REMOVABLE SET FOR HÖLDER CONTINUOUS SOLUTIONS OF $\mathcal A$ -HARMONIC FUNCTIONS ON FINSLER MANIFOLDS

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ABSTRACT. We establish that a closed set $\mathcal S$ is removable for α -Hölder continuous $\mathcal A$ -harmonic functions in a reversible Finsler manifold $(\Omega,F,\mathtt V)$ of dimension $n\geq 2$, provided that (under certain conditions on $(\Omega,F,\mathtt V)$ and the variable exponent p) for each compact subset K of $\mathcal S$, the $\mathsf n_1-p_K^++\alpha(p_K^+-1)$ -Hausdorff measure of K is zero. Here, $p_K^+=\sup_K p$ and $\mathsf n_1$ is chosen so that $\mathsf V(B(x,r))\leq \mathsf K r^{\mathsf n_1}$ for every ball.

The estimates used to remove the singularities will focus on a family $\{u_\ell\}_{\ell\in\mathcal{J}}\subset W^{1,p(x)}_{\mathrm{loc}}(\Omega;\mathtt{V})$ that converges to u in a certain sense. As a second main result of this article, we will also obtain an estimate (when $\lim_{d(x,0_\Omega)\to\infty}p=1$) for

$$\mu_{\ell}(B(x,r)) := \sup \left\{ \int_{B(x,r)} \mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) \bullet \mathcal{D} \zeta \; \mathrm{dV} \mid 0 \leq \zeta \leq 1 \text{ and } \zeta \in C_0^{\infty}(B(x,r)) \right\},$$

which is related to the measure $\mu = \text{div}(\mathcal{A}(\cdot, \nabla u))$.

1. Introduction

In this paper, we study solutions to quasilinear elliptic equations of the form:

$$-\operatorname{div}\mathcal{A}(\cdot,\nabla u) = 0 \quad \text{in} \quad \Omega. \tag{1.1}$$

Throughout this work, (Ω, F, V) is a reversible Finsler manifold satisfying the properties (1.7) - (1.12) below, and the variable exponent $p:\Omega\to(1,\infty)$ is continuous. Additionally, the operator $\mathcal{A}:T\Omega\to T\Omega$ satisfies the following conditions for all functions $u:\Omega\to\mathbb{R}$ that are differentiable in the distributional sense:

- $(a_1) \left[\mathcal{A}(\cdot, \nabla u) \bullet \mathcal{D}u \right](x) \ge \Lambda_1 |\nabla u|_F^p(x) \text{ for a.e. } x \in \Omega.$
- $(a_2) |\mathcal{A}(\cdot, \nabla u)|_F(x) \leq \Lambda_2 |\nabla u|_F^{p-1}(x) \text{ for a.e. } x \in \Omega.$
- $(a_3) \{ [\mathcal{A}(\cdot, \nabla u) \mathcal{A}(\cdot, \nabla v)] \bullet (\mathcal{D}u \mathcal{D}v) \} (x) > 0 \text{ for a.e. } x \in \Omega \text{ whenever } \mathcal{D}u(x) \neq \mathcal{D}v(x).$

Here, we write

$$F(x,Y) = |Y|_F(x)$$
 and $F^*(x,\omega) = |\omega|_{F^*}(x)$,

for $Y \in T_x\Omega$ and $\omega \in T_x^*\Omega$. Furthermore, for a function $f:\Omega \to \mathbb{R}$ that is differentiable in the distributional sense, we use the notation

$$Y \bullet \mathcal{D}f(x) = \mathcal{D}f_x(Y)$$
 for $Y \in T_x\Omega$,

where $\mathfrak{D}f_x \in T_x^*\Omega$ (or equivalently $\mathfrak{D}f(x)$) denotes the derivative of f at x.

For a relatively closed subset $E \subset \Omega' \subset \mathbb{R}^n$ with s-dimensional Hausdorff measure zero (for s > n-p) and u p-harmonic in $\Omega' \setminus E$, previous studies determined the values of s for which the extension of u is p-harmonic in all of Ω' . Classical results are given in [6, 28, 29], and problems involving lower-order terms are discussed in [25, 41]. In the nonstandard growth framework, removability problems were studied in variable exponent spaces by [14, 32, 1, 2], in Orlicz spaces by [7, 9], and in double-phase spaces by [8]. Building on these studies, we use the intrinsic capacities and Hausdorff measures introduced by [3] and [10], respectively.

Mäkäläinen in [36] studied p-harmonic functions in complete metric spaces Ω , assuming the space has a doubling measure and supports a weak (1,p)-Poincaré inequality. To control the integrability of the derivative in the metric space setting, a substitute for Sobolev space is needed. The Newtonian space, introduced by Shanmugalingam in [44], is used. The following equation is studied:

$$\int_{\tilde{\Omega}} |Du|^{p-2} Du \cdot D\varphi \, \mathrm{d}\mu = 0,$$

where 1 and <math>D denotes the derivative. The paper shows that sets of weighted $(-p + \alpha(p-1))$ -Hausdorff measure zero are removable for α -Hölder continuous Cheeger p-harmonic functions.

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Kilpeläinen & Zhong in [28] studied continuous solutions $u \in W^{1,p}_{loc}(\Omega')$ of the equation

$$-\operatorname{div} A(x, \nabla u) = 0,$$

where $\Omega' \subset \mathbb{R}^n$ is an open set and $1 . These solutions are called A-harmonic in <math>\Omega'$. An example of such operators is the p-Laplacian

$$-\Delta_p u = -\operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

They show that sets with $n-p+\alpha(p-1)$ Hausdorff measure zero, where $0<\alpha\leq 1$, are removable for α -Hölder continuous solutions to quasilinear elliptic equations. For the quasilinear case, Heinonen & Kilpeläinen [23] proved similar results for $\alpha=1$, and Trudinger & Wang [45] proved it under the assumption that u has an A-superharmonic extension to Ω' , which can be omitted for small α . Koskela & Martio [29] proved a weaker version using Minkowski content instead of Hausdorff measure. Buckley and Koskela [5] also established special cases.

In relation to Hölder continuity of solutions, Lyaghfouri in [34] showed that a solution of the equation $-\Delta_{p(x)}u=\hat{\mu}$ is Hölder continuous with exponent α if and only if the nonnegative Radon measure $\hat{\mu}$ satisfies the growth condition

$$\hat{\mu}(B_r(x)) < Cr^{n-p(x)+\alpha(p(x)-1)}.$$

for any ball $B_r(x) \subset \Omega' \subset \mathbb{R}^n$, with r small enough. This extends a result of Lewy and Stampacchia for the Laplace operator and a result of Kilpeläinen and Zhong for the p-Laplace operator with constant p.

In [33], Lewy and Stampacchia studied the equation

$$-\Delta u = \hat{\mu}.\tag{1.2}$$

They proved the equivalence between the growth condition

$$\hat{\mu}(B_r(x)) \le Cr^{n-2+\alpha}$$

on the measure $\hat{\mu}$, and the Hölder continuity of the solutions with exponent α , when n > 2. For n = 2, they proved that each solution of (1.2) is Hölder continuous with exponent β for all $\beta \in (0, \alpha)$.

In [26], Kilpeläinen considered the p-Laplace operator, where p is a constant. He proved that if a solution to the equation

$$-\Delta_n u = \hat{\mu} \tag{1.3}$$

is Hölder continuous with exponent α , then the measure $\hat{\mu}$ satisfies the growth condition

$$\hat{\mu}(B_r(x)) < Cr^{n-p+\alpha(p-1)}. \tag{1.4}$$

Conversely he proved that if $\hat{\mu}$ satisfies (1.4), then each solution of (1.3) is Hölder continuous with exponent β for all $\beta \in (0, \alpha)$.

In [28], Kilpeläinen and Zhong improved the result in [27] by showing that if $\hat{\mu}$ satisfies (1.4), then each solution of (1.3) is Hölder continuous with exponent α . Rakotoson [42] and Rakotoson & Ziemer [43] also obtained results in this direction.

Throughout this paper, a Finsler manifold will be denoted by (Ω, F, V) , where V is a measure and $F : T\Omega \to [0, \infty)$ is a function (called a Finsler structure) with the following properties:

- (F_1) F is $C^{\infty}(T\Omega\setminus\{0\})$.
- (F_2) $F(x,t\xi)=tF(x,\xi)$ for all $(x,\xi)\in T\Omega$ and $t\geq 0$.
- (F_3) For each chart $\psi: \tilde{U} \subset \mathbb{R}^n \to \psi(\tilde{U}) \subset \Omega$, the matrix

$$g_{ij}(x,y) := \frac{\partial^2}{\partial \xi_i \partial \xi_j} \left(\frac{1}{2} F^2(x, d\psi_x(\xi)) \right) \Big|_{\xi=y}$$

is uniformly elliptic in the sense that there exist positive constants κ_1 and κ_2 such that

$$\kappa_1 \sum_{k=1}^{n} (\eta^k)^2 \le g_{ij}(x,\xi) \eta^i \eta^j \le \kappa_2 \sum_{k=1}^{n} (\eta^k)^2$$

holds for all $x \in \psi(\tilde{U})$, $\xi \in T_x\Omega\setminus\{0\}$, and $\eta = \eta^i\partial_i \in T_x\Omega$, where $\{\partial_i\}_{i=1}^n$ is the basis associated with the chart ψ .

If F(x,Y) = F(x,-Y) for all $(x,Y) \in T\Omega$, we say that the Finsler manifold (Ω,F) is reversible.

Throughout this paper, we fix a measure V on Ω . For each point $x \in \Omega$, we assume there exists a neighborhood U with a local coordinate system $\psi: \tilde{U} \to U$, and a measurable function $\tilde{\mathbf{V}}: \tilde{U} \to \mathbb{R}^+$ such that for some constant C>0, we have

$$C \le \tilde{\mathbf{V}} \le C^{-1} \quad \text{in} \quad \tilde{U}. \tag{1.5}$$

Additionally, for every measurable set $E \subset U$, the following holds:

$$V(E) = \int_{\psi^{-1}(E)} \tilde{V} \, \mathrm{d}x. \tag{1.6}$$

We work with the following conditions on V and p.

(i) There is constants $K, K_0, K_1, n_1 > 0$ such that

$$V(B(x,r)) \le Kr^{\mathbf{n}_1},\tag{1.7}$$

$$[V(B(x,r))]^{p_r^- - p_r^+} \le K_0, \tag{1.8}$$

and

$$V(B(x,2r)) \le K_1 V(B(x,r)), \tag{1.9}$$

for all r > 0 and $x \in \Omega$, where $p_r^- = \inf_{B(x,r)} p$ and $p_r^+ = \sup_{B(x,r)} p$.

Remark: Under certain conditions, the doubling condition (1.9) follows from the Bishop-Gromov-type volume comparison theorem. See [37, Section 2.8] or [31, Lemma 2.2].

(ii) There are constants $1_{\rm I} > 1$ and ${\rm K_I}, {\rm K_{II}} > 0$ such that for all ball B = B(x,r):

$$\left(\int_{B} |v|^{1_{\mathrm{I}}} \, \mathrm{dV} \right)^{\frac{1}{1_{\mathrm{I}}}} \le K_{\mathrm{I}} r \int_{B} |\nabla v|_{F} \, \mathrm{dV}, \tag{1.10}$$

whenever $v \in C_0^{\infty}(B)$.

Furthermore:

$$\int_{B} |v - v_{B}| \, dV \le K_{\text{II}} r \int_{B} |\nabla v|_{F} \, dV, \tag{1.11}$$

whenever $v \in C^{\infty}(B)$ is bounded.

Here we are using the notation:

$$\int_B f \, \mathrm{d} \mathtt{V} := rac{1}{\mathtt{V}(B)} \int_B f \, \mathrm{d} \mathtt{V} \quad ext{ and } \quad f_B := rac{1}{\mathtt{V}(B)} \int_B f \, \mathrm{d} \mathtt{V}.$$

(iii) For every bounded open set U, there is a constant such that

$$||u||_{p,U,\mathbf{V}} \le C||F(\cdot, \nabla u)||_{p,U,\mathbf{V}} \quad \forall u \in C_0^{\infty}(U). \tag{1.12}$$

Our first main result is as follows:

Theorem 1.1. Let (Ω, F, V) be a reversible Finsler manifold, where V and p satisfy conditions (1.5)–(1.12). Let $S \subset \Omega$ be a closed set. Assume that $u \in C(\Omega)$ is a solution of (1.1) in $\Omega \setminus S$, and there exists $\alpha \in (0,1]$ such that for any $x \in S$ and $y \in \Omega$:

$$|u(x) - u(y)| \le Cd^{\alpha}(x, y).$$

If for each compact subset K of S, the $n_1 - p_K^+ + \alpha(p_K^+ - 1)$ -Hausdorff measure of K is zero, then u is a solution of (1.1) in Ω .

Next, we consider a family $\{\vartheta_\ell\}_{\ell\in\mathcal{J}}\subset W^{1,p(x)}_{\mathrm{loc}}(\Omega;\mathtt{V})$ and an element $\vartheta\in W^{1,p(x)}_{\mathrm{loc}}(\Omega;\mathtt{V})$. For the following result, we assume the properties:

 (P_1) For all $\ell \in \mathcal{J}$ we have

$$\int_{\Omega} \mathcal{A}(\cdot, \nabla \vartheta) \bullet \mathcal{D}(\vartheta_{\ell} - \vartheta) \, dV = 0 \tag{1.13}$$

and, for some positive constants \mathfrak{T}_0 and \mathfrak{T} ,

$$|\vartheta_{\ell} - \vartheta| \le \mathfrak{T}_0 \quad \text{and} \quad |\mathcal{A}(\cdot, \nabla \vartheta_{\ell}) - \mathcal{A}(\cdot, \nabla \vartheta)|_F \le \mathfrak{T} \le \frac{\Lambda_1}{4} \quad \text{in } \Omega.$$
 (1.14)

(P₂) There exist functions $\Upsilon_i : \mathbb{R}^+ \to [0, \infty)$, i = 1, 2, such that, for all $\ell \in \mathcal{J}$ and for all $\zeta \in C_0^{\infty}(B(x, r))$ with $B(x, r) \subset\subset \Omega$ and $0 \leq \zeta \leq 1$, the following holds:

$$\begin{split} \int_{B(x,r)} \left[\mathcal{A}(\cdot, \boldsymbol{\nabla} \vartheta) - \mathcal{A}(\cdot, \boldsymbol{\nabla} \vartheta_{\ell}) \right] \bullet \, \mathcal{D} \zeta \, \mathrm{d} \mathbf{V} &\leq \Upsilon_1(x,r) \quad \text{ and } \\ - \int_{B(x,r)} \left[\mathcal{A}(\cdot, \boldsymbol{\nabla} \vartheta) - \mathcal{A}(\cdot, \boldsymbol{\nabla} \vartheta_{\ell}) \right] \bullet \, \mathcal{D} \zeta \, \mathrm{d} \mathbf{V} &\leq \Upsilon_2(x,r) \quad \text{ if } \quad B(x,r) \subset \{\vartheta > \mathfrak{O}_1\} \end{split}$$

 (P_3) Let $0_{\Omega} \in \Omega$ be fixed. Assume that $p \in C(\Omega)$ satisfies $1 in <math>\Omega$, and that $\lim_{d(x,0_{\Omega}) \to \infty} p(x) = 1$, and, for some constant $\tilde{K}_1 > 0$,

$$\frac{\sup_{B(x_0,r)} p - 1}{\inf_{B(x_0,r)} p - 1} \le \tilde{\mathsf{K}}_1 \quad \forall B(x_0,r) \subset \Omega. \tag{1.15}$$

For every open set $V \subset\subset \Omega$, we define

$$\mu_\ell(V) := \sup \left\{ \int_V \mathcal{A}(\cdot, \boldsymbol{\nabla} \vartheta_\ell) \bullet \mathcal{D} \zeta \ \mathrm{dV} \ | \ 0 \leq \zeta \leq 1 \ \mathrm{and} \ \zeta \in C_0^\infty(V) \right\}.$$

Our second main result is the following

Theorem 1.2. Let (Ω, F, V) be a reversible Finsler manifold, where V and p satisfy conditions (1.5)–(1.11). Let K be a compact subset of Ω , and define $K_{\delta} := \{x \in \Omega \mid d(x,K) < \delta\}$, where $\overline{K_{\delta}} \subset \Omega$. Suppose that $\vartheta \in \mathcal{K}_{\mathcal{O}_1}(\Omega)$ is a solution to the obstacle problem (2.14), with the obstacle $\mathcal{O}_1 \in C(\Omega)$ satisfying

$$|\mathfrak{O}_1(x) - \mathfrak{O}_1(y)| \le \hat{\mathbf{s}} d^{\alpha}(x, y), \quad \text{for all } x \in K, y \in \Omega,$$

where $0 < \alpha \le 1$ and $\hat{s} > 0$.

Additionally, let $\{\vartheta_\ell\} \subset \mathcal{K}_{\mathcal{O}_1}(\Omega)$ be a family such that $\{\vartheta = \mathcal{O}_1\} \subset \{\vartheta_\ell = \mathcal{O}_1\}$ for all ℓ , and suppose that this family satisfies properties (P_1) and (P_2) , with $p \in C(\Omega)$ satisfying property (P_3) .

There exists $R_1 = R_1(\Lambda_1, \Lambda_2, p^+, \tilde{\mathsf{K}}_1, \mathsf{K}_1, \mathsf{K}_2, \mathsf{K}_3, \mathsf{K}_4, \mathsf{K}_{\mathrm{II}}, \mathsf{K}_0, \mathfrak{T}_0, n, s, \mathsf{n}_1, \inf_{B(x,5r)} \mathfrak{O}_1, \sup_{B(x,5r)} \mathfrak{O}_1) > 0$, where $s > \sup_{K_\delta} p - \inf_{K_\delta} p$, such that $p-1 \leq 1$ in $\Omega \backslash B(\mathfrak{O}_\Omega, R_1)$. Furthermore, for every $x \in K \cap (\Omega \backslash B(\mathfrak{O}_\Omega, R_1))$ and $0 < r < \frac{1}{41} \min\{R_*, \delta\}$, the following holds:

$$(B(x,r)) \le C_1 C_2^{\frac{\tilde{\mathsf{K}}_1}{p_{41r}^- - 1}} (p_{41r}^- 1)^{-\tilde{\mathsf{K}}_1} r^{\mathsf{n}_1 - p(x) + \alpha(p(x) - 1)} + \Upsilon_1(x, 5r) + \Upsilon_2(x, r),$$

 $if \ B(x,41r) \subset \subset \psi(\tilde{U}), \ where \ \psi: \tilde{U} \to \psi(\tilde{U}) \subset \Omega \ is \ a \ chart, \ p_{41r}^- := \inf_{B(x,41r)} p, \ C_i := C_i(\Lambda_1,\Lambda_2,p^+,\tilde{\mathtt{K}}_1,\mathtt{K}_1,\mathtt{K}_2,\mathtt{K}_3,\mathtt{K}_4,\mathtt{K}_1,\mathtt{K}_1,\mathtt{K}_0,\mathtt{K},n,s,\gamma,1_{\mathrm{I}},\mathtt{n}_1,\mathtt{n}_2,\alpha,\hat{\mathtt{s}},\inf_{B(x,5r)} \mathcal{O}_1,\sup_{B(x,5r)} \mathcal{O}_1,\hat{c}_1,\mathfrak{T}_0, \|\vartheta\|_{s,B(x,41r),\mathtt{V}}, \|\vartheta\|_{s\gamma',B(x,41r),\mathtt{V}}) > 0, \ for \ i=1,2, \ and \ 1 < \gamma < 1_{\mathrm{I}}. \ Also, \ \hat{c}_1 \ is \ a \ constant \ (depending \ on \ \psi \ and \ F) \ given \ in \ (3.1), \ and \ R_* \ is \ defined \ in \ Lemma \ 2.8 \ (i).$

In this work, we employ the techniques from [14, 35, 41], which address the case where the norm and the domain are in \mathbb{R}^n . This paper is organized as follows. In Section 2, we provide the necessary definitions of variable exponent Lebesgue and Sobolev spaces on Finsler manifolds. In this section, we also define \mathcal{A} -harmonic functions, supersolutions, and solutions to the obstacle problem. Additionally, we present some basic results that will be used throughout the paper. In Section 3, we provide estimates for solutions of the obstacle problem and for supersolutions. Finally, in Section 4, we study removable sets for \mathcal{A} -harmonic functions and prove Theorems 1.1 and 1.2.

2. Preliminary Results and Definitions

- 2.1. **Finsler Manifolds.** We recall some facts and notation about Finsler manifolds. Most properties can be found in [13, 37, 39, 46].
- 2.1.1. The Legendre Transform. The polar transform (or co-metric) $F^*: T^*\Omega \to [0, \infty)$ is defined as the dual metric of F, given by the expression:

$$F^*(x,\omega) = \sup_{v \in T_x \Omega \setminus \{0\}} \frac{\omega(v)}{F(x,v)},$$

where $T^*\Omega = \bigcup_{x \in \Omega} T_x^*\Omega$ is the cotangent bundle of Ω , and $T_x^*\Omega$ is the dual space of $T_x\Omega$.

If $x \in \psi(\tilde{U}) \subset \Omega$, where ψ is a chart. We have that for all $\omega \in T_x^*\Omega$ there exist a unique $(\omega_1, \dots, \omega_n) \in \mathbb{R}^n$ such that

$$\omega(d\psi_x(v)) = \langle (\omega_1, \dots, \omega_n), v \rangle \quad \forall v \in \mathbb{R}^n,$$

where $\langle \cdot, \cdot \rangle$ is the Euclidean metric. In these local coordinates, for simplicity, we write $F^*(x, \omega) = F^*(x, (\omega_1, \ldots, \omega_n))$.

Since $F^{*2}(x,\cdot)$ is twice differentiable on $T_x^*\Omega\setminus\{0\}$, one can define the Hessian (dual) matrix $[g_{ij}^*(x,\omega)]$, where

$$g_{ij}^*(x,\omega) := \frac{1}{2} \frac{\partial^2}{\partial \omega_i \partial \omega_j} F^{*2}(x,(\omega_1,\ldots,\omega_n)).$$

Using the strong convexity assumption on the Finsler structure F, the Legendre transform $J^*: T^*\Omega \to T\Omega$ is defined as follows: for each fixed $x \in \Omega$, J^* associates to each $\omega \in T_x^*\Omega$ the unique maximizer $v \in T_x\Omega$ of the mapping:

$$v \mapsto \omega(v) - \frac{1}{2}F^2(x, v).$$

Note that if $J^*(x,\omega) = (x,v)$, then the following relations hold:

$$F(x,v) = F^*(x,\omega)$$
 and $\omega(v) = F^*(x,\omega)F(x,v)$.

In local coordinates, we have:

$$J^*(x,\omega) = \sum_{i=1}^n \frac{\partial}{\partial \omega_i} \left(\frac{1}{2} F^{*2}(x,\omega) \right) \partial_i,$$

where $\{\partial_i\}_{i=1}^n$ is the basis associated with the chart ψ .

2.1.2. Gradient and Distance Functions. For a weakly differentiable function $u: \Omega \to \mathbb{R}$, the gradient vector at x is defined by

$$\nabla u(x) := J^*(x, \mathcal{D}u(x)),$$

for every regular point $x \in \Omega$, where the derivative $\mathcal{D}u(x) \in T_x^*\Omega$ is well-defined. By applying the properties of the Legendre transform, it follows that

$$F^*(x, \mathcal{D}v(x)) = F(x, \nabla v(x)) \tag{2.1}$$

In a local coordinate system, if $x \in \psi(\tilde{U})$, we have

$$\mathcal{D}u(x) = \sum_{i=1}^{n} \left(\partial_{i}(u)\partial_{i}^{*}\right)(x) \quad \text{ and } \quad \nabla u(x) = \sum_{i,j=1}^{n} \left(g_{ij}^{*}(\cdot, \mathcal{D}u)\partial_{i}(u)\partial_{j}\right)(x),$$

where $\partial_i^*(x) \in T_x^*\Omega$ is defined by $\partial_i^*(x)(\partial_i(x)) := \delta_{ij}$.

For a differentiable vector field $Y: \Omega \to T\Omega$ on Ω , we define its divergence $\operatorname{div} Y: \Omega \to \mathbb{R}$ through the identity

$$\int_{\Omega} u \operatorname{div} Y \, dV = - \int_{\Omega} \mathcal{D}u(Y) \, dV.$$

We observe that the nonlinearity of the Legendre transform extends to the gradient vector, namely $\nabla(u + v) \neq \nabla u + \nabla v$ in general. For the same reason, at points x where $\nabla u(x) = 0$, the gradient vector field ∇u is, in general, not differentiable (even if u is smooth), but only continuous.

We define the distance function $d: \Omega \times \Omega \to [0, \infty)$ by

$$d(x,y) = \inf_{\gamma} \int_{a}^{b} F(\gamma(t), \gamma'(t)) dt,$$

where the infimum is taken over all piecewise differentiable curves $\gamma:[a,b]\to\Omega$ with $\gamma(a)=x$ and $\gamma(b)=y$. This definition is equivalent to

$$d(x,y) := \sup \left\{ u(y) - u(x) \mid u \in C^1(\Omega), F(z, \nabla u(z)) \le 1 \text{ for all } z \in \Omega \right\}.$$

For a fixed $y \in \Omega$, the distance function $x \mapsto d(y,x)$ satisfies $F(x, \nabla d(y,x)) = 1$ for almost every $x \in \Omega$. Moreover, the distance function d possesses the following properties of a metric:

- (a) $d(x,y) \ge 0$ for all $x,y \in \Omega$, and d(x,y) = 0 if and only if x = y.
- (b) $d(x,z) \le d(x,y) + d(y,z)$ for all $x,y,z \in \Omega$.

However, in general, the distance function is not symmetric. In fact, d(x,y) = d(y,x) for all $x,y \in \Omega$ if and only if (Ω,F) is a reversible Finsler manifold.

2.2. Variable exponent Sobolev spaces. Next, we recall some key facts and notation regarding variable exponent Lebesgue and Sobolev spaces. A detailed discussion of the properties of these spaces can be found in [16, 17] for Riemannian manifolds, [15, 19, 20] for metric spaces, and [12, 30] for the Euclidean setting.

Let $p:\Omega\to(1,\infty)$ be a continuous function, and let $\Omega\subset\Omega$ be a open subset. The variable exponent, or generalized Lebesgue space $L^{p(x)}(\hat{\Omega}; V)$, is the space of all measurable functions $u: \hat{\Omega} \to \mathbb{R}$ for which the functional

$$ho_p(u) := \int_{\hat{\Omega}} |u|^p \, \mathrm{dV}$$

is finite. This is a special case of an Orlicz-Musielak space, see [38].

The functional ρ_p is convex and is sometimes called a convex modular [30]. The space $L^{p(x)}(\hat{\Omega}; V)$ is a Banach space with respect to the Luxemburg-Minkowski-type norm

$$\left\|u\right\|_{p,\hat{\Omega},\mathbf{V}}:=\inf\left\{t>0\:|\:\rho_p\left(\frac{u}{t}\right)\leq 1\right\}.$$

The space $L^{p(x)}(\Omega; \mathbf{V})$ is reflexive if $1 < p_{\hat{\Omega}}^- \le p_{\hat{\Omega}}^+ < \infty$, see [20, Proposition 2.3], where $p_{\hat{\Omega}}^- := \inf_{\hat{\Omega}} p$ and $p_{\hat{\Omega}}^+:=\sup_{\hat{\Omega}}p.$ In what follows, we will use the generalized Hölder inequality:

$$\int_{\hat{\Omega}} |uv| \, \mathrm{dV} \le 2||u||_{p,\hat{\Omega},\mathbf{V}} ||v||_{p',\hat{\Omega},\mathbf{V}},\tag{2.2}$$

as stated in [30, Theorem 2.1].

To compare the functionals $\|\cdot\|_{p,\hat{\Omega},V}$ and ρ_p , we have the following relations:

$$\min\{\rho_p(u)^{1/p^-}, \rho_p(u)^{1/p^+}\} \le \|u\|_{p,\hat{\Omega},\mathbf{V}} \le \max\{\rho_p(u)^{1/p^-}, \rho_p(u)^{1/p^+}\}. \tag{2.3}$$

The variable exponent Sobolev space $W^{1,p(x)}(\hat{\Omega}; V)$ is defined by

$$W^{1,p(x)}(\hat{\Omega}; \mathtt{V}) := \left\{ u \in L^{p(x)}(\hat{\Omega}; \mathtt{V}) \mid \int_{\hat{\Omega}} F(\cdot, \mathbf{\nabla} u)^p \, \mathsf{dV} < \infty
ight\}.$$

This space is a Banach space with the norm

$$\|u\|_{1,p,\hat{\Omega},\mathbf{V}} := \|u\|_{p,\hat{\Omega},\mathbf{V}} + \|F(\cdot, \nabla u)\|_{p,\hat{\Omega},\mathbf{V}}.$$

We also define $W_0^{1,p(x)}(\hat{\Omega}; \mathbf{V})$ as the closure of $C_0^{\infty}(\hat{\Omega})$ in $W^{1,p(x)}(\hat{\Omega}; \mathbf{V})$ with respect to the norm $\|\cdot\|_{1,p,\hat{\Omega},\mathbf{V}}$.

Note that the classes $L^{p(x)}_{\rm loc}(\Omega; \mathtt{V})$ and $W^{1,p(x)}_{\rm loc}(\Omega; \mathtt{V})$ depend only on the manifold structure of Ω (not on the Finsler structure F and the measure \mathtt{V}); see the last paragraph on page 1394 of [40]. More precisely, let $\dot{\Omega} \subset\subset \Omega$ be an open set, and consider two measures V_1 and V_2 satisfying (1.5) and (1.6). These measures are comparable in $\hat{\Omega}$. In other words, there exists a constant C>0 such that

$$CV_1(E) \le V_2(E) \le C^{-1}V_1(E),$$

for every measurable subset $E \subset \hat{\Omega}$. This implies that $\int_{\hat{\Omega}} |u|^p dV_1 < \infty$ if and only if $\int_{\hat{\Omega}} |u|^p dV_2 < \infty$. Thus, $L^p_{\mathrm{loc}}(\Omega; \mathbf{V})$ does not depend on the measure \mathbf{V} .

For $W_{loc}^{1,p(x)}(\Omega, V)$, we further observe that for any two Finsler structures F_1 and F_2 , by 1-homogeneity and property (F_3) , we have

$$cF_1(\cdot, \nabla u) \le F_2(\cdot, \nabla u) \le c^{-1}F_1(\cdot, \nabla u) \quad \text{in } \hat{\Omega},$$

for some constant c>0. Therefore, $W^{1,p(x)}_{\mathrm{loc}}(\Omega;\mathtt{V})$ does not depend on either the measure \mathtt{V} or the Finsler structure F.

We will finish this subsection with the following definitions:

Definition 2.1. We say that a continuous function $u \in W^{1,p(x)}_{loc}(\Omega \backslash \mathcal{S}; V)$ is an \mathcal{A} -harmonic function in $\Omega \backslash \mathcal{S}$ if, for every test function $\varphi \in W^{1,p(x)}(\Omega \backslash \mathcal{S}; V)$ with compact support, the following holds:

$$\int_{\Omega} \mathcal{A}(\cdot, \nabla u) \bullet \mathcal{D}\varphi \, dV = 0.$$

Definition 2.2. We say that $u \in W^{1,p(x)}_{loc}(\Omega; V)$ is a supersolution of (1.1) in Ω if, for any nonnegative test function $\varphi \in W^{1,p(x)}(\Omega; V)$ with compact support, the following holds:

$$\int_{\Omega} \mathcal{A}(\cdot, \nabla u) \bullet \mathcal{D}\varphi \, dV \ge 0. \tag{2.4}$$

Additionally, u is a subsolution in Ω if -u is a supersolution in Ω , and u is a solution in Ω if it is both a supersolution and a subsolution in Ω .

Definition 2.3. Let S be a closed subset of Ω . We say that S is a removable set for Hölder continuous A-harmonic functions if, for any Hölder continuous function $u:\Omega\to\mathbb{R}$, the following holds: If u is A-harmonic in $\Omega\setminus S$, then u is A-harmonic in Ω .

2.3. **Metric Spaces.** We assume that $X=(X,d,\mu)$ is a metric space endowed with a metric d and a positive complete Borel measure μ such that

$$\mu(B) < \infty$$
 for all balls $B \subset X$.

The measure μ is said to be *doubling* if there exists a constant $C_{\mu} \geq 1$, called the doubling constant of μ , such that for all balls B,

$$0 < \mu(2B) \le C_{\mu}\mu(B) < \infty.$$

Definition 2.4. Let

$$||f||_{\mathrm{BMO}(W;X)} := \sup_{B \subset W} \oint_B |f - f_B| \, \mathrm{d}\mu,$$

where the supremum is taken over all balls $B \subset \Omega$ (and we implicitly require that f_B is finite for all balls $B \subset \Omega$). We define the class of functions of bounded mean oscillation as

$$BMO(W; X) := \{ f \mid ||f||_{BMO(W; X)} < \infty \}.$$

Proposition 2.5. (see [4, Theorem 3.20].) Suppose μ is doubling. Let $B \subset X$ be a ball and $f \in BMO(5B; X)$. Let further $A := \log(2)/4C_{\mu}^{15}$. Then for every $0 < \epsilon < A$ there is a constant c, only depending on ϵ and C_{μ} , such that

$$\int_{B} e^{\epsilon |f - f_B| / \|f\|_{\mathrm{BMO}(5B;X)}} \, \mathrm{d}\mu \le c.$$

2.4. Integrals in Terms of Distribution Functions. Let $\{X, A, \mu\}$ be a measure space and $E \in A$.

Let $f:E\to [-\infty,\infty]$ be measurable and nonnegative. The distribution function of f relative to E is defined as

$$t \in \mathbb{R}^+ \mapsto \mu(\{f > t\}).$$

This is a nonincreasing function of t and if f is finite a.e. in E, then

$$\lim_{t\to\infty}\mu([f>t])=0 \quad \text{ unless } \quad \mu([f>t])\equiv\infty.$$

If f is integrable, such a limit can be given a quantitative form. Indeed

$$t\mu([f>t]) = \int_E t\chi_{[f>t]} \,\mathrm{d}\mu \le \int_E f \,\mathrm{d}\mu < \infty.$$

Proposition 2.6. (see [11, Proposition 15.1].) Let $\{X, A, \mu\}$ be complete and σ -finite and let $f: E \to [-\infty, \infty]$ be measurable and nonnegative. Let also v be a complete and σ -finite measure on \mathbb{R}^+ such that v([0,t]) = v([0,t]) for all t > 0. Then

$$\int_{E} \mathbf{v}(\{0, f\}) \, \mathrm{d}\mu = \int_{0}^{\infty} \mu(\{f > t\}) \, \mathrm{d}\mathbf{v}.$$

In particular if $v([0,t]) = t^q$ for some q > 0, then

$$\int_{E} f^{q} d\mu = q \int_{0}^{\infty} t^{q-1} \mu(\{f > t\}) dt,$$

where dt is the Lebesgue measure on \mathbb{R}^+ .

2.5. Auxiliary Inequalities and Estimates. From (1.9), there exist constants $K_2 > 0$ and $n_2 > 1$ such that we have

$$\frac{\operatorname{V}(B(x_1,r))}{\operatorname{V}(B(x_0,R))} \ge \operatorname{K}_2\left(\frac{r}{R}\right)^{\operatorname{n}_2},\tag{2.5}$$

whenever 0 < r < R and $x_1 \in B(x_0, R)$, see [22, pp. 103-104]

Lemma 2.7. Assume that (1.10) is satisfied. Let $v \in W_0^{1,q}(B(x,r); V) \cap L^{\infty}(B(x,r))$ with $q \ge 1$, then

$$\left(\int_{B(x,r)} |v|^{q1_{\mathrm{I}}} \, \mathrm{dV} \right)^{\frac{1}{q1_{\mathrm{I}}}} \le q \mathsf{K}_{\mathrm{I}} r \left(\int_{B(x,r)} |\nabla v|_F^q \, \mathrm{dV} \right)^{\frac{1}{q}}. \tag{2.6}$$

Proof. From (1.10) and Hölder's inequality,

$$\begin{split} \left(\int_{B(x,r)} |v|^{q1_{\mathrm{I}}} \, \mathrm{dV} \right)^{\frac{1}{1_{\mathrm{I}}}} &\leq \mathrm{K}_{\mathrm{I}} r \int_{B(x,r)} |\boldsymbol{\nabla} (|v|^q)|_F \, \mathrm{dV} \\ &\leq q \mathrm{K}_{\mathrm{I}} r \left(\int_{B(x,r)} |\boldsymbol{\nabla} v|_F^q \, \mathrm{dV} \right)^{\frac{1}{q}} \left(\int_{B(x,r)} |v|^q \, \mathrm{dV} \right)^{\frac{q-1}{q}} \\ &\leq q \mathrm{K}_{\mathrm{I}} r \left(\int_{B(x,r)} |\boldsymbol{\nabla} v|_F^q \, \mathrm{dV} \right)^{\frac{1}{q}} \left(\int_{B(x,r)} |v|^{q1_{\mathrm{I}}} \, \mathrm{dV} \right)^{\frac{q-1}{q1_{\mathrm{I}}}}. \end{split}$$

Using this inequality we obtain (2.6).

The next two lemmas are based on [21].

Lemma 2.8. Assume that (1.7) and (1.8) holds, we have

(i) If
$$0 < r \le R_* := [1/(K+1)]^{1/n_1} < 1$$
, then

$$p_r^+ - p_r^- \le K_3 := \frac{\log K_0}{\log \frac{K+1}{K}},$$
 (2.7)

where $p_r^- = \inf_{B(x,r)} p$ and $p_r^+ = \sup_{B(x,r)} p$.

(ii) If $0 < r \le R_*$, then

$$r^{-p(x)} \le K_4(K, K_0, n_1) r^{-p_W^-},$$
 (2.8)

where $x \in W \subset B(x,r)$ and $p_W^- = \inf_W p$.

Proof. (i) This is a consequence of (1.7) and (1.8).

(ii) We have $0 \le r \le R_* < 1$ and $x \in W \subset B(x, r)$. From (1.8),

$$r^{-p(x)} \le r^{-p_W^+} \le r^{-p_r^+} \le r^{-p_r^+ + p_r^- - p_W^-} \le \left(\mathbb{K}^{p_r^+ - p_r^-} \mathbb{K}_0 \right)^{\frac{1}{\mathbf{n}_1}} r^{-p_W^-}.$$

By employing (2.7), we obtain (2.8).

Lemma 2.9. Assume that (1.8) holds. Let $f \in L^s(B(x,r); V)$, with $0 < r \le R_*$ and $s > p_r^+ - p_r^-$. Then,

$$\int_{B(x,r)} |f|^{p_r^+ - p_r^-} \, d\mathbf{V} \le \mathbf{K}_0^{\frac{1}{s}} \|f\|_{s,B(x,r),\mathbf{V}}^{p_r^+ - p_r^-} \tag{2.9}$$

and

$$\int_{B(x,r)} |f|^{p-p_r^-} \, \mathrm{dV} \le \mathsf{K}_0^{\frac{1}{s}} ||f||_{s,B(x,r),\mathsf{V}}^{p_r^+ - p_r^-} + 1. \tag{2.10}$$

Proof. Let $q:=p_r^+-p_r^-$. Hölder's inequality gives

$$\int_{B(x,r)} |f|^{p_r^+ - p_r^-} \, \mathrm{dV} \le [\mathtt{V}(B(x,r))]^{-\frac{q}{s}} \left(\int_{B(x,r)} |f|^s \, \mathrm{dV} \right)^{\frac{q}{s}}.$$

Using (1.8), we get (2.9).

Inequality (2.10) follows from (2.9) by applying $|a|^{p(y)-p_r^-} \le |a|^{p_r^+-p_r^-} + 1$ for all $a \in \mathbb{R}$ and $y \in B(x,r)$.

Based on [24, Lemma 3.38], we obtain the following result.

Lemma 2.10. Assume that (1.9) is verified. Suppose $0 < q < \gamma < s$ and $\beta \ge 0$. If a nonnegative function $v \in L^s(B(x_0, r); V)$ satisfies

$$\left(\int_{B(x_0, \lambda r')} v^s \, d\mathbf{V} \right)^{\frac{1}{s}} \le \bar{c} (1 - \lambda)^{-\beta} \left(\int_{B(x_0, r')} v^{\gamma} \, d\mathbf{V} \right)^{\frac{1}{\gamma}} \tag{2.11}$$

for each $0 < r' \le r$ and $0 < \lambda < 1$, then

$$\left(\oint_{B(x_0,\lambda r)} v^s \, d\mathbf{V} \right)^{\frac{1}{s}} \le C(1-\lambda)^{-\frac{\beta}{\theta}} \left(\oint_{B(x_0,r)} v^q \, d\mathbf{V} \right)^{\frac{1}{q}}. \tag{2.12}$$

for all $0 < \lambda < 1$. Here, $C := 2(\mathtt{K}_2 2^{-n_2})^{-\frac{1}{q}} \theta \left[(1-\theta) \overline{c} 2^{\frac{\beta}{\theta}+1} \right]^{\frac{1-\theta}{\theta}}$, and $\theta \in (0,1)$ is such that $\frac{1}{\gamma} = \frac{\theta}{q} + \frac{1-\theta}{s}.$

Proof. Let

$$N := \sup_{\frac{1}{2} < \lambda < 1} (1 - \lambda)^{\hat{\beta}} \left(\oint_{B(x_0, \lambda r)} v^{\gamma} \, \mathrm{dV} \right)^{\frac{1}{\gamma}},$$

where $\hat{\beta} = \beta(1 - \theta)/\theta$.

Writing $\lambda' := 1/2(1+\lambda)$ for each $\lambda \in (0,1)$, we have, by (2.11),

$$(1-\lambda)^{\frac{\hat{\beta}}{1-\theta}} \left(\oint_{B(x,\lambda r)} v^s \, d\mathbf{V} \right)^{\frac{1}{s}} \leq \bar{c} 2^{\frac{\beta}{\theta}} (1-\lambda')^{\hat{\beta}} \left(\oint_{B(x_0,\lambda' r)} v^{\gamma} \, d\mathbf{V} \right)^{\frac{1}{\gamma}} \leq \bar{c} 2^{\frac{\beta}{\theta}} N. \tag{2.13}$$

Fix $\delta > 0$ and choose $\lambda_{\delta} \in [1/2, 1)$ such that

$$N \leq (1 - \lambda_{\delta})^{\hat{\beta}} \left(\int_{B(x_0, \lambda_{\delta} r)} v^{\gamma} \, \mathrm{dV} \right)^{\frac{1}{\gamma}} + \delta.$$

Next, we apply Young's inequality:

$$|ab| \le \epsilon^{-\frac{1-\theta}{\theta}} \theta |a|^{\frac{1}{\theta}} + \epsilon (1-\theta) |b|^{\frac{1}{1-\theta}},$$

where $\epsilon > 0$.

Since

$$1 = \theta \frac{\gamma}{a} + (1 - \theta) \frac{\gamma}{s},$$

we have

$$\begin{split} N &\leq (1-\lambda_0)^{\hat{\beta}} \left(\oint_{B(x_0,\lambda_\delta r)} v^{\gamma} \, \mathrm{dV} \right)^{\frac{1}{\gamma}} + \delta \\ &= (1-\lambda_\delta)^{\hat{\beta}} \left(\oint_{B(x_0,\lambda_\delta r)} v^{\gamma\theta} v^{\gamma(1-\theta)} \, \mathrm{dV} \right)^{\frac{1}{\gamma}} + \delta \\ &\leq (1-\lambda_\delta)^{\hat{\beta}} \left(\oint_{B(x_0,\lambda_\delta r)} v^q \, \mathrm{dV} \right)^{\frac{\theta}{q}} \left(\oint_{B(x_0,\lambda_\delta r)} v^s \, \mathrm{dV} \right)^{\frac{1-\theta}{s}} + \delta \\ &\leq \theta \epsilon^{-\frac{1-\theta}{\theta}} \left(\oint_{B(x_0,\lambda_\delta r)} v^q \, \mathrm{dV} \right)^{\frac{1}{q}} + \epsilon (1-\theta) (1-\lambda_\delta)^{\frac{\hat{\beta}}{1-\theta}} \left(\oint_{B(x_0,\lambda_\delta r)} v^s \, \mathrm{dV} \right)^{\frac{1}{s}} + \delta \\ &\leq (\mathrm{K}_2 \lambda_\delta^{\mathrm{n}_2})^{-\frac{1}{q}} \theta \epsilon^{-\frac{1-\theta}{\theta}} \left(\oint_{B(x_0,r)} v^q \, \mathrm{dV} \right)^{\frac{1}{q}} + \epsilon (1-\theta) \bar{c} 2^{\frac{\beta}{\theta}} N + \delta, \end{split}$$

where the last two inequalities follow from the property (2.5) of V and (2.13).

Choosing $\epsilon=[(1-\theta)\bar{c}2^{\frac{\beta}{\theta}+1}]^{-1}$ and letting $\delta\to 0$, we get

$$N \leq 2(\mathsf{K}_2 2^{-\mathsf{n}_2})^{-\frac{1}{q}} \theta \left[(1-\theta) \bar{c} 2^{\frac{\beta}{\theta}+1} \right]^{\frac{1-\theta}{\theta}} \left(\int_{B(x_0,r)} v^q \, \mathrm{dV} \right)^{\frac{1}{q}}.$$

Hence, from (2.13), we have

$$(1-\lambda)^{\frac{\beta}{\theta}} \left(\oint_{B(x_0,\lambda r)} v^s \,\mathrm{dV} \right)^{\frac{1}{s}} \leq 2 (\mathtt{K}_2 2^{-\mathtt{n}_2})^{-\frac{1}{q}} \theta \left[(1-\theta) \bar{c} 2^{\frac{\beta}{\theta}+1} \right]^{\frac{1-\theta}{\theta}} \left(\oint_{B(x_0,r)} v^q \,\mathrm{dV} \right)^{\frac{1}{q}}.$$

This proves (2.12).

As a consequence of Lemma 2.10, we have the

Corollary 2.11. Assume that (1.9) is verified. Suppose $0 < q < \gamma < s$ and $\beta \ge 0$. If a nonnegative function $v \in L^{\infty}(B(x_0, r))$ satisfies

$$||v||_{L^{\infty}(B(x_0,\lambda r'))} \leq \bar{c}(1-\lambda)^{-\beta} \left(\int_{B(x_0,r')} v^{\gamma} \, \mathrm{dV} \right)^{\frac{1}{\gamma}}$$

for each $0 < r' \le r$ and $0 < \lambda < 1$, then

$$||v||_{L^{\infty}(B(x_0,\lambda r))} \leq C(1-\lambda)^{-\frac{\beta}{\theta}} \left(\oint_{B(x_0,r)} v^q \, \mathrm{dV} \right)^{\frac{1}{q}}.$$

for all $0 < \lambda < 1$. Here, $C := 2(\mathtt{K}_2 2^{-\mathtt{n}_2})^{-\frac{1}{q}} \theta \left[(1-\theta) \overline{c} 2^{\frac{\beta}{\theta}+1} \right]^{\frac{1-\theta}{\theta}}$, and $\theta \in (0,1)$ is such that $\frac{1}{\gamma} = \frac{\theta}{q}.$

2.6. The existence of solutions for obstacle problems. Let $\mathcal{O}_1:\Omega\to[-\infty,+\infty]$ be a function, and let $\mathcal{O}_2\in W^{1,p(x)}(\Omega;\mathbb{V})$. Define

$$\mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega) := \left\{ v \in W^{1,p(x)}(\Omega; \mathbf{V}) \mid v \geq \mathcal{O}_1 \text{ a.e. in } \Omega, v - \mathcal{O}_2 \in W^{1,p(x)}_0(\Omega; \mathbf{V}) \right\}.$$

If $\mathcal{O}_1 = \mathcal{O}_2$, we write $\mathcal{K}_{\mathcal{O}_1,\mathcal{O}_1}(\Omega) = \mathcal{K}_{\mathcal{O}_1}(\Omega)$.

The obstacle problem is to find a function $u \in \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega)$ such that

$$\int_{\Omega} \mathcal{A}(\cdot, \nabla u) \bullet \mathcal{D}(v - u) \, dV \ge 0 \tag{2.14}$$

or

$$\int_{\Omega} \left[-\mathcal{A}(\cdot, -\nabla u) \right] \bullet \mathcal{D}(v - u) \, dV \ge 0, \tag{2.15}$$

for all $v \in \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega)$. The function $u \in W^{1,p(x)}(\Omega; V)$ is called a solution to the obstacle problem with obstacle \mathcal{O}_1 and boundary values \mathcal{O}_2 .

Similar to [40, Theorem 3.4] and [14, Theorem 3.2] (for the proof, see these works), we obtain:

Proposition 2.12. Let $\Omega_0 \subset\subset \Omega$ be a open set. Suppose that $\mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega_0) \neq \emptyset$ and conditions (a_1) - (a_3) are satisfied. Then there exists a unique solution $u \in \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega_0)$ to the obstacle problem (2.14).

The proofs of the following theorems are similar to those on pp. 61-62 in [24] and are not repeated here.

Lemma 2.13. Let $u \in \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega)$ be a solution of the obstacle problem (2.14). If $v \in W^{1,p(x)}(\Omega; V)$ is a supersolution in Ω such that $\min\{u,v\} \in \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega)$, then $v \geq u$ a.e. in Ω .

Lemma 2.14. If u and v are two supersolutions of (1.1), then $\min\{u, v\}$ is also a supersolution.

Lemma 2.15. Let $u \in \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega)$ be a solution of the obstacle problem (2.14). Then

$$u \leq \operatorname{ess\,sup\,max}\{\mathcal{O}_1(x),\mathcal{O}_2(x)\}$$
 a.e. in Ω .

Similary to [21, Theorem 4.11] (see also [24, Theorem 3.67] for the fixed exponent case) we have the

Proposition 2.16. Let $\mathcal{O}_1: \Omega \to [-\infty, \infty)$ be continuous. Then the solution $u \in \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega)$ of the obstacle problem (2.14) is continuous. Moreover, u is a solution in the open set $\{x \in \Omega \mid u(x) > \mathcal{O}_1(x)\}$.

3. Basic estimates

The main results of this section are Propositions 3.4 and 3.5, and 3.7. As a consequence of these results, we have the following:

Proposition 3.1. Let $\mathcal{O}_1: \Omega \to [-\infty, \infty)$ be continuous. Let $\{u_\ell\} \subset W^{1,p(x)}_{\mathrm{loc}}(\Omega; V) \cap \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega)$ be a family of functions, and let $u \in W^{1,p(x)}_{loc}(\Omega; V) \cap L^{\infty}_{loc}(\Omega)$ be an element. Assume that property (P_1) is satisfied and that $u \in \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega)$ is a solution to the obstacle problem (2.14). Then, for all ℓ , u_ℓ is continuous.

The proof of this proposition follows the same argument as in [14, Theorem 4.3], [24, Theorem 3.67], or [18, Theorem 4.11], and is therefore omitted.

For a chart $\psi: U \to \psi(U) \subset \Omega$, we define

$$\begin{aligned} \|d(\psi^{-1})\| &:= \sup\{|d(\psi^{-1})_x v|_{\delta_{\mathbb{R}^n}} \mid (x,v) \in T\psi(\tilde{U}), |v|_F(x) = 1\}, \\ \|d\psi\| &:= \sup\{|d\psi_{\mathbf{x}} \mathbf{z}|_F(\psi(\mathbf{x})) \mid \mathbf{x} \in \tilde{U}, |\mathbf{z}|_{\delta_{\mathbb{R}^n}} = 1\}, \end{aligned}$$

and $\hat{c}_1 > 0$ is a constant such that

$$\hat{c}_1 \ge \|d(\psi^{-1})\| \|d\psi\|. \tag{3.1}$$

Lemma 3.2. Let $0 < \sigma < \rho$ such that $B(x_0, \rho) \subset \psi(\tilde{U})$. There exists a smooth function $\eta : B(x_0, \rho) \to$ [0,1] such that $\eta=1$ in $B(x_0,\sigma)$, supp $\eta\subset B(x_0,\rho)$, and $|\mathfrak{D}\eta|_{F^*}\leq \hat{c}_0\hat{c}_1(\rho-\sigma)^{-1}$ in $B(x_0,\rho)$, where $\hat{c}_0 := \hat{c}_0(n) > 0.$

Proof. Set $V := \psi^{-1}(B(x_0, \sigma))$ and $W := \psi^{-1}(B(x_0, \rho))$. There exists a smooth function $\eta_0 : V \subset \mathbb{R}^n \to \mathbb{R}^n$ [0,1] such that $\eta_0=1$ in V, $\operatorname{supp} \eta_0\subset W$, and $|\nabla_{\delta_{\mathbb{R}^n}}\eta_0|_{\delta_{\mathbb{R}^n}}\leq C_1(n)(\operatorname{dist}(V,\mathbb{R}^n\backslash W))^{-1}$. Let $\mathbf{x}\in\partial V$ and $\mathbf{y}\in\partial W$ such that $\operatorname{dist}(V,\mathbb{R}^n\backslash W)=|\mathbf{x}-\mathbf{y}|_{\delta_{\mathbb{R}^n}}$. For $\alpha(t):=\psi((1-t)\mathbf{x}+t\mathbf{y})$, we have

$$\rho - \sigma \le \int_0^1 F(\alpha, \alpha') \, \mathrm{d}t \le \|d\psi\| |\mathbf{x} - \mathbf{y}|_{\delta_{\mathbb{R}^n}}. \tag{3.2}$$

Then $\eta = \eta_0 \circ \psi^{-1} : B(x_0, \rho) \to [0, 1]$ satisfies $\eta = 1$ in $B(x_0, \sigma)$, supp $\eta \subset B(x_0, \rho)$, and

$$\begin{split} |\mathcal{D}\eta|_{F^*}(x) &= \sup_{v \in T_x \Omega \setminus \{0\}} \frac{\mathcal{D}\eta_x(v)}{|v|_F(x)} \le ||d(\psi^{-1})|| |\nabla_{\delta_{\mathbb{R}^n}} \eta_0|_{\delta_{\mathbb{R}^n}} (\psi^{-1}(x)) \\ &\le C_1 ||d(\psi^{-1})|| |\mathbf{x} - \mathbf{y}|_{\delta_{\mathbb{R}^n}}^{-1} \quad \text{in} \quad B(x_0, \rho). \end{split}$$

Using (3.2), we prove the lemma.

3.1. Upper bound for solutions of the obstacle problem and for supersolutions. In this subsection, the objective is to prove Proposition 3.4. Before that, we show the auxiliary result:

Lemma 3.3. Let $N_0 \geq |N|$, $q \geq 0$, and $B(x_0, \mathbb{R}) \subset \Omega$ with $0 < \mathbb{R} \leq 1$. Let $\eta \in C_0^\infty(B(x_0, \mathbb{R}))$ with $0 \leq \eta \leq 1$. Consider a family $\{u_\ell\} \subset W^{1,p(x)}_{\mathrm{loc}}(\Omega; \mathbb{V})$ and an element $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega; \mathbb{V}) \cap L^\infty_{\mathrm{loc}}(\Omega)$. Then, we have:

(i) Assume that property (P_1) is satisfied and $u, u_\ell \in \mathcal{K}_{\mathcal{O}_1, \mathcal{O}_2}(\Omega)$ for all ℓ . If u is a solution of the obstacle problem (2.14) with the obstacle $O_1 \leq N$ in $B(x_0, \mathbb{R})$, then

$$\int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{q} |\nabla (u_{\ell} - N)^{+}|_{F}^{p_{R}^{-}} \eta^{p_{R}^{+}} dV
\leq C_{1} \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{p+q} |\mathcal{D}\eta|_{F^{*}}^{p} dV + C_{2} \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{q} \eta^{p_{R}^{+}} dV,$$
(3.3)

where

$$\mathcal{C}_1 := \frac{4\Lambda_2 p_{\mathtt{R}}^+}{\Lambda_1 p_{\mathtt{R}}^-} \max \left\{ \left[2\Lambda_2 (p_{\mathtt{R}}^+ - 1)/\Lambda_1 \right]^{p_{\mathtt{R}}^+ - 1}, \left[2\Lambda_2 (p_{\mathtt{R}}^+ - 1)/\Lambda_1 \right]^{p_{\mathtt{R}}^- - 1} \right\} + \frac{4\mathfrak{T} p_{\mathtt{R}}^+}{\Lambda_1 p_{\mathtt{R}}^-}$$

and

$$\mathcal{C}_2 := \frac{4\mathfrak{T}[(p_{\mathtt{R}}^+)^2 - 1]}{\Lambda_1 p_{\mathtt{R}}^+} + 1.$$

(ii) Assume that (1.14) (of property (P_1)) holds. If u is a supersolution of (1.1) in Ω , then

$$\int_{\Omega} \left[|(u_{\ell} - N)^{-}| + r \right]^{q} |\nabla(u_{\ell} - N)^{-}|_{F}^{p_{R}^{-}} \eta^{p_{R}^{+}} dV
\leq C_{1} \int_{\Omega} \left[|(u_{\ell} - N)^{-}| + r \right]^{p+q} |\mathcal{D}\eta|_{F^{*}}^{p} dV + C_{2} \int_{\Omega} \left[|(u_{\ell} - N)^{-}| + r \right]^{q} \eta^{p_{R}^{+}} dV.$$
(3.4)

In (i) and (ii), $p_{\mathbb{R}}^- := \inf_{B(x_0,\mathbb{R})} p$, $p_{\mathbb{R}}^+ := \sup_{B(x_0,\mathbb{R})} p$, $(u_{\ell} - N)^+ := \max\{u_{\ell} - N, 0\}$, and $(u_{\ell} - N)^- = \min\{u_{\ell} - N, 0\}$.

Proof. (i) Steep 1. Let $h \ge -r$ and define $\zeta = -[(u_{\ell} - N - h)^+ - r]\eta^{p_{\mathtt{R}}^+}$. Then $u_{\ell} + \zeta = u_{\ell} - [(u_{\ell} - N - h)^+ - r]\eta^{p_{\mathtt{R}}^+} \ge \mathfrak{O}_1$ in $B(x_0,\mathtt{R})$, and $u_{\ell} + \zeta - \mathfrak{O}_2 \in W_0^{1,p(x)}(\Omega;\mathtt{V})$.

We have

$$\mathcal{D}\zeta = -\eta^{p_{R}^{+}} \mathcal{D}(u_{\ell} - N - h)^{+} - p_{R}^{+} \eta^{p_{R}^{+} - 1} \left[(u_{\ell} - N - h)^{+} - r \right] \mathcal{D}\eta.$$

Taking $v := u_{\ell} + \zeta$:

$$\begin{split} & \int_{\Omega} \mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) \bullet \mathcal{D}(v - u_{\ell}) \, \mathrm{d} \mathbf{V} \\ & = \int_{\Omega} \mathcal{A}(\cdot, \boldsymbol{\nabla} u) \bullet \mathcal{D}(v - u) \\ & + \left[\mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) - \mathcal{A}(\cdot, \boldsymbol{\nabla} u) \right] \bullet \mathcal{D}(v - u) + \mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) \bullet \mathcal{D}(u - u_{\ell}) \, \mathrm{d} \mathbf{V} \\ & = \int_{\Omega} \mathcal{A}(\cdot, \boldsymbol{\nabla} u) \bullet \mathcal{D}(v - u) \, \mathrm{d} \mathbf{V} + f(\ell, N, h), \end{split}$$

where

$$f(\ell, N, h) := \int_{\Omega} \left[\mathcal{A}(\cdot, \nabla u_{\ell}) - \mathcal{A}(\cdot, \nabla u) \right] \bullet \left\{ \mathcal{D}(u_{\ell} - u) - \eta^{p_{\mathsf{R}}^{+}} \mathcal{D}(u_{\ell} - N - h)^{+} - p_{\mathsf{R}}^{+} \eta^{p_{\mathsf{R}}^{+} - 1} \left[(u_{\ell} - N - h)^{+} - r \right] \mathcal{D} \eta \right\} \, d\mathsf{V} + \int_{\Omega} \mathcal{A}(\cdot, \nabla u_{\ell}) \bullet \mathcal{D}(u - u_{\ell}) \, d\mathsf{V},$$

Since $u \in \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega)$ is a solution of the obstacle problem (2.14), and $u_\ell \in \mathcal{K}_{\mathcal{O}_1,\mathcal{O}_2}(\Omega)$:

$$\int_{\Omega} \mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) \bullet \left\{ -\eta^{p_{\mathtt{R}}^{+}} \mathcal{D}(u_{\ell} - N - h)^{+} - p_{\mathtt{R}}^{+} \eta^{p_{\mathtt{R}}^{+}} \left[(u_{\ell} - N - h)^{+} - r \right] \mathcal{D} \eta \right\} d\mathtt{V} \geq f(\ell, N, h),$$

Let $\Omega_{\ell,N,h} = \{x \in \Omega \mid u_{\ell}(x) > N+h\}$. By (a_1) and (a_2) ,

$$\Lambda_{1} \int_{\Omega_{\ell,N,h}} |\nabla u_{\ell}|_{F}^{p} \eta^{p_{R}^{+}} dV
\leq p_{R}^{+} \Lambda_{2} \int_{\Omega_{\ell,N,h}} |\nabla u_{\ell}|_{F}^{p-1} |\mathcal{D}\eta|_{F^{*}} \left| (u_{\ell} - N - h)^{+} - r \right| \eta^{p_{R}^{+} - 1} dV - f(\ell, N, h)
\leq p_{R}^{+} \Lambda_{2} \int_{\Omega_{\ell,N,h}} \frac{p_{R}^{+} - 1}{p_{R}^{+}} \epsilon |\nabla u_{\ell}|_{F}^{p} \eta^{p_{R}^{+}} + \frac{\max\{\epsilon^{-p_{R}^{+} + 1}, \epsilon^{-p_{R}^{-} + 1}\}}{p_{R}^{-}} \left| (u_{\ell} - N - h)^{+} - r \right|^{p} \eta^{p_{R}^{+} - p} |\mathcal{D}\eta|_{F^{*}} dV
- f(\ell, N, h).$$
(3.5)

Taking $\epsilon = \frac{\Lambda_1}{2\Lambda_2(p_{\rm k}^+-1)}$ in (3.6), we have

$$\frac{\Lambda_{1}}{2} \int_{\Omega_{\ell,N,h}} |\nabla u_{\ell}|_{F}^{p} \eta^{p_{R}^{+}} dV \leq \frac{\Lambda_{2} p_{R}^{+}}{p_{R}^{-}} \max \left\{ \left[2\Lambda_{2} (p_{R}^{+} - 1)/\Lambda_{1} \right]^{p_{R}^{+} - 1}, \left[2\Lambda_{2} (p_{R}^{+} - 1)/\Lambda_{1} \right]^{p_{R}^{-} - 1} \right\}
\cdot \int_{\Omega_{\ell,N,h}} \left| (u_{\ell} - N - h)^{+} - r \right|^{p} \eta^{p_{R}^{+} - p} |\mathcal{D}\eta|_{F^{*}}^{p} dV - f(\ell, N, h).$$
(3.7)

Steep 2. Now we will estimate the last term of (3.7). From (1.13) and (2.1),

$$-f(\ell, N, h)$$

$$= \int_{\Omega_{\ell, N, h}} \left[\mathcal{A}(\cdot, \nabla u_{\ell}) - \mathcal{A}(\cdot, \nabla u) \right] \bullet \left\{ \eta^{p_{\mathbb{R}}^{+}} \mathcal{D}(u_{\ell} - N - h)^{+} + p_{\mathbb{R}}^{+} \eta^{p_{\mathbb{R}}^{+} - 1} \left[(u_{\ell} - N - h)^{+} - r \right] \mathcal{D} \eta \right\} dV$$

$$\leq \int_{\Omega_{\ell, N, h}} \left| \mathcal{A}(\cdot, \nabla u_{\ell}) - \mathcal{A}(\cdot, \nabla u) \right|_{F} \eta^{p_{\mathbb{R}}^{+}} |\nabla u_{\ell}|_{F}$$

$$+ p_{\mathbb{R}}^{+} \left| \mathcal{A}(\cdot, \nabla u_{\ell}) - \mathcal{A}(\cdot, \nabla u) \right|_{F} \eta^{p_{\mathbb{R}}^{+} - 1} \left| (u_{\ell} - N + h)^{+} - r \right| |\mathcal{D} \eta|_{F^{*}} dV. \tag{3.8}$$

From (1.14),

$$|\mathcal{A}(\cdot,\boldsymbol{\nabla}u_{\ell})-\mathcal{A}(\cdot,\boldsymbol{\nabla}u)|_{F}\leq\mathfrak{T}<\frac{\Lambda_{1}p_{\mathbf{R}}^{-}}{\varLambda}\quad\text{ in }\quad\Omega.$$

We also have, in $B(x_0, \mathbb{R})$,

$$|\nabla u_\ell|_F \le \frac{1}{p_{\mathtt{R}}^-} |\nabla u_\ell|_F^p + \frac{p_{\mathtt{R}}^+ - 1}{p_{\mathtt{R}}^+}$$

and

$$\eta^{p_{\mathbf{R}}^{+}-1} \left| (u_{\ell} - N - h)^{+} - r \right| |\mathfrak{D}\eta|_{F^{*}} \leq \frac{1}{p} \left| (u_{\ell} - N - h)^{+} - r \right|^{p} |\mathfrak{D}\eta|_{F^{*}}^{p} + \frac{p-1}{p} \eta^{\frac{p_{\mathbf{R}}^{+}-1}{p-1}p}$$

$$\leq \frac{1}{p_{\mathbf{R}}^{-}} \left| (u_{\ell} - N - h)^{+} - r \right|^{p} |\mathfrak{D}\eta|_{F^{*}}^{p} + \frac{p_{\mathbf{R}}^{+}-1}{p_{\mathbf{R}}^{+}} \eta^{p_{\mathbf{R}}^{+}}.$$

Consequently, from (3.8),

$$\begin{aligned}
&-f(\ell, N, h) \\
&\leq \frac{\Lambda_{1}}{4} \int_{\Omega_{\ell, N, h}} |\nabla u_{\ell}|_{F}^{p} \eta^{p_{R}^{+}} \, dV + \frac{\mathfrak{T}(p_{R}^{+} - 1)}{p_{R}^{+}} \int_{\Omega_{\ell, N, h}} \eta^{p_{R}^{+}} \, dV \\
&+ \frac{\mathfrak{T}p_{R}^{+}}{p_{R}^{-}} \int_{\Omega_{\ell, N, h}} \left| (u_{\ell} - N - h)^{+} - r \right|^{p} |\mathcal{D}\eta|_{F^{*}}^{p} \, dV + \mathfrak{T}(p_{R}^{+} - 1) \int_{\Omega_{\ell, N, h}} \eta^{p_{R}^{+}} \, dV.
\end{aligned} \tag{3.9}$$

Steep 3. Using (3.7) and (3.9),

$$\frac{\Lambda_{1}}{4} \int_{\Omega_{\ell,N,h}} |\nabla u_{\ell}|_{F}^{p} \eta^{p_{\mathbb{R}}^{+}} \, d\mathbb{V} \leq \mathcal{C} \int_{\Omega_{\ell,N,h}} \left| (u_{\ell} - N - h)^{+} - r \right|^{p} |\mathcal{D}\eta|_{F^{*}}^{p} \, d\mathbb{V} + \frac{\mathfrak{T}[(p_{\mathbb{R}}^{+})^{2} - 1]}{p_{\mathbb{R}}^{+}} \int_{\Omega_{\ell,N,h}} \eta^{p_{\mathbb{R}}^{+}} \, d\mathbb{V}, \tag{3.10}$$

where

$$\mathcal{C} = \frac{\Lambda_2 p_{\rm R}^+}{p_{\rm R}^-} \max \left\{ \left[2\Lambda_2 (p_{\rm R}^+ - 1)/\Lambda_1 \right]^{p_{\rm R}^+ - 1}, \left[2\Lambda_2 (p_{\rm R}^+ - 1)/\Lambda_1 \right]^{p_{\rm R}^- - 1} \right\} + \frac{\mathfrak{T}p_{\rm R}^+}{p_{\rm R}^-}.$$

By Proposition 2.6 (see also [24, equation (3.31)]), we have

$$\begin{split} & \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{q} | \boldsymbol{\nabla} (u_{\ell} - N)^{+} |_{F}^{p} \eta^{p_{R}^{+}} \, \mathrm{dV} \\ & = q \int_{0}^{\infty} t^{q-1} \int_{\{(u_{\ell} - N)^{+} + r > t\}} | \boldsymbol{\nabla} (u_{\ell} - N)^{+} |_{F}^{p} \eta^{p_{R}^{+}} \, \mathrm{dV} \mathrm{d}t \\ & = q \int_{-r}^{\infty} (h + r)^{q-1} \int_{\{u_{\ell} - N > h\}} | \boldsymbol{\nabla} (u_{\ell} - N)^{+} |_{F}^{p} \eta^{p_{R}^{+}} \, \mathrm{dV} \mathrm{d}h. \end{split}$$

From (3.10),

$$\frac{\Lambda_{1}}{4} \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{q} |\nabla(u_{\ell} - N)^{+}|_{F}^{p} \eta^{p_{R}^{+}} dV
\leq q \frac{\Lambda_{1}}{4} \int_{-r}^{\infty} (h+r)^{q-1} \int_{\Omega_{\ell,N,h}} |\nabla u_{\ell}|_{F}^{p} \eta^{p_{R}^{+}} dV dh
\leq q \mathcal{C} \int_{-r}^{\infty} (h+r)^{q-1} \int_{\Omega_{\ell,N,h}} \left| (u_{\ell} - N - h)^{+} - r \right|^{p} |\mathcal{D}\eta|_{F^{*}}^{p} dV dh
+ \frac{q \mathfrak{T}[(p_{R}^{+})^{2} - 1]}{p_{R}^{+}} \int_{-r}^{\infty} (h+r)^{q-1} \int_{\Omega_{\ell,N,h}} \eta^{p_{R}^{+}} dV dh$$
(3.11)

Consequently, taking into account that $|(u_{\ell} - N - h)^+ - r| \leq (u_{\ell} - N)^+ + r$ in $\Omega_{\ell,N,h}$:

$$\frac{\Lambda_{1}}{4} \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{q} |\nabla (u_{\ell} - N)^{+}|_{F}^{p} \eta^{p_{R}^{+}} dV
\leq q \mathcal{C} \int_{0}^{\infty} t^{q-1} \int_{\Omega_{\ell,N,t-r}} [(u_{\ell} - N)^{+} + r]^{p} |\mathcal{D}\eta|_{F^{*}}^{p} dV dt
+ \frac{q \mathfrak{T}[(p_{R}^{+})^{2} - 1]}{p_{R}^{+}} \int_{0}^{\infty} t^{q-1} \int_{\Omega_{\ell,N,t-r}} \eta^{p_{R}^{+}} dV dt.$$
(3.12)

Employing $|a|^{p_{\mathbb{R}}^-} \leq |a|^p + 1 \ \forall a \in \mathbb{R}$ and Proposition 2.6. From (3.12), we have

$$\frac{\Lambda_{1}}{4} \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{q} |\nabla(u_{\ell} - N)^{+}|_{F}^{p_{R}^{-}} \eta^{p_{R}^{+}} dV
\leq \frac{\Lambda_{1}}{4} \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{q} |\nabla(u_{\ell} - N)^{+}|_{F}^{p} \eta^{p_{R}^{+}} dV + \frac{\Lambda_{1}}{4} \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{q} \eta^{p_{R}^{+}} dV
\leq C \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{p+q} |\mathcal{D}\eta|_{F^{*}}^{p} dV
+ \left\{ \frac{\mathfrak{T}[(p_{R}^{+})^{2} - 1]}{p_{R}^{+}} + \frac{\Lambda_{1}}{4} \right\} \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{q} \eta^{p_{R}^{+}} dV$$
(3.13)

Which conclude the proof of (i).

(ii) Let $h \ge -r$ and define $\zeta = -(u_\ell - N + h)^- \eta^{p_{\rm R}^+}$, then $\zeta \ge 0$ and $\zeta \in W_0^{1,p(x)}(\Omega; V)$. So ζ is a test function for (2.4), then

$$\int_{\Omega} \mathcal{A}(\cdot, \nabla u_{\ell}) \bullet \mathcal{D}\zeta \, dV = \int_{\Omega} \mathcal{A}(\cdot, \nabla u) \bullet \mathcal{D}\zeta \, dV + \int_{\Omega} \left[\mathcal{A}(\cdot, \nabla u_{\ell}) - \mathcal{A}(\cdot, \nabla u) \right] \bullet \mathcal{D}\zeta \, dV.$$

Hence,

$$\int_{\Omega} \mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) \bullet \left[-\eta^{p_{\mathtt{R}}^{+}} \mathcal{D}(u_{\ell} - N + h)^{-} - p_{\mathtt{R}}^{+} \eta^{p_{\mathtt{R}}^{+} - 1} (u_{\ell} - N + h)^{-} \mathcal{D} \eta \right] d\mathtt{V} \geq \hat{f}(\ell, N, h),$$

where

$$\begin{split} \hat{f}(\ell,N,h) := \int_{\Omega} \left[\mathcal{A}(\cdot,\boldsymbol{\nabla}u_{\ell}) - \mathcal{A}(\cdot,\boldsymbol{\nabla}u) \right] \bullet \left[-\eta^{p_{\mathtt{R}}^{+}} \mathcal{D}(u_{\ell} - N + h)^{-} \right. \\ \left. - p_{\mathtt{R}}^{+} \eta^{p_{\mathtt{R}}^{+} - 1} (u_{\ell} - N + h)^{-} \mathcal{D} \eta \right] \, \mathrm{dV}. \end{split}$$

Let $\Omega'_{\ell,N,h} := \{x \in \Omega \mid u_{\ell}(x) < N - h\}$. By (a_1) and (a_2) , we have

$$\Lambda_1 \int_{\Omega'_{\ell,N,h}} |\boldsymbol{\nabla} u_\ell|_F^p \eta^{p_{\rm R}^+} \, {\rm d} {\tt V} \leq p_{\rm R}^+ \Lambda_2 \int_{\Omega'_{\ell,N,h}} |\boldsymbol{\nabla} u_\ell|_F^{p-1} |\mathfrak{D} \eta|_{F^*} |(u_\ell - N + h)^-| \eta^{p_{\rm R}^+ - 1} \, {\rm d} {\tt V} - \hat{f}(\ell,N,h)$$

Proceeding similarly as we did for (3.5), we will obtain the following inequality (see (3.10)):

$$\frac{\Lambda_1}{4} \int_{\Omega'_{\ell,N,h}} |\nabla u_{\ell}|_F^p \eta^{p_{\mathsf{R}}^+} \, \mathrm{dV} \le \mathcal{C} \int_{\Omega'_{\ell,N,h}} \left| (u_{\ell} - N - h)^- \right|^p |\mathcal{D}\eta|_{F^*}^p \, \mathrm{dV} + \frac{\mathfrak{T}[(p_{\mathsf{R}}^+)^2 - 1]}{p_{\mathsf{R}}^+} \int_{\Omega'_{\ell,N,h}} \eta^{p_{\mathsf{R}}^+} \, \mathrm{dV}. \quad (3.14)$$

With which we obtain (3.4).

Proposition 3.4. Suppose the conditions (1.7) - (1.10) are verified. Assume that $r \leq \sigma < \rho \leq 2r \leq 2$ and $B(x_0, 2r) \subset \subset \psi(\tilde{U})$. Let $|N| \leq N_0$ and $1 < \gamma < 1_{\rm I}$. Consider a family $\{u_\ell\} \subset W^{1,p(x)}_{\rm loc}(\Omega; V)$ and an element $u \in W^{1,p(x)}_{\rm loc}(\Omega; V) \cap L^\infty_{\rm loc}(\Omega)$. Then, we have:

(i) Assume that property (P_1) is satisfied and $u, u_{\ell} \in \mathcal{K}_{\mathcal{O}_1, \mathcal{O}_2}(\Omega)$ for all ℓ . If u is a solution of the obstacle problem (2.14) with the obstacle $\mathcal{O}_1 \leq N$ in $B(x_0, 2r)$, then

$$\begin{aligned}
&\operatorname{ess\,sup}_{B(x_{0},\sigma)} \left\{ (u_{\ell} - N)^{+} + r \right\} \\
&\leq (C_{3} 1_{\mathrm{I}} / \gamma)^{C_{4}} C_{6}^{\frac{1}{p_{2r}^{-}(1_{\mathrm{I}} - \gamma)}} (p_{2r}^{-})^{\frac{p_{2r}^{+} 1_{\mathrm{I}}}{p_{2r}^{-}(1_{\mathrm{I}} - \gamma)}} \left(\frac{\rho}{\rho - \sigma} \right)^{\frac{p_{2r}^{+} 1_{\mathrm{I}}}{p_{2r}^{-}(1_{\mathrm{I}} - \gamma)}} \\
&\cdot \left\{ \int_{B(x_{0},\rho)} \left[(u_{\ell} - N)^{+} + r \right]^{\gamma p_{2r}^{-}} \, \mathrm{dV} \right\}^{\frac{1}{\gamma p_{2r}^{-}}},
\end{aligned} \tag{3.15}$$

where
$$C_3 := [2(\hat{c}_0 + 1)K_2^{-1}(K_I + 1)(K_4 + 1)]^{n_2}$$
, $C_4 := \frac{5p_{2r}^+ l_1}{p_{2r}^-(l_1 - \gamma)} + \frac{p_{2r}^+ l_1 \gamma}{p_{2r}^-(l_1 - \gamma)^2}$,

$$C_5 := \left\{ \int_{B(x_0, 2r)} \left[(u_{\ell} - N)^+ + r \right]^{(p - p_{2r}^-)\gamma'} d\mathbf{V} \right\}^{\frac{1}{\gamma'}},$$

and $C_6 := C_1 C_5 \max\{\hat{c}_1^{p_{2r}^-}, \hat{c}_1^{p_{2r}^+}\} + C_2 + (\hat{c}_1 p_{2r}^+)^{p_{2r}^-} + 1.$ (ii) Assume that (1.14) (of property (P_1)) holds. If u is a supersolution of (1.1) in Ω , then

$$\operatorname{ess\,sup}_{B(x_0,\sigma)} \left\{ |(u_{\ell} - N)^-| + r \right\}$$

$$\leq (\mathcal{C}_3 \mathbf{1}_{\mathrm{I}}/\gamma)^{\mathcal{C}_4} \mathcal{C}_8^{\frac{1}{p_{2r}^-(\mathbf{1}_{\mathrm{I}}-\gamma)}} (p_{2r}^-)^{\frac{p_{2r}^+\mathbf{1}_{\mathrm{I}}}{p_{2r}^-(\mathbf{1}_{\mathrm{I}}-\gamma)}} \left(\frac{\rho}{\rho-\sigma}\right)^{\frac{p_{2r}^+\mathbf{1}_{\mathrm{I}}}{p_{2r}^-(\mathbf{1}_{\mathrm{I}}-\gamma)}} \\ \cdot \left\{ \int_{B(x_0,\rho)} \left[|(u_\ell-N)^-|+r\right]^{\gamma p_{2r}^-} \, \mathrm{dV} \right\}^{\frac{1}{\gamma p_{2r}^-}},$$

where

$$\mathcal{C}_7 := \left\{ \int_{B(x_0, 2r)} \left[|(u_\ell - N)^-| + r \right]^{(p - p_{2r}^-)\gamma'} \, dV \right\}^{\frac{1}{\gamma'}}$$

and $C_8 := C_1 C_7 \max\{\hat{c}_1^{p_{2r}^-}, \hat{c}_1^{p_{2r}^+}\} + C_2 + (\hat{c}_1 p_{2r}^+)^{p_{2r}^-} + 1.$

In (i) and (ii), $p_{2r}^- := \inf_{B(x_0, 2r)} p$ and $p_{2r}^+ := \sup_{B(x_0, 2r)} p$

Proof. (i) We take R=2r in (3.3) and the function $\eta\in C_0^\infty(B(x_0,\rho))$ from Lemma 3.2, which satisfies: $0 \le \eta \le 1$, $\eta = 1$ in $B(x_0, \sigma)$, and $|\mathfrak{D}\eta|_{F^*} \le \hat{c}_0\hat{c}_1/(\rho - \sigma)$. We also consider $q = \beta - p_{2r}^-$ in (3.3), with $\beta \geq p_{2r}^-$.

$$\int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{\beta - p_{2r}^{-}} |\nabla (u_{\ell} - N)^{+}|_{F}^{p_{2r}^{-}} \eta^{p_{2r}^{+}} dV
\leq C_{1} \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{p + \beta - p_{2r}^{-}} |\mathcal{D}\eta|_{F^{*}}^{p} dV + C_{2} \int_{\Omega} \left[(u_{\ell} - N)^{+} + r \right]^{\beta - p_{2r}^{-}} \eta^{p_{2r}^{+}} dV$$
(3.16)

Applying inequality (2.6) to the function $[(u_{\ell}-N)^++r]^{\beta/p_{2r}^-}\eta^{p_{2r}^+/p_{2r}^-}$, we get

$$\begin{split} &\left(\int_{B(x_0,2r)} \left\{ [(u_{\ell} - N)^+ + r]^{\beta/p_{2r}^-} \eta^{p_{2r}^+/p_{2r}^-} \right\}^{p_{2r}^- 1_1} \, \mathrm{dV} \right)^{\frac{1}{1_1}} \\ & \leq (2p_{2r}^- \mathbb{K}_{\mathrm{I}} r)^{p_{2r}^-} \int_{B(x_0,2r)} \left| \boldsymbol{\nabla} \left\{ [(u_{\ell} - N)^+ + r]^{\beta/p_{2r}^-} \eta^{p_{2r}^+/p_{2r}^-} \right\} \right|_F^{p_{2r}^-} \, \mathrm{dV} \\ & \leq (2\mathbb{K}_{\mathrm{I}} r)^{p_{2r}^-} 2^{p_{2r}^- - 1} (1 + \beta)^{p_{2r}^-} \left\{ \int_{B(x_0,2r)} \left[(u_{\ell} - N)^+ + r \right]^{\beta - p_{2r}^-} \left| \boldsymbol{\nabla} (u_{\ell} - N)^+ \right|_F^{p_{2r}^-} \eta^{p_{2r}^+} \, \mathrm{dV} \right. \\ & \left. + (p_{2r}^+)^{p_{2r}^-} \int_{B(x_0,2r)} \left[(u_{\ell} - N)^+ + r \right]^{\beta} \eta^{p_{2r}^+ - p_{2r}^-} |\mathcal{D} \eta|_{F^*}^{p_{2r}^-} \, \mathrm{dV} \right\}. \end{split}$$

By (3.16) and (2.5), we can obtain

$$\left\{ \mathbb{K}_{2} 2^{-n_{2}} \int_{B(x_{0},\sigma)} \left[(u_{\ell} - N)^{+} + r \right]^{\beta 1_{1}} dV \right\}^{\frac{1}{1_{1}}} \\
\leq (2\mathbb{K}_{I} r)^{p_{2r}^{-}} 2^{p_{2r}^{-} - 1} (1 + \beta)^{p_{2r}^{-}} \left\{ \mathcal{C}_{1} \int_{B(x_{0},2r)} \left[(u_{\ell} - N)^{+} + r \right]^{p + \beta - p_{2r}^{-}} |\mathfrak{D}\eta|_{F^{*}}^{p} dV \right. \\
+ \mathcal{C}_{2} \int_{B(x_{0},\rho)} \left[(u_{\ell} - N)^{+} + r \right]^{\beta - p_{2r}^{-}} \eta^{p_{2r}^{+}} dV \\
+ (p_{2r}^{+})^{p_{2r}^{-}} \int_{B(x_{0},\rho)} \left[(u_{\ell} - N)^{+} + r \right]^{\beta} |\mathfrak{D}\eta|_{F^{*}}^{p_{2r}^{-}} dV \right\}.$$
(3.17)

Steep 2. Next, we estimate the right-hand side of (3.17). By Hölder's inequality, for $\gamma \in (1, 1_{\rm I})$,

$$\int_{B(x_0,\rho)} \left[(u_{\ell} - N)^+ + r \right]^{\beta} |\mathcal{D}\eta|_{F^*}^{p_{2r}^-} \, dV \le r^{-p_{2r}^-} \left(\frac{\hat{c}_0 \hat{c}_1 \rho}{\rho - \sigma} \right)^{p_{2r}^-} \left\{ \int_{B(x_0,\rho)} \left[(u_{\ell} - N)^+ + r \right]^{\beta \gamma} \, dV \right\}^{\frac{1}{\gamma}} \tag{3.18}$$

and

$$\oint_{B(x_0,\rho)} \left[(u_{\ell} - N)^+ + r \right]^{\beta - p_{2r}^-} \, dV \le r^{-p_{2r}^-} \left\{ \oint_{B(x_0,\rho)} \left[(u_{\ell} - N)^+ + r \right]^{\beta \gamma} \, dV \right\}^{\frac{1}{\gamma}}, \tag{3.19}$$

since $r \leq (u_{\ell} - N)^+ + r$.

By (2.8), we have

$$|\mathcal{D}\eta|_{F^*}^p \le r^{-p} \left(\frac{\hat{c}_0 \hat{c}_1 r}{\rho - \sigma}\right)^p \le r^{-p} \left(\frac{\hat{c}_0 \hat{c}_1 \rho}{\rho - \sigma}\right)^p \le \mathsf{K}_4 r^{-p_{2r}^-} \left(\frac{\hat{c}_0 \hat{c}_1 \rho}{\rho - \sigma}\right)^p \le c_2 r^{-p_{2r}^-} \left(\frac{\rho}{\rho - \sigma}\right)^{p_{2r}^+},$$

where $c_2 := K_4 \max\{(\hat{c}_0 \hat{c}_1)^{p_{2r}^-}, (\hat{c}_0 \hat{c}_1)^{p_{2r}^+}\}$

Then,

$$\int_{B(x_{0},2r)} \left[(u_{\ell} - N)^{+} + r \right]^{\beta - p_{2r}^{-} + p} |\mathcal{D}\eta|_{F^{*}}^{p} \, dV$$

$$\leq \frac{c_{2}r^{-p_{2r}^{-}}}{\mathsf{V}(B(x_{0},2r))} \left(\frac{\rho}{\rho - \sigma} \right)^{p_{2r}^{+}} \int_{B(x_{0},\rho)} \left[(u_{\ell} - N)^{+} + r \right]^{\beta - p_{2r}^{-} + p} \, dV$$

$$\leq c_{2}r^{-p_{2r}^{-}} \left(\frac{\rho}{\rho - \sigma} \right)^{p_{2r}^{+}} \left\{ \int_{B(x_{0},2r)} \left[(u_{\ell} - N)^{+} + r \right]^{(p - p_{2r}^{-})\gamma'} \, dV \right\}^{\frac{1}{\gamma'}}$$

$$\cdot \left\{ \int_{B(x_{0},\rho)} \left[(u_{\ell} - N)^{+} + r \right]^{\beta\gamma} \, dV \right\}^{\frac{1}{\gamma}}$$

$$= c_{2}c_{3}r^{-p_{2r}^{-}} \left(\frac{\rho}{\rho - \sigma} \right)^{p_{2r}^{+}} \left\{ \int_{B(x_{0},\rho)} \left[(u_{\ell} - N)^{+} + r \right]^{\beta\gamma} \, dV \right\}^{\frac{1}{\gamma}}, \tag{3.20}$$

where

$$c_3 := \left\{ \int_{B(x_0, 2r)} \left[(u_\ell - N)^+ + r \right]^{(p - p_{2r}^-)\gamma'} \, d\mathbf{V} \right\}^{\frac{1}{\gamma'}}.$$

From (3.17) - (3.20), we can get

$$\begin{split} &\left\{ \mathsf{K}_{2} 2^{-\mathsf{n}_{2}} \int_{B(x_{0},\sigma)} \left[(u_{\ell} - N)^{+} + r \right]^{\beta 1_{\mathsf{I}}} \, \mathrm{dV} \right\}^{\frac{1}{\mathsf{I}_{\mathsf{I}}}} \\ & \leq (2\mathsf{K}_{\mathsf{I}})^{p_{2r}^{-}} 2^{p_{2r}^{-} - 1} \left\{ \mathcal{C}_{1} c_{2} c_{3} + \mathcal{C}_{2} + (\hat{c}_{0} \hat{c}_{1} p_{2r}^{+})^{p_{2r}^{-}} \right\} \\ & \cdot (1 + \beta)^{p_{2r}^{-}} \left(\frac{\rho}{\rho - \sigma} \right)^{p_{2r}^{+}} \left\{ \int_{B(x_{0},\rho)} \left[(u_{\ell} - N)^{+} + r \right]^{\beta \gamma} \, \mathrm{dV} \right\}^{\frac{1}{\gamma}}. \end{split}$$

Define

$$\Psi(f,q,D) := \left(\oint_D f^q \, \mathrm{dV} \right)^{\frac{1}{q}}.$$

Consequently,

$$\Psi((u_{\ell} - N)^{+} + r, \beta 1_{\mathrm{I}}, B(x_{0}, \sigma))$$

$$\leq c_{4}^{\frac{1}{\beta}} (1 + \beta)^{\frac{p_{2r}^{+}}{\beta}} \left(\frac{\rho}{\rho - \sigma}\right)^{\frac{p_{2r}^{+}}{\beta}} \Psi((u_{\ell} - N)^{+} + r, \beta \gamma, B(x_{0}, \rho)), \tag{3.21}$$

if $r \leq \sigma < \rho \leq 2r$, where

$$c_4 := (\mathtt{K}_2^{-1} 2^{\mathtt{n}_2})^{1/1_{\mathrm{I}}} (2 \mathtt{K}_{\mathrm{I}})^{p_{2r}^-} 2^{p_{2r}^- - 1} [\mathcal{C}_1 c_2 c_3 + \mathcal{C}_2 + (\hat{c}_0 \hat{c}_1 p_{2r}^+)^{p_{2r}^-}].$$

Taking
$$r_j = \sigma + 2^{-j}(\rho - \sigma)$$
, $\xi_j = (1_{\rm I}/\gamma)^j \gamma p_{2r}^-$, and $\beta = (1_{\rm I}/\gamma)^j p_{2r}^-$ in (3.21), we have
$$\Psi((u_\ell - N)^+ + r, \xi_{j+1}, B(x_0, r_{j+1}))$$

$$\leq c_4^{\frac{\gamma}{\xi_j}} \left(1 + \frac{\xi_j}{\gamma}\right)^{\frac{\gamma p_{2r}^+}{\xi_j}} \left(\frac{r_j}{r_j - r_{j+1}}\right)^{\frac{\gamma p_{2r}^+}{\xi_j}} \Psi((u_\ell - N)^+ + r, \xi_j, B(x_0, r_j)).$$

By iterating this inequality, we have

ess sup
$$\{(u_{\ell} - N)^{+} + r\}$$

$$\leq \prod_{j=0}^{\infty} \left[c_{4}^{\gamma/\xi_{j}} \left(1 + \frac{\xi_{j}}{\gamma} \right)^{\gamma p_{2r}^{+}/\xi_{j}} \left(\frac{2^{j+1}\rho}{\rho - \sigma} \right)^{\gamma p_{2r}^{+}/\xi_{j}} \right] \Psi((u_{\ell} - N)^{+} + r, \gamma p_{2r}^{-}, B(x_{0}, \rho))$$

$$\leq c_{4}^{\sum_{j=0}^{\infty} \gamma/\xi_{j}} 2^{\sum_{j=0}^{\infty} (j+1)\gamma p_{2r}^{+}/\xi_{j}} \left(\frac{\rho}{\rho - \sigma} \right)^{\sum_{j=0}^{\infty} \gamma p_{2r}^{+}/\xi_{j}} \prod_{j=0}^{\infty} \left(1 + \frac{\xi_{j}}{\gamma} \right)^{\gamma p_{2r}^{+}/\xi_{j}}$$

$$\cdot \Psi((u_{\ell} - N)^{+} + r, \gamma p_{2r}^{-}, B(x_{0}, \rho)).$$

We note that

$$\sum_{j=0}^{\infty}\frac{\gamma}{\xi_j}=\frac{1_{\mathrm{I}}}{p_{2r}^-(1_{\mathrm{I}}-\gamma)}\quad\text{ and }\quad \sum_{j=0}^{\infty}j\frac{\gamma}{\xi_j}=\frac{1_{\mathrm{I}}\gamma}{p_{2r}^-(1_{\mathrm{I}}-\gamma)^2}.$$

Then,

$$\prod_{j=0}^{\infty} \left(1 + \frac{\xi_j}{\gamma} \right)^{\gamma p_{2r}^+/\xi_j} \le 2^{\sum_{j=0}^{\infty} \gamma p_{2r}^+/\xi_j} \prod_{j=0}^{\infty} \left(\frac{\xi_j}{\gamma} \right)^{\gamma p_{2r}^+/\xi_j}
= (2p_{2r}^-)^{\sum_{j=0}^{\infty} \gamma p_{2r}^+/\xi_j} \left(\frac{1_{\mathbf{I}}}{\gamma} \right)^{\sum_{j=0}^{\infty} j \gamma p_{2r}^+/\xi_j} .$$

Which implies,

ess sup
$$\{(u_{\ell} - N)^{+} + r\}$$

$$\leq (c_{5}1_{I}/\gamma)^{c_{6}} c_{7}^{\frac{1}{1_{1}-\gamma}} (p_{2r}^{-})^{\frac{p_{2r}^{+}1_{I}}{p_{2r}^{-}(1_{I}-\gamma)}} \left(\frac{\rho}{\rho - \sigma}\right)^{\frac{p_{2r}^{+}1_{I}}{p_{2r}^{-}(1_{I}-\gamma)}}$$

$$\cdot \Psi((u_{\ell} - N)^{+} + r, \gamma p_{2r}^{-}, B(x_{0}, \rho)),$$

where $c_5 := [2(\hat{c}_0 + 1)\mathsf{K}_2^{-1}(\mathsf{K}_{\mathrm{I}} + 1)(\mathsf{K}_4 + 1)]^{n_2}, c_6 = \frac{5p_{2r}^+ 1_{\mathrm{I}}}{p_{2r}^- (1_{\mathrm{I}} - \gamma)} + \frac{p_{2r}^+ 1_{\mathrm{I}} \gamma}{p_{2r}^- (1_{\mathrm{I}} - \gamma)^2}, \text{ and } c_7 := \mathcal{C}_1 c_3 \max\{\hat{c}_1^{p_{2r}^-}, \hat{c}_1^{p_{2r}^+}\} + \mathcal{C}_2 + (\hat{c}_1 p_{2r}^+)^{p_{2r}^-} + 1. \text{ This completes the proof.}$

- (ii) Performing the procedure of the proof of (i), we can easily obtain the result.
- 3.2. **Lower bound of supersolutions.** In this subsection, we obtain a lower bound for a supersolution of (1.1) (Proposition 3.5). We also prove a reverse weak Hölder inequality (Lemma 3.7).

Proposition 3.5. Let $\{u_\ell\} \subset W^{1,p(x)}_{loc}(\Omega; \mathbb{V})$ be a family of functions, and let $u \in W^{1,p(x)}_{loc}(\Omega; \mathbb{V})$ be an element. Assume that (1.7) - (1.10) and (1.14) (of property (P_1)) are verified. Let $r \leq \sigma < \rho \leq 2r \leq 2$, $B(x_0, 2r) \subset \psi(\tilde{U})$, $1 < \gamma < 1_{\mathrm{I}}$, and $\gamma \geq s_0 > 0$. Suppose that u is a supersolution of (1.1) with $u_\ell \geq N$ in $B(x_0, 2r)$. Then, we have

$$\begin{split} & \left[\int_{B(x_0,\rho)} (u_{\ell} - N + r)^{-s_0} \, \mathrm{dV} \right]^{\frac{1}{-s_0}} \\ & \leq \left(\mathcal{C}_3 \mathbf{1}_{\mathrm{I}} / \gamma \right)^{\frac{\mathcal{C}_{10}}{s_0}} \mathcal{C}_{14}^{\frac{1_1 \gamma}{s_0 (1_1 - \gamma)}} \left(\frac{\rho}{\rho - \sigma} \right)^{\frac{p_{2r}^+ \mathbf{1}_1 \gamma}{s_0 (1_1 - \gamma)}} \underset{B(x_0,\sigma)}{\mathrm{ess inf}} \, \left\{ u_{\ell} - N + r \right\}, \end{split}$$

 $\textit{where $p_{2r}^-:=\inf_{B(x_0,2r)}p$, $p_{2r}^+:=\sup_{B(x_0,2r)}p$, $\mathcal{C}_3=[2(\hat{c}_0+1)\mathtt{K}_2^{-1}(\mathtt{K}_{\mathrm{I}}+1)(\mathtt{K}_4+1)]^{\mathtt{n}_2}$, $\mathcal{C}_{10}=p_{2r}^+\left[\frac{51_1\gamma}{1_1-\gamma}+\frac{1_1\gamma^2}{(1_1-\gamma)^2}\right]$, $p_{2r}^+:=\sup_{B(x_0,2r)}p$, $$

$$\begin{split} \mathcal{C}_{11} &:= \frac{4\Lambda_2 p_{2r}^+}{\Lambda_1(p_{2r}^- - 1)p_{2r}^-} \left[\frac{2\Lambda_2(p_{2r}^+ - 1)}{\Lambda_1(p_{2r}^- - 1)} \right]^{p_{2r}^+ - 1} + \frac{\mathfrak{T}4p_{2r}^+}{\Lambda_1(p_{2r}^- - 1)p_{2r}^-} \\ & \mathcal{C}_{12} := 4\mathfrak{T} \frac{p_{2r}^+ - 1}{\Lambda_1 p_{2r}^+} \left(\frac{p_{2r}^+}{p_{2r}^- - 1} + 1 \right), \\ & \mathcal{C}_{13} := \left\{ \int_{B(x_0, 2r)} (u_\ell - N + r)^{(p - p_{2r}^-)\gamma'} \, \mathrm{d}\mathbf{V} \right\}^{\frac{1}{\gamma'}}, \end{split}$$

and $C_{14} := C_{11}C_{13} \max\{\hat{c}_1^{p_{2r}^-}, \hat{c}_1^{p_{2r}^+}\} + C_{12} + (\hat{c}_1p_{2r}^+)^{p_{2r}^-} + 1.$

Proof. Fix r>0. Let $\zeta:=(u_\ell-N+r)^q\eta^{p_{\rm R}^+}$, with q<0, and let $\eta\in C_0^\infty(B(x_0,{\bf R}))$ such that $0\leq\eta\leq 1$. Note that $\varphi\in W_0^{1,p(x)}(B(x_0,{\bf R});{\bf V})$.

We have

$$\int_{\Omega} \mathcal{A}(\cdot, \nabla u_{\ell}) \bullet \mathcal{D}\zeta \, dV = \int_{\Omega} g(\mathcal{A}(\cdot, \nabla u), \nabla \zeta) \, dV$$
$$+ \int_{\Omega} \left[\mathcal{A}(\cdot, \nabla u_{\ell}) - \mathcal{A}(\cdot, \nabla u) \right] \bullet \mathcal{D}\zeta \, dV.$$

Step 1. Since u is a supersolution and

$$\mathcal{D}\zeta = q(u_{\ell} - N + r)^{q-1} \eta^{p_{\mathbf{R}}^+} \mathcal{D}u_{\ell} + p_{\mathbf{R}}^+ (u_{\ell} - N + r)^q \eta^{p_{\mathbf{R}}^+ - 1} \mathcal{D}\eta$$

we have

$$\int_{\Omega} \mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) \bullet \left[q(u_{\ell} - N + r)^{q-1} \boldsymbol{\eta}^{p_{\mathrm{R}}^{+}} \mathcal{D} u_{\ell} + p_{\mathrm{R}}^{+} (u_{\ell} - N + r)^{q} \boldsymbol{\eta}^{p_{\mathrm{R}}^{+} - 1} \mathcal{D} \boldsymbol{\eta} \right] \, \mathrm{dV} \geq E(\ell, N, q),$$

where

$$E(\ell,N,q) := \int_{\Omega} \left[\mathcal{A}(\cdot,\boldsymbol{\nabla}\boldsymbol{u}_{\ell}) - \mathcal{A}(\cdot,\boldsymbol{\nabla}\boldsymbol{u}) \right] \bullet \left[q(\boldsymbol{u}_{\ell} - N + r)^{q-1} \boldsymbol{\eta}^{\boldsymbol{p}_{\mathrm{R}}^{+}} \mathcal{D}\boldsymbol{u}_{\ell} + \boldsymbol{p}_{\mathrm{R}}^{+} (\boldsymbol{u}_{\ell} - N + r)^{q} \boldsymbol{\eta}^{\boldsymbol{p}_{\mathrm{R}}^{+} - 1} \mathcal{D}\boldsymbol{\eta} \right] \, \mathrm{dV}.$$

By (a_1) and (a_2) , we have

$$\begin{split} & \Lambda_{1}|q| \int_{\Omega} |\boldsymbol{\nabla} u_{\ell}|_{F}^{p} (u_{\ell} - N + r)^{q-1} \eta^{p_{\mathrm{R}}^{+}} \, \mathrm{dV} \\ & \leq \Lambda_{2} p_{\mathrm{R}}^{+} \int_{\Omega} |\boldsymbol{\nabla} u_{\ell}|_{F}^{p-1} (u_{\ell} - N + r)^{q} \eta^{p_{\mathrm{R}}^{+} - 1} |\mathcal{D} \eta|_{F^{*}} \, \mathrm{dV} - E(\ell, N, q) \\ & \leq \Lambda_{2} p_{\mathrm{R}}^{+} \int_{\Omega} (u_{\ell} - N + r)^{q-1} \eta^{p_{\mathrm{R}}^{+}} \left[\frac{p_{\mathrm{R}}^{+} - 1}{p_{\mathrm{R}}^{+}} \epsilon |\boldsymbol{\nabla} u_{\ell}|_{F}^{p} \right. \\ & \left. + \frac{1}{p_{\mathrm{R}}^{-}} \max\{\epsilon^{-p_{\mathrm{R}}^{-} + 1}, \epsilon^{-p_{\mathrm{R}}^{+} + 1}\} (u_{\ell} - N + r)^{p} \eta^{-p} |\mathcal{D} \eta|_{F^{*}}^{p} \right] \, \mathrm{dV} - E(\ell, N, q). \end{split}$$

Taking $\epsilon = \frac{\Lambda_1|q|}{2\Lambda_2(p_{\mathtt{R}}^+-1)}$, we have

$$\frac{\Lambda_{1}|q|}{2} \int_{\Omega} |\nabla u_{\ell}|_{F}^{p} (u_{\ell} - N + r)^{q-1} \eta^{p_{R}^{+}} dV
\leq \frac{\Lambda_{2} p_{R}^{+}}{p_{R}^{-}} \max \left\{ [2\Lambda_{2} (p_{R}^{+} - 1)/\Lambda_{1}|q|]^{p_{R}^{-} - 1}, [2\Lambda_{2} (p_{R}^{+} - 1)/\Lambda_{1}|q|]^{p_{R}^{+} - 1} \right\}
\cdot \int_{\Omega} (u_{\ell} - N + r)^{q-1+p} \eta^{p_{R}^{+} - p} |\mathcal{D}\eta|_{F^{*}}^{p} dV - E(\ell, N, q).$$
(3.22)

Steep 2. We estimate the term $E(\ell, N, q)$:

$$\begin{aligned} |E(\ell,N,q)| \\ & \leq \left| \int_{\Omega} \left[\mathcal{A}(\cdot,\nabla u_{\ell}) - \mathcal{A}(\cdot,\nabla u) \right] \bullet \left[q(u_{\ell} - N + r)^{q-1} \eta^{p_{\mathsf{R}}^{+}} \mathcal{D} u_{\ell} + p_{\mathsf{R}}^{+} (u_{\ell} - N + r)^{q} \eta^{p_{\mathsf{R}}^{+} - 1} \mathcal{D} \eta \right] \, \mathrm{d} \mathbb{V} \right| \\ & \leq |q| \int_{\Omega} \left| \mathcal{A}(\cdot,\nabla u_{\ell}) - \mathcal{A}(\cdot,\nabla u) \right|_{F} (u_{\ell} - N + r)^{q-1} \eta^{p_{\mathsf{R}}^{+}} |\mathcal{D} u_{\ell}|_{F^{*}} \\ & + p_{\mathsf{R}}^{+} \left| \mathcal{A}(\cdot,\nabla u_{\ell}) - \mathcal{A}(\cdot,\nabla u) \right|_{F} (u_{\ell} - N + r)^{q} \eta^{p_{\mathsf{R}}^{+} - 1} |\mathcal{D} \eta|_{F^{*}} \, \mathrm{d} \mathbb{V}. \end{aligned} \tag{3.23}$$

From (1.14),

$$|\mathcal{A}(\cdot, \nabla u_\ell) - \mathcal{A}(\cdot, \nabla u)|_F \le \mathfrak{T} < \frac{\Lambda_1 p_{\mathbf{R}}^-}{4} \quad \text{ in } \quad \Omega.$$

We also have, in $B(x_0, \mathbb{R})$,

$$|\mathcal{D}u_{\ell}|_{F^*} = |\nabla u_{\ell}|_F \le \frac{1}{p_{\mathsf{R}}^-} |\nabla u_{\ell}|_F^p + \frac{p_{\mathsf{R}}^+ - 1}{p_{\mathsf{R}}^+}$$

and

$$(u_{\ell} - N + r)^{q} \eta^{p_{\mathbf{R}}^{+} - 1} |\mathfrak{D}\eta|_{F^{*}}$$

$$\leq (u_{\ell} - N + r)^{q - 1} \left[\frac{1}{p} (u_{\ell} - N + r)^{p} |\mathfrak{D}\eta|_{F^{*}}^{p} + \frac{p - 1}{p} \eta^{\frac{p_{\mathbf{R}}^{+} - 1}{p - 1}} p \right]$$

$$\leq \frac{1}{p_{\mathbf{R}}^{-}} (u_{\ell} - N + r)^{q - 1 + p} |\mathfrak{D}\eta|_{F}^{p} + \frac{p_{\mathbf{R}}^{+} - 1}{p_{\mathbf{R}}^{+}} (u_{\ell} - N + r)^{q - 1} \eta^{p_{\mathbf{R}}^{+}}.$$

Consequently, from (3.23),

$$\begin{split} |E(\ell,N,q)| & \leq \frac{\Lambda_{1}|q|}{4} \int_{\Omega} (u_{\ell} - N + r)^{q-1} |\nabla u_{\ell}|_{F}^{p} \eta^{p_{\mathsf{R}}^{+}} \, \mathrm{dV} + \frac{\mathfrak{T}|q|(p_{\mathsf{R}}^{+} - 1)}{p_{\mathsf{R}}^{+}} \int_{\Omega} (u_{\ell} - N + r)^{q-1} \eta^{p_{\mathsf{R}}^{+}} \, \mathrm{dV} \\ & + \frac{\mathfrak{T}p_{\mathsf{R}}^{+}}{p_{\mathsf{R}}^{-}} \int_{\Omega} (u_{\ell} - N + r)^{q-1+p} |\mathcal{D}\eta|_{F^{*}}^{p} \, \mathrm{dV} + \mathfrak{T}(p_{\mathsf{R}}^{+} - 1) \int_{\Omega} (u_{\ell} - N + r)^{q-1} \eta^{p_{\mathsf{R}}^{+}} \, \mathrm{dV}. \end{split} \tag{3.24}$$

Steep 3. From (3.22) and (3.24),

$$\frac{\Lambda_{1}|q|}{4} \int_{\Omega} |\nabla u_{\ell}|_{F}^{p} (u_{\ell} - N + r)^{q-1} \eta^{p_{R}^{+}} dV
\leq C_{0} \int_{\Omega} (u_{\ell} - N + r)^{q-1+p} |\mathcal{D}\eta|_{F^{*}}^{p} dV + \mathfrak{T} \frac{(p_{R}^{+} - 1)(p_{R}^{+} + |q|)}{p_{R}^{+}} \int_{\Omega} (u_{\ell} - N + r)^{q-1} \eta^{p_{R}^{+}} dV,$$
(3.25)

where

$$\mathcal{C}_0 := \frac{\Lambda_2 p_{\mathtt{R}}^+}{p_{\mathtt{R}}^-} \max \left\{ [2\Lambda_2 (p_{\mathtt{R}}^+ - 1)/\Lambda_1 |q_0|]^{p_{\mathtt{R}}^- - 1}, [2\Lambda_2 (p_{\mathtt{R}}^+ - 1)/\Lambda_1 |q_0|]^{p_{\mathtt{R}}^+ - 1} \right\} + \frac{\mathfrak{T} p_{\mathtt{R}}^+}{p_{\mathtt{R}}^-}$$

and $0 > q_0 \ge q$.

Taking $q_0=1-p_{\rm R}^-$ and $q=\beta-p_{\rm R}^-+1$ with $\beta<0$ in (3.25), we have

$$\int_{\Omega} |\nabla u_{\ell}|_{F}^{p} (u_{\ell} - N + r)^{\beta - p_{\mathbb{R}}^{-}} \eta^{p_{\mathbb{R}}^{+}} dV
\leq \mathbf{c}_{1} \int_{\Omega} (u_{\ell} - N + r)^{\beta - p_{\mathbb{R}}^{-} + p} |\mathcal{D}\eta|_{F^{*}}^{p} dV + \mathbf{c}_{2} \int_{\Omega} (u_{\ell} - N + r)^{\beta - p_{\mathbb{R}}^{-}} \eta^{p_{\mathbb{R}}^{+}} dV, \tag{3.26}$$

where

$$\mathbf{c}_{1} := \frac{4\Lambda_{2}p_{\mathsf{R}}^{+}}{\Lambda_{1}(p_{\mathsf{R}}^{-} - 1)p_{\mathsf{R}}^{-}} \max \left\{ \left[\frac{2\Lambda_{2}(p_{\mathsf{R}}^{+} - 1)}{\Lambda_{1}(p_{\mathsf{R}}^{-} - 1)} \right]^{p_{\mathsf{R}}^{-} - 1}, \left[\frac{2\Lambda_{2}(p_{\mathsf{R}}^{+} - 1)}{\Lambda_{1}(p_{\mathsf{R}}^{-} - 1)} \right]^{p_{\mathsf{R}}^{+} - 1} \right\} + \frac{\mathfrak{T}4p_{\mathsf{R}}^{+}}{\Lambda_{1}(p_{\mathsf{R}}^{-} - 1)p_{\mathsf{R}}^{-}}$$

$$\mathbf{c}_{2} := 4\mathfrak{T}\frac{p_{\mathsf{R}}^{+} - 1}{\Lambda_{1}p_{\mathsf{R}}^{+}} \left[\frac{p_{\mathsf{R}}^{+}}{p_{\mathsf{R}}^{-} - 1} + 1 \right].$$

Steep 4. Let $r \leq \sigma < \rho \leq 2r$. Next, take R = 2r in (3.26) and consider the function $\eta \in C_0^{\infty}(B(x_0, \rho))$ from Lemma 3.2, where η satisfies: $0 \leq \eta \leq 1$, $\eta = 1$ in $B(x_0, \sigma)$, and $|\mathfrak{D}\eta|_{F^*} \leq \hat{c}_0\hat{c}_1/(\rho - \sigma)$.

Applying inequality (2.6) to the function $(u_{\ell} - N + r)^{\beta/p_{2r}^-} \eta^{p_{2r}^+/p_{2r}^-}$, we have

$$\begin{split} &\left\{ \int_{B(x_0,2r)} \left[(u_{\ell} - N + r)^{\beta/p_{2r}^-} \eta^{p_{2r}^+/p_{2r}^-} \right]^{p_{2r}^- 1_{\mathrm{I}}} \, \mathrm{d} \mathbb{V} \right\}^{\frac{1}{1_{\mathrm{I}}}} \\ & \leq (2p_{2r}^- \mathbb{K}_{\mathrm{I}} r)^{p_{2r}^-} \int_{B(x_0,2r)} \left| \nabla \left[(u_{\ell} - N + r)^{\beta/p_{2r}^-} \eta^{p_{2r}^+/p_{2r}^-} \right] \right|_F^{p_{2r}^-} \, \mathrm{d} \mathbb{V} \\ & \leq (2\mathbb{K}_{\mathrm{I}} r)^{p_{2r}^-} 2^{p_{2r}^- - 1} (1 + |\beta|)^{p_{2r}^-} \left\{ \int_{B(x_0,2r)} (u_{\ell} - N + r)^{\beta - p_{2r}^-} \left| \nabla u_{\ell} \right|_F^{p_{2r}^-} \eta^{p_{2r}^+} \, \mathrm{d} \mathbb{V} \right. \\ & \left. + (p_{2r}^+)^{p_{2r}^-} \int_{B(x_0,2r)} (u_{\ell} - N + r)^{\beta} \, \eta^{p_{2r}^+ - p_{2r}^-} |\mathcal{D} \eta|_{F^*}^{p_{2r}^-} \, \mathrm{d} \mathbb{V} \right\}. \end{split}$$

By (3.26) and (2.5), we can obtain

$$\left\{ K_{2} 2^{-n_{2}} \oint_{B(x_{0},\sigma)} (u_{\ell} - N + r)^{\beta 1_{I}} dV \right\}^{\frac{1}{1_{I}}} \\
\leq (2K_{I}r)^{p_{2r}^{-}} 2^{p_{2r}^{-} - 1} (1 + |\beta|)^{p_{2r}^{-}} \left\{ \mathbf{c}_{1} \oint_{B(x_{0},2r)} (u_{\ell} - N + r)^{p + \beta - p_{2r}^{-}} |\mathcal{D}\eta|_{F^{*}}^{p} dV \right. \\
+ \mathbf{c}_{2} \oint_{B(x_{0},\rho)} (u_{\ell} - N + r)^{\beta - p_{2r}^{-}} \eta^{p_{2r}^{+}} dV \\
+ (p_{2r}^{+})^{p_{2r}^{-}} \oint_{B(x_{0},\rho)} (u_{\ell} - N + r)^{\beta} |\mathcal{D}\eta|_{F^{*}}^{p_{2r}^{-}} dV \right\}.$$
(3.27)

Steep 2. Next, we estimate the right-hand side of (3.27). By Hölder's inequality, for $\gamma \in (1, 1_I)$,

$$\int_{B(x_0,\rho)} (u_{\ell} - N + r)^{\beta} |\mathcal{D}\eta|_{F^*}^{p_{2r}^-} \, dV \le r^{-p_{2r}^-} \left(\frac{\hat{c}_0 \hat{c}_1 \rho}{\rho - \sigma}\right)^{p_{2r}^-} \left\{ \int_{B(x_0,\rho)} (u_{\ell} - N + r)^{\beta \gamma} \, dV \right\}^{\frac{1}{\gamma}}$$
(3.28)

and

$$\int_{B(x_0,\rho)} (u_{\ell} - N + r)^{\beta - p_{2r}^-} \, dV \le r^{-p_{2r}^-} \left\{ \int_{B(x_0,\rho)} (u_{\ell} - N + r)^{\beta\gamma} \, dV \right\}^{\frac{1}{\gamma}},$$
(3.29)

since $r \leq u_{\ell} - N + r$.

By (2.8), we have

$$|\mathcal{D}\eta|_{F^*}^p \leq r^{-p} \left(\frac{\hat{c}_0\hat{c}_1r}{\rho-\sigma}\right)^p \leq r^{-p} \left(\frac{\hat{c}_0\hat{c}_1\rho}{\rho-\sigma}\right)^p \leq \mathsf{K}_4r^{-p_{2r}^-} \left(\frac{\hat{c}_0\hat{c}_1\rho}{\rho-\sigma}\right)^p \leq \mathbf{c}_3r^{-p_{2r}^-} \left(\frac{\rho}{\rho-\sigma}\right)^{p_{2r}^+},$$

where $\mathbf{c}_3 := K_4 \max\{(\hat{c}_0 \hat{c}_1)^{p_{2r}^-}, (\hat{c}_0 \hat{c}_1)^{p_{2r}^+}\}.$

Then,

$$\int_{B(x_{0},2r)} (u_{\ell} - N + r)^{\beta - p_{2r}^{-} + p} |\mathcal{D}\eta|_{F^{*}}^{p} dV
\leq \frac{\mathbf{c}_{3} r^{-p_{2r}^{-}}}{\mathbf{V}(B(x_{0},2r))} \left(\frac{\rho}{\rho - \sigma}\right)^{p_{2r}^{+}} \int_{B(x_{0},\rho)} (u_{\ell} - N + r)^{\beta - p_{2r}^{-} + p} dV
\leq \mathbf{c}_{3} r^{-p_{2r}^{-}} \left(\frac{\rho}{\rho - \sigma}\right)^{p_{2r}^{+}} \left\{ \int_{B(x_{0},2r)} (u_{\ell} - N + r)^{(p-p_{2r}^{-})\gamma'} dV \right\}^{\frac{1}{\gamma'}}
\cdot \left\{ \int_{B(x_{0},\rho)} (u_{\ell} - N + r)^{\beta\gamma} dV \right\}^{\frac{1}{\gamma}}
\leq \mathbf{c}_{3} \mathbf{c}_{4} r^{-p_{2r}^{-}} \left(\frac{\rho}{\rho - \sigma}\right)^{p_{2r}^{+}} \left\{ \int_{B(x_{0},\rho)} (u_{\ell} - N + r)^{\beta\gamma} dV \right\}^{\frac{1}{\gamma}},$$
(3.30)

where

$$\mathbf{c}_4 := \left\{ \int_{B(x_0,2r)} (u_\ell - N + r)^{(p - p_{2r}^-)\gamma'} \, \mathrm{dV} \right\}^{\frac{1}{\gamma'}}.$$

From (3.27) - (3.30), we can get

$$\begin{split} & \left[\mathbf{K}_{2} 2^{-\mathbf{n}_{2}} \oint_{B(x_{0},\sigma)} (u_{\ell} - N + r)^{\beta \mathbf{1}_{\mathbf{I}}} \, \mathrm{d} \mathbf{V} \right]^{\frac{1}{\mathbf{1}_{\mathbf{I}}}} \\ & \leq (2\mathbf{K}_{\mathbf{I}})^{p_{2r}^{-}} 2^{p_{2r}^{-} - 1} \left[\mathbf{c}_{1} \mathbf{c}_{3} \mathbf{c}_{4} + \mathbf{c}_{2} + (\hat{c}_{0} \hat{c}_{1} p_{2r}^{+})^{p_{2r}^{-}} \right] \\ & \cdot (1 + |\beta|)^{p_{2r}^{-}} \left(\frac{\rho}{\rho - \sigma} \right)^{p_{2r}^{+}} \left[\oint_{B(x_{0},\rho)} (u_{\ell} - N + r)^{\beta \gamma} \, \mathrm{d} \mathbf{V} \right]^{\frac{1}{\gamma}}. \end{split}$$

Then

$$\left[\int_{B(x_0,\rho)} (u_{\ell} - N + r)^{\beta\gamma} \, d\mathbb{V} \right]^{\frac{1}{\beta\gamma}} \\
\leq \mathbf{c}_{5}^{\frac{1}{|\beta|}} (1 + |\beta|)^{\frac{p_{2r}^{-}}{|\beta|}} \left(\frac{\rho}{\rho - \sigma} \right)^{\frac{p_{2r}^{-}}{|\beta|}} \left[\int_{B(x_0,\sigma)} (u_{\ell} - N + r)^{\beta 1_{\mathrm{I}}} \, d\mathbb{V} \right]^{\frac{1}{\beta 1_{\mathrm{I}}}}, \tag{3.31}$$

where

$$\mathbf{c}_5 := (\mathtt{K}_2^{-1} 2^{\mathtt{n}_2})^{1/1_{\mathrm{I}}} (2 \mathtt{K}_{\mathrm{I}})^{p_{2r}^-} 2^{p_{2r}^- - 1} \left[\mathbf{c}_1 \mathbf{c}_3 \mathbf{c}_4 + \mathbf{c}_2 + (\hat{c}_0 \hat{c}_1 p_{2r}^+)^{p_{2r}^-} \right].$$

Substituting $r_j = \sigma + 2^{-j}(\rho - \sigma)$, $\xi_j = -(1_{\rm I}/\gamma)^j s_0$, and $\beta = -(1_{\rm I}/\gamma)^j (s_0/\gamma)$ in (3.31), we obtain the iterative inequality

$$\Psi(u_{\ell} - N + r, \xi_{j}, B(x_{0}, r_{j})) \\
\leq \mathbf{c}_{5}^{\frac{\gamma}{|\xi_{j}|}} \left(1 + \frac{|\xi_{j}|}{\gamma}\right)^{\frac{\gamma p_{2r}^{+}}{|\xi_{j}|}} \left(\frac{r_{j}}{r_{j} - r_{j+1}}\right)^{\frac{\gamma p_{2r}^{+}}{|\xi_{j}|}} \Psi(u_{\ell} - N + r, \xi_{j+1}, B(x_{0}, r_{j+1})),$$

where $\Psi(f,q,D) = (\int_D f^q \, dV)^{1/q}$.

Hence,

$$\begin{split} &\Psi(u_{\ell} - N + r, -s_{0}, B(x_{0}, \rho)) \\ &\leq \prod_{j=0}^{\infty} \left[\mathbf{c}_{5}^{\gamma/|\xi_{j}|} \left(1 + \frac{|\xi_{j}|}{\gamma} \right)^{\gamma p_{2r}^{+}/|\xi_{j}|} \left(\frac{2^{j+1}\rho}{\rho - \sigma} \right)^{\gamma p_{2r}^{+}/|\xi_{j}|} \right] \underset{B(x_{0}, \sigma)}{\operatorname{ess inf}} \left\{ u_{\ell} - N + r \right\} \\ &\leq \mathbf{c}_{5}^{\sum_{i=0}^{\infty} \gamma/|\xi_{j}|} 2^{\sum_{j=0}^{\infty} (j+1)\gamma p_{2r}^{+}/|\xi_{j}|} \left(\frac{\rho}{\rho - \sigma} \right)^{\sum_{i=0}^{\infty} \gamma p_{2r}^{+}/|\xi_{j}|} \prod_{j=0}^{\infty} \left(1 + \frac{|\xi_{j}|}{\gamma} \right)^{\gamma p_{2r}^{+}/|\xi_{j}|} \\ &\cdot \underset{B(x_{0}, \sigma)}{\operatorname{ess inf}} \left\{ u_{\ell} - N + r \right\}. \end{split}$$

We have

$$\sum_{j=0}^{\infty} \frac{\gamma}{|\xi_j|} = \frac{1_{\mathrm{I}} \gamma}{s_0 (1_{\mathrm{I}} - \gamma)} \quad \text{ and } \quad \sum_{j=0}^{\infty} j \frac{\gamma}{|\xi_j|} = \frac{1_{\mathrm{I}} \gamma^2}{s_0 (1_{\mathrm{I}} - \gamma)^2}.$$

Let $j_0 \in \mathbb{N}$ such that

$$\frac{|\xi_j|}{\gamma} \geq 1 \text{ if } j \geq j_0 + 1 \quad \text{ and } \quad \frac{|\xi_j|}{\gamma} < 1 \text{ if } j \leq j_0.$$

Then,

$$\begin{split} & \prod_{j=0}^{\infty} \left(1 + \frac{|\xi_{j}|}{\gamma} \right)^{\gamma p_{2r}^{+}/|\xi_{j}|} \leq \prod_{j=0}^{j_{0}} 2^{\gamma p_{2r}^{+}/|\xi_{j}|} \prod_{j=j_{0}+1}^{\infty} \left(2 \frac{|\xi_{j}|}{\gamma} \right)^{\gamma p_{2r}^{+}/|\xi_{j}|} \\ & \leq \prod_{j=0}^{j_{0}} 2^{\gamma p_{2r}^{+}/|\xi_{j}|} \prod_{j=j_{0}+1}^{\infty} \left(2 \frac{|\xi_{j}|}{\gamma} \right)^{\gamma p_{2r}^{+}/|\xi_{j}|} \\ & \leq 2^{\sum_{j=0}^{\infty} \gamma p_{2r}^{+}/|\xi_{j}|} \prod_{j=j_{0}+1}^{\infty} \left[\left(\frac{1_{I}}{\gamma} \right)^{j} \frac{s_{0}}{\gamma} \right]^{\gamma p_{2r}^{+}/|\xi_{j}|} \\ & \leq 2^{\sum_{j=0}^{\infty} \gamma p_{2r}^{+}/|\xi_{j}|} \left(\frac{1_{I}}{\gamma} \right)^{\sum_{j=0}^{\infty} j \gamma p_{2r}^{+}/|\xi_{j}|}, \end{split}$$

since $\gamma \geq s_0$.

This implies,

$$\Psi(u_{\ell} - N + r, -s_0, B(x_0, \rho))$$

$$\leq (\mathbf{c}_6 \mathbf{1}_{\mathrm{I}} / \gamma)^{\frac{\mathbf{c}_7}{s_0}} \mathbf{c}_8^{\frac{\mathbf{1}_{\mathrm{I}} \gamma}{s_0 (\mathbf{1}_{\mathrm{I}} - \gamma)}} \left(\frac{\rho}{\rho - \sigma} \right)^{\frac{p_{2r}^+ \mathbf{1}_{\mathrm{I}} \gamma}{s_0 (\mathbf{1}_{\mathrm{I}} - \gamma)}}$$

$$\cdot \text{ ess inf }_{B(x_0, \sigma)} \left\{ u_{\ell} - N + r \right\},$$

where $\mathbf{c}_6 := [2(\hat{c}_0+1)\mathbf{K}_2^{-1}(\mathbf{K}_{\mathrm{I}}+1)(\mathbf{K}_4+1)]^{\mathbf{n}_2}, \mathbf{c}_7 = p_{2r}^+ \left[\frac{5\mathbf{1}_{\mathrm{I}}\gamma}{\mathbf{1}_{\mathrm{I}}-\gamma} + \frac{\mathbf{1}_{\mathrm{I}}\gamma^2}{(\mathbf{1}_{\mathrm{I}}-\gamma)^2}\right], \text{ and } \mathbf{c}_8 := \mathbf{c}_1\mathbf{c}_4 \max\{\hat{c}_1^{p_{2r}^-}, \hat{c}_1^{p_{2r}^+}\} + \mathbf{c}_2 + (\hat{c}_1p_{2r}^+)^{p_{2r}^-} + 1. \text{ This completes the proof.}$

Before proving the weak reverse Hölder inequality (Lemma 3.7), we will first prove the following

Lemma 3.6. Let $\{u_\ell\} \subset W^{1,p(x)}_{\mathrm{loc}}(\Omega; V)$ be a family of functions, and let $u \in W^{1,p(x)}_{\mathrm{loc}}(\Omega; V)$ be an element. Assume that (1.14) (of property (P_1)) holds. Suppose that u is a supersolution of (1.1) and $u_\ell \geq 0$. Let W be a measurable subset of $B(x_0, R) \subset \psi(\tilde{U})$, and let $\eta \in C_0^{\infty}(B(x_0, R))$ such that $0 \leq \eta \leq 1$. Let $\gamma < 0$. Then, we have

$$\int_{W} |\boldsymbol{\nabla} u_{\ell}|_{F}^{p_{W}^{-}} \eta^{p_{R}^{+}} u_{\ell,\alpha}^{\gamma-1} \, dV$$

$$\leq c_{1} \int_{\Omega} u_{\ell,\alpha}^{\gamma+p-1} |\mathcal{D}\eta|_{F^{*}}^{p} \, dV + c_{2} \int_{\Omega} \eta^{p_{R}^{+}} u_{\ell,\alpha}^{\gamma-1} \, dV, \tag{3.32}$$

 $\textit{where } u_{\ell,\alpha} := u_\ell + \alpha, \, \alpha \geq 0, \, \, p_W^- := \inf_W p, \, p_{\mathtt{R}}^- := \inf_{B(x_0,\mathtt{R})} p, \, p_{\mathtt{R}}^+ := \sup_{B(x_0,\mathtt{R})} p, \, p_{\mathtt{R}}^+ := \sup_{B(x_0,$

$$c_1 := \frac{4\Lambda_2 p_{\mathtt{R}}^+}{|\gamma|\Lambda_1 p_{\mathtt{R}}^-} \max\left\{ \left[2\Lambda_2 (p_{\mathtt{R}}^+ - 1)/(|\gamma|\Lambda_1) \right]^{p_{\mathtt{R}}^- - 1}, \left[2\Lambda_2 (p_{\mathtt{R}}^+ - 1)/(|\gamma|\Lambda_1) \right]^{p_{\mathtt{R}}^+ - 1} \right\} + \mathfrak{T} \frac{4\Lambda_2 p_{\mathtt{R}}^+}{|\gamma|\Lambda_1 p_{\mathtt{R}}^-}$$

and

$$c_2 := \mathfrak{T} \frac{4(p_{\mathtt{R}}^+ - 1)(p_{\mathtt{R}}^+ + |\gamma|)}{\Lambda_1 p_{\mathtt{R}}^+ |\gamma|} + 1.$$

Proof. We have,

$$\begin{split} \int_{\Omega} \mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) \bullet \mathcal{D}\zeta \, \mathrm{d} \boldsymbol{\mathsf{V}} &= \int_{\Omega} \mathcal{A}(\cdot, \boldsymbol{\nabla} u) \bullet \mathcal{D}\zeta \, \mathrm{d} \boldsymbol{\mathsf{V}} \\ &+ \int_{\Omega} \left[\mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) - \mathcal{A}(\cdot, \boldsymbol{\nabla} u) \right] \bullet \mathcal{D}\zeta \, \mathrm{d} \boldsymbol{\mathsf{V}}. \end{split}$$

Let $\zeta:=u_{\ell,\alpha}^{\gamma}\eta^{p_{\rm R}^+}$. Since u is a supersolution and

$$\mathfrak{D}\zeta = \gamma u_{\ell,\alpha}^{\gamma-1} \eta^{p_{\mathbf{R}}^+} \mathfrak{D}u + p_{\mathbf{R}}^+ u_{\ell,\alpha}^{\gamma} \eta^{p_{\mathbf{R}}^+ - 1} \mathfrak{D}\eta,$$

we have

$$\int_{\Omega} \mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) \bullet \left[\gamma u_{\ell,\alpha}^{\gamma-1} \boldsymbol{\eta}^{p_{\mathrm{R}}^{+}} \mathfrak{D} u + p_{\mathrm{R}}^{+} u_{\ell,\alpha}^{\gamma} \boldsymbol{\eta}^{p_{\mathrm{R}}^{+}-1} \mathfrak{D} \boldsymbol{\eta} \right] \, \mathrm{dV} \geq E(\ell),$$

where

$$E(\ell) := \int_{\Omega} \left[\mathcal{A}(\cdot, \boldsymbol{\nabla} u_{\ell}) - \mathcal{A}(\cdot, \boldsymbol{\nabla} u) \right] \bullet \left[\gamma u_{\ell, \alpha}^{\gamma-1} \boldsymbol{\eta}^{p_{\mathrm{R}}^{+}} \mathcal{D} u_{\ell} + p_{\mathrm{R}}^{+} u_{\ell, \alpha}^{\gamma} \boldsymbol{\eta}^{p_{\mathrm{R}}^{+} - 1} \mathcal{D} \boldsymbol{\eta} \right] \, \mathrm{dV}.$$

From (a_1) and (a_2) , and since γ is a negative.

$$|\gamma|\Lambda_1 \int_{\Omega} |\boldsymbol{\nabla} u_{\ell}|_F^p \eta^{p_{\mathsf{R}}^+} u_{\ell,\alpha}^{\gamma-1} \, \mathrm{dV} \leq p_{\mathsf{R}}^+ \Lambda_2 \int_{\Omega} u_{\ell,\alpha}^{\gamma} \eta^{p_{\mathsf{R}}^+ - 1} |\boldsymbol{\nabla} u_{\ell}|_F^{p-1} |\mathfrak{D} \eta|_{F^*} \, \mathrm{dV} - E(\ell). \tag{3.33}$$

Next, we estimate the first term of the right-hand side of (3.33). Using the Young's inequality,

$$\begin{split} p_{\mathrm{R}}^{+} & \int_{\Omega} |\nabla u_{\ell}|_{F}^{p-1} u_{\ell,\alpha}^{\gamma} \eta^{p_{\mathrm{R}}^{+}-1} |\mathcal{D}\eta|_{F^{*}} \, \mathrm{dV} \\ & \leq p_{\mathrm{R}}^{+} \int_{\Omega} \frac{\epsilon^{-p+1}}{p} \left[u_{\ell,\alpha}^{(\gamma+p-1)/p} |\mathcal{D}\eta|_{F^{*}} \eta^{p_{\mathrm{R}}^{+}-1-p_{\mathrm{R}}^{+}(p-1)/p} \right]^{p} \\ & + \frac{\epsilon(p-1)}{p} \left[|\nabla u_{\ell}|_{F}^{p-1} \eta^{p_{\mathrm{R}}^{+}(p-1)/p} u_{\ell,\alpha}^{\gamma-(\gamma+p-1)/p} \right]^{p/(p-1)} \, \mathrm{dV} \\ & \leq \frac{p_{\mathrm{R}}^{+}}{p_{\mathrm{R}}^{-}} \max\{\epsilon^{-p_{\mathrm{R}}^{-}+1}, \epsilon^{-p_{\mathrm{R}}^{+}+1}\} \int_{\Omega} u_{\ell,\alpha}^{\gamma+p-1} |\mathcal{D}\eta|_{F^{*}}^{p} \eta^{p_{\mathrm{R}}^{+}-p} \, \mathrm{dV} \\ & + (p_{\mathrm{R}}^{+}-1)\epsilon \int_{\Omega} |\nabla u|_{F}^{p} \eta^{p_{\mathrm{R}}^{+}} u_{\ell,\alpha}^{\gamma-1} \, \mathrm{dV}. \end{split} \tag{3.34}$$

Choosing $\epsilon = |\gamma|\Lambda_1/[2\Lambda_2(p_{\rm R}^+-1)]$, from (3.33) and (3.34),

$$\begin{split} &\frac{|\gamma|\Lambda_{1}}{2} \int_{\Omega} |\nabla u_{\ell}|_{F}^{p} \eta^{p_{\mathsf{R}}^{+}} u_{\ell,\alpha}^{\gamma-1} \, \mathrm{dV} \\ &\leq \Lambda_{2} \frac{p_{\mathsf{R}}^{+}}{p_{\mathsf{R}}^{-}} \max \left\{ \left[2\Lambda_{2} (p_{\mathsf{R}}^{+} - 1)/(|\gamma|\Lambda_{1}) \right]^{p_{\mathsf{R}}^{-} - 1}, \left[2\Lambda_{2} (p_{\mathsf{R}}^{+} - 1)/(|\gamma|\Lambda_{1}) \right]^{p_{\mathsf{R}}^{+} - 1} \right\} \\ &\cdot \int_{\Omega} u_{\ell,\alpha}^{\gamma+p-1} |\mathfrak{D}\eta|_{F^{*}}^{p} \eta^{p_{\mathsf{R}}^{+} - p} \, \mathrm{dV} - E(\ell). \end{split} \tag{3.35}$$

Now, we estimate the term $E(\ell)$. From (1.14),

$$|\mathcal{A}(\cdot, \nabla u_{\ell}) - \mathcal{A}(\cdot, \nabla u)|_F \le \mathfrak{T} < \frac{\Lambda_1 p_{\mathsf{R}}^-}{4} \quad \text{in} \quad \Omega.$$

We also have

$$|\mathcal{D}u_{\ell}|_{F^*} = |\nabla u_{\ell}|_F \le \frac{1}{p_{\mathbf{R}}^-} |\nabla u_{\ell}|_F^p + \frac{p_{\mathbf{R}}^+ - 1}{p_{\mathbf{R}}^+} \quad \text{in} \quad B(x_0, \mathbf{R}),$$

and

$$\begin{split} u_{\ell,\alpha}^{\gamma} \eta^{p_{\mathsf{R}}^{+}-1} | \mathfrak{D} \eta |_{F^*} \\ & \leq \frac{1}{p_{\mathsf{R}}^{-}} \left[u_{\ell,\alpha}^{(\gamma+p-1)/p} | \mathfrak{D} \eta |_{F^*} \eta^{p_{\mathsf{R}}^{+}-1-p_{\mathsf{R}}^{+}(p-1)/p} \right]^p + \frac{(p_{\mathsf{R}}^{+}-1)}{p_{\mathsf{R}}^{+}} \left[\eta^{p_{\mathsf{R}}^{+}(p-1)/p} u_{\ell,\alpha}^{\gamma-(\gamma+p-1)/p} \right]^{p/(p-1)} \\ & = \frac{1}{p_{\mathsf{R}}^{-}} u_{\ell,\alpha}^{\gamma+p-1} | \mathfrak{D} \eta |_{F^*}^p \eta^{p_{\mathsf{R}}^{+}-p} + \frac{(p_{\mathsf{R}}^{+}-1)}{p_{\mathsf{R}}^{+}} \eta^{p_{\mathsf{R}}^{+}} u_{\ell,\alpha}^{\gamma-1}. \end{split}$$

Therefore,

$$\begin{split} |E(\ell)| &\leq \mathfrak{T} \int_{\Omega} \left| \gamma u_{\ell,\alpha}^{\gamma-1} \eta^{p_{\mathrm{R}}^{+}} \mathcal{D} u + p_{\mathrm{R}}^{+} u_{\ell,\alpha}^{\gamma} \eta^{p_{\mathrm{R}}^{+}-1} \mathcal{D} \eta \right|_{F^{*}} \, \mathrm{d} \mathbb{V} \\ &\leq \frac{|\gamma| \Lambda_{1}}{4} \int_{\Omega} |\boldsymbol{\nabla} u_{\ell}|_{F}^{p} \eta^{p_{\mathrm{R}}^{+}} u_{\ell,\alpha}^{\gamma-1} \, \mathrm{d} \mathbb{V} \\ &+ \mathfrak{T} \frac{p_{\mathrm{R}}^{+}}{p_{\mathrm{R}}^{-}} \int_{\Omega} u_{\ell,\alpha}^{\gamma+p-1} |\mathcal{D} \eta|_{F^{*}}^{p} \eta^{p_{\mathrm{R}}^{+}-p} \, \mathrm{d} \mathbb{V} + \mathfrak{T} \frac{(p_{\mathrm{R}}^{+}-1)(p_{\mathrm{R}}^{+}+|\gamma|)}{p_{\mathrm{R}}^{+}} \int_{\Omega} \eta^{p_{\mathrm{R}}^{+}} u_{\ell,\alpha}^{\gamma-1} \, \mathrm{d} \mathbb{V}. \end{split}$$

By combining this with (3.35) we arrive at

$$\begin{split} & \int_{\Omega} |\boldsymbol{\nabla} u_{\ell}|_F^p \boldsymbol{\eta}^{p_{\mathrm{R}}^+} \boldsymbol{u}_{\ell,\alpha}^{\gamma-1} \, \mathrm{d} \mathbf{V} \\ & \leq c_1 \int_{\Omega} \boldsymbol{u}_{\ell,\alpha}^{\gamma+p-1} |\mathfrak{D} \boldsymbol{\eta}|_{F^*}^p \, \mathrm{d} \mathbf{V} + c_2 \int_{\Omega} \boldsymbol{\eta}^{p_{\mathrm{R}}^+} \boldsymbol{u}_{\ell,\alpha}^{\gamma-1} \, \mathrm{d} \mathbf{V}, \end{split}$$

where

$$c_1 := \frac{4\Lambda_2 p_{\rm R}^+}{|\gamma| \Lambda_1 p_{\rm R}^-} \max \left\{ \left[2\Lambda_2 (p_{\rm R}^+ - 1)/(|\gamma| \Lambda_1) \right]^{p_{\rm R}^- - 1}, \left[2\Lambda_2 (p_{\rm R}^+ - 1)/(|\gamma| \Lambda_1) \right]^{p_{\rm R}^+ - 1} \right\} + \mathfrak{T} \frac{4\Lambda_2 p_{\rm R}^+}{|\gamma| \Lambda_1 p_{\rm R}^-}$$

and

$$c_2 := \mathfrak{T} \frac{4(p_{\mathbf{R}}^+ - 1)(p_{\mathbf{R}}^+ + |\gamma|)}{\Lambda_1 p_{\mathbf{R}}^+ |\gamma|}.$$

Using $|a|^{p_W^-} \le |a|^{p(x)} + 1$ for all $x \in W$ and $a \in \mathbb{R}$, we obtain the desired estimate in (3.32).

Lemma 3.7. Let $\{u_\ell\} \subset W^{1,p(x)}_{loc}(\Omega; V)$ be a family of functions, and let $u \in W^{1,p(x)}_{loc}(\Omega; V) \cap L^{\infty}_{loc}(\Omega)$ be an element. Assume that (1.7), (1.8), (1.11), (1.15), and (1.14) (of property (P_1)) are satisfied. Let $B(x_0, 20r) \subset \psi(\tilde{U})$, $0 < 20r < R_*$, and $1 \ge \alpha \ge r$. Suppose that u is a supersolution of (1.1), with $u_\ell \ge N \ge 0$ and $N_0 \ge N$. Then, we have

$$\left[\int_{B(x_0,2r)} (u_\ell - N + \alpha)^{c_2 c_3} \, \mathrm{dV} \right]^{\frac{1}{c_2 c_3}} \le c_1^{1/c_3} \left[\int_{B(x_0,2r)} (u_\ell - N + \alpha)^{-c_2 c_3} \, \mathrm{dV} \right]^{\frac{1}{-c_2 c_3}},$$

$$c_3 := \left\{ \left[\left(p_{20r}^- - 1 \right)^{-1} + 1 \right] \left(\|u\|_{s, B(x_0, 20r), \mathbf{V}} + 1 \right)^{\mathbf{K}_3} \left(\hat{c}_1 + 1 \right)^{p^+} \right\}^{-1}.$$

Proof. Choose a ball $B(x_1,2\mathtt{R})\subset B(x_0,20r)$ and a cutoff function $\eta\in C_0^\infty(B(x_1,2\mathtt{R}))$ given in Lemma 3.2: $\eta=1$ in $B(x_1,\mathtt{R})$ and $|\mathcal{D}\eta|_{F^*}\leq \hat{c}_0\hat{c}_1\mathtt{R}^{-1}$.

Let $p_{1,2R}^- = \inf_{B(x_1,2R)} p$. Taking $W = B(x_1,2R)$ and $\gamma = 1 - p_{1,2R}^-$ in (3.32), we have

$$\begin{split} & \oint_{B(x_1,\mathbf{R})} |\nabla \log w_{\ell,\alpha}|_F^{p_{1,2\mathbf{R}}^-} \, \mathrm{dV} \\ & \leq (c_1 + c_2) \frac{\mathbf{V}(B(x_1,2\mathbf{R}))}{\mathbf{V}(B(x_1,\mathbf{R}))} \left(\oint_{B(x_1,2\mathbf{R})} w_{\ell,\alpha}^{-p_{1,2\mathbf{R}}^-} \, \mathrm{dV} + \oint_{B(x_1,2\mathbf{R})} (\hat{c}_0 \hat{c}_1)^p w_{\ell,\alpha}^{p-p_{1,2\mathbf{R}}^-} \mathbf{R}^{-p} \, \mathrm{dV} \right), \end{split}$$

where $w_{\ell,\alpha} := u_{\ell,\alpha} - N = u_{\ell} - N + \alpha$ and $r \leq \alpha \leq 1$.

From (1.15), (2.7), (1.14), and (2.5),

$$c_1 + c_2 \leq (p_{1,2\mathtt{R}}^- - 1)^{-1} C_1(\Lambda_1, \Lambda_2, p^+, \tilde{\mathtt{K}}_1, \mathtt{K}_2, \mathtt{K}_3, \mathtt{n}_2).$$

Using (2.8) and the estimate $w_{\ell,\alpha}^{-p_{1,2R}^-} \leq \alpha^{-p_{1,2R}^-} \leq r^{-p_{1,2R}^-} \leq (R/10)^{-p_{1,2R}^-}$, we get

$$\int_{B(x_1,\mathbf{R})} |\boldsymbol{\nabla} \log w_{\ell,\alpha}|_F^{p_{1,2\mathbf{R}}^-} \, \mathrm{dV} \leq (p_{1,2\mathbf{R}}^- - 1)^{-1} C_1 \left[\left(\frac{\mathbf{R}}{10} \right)^{-p_{1,2\mathbf{R}}^-} + C_2 \mathbf{R}^{-p_{1,2\mathbf{R}}^-} \int_{B(x_1,2\mathbf{R})} w_{\ell,\alpha}^{p-p_{1,2\mathbf{R}}^-} \, \mathrm{dV} \right],$$

where $C_2 := K_4 \max\{(\hat{c}_0 \hat{c}_1)^{p_{1,2R}^-}, (\hat{c}_0 \hat{c}_1)^{p_{1,2R}^+}\}.$

Let $f = \log w_{\ell,\alpha}$. By (1.11), (2.9), (1.14), and the above estimate, we obtain

$$\int_{B(x_{1},\mathbf{R})} \left| f - f_{B(x_{1},\mathbf{R})} \right| \, d\mathbf{V} \leq \mathbf{K}_{\mathrm{II}} \left(\mathbf{R}^{p_{1,2\mathbf{R}}^{-}} \int_{B(x_{1},\mathbf{R})} \left| \boldsymbol{\nabla} f \right|_{F}^{p_{1,2\mathbf{R}}^{-}} \, d\mathbf{V} \right)^{\frac{1}{p_{1,2\mathbf{R}}^{-}}} \\
\leq \left(p_{1,2\mathbf{R}}^{-} - 1 \right)^{-\frac{1}{p_{1,2\mathbf{R}}^{-}}} C_{3} C_{4} \left[1 + \int_{B(x_{1},2\mathbf{R})} (u - N + \mathfrak{T}_{0} + 1)^{p - p_{1,2\mathbf{R}}^{-}} \, d\mathbf{V} \right]^{\frac{1}{p_{1,2\mathbf{R}}^{-}}} \\
\leq \left[\left(p_{20r}^{-} - 1 \right)^{-1} + 1 \right] C_{3} C_{4} \\
\cdot \left\{ 1 + 2^{p_{20r}^{+} - p_{20r}^{-}} \left[\left(\mathbf{K}_{0}^{1/s} + 1 \right) \left(\| u \|_{s,B(x_{0},20r),\mathbf{V}} + 1 \right)^{p_{20r}^{+} - p_{20r}^{-}} + \left(N + \mathfrak{T}_{0} + 1 \right)^{p_{20r}^{+} - p_{20r}^{-}} \right] \right\}, \tag{3.36}$$

if $B(x_1, \mathbb{R}) \subset B(x_0, 10r)$, where $C_3 := C_3(\Lambda_1, \Lambda_2, p^+, \tilde{\mathbb{K}}_1, \mathbb{K}_2, \mathbb{K}_3, \mathbb{K}_4, \mathbb{K}_{II}, N_0, n_2, n) > 0$ and $C_4 := (\hat{c}_1 + 1)^{p^+}$. Since (3.36) holds for all balls $B(x_1, \mathbb{R}) \subset B(x_0, 10r)$, by Proposition 2.5:

where $C_5:=C_5(\Lambda_1,\Lambda_2,p^+,\tilde{\mathtt{K}}_1,\mathtt{K}_1,\mathtt{K}_2,\mathtt{K}_3,\mathtt{K}_4,\mathtt{K}_{\mathrm{II}},\mathtt{K}_0,N_0,\mathfrak{T}_0,\mathtt{n}_2,n,s)>0,$ $C_7:=C_7(\mathtt{K}_1)>0$ and

$$C_6 := \left\{ \left[(p_{20r}^- - 1)^{-1} + 1 \right] (\|u\|_{s, B(x_0, 20r), \mathbf{V}} + 1)^{\mathbf{K}_3} (\hat{c}_1 + 1)^{p^+} \right\}^{-1}.$$

Using (3.37), we can conclude that

$$\begin{split} &\left(\int_{B(x_0,2r)} e^{C_5 C_6 f} \, \mathrm{dV} \right) \left(\int_{B(x_0,2r)} e^{-C_5 C_6 f} \, \mathrm{dV} \right) \\ &= \left(\int_{B(x_0,2r)} e^{C_5 C_6 [f - f_{B(x_0,10r)}]} \, \mathrm{dV} \right) \left(\int_{B(x_0,2r)} e^{-C_5 C_6 [f - f_{B(x_0,10r)}]} \, \mathrm{dV} \right) \\ &\leq C_7^2 \end{split}$$

which implies

$$\begin{split} \left(\oint_{B(x_0,2r)} w_{\ell,\alpha}^{C_5 C_6} \, \mathrm{dV} \right)^{\frac{1}{C_5 C_6}} &= \left(\oint_{B(x_0,2r)} e^{C_5 C_6 f} \, \mathrm{dV} \right)^{\frac{1}{C_5 C_6}} \\ &\leq C_7^{\frac{2}{C_5 C_6}} \left(\oint_{B(x_0,2r)} e^{-C_5 C_6 f} \, \mathrm{dV} \right)^{\frac{-1}{C_5 C_6}} \\ &= C_7^{\frac{2}{C_5 C_6}} \left(\oint_{B(x_0,2r)} w_{\ell,\alpha}^{-C_5 C_6} \, \mathrm{dV} \right)^{\frac{-1}{C_5 C_6}}. \end{split}$$

Thus we conclude the proof of the lemma.

4. Removable sets for Hölder continuous solutions

In this section, we prove the two main results of this paper.

Definition 4.1. Let (X, d) be a metric space, and fix s > 0. For each $\delta > 0$ and $E \subset X$, define

$$H^s_{\delta}(E) := \inf \left\{ \sum_{j=1}^{\infty} \beta(s) \left[\frac{\operatorname{diam}(C_j)}{2} \right]^s \mid E \subset \cup_{j=1}^{\infty} C_j, \operatorname{diam}(C_j) \leq \delta \right\},\,$$

where $\beta(s)=\frac{\pi^{\frac{s}{2}}}{\Gamma(\frac{s}{2}+1)}$ and $\Gamma(s)=\int_0^\infty e^{-t}t^{s-1}\mathrm{d}t$ is the usual Gamma function.

The s-Hausdorff measure of E is the number

$$H^{s}(E) := \lim_{\delta \to 0} H^{s}_{\delta}(E) = \sup_{\delta > 0} H^{s}_{\delta}(E).$$

4.1. Proof of Theorem 1.2.

Lemma 4.2. Let $\{u_\ell\}\subset W^{1,p(x)}_{loc}(\Omega;\mathbb{V})$ be a family of functions, and let $u\in W^{1,p(x)}_{loc}(\Omega;\mathbb{V})\cap L^\infty_{loc}(\Omega)$ be an element. Suppose that (1.7) - (1.10) and property (P_3) are satisfied. Assume that $r\leq\sigma<\rho\leq 2r< R_*/10$ and $B(x_0,20r)\subset\subset\psi(\tilde{U})$. Let $|N|\leq N_0$ and $1<\gamma<1_{\mathrm{I}}$. There is $R_1>0$ such that $(c_2+1)(p-1)\leq 1$ in $\Omega\setminus B(0_\Omega,R_1)$, where $c_2=c_2(\Lambda_1,\Lambda_2,p^+,\tilde{\mathrm{K}}_1,\mathrm{K}_1,\mathrm{K}_2,\mathrm{K}_3,\mathrm{K}_4,\mathrm{K}_{\mathrm{II}},\mathrm{K}_0,N_0,\mathfrak{T}_0,\mathfrak{n}_2,n,s)$ is as defined in Lemma 3.7. Furthermore, we have

(i) Assume that property (P_1) is satisfied and $u, u_{\ell} \in \mathcal{K}_{\mathcal{O}_1, \mathcal{O}_2}(\Omega)$ for all ℓ . If u is a solution of the obstacle problem (2.14) with the obstacle $\mathcal{O}_1 \leq N$ in $B(x_0, 2r)$ and $x_0 \in \Omega \setminus B(\mathcal{O}_{\Omega}, R_1)$, then

ess sup
$$\left\{ (u_{\ell} - N)^{+} \right\}$$

$$\leq C_{1}^{\frac{1}{(p_{20r}^{-}-1)^{2}}} \left\{ \int_{B(x_{0},2r)} \left[(u_{\ell} - N)^{+} + r \right]^{c_{2}c_{3}} \, dV \right\}^{\frac{1}{c_{2}c_{3}}}, \tag{4.1}$$

 $\begin{aligned} \textit{where} \ C_1 &:= C_1(\Lambda_1, \Lambda_2, p^+, \tilde{\mathbf{K}}_1, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4, \mathbf{K}_{\mathrm{I}}, \mathbf{K}_{\mathrm{II}}, \mathbf{K}_0, \mathbf{n}_2, n, s, \gamma, \mathbf{1}_{\mathrm{I}}, N_0, \mathfrak{T}_0, \hat{c}_1, \|u\|_{s, B(x_0, 20r), \mathbf{V}}, \|u\|_{s\gamma', B(x_0, 20r), \mathbf{V}}) > 0, \ s > p_{20r}^+ - p_{20r}^-, \ 1 < \gamma < \mathbf{1}_{\mathrm{I}}, \ \textit{and} \end{aligned}$

$$c_3 := \left\{ \left[(p_{20r}^- - 1)^{-1} + 1 \right] (\|u\|_{s, B(x_0, 20r), V} + 1)^{\mathsf{K}_3} (\hat{c}_1 + 1)^{p^+} \right\}^{-1}.$$

(ii) Assume that (1.14) (of property (P_1)) holds. If u is a supersolution of (1.1), then, for $x_0 \in \Omega \setminus B(0_\Omega, R_1)$:

ess sup
$$\{|(u_{\ell} - N)^{-}|\}$$

$$\leq C_2^{\frac{1}{(p_{20r}^--1)^2}} \left\{ \oint_{B(x_0,2r)} \left[|(u_\ell-N)^-| + r \right]^{c_2c_3} \, \mathrm{dV} \right\}^{\frac{1}{c_2c_3}},$$

where $C_2 := C_2(\Lambda_1, \Lambda_2, p^+, \tilde{K}_1, K_1, K_2, K_3, K_4, K_I, K_{II}, K_0, n_2, n, s, \gamma, 1_I, N_0, \mathfrak{T}_0, \hat{c}_1, \|u\|_{s, B(x_0, 20r), \mathbf{v}}, \|u\|_{s\gamma', B(x_0, 20r), \mathbf{v}}) > 0.$

In (i) and (ii), $p_{20r}^- := \inf_{B(x_0, 20r)} p$ and $p_{20r}^+ := \sup_{B(x_0, 20r)} p$.

Proof. (i) Choose $R_0 > 0$ such that

$$(c_2+1)(p-1)<1$$
 in $\Omega \setminus B(0_{\Omega}, R_0)$,

where $c_2 = c_2(\Lambda_1, \Lambda_2, p^+, \tilde{K}_1, K_1, K_2, K_3, K_4, K_{II}, N_0, \mathfrak{T}_0, K_0, n_2, n, s) > 0$. Next, from Corollary 2.11 and Proposition 3.4 (i), we obtain (4.1).

Similarly to (i), we conclude (ii).
$$\Box$$

Lemma 4.3. Let $\{u_\ell\}\subset W^{1,p(x)}_{loc}(\Omega;\mathbb{V})$ be a family of functions, and let $u\in W^{1,p(x)}_{loc}(\Omega;\mathbb{V})\cap L^\infty_{loc}(\Omega)$ be an element. Assume that (1.7)–(1.11) and (1.14) (from property (P_1)) are verified, with $p\in C(\Omega)$ satisfying property (P_3) . Let $r\leq \sigma<\rho\leq 2r\leq R_*/10$, $B(x_0,20r)\subset \psi(\tilde{U})$, and $1<\gamma<1_{\mathrm{I}}$. Suppose that u is a supersolution of (1.1) with $u_\ell\geq N$ in $B(x_0,2r)$ and $0\leq N\leq N_0$. There is $R_1>0$ such that $(c_2+1)(p-1)\leq 1$ in $\Omega\setminus B(0_\Omega,R_1)$, where $c_2=c_2(\Lambda_1,\Lambda_2,p^+,\tilde{\mathrm{K}}_1,\mathrm{K}_1,\mathrm{K}_2,\mathrm{K}_3,\mathrm{K}_4,\mathrm{K}_{\mathrm{II}},\mathrm{K}_0,N_0,\mathfrak{T}_0,\mathrm{n}_2,n,s)$ is as defined in Lemma 3.7. Furthermore, we have

$$\left[\int_{B(x_0,2r)} (u_{\ell} - N + r)^{c_2 c_3} \, d\mathbf{V} \right]^{\frac{1}{c_2 c_3}} \\
\leq \left[C_3(p_{20r}^- - 1) \right]^{-\frac{1}{p_{20r}^- - 1}} \underset{B(x_0,r)}{\operatorname{ess inf}} \left\{ u_{\ell} - N + r \right\}, \tag{4.2}$$

$$c_3 := \left\{ \left[\left(p_{20r}^- - 1 \right)^{-1} + 1 \right] \left(\|u\|_{s, B(x_0, 20r), V} + 1 \right)^{\mathsf{K}_3} \left(\hat{c}_1 + 1 \right)^{p^+} \right\}^{-1}.$$

Proof. The proof follow from of Lemma 3.7 and Proposition 3.5, taking $s_0 = c_2 c_3$.

Remember that:

Theorem 1.2. Let (Ω, F, V) be a reversible Finsler manifold, where V and p satisfy conditions (1.5)–(1.11). Let K be a compact subset of Ω , and define $K_{\delta} := \{x \in \Omega \mid d(x, K) < \delta\}$, where $\overline{K_{\delta}} \subset \Omega$. Suppose that $\vartheta \in \mathcal{K}_{\mathcal{O}_1}(\Omega)$ is a solution to the obstacle problem (2.14), with the obstacle $\mathcal{O}_1 \in C(\Omega)$ satisfying

$$|\mathcal{O}_1(x) - \mathcal{O}_1(y)| \le \hat{\mathbf{s}} d^{\alpha}(x, y), \quad \text{for all } x \in K, y \in \Omega,$$
 (4.3)

where $0 < \alpha \le 1$ and $\hat{s} > 0$.

Additionally, let $\{\vartheta_\ell\} \subset \mathfrak{K}_{\mathfrak{O}_1}(\Omega)$ be a family such that $\{\vartheta = \mathfrak{O}_1\} \subset \{\vartheta_\ell = \mathfrak{O}_1\}$ for all ℓ , and suppose that this family satisfies properties (P_1) and (P_2) , with $p \in C(\Omega)$ satisfying property (P_3) .

There exists $R_1 = R_1(\Lambda_1, \Lambda_2, p^+, \tilde{K}_1, K_1, K_2, K_3, K_4, K_{II}, K_0, \mathfrak{T}_0, n, s, n_1, \inf_{B(x,5r)} \mathfrak{O}_1, \sup_{B(x,5r)} \mathfrak{O}_1) > 0$, where $s > \sup_{K_\delta} p - \inf_{K_\delta} p$, such that $p - 1 \leq 1$ in $\Omega \backslash B(\mathfrak{O}_\Omega, R_1)$. Furthermore, for every $x \in K \cap (\Omega \backslash B(\mathfrak{O}_\Omega, R_1))$ and $0 < r < \frac{1}{41} \min\{R_*, \delta\}$, the following holds:

$$\mu_{\ell}(B(x,r))$$

$$\leq C_1 C_2^{\frac{\tilde{\mathsf{K}}_1}{p^-_{41r}-1}} (p^-_{41r}-1)^{-\tilde{\mathsf{K}}_1} r^{\mathsf{n}_1-p(x)+\alpha(p(x)-1)} + \Upsilon_1(x,5r) + \Upsilon_2(x,r),$$

 $if \ B(x,41r) \subset \subset \psi(\tilde{U}), \ where \ \psi: \tilde{U} \to \psi(\tilde{U}) \subset \Omega \ is \ a \ chart, \ p_{41r}^- := \inf_{B(x,41r)} p, \ C_i := C_i(\Lambda_1,\Lambda_2,p^+,\tilde{\mathsf{K}}_1,\mathsf{K}_1,\mathsf{K}_2,\mathsf{K}_3,\mathsf{K}_4,\mathsf{K}_1,\mathsf{K}_1,\mathsf{K}_0,\mathsf{K},n,s,\gamma,1_1,n_1,n_2,\alpha,\hat{\mathsf{s}},\inf_{B(x,5r)} \mathfrak{O}_1,\sup_{B(x,5r)} \mathfrak{O}_1,\hat{c}_1,\mathfrak{T}_0, \|\vartheta\|_{s,B(x,41r),\mathtt{V}}, \|\vartheta\|_{s\gamma',B(x,41r),\mathtt{V}}) > 0, \ for \ i=1,2, \ and \ 1 < \gamma < 1_{\mathrm{I}}. \ Also, \ \hat{c}_1 \ is \ a \ constant \ (depending \ on \ \psi \ and \ F) \ given \ in \ (3.1), \ and \ R_* \ is \ defined \ in \ Lemma \ 2.8 \ (i).$

Proof. Steep 1. Since ϑ is a solution to the obstacle problem (2.14) in $\mathfrak{K}_{\mathcal{O}_1}(\Omega)$, it follows that ϑ is a supersolution of (1.1). Moreover, since the obstacle $\mathcal{O}_1 \in C(\Omega)$, Proposition 2.16 ensures that ϑ is continuous.

If $B(x,r) \cap \{y \in \Omega \mid \vartheta(y) = \mathcal{O}_1(y)\} = \emptyset$, by Proposition 2.16 and property (P_2) we have

$$\int_{\Omega} \mathcal{A}(\cdot,\boldsymbol{\nabla}\boldsymbol{\vartheta}_{\ell}) \bullet \mathcal{D}\boldsymbol{\zeta} \, \mathrm{d} \mathbf{V} = \int_{\{\boldsymbol{\vartheta}>\mathcal{O}_1\}} \left[\mathcal{A}(\cdot,\boldsymbol{\nabla}\boldsymbol{\vartheta}_{\ell}) - \mathcal{A}(\cdot,\boldsymbol{\nabla}\boldsymbol{\vartheta})\right] \bullet \mathcal{D}\boldsymbol{\zeta} \, \mathrm{d} \mathbf{V} \leq \Upsilon_2(\boldsymbol{x},\boldsymbol{r}),$$

where $\zeta \in C_0^{\infty}(B(x,r))$ and $0 \le \zeta \le 1$.

If $B(x,r) \cap \{y \in \Omega \mid \vartheta(y) = \mathcal{O}_1(y)\} \neq \emptyset$, then let $x_0 \in B(x,r) \cap \{y \in \Omega \mid \vartheta(y) = \mathcal{O}_1(y)\}$. As a consequence of the hypotheses, $x_0 \in B(x,r) \cap \{y \in \Omega \mid \vartheta_\ell(y) = \mathcal{O}_1(y)\}$. Then,

$$\mu_{\ell}(B(x,r)) < \mu_{\ell}(B(x_0,2r)).$$
 (4.4)

Next, we proceed to estimate the right hand side of (4.4).

There exist points $x_1, x_2 \in B(x_0, 8r)$ such that $\sup_{B(x_0, 8r)} \mathcal{O}_1 = \mathcal{O}_1(x_1)$ and $\inf_{B(x_0, 8r)} \mathcal{O}_1 = \mathcal{O}_1(x_2)$. Then, (4.3) implies that

$$\sup_{B(x_{0},8r)} \mathcal{O}_{1} - \inf_{B(x_{0},8r)} \mathcal{O}_{1} = \mathcal{O}_{1}(x_{1}) - \mathcal{O}_{1}(x) + \mathcal{O}_{1}(x) - \mathcal{O}_{1}(x_{2})$$

$$\leq \hat{s}d^{\alpha}(x_{1},x) + \hat{s}d^{\alpha}(x,x_{2})$$

$$\leq \hat{s}2^{\alpha} \left(d^{\alpha}(x_{1},x_{0}) + d^{\alpha}(x_{0},x) + d^{\alpha}(x,x_{0}) + d^{\alpha}(x_{0},x_{2})\right)$$

$$\leq \hat{s}2^{\alpha} \left[(8r)^{\alpha} + r^{\alpha} + r^{\alpha} + (8r)^{\alpha}\right] \leq C_{4}r^{\alpha},$$
(4.5)

where $C_4 := 2^{\alpha+1} \hat{s}(8^{\alpha} + 2)$.

We define,

$$\alpha_{1} = \sup_{B(x_{0},4r)} \mathcal{O}_{1} - \inf_{B(x_{0},4r)} \mathcal{O}_{1} + \inf_{B(x_{0},4r)} \vartheta_{\ell},$$

$$\alpha_{2} = \sup_{B(x_{0},4r)} \mathcal{O}_{1} - \inf_{B(x_{0},4r)} \mathcal{O}_{1} - \inf_{B(x_{0},4r)} \vartheta_{\ell}.$$

We have

$$\inf_{B(x_0,4r)} \mathcal{O}_1 \le \inf_{B(x_0,4r)} \vartheta_{\ell} \le \vartheta_{\ell}(x_0) = \mathcal{O}_1(x_0) \le \sup_{B(x_0,4r)} \mathcal{O}_1. \tag{4.6}$$

Thus, we obtain

$$|\alpha_1| \le 2 \left(\sup_{B(x,5r)} \mathcal{O}_1\right)^+ - \inf_{B(x,5r)} \mathcal{O}_1.$$

Furthermore, since $O_1 \le \alpha_1$ in $B(x_0, 4r)$, we take $N = \alpha_1$ and $N_0 = 2(\sup_{B(x,5r)} O_1)^+ - \inf_{B(x,5r)} O_1$ in (4.1). Then,

$$\sup_{B(x_0,2r)} \left\{ (\vartheta_\ell - \alpha_1)^+ \right\} \leq C_1^{\frac{1}{(p_{40r}^- - 1)^2}} \left\{ \oint_{B(x_0,4r)} \left[(\vartheta_\ell - \alpha_1)^+ + 2r \right]^{c_2 c_3} \, \mathrm{dV} \right\}^{\frac{1}{c_2 c_3}},$$

where $p_{40r}^- := \inf_{B(x_0, 40r)} p$.

$$\vartheta_{\ell} - \alpha_{1} = \vartheta_{\ell} - \left(\sup_{B(x_{0},4r)} \mathcal{O}_{1} - \inf_{B(x_{0},4r)} \mathcal{O}_{1}\right) - \inf_{B(x_{0},4r)} \vartheta_{\ell}$$

$$\leq \vartheta_{\ell} + \left(\sup_{B(x_{0},4r)} \mathcal{O}_{1} - \inf_{B(x_{0},4r)} \mathcal{O}_{1}\right) - \inf_{B(x_{0},4r)} \vartheta_{\ell} = \vartheta_{\ell} + \alpha_{2}$$

and $\vartheta_{\ell} + \alpha_2 \geq 0$ in $B(x_0, 4r)$, we have

$$\sup_{B(x_0,2r)} \left\{ (\vartheta_{\ell} - \alpha_1)^+ \right\} \le C_1^{\frac{1}{(p_{40r}^- - 1)^2}} \left[\oint_{B(x_0,4r)} (\vartheta_{\ell} + \alpha_2 + 2r)^{c_2 c_3} \, dV \right]^{\frac{1}{c_2 c_3}}. \tag{4.7}$$

If $\alpha_2 \geq 0$, then $\vartheta_\ell + \alpha_2$ is a nonnegative supersolution of (1.1) in $B(x_0, 4r)$, then we take $N = N_0 = 0$ in (4.2). If $\alpha_2 < 0$, we take $N = -\alpha_2$ and $N_0 = \sup_{B(x,5r)} \mathcal{O}_1$ in (4.2). Then, from (4.7) and (4.6),

$$\sup_{B(x_0,2r)} \left\{ (\vartheta_{\ell} - \alpha_1)^+ \right\} \leq C_5 \inf_{B(x_0,2r)} \{ \vartheta_{\ell} + \alpha_2 + 2r \}
\leq C_5 \left(\vartheta_{\ell}(x_0) + \sup_{B(x_0,4r)} \mathcal{O}_1 - \inf_{B(x_0,4r)} \mathcal{O}_1 - \inf_{B(x_0,4r)} \vartheta_{\ell} + 2r \right)
\leq 2C_5 \left(\sup_{B(x_0,4r)} \mathcal{O}_1 - \inf_{B(x_0,4r)} \mathcal{O}_1 + r \right),$$

 $\text{ where } C_5 := C_1^{\frac{1}{(p_{40r}^- - 1)^2}} \left[C_3(p_{40r}^- - 1) \right]^{-\frac{1}{p_{40r}^- - 1}}, \text{ and } C_i := C_i(\Lambda_1, \Lambda_2, p^+, \tilde{\mathbf{K}}_1, \mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3, \mathbf{K}_4, \mathbf{K}_{\text{II}}, \mathbf{K}_0, \mathbf{n}_2, n, s, \gamma, \mathbf{1}_{\text{I}}, \\ \inf_{B(x,5r)} \mathcal{O}_1, \sup_{B(x,5r)} \mathcal{O}_1, \mathfrak{T}_0, \hat{c}_1, \|\vartheta\|_{s, B(x_0, 40r), \mathbf{V}}, \|\vartheta\|_{s\gamma', B(x_0, 40r), \mathbf{V}}) \ > \ 0, \ i = 1, 3, \ C_1, C_3^{-1} \ > \ 1, \ s > p_{40r}^+ - p_{40r}^-, \ 1 < \gamma < \mathbf{1}_{\text{I}}. \ \text{Hence},$

$$\begin{split} \sup_{x \in B(x_0, 2r)} \left\{ \left(\vartheta_\ell(x) - \sup_{B(x_0, 4r)} \mathcal{O}_1 + \inf_{B(x_0, 4r)} \mathcal{O}_1 - \inf_{B(x_0, 4r)} \vartheta_\ell \right)^+ \right\} \\ \leq 2C_5 \left(\sup_{B(x_0, 4r)} \mathcal{O}_1 - \inf_{B(x_0, 4r)} \mathcal{O}_1 + r \right). \end{split}$$

So,

$$\sup_{B(x_0,2r)} \vartheta_{\ell} - \inf_{B(x_0,2r)} \vartheta_{\ell} \le 3C_5 \left(\sup_{B(x_0,4r)} \mathcal{O}_1 - \inf_{B(x_0,4r)} \mathcal{O}_1 + r \right). \tag{4.8}$$

Steep 2. Let $\eta \in C_0^{\infty}(B(x_0, 4r))$ with $0 < \eta \le 1$, $\eta = 1$ in $B(x_0, 2r)$, and $|\mathcal{D}\eta|_{F^*} \le \hat{c}_0\hat{c}_1(2r)^{-1}$, as stated in Lemma 3.2.

Define $p_{4r}^+ := \sup_{B(x_0, 4r)} p$. If $\zeta \in C_0^{\infty}(B(x_0, 2r))$ with $0 \le \zeta \le 1$, we have

$$\begin{split} &\int_{B(x,5r)} \mathcal{A}(\cdot,\boldsymbol{\nabla}\vartheta_{\ell}) \bullet \mathcal{D}\left(\eta^{p_{4r}^{+}} - \zeta\right) \, \mathrm{d}\mathbb{V} \\ &= \int_{B(x,5r)} \left[\mathcal{A}(\cdot,\boldsymbol{\nabla}\vartheta_{\ell}) - \mathcal{A}(\cdot,\boldsymbol{\nabla}\vartheta)\right] \bullet \mathcal{D}\left(\eta^{p_{4r}^{+}} - \zeta\right) \, \mathrm{d}\mathbb{V} + \int_{B(x,5r)} \mathcal{A}(\cdot,\boldsymbol{\nabla}\vartheta) \bullet \mathcal{D}\left(\eta^{p_{4r}^{+}} - \zeta\right) \, \mathrm{d}\mathbb{V} \\ &\geq -\Upsilon_{1}(x,5r), \end{split}$$

due to property (P_2) , the fact that ϑ is a supersolution of (1.1), and $\eta^{p_{4r}^+} - \zeta \ge 0$ in $B(x_0, 4r)$.

Using property (a_2) , we get:

$$\mu_{\ell}(B(x_{0},2r)) \leq p_{4r}^{+} \int_{B(x_{0},4r)} \eta^{p_{4r}^{+}-1} \mathcal{A}(\cdot, \nabla \vartheta_{\ell}) \bullet \mathcal{D}\eta \, dV + \Upsilon_{1}(x,5r)$$

$$\leq p_{4r}^{+} \Lambda_{2} \int_{B(x_{0},4r)} \eta^{p_{4r}^{+}-1} |\nabla \vartheta_{\ell}|_{F}^{p-1} |\mathcal{D}\eta|_{F^{*}} \, dV + \Upsilon_{1}(x,5r)$$

$$\leq 2p_{4r}^{+} \Lambda_{2} |||\nabla \vartheta_{\ell}|_{F}^{p-1} \eta^{p_{4r}^{+}-1}||_{p/(p-1),B(x_{0},4r),V}|||\mathcal{D}\eta|_{F^{*}}||_{p,B(x_{0},4r),V} + \Upsilon_{1}(x,5r)$$

$$\leq 2p_{4r}^{+} \Lambda_{2} \max \left\{ \left(\int_{B(x_{0},4r)} |\nabla \vartheta_{\ell}|_{F}^{p} \eta^{p_{4r}^{+}} \, dV \right)^{\frac{p_{4r}^{+}-1}{p_{4r}^{+}}}, \left(\int_{B(x_{0},4r)} |\nabla \vartheta_{\ell}|_{F}^{p} \eta^{p_{4r}^{+}} \, dV \right)^{\frac{p_{4r}^{-}-1}{p_{4r}^{-}}} \right\}$$

$$\cdot \max \left\{ \left(\int_{B(x_{0},4r)} |\mathcal{D}\eta|_{F^{*}}^{p} \, dV \right)^{\frac{1}{p_{4r}^{+}}}, \left(\int_{B(x_{0},4r)} |\mathcal{D}\eta|_{F^{*}}^{p} \, dV \right)^{\frac{1}{p_{4r}^{-}}} \right\} + \Upsilon_{1}(x,5r).$$

$$(4.10)$$

We take $N = \sup_{B(x_0,4r)} \vartheta_{\ell}$, h = 0, and R = 4r in (3.14). By (1.7), (4.8), (4.5), and (2.8), we obtain

$$\int_{B(x_{0},4r)} |\nabla \vartheta_{\ell}|_{F}^{p} \eta^{p_{4r}^{+}} \, dV \leq C_{6} \int_{B(x_{0},4r)} \left(\sup_{B(x_{0},4r)} \vartheta_{\ell} - \vartheta_{\ell} \right)^{p} |\mathcal{D}\eta|_{F^{*}}^{p} \, dV + C_{6} \int_{B(x_{0},4r)} \eta^{p_{4r}^{+}} \, dV \\
\leq C_{6} \int_{B(x_{0},4r)} \left(\sup_{B(x_{0},4r)} \vartheta_{\ell} - \inf_{B(x_{0},4r)} \vartheta_{\ell} \right)^{p} \left(\frac{\hat{c}_{0}\hat{c}_{1}}{2r} \right)^{p} \, dV + C_{6} K(4r)^{n_{1}} \\
\leq C_{6} \int_{B(x_{0},4r)} (3C_{5})^{p} \left(\sup_{B(x_{0},8r)} \mathcal{O}_{1} - \inf_{B(x_{0},8r)} \mathcal{O}_{1} + r \right)^{p} \left(\frac{\hat{c}_{0}\hat{c}_{1}}{2r} \right)^{p} \, dV + C_{6} K(4r)^{n_{1}} \\
\leq C_{6} \int_{B(x_{0},4r)} (3C_{5})^{p} \left(C_{4}r^{\alpha} + r \right)^{p} \left(\frac{\hat{c}_{0}\hat{c}_{1}}{2r} \right)^{p} \, dV + C_{6} K(4r)^{n_{1}} \\
\leq C_{7} C_{5}^{p_{4r}^{+}} r^{n_{1} - (1-\alpha)p_{4r}^{-}}, \tag{4.11}$$

where $C_6:=C_6(\Lambda_1,\Lambda_2,p^+)$ and $C_7:=C_7(\Lambda_1,\Lambda_2,p^+,n,\hat{c}_1,\mathtt{K},\mathtt{K}_4,\mathtt{n}_1,\alpha,\hat{\mathbf{s}})>0.$ From (4.10), (4.11), and (1.7), we conclude $\mu_\ell(B(x_0,2r))$

$$\leq 2p^{+}\Lambda_{2}(2^{-1}\hat{c}_{0}\hat{c}_{1}+1)^{p_{4r}^{+}/p_{4r}^{-}} \max \left\{ \left[C_{7}C_{5}^{p_{4r}^{+}}r^{\mathbf{n}_{1}-(1-\alpha)p_{4r}^{-}} \right]^{\frac{p_{4r}^{+}-1}{p_{4r}^{+}}}, \left[C_{7}C_{5}^{p_{4r}^{+}}r^{\mathbf{n}_{1}-(1-\alpha)p_{4r}^{-}} \right]^{\frac{p_{4r}^{-}-1}{p_{4r}^{-}}} \right\} \\ \cdot \max \left\{ \left[\mathbf{K}(4r)^{\mathbf{n}_{1}}\mathbf{K}_{4}r^{-p_{4r}^{-}} \right]^{\frac{1}{p_{4r}^{+}}}, \left[\mathbf{K}(4r)^{\mathbf{n}_{1}}\mathbf{K}_{4}r^{-p_{4r}^{-}} \right]^{\frac{1}{p_{4r}^{-}}} \right\} + \Upsilon_{1}(x,5r),$$

since (2.8) holds. Additionally, from (1.8), we have

$$\leq C_8 C_1^{\frac{\tilde{\mathsf{K}}_1}{p_{40r}^- - 1}} \left[C_3(p_{40r}^- - 1) \right]^{-\tilde{\mathsf{K}}_1} r^{\mathsf{n}_1 - p_{4r}^- + \alpha(p_{4r}^- - 1) - \frac{p_{4r}^+ - p_{4r}^-}{p_{4r}^+} \max \left\{ \frac{\mathsf{n}_1}{p_{4r}^-}, 1 \right\} + \Upsilon_1(x, 5r) \\
\leq C_9 C_1^{\frac{\tilde{\mathsf{K}}_1}{p_{x,41r}^- - 1}} \left[C_3(p_{x,41r}^- - 1) \right]^{-\tilde{\mathsf{K}}_1} r^{\mathsf{n}_1 - p(x) + \alpha(p(x) - 1)} + \Upsilon_1(x, 5r),$$

where $p_{x,41r}^- := \inf_{B(x,41r)} p$ and $C_i := C_i(\Lambda_1, \Lambda_2, p^+, n, \hat{c}_1, \mathsf{K}, \mathsf{K}_0, \mathsf{K}_3, \mathsf{K}_4, \mathsf{n}_1, \mathsf{n}_2, \alpha, \hat{\mathsf{s}}) > 0$ for i = 8, 9. Therefore, from (4.4), the proof of the theorem is complete.

For every open set $V \subset\subset \Omega$, we write

$$\mu(V) := \sup \left\{ \int_V \mathcal{A}(\cdot, \boldsymbol{\nabla} \boldsymbol{\vartheta}) \bullet \mathcal{D} \boldsymbol{\zeta} \, \mathrm{d} \mathbf{V} \, | \, 0 \leq \boldsymbol{\zeta} \leq 1 \text{ and } \boldsymbol{\zeta} \in C_0^\infty(V) \right\}.$$

Corollary 4.4. Let K be a compact subset of Ω , and define $K_{\delta} := \{x \in \Omega \mid d(x,K) < \delta\}$ such that $\overline{K_{\delta}} \subset \Omega$. Suppose that $\vartheta \in \mathcal{K}_{\mathfrak{O}_1}(\Omega)$ is a solution to the obstacle problem (2.14). Additionally, assume that the obstacle $\mathfrak{O}_1 \in C(\Omega)$ satisfies (4.3).

For every $x \in K$ and $0 < r < \frac{1}{41} \min\{R_*, \delta\}$, the following inequality holds:

$$\mu(B(x,r)) \le Cr^{\mathsf{n}_1 - p(x) + \alpha(p(x) - 1)},$$

 $\textit{where } B(x,41r) \subset \subset \psi(\tilde{U}) \textit{ and } C := C(\Lambda_1,\Lambda_2,p_{K_\delta}^-,p_{K_\delta}^+,\tilde{\mathtt{K}}_1,\mathtt{K}_1,\mathtt{K}_2,\mathtt{K}_3,\mathtt{K}_4,\mathtt{K}_{\mathrm{II}},\mathtt{K}_0,\mathtt{K},n,s,\gamma,1_{\mathrm{I}},\mathtt{n}_1,\mathtt{n}_2,\alpha,\hat{\mathtt{s}},\hat{c}_1,\|\vartheta\|_{s,B(x,41r),\mathtt{V}}, \\ \|\vartheta\|_{s\gamma',B(x,41r),\mathtt{V}}, \inf_{B(x,5r)} \mathfrak{O}_1, \sup_{B(x,5r)} \mathfrak{O}_1, 0 > 0. \textit{ Here, } s > p_{41r}^+ - p_{41r}^- \ 1 < \gamma < 1_{\mathrm{I}}, p_{K_\delta}^- = \inf_{K_\delta} p \textit{ and } p_{K_\delta}^+ = \sup_{K_\delta} p. \\ \|\varphi\|_{s\gamma',B(x,41r),\mathtt{V}}, \lim_{K_\delta} \|\varphi\|_{s,B(x,5r)} + \|\varphi\|_{s,B(x,5r)$

4.2. **Proof of Theorem 1.1.** Recall that:

Theorem 1.1. Let (Ω, F, V) be a reversible Finsler manifold, where V and p satisfy conditions (1.5)–(1.12). Let $S \subset \Omega$ be a closed set. Assume that $u \in C(\Omega)$ is a solution of (1.1) in $\Omega \setminus S$, and there exists $\alpha \in (0,1]$ such that for any $x \in S$ and $y \in \Omega$:

$$|u(x) - u(y)| \le Cd^{\alpha}(x, y).$$

If for each compact subset K of S, the $n_1 - p_K^+ + \alpha(p_K^+ - 1)$ -Hausdorff measure of K is zero, then u is a solution of (1.1) in Ω .

Proof. Steep 1. Fix open sets Ω_0 , $\hat{\Omega} \subset \Omega$ such that $\Omega_0 \subset\subset \hat{\Omega} \subset\subset \Omega$. From Propositions 2.12 and 2.16, there exists a unique continuous solution $u_1 \in \mathcal{K}_u(\hat{\Omega})$ to the obstacle problem (2.14).

For an open set $V \subset \Omega_0$, define

$$\mu_1(V) := \sup \left\{ \int_V \mathcal{A}(\cdot, \boldsymbol{\nabla} u_1) \bullet \mathcal{D}\zeta \, \mathrm{dV} \, | \, 0 \leq \zeta \leq 1 \text{ and } \zeta \in C_0^\infty(V) \right\}.$$

Also, for an arbitrary set $E \subset \Omega_0$, define

$$\mu_1(E) := \inf \{ \mu_1(V) \mid V \subset \Omega_0 \text{ is an open set such that } E \subset V \}.$$

Let K be a compact set in $S \cap \Omega_0$. From Corollary 4.4, for any $x \in K$ and $0 < r < \frac{1}{41} \min\{R_*, \delta\}$, we have

$$\mu(B(x,r)) \le Cr^{\mathsf{n}_1 - p_K^+ + \alpha(p_K^+ - 1)}.$$

Since the $n_1 - p_K^+ + \alpha(p_K^+ - 1)$ -Hausdorff measure of K is zero, for any $\epsilon > 0$, there exists $\delta_0 > 0$ such that for any $0 < \delta_1 < \delta_0$,

$$H^s_{\delta_1}(K) < \epsilon,$$

where $s = \mathbf{n}_1 - p_K^+ + \alpha(p_K^+ - 1)$.

Taking δ_1 small enough, there exists a family of sets $C_j^{\delta_1}$ such that $K \subset \bigcup_{j=1}^{\infty} C_j^{\delta_1}$, $\operatorname{diam}(C_j^{\delta_1}) < \delta_1$, and

$$H^s_{\delta_1}(K) \le \sum_{i=1}^{\infty} \beta(s) \left\lceil \frac{\operatorname{diam}(C_j^{\delta_1})}{2} \right\rceil^s < \epsilon.$$

Hence,

$$\sum_{j=1}^{\infty} \left[\operatorname{diam}(C_j^{\delta_1}) \right]^s < \left(\frac{2^s}{\beta(s)} \right) \epsilon.$$

Take a family of balls $\{B(x_j,r_j)\}$ such that $C_j^{\delta_1}\subset B(x_j,r_j)$, where $x_j\in K$ and $r_j=\mathrm{diam}(C_j^{\delta_1})$. Then

$$\mu_1(K) \le \sum_{i=1}^{\infty} \mu_1(C_j^{\delta_1}) \le \sum_{j=1}^{\infty} \mu_1(B(x_j, r_j)) \le C \sum_{j=1}^{\infty} r_j^{\mathsf{n}_1 - p_K^+ + \alpha(p_K^+ - 1)} \le C \left(\frac{2^s}{\beta(s)}\right) \epsilon,$$

where the constant C is given in Corollary 4.4.

Since ϵ is arbitrary, we conclude that $\mu_1(K) = 0$. Hence, $\mu_1(S \cap \Omega_0) = 0$.

Steep 2. Next, we prove that $\mu_1(\Omega_0 \backslash S) = 0$.

Let $\epsilon > 0$ and $\zeta \in C_0^{\infty}(\Omega_0 \backslash \mathcal{S})$ with $\zeta \geq 0$. Define $\zeta_{\epsilon} = \min\{\zeta, \frac{u_1 - u}{\epsilon}\}$, so that $\zeta_{\epsilon} \in W_0^{1, p(x)}(\Omega_0 \backslash \mathcal{S}; V)$.

Define $D_1 := \{x \in \Omega_0 \mid u_1(x) > u(x)\}$ and $D_2 := \{x \in \Omega_0 \mid u_1(x) = u(x)\}$. Since the obstacle u is continuous, by Proposition 2.16, we know that u_1 is a solution of (1.1) in D_1 . Then,

$$\int_{\Omega_0 \setminus \mathcal{S}} \mathcal{A}(\cdot, \nabla u_1) \bullet \mathcal{D}\zeta_{\epsilon} \, dV$$

$$= \int_{D_1} \mathcal{A}(\cdot, \nabla u_1) \bullet \mathcal{D}\zeta_{\epsilon} \, dV + \int_{D_2} \mathcal{A}(\cdot, \nabla u_1) \bullet \mathcal{D}\zeta_{\epsilon} \, dV = 0.$$
(4.12)

Additionally, since u is a solution of (1.1) in $\Omega_0 \setminus \mathcal{S}$, we have

$$\int_{\Omega_0 \setminus \mathcal{S}} \mathcal{A}(\cdot, \nabla u) \bullet \mathcal{D}\zeta_{\epsilon} \, dV = 0. \tag{4.13}$$

By (4.12) and (4.13),

$$\int_{\Omega_0 \setminus \mathcal{S}} \left[\mathcal{A}(\cdot, \nabla u_1) - \mathcal{A}(\cdot, \nabla u) \right] \bullet \mathcal{D}\zeta_{\epsilon} \, dV = 0.$$
 (4.14)

Define $(\Omega_0 \backslash \mathcal{S})_1 = \{x \in \Omega_0 \backslash \mathcal{S} \mid \zeta(x) \leq (u_1(x) - u(x))/\epsilon\}$ and $(\Omega_0 \backslash \mathcal{S})_2 = \{x \in \Omega_0 \backslash \mathcal{S} \mid \zeta(x) > (u_1(x) - u(x))/\epsilon\}$. From (4.14) and (a_3) , we have

$$\int_{(\Omega_0 \backslash \mathcal{S})_1} \left[\mathcal{A}(\cdot, \boldsymbol{\nabla} u_1) - \mathcal{A}(\cdot, \boldsymbol{\nabla} u) \right] \bullet \mathcal{D}\zeta \, \mathrm{dV} = -\frac{1}{\epsilon} \int_{(\Omega_0 \backslash \mathcal{S})_2} \left[\mathcal{A}(\cdot, \boldsymbol{\nabla} u_1) - \mathcal{A}(\cdot, \boldsymbol{\nabla} u) \right] \bullet \mathcal{D}(u_1 - u) \, \mathrm{dV} \leq 0.$$

Taking the limit as $\epsilon \to 0$, we obtain

$$\int_{(\Omega_0 \backslash \mathcal{S})_1} \left[\mathcal{A}(\cdot, \boldsymbol{\nabla} u_1) - \mathcal{A}(\cdot, \boldsymbol{\nabla} u) \right] \bullet \mathcal{D} \zeta \, \mathrm{dV} \leq 0.$$

Further, since

$$\int_{\Omega_0 \setminus \mathcal{S}} \mathcal{A}(\cdot, \nabla u) \bullet \mathcal{D}\zeta \, dV = 0,$$

we conclude

$$\int_{\Omega_0 \backslash \mathcal{S}} \mathcal{A}(\cdot, \boldsymbol{\nabla} u_1) \bullet \mathcal{D} \zeta \, \mathrm{dV} \leq 0,$$

it implies that $\mu_1(\Omega_0 \backslash S) \leq 0$, so $\mu_1(\Omega_0 \backslash S) = 0$.

From the above, we conclude that $\mu_1(\Omega_0)=0$. That is, for any $\zeta\in W^{1,p(x)}_0(\Omega_0;V)$, we have

$$\int_{\Omega_0} \mathcal{A}(\cdot, \nabla u_1) \bullet \mathcal{D}\zeta \, dV = 0.$$

Steep 3. Let u_2 be a solution to the obstacle problem (2.15) in $\mathcal{K}_{-u}(\hat{\Omega})$. Similarly to u_1 , we find that for all $\zeta \in C_0^{\infty}(\Omega_0)$,

$$\int_{\Omega_0} \left[-\mathcal{A}(\cdot, -\nabla u_2) \right] \bullet \mathcal{D}\zeta \, dV = 0.$$

Next, we will prove that $u_1=-u_2=u$ a.e. in Ω_0 . Indeed, since $-u_2\leq u\leq u_1$ a.e. in Ω_0 , $u_1-u\in W_0^{1,p(x)}(\Omega_0; \mathtt{V})$, and $u_2+u\in W_0^{1,p(x)}(\Omega_0; \mathtt{V})$, it follows that $u_1+u_2\in W_0^{1,p(x)}(\Omega_0; \mathtt{V})$. Therefore,

$$\int_{\Omega_0} \mathcal{A}(\cdot, \nabla u_1) \bullet (\mathcal{D}u_1 + \mathcal{D}u_2) \ dV = 0$$
$$\int_{\Omega_0} \mathcal{A}(\cdot, -\nabla u_2) \bullet (\mathcal{D}u_1 + \mathcal{D}u_2) \ dV = 0.$$

Thus,

$$\int_{\Omega_0} \left[\mathcal{A}(\cdot, \nabla u_1) - \mathcal{A}(\cdot, -\nabla u_2) \right] \bullet (\mathcal{D}u_1 + \mathcal{D}u_2) \ \mathrm{dV} = 0.$$

Therefore, by (a_3) , $\mathcal{D}(u_1+u_2)=0$ a.e. in Ω_0 . This implies $u_1+u_2=0$ a.e. in Ω_0 , so $u_1=-u_2=u$ a.e. in Ω_0 . Therefore, u is the solution to (1.1) in Ω_0 , meaning that for any $\zeta\in W_0^{1,p(x)}(\Omega_0; V)$, we have

$$\int_{\Omega_0} \mathcal{A}(\cdot, \nabla u) \bullet \mathcal{D}\zeta \, dV = 0.$$

Thus, u is a solution of (1.1) in Ω , and the closed set S is removable.

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