Various form closures associated with a fixed non-semibounded self-adjoint operator

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Abstract. If T is a semibounded self-adjoint operator in a Hilbert space $(H, (\cdot, \cdot))$ then the closure of the sesquilinear form $(T \cdot, \cdot)$ is a unique Hilbert space completion. In the non-semibounded case a closure is a Kreın space completion and generally, it is not unique. Here, all such closures are studied. A one-to-one correspondence between all closed symmetric forms (with "gap point" 0) and all J-non-negative, J-self-adjoint and boundedly invertible Kreın space operators is observed. Their eigenspectral functions are investigated, in particular near the critical point infinity. An example for infinitely many closures of a fixed form $(T \cdot, \cdot)$ is discussed in detail using a non-semibounded self-adjoint multiplication operator T in a model Hilbert space. These observations indicate that closed symmetric forms may carry more information than self-adjoint Hilbert space operators.

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1. Introduction

It is a classical result that semibounded self-adjoint operators in Hilbert spaces and closed semibounded symmetric sesquilinear forms are in one-to-one correspondence and hence, include similar information. In particular, if T is a positive and boundedly invertible operator in the Hilbert space $(H, (\cdot, \cdot))$ then the associated closed form $\mathfrak{t}[\cdot, \cdot]$ is obtained as the inner product of the Hilbert space completion (or *form closure*) of the positive definite inner product $(T\cdot, \cdot)$ defined on the domain dom T. Note that this Hilbert space is continuously embedded in $(H, (\cdot, \cdot))$. Conversely, T is recovered from $\mathfrak{t}[\cdot, \cdot]$ by Kato's First representation Theorem (see e.g. [15, VI-§ 2.1]).

A first approach to non-semibounded sesqulilinear forms was initiated by McIntosh (see e.g. [17, 18, 13]). However, here we follow a different approach via Kreı̆n space methods outlined e.g. in [8, 10, 11, 12]. To this end consider a generally non-semibounded self-adjoint operator T in $(H, (\cdot, \cdot))$ which is boundedly invertible. Then, the inner product $(T \cdot, \cdot)$ on dom T allows a Kreı̆n space completion, i.e. there is a Kreı̆n space $(\text{dom } \mathfrak{t}, \mathfrak{t}[\cdot, \cdot])$ continuously embedded in $(H, (\cdot, \cdot))$ and including dom T as a dense subspace such that $\mathfrak{t}[\cdot, \cdot]$ coincides with $(T \cdot, \cdot)$ on dom T (cf. [10, Section 5]). In the terminology of [10] this means that $\mathfrak{t}[\cdot, \cdot]$ is a closure of $(T \cdot, \cdot)$ in $(H, (\cdot, \cdot))$ and obviously, this is a generalization of the positive setting from above. Similarly, the terminology of a closed symmetric form is generalized to the non-semibounded situation. Again, T can be recovered from $\mathfrak{t}[\cdot, \cdot]$ by a generalization of Kato's First Representation Theorem (cf. [8, Theorem 1] or [10, Theorem 3.3]).

However, a Kreı̆n space completion is generally not unique. In [6] Ćurgus and Langer studied Kreı̆n space completions in detail for a quite general setting. In the present paper we investigate such completions from the point of view of non-semibounded form closures. In particular, we study all form closures of $(T\cdot,\cdot)$ for a fixed self-adjoint operator T given as above.

So far, in the previous papers like [8, 10, 11, 12] the main focus was on so-called regular closed forms $\mathfrak{t}[\cdot,\cdot]$ which are defined on $\mathrm{dom}\,|T|^{\frac{1}{2}}$ and hence, allow a generalization of the Second Representation Theorem (cf. [8, Theorem 3] or [10, Theorem 4.2]). In particular, in [10, Theorem 5.2] the "semibounded result" from the beginning was generalized: There is a one-to-one correspondence between all regular closed forms and all self-adjoint operators with a gap in the real spectrum. In [10] the above restriction to $0 \in \rho(T)$ is shifted to this gap and this is reflected by the introduction of so-called "gap points" of a closed form.

In some sense this "one-to-one result" is not complete since we know that there are also non-regular closed forms (see e.g. [11, Theorem 6.9]). In the present paper our main interest is in non-regularity where the form $\mathfrak{t}[\cdot,\cdot]$ has no representation by means of $|T|^{\frac{1}{2}}$. A different one-to-one correspondence is presented allowing all (not only regular) closed forms:

To this end we start with an arbitrary J-non-negative, J-self-adjoint and boundedly inverible operator A in a $Kre\Billing n$ space $(K, [\cdot, \cdot])$ and construct a $Hilbert\ space\ (K_-, \{\cdot, \cdot\}_-)$ such that $K \subset K_-$ and the inclusion is dense and continuous. Then, $\mathfrak{t}[\cdot, \cdot] := [\cdot, \cdot]$ is a closed symmetric form in $(K_-, \{\cdot, \cdot\}_-)$ (with gap point 0) and A appears as the range restriction of the representing self-adjoint operator in $(K_-, \{\cdot, \cdot\}_-)$ according to the generalized First Representation Theorem. Now, the mapping $A \longrightarrow \mathfrak{t}[\cdot, \cdot]$ defines a one-to-one correspondence between all J-non-negative, J-self-adjoint and boundedly inverible Kre\Billing n space operators and all closed forms with a gap point 0 (Theorem 4.1).

In the next step we consider the restriction of the inverse of this mapping to the form closures of (T,\cdot) with a fixed self-adjoint and boundedly

invertible operator T in a Hilbert space $(H, (\cdot, \cdot))$ (Theorem 5.2). For the associated J-non-negative, J-self-adjoint and boundedly inverible operators infinity is the only possible critical point in the sense of Langer's theory of definitizable operators from [16]. There is precisely one regular closure $\mathfrak{t}_0[\cdot,\cdot]$ and this is characterized by the regularity of the critical point infinity of the associated Krein space operator A_0 (Theorem 5.4). This means that the eigenspectral function of A_0 according to [16, Theorem II.3.1] is bounded near infinity. Since for all other closures and operators we have nonregularity, $\mathfrak{t}_0[\cdot,\cdot]$ and A_0 will be called regularizations of the other closures and operators, respectively. From the Krein space point of view this regularization is a "small" modification of the operator and of the Krein space as well. Although this modification makes a big difference for the norm, the eigenspectral functions themselves do not differ essentially. More precisely, each eigenspectral function appears as the restriction of the spectral measure of T to the form domain (Theorem 5.6). Furthermore, a consequence of [6,Theorem 2.7 is mentioned: The form closure of (T,\cdot) is unique if and only if T is semibounded (Theorem 5.9).

Some of the above results do only make certain ideas more precise which had already been in the background of previous papers (see again [8, 10, 11, 12]). However, when we leave this abstract level and turn to simple example operators the difficulties increase in describing different form closures explicitly. Also from the Kreın space point of view so far there are not many examples of different completions known (see e.g. [14, 6]). Here, we first recall an example from [9] where a non-regular closed form associated with an indefinite Sturm-Liouville operator is constructed. Unfortunately, we cannot describe the regularization explicitly in this case.

In [6, Theorem 5.2, proof part II] infinitely many Kreı̆n space completions are constructed on a certain abstract level. Here, using some of these ideas we explicitly construct a family of infinitely many form closures or Kreı̆n space completions associated with the self-adjoint multiplication operator (Tf)(x) := xf(x) in a model Hilbert space $L^2_{r_-}(\mathbb{R})$ (Theorem 6.9, Corollary 6.10). The inner product in this space is given by

$$(f,g)_{r_{-}} := \int_{-\infty}^{\infty} f\overline{g} r_{-} dx, \quad (f,g \in L_{r_{-}}^{2}(\mathbb{R}))$$

where the weight function $r_{-} \in L^{1}_{loc}(\mathbb{R})$ is non-negative, vanishes around 0 and satisfies some additional conditions. With the function $r(x) := xr_{-}(x)$ the regular closure of $(T_{\cdot}, \cdot)_{r_{-}}$ is given by the Kreın space inner product

$$\mathfrak{t}_0[f,g] := [f,g]_r := \int_{-\infty}^{\infty} f\overline{g} \, r \, dx, \quad (f,g \in \mathrm{dom}\, \mathfrak{t}_0 := L^2_r(\mathbb{R})).$$

(Note that r changes its sign.) Now, let $\alpha \in (0,2]$ and consider the functions

$$\eta_\alpha(x):=(\sqrt{|x|^\alpha+1}-\sqrt{|x|^\alpha})|r(x)|,\quad \omega_\alpha(x):=\sqrt{|x|^\alpha}\,|r(x)|\quad (x\in\mathbb{R}).$$

If we denote the even part of a function f by f_e and the odd part by f_o then, for each $\alpha \in (0, 2]$ an additional non-regular closure can be defined on

$$\operatorname{dom}\mathfrak{t}_{\alpha}:=\{f\in L^2_{r_{-}}(\mathbb{R})\,|\,f_e\in L^2_{\eta_{\alpha}}(\mathbb{R}),\,f_o\in L^2_{\omega_{\alpha}}(\mathbb{R})\}$$

by means of the Kreĭn space inner product

$$\mathfrak{t}_{\alpha}[f,g] := \lim_{k \to \infty} \int_{-k}^{k} f\overline{g} \, r \, dx \quad (f,g \in \mathrm{dom}\,\mathfrak{t}_{\alpha})$$

(Proposition 6.8). It is observed that this limit always exists. Furthermore, for all $\alpha \in [0, 2]$ the associated eigenspectral functions are studied near infinity.

Finally, note that here we have an explicit example showing that each of the infinitely many form closures carries more information $(r \text{ and } \alpha)$ than the original self-adjoint operator (only r). This seems to imply that in the non-semibounded case the concept of closed symmetric sesquilinear forms is more suitable than the concept of self-adjoint Hilbert space operators. An interpretation in physics would be interesting.

2. Preliminaries

For further use we recall some basic definitions and facts from known theories.

2.1. J-non-negative and boundedly invertible operators in Kreĭn spaces

An indefinite inner product space $(K, [\cdot, \cdot])$ is called a $Kreĭn\ space$ if it allows a so-called $fundamental\ decomposition\ K = K^+ \oplus K^-$ with a direct and orthogonal (with respect to $[\cdot, \cdot]$) sum of a Hilbert space $(K^+, [\cdot, \cdot])$ and an anti Hilbert space $(K^-, [\cdot, \cdot])$. If P^\pm denotes the orthogonal projection on K^\pm then $J := P^+ - P^-$ is called a $fundamental\ symmetry$ and $\{\cdot, \cdot\} := [J\cdot, \cdot]$ defines a positive definite inner product such that $(K, \{\cdot, \cdot\})$ is a Hilbert space. The induced topology also serves as the topology of the Kreĭn space. A densely defined operator A in the Kreĭn space $(K, [\cdot, \cdot])$ is called J-self-adjoint or J-non-negative if JA has the corresponding property in the Hilbert space $(K, \{\cdot, \cdot\})$. Note that these definitions do not depend on the choice of J. Furthermore, a $Kreĭn\ space\ (K, [\cdot, \cdot])$ is called a $Kreĭn\ space\ completion$ of an inner product space $(\widetilde{K}, [\cdot, \cdot])$ if \widetilde{K} is a dense subspace of K and $[\cdot, \cdot]$ and $[\cdot, \cdot]$ coincide on \widetilde{K} . For more details on Kreĭn spaces we refer to [1].

According to [16] a J-self-adjoint operator A is called definitizable if the resolvent set $\rho(A)$ is not empty and there is a real so-called definitizing polynomial p such that p(A) is J-non-negative. The zeroes of p which also belong to the real part of the spectrum $\sigma(A) \cap \mathbb{R}$ are called critical points of A. Additionally, ∞ is called a critical point of A if p has an odd degree (i.e. p has a sign change at ∞) and $\sigma(A) \cap \mathbb{R}$ is neither bounded from above nor from below.

In the following we restrict ourselves to the case of a J-self-adjoint operator A such that $A - \lambda$ is J-non-negative and boundedly invertible for some $\lambda \in \mathbb{R}$. Then clearly, A is definitizable with definitizing polynomial $p(t) = t - \lambda$. Furthermore, ∞ is the only possible critical point of A.

Now, we recall Langer's result from [16, Theorem II.3.1] on eigenspectral functions for the case of a J-self-adjoint, J-non-negative and boundedly invertible operator A in a Kreı̆n space $(K, [\cdot, \cdot])$ (i.e. $\lambda = 0$). To this end let Σ denote the semiring of all bounded intervalls and its complements in \mathbb{R} . A mapping E from Σ to L(K) (the space of all bounded operators in $(K, [\cdot, \cdot])$) is constructed which is called the eigenspectral function of A.

For a bounded interval $\emptyset \neq \Delta \in \Sigma$ and $f \in K$ the limit

$$E(\Delta)f := \lim_{\epsilon \searrow 0} \lim_{\delta \searrow 0} -\frac{1}{2\pi i} \int_{C_{\Delta}^{\delta}} (A - \lambda)^{-1} f \, d\lambda \tag{2.1}$$

exists in $(K, [\cdot, \cdot])$ where $C_{\Delta_{\epsilon}}^{\delta}$ is defined for $\delta, \epsilon \in (0, 1)$ in the following way: For $\Delta = [\alpha, \beta]$ with $-\infty < \alpha < \beta < \infty$ consider the positive oriented curve C_{Δ} in $\mathbb C$ consisting of the line segments which connect the points

$$\beta + i$$
, $\alpha + i$, $\alpha - i$, $\beta - i$, $\beta + i$.

In order to define C_{Δ}^{δ} we take off from C_{Δ} the segments between

$$\alpha + i\delta$$
, $\alpha - i\delta$, and $\beta - i\delta$, $\beta + i\delta$.

For an arbitrary bounded interval $\emptyset \neq \Delta \in \Sigma$ with endpoints $\alpha \leq \beta$ put

$$\Delta_{\epsilon} := [\alpha \mp \epsilon, \beta \pm \epsilon] \quad (:= \emptyset \text{ if } \alpha \mp \epsilon > \beta \pm \epsilon)$$

where the upper (lower) sign must be taken whenever the corresponding endpoint is (not, respectively) a part of Δ . Furthermore, put $E(\emptyset) := 0$ and $E(\Delta) := I_K - E(\mathbb{R} \setminus \Delta)$ for an unbounded $\Delta \in \Sigma$ where I_K denotes the identity on K. Then, the following Theorem is obtained from [16, Theorem II.3.1] (see also [7, Appendix B]).

Theorem 2.1 (Langer). Let A be a J-self-adjoint, J-non-negative and boundedly invertible operator in the Kreĭn space $(K, [\cdot, \cdot])$ and let $E(\Delta)$ be given as constructed above for $\Delta \in \Sigma$. Then, E defines a mapping from Σ to L(K) with the following properties $(\Delta, \Delta' \in \Sigma)$:

$$\begin{split} E(\Delta) \ \ is \ J\text{-self-adjoint}, \\ E(\Delta)E(\Delta') &= E(\Delta \cap \Delta'), \\ E(\Delta \cup \Delta') &= E(\Delta) + E(\Delta') \ \ if \ \Delta \cup \Delta' \in \Sigma, \ \Delta \cap \Delta' = \emptyset, \\ E(\mathbb{R}) &= I_K, \ E(\emptyset) = 0 \\ E(\Delta) \ \ is \ J\text{-non-negative}, \ \ if \ \overline{\Delta} \subset (0,\infty), \\ E(\Delta) \ \ is \ J\text{-non-positive}, \ \ if \ \overline{\Delta} \subset (-\infty,0), \\ if \ \Delta \ \ is \ bounded \ \ then \\ E(\Delta)K \subset \operatorname{dom} A, \quad AE(\Delta)f = E(\Delta)Af \quad (f \in \operatorname{dom} A) \\ \ \ \ and \ \ the \ \ restriction \ A|_{E(\Delta)K} \ \ is \ bounded, \\ \sigma(A|_{E(\Delta)K}) \subset \overline{\Delta} \end{split}$$

where σ denotes the spectrum.

According to [16, Section II.5] ∞ is called a *singular critical point* of A if for some $\varepsilon > 0$ and some $f \in K$ one of the limits

$$\lim_{\lambda \to \infty} E[\varepsilon, \lambda] f \quad \text{and} \quad \lim_{\lambda \to \infty} E[-\lambda, -\varepsilon] f \tag{2.2}$$

does not exist with convergence in $(K, [\cdot, \cdot])$. By [16, Proposition II.5.6] this is equivalent to the property that the operator norms $||E[\varepsilon, \lambda]||$ or $||E[-\lambda, -\varepsilon]||$ are unbounded for $\lambda \to \infty$.

2.2. Closed symmetric sesquilinear forms

Let $\mathfrak{t}[\cdot,\cdot]$ be a densely defined symmetric sesquilinear form (short only *form*) in a Hilbert space $(H,(\cdot,\cdot))$ with domain dom \mathfrak{t} . Then, $\mathfrak{t}[\cdot,\cdot]$ is semibounded from below if for some $\lambda \in \mathbb{R}$ the form

$$\mathfrak{t}_{\lambda}[\cdot,\cdot] := \mathfrak{t}[\cdot,\cdot] - \lambda(\cdot,\cdot) \tag{2.3}$$

is non-negative, i.e. $\mathfrak{t}_{\lambda}[f,f] \geq 0$ for all $f \in \text{dom }\mathfrak{t}$. Furthermore, according to [15, Theorem VI 1.11] the form $\mathfrak{t}[\cdot,\cdot]$ is closed if and only if $(\text{dom }\mathfrak{t},\,\mathfrak{t}_{\lambda}[\cdot,\cdot])$ is a Hilbert space which is continuously embedded in $(H,\,(\cdot,\cdot))$ for some $\lambda \in \mathbb{R}$. The setting is similar if $\mathfrak{t}[\cdot,\cdot]$ is semibounded from above, i.e. $-\mathfrak{t}[\cdot,\cdot]$ is semibounded from below.

However, in the following we do not assume that $\mathfrak{t}[\cdot,\cdot]$ is semibounded (from below or above). Using (2.3), according to [10] the form $\mathfrak{t}[\cdot,\cdot]$ is called closed if $(\operatorname{dom}\mathfrak{t},\mathfrak{t}_{\lambda}[\cdot,\cdot])$ is a Kreın space which is continuously embedded in $(H,(\cdot,\cdot))$ for some $\lambda\in\mathbb{R}$ (called gap point of $\mathfrak{t}[\cdot,\cdot]$). A closed extension $\mathfrak{t}[\cdot,\cdot]$ of a form $\mathfrak{t}[\cdot,\cdot]$ is called a closure of $\mathfrak{t}[\cdot,\cdot]$ if dom \mathfrak{t} is dense in the Kreın space $(\operatorname{dom}\mathfrak{t},\mathfrak{t}_{\lambda}[\cdot,\cdot])$ for some gap point λ . These definitions do not depend on the choice of the gap point (cf. [10, Lemma 3.1]) and are obviously generalizations from the semibounded case. We recall the following generalizations of Kato's Representation Theorems [15, Theorem VI-2.1, Theorem VI-2.23] according to [10, Theorem 3.3, Theorem 4.2].

Theorem 2.2 (First Representation Theorem). Let $\mathfrak{t}[\cdot,\cdot]$ be a closed symmetric sesquilinear form in the Hilbert space $(H,(\cdot,\cdot))$. Then there exists a unique self-adjoint operator $T_{\mathfrak{t}}$ in $(H,(\cdot,\cdot))$ such that $\operatorname{dom} T_{\mathfrak{t}} \subset \operatorname{dom} \mathfrak{t}$ and

$$\mathfrak{t}[u,v] = (T_{\mathfrak{t}}u,v), \quad u \in \operatorname{dom} T_{\mathfrak{t}}, \quad v \in \operatorname{dom} \mathfrak{t}.$$

All gap points λ of $\mathfrak{t}[\cdot,\cdot]$ belong to the resolvent set of $T_\mathfrak{t}$ and $\operatorname{dom} T_\mathfrak{t}$ is dense in the Kreĭn space $(\operatorname{dom}\mathfrak{t},\mathfrak{t}_\lambda[\cdot,\cdot])$.

Associated with a closed form $\mathfrak{t}[\cdot,\cdot]$ is also the range restriction $A_{\mathfrak{t}}$

$$\operatorname{dom} A_{\mathfrak{t}} := T_{\mathfrak{t}}^{-1}(\operatorname{dom} \mathfrak{t}), \qquad A_{\mathfrak{t}} f := T_{\mathfrak{t}} f \quad (f \in \operatorname{dom} A_{\mathfrak{t}}) \tag{2.4}$$

of $T_{\mathfrak{t}}$ to dom \mathfrak{t} . If $\lambda \in \mathbb{R}$ is a gap point of $\mathfrak{t}[\cdot,\cdot]$ then by [10, Lemma 4.1] $A_{\mathfrak{t}}$ is J-self-adjoint and $A_{\mathfrak{t}} - \lambda$ is J-non-negative and boundedly invertible in the Kreın space (dom $\mathfrak{t}, \mathfrak{t}_{\lambda}[\cdot,\cdot]$). Consequently, $A_{\mathfrak{t}}$ is definitizable and ∞ is the only possible critical point of $A_{\mathfrak{t}}$.

Theorem 2.3 (Second Representation Theorem). Let $\mathfrak{t}[\cdot,\cdot]$ be a closed symmetric sesquilinear form in the Hilbert space $(H,(\cdot,\cdot))$ and let $T_\mathfrak{t}$ and $A_\mathfrak{t}$ be the associated operators. Then

$$\operatorname{dom} \mathfrak{t} = \operatorname{dom} |T_{\mathfrak{t}}|^{\frac{1}{2}} \tag{2.5}$$

if and only if ∞ is not a singular critical point of A_t . In this case the topology of the Kreĭn space (dom t, $t_{\lambda}[\cdot,\cdot]$) is induced by the inner product $(|T_t - \lambda|^{\frac{1}{2}} \cdot, |T_t - \lambda|^{\frac{1}{2}} \cdot)$ for each gap point $\lambda \in \mathbb{R}$

According to [10] a closed symmetric form $\mathfrak{t}[\cdot,\cdot]$ is said to be *regular* if (2.5) is satisfied. In this case also a representation of $\mathfrak{t}[\cdot,\cdot]$ by means of $|T_{\mathfrak{t}}|^{\frac{1}{2}}$ in analogy to the classical Second Representation Theorem [15, Theorem VI-2.23] can be found in [10, Theorem 4.2]. The following result from [10, Theorem 5.2] was already mentioned in the Introduction:

Theorem 2.4. The mapping $\mathfrak{t}[\cdot,\cdot] \to T_{\mathfrak{t}}$ defines a one-to-one correspondence between all regular closed symmetric forms in $(H, (\cdot, \cdot))$ and all self-adjoint operators in $(H, (\cdot, \cdot))$ with spectrum different from the whole real axis \mathbb{R} .

By [12, Proposition 2.5] the definition of regularity can be weakened:

Proposition 2.5. Let $\mathfrak{t}[\cdot,\cdot]$ be a closed symmetric sesquilinear form in the Hilbert space $(H,(\cdot,\cdot))$ and let $T_\mathfrak{t}$ be the associated operator. Then the following statements are equivatent:

- (i) dom $\mathfrak{t} = \operatorname{dom} |T_{\mathfrak{t}}|^{\frac{1}{2}}$,
- (ii) dom $\mathfrak{t} \subset \operatorname{dom} |T_{\mathfrak{t}}|^{\frac{1}{2}}$,
- (iii) $\operatorname{dom} \mathfrak{t} \supset \operatorname{dom} |T_{\mathfrak{t}}|^{\frac{1}{2}}$.

This result can be improved by a slight extension of [12, Lemma 2.4]:

Lemma 2.6. If we have two closed forms $\mathfrak{t}_1[\cdot,\cdot]$ and $\mathfrak{t}_2[\cdot,\cdot]$ associated with the same self-adjoint representing operator T in the Hilbert space $(H,(\cdot,\cdot))$ then $dom \mathfrak{t}_1 \subset dom \mathfrak{t}_2$ implies $\mathfrak{t}_1[\cdot,\cdot] = \mathfrak{t}_2[\cdot,\cdot]$.

Proof. From [12, Lemma 2.4] we can conclude that $\operatorname{dom} \mathfrak{t}_1 = \operatorname{dom} \mathfrak{t}_2$ and (looking in the proof of [12, Lemma 2.4]) that the embeddings of the associated Kreĭn spaces are continuous (using two gap points). Now, let $f \in \operatorname{dom} \mathfrak{t}_1 (= \operatorname{dom} \mathfrak{t}_2)$, $f_n \in \operatorname{dom} T$ such that $f_n \to f(n \to \infty)$ with respect to this Kreĭn space topology. Then, according to Theorem 2.2 we have

$$\mathfrak{t}_1[f,f] = \lim_{n \to \infty} \mathfrak{t}_1[f_n, f_n] = \lim_{n \to \infty} (Tf_n, f_n) = \lim_{n \to \infty} \mathfrak{t}_2[f_n, f_n] = \mathfrak{t}_2[f, f].$$

This implies $\mathfrak{t}_1[\cdot,\cdot] = \mathfrak{t}_2[\cdot,\cdot]$ by the polarization identity.

A space triplet associated with a J-non-negative, J-self-adjoint and boundedly invertible Krein space operator

Here a construction from [7, Section 4.1] is recalled for the general case of a J-non-negative, J-self-adjoint and boundedly invertible operator A in a Kreĭn space $(K, [\cdot, \cdot])$ (rather than for an indefinite Kreĭn-Feller operator).

Let J be a fundamental symmetry of $(K, [\cdot, \cdot])$ and denote the associated Hilbert space inner product by $\{\cdot, \cdot\} := [J \cdot, \cdot]$. Furthermore, put

$$K_{+} := \operatorname{dom}(JA)^{\frac{1}{2}}, \qquad \{f, g\}_{+} := \{(JA)^{\frac{1}{2}}f, (JA)^{\frac{1}{2}}g\} \quad (f, g \in K_{+}).$$

Then $(K_+, \{\cdot, \cdot\}_+)$ is a Hilbert space which is dense and continuously embedded in $(K, [\cdot, \cdot])$. Note that by [3, Remark 1.5] the form $\{\cdot, \cdot\}_+$ on K_+ coincides with the closure of the positive definite sesquilinear form $[A \cdot, \cdot]$ defined on dom A. Let K_- be the dual space of $(K_+, \{\cdot, \cdot\}_+)$, i.e. the space of all linear functionals $\varphi : K_+ \longrightarrow \mathbb{C}$ which are continuous with respect to $\{\cdot, \cdot\}_+$. This is a complex Banach space with

$$(\varphi + \psi)(f) := \varphi(f) + \psi(f), \quad (\alpha \cdot \varphi)(f) := \overline{\alpha} \cdot \varphi(f) \quad (f \in K_+)$$

and with the norm

$$\{\varphi\}_- := \sup_{f \in K_+, \{f, f\}_+ \le 1} |\varphi(f)|.$$

Each element $f \in K$ defines a unique continuous linear functional $[\cdot, f]$ on K_+ . In this sense we have the inclusion $K \subset K_-$ and the inner product $[\cdot, \cdot]$ on K extends naturally to

$$[\cdot,\cdot]: K_+ \times K_- \longrightarrow \mathbb{C}, \quad [f,\varphi]:=\varphi(f) \quad (f \in K_+, \varphi \in K_-).$$

Now, let A_{-} denote the operator from the Riesz Representation Theorem, i.e.

$$A_{-}g := \{\cdot, g\}_{+}, \quad (g \in \text{dom } A_{-} := K_{+}).$$

This is an isometric isomorphism between $(K_+, \{\cdot, \cdot\}_+)$ and the dual space $(K_-, \{\cdot\}_-)$. By definition we have for $g \in \text{dom } A \subset K_+$

$$[f,A_-g] = \{f,g\}_+ = \{(JA)^{\frac{1}{2}}f,(JA)^{\frac{1}{2}}g\} = [f,Ag] \quad (f \in K_+)$$

and hence, $A_-g = Ag$. Therefore, A is the restriction of A_- to dom A and the inverse A^{-1} is the restriction of A_-^{-1} to K. By the isometric property of A_- we have for $\varphi \in K_-$

$$[A_{-}^{-1}\varphi,\varphi]=[A_{-}^{-1}\varphi,A_{-}(A_{-}^{-1}\varphi)]=\{A_{-}^{-1}\varphi,A_{-}^{-1}\varphi\}_{+}=\{\varphi\}_{-}^{2}.$$

Therefore, K_{-} turns into a Hilbert space with the inner product

$$\{\varphi,\psi\}_- := [A_-^{-1}\varphi,\psi] (= \{A_-^{-1}\varphi,A_-^{-1}\psi\}_+) \quad (\varphi,\psi\in K_-).$$

Lemma 3.1. (i) $(K, [\cdot, \cdot])$ is dense and continuously embedded in the Hilbert space $(K_-, \{\cdot, \cdot\}_-)$.

- (ii) Considered in the Hilbert space $(K_-, \{\cdot, \cdot\}_-)$, the operator A_- is self-adjoint and boundedly invertible.
- (iii) Considered in the Hilbert space $(K_+, \{\cdot, \cdot\}_+)$, the range restriction A_+

$$dom A_+ := A^{-1}(K_+), \quad A_+ f := Af \quad (f \in dom A_+)$$

is self-adjoint and boundedly invertible.

(iv) The operator A_{-} allows the representation

$$[f,g] = \{A_-f,g\}_ (f \in \text{dom } A_-, g \in K).$$

Proof. (i) Let $\varphi \in K_{-}$ be orthogonal to K with respect to $\{\cdot, \cdot\}_{-}$, i.e.

$$0 = \{\varphi, f\}_- = [A_-^{-1}\varphi, f] \quad \text{for all} \quad f \in K.$$

Then, $A_{-}^{-1}\varphi \in K_{+}$ is orthogonal to K with respect to $[\cdot,\cdot]$. Consequently, we have $A_{-}^{-1}\varphi = 0$ and hence, $\varphi = 0$. This implies the density of K in $(K_{-}, \{\cdot,\cdot\}_{-})$. Furthermore, for $f \in K$ we can estimate

$$\{f,f\}_{-}^{2} = [A^{-1}f,f]^{2} \le \{A^{-1}f,A^{-1}f\}\{f,f\} \le ||A^{-1}||\{f,f\}^{2}$$

where $||A^{-1}||$ denotes the operator norm of A^{-1} in $(K, \{\cdot, \cdot\})$. Therefore, the embedding is continuous.

(ii) First, for $f, g \in \text{dom } A_- = K_+$ we observe

$${A_{-}f,g}_{-} = [A_{-}^{-1}(A_{-}f),g] = [f,g] = \overline{\{A_{-}g,f\}_{-}} = \{f,A_{-}g\}_{-}.$$

Therefore, A_{-} is symmetric with respect to $\{\cdot,\cdot\}_{-}$ and hence, self-adjoint since the range of A_{-} is the whole space K_{-} .

(iii) Similarly, for $f, g \in \text{dom } A_+$ we observe

$${A_+f,g}_+ = {Af,g}_+ = {Af,A_-g} = {\overline{Af,Ag}} = {\overline{A_+g,f}_+} = {f,A_+g}_+.$$

Therefore, A_+ is symmetric with respect to $\{\cdot,\cdot\}_+$ and hence, self-adjoint since the range of A_+ is the whole space K_+ .

(iv) immediately follows from the definition of
$$\{\cdot,\cdot\}_{-}$$
.

Now, we have constructed the "space triplet"

$$K_{+} \subset K \subset K_{-} \tag{3.1}$$

where each inclusion is dense and continuous with respect to the associated topologies. Finally, as an example, we present "model spaces" which appear e.g. as the images of Fourier transformations associated with indefinite Sturm-Liouville operators or more generally, of indefinite Kreĭn-Feller operators (see e.g. [7, Section 4.4, Section 4.5]).

Example 1. Let the real function $r \in L^1_{loc}(\mathbb{R})$ satisfy the sign conditions

$$r(x) = 0$$
 a.e. on $[-\varepsilon, \varepsilon]$, $xr(x) > 0$ a.e. on $(-\infty, -\varepsilon) \cup (\varepsilon, \infty)$ (3.2)

with some $\varepsilon > 0$ and assume that r is odd, i.e.

$$r(-x) = -r(x) \text{ a.e. on } \mathbb{R}. \tag{3.3}$$

Furthermore, consider the functions

$$r_{+}(x) := xr(x), \quad r_{-}(x) := \frac{1}{x}r(x) \qquad (x \in \mathbb{R} \setminus \{0\})$$
 (3.4)

which are well defined a.e. on \mathbb{R} and non-negative. On the weighted space $L_r^2(\mathbb{R})$ we define the following inner products

$$(f,g)_r := \int_{-\infty}^{\infty} f\overline{g} |r| dx, \quad [f,g]_r := \int_{-\infty}^{\infty} f\overline{g} r dx \quad (f,g \in L_r^2(\mathbb{R}))$$
 (3.5)

and on $L^2_{r+}(\mathbb{R})$

$$(f,g)_{r_{\pm}} := \int_{-\infty}^{\infty} f\overline{g} \, r_{\pm} \, dx \quad (f,g \in L^2_{r_{\pm}}(\mathbb{R})).$$

Then, $(L_{r_+}^2(\mathbb{R}), (\cdot, \cdot)_{r_+})$ and $(L_{r_-}^2(\mathbb{R}), (\cdot, \cdot)_{r_-})$ are Hilbert spaces. Furthermore, $(L_r^2(\mathbb{R}), [\cdot, \cdot]_r)$ is a Kreĭn space with the fundamental symmetry

$$(Jf)(x) := \operatorname{sgn}(x)f(x) \quad (x \in \mathbb{R})$$
(3.6)

satisfying $[J\cdot,\cdot]_r=(\cdot,\cdot)_r$ and hence, $(L_r^2(\mathbb{R}),\,(\cdot,\cdot)_r)$ is the associated Hilbert space. From (3.2) we can conclude the inclusions

$$L_{r_{+}}^{2}(\mathbb{R}) \subset L_{r}^{2}(\mathbb{R}) \subset L_{r_{-}}^{2}(\mathbb{R}) \tag{3.7}$$

and each inclusion is dense and continuous with respect to the associated topologies. Moreover, $L_{r_{-}}^{2}(\mathbb{R})$ can be regarded as the dual space of the Hilbert space $(L_{r_{+}}^{2}(\mathbb{R}), (\cdot, \cdot)_{r_{+}})$ if we identify each element $g \in L_{r_{-}}^{2}(\mathbb{R})$ with the functional $[\cdot, g]_{r}$ on $L_{r_{+}}^{2}(\mathbb{R})$, i.e. with

$$\varphi_g(f) := [f, g]_r := \int_{-\infty}^{\infty} f\overline{g} \, r \, dx \quad (f \in L^2_{r_+}(\mathbb{R})).$$

(Here, we use an obvious extension of the inner product $[\cdot,\cdot]_r$ from (3.5) to $L^2_{r_+}(\mathbb{R}) \times L^2_{r_-}(\mathbb{R})$.) Note, that indeed this integral exists and the functional φ_g is continuous with respect to $(\cdot,\cdot)_{r_+}$ since

$$\begin{aligned} |\varphi_{g}(f)|^{2} & \leq & \left(\int_{-\infty}^{\infty} \sqrt{|x|} |f(x)| |g(x)| \frac{1}{\sqrt{|x|}} |r(x)| \, dx \right)^{2} \\ & \leq & \left(\int_{-\infty}^{\infty} |x| |f(x)|^{2} |r(x)| \, dx \right) \left(\int_{-\infty}^{\infty} |g(x)|^{2} \frac{1}{|x|} |r(x)| \, dx \right) \\ & = & (f, f)_{r_{+}}(g, g)_{r_{-}}. \end{aligned}$$

Next, in the Krein space $(L_r^2(\mathbb{R}), [\cdot, \cdot]_r)$ we define the operator A by

$$dom A := \{ f \in L_r^2(\mathbb{R}) \mid \int_{-\infty}^{\infty} |x|^2 |f(x)|^2 |r(x)| dx < \infty \},$$

$$(Af)(x) := xf(x) \quad (f \in dom A)$$

and in the Hilbert spaces $(L^2_{r_{\pm}}(\mathbb{R}),\,(\cdot,\cdot)_{r_{\pm}})$ the operators A_{\pm} by

$$\operatorname{dom} A_{\pm} := \{ f \in L^{2}_{r_{\pm}}(\mathbb{R}) \mid \int_{-\infty}^{\infty} |x|^{2} |f(x)|^{2} r_{\pm}(x) \, dx < \infty \},$$

$$(A_{+} f)(x) := x f(x) \quad (f \in \operatorname{dom} A_{+}).$$

Then, by (3.2) A is J-non-negative, J-self-adjoint and boundedly invertible and A_+ and A_- are self-adjoint and boundedly invertible (all with respect to the corresponding spaces). In particular, we have dom $A_- = L^2_{r_+}(\mathbb{R})$ and also dom $(JA)^{\frac{1}{2}} = L^2_{r_+}(\mathbb{R})$ and for $f, g \in L^2_{r_+}(\mathbb{R})$

$$((JA)^{\frac{1}{2}}f,(JA)^{\frac{1}{2}}g)_r = \int_{-\infty}^{\infty} \sqrt{|x|}f(x)\sqrt{|x|} \ \overline{g(x)} |r(x)| \, dx = (f,g)_{r_+}.$$

Finally, we observe that with

$$K:=L^2_r(\mathbb{R}), \; [\cdot,\cdot]:=[\cdot,\cdot]_r \quad \text{ and } \quad K_\pm:=L^2_{r_+}(\mathbb{R}), \; \{\cdot,\cdot\}_\pm:=(\cdot,\cdot)_{r_\pm} \quad (3.8)$$

and with the associated operators A and A_{\pm} the general setting of Section 3 is given and (3.7) coincides with the space triplet (3.1) in this "model situation".

4. Relations between closed symmetric sesquilinear forms and J-self-adjoint, J-non-negative operators in Kreĭn spaces

In Theorem 2.4 the one-to-one relation between all regular closed symmetric forms and all self-adjoint Hilbert space operators with a gap in its real spectrum was recalled from [10]. Now, we get rid of the restriction to regular closed forms and in return we consider J-non-negative Kreĭn space operators. Using the construction from Section 3 we shall see that all closed symmetric forms with a gap point 0 are in one-to-one correspondance with all J-non-negative, J-self-adjoint and boundedly invertible Kreĭn space operators.

This result is now presented in the following Theorem 4.1. Here, we use certain identifications without stating them explictly in the Theorem in order to avoid a too technical overhead. Two Hilbert spaces with closed forms $(H_1, (\cdot, \cdot)_1, \mathfrak{t}_1[\cdot, \cdot])$ and $(H_2, (\cdot, \cdot)_2, \mathfrak{t}_2[\cdot, \cdot])$ (both with gap point 0) are identified if there is an isometric Hilbert space isomorphisms ϕ from $(H_1, (\cdot, \cdot)_1)$ to $(H_2, (\cdot, \cdot)_2)$ such that dom $\mathfrak{t}_2 = \phi(\operatorname{dom} \mathfrak{t}_1)$ and $\mathfrak{t}_1[f, g] = \mathfrak{t}_2[\phi(f), \phi(g)]$ for all $f, g \in \operatorname{dom} \mathfrak{t}_1$. Similarly, two Kreın spaces with J-non-negative, J-self-adjoint and boundedly invertible operators $(K_1, [\cdot, \cdot]_1, A_1)$ and $(K_2, [\cdot, \cdot]_2, A_2)$ are identified if there is an isometric Kreın space isomorphisms ψ from $(K_1, [\cdot, \cdot]_1)$ to $(K_2, [\cdot, \cdot]_2)$ such that $\psi(\operatorname{dom} A_1) = \operatorname{dom} A_2$ and $A_1 = \psi^{-1}A_2\psi$. In this sense the tuples in the formulation of the following Theorem must be regarded as the corresponding equivalence classes.

Theorem 4.1. The mapping Φ from the set

$$\mathcal{M} := \{ (H, (\cdot, \cdot), \mathfrak{t}[\cdot, \cdot]) \mid (H, (\cdot, \cdot)) \text{ Hilbert space, } \mathfrak{t}[\cdot, \cdot] \text{ closed symmetric}$$

$$form \text{ in this space with a qap point } 0 \}$$

to the set

$$\mathcal{N} := \{ (K, [\cdot, \cdot], A) \mid (K, [\cdot, \cdot]) \text{ Kreĭn space, } A \text{ J-non-negative, J-self-adjoint and boundedly invertible in this space} \}$$

given by the rule

$$K := \operatorname{dom} \mathfrak{t}, \quad [\cdot, \cdot] := \mathfrak{t}[\cdot, \cdot], \quad A := A_{\mathfrak{t}} \tag{4.1}$$

(using the range restriction A_t from (2.4)) is bijective. Its inverse Φ^{-1} is given by the rule

$$H := K_{-}, \quad (\cdot, \cdot) := \{\cdot, \cdot\}_{-}, \quad \operatorname{dom} \mathfrak{t} := K, \quad \mathfrak{t}[\cdot, \cdot] := [\cdot, \cdot] \tag{4.2}$$

(using the construction from Section 3).

Proof. Starting with $(H, (\cdot, \cdot), \mathfrak{t}[\cdot, \cdot]) \in \mathcal{M}$ it is clear by Section 2.2 that $(K, [\cdot, \cdot], A)$ given by (4.1) belongs to \mathcal{N} . Conversely, we show that the Hilbert

space $(K_-, \{\cdot, \cdot\}_-)$ given by Section 3 is isometrically isomorphic to the original space $(H, (\cdot, \cdot))$ such that also $[\cdot, \cdot]$ and $\mathfrak{t}[\cdot, \cdot]$ are connected by this isomorphism. To this end, first observe that the space K_+ according to Section 3 coincides with the space $\operatorname{dom} T$ where $T = T_{\mathfrak{t}}$ is the associated representing operator. Indeed, $\operatorname{dom} T$ is a Hilbert space with the inner product $(T\cdot, T\cdot)$ since $0 \in \rho(T_{\mathfrak{t}})$ and on $\operatorname{dom} A$ this inner product coincides with $\mathfrak{t}[A\cdot,\cdot]$. Therefore, $(T\cdot, T\cdot)$ on $\operatorname{dom} T$ is the closure of $\mathfrak{t}[A\cdot,\cdot]$ and hence, $(T\cdot, T\cdot)$ coincides with the inner product $\{\cdot,\cdot\}_+$ on $\operatorname{dom} T (=K_+)$ according to Section 3. Now, for $f \in H$ consider the linear functional $\phi_f := (T\cdot, f)$ on $\operatorname{dom} T$. Obviously, this functional is continuous with respect to $\{\cdot,\cdot\}_+$ (= $(T\cdot, T\cdot)$) and for $f \in K$ (= $\operatorname{dom} \mathfrak{t}$) we have $\phi_f = \mathfrak{t}[\cdot, f]$. Consequently, by $\phi : f \to \phi_f$ the function ϕ maps H into the dual space K_- of $(K_+, \{\cdot,\cdot\}_+)$ and on K it coincides with the embedding of K into K_- according to Section 3. Furthermore, for $f \in H$ we have

$$\{\phi_f\}_- = \sup_{g \in K_+, \{g,g\}_+ \le 1} |\phi_f(g)| = \sup_{g \in \text{dom } T, \, (Tg,Tg) \le 1} |(Tg,f)| = \sqrt{(f,f)}$$

since $0 \in \rho(T)$. Therefore, ϕ maps H isometrically into a (closed) subspace of K_{-} and since K is dense in K_{-} the range of ϕ is the whole space K_{-} . Consequently, ϕ is the required isomorphism.

On the other hand, starting with $(K, [\cdot, \cdot], A) \in \mathcal{N}$ it is clear by Section 3 that $(H, (\cdot, \cdot), \mathfrak{t}[\cdot, \cdot])$ given by (4.2) belongs to \mathcal{M} . Conversely, with (4.1) we return to the original element $(K, [\cdot, \cdot], A) \in \mathcal{N}$ by definition since by Lemma 3.1 A_- is the representing operator for the form $\mathfrak{t}[\cdot, \cdot] = [\cdot, \cdot]$ and A is its range restriction to K.

It remains to show that Φ maps the equivalence classes according to the indicated identifications into each other and hence, the mapping Φ is well defined. To this end, first consider two Hilbert spaces with closed forms $(H_1, (\cdot, \cdot)_1, \mathfrak{t}_1[\cdot, \cdot])$ and $(H_2, (\cdot, \cdot)_2, \mathfrak{t}_2[\cdot, \cdot])$ (both with gap point 0) which are identified by means of an isomorphisms $\phi: H_1 \to H_2$. Then, by $\mathfrak{t}_1[f, g] = \mathfrak{t}_2[\phi(f), \phi(g)]$ it is clear that also the Kreın spaces (dom $\mathfrak{t}_1, \mathfrak{t}_1[\cdot, \cdot]$) and (dom $\mathfrak{t}_2, \mathfrak{t}_2[\cdot, \cdot]$) are isometrically isomorph by means of ϕ . Furthermore, with the representing operators $T_1 = T_{\mathfrak{t}_1}$ and $T_2 = T_{\mathfrak{t}_2}$ we have

$$\mathfrak{t}_2[\phi(f),\phi(g)] = \mathfrak{t}_1[f,g] = (T_1f,g)_1 = (\phi(T_1f),\phi(g))_2$$

for all $f \in \text{dom } T_1$, $g \in \text{dom } \mathfrak{t}_1$ which implies $\phi(f) \in \text{dom } T_2$ and $T_2\phi(f) = \phi(T_1f)$. This gives $T_1 \subset \phi^{-1}T_2\phi$ and hence, $T_1 = \phi^{-1}T_2\phi$ by the self-adjointness. Consequently also the range restrictions satisfy $A_{\mathfrak{t}_1} = \phi^{-1}A_{\mathfrak{t}_2}\phi$ and therefore, also the Kreın spaces and its operators must be identified.

On the other hand consider two Kreı̆n spaces with J-non-negative, J-self-adjoint and boundedly invertible operators $(K_1, [\cdot, \cdot]_1, A_1)$ and $(K_2, [\cdot, \cdot]_2, A_2)$ which are identified by means of an isomorphisms $\psi : K_1 \to K_2$ such that $A_1 = \psi^{-1}A_2\psi$. Then, also the positive sesquilinear forms $[A_1\cdot, \cdot]_1$ and $[A_2\cdot, \cdot]_2$ are connected by ψ , i.e. $[A_2\psi\cdot, \psi\cdot]_2 = [A_1\cdot, \cdot]_1$ and hence, for its closures we have $K_{2+} = \psi(K_{1+})$ and $\{\cdot, \cdot\}_{1+} = \{\psi\cdot, \psi\cdot\}_{2+}$. Therefore, φ is a continuous liner functional on $(K_{2+}, \{\cdot, \cdot\}_{2+})$ (i.e. $\varphi \in K_{2-}$) if and only if $\varphi \circ \psi$ is a

continuous liner functional on $(K_{1+}, \{\cdot, \cdot\}_{1+})$ (i.e. $\varphi \circ \psi \in K_{1-}$) and we have

$$\{\varphi\}_{2_-} = \sup_{g \in K_{2_+}, \{g,g\}_{2_+} \leq 1} |\varphi(g)| = \sup_{h \in K_{1_+}, \{h,h\}_{1_+} \leq 1} |\varphi(\psi(h))| = \{\varphi \circ \psi\}_{1_-}.$$

Consequently, $\varphi \to \varphi \circ \psi$ defines an isometric isomorhism ψ_- from the Hilbert space $(K_{2-}, \{\cdot, \cdot\}_{2-})$ to $(K_{1-}, \{\cdot, \cdot\}_{1-})$ such that the corresponding forms satisfy

$$[\psi_{-}(f), \psi_{-}(g)]_1 = [f \circ \psi, g \circ \psi]_1 = [f, g]_2$$

for all $f, g \in K_2 (= \psi(K_1))$. Then, also these Hilbert spaces and forms must be identified.

With a shift we can get rid of the restriction to the gap point 0.

Corollary 4.2. Let $\lambda \in \mathbb{R}$. Then, the mapping Φ_{λ} from the set

$$\{(H, (\cdot, \cdot), \mathfrak{t}[\cdot, \cdot]) \mid (H, (\cdot, \cdot)) \text{ Hilbert space, } \mathfrak{t}[\cdot, \cdot] \text{ closed symmetric}$$

form in this space with gap point $\lambda\}$

to the set

$$\{(K, [\cdot, \cdot], A) \mid (K, [\cdot, \cdot]) \text{ Kreĭn space, A J-self-adjoint, } A - \lambda$$

$$J\text{-non-negative and boundedly invertible in this space}\}$$

given by the rule

$$K := \operatorname{dom} \mathfrak{t}, \quad [\cdot, \cdot] := \mathfrak{t}[\cdot, \cdot] - \lambda(\cdot, \cdot), \quad A := A_{\mathfrak{t}}$$

(using the range restriction A_t from (2.4)) is bijective. Its inverse Φ_{λ}^{-1} is given by the rule

$$H:=K_-, \quad (\cdot,\cdot):=\{\cdot,\cdot\}_-, \quad \operatorname{dom}\mathfrak{t}:=K, \quad \mathfrak{t}[\cdot,\cdot]:=[\cdot,\cdot]+\lambda\{\cdot,\cdot\}_-$$
 (using the construction from Section 3 for $A-\lambda$).

Proof. A form $\mathfrak{t}[\cdot,\cdot]$ is closed with gap point λ if and only if $\mathfrak{t}_{\lambda}[\cdot,\cdot]$ from (2.3) is closed with gap point 0. In this case the representing operators $T_{\mathfrak{t}}$ and $T_{\mathfrak{t}_{\lambda}}$ are connected via $T_{\mathfrak{t}_{\lambda}} = T_{\mathfrak{t}} - \lambda$. This implies $A_{\mathfrak{t}_{\lambda}} = A_{\mathfrak{t}} - \lambda$ for the corresponding range restrictions. Therefore, the statement follows from Theorem 4.1 applied to $\mathfrak{t}_{\lambda}[\cdot,\cdot]$ and $A_{\mathfrak{t}_{\lambda}}$.

Note that this Section can be regarded as a more detailed discussion of a setting from [10, Section 5], in particular from [10, Proposition 5.3].

5. Families of form closures and of Kreĭn space completions associated with a fixed form $(T \cdot, \cdot)$

First, we mention the following improvement of [12, Proposition 2.5] (where a similar statement was obtained only for *regular* closed forms).

Proposition 5.1. The set of gap points of a closed form $\mathfrak{t}[\cdot,\cdot]$ in a Hilbert space $(H,(\cdot,\cdot))$ coincides with the real part of the resolvent set of its representing operator, i.e. with $\mathbb{R} \cap \rho(T_t)$.

Proof. By the First Representation Theorem 2.2 each gap point belongs to $\rho(T_{\mathfrak{t}})$. Conversely, let $\lambda_0 \in \mathbb{R} \cap \rho(T_{\mathfrak{t}})$ and let $\lambda_1 \in \mathbb{R}$ be a gap point of $\mathfrak{t}[\cdot, \cdot]$. Then, we show that $\mathfrak{t}_{\lambda_0}[\cdot, \cdot]$ defines a Kreı̆n space structure on dom \mathfrak{t} with the Hilbert space topology induced by $\mathfrak{t}_{\lambda_1}[\cdot, \cdot]$, i.e. also λ_0 is a gap point of $\mathfrak{t}[\cdot, \cdot]$.

We have $\lambda_0, \lambda_1 \in \rho(T_{\mathfrak{t}})$ for the representing operator $T_{\mathfrak{t}}$ and hence, also for its range restriction $A := A_{\mathfrak{t}}$ the operators $A - \lambda_0$ and $A - \lambda_1$ are boundedly invertible in the Kreı̆n space (dom $\mathfrak{t}, \mathfrak{t}_{\lambda_1}[\cdot, \cdot]$). Let J_{λ_1} denote a fundamental symmetry in this Kreı̆n space and let $\mathfrak{t}_{\lambda_1}(\cdot, \cdot) := \mathfrak{t}_{\lambda_1}[J_{\lambda_1} \cdot, \cdot]$ be the associated Hilbert space inner product. Then, for $f, g \in \text{dom } \mathfrak{t}$ we can calculate

$$\begin{array}{lll} \mathfrak{t}_{\lambda_0}[f,g] &=& \mathfrak{t}_{\lambda_0}[(A-\lambda_1)(A-\lambda_1)^{-1}f,g] \\ &=& \mathfrak{t}_{\lambda_0}[A(A-\lambda_1)^{-1}f,g] - \lambda_1\mathfrak{t}_{\lambda_0}[(A-\lambda_1)^{-1}f,g] \\ &=& \mathfrak{t}[A(A-\lambda_1)^{-1}f,g] - \lambda_0(A(A-\lambda_1)^{-1}f,g) \\ && -\lambda_1\mathfrak{t}[(A-\lambda_1)^{-1}f,g] + \lambda_1\lambda_0((A-\lambda_1)^{-1}f,g) \\ &=& \mathfrak{t}[A(A-\lambda_1)^{-1}f,g] - \lambda_0\mathfrak{t}[(A-\lambda_1)^{-1}f,g] \\ && -\lambda_1(A(A-\lambda_1)^{-1}f,g) + \lambda_1\lambda_0((A-\lambda_1)^{-1}f,g) \\ &=& \mathfrak{t}[(A-\lambda_0)(A-\lambda_1)^{-1}f,g] - \lambda_1((A-\lambda_0)(A-\lambda_1)^{-1}f,g) \\ &=& \mathfrak{t}_{\lambda_1}[(A-\lambda_0)(A-\lambda_1)^{-1}f,g] \\ &=& \mathfrak{t}_{\lambda_1}(J_{\lambda_1}(A-\lambda_0)(A-\lambda_1)^{-1}f,g). \end{array}$$

By the symmetry of $\mathfrak{t}_{\lambda_0}[\cdot,\cdot]$ the operator $G:=J_{\lambda_1}(A-\lambda_0)(A-\lambda_1)^{-1}$ is symmetric and everywhere defined in the Hilbert space $(\operatorname{dom}\mathfrak{t},\mathfrak{t}_{\lambda_1}(\cdot,\cdot))$ and hence, G is self-adjoint and bounded. Furthermore, G is invertible and its inverse $G^{-1}=(A-\lambda_1)(A-\lambda_0)^{-1}J_{\lambda_1}$ is also bounded. Therefore, by [1,1.6.13] $\mathfrak{t}_{\lambda_0}[\cdot,\cdot]=\mathfrak{t}_{\lambda_1}(G\cdot,\cdot)$ defines a suitable Kreın space inner product on $(\operatorname{dom}\mathfrak{t},\mathfrak{t}_{\lambda_1}(\cdot,\cdot))$.

Now, we consider a fixed Hilbert space $(H, (\cdot, \cdot))$ and a fixed self-adjoint and boundedly invertible operator T in this space. Then, Proposition 5.1 allows the following characterization of the restriction of the mapping Φ from Theorem 4.1 to the family of all closed forms $\mathfrak{t}[\cdot, \cdot]$ in $(H, (\cdot, \cdot))$ which are represented by T, i.e. $T_{\mathfrak{t}} = T$.

Theorem 5.2. Let T be a self-adjoint and boundedly invertible operator in the Hilbert space $(H, (\cdot, \cdot))$. Then, the mapping Φ_T from the set

$$\mathcal{M}_T = \{\mathfrak{t}[\cdot, \cdot] \mid \mathfrak{t}[\cdot, \cdot] \text{ is a closure of } (T \cdot, \cdot) \text{ in } (H, (\cdot, \cdot))\}$$

to the set

$$\mathcal{N}_T = \{(K, [\cdot, \cdot], A) \mid (K, [\cdot, \cdot]) \text{ is a Kreĭn space completion of}$$

$$(\operatorname{dom} T, (T \cdot, \cdot)) \text{ and continuously embedded in}$$

$$(H, (\cdot, \cdot)), A = T|_{\operatorname{dom} A} \text{ with } \operatorname{dom} A = T^{-1}(K)\}$$

given by the rule (4.1) is bijective. We have

$$\mathcal{N}_T \subset \mathcal{N}, \qquad \mathcal{M}_T = \{\mathfrak{t}[\cdot, \cdot] \mid (H, (\cdot, \cdot), \mathfrak{t}[\cdot, \cdot]) \in \mathcal{M}, \ T_{\mathfrak{t}} = T\}$$

where the sets $\mathcal N$ and $\mathcal M$ are given by Theorem 4.1.

Proof. The statements are clear by definition and by the First Representation Theorem 2.2 when we observe that T is the representing operator of each closure $\mathfrak{t}[\cdot,\cdot]$ of the sesquiliear form $(T\cdot,\cdot)$ and hence, by Proposition 5.1 $\mathfrak{t}[\cdot,\cdot]$ has the gap point 0.

The following observation is a consequence of Theorem 4.1:

Remark 5.3. If we start with a closure $\mathfrak{t}[\cdot,\cdot]\in\mathcal{M}_T$ of $(T\cdot,\cdot)$ in $(H,(\cdot,\cdot))$ and consider the element $(K,[\cdot,\cdot],A):=\Phi_T(\mathfrak{t}[\cdot,\cdot])\in\mathcal{N}_T$ and finally construct the associated Hilbert space $(K_-,\{\cdot,\cdot\}_-)$ according to Section 3 then $(H,(\cdot,\cdot))$ and $(K_-,\{\cdot,\cdot\}_-)$ are isometrically isomorphic. Furthermore, from the proof of Theorem 4.1 we can conclude that also the associated operators T in $(H,(\cdot,\cdot))$ and A_- in $(K_-,\{\cdot,\cdot\}_-)$ are connected by this isomorpism, say $\phi:H\to K_-$: We have $T=\phi^{-1}A_-\phi$.

In the next step, the regularity of forms is reformulated in order to identify precisely one exceptional element in the family \mathcal{M}_T and one in \mathcal{N}_T .

Theorem 5.4. Let T be a self-adjoint and boundedly invertible operator in the Hilbert space $(H, (\cdot, \cdot))$. Then, the following statements hold true using the sets \mathcal{M}_T and \mathcal{N}_T and the mapping Φ_T from Theorem 5.2:

- (i) One closure $\mathfrak{t}_0[\cdot,\cdot] \in \mathcal{M}_T$ is regular. It is characterized by the property $\operatorname{dom} \mathfrak{t}_0 = \operatorname{dom} |T|^{\frac{1}{2}}$. All other closures are not regular.
- (ii) For one element $(K_0, [\cdot, \cdot]_0, A_0) \in \mathcal{N}_T$ infinity is not a singular critical point of the J-self-adjoint, J-non-negative and boundedly invertible operator A_0 . For all others infinity is a singular critical point.
- (iii) We have $(K_0, [\cdot, \cdot]_0, A_0) = \Phi_T(\mathfrak{t}_0[\cdot, \cdot])$, i.e. the two exceptional elements in \mathcal{M}_T and \mathcal{N}_T are connected via (4.1).
- (iv) For two closures $\mathfrak{t}_1[\cdot,\cdot],\mathfrak{t}_2[\cdot,\cdot]\in\mathcal{M}_T$ the relation $\mathrm{dom}\,\mathfrak{t}_1\subset\mathrm{dom}\,\mathfrak{t}_2$ implies $\mathfrak{t}_1[\cdot,\cdot]=\mathfrak{t}_2[\cdot,\cdot].$
- (v) For two Kreĭn space completions $(K_1, [\cdot, \cdot]_1), (K_2, [\cdot, \cdot]_2)$ of $(\text{dom } T, (T \cdot, \cdot))$ which are continuously embedded in $(H, (\cdot, \cdot))$ the relation $K_1 \subset K_2$ implies $K_1 = K_2$ and $[\cdot, \cdot]_1 = [\cdot, \cdot]_2$.

Proof. Again, in view of Proposition 5.1 the statements are clear by definition and by Section 2.2, in particular, by Lemma 2.6. \Box

Theorem 5.4 justifies to call the regular closure $\mathfrak{t}_0[\cdot,\cdot]\in\mathcal{M}_T$ according to Theorem 5.4 (i) the regularization of each closure $\mathfrak{t}[\cdot,\cdot]\in\mathcal{M}_T$. Similarly, among all J-self-adjoint, J-non-negative and boundedly invertible operators which are connected with T by \mathcal{N}_T the exceptional operator will be called regularization. More precisely, we call the exceptional element $(K_0, [\cdot, \cdot]_0, A_0) \in \mathcal{N}_T$ according to Theorem 5.4 (ii) the regularization of each element $(K, [\cdot, \cdot], A) \in \mathcal{N}_T$.

Now, we reformulate some of the above statements for a fixed J-self-adjoint, J-non-negative and boundedly invertible operator in a Kreĭn space.

Corollary 5.5. Let A be a J-self-adjoint, J-non-negative and boundedly invertible operator in a Kreĭn space $(K, [\cdot, \cdot])$. Let the Hilbert space $(H, (\cdot, \cdot))$ and

the form $\mathfrak{t}[\cdot,\cdot]$ be given by (4.2). Furthermore, let $T:=T_\mathfrak{t}$ be the associated self-adjoint operator in $(H,\,(\cdot,\cdot))$. Then, for all elements $(\widetilde{K},\,[\cdot,\cdot],\,\widetilde{A})\in\mathcal{N}_T$ the spaces $(\widetilde{K},\,[\cdot,\cdot])$ have dom T^2 as a common dense subspace and the operators \widetilde{A} coincide on this subspace:

$$\operatorname{dom} T^2 \subset \operatorname{dom} \widetilde{A}, \qquad \widetilde{A}f = Tf \quad (f \in \operatorname{dom} T^2). \tag{5.1}$$

For all but one of the elements $(\widetilde{K}, [\cdot, \cdot], \widetilde{A}) \in \mathcal{N}_T$ infinity is a singular critical point of the J-self-adjoint, J-non-negative and boundedly invertible operator \widetilde{A} . For the exceptional element $(K_0, [\cdot, \cdot]_0, A_0) \in \mathcal{N}_T$ infinity is not a singular critical point of A_0 . This element is the regularization of $(K, [\cdot, \cdot], A) \in \mathcal{N}_T$) and it is characterized by the property $K_0 = \text{dom} |T|^{\frac{1}{2}}$.

Proof. In view of Theorem 5.4 it remains to show (5.1) and the density of $\operatorname{dom} T^2$ in $(\widetilde{K}, [\cdot, \cdot])$. Indeed, if $f \in \operatorname{dom} T^2$ then $Tf \in \operatorname{dom} T \subset \widetilde{K}$ and hence, $f \in \operatorname{dom} \widetilde{A} (= T^{-1}(\widetilde{K}))$. Furthermore, for the dense subspace $\operatorname{dom} \widetilde{A}^2$ of $(\widetilde{K}, [\cdot, \cdot])$ we have $\operatorname{dom} \widetilde{A}^2 \subset \operatorname{dom} T^2$.

Note that by (5.1) the regularization of a J-self-adjoint, J-non-negative and boundedly invertible operator in a Kreĭn space can only be a "small" modification of the space and of the operator. In particular, the eigenvalues and eigenfunctions of the operators coincide.

Next, the eigenspectral functions associated with all operators from \mathcal{N}_T are determined by the spectral measure associated with T.

Theorem 5.6. Let T be a self-adjoint and boundedly invertible operator in the Hilbert space $(H, (\cdot, \cdot))$ and let E denote the corresponding spectral measure allowing the representation

$$T = \int_{-\infty}^{\infty} \lambda \, dE(\lambda).$$

Then, for each element $(K, [\cdot, \cdot], A) \in \mathcal{N}_T$ the restrictions

$$E_K(\Delta) := E(\Delta)|_K \quad (\Delta \in \Sigma)$$
 (5.2)

define the eigenspectral function E_K of A in $(K, [\cdot, \cdot])$ according to (2.1).

Proof. Let \widetilde{E}_K denote the eigenspectral function of the J-self-adjoint, J-nonnegative and boundedly invertible operator A according to (2.1). For $f \in K$ the limits in (2.1) must be understood with respect to the topology in $(K, [\cdot, \cdot])$ (including the integral itself). By the continuous embedding this also implies convergence in $(H, (\cdot, \cdot))$ and additionally, we have

$$(A - \lambda)^{-1} f = (T - \lambda)^{-1} f$$
 for all $\lambda \in \mathbb{C} \setminus \mathbb{R}, f \in K$

since A is the range restriction of T. Then, we can conclude $\widetilde{E}_K(\Delta)f = E(\Delta)f$ for a bounded interval $\emptyset \neq \Delta \in \Sigma$ by Stone's formula for the spectral measure of a self-adjoint Hilbert space operator (see e.g. [15, Problem VI 5.7] or [2, (1.5.4)]). This implies $\widetilde{E}_K(\Delta) = E(\Delta)|_K$ for all $\Delta \in \Sigma$ and hence, $\widetilde{E}_K = E_K$.

Of course, Theorem 5.6 also applies to the approach from Corollary 5.5:

Corollary 5.7. Let A be a J-self-adjoint, J-non-negative and boundedly invertible operator in a Kreĭn space $(K, [\cdot, \cdot])$. Let the Hilbert space $(K_-, \{\cdot, \cdot\}_-)$ and the self-adjoint operator A_- be given according to Section 3. Furthermore, denote by E the spectral measure of A_- . Then, the eigenspectral function E_K of A satisfies (5.2).

Using again the terminology of Section 3 and Corollary 5.7 Ćurgus' result from [4, Proposition 3.1] has the form

$$f \in K_+ \quad \Leftrightarrow \quad \int_{-\infty}^{\infty} \lambda \ d[E_K(\lambda)f, f] < \infty.$$

Corollary 5.7 allows the following slight improvement (which however, can also be obtained directly from the proof in [4, Proposition 3.1]).

Corollary 5.8. In the setting of Corollary 5.7 we have for $f \in K_+$

$$\{f, f\}_+ = \int_{-\infty}^{\infty} \lambda \ d[E_K(\lambda)f, f]$$

Proof. By Corollary 5.7 E is an extension of E_K . Therefore we can calculate

$$\int_{-\infty}^{\infty} \lambda \ d[E_K(\lambda)f, f] = \int_{-\infty}^{\infty} \lambda \ d\{E(\lambda)f, A_-f\}_- = \{A_-f, A_-f\}_- = \{f, f\}_+$$

since $A_-: K_+ \to K_-$ is isometric.

Finally, it is mentioned that the results in this Section are related to the study of Kreĭn space completions by Ćurgus and Langer in [6]. In the terminology of [6] the exceptional element $(K_0, [\cdot, \cdot]_0, A_0) \in \mathcal{N}_T$ is related to the so-called "canonical" Kreĭn space completion of $(\text{dom } T, (T \cdot, \cdot))$. Furthermore, translating [6, Theorem 2.7] into the present setting we obtain the following result:

Theorem 5.9. Let T be a self-adjoint and boundedly invertible operator in the Hilbert space $(H, (\cdot, \cdot))$. Then, the form $(T \cdot, \cdot)$ has a unique closure (i.e. the sets \mathcal{M}_T and \mathcal{N}_T from Theorem 5.2 both have only one element) if and only if the operator T is semibounded.

Proof. By the assumption, the operator $S:=T^{-1}$ is self-adjoint and bounded in $(H, (\cdot, \cdot))$. The operator T is semibounded if and only if (c, ∞) or $(-\infty, -c)$ belongs to the resolvent set $\rho(T)$ for some c>0. With $\varepsilon:=\frac{1}{c}$ this is further equivalent to $(0, \varepsilon) \subset \rho(S)$ or $(-\varepsilon, 0) \subset \rho(S)$. Note that the inner product $(\cdot, \cdot)_S:=(T\cdot, \cdot)$ on dom T (i.e. the range of S) satisfies

$$(f,g)_S = (f,Tg) = (Sx,y)$$
 if $f = Sx, g = Sy, x,y \in H$

and hence, $(\cdot, \cdot)_S$ coincides with the inner product defined in [6, (2.2)]. Therefore, [6, Theorem 2.7] can be applied to S and $(\cdot, \cdot)_S$ and it turns out that $(\text{dom } T, (\cdot, \cdot)_S)$ has a unique Kreĭn space completion which is continuously embedded in $(H, (\cdot, \cdot))$ if and only if $(0, \varepsilon) \subset \rho(S)$ or $(-\varepsilon, 0) \subset \rho(S)$ for some

 $\varepsilon > 0$. This means that the set \mathcal{N}_T has precisely one element if and only if T is semibounded. Then, the statement follows by means of Theorem 5.2. \square

By Theorem 5.9 (or by classical arguments) in the semibounded case each of the sets \mathcal{M}_T and \mathcal{N}_T from Theorem 5.2 has only one element, namely the exceptional one. From [6, Theorem 5.2, proof part II] one can conclude that in the non-semibounded case the sets \mathcal{M}_T and \mathcal{N}_T have even infinitely many elements. However, it is generally a non-trivial task to identify at least two different elements in these sets explicitly. In the (two different) proofs of [6, Theorem 5.2] such constructions are presented on a certain abstract level. Here, we try an approach to explicit examples. First, we recall an example from [9].

Example 2. In the usual Hilbert space $L^2[-1,1]$ with inner product $(u,v) := \int_{-1}^1 u\overline{v} \, dx$ consider the form $\mathfrak{t}[\cdot,\cdot]$ given by

$$\operatorname{dom} \mathfrak{t} := \{ u \in AC[-1,1] \mid \int_{-1}^{1} |p| |u'|^2 \, dx < \infty, \, u(-1) = u(1) = 0 \},$$

$$\mathfrak{t}[u,v] := \int_{-1}^{1} u' \overline{v}' p \, dx \quad (u,v \in \operatorname{dom} \mathfrak{t})$$

where the indefinite weight function p is defined by

$$p(x) := \begin{cases} -\frac{2}{e}(1 - \log 2)^3 & (x \in [-1, -\frac{2}{e}]) \\ x |\log |x||^3 & (x \in (-\frac{2}{e}, \frac{1}{e}) \setminus \{0\}) \\ 0 & (x = 0) \\ \frac{1}{2} & (x \in [\frac{1}{2}, 1]) \end{cases}$$

with Euler's constant $e \approx 2,718...$ In [9, Section 2, Theorem 6.4] it is shown that $\mathfrak{t}[\cdot,\cdot]$ is closed with gap point 0 and for the function

$$u_0(x) := \begin{cases} 0 & (x \in [-1, 0]) \\ \frac{8}{9 \mid \log x \mid^{\frac{9}{8}}} & (x \in (0, \frac{1}{e}) \\ \frac{8}{9} - \frac{8(ex - 1)}{9(e - 1)} & (x \in [\frac{1}{e}, 1]) \end{cases}$$
 (5.3)

we have $u_0 \in \text{dom } \mathfrak{t} \setminus \text{dom } |T|^{\frac{1}{2}}$. Here, $T := T_{\mathfrak{t}}$ is the associated operator in $L^2[-1,1]$ given by Tu = -(pu')' defined on

$$\operatorname{dom} T = \{u \in L^2[-1,1] \mid u, pu' \in AC[-1,1], (pu')' \in L^2[-1,1], \\ u(-1) = u(1) = 0\}.$$

Consequently, the form $\mathfrak{t}[\cdot,\cdot]$ is not regular and hence, we can identify at least two elements in \mathcal{M}_T in this example: $\mathfrak{t}[\cdot,\cdot]$ and its regularization. However, here we cannot give an explict description of this regularization of $\mathfrak{t}[\cdot,\cdot]$.

Furthermore, it is observed in [9] that the spectrum of T is discrete and consists of a sequence of simple eigenvalues

$$-\infty < \dots < \lambda_{-2} < \lambda_{-1} < 0 < \lambda_1 < \lambda_2 < \dots < \infty$$

accumulating only at ∞ and $-\infty$. Let u_n denote the corresponding eigenfunctions normed by

$$1 = |\lambda_n|(u_n, u_n) = (|T|^{\frac{1}{2}}u_n, |T|^{\frac{1}{2}}u_n) \quad (n \in \mathbb{Z} \setminus \{0\})$$

and let $A_{\mathfrak{t}}$ be the range restriction of T to the Kreĭn space (dom \mathfrak{t} , $\mathfrak{t}[\cdot,\cdot]$). Then, for the eigenspectral function $E_{\mathrm{dom}\,\mathfrak{t}}$ of $A_{\mathfrak{t}}$ we have

$$E_{\text{dom }\mathfrak{t}}([-m,m])u = \sum_{|\lambda_n| < m, n \neq 0} \operatorname{sgn}(\lambda_n)\mathfrak{t}[u,u_n]u_n \tag{5.4}$$

according to [9, Section 7]. Since the form $\mathfrak{t}[\cdot,\cdot]$ is not regular infinity is a singular critical point of $A_{\mathfrak{t}}$ by Theorem 2.3. Indeed, by [9, Theorem 6.4] for the function u_0 from (5.3) the "eigenfunction expansion"

$$\sum_{n \in \mathbb{Z} \setminus \{0\}} \operatorname{sgn}(\lambda_n) \mathfrak{t}[u_0, u_n] u_n \tag{5.5}$$

does not converge unconditionally in $(\text{dom }\mathfrak{t}, \mathfrak{t}[\cdot, \cdot])$. In particular, by [9, Theorem 6.4, Theorem 3.3] at least one of the two series

$$\begin{split} & \Sigma_{n=1}^{\infty} \ \mathfrak{t}[u_0, u_n] u_n \ (= \lim_{m \to \infty} E_{\text{dom} \, \mathfrak{t}} ([0, m]) u_0), \\ & \Sigma_{n=1}^{\infty} \ \mathfrak{t}[u_0, u_{-n}] u_{-n} \ (= \lim_{m \to \infty} -E_{\text{dom} \, \mathfrak{t}} ([-m, 0]) u_0) \end{split}$$

does not converge in $(\operatorname{dom} \mathfrak{t}, \mathfrak{t}[\cdot, \cdot])$. However, it remains as an open question in [9] whether the rearrangement of the series (5.5) according to (5.4) does converge, i.e. whether the limit $\lim_{m\to\infty} E_{\operatorname{dom} \mathfrak{t}}([-m,m])u_0$ exists. Below, we shall study a similar question in a different setting.

6. Examples with multiplication operators in "model spaces"

6.1. A regular closed form

We return to the setting of Examle 1 and change to the notations according to (4.2). In this notation we consider the Hilbert space $(H, (\cdot, \cdot))$ given by

$$H := K_{-} = L_{r_{-}}^{2}(\mathbb{R}), \qquad (f,g) := \{f,g\}_{-} = (f,g)_{r_{-}} = \int_{-\infty}^{\infty} f\overline{g} r_{-} dx$$

(according to (3.8)) and the form $\mathfrak{t}[\cdot,\cdot]$ given by

$$\operatorname{dom} \mathfrak{t} := K = L_r^2(\mathbb{R}), \qquad \mathfrak{t}[f, g] := [f, g] = [f, g]_r = \int_{-\infty}^{\infty} f \overline{g} \, r \, dx. \tag{6.1}$$

This form is closed with gap point 0 and by Lemma 3.1(iv) the associated self-adjoint Hilbert space operator $T := T_{\mathfrak{t}}$ in $H = L^2_{r_-}(\mathbb{R})$ is given by

$$dom T = dom A_{-} = L_{r_{+}}^{2}(\mathbb{R}), \qquad (Tf)(x) = (A_{-}f)(x) = xf(x). \tag{6.2}$$

Here, we can explicitly describe the square root operator $|T|^{\frac{1}{2}}$ by

$$dom |T|^{\frac{1}{2}} = L_r^2(\mathbb{R}), \qquad (|T|^{\frac{1}{2}}f)(x) = \sqrt{|x|}f(x).$$

Therefore, we have dom $\mathfrak{t}=\mathrm{dom}\,|T|^{\frac{1}{2}}$ and hence, in this situation the form $\mathfrak{t}[\cdot,\cdot]$ is a regular closed form, i.e. a regular closure of $(T\cdot,\cdot)$. This means that for the associated J-self-adjoint, J-non-negative and boundedly invertible range restriction $A_{\mathfrak{t}}=A$ in the Kreın space $(L_r^2(\mathbb{R}), [\cdot,\cdot]_r)$ from Examle 1 infinity is not a singular critical point. This is also evident in an explicit way:

Indeed, it is well known that the spectral measure E of the multiplication operator T is given by the multiplication operators

$$E(\Delta)f := \chi_{\Delta}f \qquad (f \in L_r^2(\mathbb{R})) \tag{6.3}$$

for each measurable set $\Delta \subset \mathbb{R}$. Here, χ_{Δ} denotes the characteristic function of the set Δ . Then, by Theorem 5.6 the eigenspectral function $E_{\text{dom }\mathfrak{t}}$ of $A_{\mathfrak{t}}$ is given by the restriction of the multiplication operator $E(\Delta)$ to dom $\mathfrak{t}=L^2_r(\mathbb{R})$ for each $\Delta \in \Sigma$. Therefore, for each $f \in L^2_r(\mathbb{R})$ the limits

$$\lim_{\lambda \to \infty} E_{\text{dom }\mathfrak{t}}([\varepsilon, \lambda]) f = \lim_{\lambda \to \infty} \chi_{[\varepsilon, \lambda]} f = \chi_{[\varepsilon, \infty)} f,$$

$$\lim_{\lambda \to \infty} E_{\text{dom }\mathfrak{t}}([-\lambda, \varepsilon]) f = \lim_{\lambda \to \infty} \chi_{[-\lambda, \varepsilon]} f = \chi_{(-\infty, \varepsilon]} f$$

exist in the Kreĭn space $(L_r^2(\mathbb{R}), [\cdot, \cdot]_r)$ (using $\varepsilon > 0$ from (3.2)). Again, this shows that infinity is not a singular critical point of A_t .

6.2. Non-regular closed forms

We proceed with the setting of Section 6.1. However, we now identify an infinite number of non-regular closures (or Kreın space completions) of the form $(T\cdot,\cdot)$ defined on dom $T=L^2_{r_+}(\mathbb{R})$. This construction is inspired by [6, Theorem 5.2, proof part II]. To this end, for $\alpha \in [0,2]$ we first introduce the even functions

$$\eta_{\alpha}(x) := (\sqrt{|x|^{\alpha} + 1} - \sqrt{|x|^{\alpha}})|r(x)|, \quad \omega_{\alpha}(x) := \sqrt{|x|^{\alpha}}|r(x)| \quad (x \in \mathbb{R}).$$

Furthermore, for a complex function f on \mathbb{R} we define the function

$$(\mathcal{Q}_{\alpha}f)(x) := \sqrt{|x|^{\alpha} + 1} f(x) - \sqrt{|x|^{\alpha}} f(-x) \quad (x \in \mathbb{R})$$
(6.4)

and the even and odd part

$$f_e(x) := \frac{1}{2}(f(x) + f(-x)), \quad f_o(x) := \frac{1}{2}(f(x) - f(-x)) \quad (x \in \mathbb{R}).$$

Since η_{α} and ω_{α} are non-negative functions on \mathbb{R} we can consider the associated weighted Hilbert spaces $L^2_{\eta_{\alpha}}(\mathbb{R})$ and $L^2_{\omega_{\alpha}}(\mathbb{R})$ equipped with the inner products

$$(f,g)_{\eta_{\alpha}} := \int_{-\infty}^{\infty} f\overline{g} \, \eta_{\alpha} \, dx \qquad (f,g \in L^{2}_{\eta_{\alpha}}(\mathbb{R})),$$

$$(f,g)_{\omega_{\alpha}} := \int_{-\infty}^{\infty} f\overline{g} \, \omega_{\alpha} \, dx \qquad (f,g \in L^{2}_{\omega_{\alpha}}(\mathbb{R})).$$

In a first step these spaces will be used in order to construct a Hilbert space structure on the space

$$\operatorname{dom} \mathfrak{t}_{\alpha} := \{ f \in L^{2}_{r_{-}}(\mathbb{R}) \mid f_{e} \in L^{2}_{\eta_{\alpha}}(\mathbb{R}), f_{o} \in L^{2}_{\omega_{\alpha}}(\mathbb{R}) \}.$$
 (6.5)

To this end, we first identify the "extremal" cases $\alpha = 0$ and $\alpha = 2$ and observe that η_{α} behaves like

$$\widetilde{\eta}_{\alpha}(x) := \frac{|r(x)|}{\sqrt{|x|^{\alpha}}} \quad (x \in \mathbb{R} \setminus \{0\})$$

for $\alpha \in [0, 2]$. Note that by (3.2) the functions ω_{α} , η_{α} , $\widetilde{\eta}_{\alpha}$ vanish on $[-\varepsilon, \varepsilon]$.

Lemma 6.1. The following statements hold true

- $\frac{\eta_{\alpha}(x)}{\widetilde{\eta}_{-}(x)} \longrightarrow \frac{1}{2} \quad (|x| \longrightarrow \infty) ,$ (i) for $\alpha \in (0,2]$:
- (ii) for $\alpha = 0$: $\eta_0(x) = (\sqrt{2} 1)\widetilde{\eta}_0(x) = (\sqrt{2} 1)|r(x)|, \quad \omega_0(x) = |r(x)|,$
- (iii) for $\alpha \in [0,2]$: $L^2_{\eta_{\alpha}}(\mathbb{R}) = L^2_{\widetilde{\eta}_{\alpha}}(\mathbb{R}),$

- (iv) for $\alpha = 0$: $L^{2}_{\eta_{0}}(\mathbb{R}) = L^{2}_{\omega_{0}}(\mathbb{R}) = \text{dom }\mathfrak{t}_{0} = L^{2}_{r}(\mathbb{R}) = \text{dom }\mathfrak{t},$ (v) for $\alpha = 2$: $\widetilde{\eta}_{2}(x) = r_{-}(x)$, $\omega_{2}(x) = r_{+}(x)$, (vi) for $\alpha = 2$: $L^{2}_{\eta_{2}}(\mathbb{R}) = L^{2}_{r_{-}}(\mathbb{R})$, $L^{2}_{\omega_{2}}(\mathbb{R}) = L^{2}_{r_{+}}(\mathbb{R})$.

Proof. By the mean value theorem (applied to $y(t) := \sqrt{t}$) we know that

$$\sqrt{|x|^{\alpha} + 1} - \sqrt{|x|^{\alpha}} = \frac{1}{2\sqrt{\xi}}$$

with some $\xi \in [|x|^{\alpha}, |x|^{\alpha} + 1]$. Therefore, for all $x > \epsilon$ we can estimate

$$\frac{\eta_{\alpha}(x)}{\widetilde{\eta}_{\alpha}(x)} = \sqrt{|x|^{\alpha}} (\sqrt{|x|^{\alpha} + 1} - \sqrt{|x|^{\alpha}}) \le \frac{1}{2}$$

and

$$\frac{\eta_{\alpha}(x)}{\widetilde{\eta}_{\alpha}(x)} \ge \frac{\sqrt{|x|^{\alpha}}}{2\sqrt{|x|^{\alpha}+1}} = \frac{1}{2\sqrt{1+|x|^{-\alpha}}} \longrightarrow \frac{1}{2} \quad (x \longrightarrow \infty)$$

for $\alpha \neq 0$. This implies (i) and (iii) follows immediately. The other statements are obvious.

Next, we study the relations between these spaces for $\alpha \in [0, 2]$.

(i) We have Lemma 6.2.

$$L^2_{r_+}(\mathbb{R})\subset L^2_{\omega_\alpha}(\mathbb{R})\subset L^2_{\eta_\alpha}(\mathbb{R})\subset L^2_{r_-}(\mathbb{R}).$$

(ii) There is a constant c > 0 such that

$$(f,f)_{\omega_\alpha} \leq c(f,f)_{r_+}, \quad (g,g)_{\eta_\alpha} \leq c(g,g)_{\omega_\alpha}, \quad (h,h)_{r_-} \leq c(h,h)_{\eta_\alpha}$$

for all $f \in L^2_{r_+}(\mathbb{R}), g \in L^2_{\omega_{\alpha}}(\mathbb{R}), h \in L^2_{\eta_{\alpha}}(\mathbb{R}).$

(iii) For $f \in L^2_{\omega_\alpha}(\mathbb{R})$ also the function f(-x) belongs to $L^2_{\omega_\alpha}(\mathbb{R})$ and we have

$$\int_{-\infty}^{\infty} |f(-x)|^2 \,\omega_{\alpha}(x) \,dx = \int_{-\infty}^{\infty} |f(x)|^2 \,\omega_{\alpha}(x) \,dx,$$
$$\int_{-\infty}^{\infty} f(-x) \overline{f(x)} \,\omega_{\alpha}(x) \,dx = \int_{-\infty}^{\infty} f(x) \overline{f(-x)} \,\omega_{\alpha}(x) \,dx.$$

(iv) The statement of (iii) remains true with ω_{α} replaced by η_{α} .

(v) For $f,g\in L^2_{\omega_\alpha}(\mathbb{R})$ the following intergal exists and we have

$$\int_{-\infty}^{\infty} (\mathcal{Q}_{\alpha} f) \overline{g} |r| dx = 2(f_o, g_o)_{\omega_{\alpha}} + (f, g)_{\eta_{\alpha}}. \tag{6.6}$$

Proof. (i), (ii): For $\alpha \in (0,2)$ we obtain the convergence results

$$\frac{\omega_{\alpha}(x)}{r_{+}(x)} = |x|^{\frac{\alpha-2}{2}} \searrow 0, \quad \frac{\eta_{\alpha}(x)}{\omega_{\alpha}(x)} = \sqrt{1+|x|^{-\alpha}} - 1 \searrow 0 \quad (x \longrightarrow \infty)$$

and by Lemma 6.1(i)

$$\frac{r_-(x)}{\eta_\alpha(x)} \leq d\frac{r_-(x)}{\widetilde{\eta}_\alpha(x)} = d\frac{\sqrt{|x|^\alpha}}{|x|} = d|x|^{\frac{\alpha-2}{2}} \searrow 0 \quad (x \longrightarrow \infty)$$

with some d>0. Therefore, by (3.2) these fractions are bounded and for $\alpha\in\{0,2\}$ this is also true by similar calculations. This implies (i) und (ii) with a common upper bound c>0. Indeed, e.g. for $f\in L^2_{r_+}(\mathbb{R})$ we have

$$\int_{-\infty}^{\infty} |f(x)|^2 \,\omega_{\alpha}(x) \,dx$$

$$= \int_{-\infty}^{-\varepsilon} |f(x)|^2 \, \frac{\omega_{\alpha}(x)}{r_+(x)} r_+(x) \,dx + \int_{\varepsilon}^{\infty} |f(x)|^2 \, \frac{\omega_{\alpha}(x)}{r_+(x)} r_+(x) \,dx$$

$$\leq c \int_{-\infty}^{\infty} |f(x)|^2 \, r_+(x) \,dx.$$

(iii) For $f \in L^2_{\omega_\alpha}(\mathbb{R})$ we have

$$\int_{-\infty}^{\infty} |f(-x)|^2 \,\omega_{\alpha}(x) \,dx = \int_{-\infty}^{\infty} |f(t)|^2 \,\omega_{\alpha}(-t) \,dt = \int_{-\infty}^{\infty} |f(t)|^2 \,\omega_{\alpha}(t) \,dt.$$

since ω_{α} is an even function. The second equation in (iii) follows similarly.

- (iv) can be shown by the same arguments as in (iii) since also η_{α} is even.
- (v) First, for g = f we calculate

$$\int_{-\infty}^{\infty} \left(\sqrt{|x|^{\alpha} + 1} f(x) - \sqrt{|x|^{\alpha}} f(-x) \right) \overline{f(x)} |r(x)| dx$$

$$= \int_{-\infty}^{\infty} \left(\eta_{\alpha}(x) f(x) + \omega_{\alpha}(x) f(x) - \omega_{\alpha}(x) f(-x) \right) \overline{f(x)} dx$$

$$= \int_{-\infty}^{\infty} \left(|f(x)|^{2} - f(-x) \overline{f(x)} \right) \omega_{\alpha}(x) dx + \int_{-\infty}^{\infty} |f(x)|^{2} \eta_{\alpha}(x) dx.$$

Writing $|f(x)|^2 - f(-x)\overline{f(x)}$ as $\frac{1}{2}(|f(x)|^2 - 2f(-x)\overline{f(x)} + |f(x)|^2)$ and using (iii) we end up with

$$\int_{-\infty}^{\infty} (\mathcal{Q}_{\alpha}f)\overline{f} |r| dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} (|f(x)|^2 - f(-x)\overline{f(x)} - f(x)\overline{f(-x)} + |f(-x)|^2) \omega_{\alpha}(x) dx$$

$$+ \int_{-\infty}^{\infty} |f(x)|^2 \eta_{\alpha}(x) dx$$

$$= \frac{1}{2} \int_{-\infty}^{\infty} |f(x) - f(-x)|^2 \omega_{\alpha}(x) dx + \int_{-\infty}^{\infty} |f(x)|^2 \eta_{\alpha}(x) dx$$

$$= 2(f_o, f_o)_{\omega_{\alpha}} + (f, f)_{\eta_{\alpha}}.$$

Finally, the statement (v) follows by the polarization identity. \Box

By means of (6.6) we can now define a Hilbert space structure on dom \mathfrak{t}_{α} :

Proposition 6.3. Let $\alpha \in [0,2]$. Then, the following statements hold true:

(i) dom \mathfrak{t}_{α} is a Hilbert space with the inner product

$$\mathfrak{t}_{\alpha}(f,g) := 2(f_o, g_o)_{\omega_{\alpha}} + (f,g)_{\eta_{\alpha}} \quad (f,g \in \mathrm{dom}\,\mathfrak{t}_{\alpha}). \tag{6.7}$$

(ii) We have the space triplet

$$L^2_{r_+}(\mathbb{R}) \subset \operatorname{dom} \mathfrak{t}_{\alpha} \subset L^2_{r_-}(\mathbb{R}).$$

(iii) Each of the inclusions in (ii) is continuous with respect to the corresponding Hilbert space inner products, i.e. the inclusion of $(L^2_{r_+}(\mathbb{R}), (\cdot, \cdot)_{r_+})$ in $(\operatorname{dom} \mathfrak{t}_{\alpha}, \mathfrak{t}_{\alpha}(\cdot, \cdot))$ and of this space in $(L^2_{r_-}(\mathbb{R}), (\cdot, \cdot)_{r_-})$.

Proof. (i) First, we observe that $L_o := \{ f \in L^2_{\omega_\alpha}(\mathbb{R}) \mid f \text{ is odd } \}$ is a Hilbert space with $(\cdot, \cdot)_{\omega_\alpha}$. Indeed, L_o is the kernel of I + S where S is the self-adjoint and bounded operator (Sf)(x) := f(-x) in $(L^2_{\omega_\alpha}(\mathbb{R}), (\cdot, \cdot)_{\omega_\alpha})$ (cf. Lemma 6.2 (iii)). Similarly, also $L_e := \{ f \in L^2_{\eta_\alpha}(\mathbb{R}) \mid f \text{ is even } \}$ is a Hilbert space with $(\cdot, \cdot)_{\eta_\alpha}$ (cf. Lemma 6.2 (iv)). Therefore, the orthogonal sum dom $\mathfrak{t}_\alpha = L_e \oplus L_o$ is also a Hilbert space with the corresponding inner product $(f_e, g_e)_{\eta_\alpha} + (f_o, g_o)_{\omega_\alpha}$ defined for $f_e, g_e \in L_e$ and $f_o, g_o \in L_o$. However, this inner product is equivalent to $\mathfrak{t}_\alpha(\cdot, \cdot)$. Indeed, for all $f = f_e + f_o \in \operatorname{dom} \mathfrak{t}_\alpha$ we have

$$(f_e, f_e)_{\eta_\alpha} \le (f_e, f_e)_{\eta_\alpha} + (f_o, f_o)_{\eta_\alpha} = (f, f)_{\eta_\alpha}$$

by the orthogonality of f_e and f_o with respect to $(\cdot, \cdot)_{\eta_\alpha}$ and similarly, using Lemma 6.2 (ii) we have

$$(f,f)_{\eta_{\alpha}} = (f_e, f_e)_{\eta_{\alpha}} + (f_o, f_o)_{\eta_{\alpha}} \le (f_e, f_e)_{\eta_{\alpha}} + c(f_o, f_o)_{\omega_{\alpha}}.$$

(ii) and (iii) follow similarly from Lemma 6.2. First, for $f=f_e+f_o\in L^2_{r_+}(\mathbb{R})$ we have by Lemma 6.2 (ii)

$$\mathfrak{t}_{\alpha}(f,f) \le 2c(f_o,f_o)_{r_+} + c^2(f,f)_{r_+} \le (2c+c^2)(f,f)_{r_+}$$

since f_e and f_o are also othogonal with respect to $(\cdot,\cdot)_{r_+}$. Furthermore, for $f = f_e + f_o \in \text{dom }\mathfrak{t}_{\alpha}$ we have by Lemma 6.2 (ii)

$$(f, f)_{r_{-}} \leq (f_{o}, f_{o})_{r_{-}} + (f, f)_{r_{-}} \leq c^{2}(f_{o}, f_{o})_{\omega_{\alpha}} + c(f, f)_{\eta_{\alpha}} \leq d \, \mathfrak{t}_{\alpha}(f, f)$$
with $d := \max\{c, \frac{c^{2}}{2}\}.$

Now, we have a closer look at some set differences using the functions

$$f_{\tau}(x) := \frac{\operatorname{sgn}(x)}{\sqrt{|r(x)|}|x|^{\frac{\tau+2}{4}}}, \quad g_{\tau}(x) := \frac{1}{\sqrt{|r(x)|}|x|^{\frac{2-\tau}{4}}} \quad (x \in \mathbb{R} \setminus [-\varepsilon, \varepsilon]) \quad (6.8)$$

for $\tau \in [0,2]$ and $\varepsilon > 0$ according to (3.2). Formally, these functions are extended by $f_{\tau}(x) := 0 =: g_{\tau}(x)$ for $x \in [-\varepsilon, \varepsilon]$. In particular, f_{τ} is odd and g_{τ} is even.

Lemma 6.4. For $0 \le \alpha < \beta \le 2$ the following statements hold true:

- (i) $L^2_{\omega_\beta}(\mathbb{R}) \subset L^2_{\omega_\alpha}(\mathbb{R}) \subset L^2_r(\mathbb{R},$
- (ii) $f_{\beta} \in L^2_{\omega_{\alpha}}(\mathbb{R}) \setminus L^2_{\omega_{\beta}}(\mathbb{R})$,
- (iii) $f_{\beta} \in \text{dom } \mathfrak{t}_{\alpha} \setminus \text{dom } \mathfrak{t}_{\beta}$, (iv) $L_r^2(\mathbb{R} \subset L_{\eta_{\alpha}}^2(\mathbb{R}) \subset L_{\eta_{\beta}}^2(\mathbb{R})$,
- (v) $g_{\alpha} \in L^{2}_{\eta_{\beta}}(\mathbb{R}) \setminus L^{2}_{\eta_{\alpha}}(\mathbb{R}),$
- (vi) $g_{\alpha} \in \operatorname{dom} \mathfrak{t}_{\beta} \setminus \operatorname{dom} \mathfrak{t}_{\alpha}$.

Proof. (i) For $\alpha = 0$ we have the identity $L^2_{\omega_0}(\mathbb{R}) = L^2_r(\mathbb{R})$. Therefore, the inclusions in (i) follow like in Lemma 6.2 (i) from the convergence

$$\frac{\omega_{\alpha}(x)}{\omega_{\beta}(x)} = |x|^{\frac{\alpha-\beta}{2}} \searrow 0 \quad (x \longrightarrow \infty).$$

(ii) For $0 \le \gamma \le 2$ we have

$$\int_{-\infty}^{\infty} |f_{\beta}(x)|^2 \,\omega_{\gamma}(x) \,dx = \int_{-\infty}^{-\varepsilon} |x|^{\frac{\gamma-\beta-2}{2}} \,dx + \int_{\varepsilon}^{\infty} |x|^{\frac{\gamma-\beta-2}{2}} \,dx$$

which is a finite number if and only if $\gamma < \beta$.

- (iii) follows from (ii) since f_{β} is odd.
- (iv) For $\alpha=0$ we have the identity $L^2_{\eta_0}(\mathbb{R})=L^2_r(\mathbb{R})$. Therefore, the inclusions in (iv) follow like in Lemma 6.2 (i) from the convergence

$$\frac{\eta_{\beta}(x)}{\eta_{\alpha}(x)} \leq d\frac{\widetilde{\eta}_{\beta}(x)}{\widetilde{\eta}_{\alpha}(x)} = d|x|^{\frac{\alpha-\beta}{2}} \searrow 0 \quad (x \longrightarrow \infty)$$

with some d > 0 using Lemma 6.1(i)(ii).

(v) For $0 \le \gamma \le 2$ we have

$$\int_{-\infty}^{\infty} |g_{\alpha}(x)|^2 \, \widetilde{\eta}_{\gamma}(x) \, dx = \int_{-\infty}^{-\varepsilon} |x|^{\frac{\alpha - 2 - \gamma}{2}} \, dx + \int_{\varepsilon}^{\infty} |x|^{\frac{\alpha - 2 - \gamma}{2}} \, dx$$

which is a finite number if and only if $\gamma > \alpha$. Then, we use Lemma 6.1(iii).

(vi) follows from (v) since g_{α} is even.

In the next step we shall construct a Kreın space structure on the Hilbert space $(\text{dom }\mathfrak{t}_{\alpha}, \mathfrak{t}_{\alpha}(\cdot, \cdot))$ by means of the representation (6.6) of $\mathfrak{t}_{\alpha}(\cdot, \cdot)$. To this end, we first introduce two operators on the linear space of weighted functions with compact support

$$L^2_{r,0}=\{f\in L^2_r(\mathbb{R})\,|\, f(x)=0 \text{ a.e. on } \mathbb{R}\setminus [-k,k] \text{ with some } k>0\}.$$

Using the expression $Q_{\alpha}f$ from (6.4) the operators Q_{α} and S_{α} given by

$$Q_{\alpha}f := \mathcal{Q}_{\alpha}f, \quad (S_{\alpha}f)(x) := \operatorname{sgn}(x)(\mathcal{Q}_{\alpha}f)(x) \quad (f \in L^{2}_{r,0})$$

are well defined in $L_{r,0}^2$ since for each $f \in L_{r,0}^2$ also $\mathcal{Q}_{\alpha}f$ has a compact support. Furthermore, we consider the kernels

$$\mathcal{M}_{\alpha}^{+} := \ker(I - S_{\alpha}), \quad \mathcal{M}_{\alpha}^{-} := \ker(I + S_{\alpha})$$
 (6.9)

where I denotes the identity on $L^2_{r,0}$. Note that obviously, we have $L^2_{r,0} \subset L^2_{r_+}(\mathbb{R})$ and hence, $L^2_{r,0} \subset L^2_{\omega_{\alpha}}(\mathbb{R})$, $L^2_{r,0} \subset \text{dom }\mathfrak{t}_{\alpha}$ by Lemma 6.2 and Proposition 6.3, respectively. Moreover, (6.6) and (6.7) allow the representation

$$\mathfrak{t}_{\alpha}(f,g) = (Q_{\alpha}f,g)_r \qquad (f,g \in L_{r\,0}^2).$$
 (6.10)

Lemma 6.5. (i) The operator Q_{α} is symmetric with respect to $(\cdot, \cdot)_r$, i.e.

$$(Q_{\alpha}f, g)_r = (f, Q_{\alpha}g)_r \quad (f, g \in L^2_{r,0})$$

- (ii) For $f \in L^2_{r,0}$ we have $S^2_{\alpha}f = f$.
- (iii) The space $L_{r,0}^2$ allows the decomposition

$$L_{r,0}^2 = \mathcal{M}_{\alpha}^+ \oplus \mathcal{M}_{\alpha}^-. \tag{6.11}$$

- (iv) The spaces \mathcal{M}_{α}^+ and \mathcal{M}_{α}^- are orthogonal with respect to $\mathfrak{t}_{\alpha}(\cdot,\cdot)$, i.e. the decomposition (6.11) is a direct and orthogonal sum in this sense.
- (v) For $f_+ \in \mathcal{M}_{\alpha}^+$, $f_- \in \mathcal{M}_{\alpha}^-$ we have

$$\mathfrak{t}_{\alpha}(f_{+}, f_{+}) = [f_{+}, f_{+}]_{r}, \quad \mathfrak{t}_{\alpha}(f_{-}, f_{-}) = -[f_{-}, f_{-}]_{r}.$$

- (vi) The spaces \mathcal{M}_{α}^+ and \mathcal{M}_{α}^- are also orthogonal with respect to $[\cdot,\cdot]_r$.
- (vii) For $f \in \text{dom } \mathfrak{t}_{\alpha}$, $k \in \mathbb{N}$ we have $\chi_{[-k,k]} f \in L^2_{r,0}$ and

$$\chi_{[-k,k]}f \longrightarrow f \quad (k \longrightarrow \infty) \quad w.r.t. \ \mathfrak{t}_{\alpha}(\cdot,\cdot).$$

(viii) For $g \in L^2_{r,+}(\mathbb{R})$, $k \in \mathbb{N}$ we have $\chi_{[-k,k]}g \in L^2_{r,0}$ and

$$\chi_{[-k,k]}g \longrightarrow g \quad (k \longrightarrow \infty) \quad w.r.t. \ (\cdot,\cdot)_{r_+}.$$

Proof. (i) From Lemma 6.2 (iii) (or by an immediate calculation) we obtain

$$\int_{-\infty}^{\infty} \sqrt{|x|^{\alpha}} f(-x)\overline{g}(x)|r(x)| dx = \int_{-\infty}^{\infty} \sqrt{|t|^{\alpha}} f(t)\overline{g}(-t)|r(t)| dx$$

for $f, g \in L^2_{r,0}$. This implies (i) by the definition of $\mathcal{Q}_{\alpha}f$ in (6.4). (Of course, (i) can also be deduced from (6.10).)

(ii) Using sgn (-x) = -sgn(x) we can calculate for $f \in L^2_{r,0}$

$$(S_{\alpha}^{2}f)(x) = \operatorname{sgn}(x)(\sqrt{|x|^{\alpha} + 1}(S_{\alpha}f)(x) - \sqrt{|x|^{\alpha}}(S_{\alpha}f)(-x))$$

$$= \sqrt{|x|^{\alpha} + 1}(\sqrt{|x|^{\alpha} + 1}f(x) - \sqrt{|x|^{\alpha}}f(-x))$$

$$+ \sqrt{|x|^{\alpha}}(\sqrt{|x|^{\alpha} + 1}f(-x) - \sqrt{|x|^{\alpha}}f(x)) = f(x).$$

(iii) Each $f \in L^2_{r,0}$ can be written as $f = f_+ + f_-$ with

$$f_{+} := \frac{1}{2}(f + S_{\alpha}f), \quad f_{-} := \frac{1}{2}(f - S_{\alpha}f) \in L^{2}_{r,0}$$

and we have

$$(I - S_{\alpha})f_{+} = \frac{1}{2}(f - S_{\alpha}^{2}f) = 0, \qquad (I + S_{\alpha})f_{-} = \frac{1}{2}(f - S_{\alpha}^{2}f) = 0.$$

Therefore, f_{\pm} belongs to $\mathcal{M}_{\alpha}^{\pm}$ and we have proved $L_{r,0}^2 = \mathcal{M}_{\alpha}^+ + \mathcal{M}_{\alpha}^-$.

(iv) For $f_+ \in \mathcal{M}_{\alpha}^+$, $f_- \in \mathcal{M}_{\alpha}^-$ we know that $S_{\alpha}f_+ = f_+$ and $S_{\alpha}f_- = -f_-$. Consequently, we can first calculate by (6.10)

$$\mathfrak{t}_{\alpha}(f_+, f_-) = (Q_{\alpha}f_+, f_-)_r = (JS_{\alpha}f_+, f_-)_r = (Jf_+, f_-)_r = [f_+, f_-]_r$$

using the fundamental symmetry J from (3.6). Similarly, we obtain by (i)

$$\mathfrak{t}_{\alpha}(f_+, f_-) = (f_+, Q_{\alpha}f_-)_r = (f_+, JS_{\alpha}f_-)_r = -(f_+, Jf_-)_r = -[f_+, f_-]_r.$$

Therefore, we have $\mathfrak{t}_{\alpha}(f_+, f_-) = 0$ which implies (iv).

- (v) follows by a similar calculation as in (iv).
- (vi) also follows from the calculation in (iv).
- (vii) Put $f_k := \chi_{[-k,k]} f$. Then, we have $f_k \in L^2_{r,0}$ and we can estimate

$$\mathfrak{t}_{\alpha}(f - f_{k}, f - f_{k}) = 2(f_{o} - f_{k_{o}}, f_{o} - f_{k_{o}})_{\omega_{\alpha}} + (f - f_{k}, f - f_{k})_{\eta_{\alpha}}
= \frac{1}{2} \int_{-\infty}^{-k} |f(x) - f(-x)|^{2} \omega_{\alpha}(x) dx + \frac{1}{2} \int_{k}^{\infty} |f(x) - f(-x)|^{2} \omega_{\alpha}(x) dx
+ \int_{-\infty}^{-k} |f(x)|^{2} \eta_{\alpha}(x) dx + \int_{k}^{\infty} |f(x)|^{2} \eta_{\alpha}(x) dx \longrightarrow 0 \quad (k \longrightarrow \infty)$$

where f_o and f_{ko} denote the odd parts of f and f_k , respectively.

(viii) Put $g_k := \chi_{[-k,k]}g$. Then, we have $g_k \in L^2_{r,0}$ and we can estimate

$$(g - g_k, g - g_k)_{r_+} = \int_{-\infty}^{-k} |g|^2 |r_+| dx + \int_k^{\infty} |g|^2 |r_+| dx \longrightarrow 0 \quad (k \longrightarrow \infty).$$

An orthogonal decomposition is preserved when we go over to the closure and by Lemma 6.5(vii) $L_{r,0}^2$ is dense in the Hilbert space (dom \mathfrak{t}_{α} , $\mathfrak{t}_{\alpha}(\cdot,\cdot)$). Therefore, (6.11) implies

$$\operatorname{dom} \mathfrak{t}_{\alpha} = \overline{L_{r,0}^2} = \overline{\mathcal{M}_{\alpha}^+} \oplus \overline{\mathcal{M}_{\alpha}^-}$$
 (6.12)

with a direct and othogonal sum in $(\operatorname{dom} \mathfrak{t}_{\alpha}, \mathfrak{t}_{\alpha}(\cdot, \cdot))$ where $\overline{L_{r,0}^2}, \overline{\mathcal{M}_{\alpha}^+}, \overline{\mathcal{M}_{\alpha}^-}$ denote the closures in this space. If P_{α}^{\pm} denotes the orthogonal projection onto $\overline{\mathcal{M}_{\alpha}^{\pm}}$ in this space then, with

$$J_{\alpha} := P_{\alpha}^+ - P_{\alpha}^-$$

the sesquilinear form

$$\mathfrak{t}_{\alpha}[f,g] := \mathfrak{t}_{\alpha}(J_{\alpha}f,g) \qquad (f,g \in \mathrm{dom}\,\mathfrak{t}_{\alpha})$$
 (6.13)

defines a Kreĭn space inner product on dom \mathfrak{t}_{α} . Furthermore, (6.12) is a corresponding fundamental decomposition, J_{α} is a corresponding fundamental symmetry and $\mathfrak{t}_{\alpha}(\cdot,\cdot)$ is the corresponding Hilbert space inner product. On $L_{r,0}^2$ we can make this structure more explicit.

Lemma 6.6. For $f, g \in L^2_{r,0}$ we have

$$P_{\alpha}^{+}f = \frac{1}{2}(f + S_{\alpha}f), \qquad P_{\alpha}^{-}f = \frac{1}{2}(f - S_{\alpha}f), \quad J_{\alpha}f = S_{\alpha}f, \quad (6.14)$$

 $\mathfrak{t}_{\alpha}[f, g] = [f, g]_{r}. \quad (6.15)$

Proof. By Lemma 6.5(ii) we observe that

$$(I - S_{\alpha})(f + S_{\alpha}f) = f - S_{\alpha}^{2}f = 0, \quad (I + S_{\alpha})(f - S_{\alpha}f) = f - S_{\alpha}^{2}f = 0$$

and hence $f \pm S_{\alpha} f \in \mathcal{M}_{\alpha}^{\pm}$. Therefore, the decomposition

$$f = \frac{1}{2}(f + S_{\alpha}f) + \frac{1}{2}(f - S_{\alpha}f)$$

corresponds to (6.11). This implies the representation (6.14) of $P_{\alpha}^{\pm}f$ and hence,

$$J_{\alpha}f = \frac{1}{2}(f + S_{\alpha}f) - \frac{1}{2}(f - S_{\alpha}f) = S_{\alpha}f.$$

Finally, by Lemma 6.2(v) and Lemma 6.5(ii) we obtain

$$\mathfrak{t}_{\alpha}[f,g] = \mathfrak{t}_{\alpha}(S_{\alpha}f,g) = (Q_{\alpha}S_{\alpha}f,g)_r = (JS_{\alpha}S_{\alpha}f,g)_r = [f,g]_r$$

using the fundamental symmetry J in $(L_r^2(\mathbb{R}), [\cdot, \cdot]_r)$ from (3.6).

The indefinite inner product $\mathfrak{t}_{\alpha}[\cdot,\cdot]$ can be characterized more explicitly:

Lemma 6.7. For $f, g \in \text{dom } \mathfrak{t}_{\alpha}$ the following limit exists and we have

$$\lim_{k \to \infty} \int_{-k}^{k} f \overline{g} r \, dx = \mathfrak{t}_{\alpha}[f, g]. \tag{6.16}$$

Proof. By Lemma 6.5(vii) we know that $f_k := \chi_{[-k,k]} f$, $g_k := \chi_{[-k,k]} g \in L^2_{r,0}$ satisfy $f_k \longrightarrow f$ and $g_k \longrightarrow g$ $(k \longrightarrow \infty)$ with respect to $\mathfrak{t}_{\alpha}(\cdot,\cdot)$. Since $\mathfrak{t}_{\alpha}[\cdot,\cdot]$ is continuous with respect to $\mathfrak{t}_{\alpha}(\cdot,\cdot)$ this implies

$$\mathfrak{t}_{\alpha}[f,g] = \lim_{k \to \infty} \mathfrak{t}_{\alpha}[f_k,g_k] = \lim_{k \to \infty} [f_k,g_k]_r = \lim_{k \to \infty} \int_{-k}^k f\overline{g} \, r \, dx$$
 by (6.15).

Finalle, we collect some of the above results:

Proposition 6.8. For $\alpha \in [0,2]$ the space dom \mathfrak{t}_{α} from (6.5) is a Kreĭn space with the indefinite inner product $\mathfrak{t}_{\alpha}[\cdot,\cdot]$ satisfying (6.16). A fundamental decomposition is given by (6.12) and $\mathfrak{t}_{\alpha}(\cdot,\cdot)$ from (6.7) is the corresponding Hilbert space inner product.

6.3. Conclusions for forms associated with the multiplication operator

Now, we bring together the results from the Sections 6.1 and 6.2. We start with the "Kreĭn space point of view".

Theorem 6.9. Let T be the self-adjoint multiplication operator in the Hilbert space $(L_{r_{-}}^{2}(\mathbb{R}), (\cdot, \cdot)_{r_{-}})$ according to (6.2) and let \mathcal{N}_{T} be given according to Theorem 5.2. Furthermore, for $\alpha \in [0, 2]$ consider the space $\operatorname{dom} \mathfrak{t}_{\alpha}$ from (6.5) and the inner products $\mathfrak{t}_{\alpha}[\cdot, \cdot]$, $\mathfrak{t}_{\alpha}(\cdot, \cdot)$ and $\mathfrak{t}[\cdot, \cdot]$ satisfying (6.16), (6.7) and (6.1), respectively. Then the following statements hold true:

- (i) The form $\mathfrak{t}[\cdot,\cdot]$ coincides with the form $\mathfrak{t}_0[\cdot,\cdot]$, i.e $\mathfrak{t}_{\alpha}[\cdot,\cdot]$ with $\alpha=0$.
- (ii) All spaces $(\text{dom }\mathfrak{t}_{\alpha}, \mathfrak{t}_{\alpha}[\cdot, \cdot])$ with $\alpha \in [0, 2]$ are Kreĭn space completions of $(\text{dom }T, (T\cdot, \cdot)_{r_{-}})$ which are continuously embedded in $(L^{2}_{r_{-}}(\mathbb{R}), (\cdot, \cdot)_{r_{-}})$. An associated Hilbert space inner product is given by $\mathfrak{t}_{\alpha}(\cdot, \cdot)$.
- (iii) For all $\alpha \in (0,2]$ the J-self-adjoint, J-non-negative and boundedly invertible range restriction $A_{t_{\alpha}}$ of T to dom t_{α} is given by

$$\operatorname{dom} A_{\mathfrak{t}_{\alpha}} = \{ f \in L^{2}_{r_{+}}(\mathbb{R}) \mid Tf \in \operatorname{dom} \mathfrak{t}_{\alpha} \}, \quad (A_{\mathfrak{t}_{\alpha}}f)(x) = xf(x)$$

and infinity is a singular critical point of $A_{\mathfrak{t}_{\alpha}}$ in $(\operatorname{dom} \mathfrak{t}_{\alpha}, \mathfrak{t}_{\alpha}[\cdot, \cdot])$.

- (iv) For $\alpha = 0$ infinity is not a singular critical point of $A_{\mathfrak{t}_0} = A$ in $(L^2_r(\mathbb{R}), [\cdot, \cdot]_r)$ from Examle 1.
- (v) $(L_r^2(\mathbb{R}), [\cdot, \cdot]_r, A) \in \mathcal{N}_T$ from Examle 1 is the regularization of all elements $(\text{dom }\mathfrak{t}_{\alpha}, \mathfrak{t}_{\alpha}[\cdot, \cdot], A_{\mathfrak{t}_{\alpha}}) \in \mathcal{N}_T$ with $\alpha \in (0, 2]$.
- *Proof.* (i) follows from Lemma 6.1(iv) and Lemma 6.7.
- (ii) By Proposition 6.3 the space $L^2_{r_+}(\mathbb{R})$ (= dom T) is dense in the Kreı̆n space (dom \mathfrak{t}_{α} , $\mathfrak{t}_{\alpha}[\cdot,\cdot]$) and by Lemma 6.7 on dom T the forms $\mathfrak{t}_{\alpha}[\cdot,\cdot]$ and $[\cdot,\cdot]_r$ and hence, also $(T\cdot,\cdot)_{r_-}$ coincide. The continuity of the embedding was also already shown in Proposition 6.3.
 - (iv) was already observed in Section 6.1.
 - (v) and (iii) are immediate consequences of (iv) by Theorem 5.4(ii). \Box

In terms of form closures the above results have the following form:

Corollary 6.10. Let T be the self-adjoint multiplication operator in the Hilbert space $(L^2_{r_-}(\mathbb{R}), (\cdot, \cdot)_{r_-})$ according to (6.2). Furthermore, let the functions f_{τ} , g_{τ} be given by (6.8) for $\tau \in [0, 2]$. Then the following statements hold true:

- (i) All forms $\mathfrak{t}_{\alpha}[\cdot,\cdot]$ with $\alpha \in [0,2]$ are closures of $(T\cdot,\cdot)_{r_{-}}$ in the Hilbert space $(L^{2}_{r_{-}}(\mathbb{R}), (\cdot,\cdot)_{r_{-}})$ with gap point 0.
- (ii) The forms $\mathfrak{t}_{\alpha}[\cdot,\cdot]$ with $\alpha \in (0,2]$ are not regular.
- (iii) The form $\mathfrak{t}_0[\cdot,\cdot]$ (= $\mathfrak{t}[\cdot,\cdot]$) is regular and the regularization of all forms $\mathfrak{t}_{\alpha}[\cdot,\cdot]$ with $\alpha \in (0,2]$.

(iv) For $\alpha \in (0,2]$ we have $f_{\alpha} \in \text{dom } \mathfrak{t}_0 \setminus \text{dom } \mathfrak{t}_{\alpha}$ and $g_0 \in \text{dom } \mathfrak{t}_{\alpha} \setminus \text{dom } \mathfrak{t}_0$ (where $g_0 = g_{\tau}$ with $\tau = 0$).

Proof. (iv) follows from Lemma 6.4. The other statements follow immediately from Theorem 6.9 by the definitions. \Box

In the present example we have identified infinitely many elements in the sets \mathcal{M}_T and \mathcal{N}_T from Theorem 5.2:

Corollary 6.11. In the setting of Theorem 6.9 we have

$$\{\mathfrak{t}_{\alpha}[\cdot,\cdot] \mid \alpha \in [0,2] \} \subset \mathcal{M}_{T},$$

$$\{(\operatorname{dom} \mathfrak{t}_{\alpha}, \, \mathfrak{t}_{\alpha}[\cdot,\cdot], \, A_{\mathfrak{t}_{\alpha}}) \mid \alpha \in [0,2] \} \subset \mathcal{N}_{T}.$$

For $0 \le \alpha < \beta \le 2$ the set differences $\operatorname{dom} \mathfrak{t}_{\alpha} \setminus \operatorname{dom} \mathfrak{t}_{\beta}$ and $\operatorname{dom} \mathfrak{t}_{\beta} \setminus \operatorname{dom} \mathfrak{t}_{\alpha}$ are not empty (and Lemma 6.4 presents elements in these set differences).

Finally, we have a look at the eigenspectral function $E_{\alpha} := E_{\text{dom }\mathfrak{t}_{\alpha}}$ of $A_{\mathfrak{t}_{\alpha}}$ in the Kreı̆n space (dom \mathfrak{t}_{α} , $\mathfrak{t}_{\alpha}[\cdot,\cdot]$). Using again Theorem 5.6 as in Section 6.1 this function is given by the restriction of the spectral measure of T in (6.3) to dom \mathfrak{t}_{α} , i.e.

$$E_{\alpha}(\Delta)f = \chi_{\Delta}f \qquad (f \in \text{dom }\mathfrak{t}_{\alpha})$$
 (6.17)

for each set $\Delta \in \Sigma$. Since for $\alpha \in (0,2]$ infinity is a singular critical point we know that $E_{\alpha}([\varepsilon,\lambda])f$ or $E_{\alpha}([-\lambda,-\varepsilon])f$ does not converge in $(\operatorname{dom}\mathfrak{t}_{\alpha},\mathfrak{t}_{\alpha}[\cdot,\cdot])$ with $\lambda \longrightarrow \infty$ for some $f \in \operatorname{dom}\mathfrak{t}_{\alpha}$. Now, using the function g_{τ} for $\tau = 0$

$$g_0(x) = \frac{1}{\sqrt{r(x)x}} \quad (x \in \mathbb{R} \setminus [-\varepsilon, \varepsilon])$$

from (6.8) (with $\varepsilon > 0$ from (3.2)) we can make this result more explicit.

Proposition 6.12. For $\alpha \in (0,2]$ denote the norm for operators in the Hilbert space $(\text{dom }\mathfrak{t}_{\alpha},\mathfrak{t}_{\alpha}(\cdot,\cdot))$ by $||\cdot||_{\alpha}$. Then, the following statements hold true:

(i) For each $f \in \text{dom } \mathfrak{t}_{\alpha}$ we have

$$E_{\alpha}([-k,k])f \longrightarrow f \quad (k \longrightarrow \infty)$$

with convergence in $(\operatorname{dom} \mathfrak{t}_{\alpha}, \mathfrak{t}_{\alpha}(\cdot, \cdot))$.

(ii) $E_{\alpha}((\varepsilon,k])g_0$ does not converge in this space for $k \longrightarrow \infty$. More precisely, we have

$$g_0 \in \text{dom } \mathfrak{t}_{\alpha}, \quad \mathfrak{t}_{\alpha}(E_{\alpha}((\varepsilon, k])g_0, E_{\alpha}((\varepsilon, k])g_0) \longrightarrow \infty \quad (k \longrightarrow \infty).$$
 (6.18)

(iii) For all $k \in \mathbb{N}$, $k > \varepsilon$ we have

$$||E_{\alpha}([-k,k])||_{\alpha} = 1, \qquad ||E_{\alpha}((\varepsilon,k])||_{\alpha} \ge \frac{2(\sqrt{k^{\alpha}} - \sqrt{\varepsilon^{\alpha}})}{\alpha(q_0,q_0)_{p_{\alpha}}}.$$

Proof. (i) was already shown in Lemma 6.5(vii).

(ii) In Corollary 6.10(iv) (or Lemma 6.4) it was already mentioned that $g_0 \in \text{dom }\mathfrak{t}_{\alpha}$. For the even function g_0 we introduce the notation $g_k := \chi_{(\varepsilon,k]} g_0$ and

$$\begin{array}{lcl} g_{k,o}(x) & := & \frac{1}{2}(\chi_{(\varepsilon,k]}(x)g_0(x) - \chi_{(\varepsilon,k]}(-x)g_0(-x)) \\ \\ & = & \frac{1}{2}(\chi_{(\varepsilon,k]}(x)g_0(x) - \chi_{[-k,-\varepsilon)}(x)g_0(x)) \end{array} \quad (x \in \mathbb{R}) \end{array}$$

for the odd part of g_k . Then, we can estimate

$$\mathbf{t}_{\alpha}(g_{k}, g_{k}) = 2(g_{k,o}, g_{k,o})_{\omega_{\alpha}} + (g_{k}, g_{k})_{\eta_{\alpha}} \ge 2(g_{k,o}, g_{k,o})_{\omega_{\alpha}}
= \frac{1}{2} \int_{-k}^{-\varepsilon} |g_{0}|^{2} \omega_{\alpha} dx + \frac{1}{2} \int_{\varepsilon}^{k} |g_{0}|^{2} \omega_{\alpha} dx
= \frac{1}{2} \int_{-k}^{-\varepsilon} |x|^{\frac{\alpha-2}{2}} dx + \frac{1}{2} \int_{\varepsilon}^{k} |x|^{\frac{\alpha-2}{2}} dx
= \frac{2}{\alpha} (\sqrt{k^{\alpha}} - \sqrt{\varepsilon^{\alpha}}) \longrightarrow \infty \qquad (k \longrightarrow \infty).$$
(6.19)

This implies (6.18).

(iii) For $f \in \text{dom }\mathfrak{t}_{\alpha}$ we can estimate

$$\mathfrak{t}_{\alpha}(E_{\alpha}([-k,k])f, E_{\alpha}([-k,k])f) = 2\int_{-k}^{k} |f_{o}|^{2} \omega_{\alpha} dx + \int_{-k}^{k} |f|^{2} \eta_{\alpha} dx \le \mathfrak{t}_{\alpha}(f,f)$$

and here, we have "=" if $f = \chi_{[-k,k]}f$. This implies $||E_{\alpha}([-k,k])||_{\alpha} = 1$. With $g_k = E_{\alpha}((\varepsilon,k])g_0$ from (ii) we can further estimate

$$\mathfrak{t}_{\alpha}(g_k, g_k) \leq ||E_{\alpha}((\varepsilon, k])||_{\alpha} \mathfrak{t}_{\alpha}(g_0, g_0).$$

Therefore, using again the calculation in (6.19) we have

$$||E_{\alpha}((\varepsilon,k])||_{\alpha} \ge \frac{\mathfrak{t}_{\alpha}(g_k,g_k)}{\mathfrak{t}_{\alpha}(g_0,g_0)} \ge \frac{2(\sqrt{k^{\alpha}}-\sqrt{\varepsilon^{\alpha}})}{\alpha(g_0,g_0)_{\eta_{\alpha}}}.$$

since g_0 is even and hence $\mathfrak{t}_{\alpha}(g_0,g_0)=(g_0,g_0)_{\eta_{\alpha}}$.

Proposition 6.12(iii) can be regarded as an addition (and partly sharpening) for the norm estimates of the eigenspectral function from [5]. Note that Proposition 6.12 studies a similar question as at the end of Example 2 and (in contrast to Example 2) indeed, gives an answer in the present setting.

Furthermore, it should be mentioned that in the definition of the functions η_{α} and ω_{α} the term $|x|^{\alpha}$ can also be replaced by a more general even function. This observation shows that in Corollary 6.11 we have " \subseteq " but not "=". In other words, there are many other non-regular closures of $(T \cdot, \cdot)_{r_{-}}$ in the Hilbert space $(L_{r_{-}}^{2}(\mathbb{R}), (\cdot, \cdot)_{r_{-}})$. Therefore, the value $\alpha \in [0, 2]$ is certainly not the only information which is stored in the closures of $(T \cdot, \cdot)_{r_{-}}$ in addition to the information of T itself.

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