.3}$$

for all k = 0, ..., L. The constant C is independent of step size Δt .

Remark 6.3. In [34] the authors mentioned, that for the implementation on a computer we use fEMM in $\mathcal{A} = \mathcal{M}_N(\mathbb{C})$. The definition of fEMM Definition 5.1, the strong convergence property of $\gamma = \frac{1}{2}$ and weak order of convergence $\gamma = 1$ (see [34]) can be directly carried over to the von Neumann algebra of $N \times N$ random matrices $\mathcal{M}_N(\mathbb{C})$.

Proof of Proposition 6.2. The proof carries over from [34, Proposition 6.2]. Since we use the technique later in section 7 for the fMM scheme, we present the proof in detail. From the fEMM approximation \overline{X}_k at the time point t_k , $k=0,\ldots L$ we define a step process $\overline{X}_t = \overline{X}_k$ for $t_k \leq t < t_{k+1}$. We futher use the abbreviations $a(X_s) = a_s, a(\overline{X}_s) = \overline{a}_s$. Analog notation is used for b, c.

Consider a point $t \in [0, T]$. Let $n_t \in \mathbb{N}$ such that $t \in [t_{n_t}, t_{n_t+1}]$. Then

$$\overline{X}_{t} - X_{t} = \overline{X}_{n_{t}} - X_{t} = \overline{X}_{n_{t}} - \left(X_{0} + \int_{0}^{t} a_{s} ds + \int_{0}^{t} b_{s} dW_{s} c_{s}\right) =
= \sum_{k=0}^{n_{t}-1} (\overline{X}_{k+1} - \overline{X}_{k}) - \int_{0}^{t} a_{s} ds - \int_{0}^{t} b_{s} dW_{s} c_{s} =
= \sum_{k=0}^{n_{t}-1} \overline{a}_{k} \Delta t + \sum_{k=0}^{n_{t}-1} \overline{b}_{k} \Delta W_{k} \overline{c}_{k} - \int_{0}^{t} a_{s} ds - \int_{0}^{t} b_{s} dW_{s} c_{s} \quad (6.4)$$

Due to the definition of the step-wise constant process \overline{X}_t above, we can reformulate the terms $\overline{a}_k \Delta t$ and $\overline{b}_k dW_s \overline{c}_k \Delta t$ as an integrals as follows. We deduce

$$\overline{a}_k \Delta t = a(\overline{X}_k) \Delta t = a(\overline{X}(t_k)) \Delta t = \int_{\Delta_t} \overline{a}_s ds$$

and

$$\overline{b}_k dW_s \overline{c}_k \Delta t = \int_{\Delta t} \overline{b}_s dW_s \overline{c}_s.$$

Note that $\overline{a}_s, \overline{b}_s, \overline{c}_s$ are constant over $[t_k, t_{k+1}[$. Continuing the last line of (6.4) we obtain

$$\overline{X}_t - X_t = \int_0^{t_{n_t}} \overline{a}_s ds + \int_0^{t_{n_t}} \overline{b}_s dW_s \overline{c}_s - \int_0^t a_s ds - \int_0^t b_s dW_s c_s$$

and further

$$\overline{X}_t - X_t = \int_0^{t_{n_t}} (\overline{a}_s - a_s) ds - \int_{t_{n_t}}^t a_s ds + \int_0^{t_{n_t}} \overline{b}_s dW_s \overline{c}_s - \int_0^{t_{n_t}} b_s dW_s c_s - \int_{t_{n_t}}^t b_s dW_s c_s.$$

Then the square of the operator-norm of the difference $\overline{X}_t - X_t$ is

$$\|\overline{X}_{t} - X_{t}\|^{2} = \|\int_{0}^{t_{n_{t}}} (\overline{a}_{s} - a_{s}) ds - \int_{t_{n_{t}}}^{t} a_{s} ds + \int_{0}^{t_{n_{t}}} \overline{b}_{s} dW_{s} \overline{c}_{s} - \int_{0}^{t_{n_{t}}} b_{s} dW_{s} c_{s} - \int_{t_{n_{t}}}^{t} b_{s} dW_{s} c_{s}\|^{2}.$$
 (6.5)

By applying the inequality

$$||X_1 + X_2 + X_3 + X_4||^2 \le 4(||X_1||^2 + ||X_2||^2 + ||X_3||^2 + ||X_4||^2)$$

for $X_1, X_2, X_3, X_4 \in \mathcal{A}^{sa}$, we deduce from (6.5)

$$\|\overline{X}_{t} - X_{t}\|^{2} \leq 4\|\int_{0}^{t_{n_{t}}} (\overline{a}_{s} - a_{s}) ds\|^{2} + 4\|\int_{t_{n_{t}}}^{t} a_{s} ds\|^{2} + 4\|\int_{0}^{t_{n_{t}}} \overline{b}_{s} dW_{s} \overline{c}_{s} - \int_{0}^{t_{n_{t}}} b_{s} dW_{s} c_{s}\|^{2} + 4\|\int_{t_{n_{t}}}^{t} b_{s} dW_{s} c_{s}\|^{2}.$$
 (6.6)

Using the abbreviation

$$v(t) = \left\| \overline{X}_t - X_t \right\|^2$$

and applying Jensen's inequality it follows from (6.6) that

$$v(t) \leq 4T \int_{0}^{t_{n_{t}}} \| (\overline{a}_{s} - a_{s}) \|^{2} ds + 4\Delta t \int_{t_{n_{t}}}^{t} \| a_{s} \|^{2} ds + 4 \| \int_{0}^{t_{n_{t}}} \overline{b}_{s} dW_{s} \overline{c}_{s} - \int_{0}^{t_{n_{t}}} b_{s} dW_{s} c_{s} \|^{2} + 4 \| \int_{t_{n_{t}}}^{t} b_{s} dW_{s} c_{s} \|^{2}.$$
 (6.7)

Estimating the first integral in (6.7) gives (using (6.2))

$$\int_{0}^{t_{n_t}} \|\overline{a}_s - a_s\|^2 ds \le \int_{0}^{t_{n_t}} L_a^2 \|\overline{X}_s - X_s\|^2 ds = L_a^2 \int_{0}^{t_{n_t}} v(s) ds. \tag{6.8}$$

The second integral in (6.7) is an $O(\Delta t)$, since (using (6.2))

$$\int_{t_{n_t}}^{t} \|a_s\|^2 ds \le C_a \int_{t_{n_t}}^{t} (1 + \|X_s\|^2) ds \le C_1 (t - t_{n_t}) \le C_1 \Delta t. \tag{6.9}$$

The constant $C_1 \in \mathbb{R}$ does not depend on Δt . The third integral in (6.7) is estimated as follows.

$$\| \int_{0}^{tn_{t}} \overline{b}_{s} dW_{s} \overline{c}_{s} - \int_{0}^{tn_{t}} b_{s} dW_{s} c_{s} \|^{2} =$$

$$\leq \left\| \int_{0}^{tn_{t}} (\overline{b}_{s} - b_{s}) dW_{s} \overline{c}_{s} \right\|^{2} + \left\| \int_{0}^{tn_{t}} b_{s} dW_{s} (\overline{c}_{s} - c_{s}) \right\|^{2}$$
(6.10)

The two integrals in the last line are further handled by applying the free Burkholder-Gundy inequality of the stochastic integral (see (2.1)). Considering the first integral in the last line above we obtain the following inequality,

$$\left\| \int_{0}^{t_{n_{t}}} (\overline{b}_{s} - b_{s}) dW_{s} \overline{c}_{s} \right\|^{2} \leq 8 \int_{0}^{t_{n_{t}}} \|\overline{b}_{s} - b_{s}\|^{2} \|\overline{c}_{s}\|^{2} ds. \tag{6.11}$$

Using the assumption that a, b, c are locally operator Lipschitz and the boundedness of X_t and \overline{X}_t we estimate further,

$$\left\| \int_{0}^{t_{n_{t}}} (\overline{b}_{s} - b_{s}) dW_{s} \overline{c}_{s} \right\|^{2} \leq 8 \int_{0}^{t_{n_{t}}} \|\overline{b}_{s} - b_{s}\|^{2} \|\overline{c}_{s}\|^{2} ds \leq$$

$$\leq \int_{0}^{t_{n_{t}}} L_{b} L_{c} \|\overline{X}_{s} - X_{s}\|^{2} (1 + \|\overline{X}_{s}\|^{2}) ds \leq K_{1} \int_{0}^{t_{n_{t}}} \|\overline{X}_{s} - X_{s}\|^{2} ds. \quad (6.12)$$

The constant K_1 depends on the Lipschitz constants L_b, L_c and the operator norm of $\|X_s\|^2$ and $\|\overline{X}_s\|^2$, which are uniformly bounded. The constant K_1 does not depend on Δt . Estimating the second integral in the last line of (6.10) in the same way as above we finally obtain

$$\| \int_{0}^{t_{n_{t}}} \overline{b}_{s} dW_{s} \overline{c}_{s} - \int_{0}^{t_{n_{t}}} b_{s} dW_{s} c_{s} \|^{2} \le$$

$$\le K \int_{0}^{t_{n_{t}}} \| \overline{X}_{s} - X_{s} \|^{2} ds = K \int_{0}^{t_{n_{t}}} v(s) ds. \quad (6.13)$$

As for the constant K_1 above, the constant K is independent of Δt . The last stochastic integral in (6.7) is handled by the free Burkholder-Gundy inequality of the stochastic integral to obtain,

$$\|\int_{t_{n_t}}^{t} b_s dW_s c_s\|^2 = \int_{t_{n_t}}^{t} \|b_s\|^2 \|c_s\|^2 ds \le$$

$$\le C_3 \int_{t_{n_t}}^{t} (1 + \|X_s\|^2)^2 ds \le C_4 (t_{n_t} - t) \le C_4 \Delta t. \quad (6.14)$$

Again, since $\sup_{s \in [0,T]} ||X_s||_2 < \infty$ by definition, we have $C_4 < \infty$ and does not depend on Δt . Inserting (6.8), (6.9), (6.10), (6.14) into (6.7) yields

$$v(t) \le 4(TL_a^2 + K) \int_0^{t_{n_t}} v(s)ds + 4C_1L_a\Delta t^2 + 4C_4\Delta t$$

For Δt small enough v(t) fulfills the inequality

$$v(t) \le D\Delta t + E \int_0^{t_{n_t}} v(s)ds.$$

The Gronwall inequality implies $v(t) \leq F\Delta t, F < \infty, t \in [0,T]$. The supremum of the $L_1(\varphi)$ -norm over [0,T] of the error $\overline{X}_t - X_t$ is first estimated for all $1 \leq p < \infty$ by

$$\sup_{0 \le s \le T} \varphi(|\overline{X}_s - X_s|^p)^{\frac{1}{p}} \le \sup_{0 \le s \le T} ||\overline{X}_s - X_s|| \le \sqrt{F} \sqrt{\Delta t}.$$

As written at the beginning of the proof, \overline{X}_t is the stepwise constant process constructed from the numerical solution, i.e. $\overline{X}_t = \overline{X}_k$ for all $t_k \leq t < t_{k+1}$. As stated in Definition 6.1 X_k is the solution process of the fSDE evaluated at t_k . Therefore we have in particular (see [34])

$$\varphi(|\overline{X}_k - X_k|) < C\sqrt{\Delta t}.$$

for all k = 0, ..., L and $L \in \mathbb{N}$.

We now consider the assumption in Proposition 6.2 that the numerical solutions are uniformly bounded. We first formulate the main statement in Theorem 6.4. The proof of Theorem 6.4 is given after Lemma 6.5 and Lemma 6.6. The proof Theorem 6.4 is realized as a mixture of Picard's method as applied in [20, Theorem 3.1] and a step-wise estimation following application of the Gronwall inequality. Lemma 6.6 is important in the proof of Theorem 6.4. Roughly speaking Lemma 6.6 states that we always find a certain time interval, such that the bound of the numerical solution is independent of the discretization. We use this fact to proof Theorem 6.4. The boundedness of the exact solution is again of major importance.

Theorem 6.4. Consider a fSDE (3.1) with the solution X_t on [0,T] (see Remark 3.3). Let \overline{X}_k be a numerical solution calculated by fEMM on [0,T] with a discretization $T=L\Delta t$. a, b and c are operator functions which are locally operator Lipschitz as in Proposition 6.2. Then the numerical solution is uniformly bounded for each $k=0,\ldots,L, L\in\mathbb{N}$, i.e. there is a constant $\overline{M}>0$ such that $\|\overline{X}_k\|<\overline{M}$, where \overline{M} does not depend on L resp. Δt .

Then the fEMM approximation (5.1) has strong convergence order of $\gamma = \frac{1}{2}$, i.e.

$$\|\overline{X}_k - X_k\| \le C(\Delta t)^{\frac{1}{2}}. (6.15)$$

for all k = 0, ..., L. The constant C is independent of step size Δt .

The following two Lemmas are needed in the proof of Theorem 6.4.

Lemma 6.5. Let b, c be two locally operator Lipschitz functions $\mathcal{A}^{sa} \to \mathcal{A}^{sa}$. If $X \in \mathcal{A}^{sa}$ is free from the increment of a Brownian motion $\Delta W = W_{t+\Delta t} - W_t$, then there is a constant $L_{bc} > 0$, such that the estimation

$$||b(X)\Delta Wc(X)||^2 \le 8L_{bc}(1+||X||^2)^2\Delta t$$

holds.

Proof. An easy calculation shows, that the pointwise product of c and b implies that their product is locally operator Lipschitz. We shorten b = b(X), analog for c. Considering that the product $b\Delta Wc$ is self-adjoint, further applying [20, Lemma 3.3] and using free Burkholder-Gundy

$$\|b\Delta W_t c\|^2 = \left\| \int_{\Delta t} b dW_s c \right\|^2 \le 8 \int_{\Delta t} \|b\|^2 \|c\|^2 ds \le 8L_b L_c \left(1 + \|X\|^2\right)^2 \Delta t.$$
 (6.16)

For the following, we construct a piecewise constant process \overline{X}_t defined by $\overline{X}_t = \overline{X}_k$ for $t_k \leq t < t_{k+1}$, where \overline{X}_k is a numerical solution on [0,T] calculated by a stepsize Δt .

Lemma 6.6. Let \overline{X}_k be a numerical solution of a fSDE (3.1) calculated by fEMM given a discretization of [0,T] with $T=L\Delta t$ and \overline{X}_t the piecewise constant process defined above. Let $\tilde{M}=3M^2+1$, where (see (3.3) $M=\sup_{s\in[0,T]}\|X_s\|<\infty$). Suppose

the coefficient functions a,b,c in (3.1) are locally Lipschitz in operator norm. Then there exists a time point $0 < \tilde{T} \leq T$ and a number $K(\tilde{M})$ such that the estimation

$$\|\overline{X}_t\|^2 < 3M^2 + K(\tilde{M})t < \tilde{M}, t \in [0, \tilde{T}]$$
 (6.17)

is valid independent of the discretization. The term $K(\tilde{M})$ is independent of the discretization. The time point \tilde{T} depends on \tilde{M} and the Lipschitz constants of a,b,c.

Proof. Since $||X_0|| \leq M < \tilde{M}$, it is possible to find a timepoint $0 < T^{(L)} \leq T$ such, that for all $t \in [0, T^{(L)}]$ the estimation $||\overline{X}_t|| \leq \tilde{M}$ holds. $T^{(L)}$ depends on the discretization, which is denoted by the superscript L. $T^{(L)}$ also depends on the constant \tilde{M} . Let $\tilde{n} \in \mathbb{N}$ such, that $t_{\tilde{n}} \leq t \leq t_{\tilde{n}+1}$. Then

$$\|\overline{X}_{t}\|^{2} = \left\| \sum_{k=0}^{\tilde{n}-1} (\overline{X}_{k+1} - \overline{X}_{k}) + X_{0} \right\|^{2} = \left\| X_{0} + \int_{0}^{t} \overline{a}_{s} + \int_{0}^{t} \overline{b}_{s} dW_{s} \overline{c}_{s} \right\|^{2}$$

where $\overline{a}_s = a(\overline{X}_s)$ and \overline{X}_s is the piecewise constant process (identical notation for b). Applying the inequality $(u+v+w)^2 \leq 3(u^2+v^2+w^2)$, $u,v,w \in \mathbb{R}$, we obtain the following estimation.

$$\|\overline{X}_{t}\|^{2} \leq 3\|X_{0}\|^{2} + 3\left\|\int_{0}^{t} a\left(\overline{X}_{s}\right) ds\right\|^{2} + 3\left\|\int_{0}^{t} b\left(\overline{X}_{s}\right) dW_{s} c\left(\overline{X}_{s}\right)\right\|^{2} \leq$$

$$\leq 3M^{2} + 3L_{a}(1 + \|\overline{X}_{s}\|^{2})t + 3L_{b}L_{c}(1 + \|\overline{X}_{s}\|^{2})^{2}t \leq$$

$$\leq 3M^{2} + K(\tilde{M})t.$$

Setting $\tilde{T} = (\tilde{M} - 3M^2)/K(\tilde{M}) > 0$ it follows that $\|\overline{X}_t\|^2 \leq \tilde{M}$ for $t \in [0, \tilde{T}]$ independent of the discretization.

We are now in the position to prove Theorem 6.4.

Proof. As Lemma 6.6 states, there is a $0 < \tilde{T} \le T$ independent of the discretization, such that on $[0,\tilde{T}]$ the numerical solution is uniformly bounded. If $\tilde{T} = T$ define $\overline{M} = \tilde{M}$ and we are finished. Now assume $\tilde{T} < T$. Then on $\tilde{I} = [0,\tilde{T}]$ we can apply Proposition 6.2 to conclude $\|\overline{X}_t - X_t\| \to 0$, $N \to \infty$. This implies that for $t = \tilde{T}$ we have $\|\overline{X}_{\tilde{T}}\|^2 \to \|X_{\tilde{T}}\|^2$. This shows that $\|\overline{X}_{\tilde{T}}\|^2 < \tilde{M}$ in contrast to $\|\overline{X}_{\tilde{T}}\|^2 = \tilde{M}$. We conclude, that there is a constant $\overline{M} > 0$, such that $\|\overline{X}_t\| \le \overline{M}$ for $t \in [0,T]$ independent of the discretization. Equation (6.15) follows from Proposition 6.2.

7 Strong Convergence of fMM in $\|\cdot\|_1$ of order 1.

We now turn to the free variant of the Milstein Method in (5.15). We will show strong convergence order of $\gamma = 1$ under conditions given in Proposition 7.1 resp. Theorem 7.3. We start with the following proposition.

Proposition 7.1. Consider the fSDE (3.1). Let $a: A \to A$ be an operator function with $a(A^{sa}) \subset A^{sa}$. Additionally let the function $a \in W_3(\mathbb{R})$. The functions b, c are operator functions with the same properties as a. Let $X_t \in A^{sa}$ be a solution to the fSDE on [0,T]. Furthermore assume that there is a constant $\overline{M} > 0$ such that $\|\overline{X}_k\| < \overline{M}$ for $k = 0, \ldots, L$ and $L \in \mathbb{N}$, i.e. \overline{M} is independent of the discretization. Then the fMM approximation defined in Definition 5.2 has strong order of convergence $\gamma = 1$, i.e.

$$||X_k - \overline{X}_k||_1 \le M\Delta t. \tag{7.1}$$

for any $L \in \mathbb{N}$ and k = 0, ..., L The constant M > 0 is independent of step size Δt

Proof. Consider a discretization of [0,T] as described in Definition 5.2. We first build up a continuous reconstruction of X_t out of the discrete values \overline{X}_k obtained from fMM. Let's define the order one reconstruction

$$Z_{\tau} = Z_k + \overline{a}_k(\tau - t_k) + \overline{b}_k(W_{\tau} - W_{t_k})\overline{c}_k + \overline{m}_k(\tau)$$

$$(7.2)$$

on the interval $[t_k, \underline{t}], t_k \leq \tau \leq t_{k+1}, \ k = 0, \dots, L$. Note that Z_{t_k} is written as Z_k and Z_{τ} coincides with \overline{X}_k at the discretization point $\tau = t_k$, i.e. $Z_k = \overline{X}_k$. The term \overline{m}_k is the abbrevation for

$$\overline{m}_k(\tau) = b(\overline{X}_k) \left(b(\hat{X}_\tau) - b(\overline{X}_k) \right) b(\overline{X}_k)$$

where $\hat{X}_t = \overline{X}_k + (W_\tau - W_{t_k})b(\overline{X}_k)(W_\tau - W_{t_k}) - \varphi(b(\overline{X}_k))(\tau - t_k)$ (see also the definition of fMM in (5.15)). To continue with the proof take an arbitrary point $t \in [0, T]$. Let $n_t \in \mathbb{N}$ such that $t \in [t_{n_t}, t_{n_{t+1}}]$ (both are discretization points).

Rewriting the difference $X_t - Z_t$ by help of (5.7), (5.14) and rearranging the terms yield

$$X_{t} - Z_{t} = \sum_{k=0}^{n_{t}-1} (X_{k+1} - X_{k}) - \sum_{k=0}^{n_{t}-1} (Z_{k+1} - Z_{k}) + (X_{t} - X_{n_{t}}) - (Z_{t} - Z_{n_{t}}) =$$

$$= \underbrace{\sum_{k=0}^{n_{t}-1} (a_{k} - \overline{a}_{k}) \Delta t}_{S_{1}} + \underbrace{\sum_{k=0}^{n_{t}-1} (b_{k} \Delta W_{k} b_{k} + \overline{b}_{k} \Delta W_{k} \overline{b}_{k})}_{S_{2}} + \underbrace{\sum_{k=0}^{n_{t}-1} (m_{k}(t_{k+1}) - \overline{m}_{k}(t_{k+1})) + R_{k}}_{S_{3}} + \underbrace{(X_{t} - X_{n_{t}})}_{S_{4}} - \underbrace{(Z_{t} - Z_{n_{t}})}_{S_{5}} + \underbrace{\sum_{k=0}^{n_{t}-1} \rho_{k}}_{S_{6}}$$
 (7.3)

The term R_k is the remainder in (5.14) over the time interval $[t_k, t_{k+1}]$. Applying the $\|\cdot\|_2^2$ to (7.3) followed by the triangle inequality and $(u_1 + \cdots + u_6)^2 \le 6(u_1^2 + \ldots u_6^2)$ leaves the task to estimate $\|S_i\|_2^2$, $i = 1 \ldots 6$ in order to obtain the inequality (7.11). Now define

$$v(u) = \sup_{0 \le s \le u} \|X_s - Z_s\|_2^2.$$

Since a is locally operator Lipschitz in $L_2(\varphi)$ and $Z_k = \overline{X}_k$ we obtain

$$||S_1||_2^2 = \left\| \sum_{k=0}^{n_t - 1} (a_k - \overline{a}_k) \Delta t \right\|_2^2 \le \sum_{k=0}^{n_t - 1} L_a ||X_k - Z_k||_2^2 \Delta t \le$$

$$\le L_a \int_0^{t_{n_t}} v(u) du. \quad (7.4)$$

Note that $\Delta t = t_{k+1} - t_k \ge t - t_k$. In a similar way (see also the proof of Proposition 6.2) we estimate the second summand in (7.3) as

$$||S_2||_2^2 \le K_2 \int_0^{t_{n_t}} v(u) du.$$
 (7.5)

Since
$$\rho_k = \mathcal{O}((t - t_k)^3)$$
 (see (5.5)) we get $||S_6||_2^2 < K_6 \Delta t^2$.

Now we turn to the terms in S_3 , which determine the strong order of one. From (5.10), (5.11) and (5.15) it follows that

(7.6)

$$m_{k}(t_{k+1}) - \overline{m}_{k}(t_{k+1}) = b_{k} \left(T_{b_{1}}^{X_{k}, X_{k}} \left(\tilde{X}_{k+1} - X_{k} \right) \right) b_{k} - \overline{b}_{k} \left(b(\hat{X}_{k+1}) - b(\overline{X}_{k}) \right) \overline{b}_{k}$$
 (7.7)

where

$$\hat{X}_{k+1} = \overline{X}_k + \Delta W_k \overline{b}_k \Delta W_k - \varphi(\overline{b}_k) \Delta t$$

and

$$\tilde{X}_{k+1} = X_k + \Delta W_k b_k \Delta W_k - \varphi(b_k) \Delta t.$$

Rewriting (7.7) by applying (5.13) gives the expression

$$m_{k}(t_{k+1}) - \overline{m}_{k}(t_{k+1}) + R_{k} =$$

$$= (b_{k} - \overline{b}_{k}) \left(\tilde{b}_{k+1} - b_{k} \right) b_{k} + (b_{k} - \overline{b}_{k}) \mathcal{O} \left(\|\tilde{X}_{k+1} - X_{k}\|_{4}^{2} \right) b_{k} +$$

$$+ \overline{b}_{k} \left(\hat{b}_{k+1} - \overline{b}_{k} \right) (b_{k} - \overline{b}_{k}) + \overline{b}_{k} \mathcal{O} \left(\|\hat{X}_{k+1} - \overline{X}_{k}\|_{4}^{2} \right) (b_{k} - \overline{b}_{k}) + R_{k}, \quad (7.8)$$

where we applied the short notation $\tilde{b}_{k+1} = b(\tilde{X}_{k+1})$ and $\hat{b}_{k+1} = b(\hat{X}_{k+1})$. Since X_t and \overline{X}_k are uniformly bounded, b is locally operator Lipschitz, due to Theorem 5.5 we obtain for Δt small enough

$$||S_{3}||_{2}^{2} \leq \sum_{k=0}^{n_{t}} ||m_{k}(t_{k+1}) - \overline{m}_{k}(t_{k+1}) + R_{k}||_{2}^{2} \leq$$

$$\leq \sum_{k=0}^{n_{t}} (K_{1} \Delta t ||X_{k} - \overline{X}_{k}||_{2}^{2} + K_{2} \Delta t^{3}) \leq$$

$$\leq KT \int_{0}^{t_{n_{t}}} v(u) du + K_{2} \Delta t^{2}. \quad (7.9)$$

We now turn to S_4 and S_5 . We start by the difference

$$S_4 - S_5 = (a_{n_t} - \overline{a}_{n_t})\Delta\tau + b_{n_t}\Delta W_{n_t}c_{n_t} - \overline{b}_{n_t}\Delta W_{n_t}\overline{c}_{n_t} - m_{n_t}(t) - \overline{m}_{n_t}(t) + \rho_{n_t}(t)$$

A similar estimation as for S_1, S_2, S_3 and S_6 yields

$$||S_4 - S_5||_2^2 \le K_{45} \int_{n_t}^t v(u) du + C_{45} \Delta t^3.$$
 (7.10)

Collecting S_1 to S_6 gives the inequality (for Δt small enough)

$$v(t) \le C_1 \int_0^t v(u)du + C_2 \Delta t^2$$
 (7.11)

A Gronwall argument results in the following estimation of v,

$$v(t) = \sup_{0 \le s \le t} ||X_s - Z_s||_2^2 \le C_3 \Delta t^2$$

This implies

$$||X_k - \overline{X}_k||_1 < C\Delta t \tag{7.12}$$

for any
$$k = 0, \dots, L$$
.

Lemma 7.2. Let \overline{X}_k be a numerical solution of a fSDE (3.1) calculated by fMM (5.15) given a discretization of [0,T] with $T=L\Delta t$. Let $\tilde{M}=4M^2+1$, where $M=\sup_{s\in[0,T]}\|X_s\|_2<\infty$ (see Remark 3.3). Suppose the coefficient functions a,b,c in

(3.1) are locally Lipschitz in $L_2(\varphi)$ -norm. Then there exists a time point $0 < \tilde{T} \le T$ and a constant $K(\tilde{M})$ such that the estimation

$$\|\overline{X}_t\|_2^2 \le 4M^2 + K(\tilde{M})t \le \tilde{M}, t \in [0, \tilde{T}]$$
 (7.13)

is valid independent of the discretization. The term $K(\tilde{M})$ is independent of the discretization. The time point \tilde{T} depends on \tilde{M} and the Lipschitz constants of a,b,c.

Proof. \overline{X}_t denotes the piecewise constant process $\overline{X}_t = \overline{X}_k$, $t_k \leq t < t_{k+1}$. We proceed in the same way as for Lemma 6.6. Using the definition of fMM (5.15) we get

$$\|\overline{X}_t\|_2^2 \leq 4M^2 + 4L_a(1+\tilde{M}^2)t + 4L_bL_c(1+\tilde{M}^2)^2t + \sum_{k=0}^{\tilde{n}-1} \|\overline{b}_k(\hat{b}_{k+1}-\overline{b}_k)\overline{b}_k\|_2^2$$

The last summand can be estimated by

$$\sum_{k=0}^{\tilde{n}-1} \|\overline{b}_k (\hat{b}_{k+1} - \overline{b}_k) \overline{b}_k \|_2^2 \le L_b^3 (1 + \tilde{M}^2)^2 \sum_{k=0}^{\tilde{n}} \|\hat{X}_{k+1} - \overline{X}_k\|_2^2 \le L_b^3 (1 + \tilde{M}^2)^2 Tt$$

since $\|\hat{X}_{k+1} - \overline{X}_k\|_2^2 = \mathcal{O}(\Delta t^2)$ and $\Delta t < T$. Finally we obtain

$$\|\overline{X}_t\|_2^2 \le 4M^2 + 4L_a(1+\tilde{M}^2)t + 4L_bL_c(1+\tilde{M}^2)^2t + L_b^3(1+\tilde{M}^2)^2Tt \le$$

$$< 4M^2 + K(\tilde{M})t$$

Now choose $\tilde{T} = \frac{\tilde{M} - 4M^2}{K(\tilde{M})}$ to obtain the inequality $\|\overline{X}_t\|_2^2 \leq \tilde{M}$ for $t \in [0, \tilde{T}]$ independent of the discretization.

To get the final result, we can follow the arguments similar to Theorem 6.4 and state

Theorem 7.3. Consider a fSDE (3.1) with the solution X_t on [0,T] (see Remark 3.3). Let \overline{X}_k be a numerical solution calculated by fMM on [0,T] with a discretization $T=L\Delta t$. Assume $a,b,c\in W_3(\mathbb{R})$. Then the numerical solution is uniformly bounded for each $k=0,\ldots,L,\ L\in\mathbb{N}$, i.e. there is a constant $\overline{M}>0$ such that $\|\overline{X}_k\|_2<\overline{M}$, where \overline{M} does not depend on L resp. Δt .

Then the fMM approximation (5.2) has strong convergence order of $\gamma = 1$, i.e.

$$\|\overline{X}_k - X_k\|_1 \le C \,\Delta t. \tag{7.14}$$

The constant C is independent of step size Δt .

Remark 7.4. If b = const, as for the Ornstein-Uhlenbeck process, it immediately follows from Theorem 7.3 that fEMM has order of convergence 1. Numerical tests in [34] do numerically reproduce this behavior.

8 Numerical Examples

In the following examples we present different cases combining d=1,d>1 with different functions b,c. The methodology follows [34]. Additionally we show that an a posteriori estimation of the convergence order gives the expected results.

8.1 A Simple Example

We consider the fSDE

$$dX_t = X_t dW_t X_t \tag{8.1}$$

with start value $X_0 = I$. For this equation R in (5.14) vanishes and fMM is expected to show convergence order of 1. In [20, Proposition 3.9] it is shown, that the spectral distribution of the solution $X_T \in \mathcal{A}^{sa}$ exists of all $t \leq 1$ and is supported on the interval

$$\left[\frac{(1-\sqrt{t})^2}{(1-t)^2}, \frac{(1+\sqrt{t})^2}{(1-t)^2}\right]. \tag{8.2}$$

For $T \in]0, 1]$, the density of the spectral distribution is given by

$$f(x) = \frac{\sqrt{-(1-T)^2x^2 + 2(1+T)x - 1}}{2\pi Tx^3}.$$
 (8.3)

For T=1 the density is supported on $[1/4, \infty[$.

Similar to [34] we implement the method on matrix level. Applying fMM (5.15), the numerical solution lies in $\mathcal{M}_N(\mathbb{C})$. The probability density function of the eigenvalues of the numerical solution are then an approximation of (8.3). Since we do not know the exact solution of the fSDE in $\mathcal{M}_N(\mathbb{C})$, we apply a small time step $\Delta t_{min} = \frac{1}{L}$ with $L = 2^{13}$. The convergence order γ is then calculated with timesteps $\Delta t = R\Delta t_{min}$ with R = 8, 9, 10. The comparison of the convergence order of fEMM and fMM is shown in Figure 1.

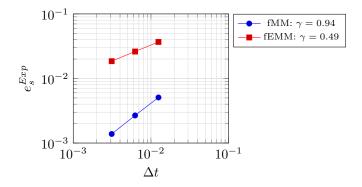
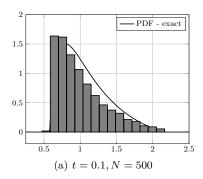
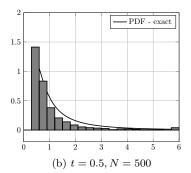


Figure 1: Strong convergence properties of $N=10,\,T=0.1,\,n=2570000$ of fSDE (8.1).

Figure Section 8.1 shows the probability PDF (8.3) of the spectral distribution of the exact solution X_t recovered from it's Cauchy transform at different timepoints and their approximation calculated by (8.1) in $\mathcal{M}_{500}(\mathbb{R})$. The bars show an estimation of the PDF via the eigenvalues of the numerical solution \overline{X}_t , calculated by fMM at different time points. The line is the PDF of the exact solution X_t .





Δt	e_s^{Exp} fEMM	e_s^{Exp} fMM
1e-3	9.184364e-04	1.074788e-04
2e-3	1.302557e-03	2.014683e-04
4e-3	1.833899e-03	3.846628e-04

Table 1: Error measures (see [34]) for fEMM and fMM visualized in Figure 2.

8.2 Geometric Brownian Motion 1

Consider the following example

$$dX_t = \theta X_t dt + \sqrt{X_t} dW_t \sqrt{X_t}, \tag{8.4}$$

with start value $X_0 = I$. For analytical insights to the spectral distribution of the solution X_t we refer to [20]. We run the simulation with N=2, $\theta=2$ and $\sigma=0.01$. The exact solution is unkown. We simulate the exact solution with a small time step of $\Delta t_{min} = 10^{-5}$. Then convergence order is then confirmed with time steps $\Delta t = 10^{-3}, 2*10^{-3}, 4*10^{-3}$. We follow the methodology of [34]. The results are listed in table 1, which is visualized in figure 2. The results show the expected strong convergence orders of 0.5 for fEMM and 1 for fMM.

It is also possible to give a posteriori estimate of the convergence order. Applying fEMM resp. fMM with timesteps $\Delta t, \Delta t/2, \Delta t/4$, one can estimate the convergence order by

$$p \approx \frac{\log\left(\frac{\overline{X}_{\Delta t} - \overline{X}_{\Delta t/2}}{\overline{X}_{\Delta t/2} - \overline{X}_{\Delta t/4}}\right)}{\log(2)}$$
(8.5)

Setting $\Delta t = 1e-3$ with N=2 and n=2000 we obtain estimations $p\approx 0.55$ for fEMM and $p\approx 0.90$ for fMM.

8.3 Geometric Brownian Motion 2

Consider the case d > 1 with smooth coefficient functions b, c.

$$dX_t = \theta X_t dt + \sigma X_t dW_t + dW_t \sigma X_t \tag{8.6}$$

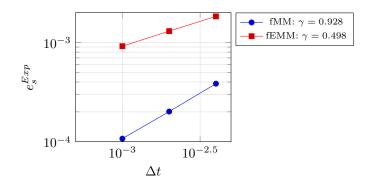


Figure 2: Strong convergence properties of N=2, T=0.1, n=20000 of fSDE (8.4).

Δt	e_s^{Exp} fEMM	e_s^{Exp} fMM
1e-3	3.861614e-02	2.707323e-04
2e-3	5.481944e-02	5.037829e-04
4e-3	7.717823e-02	9.805371e-04

Table 2: Strong convergence properties of N=2, T=0.1, n=20000 of fSDE (8.6) visualized in figure 3.

with start value $X_0 = I$. For analytical insights to the spectral distribution of the solution X_t we again refer to [20]. We run the simulation with N=2, $\theta=2$ and $\sigma=0.01$. The exact solution is unkown. We simulate the exact solution with a small time step of $\Delta t_{min} = 10^{-5}$. The convergence order is then confirmed with time steps $\Delta t = 10^{-3}, 2*10^{-3}, 4*10^{-3}$. We follow the methodology of [34]. The results are listed in table 2, which is visualized in figure 3. The results show the expected strong convergence orders of 0.5 for fEMM and 1 for fMM.

8.4 CIR Equation

Consider the case d > 1 with non-smooth coefficient functions b, c.

$$dX_t = (a - bX_t)dt + \frac{\sigma}{2}\sqrt{X_t}dW_t + dW_t\frac{\sigma}{2}\sqrt{X_t}$$
(8.7)

with start value $X_0 = I$.

For analytical insights to the spectral distribution and the existence of the solution X_t we refer to [16]. We run the simulation with N=2, a=1, b=0.1 and $\sigma=0.1$. The exact solution is unkown. We simulate the exact solution with a small time step of $\Delta t_{min}=10^{-5}$. Then convergence order is then confirmed with time steps $\Delta t=10^{-3}, 2*10^{-3}, 4*10^{-3}$. We follow the methodology of [34]. The results are listed in table 3, which is visualized in figure 4. The results show the expected strong convergence orders of 0.5 for fEMM and 1 for fMM.

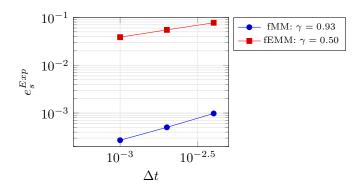


Figure 3: Strong convergence properties of $N=2,\,T=0.1,\,n=20000$ of fSDE (8.6).

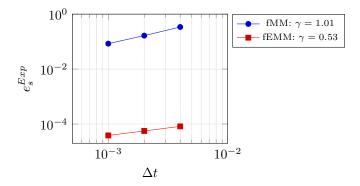


Figure 4: Strong convergence properties of $N=2,\,T=0.1,\,n=20000$ of fSDE (8.7). We obtain p=1.0073.

Δt	e_s^{Exp} fEMM	e_s^{Exp} fMM
1e-3	3.844555e-05	8.418441e-02
2e-3	5.579574e-05	1.650325e-01
4e-3	8.220341e-05	3.401413e-01

Table 3: Data of figure 4.

An a posteriori esimation of the convergence order can be calculated by (8.5). Setting $\Delta t = 1e - 3$ with N = 2 and n = 20000 we obtain estimations $p \approx 0.53$ for fEMM and $p \approx 1.12$ for fMM.

9 Backmatter

Acknowledgment

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