ON THE STRUCTURE OF MODULAR LATTICES — UNIQUE GLUING AND DISSECTION

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Abstract. This work proves that the process of gluing finite lattices to form a larger lattice is bijective, that is each lattice is the glued sum of a unique system of finite lattices, provided the class of lattices is constrained to modular, locally-finite lattices with finite covers. The results of this work are not surprising given the prior literature, but this seems to be the first proof that the processes of gluing and dissection can be made inverses, and hence that gluing is bijective.

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1. Introduction

This work proves that the process of gluing lattices to form a larger lattice can be made bijective, that is for each lattice there is a unique (up to isomorphism) connected system whose sum is the given lattice, provided that the class of connected systems and the class of sum lattices are suitably constrained.

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The foundation work on lattice gluing is Herrmann [Herr1973a] (with English translation [Herr1973a-en]). Further development and exposition are in Day and Herrmann [DayHerr1988a], Day and Frees [DayFrees1990a], and Haiman [Haim1991a, sec. 1]. The results of this work are not surprising given the prior literature, but this seems to be the first proof that the processes of gluing and dissection can be made inverses, and hence that gluing can be made bijective.

Note that the theorems of [Herr1973a] are stated only for lattices of finite length. We do not limit lattices to finite length. Many of the proofs of [Herr1973a] do not depend on finite length, and generally those proofs have been copied or included by reference here.

The class of lattices we work with here is chosen for its utility within the combinatorics of tableaux. Specifically, we are interested in modular, l.f.f.c. (locally-finite and with finite covers) lattices because they are used in Fomin's growth diagram construction.[Fom1994a][Fom1995a] The corresponding constraints on connected systems are that (1) the skeleton lattice is l.f.f.c.; (2) the blocks are finite, modular, and complemented; and (3) the connected system is monotone. (The monotone property is that no block is "contained in" a block either above or below it, in a suitable sense.)

This work is a first step in the program of [Herr1973a, sec. 0][Herr1973a-en, sec. 0], "Als entscheidende Anwendung ergibt sich eine Möglichkeit, modulare Verbände endlicher Länge durch projektive Geometrien darzustellen ..." ("A crucial application is representing finite-length modular lattices using projective geometries ..."), which is further elaborated in [Wor2024c].

2. Preliminaries

2.1. Conventions for definitions. In a sections we may define a symbol for an arbitrarily chosen object of some class. Consequently, some further definitions in the section will be implicitly parameterized by the chosen object because they explicitly depend on the chosen object or on another object parameterized by the chosen object. Similarly, when the constructions of such a section are "imported" into another section, a value for the chosen object will be fixed, and not only the symbol for that object becomes defined but implicitly all of the definitions parameterized by that object as well.

2.2. Finite covers.

Definition 2.1. *We define that a lattice has* finite covers *if every element in the lattice has a finite number of upper and lower covers.*

Definition 2.2. *We define that a lattice is* locally finite with finite covers — *abbreviated* l.f.f.c. — *if it is both locally finite and has finite covers (both upper and lower).*

Remark 2.3. Note that finite covers is "locally finite width" property, but does not imply finite width. For instance, $\mathbb{Z} \times \mathbb{Z}$ is modular and l.f.f.c., but all its ranks are infinite.

Definition 2.4. *We define that a lattice is* finitary *if all of its principal (downward) ideals are finite.*

Lemma 2.5. *If a lattice is finitary and has finite upper covers, then it is l.f.f.c.*

2.3. Tolerances.

Definition 2.6. [Band1981a, sec. 1] *We define that a reflexive and symmetric binary relation ξ on a lattice L is a* tolerance *if ξ is compatible with the meet and join of L, that is,*

(1) *$a \mathrel{\xi} b$ and $c \mathrel{\xi} d$ imply $a \vee c \mathrel{\xi} b \vee d$, and*
(2) *$a \mathrel{\xi} b$ and $c \mathrel{\xi} d$ imply $a \wedge c \mathrel{\xi} b \wedge d$.*

Lemma 2.7. [Band1981a, Lem. 1.1] *If ξ is a tolerance on a lattice L, then:*

(1) *$x \mathrel{\xi} z$ and $x \leq y \leq z$ imply $x \mathrel{\xi} y$ and $y \mathrel{\xi} z$.*
(2) *$x \mathrel{\xi} y$ iff $x \vee y \mathrel{\xi} x \wedge y$,*
(3) *$t \mathrel{\xi} x$, $t \mathrel{\xi} y$, and $t \leq x \wedge y$ imply $t \mathrel{\xi} x \vee y$.*
(4) *$t \mathrel{\xi} x$, $t \mathrel{\xi} y$, and $t \geq x \vee y$ imply $t \mathrel{\xi} x \wedge y$.*
(5) *$x \mathrel{\xi} x \vee y$ and $y \mathrel{\xi} x \vee y$ imply $x \vee y \mathrel{\xi} x \wedge y$, $x \mathrel{\xi} y$, and $x \wedge y \mathrel{\xi} x, y$, that is, ξ holds among every pair of x, y, $x \vee y$, and $x \wedge y$.*

(6) *$x \mathrel{\xi} x \wedge y$ and $y \mathrel{\xi} x \wedge y$ imply $x \wedge y \mathrel{\xi} x \vee y$, $x \mathrel{\xi} y$, and $x \vee y \mathrel{\xi} x, y$, that is, ξ holds among every pair of x, y, $x \wedge y$, and $x \vee y$.*

Proof.

Regarding (1): Since $x \mathrel{\xi} z$ and $y \mathrel{\xi} y$, $x \vee y \mathrel{\xi} z \vee y$, which is equivalent to $y \mathrel{\xi} z$. Similarly, $x \wedge y \mathrel{\xi} z \wedge y$, which is equivalent to $x \mathrel{\xi} y$.

Regarding (2), (3), and (4): see Bandelt [Band1981a, Lem. 1.1(1–3) Proof].

Regarding (5): By (4), $x \vee y \mathrel{\xi} x \wedge y$. By (2), $x \mathrel{\xi} y$. By (1), $x \wedge y \mathrel{\xi} x, y$.

Regarding (6): Proved dually to (5). □

Definition 2.8. *If ξ is a tolerance on a lattice L, for $x, y \in L$, we define $x \leq_\xi y$ iff $x \leq y$ and $x \mathrel{\xi} y$. We define $x \geq_\xi y$ iff $x \geq y$ and $x \mathrel{\xi} y$.*

Note that $\leq_\xi$ is *not* transitive.

2.4. Properties of finite modular lattices.

Theorem 2.9. *If M is a finite modular lattice, the following are equivalent:*

(1) *$\hat{1}$ is the join of atoms,*
(2) *$\hat{0}$ is the meet of coatoms,*
(3) *M is complemented,*
(4) *M relatively complemented,*
(5) *M is atomistic, and*
(6) *M is coatomistic.*

Proof.

Regarding $1 \Rightarrow 3$: [Birk1967a, Th. IV.6] Since M is a finite modular lattice, if $\hat{1}$ is the join of atoms, M is complemented.

Regarding $2 \Rightarrow 3$: Dually to $1 \Rightarrow 3$, if $\hat{0}$ is the meet of coatoms, the dual of M is complemented. But the complemented property is self-dual, so M is also complemented.

Regarding $3 \Rightarrow 4$: [Birk1967a, Th. I.14] Any complemented modular lattice is relatively complemented.

Regarding $3 \Rightarrow 5$: [Birk1967a, Th. I.15 Cor.] In a finite complemented modular lattice, every element is the join of those atoms $\leq$ it.

Regarding $3 \Rightarrow 6$: The complemented property is self-dual, so the dual of M is also complemented. Thus in the dual of M, every element is the join of those atoms $\leq$ it. And so in M, every element is the meet of those coatoms $\geq$ it.

Regarding $4 \Rightarrow 3$, $5 \Rightarrow 1$, and $6 \Rightarrow 2$: These are immediate. □

3. Modular connected systems

In this section we define *modular connected systems*, a variant of the constructions introduced by Herrmann in [Herr1973a, sec. 4][Herr1973a-en, sec. 4] and Day and Herrmann in [DayHerr1988a, Def. 4.7]. We also define isomorphism between modular connected systems.

Definition 3.1. *We define a* modular connected system[1] — *abbreviated a* m.c. system — *to be comprised of:*

- *a* skeleton *lattice S,*
- *an* overlap tolerance *γ, which is a tolerance on S,*
- *a family of* blocks *$(L_x)_{x \in S}$, which are lattices, and*
- *a family of* connections *$(\phi_x^y)_{x,y \in S, x \leq_\gamma y}$, which are mappings,*

that obey these axioms:

(MC1) *The skeleton S is l.f.f.c.*
(MC2) *The blocks L_x are finite modular complemented lattices,*
(MC3) *Each ϕ_x^y is a lattice isomorphism from a filter F_x^y of L_x to an ideal I_x^y of L_y.*[2]

[1]We dispense with the ubiquitous "S-" prefixes when defining terms for our gluing construction.

[2]Note that these are *lattice* filters and ideals, not *poset* filters and ideals; the filters are closed under meet and the ideals are closed under join. Since each $L_\bullet$ is finite, the filters $F_\bullet^\bullet$ and ideals $I_\bullet^\bullet$ are all principal.

(MC4) *For any* $x \in S$, $F_x^x = I_x^x = L_x$ *and* ϕ_x^x *is the identity map on* L_x.
(MC5) *If* $x \leq_\gamma y \leq_\gamma z$ *in* S *and* $I_x^y \cap F_y^z \neq \emptyset$, *then* $x \;\gamma\; z$.
(MC6) *For every* $x \leq z \leq y$ *in* S *where* $x \;\gamma\; y$, *then* $F_x^y = {\phi_x^z}^{-1}(I_x^z \cap F_z^y)$, $I_x^y = \phi_z^y(I_x^z \cap F_z^y)$, *and* $\phi_x^y = \phi_z^y \circ \phi_x^z|_{F_x^y}$.[3]
(MC7) *For every* $x, y \in S$ *for which* $x \;\gamma\; y$,[4]
 (a) $I_x^{x\vee y} \cap I_y^{x\vee y} \subset I_{x\wedge y}^{x\vee y}$ *and*
 (b) $F_{x\wedge y}^x \cap F_{x\wedge y}^y \subset F_{x\wedge y}^{x\vee y}$.
(MC8) *If* $x \lessdot y$ *in* S, *then* $x \;\gamma\; y$ *(thus* I_x^y *and* F_x^y *exist),* $F_x^y \neq L_x$, *and* $I_x^y \neq L_y$.[5]

By abuse of language, we say that the blocks L_x *"are a m.c. system" when the remaining parts of the m.c. system are implicit.*

Note that many of our symbols have both a subscript and a superscript that are elements of S — ϕ_x^y, F_x^y, I_x^y, and others — In all such cases, the symbol is defined only for $x \leq y$ (and usually only if $x \leq_\gamma y$); the subscript is the *lower* element and the superscript is the *upper* element of S. This convention is regardless of exactly how the symbol is related to x and y. E.g., $F_x^y \subset L_x$ but $I_x^y \subset L_y$.

Including the overlap tolerance in the definition of a m.c. system is annoying, but the distinction between pairs of elements x and y in S for which $x \;\gamma\; y$ and $x \not\gamma\; y$ seems to be fundamental — all alternative definitions in [DayHerr1988a] introduce the same distinction in some other guise.[6]

We will show in th. 6.7 that monotony, (MC8), is necessary to prove there to be only one m.c. system with a particular sum lattice.

Remark 3.2. Note that we do not require S to be modular. In particular, [Herr1973a, Satz 7.2 and 7.3] [Herr1973a-en, Th. 7.2 and 7.3] shows that any *finite* lattice is the dissection skeleton of some finite (and hence l.f.f.c.) modular lattice. It is an open question whether every l.f.f.c. lattice is the skeleton of some modular, l.f.f.c. lattice.

Definition 3.3. *The* dual *of a m.c. system is defined as the construction:*
 (1) *a skeleton lattice* S^δ *(the dual lattice of* S*),*
 (2) *the overlap tolerance* γ, *which operates on the elements of* S^δ, *which are the same as the elements of* S,
 (3) *the blocks* $({L_x}^\delta)_{x\in S^\delta}$ *(the dual lattices of the* L_x*),*
 (4) *the connections* $({\phi_y^x}^{-1})_{x,y\in S^\delta, x\leq_\gamma y}$ *(*$x, y \in S^\delta, x \leq_\gamma y$ *is equivalent to (*$y, x \in S, y \leq_\gamma x$*).*

Theorem 3.4. *The dual of a m.c. system is a m.c. system. The dual of the dual of a m.c. system is itself.*

We will use this fact often in proofs to avoid providing dual arguments for the duals of facts that we have proven.

Definition 3.5. *We define an isomorphism* χ *between a m.c. systems* $\mathcal{C}$ *(comprised of skeleton* S, *overlap tolerance* γ, *blocks* $L_\bullet$, *connections* $\phi_\bullet^\bullet$, *connection sources* $F_\bullet^\bullet$, *and connection targets* $I_\bullet^\bullet$*) and a m.c. system* $\mathcal{C}'$ *(comprised of skeleton* S', *overlap tolerance* γ', *blocks* $L'_\bullet$, *connections* ${\phi_\bullet^\bullet}'$, *connection sources* ${F_\bullet^\bullet}'$, *and connection targets* ${I_\bullet^\bullet}'$*) to be comprised of:*
 (1) *a lattice isomorphism* χ_S *from* S *to* S' *and*
 (2) *a family of lattice isomorphisms* χ_{Bx} *for every* $x \in S$, *with* χ_{Bx} *being an isomorphism from* L_x *to* $L'_{\chi_S(x)}$,

for which:
 (1) *for* $x, y \in S$, $x \;\gamma\; y$ *iff* $\chi_S(x) \;\gamma'\; \chi_S(y)$,
 (2) *for* $x \leq_\gamma y \in S$, ${F_{\chi_S(x)}^{\chi_S(y)}}' = \chi_{Bx}(F_x^y)$ *and* ${I_{\chi_S(x)}^{\chi_S(y)}}' = \chi_{By}(I_x^y)$,
 (3) *for* $x \leq_\gamma y \in S$, $\chi_{By} \circ \phi_x^y = {\phi_{\chi_S(x)}^{\chi_S(y)}}' \circ \chi_{Bx}|_{F_x^y}$.[7]

Theorem 3.6. *Isomorphism between m.c. systems is an equivalence relation.*

[3] That is, $\phi_x^y = \phi_z^y \circ \phi_x^z$ if ϕ_x^z is considered as a partial function from L_x to L_z and ϕ_z^y is considered as a partial function from L_z to L_y.

[4] And thus γ holds among every pair of x, y, $x \vee y$, and $x \wedge y$ by lem. 2.7(1)(2).

[5] Thus, the m.c. system is monotone in the stronger, self-dual sense.[Herr1973a-en, Def. T.1]

[6] A parallel is the distinction in set theory between properties of objects that can be collected into a set and those which can only be collected into a proper class.

[7] That is, for all $a \in F_x^y$, $\chi_{By}(\phi_x^y(a)) = {\phi_{\chi_S(x)}^{\chi_S(y)}}'(\chi_{Bx}(a))$.

4. Gluing

The gluing of a set of finite lattices into a sum lattice was introduced in Herrmann [Herr1973a, sec. 1][Herr1973a-en, sec. 1].

4.1. Basic properties of modular connected systems.

Definition 4.1. *In this section, we define $\mathcal{C}$ to be an arbitrarily chosen m.c. system comprised of skeleton S, overlap tolerance γ, blocks $L_\bullet$, connections $\phi^\bullet_\bullet$, connection sources $F^\bullet_\bullet$, and connection targets $I^\bullet_\bullet$).*

Definition 4.2. *If $x \leq_\gamma y$, we define 0^y_x to be the minimum of F^y_x and 1^y_x to be the maximum of I^y_x.*

Lemma 4.3. *If $x \leq_\gamma y$,*
1. $1^y_x = \phi^y_x\, \hat{1}_{L_x}$ *and*
2. $0^y_y = \phi^{y\,-1}_x\, \hat{0}_{L_y}$.

Thus we can say that 0^y_x is $\hat{0}_{L_y}$ "transported down" to L_x and 1^y_x is $\hat{1}_{L_x}$ "transported up" to L_y.

Lemma 4.4. *If $x < y$ and $x \;\gamma\; y$,*
1. $1^y_x < \hat{1}_{L_y}$, *and*
2. $0^y_x > \hat{0}_{L_x}$.

Proof.
Regarding $1^y_x < \hat{1}_{L_y}$: Of necessity, $1^y_x \leq \hat{1}_{L_y}$. But if $1^y_x = \hat{1}_{L_y}$, then $I^y_x = L_y$, which violates (MC8). $0^y_x > \hat{0}_{L_x}$ is proved dually. □

Lemma 4.5. [Herr1973a, (6)][Herr1973a-en, (6)] *If $x, y, z \in S$, $x \leq z \leq y$, and $x \;\gamma\; y$ (implying by lem. 2.7(1), $x \;\gamma\; z$ and $z \;\gamma\; y$), then $F^y_x \subset F^z_x$ and $I^y_x \subset I^y_z$.*

Proof.
Of necessity,

$$\hat{0}_{L_z} \leq \phi^{y\,-1}_z\, \hat{0}_{L_y}$$

Then

$$\begin{aligned} 0^z_x &= \phi^{z\,-1}_x\, \hat{0}_{L_z} \\ &\leq \phi^{z\,-1}_x(\phi^{y\,-1}_z\, \hat{0}_{L_y}) \end{aligned}$$

by (MC6),

$$\begin{aligned} &= \phi^{y\,-1}_x\, \hat{0}_{L_y} \\ &= 0^y_x \end{aligned}$$

Which shows that $F^y_x \subset F^z_x$.
We prove $I^y_x \subset I^y_z$ dually. □

(MC7) can be strengthened:

Lemma 4.6. [Herr1973a, (7)][Herr1973a-en, (7)] *For any $x, y \in S$ for which $x \;\gamma\; y$ (and thus $x \vee y \;\gamma\; x \wedge y$),*
1. $I^{x\vee y}_x \cap I^{x\vee y}_y = I^{x\vee y}_{x\wedge y}$ *and*
2. $F^x_{x\wedge y} \cap F^y_{x\wedge y} = F^{x\vee y}_{x\wedge y}$.

Proof.
This is a direct consequence of (MC7) and lem. 4.5. □

Lemma 4.7. *For any $z \leq_\gamma x, y$ in S, $F^x_z \cap F^y_z = F^{x\vee y}_z$. For any $x, y \leq_\gamma z$ in S, $I^z_x \cap I^z_y = I^z_{x\wedge y}$.*

Proof.
Regarding $F^x_z \cap F^y_z = F^{x\vee y}_z$:

By (MC6),

$$F^x_z \cap F^y_z = \phi^{x\wedge y\,-1}_z (I^{x\wedge y}_z \cap F^x_{x\wedge y}) \cap \phi^{x\wedge y\,-1}_z (I^{x\wedge y}_z \cap F^y_{x\wedge y})$$

because the $\phi_\bullet^{\bullet\,-1}$ are one-to-one,

$$= \phi_z^{x \wedge y\,-1}(I_z^{x \wedge y} \cap F_{x \wedge y}^x \cap F_{x \wedge y}^y)$$

by lem. 4.6,

$$= \phi_z^{x \wedge y\,-1}(I_z^{x \wedge y} \cap F_{x \wedge y}^{x \vee y})$$

by (MC6) again,

$$= F_z^{x \vee y}$$

$I_x^z \cap I_y^z = I_{x \wedge y}^z$ is proved dually. ☐

Lemma 4.8. *If $x, y \leq z \leq w$ in S, $x, y \;\gamma\; w$, $a \in F_x^w$, $b \in F_y^w$, and $\phi_x^w\, a = \phi_y^w\, b$, then $\phi_x^z\, a = \phi_y^z\, b$.*

Proof.
By (MC6),

$$\phi_z^w(\phi_x^z\, a) = \phi_x^w\, a = \phi_y^w\, b = \phi_z^w(\phi_y^z\, b)$$

Since ϕ_z^w is one-to-one, $\phi_x^z\, a = \phi_y^z\, b$. ☐

Lemma 4.9. *If $w \leq z \leq x, y$ in S, $x, y \;\gamma\; w$, $a \in I_w^x$, $b \in I_w^y$, and $\phi_w^{x\,-1}\, a = \phi_w^{y\,-1}\, b$, then $\phi_z^{x\,-1}\, a = \phi_z^{y\,-1}\, b$.*

Proof.
This lemma is proved dually to lem. 4.8. ☐

4.2. The sum.

Lemma 4.10. [Herr1973a, Hilfs. 4.1][Herr1973a-en, Prop. 4.1] *Let $x, y \in S$, $a \in L_x$, and $b \in L_y$. Then the following are equivalent:*

(1) *There is a $z \in S$ with $z \geq_\gamma x, y$; $a \in F_x^z$; $b \in F_y^z$; and $\phi_x^z\, a = \phi_y^z\, b$.*
(2) *$x \;\gamma\; y$, $a \in F_x^{x \vee y}$, $b \in F_y^{x \vee y}$, and $\phi_x^{x \vee y}\, a = \phi_y^{x \vee y}\, b$.*
(3) *There is a $z \in S$ with $z \leq_\gamma x, y$; $a \in I_x^z$; $b \in I_y^z$; and $\phi_z^{x\,-1}\, a = \phi_z^{y\,-1}\, b$.*
(4) *$x \;\gamma\; y$, $a \in I_{x \wedge y}^x$, $b \in I_{x \wedge y}^y$, and $\phi_{x \wedge y}^{x}{}^{-1}\, a = \phi_{x \vee y}^{y}{}^{-1}\, b$.*

Proof.
Regarding (2) ⇒ (1) and (4) ⇒ (3): Lem. 2.7(1)(2) shows that $x \;\gamma\; y$ implies $x \vee y \;\gamma\; x, y$ and $x \wedge y \;\gamma\; x, y$ and thus we can choose $z = x \vee y$ for (1) or $z = z \wedge y$ for (3).
Regarding (1) ⇒ (2): Applying lem. 4.8 to $\phi_x^z\, a = \phi_y^z\, b$ shows that $\phi_x^{x \vee y}\, a = \phi_y^{x \vee y}\, b$.
Regarding (3) ⇒ (4): Applying lem. 4.9 to $\phi_z^{x\,-1}\, a = \phi_z^{y\,-1}\, b$ shows that $\phi_{x \vee y}^{x}{}^{-1}\, a = \phi_{x \vee y}^{y}{}^{-1}\, b$.
Regarding (2) ⇒ (4): By (MC7)(a), there is a $c \in F_{x \wedge y}^{x \vee y} \subset L_{x \wedge y}$ such that $\phi_{x \wedge y}^{x \vee y}\, c = \phi_x^{x \vee y}\, a = \phi_y^{x \vee y}\, b$. By lem. 4.8, $\phi_{x \wedge y}^x\, c = a$ and so $c = \phi_{x \wedge y}^{x}{}^{-1}\, a$. Likewise, $c = \phi_{x \wedge y}^{y}{}^{-1}\, b$, so $\phi_{x \wedge y}^{x}{}^{-1}\, a = \phi_{x \wedge y}^{y}{}^{-1}\, b$.
The implication (4) ⇒ (2) is proved dually using (MC7)(b) and lem. 4.9. ☐

Definition 4.11. *Given a m.c. system, we define $M = \{(x, a) \mid x \in S, a \in L_x\}$. We define the relation $\sim$ on M as: $(x, a) \sim (y, b)$ iff $x \;\gamma\; y$, $a \in F_x^{x \vee y}$, $b \in F_y^{x \vee y}$, and $\phi_x^{x \vee y}\, a = \phi_y^{x \vee y}\, b$.*

Lemma 4.12. *Given $(x, a), (y, b) \in M$, $(x, a) \sim (y, b)$ iff $x \;\gamma\; y$, $a \in I_x^{x \wedge y}$, $b \in I_y^{x \wedge y}$, and $\phi_{x \wedge y}^{x}{}^{-1}\, a = \phi_{x \wedge y}^{y}{}^{-1}\, b$.*

Proof.
This follows directly by applying lem. 4.10 to def. 4.11. ☐

Lemma 4.13. *If $x \leq y$ in S:*

(1) *$(x, a) \sim (y, b)$ iff $x \;\gamma\; y$, $a \in F_x^y$, and $\phi_x^y\, a = b$.*
(2) *$(x, a) \sim (y, b)$ iff $x \;\gamma\; y$, $b \in I_x^y$, and $\phi_x^{y\,-1}\, b = a$.*

Proof.
This follows directly by applying (MC4) to def. 4.11 and lem. 4.12. ☐

Theorem 4.14. [Herr1973a, (21)][Herr1973a-en, (21)] *$\sim$ is an equivalence relation on M.*

Proof.
Regarding transitivity of $\sim$: Let $(x_0, a_0) \sim (x_1, a_1)$ and $(x_1, a_1) \sim (x_2, a_2)$, so

$$x_0 \vee x_1 \;\gamma\; x_0, x_1 \text{ and } x_1 \vee x_2 \;\gamma\; x_1, x_2$$
$$\phi_{x_0}^{x_0\vee x_1}\, a_0 = \phi_{x_1}^{x_0 \vee x_1}\, a_1 \text{ and } \phi_{x_1}^{x_1\vee x_2}\, a_1 = \phi_{x_2}^{x_1\vee x_2}\, a_2.$$

Define $x = (x_0 \vee x_1) \wedge (x_1 \vee x_2)$ and $y = x_0 \vee x_1 \vee x_2$. Since $x_1 \le x \le x_0 \vee x_1, x_1 \vee x_2$, we know $x_1 \;\gamma\; x$, $x \;\gamma\; x_0 \vee x_1$, $x \;\gamma\; x_1 \vee x_2$, and by lem. 2.7(6), $x \;\gamma\; (x_0 \vee x_1) \vee (x_1 \vee x_2) = y$. By (MC6), $a_1 \in F_{x_1}^{x}$, $\phi_{x_1}^{x}\, a_1 \in F_x^{x_0\vee x_1}$, and $\phi_{x_1}^{x}\, a_1 \in F_x^{x_1 \vee x_2}$, and so by (MC7), $\phi_{x_1}^{x}\, a_1 \in F_x^{y}$. This means that $\phi_{x_1}^{x}\, a_1 \in I_{x_1}^{x} \cap F_x^{y} \neq \emptyset$ and by (MC5), $x_1 \;\gamma\; y$ and then by lem. 2.7(1), $x_0 \vee x_1 \;\gamma\; y$ and $x_1 \vee x_2 \;\gamma\; y$. Since $x_1 \le x \le y$, by (MC6), $a_1 \in F_{x_1}^{y}$. Using (MC6) again, $\phi_{x_1}^{x_0\vee x_1}\, a_1 \in F_{x_0 \vee x_1}^{y}$. Similarly, $\phi_{x_1}^{x_1 \vee x_2}\, a_1 \in F_{x_1\vee x_2}^{y}$. By the premises, $\phi_{x_0}^{x_0\vee x_1}\, a_0 = \phi_{x_1}^{x_0\vee x_1}\, a_1 \in F_{x_0 \vee x_1}^{y}$, so

$$\begin{aligned}
\phi_{x_0}^{y}\, a_0 &= \phi_{x_0\vee x_1}^{y}(\phi_{x_0}^{x_0\vee x_1}\, a_0)\\
&= \phi_{x_0\vee x_1}^{y}(\phi_{x_1}^{x_0\vee x_1}\, a_1)\\
&= \phi_{x_1}^{y}\, a_1\\
&= \phi_{x_1\vee x_2}^{y}(\phi_{x_1}^{x_1\vee x_2}\, a_1)
\end{aligned}$$

because of the premise $(x_1, a_1) \sim (x_2, a_2)$, $\phi_{x_1}^{x_1\vee x_2}\, a_1 = \phi_{x_2}^{x_1\vee x_2}\, a_2$, so

$$\begin{aligned}
&= \phi_{x_1\vee x_2}^{y}(\phi_{x_2}^{x_1\vee x_2}\, a_2)\\
&= \phi_{x_2}^{y}\, a_2
\end{aligned}$$

By lem. 4.8, since $x_0, x_2 \le x_0 \vee x_2 \le y$, $\phi_{x_0}^{x_0\vee x_2}\, a_0 = \phi_{x_2}^{x_0\vee x_2}\, a_2$ and thus $(x_0, a_0) \sim (x_2, a_2)$.
Reflexivity and symmetry are trivially satisfied. □

Definition 4.15. *The* sum L *of a m.c. system* $\mathcal{C}$ *is defined to be the set* M *modulo the equivalence relation* $\sim$*.*

Definition 4.16. *The map* κ *is defined to be the canonical projection of* M *onto* L *for the equivalence relation* $\sim$*.*

Definition 4.17. *For each* $x \in S$*, we define the canonical mapping* $\pi_x : L_x \to L : y \mapsto \kappa\,(x, y)$*.*

Definition 4.18. *We define* $\Lambda_x = \pi_x\, L_x$*. For* $x \in S$*, we define* $0_x = \pi_x\, \hat{0}_{L_x}$ *and* $1_x = \pi_x\, \hat{1}_{L_x}$*.*

Lemma 4.19. *The mappings* π_x *are bijective between* L_x *and* Λ_x*.*

Proof.
By the definition of Λ_x, π_x is onto. To prove that π_x is one-to-one: If $\pi_x(a) = \pi_x(b)$ for $a, b \in L_x$, then $\kappa\,(x, a) = \kappa\,(x, b)$ and $(x, a) \sim (x, b)$. Then by def. 4.11, $\phi_x^x\, a = \phi_x^x\, b$, which, since ϕ_x^x is the identity on L_x, means $a = b$. □

Lemma 4.20. [Herr1973a, (6)][Herr1973a-en, (6)] *If* $x, y, z \in S$*,* $x \le z \le y$*, and* $x \;\gamma\; y$ *(implying by lem. 2.7(1),* $x \;\gamma\; z$ *and* $z \;\gamma\; y$*). Then* $\Lambda_x \cap \Lambda_y \subset \Lambda_z$*.*

Proof.
Given $a \in \Lambda_x \cap \Lambda_y$,

$$\kappa\,(x, \pi_x^{-1}\, a) = a = \kappa\,(y, \pi_y^{-1}\, a)$$
$$(x, \pi_x^{-1}\, a) \sim (y, \pi_y^{-1}\, a)$$

by lem. 4.13,

$$\begin{aligned}
\pi_y^{-1}\, a &= \phi_x^{y}(\pi_x^{-1}\, a)\\
&= \phi_z^{y}(\phi_x^{z}(\pi_x^{-1}\, a))
\end{aligned}$$

again by lem. 4.13,

$$(z, \phi_x^{z}(\pi_x^{-1}\, a)) \sim (y, \pi_y^{-1}\, a)$$
$$\kappa\,(z, \phi_x^{z}(\pi_x^{-1}\, a)) = \kappa\,(y, \pi_y^{-1}\, a) = a$$

Thus, $a \in \Lambda_z$. □

Lemma 4.21. *If $x \leq_\gamma y$ in S and $a \in L_x, F_x^y$, then $\pi_y(\phi_x^y\, a) = \pi_x\, a$. If $x \leq_\gamma y$ in S and $b \in L_y, I_x^y$, then $\pi_x(\phi_x^{y\,-1}\, b) = \pi_y\, b$.*

Proof.

By (MC7),

$$\phi_y^{x\vee y}(\phi_x^y\, a) = \phi_x^{x\vee y}\, a$$

by def. 4.11,

$$(y, \phi_x^y\, a) \sim (x, a)$$

by def. 4.16,

$$\kappa\,(y, \phi_x^y\, a) = \kappa\,(x, a)$$

by def. 4.17,

$$\pi_y(\phi_x^y\, a) = \pi_x\, a$$

Dually, we show that $\pi_x(\phi_x^{y\,-1}\, b) = \pi_y\, b$. ☐

4.3. The sum is a poset.

Definition 4.22. *For every $v \in S$, we define on L the binary relation $\leq_v$: Given $x, y \in L$, $x \leq_v y$ iff $x, y \in \Lambda_v$ and $\pi_v^{-1}\, x \leq \pi_v^{-1}\, y$ (as elements of L_v). We define $x \geq_v y$ iff $y \leq_v x$. We define $x <_v y$ iff $\pi_v^{-1}\, x < \pi_v^{-1}\, y$, or equivalently $x \leq_v y$ and $\pi_v^{-1}\, x \neq \pi_v^{-1}\, y$, or equivalently (since π_v is bijective) $x \leq_v y$ and $x \neq y$.*

Lemma 4.23. *For every $x \in S$, the relation $\leq_x$ is a partial ordering on Λ_x.*

Proof.
This follows from def. 4.22 and lem. 4.19 (π_x^{-1} is bijective). ☐

Lemma 4.24. [Herr1973a, (9)][Herr1973a-en, (9)] *For $a, b \in L$ and $x, y \in S$, if $a \leq_x b$ and $a \in \Lambda_y$, then $a \leq_{x\vee y} b$ (which implies $a, b \in \Lambda_{x\vee y}$).*

Proof.

Since $a \in \Lambda_x$ and $a \in \Lambda_y$, $\pi_x^{-1}\, a$ and $\pi_y^{-1}\, a$ exist and by def. 4.15, 4.16, and 4.17,

$$\kappa\,(x, \pi_x^{-1}\, a) = a = \kappa(y, \pi_y^{-1}\, a)$$

by def. 4.16,

$$(x, \pi_x^{-1}\, a) \sim (y, \pi_y^{-1}\, a)$$

by def. 4.11, (a) $\pi_x^{-1}\, a \in F_x^{x\vee y}$, $\pi_y^{-1}\, a \in F_y^{x\vee y}$, and

$$\phi_x^{x\vee y}(\pi_x^{-1}\, a) = \phi_y^{x\vee y}(\pi_y^{-1}\, a)$$

since $\phi_{x\vee y}^{x\vee y}$ is the identity,

$$\phi_x^{x\vee y}(\pi_x^{-1}\, a) = \phi_{x\vee y}^{x\vee y}(\phi_y^{x\vee y}(\pi_y^{-1}\, a))$$

by def. 4.11,

$$(x, \pi_x^{-1} a) \sim (x \vee y, \phi_y^{x\vee y}(\pi_y^{-1}\, a))$$

by def. 4.16,

$$\kappa\,(x, \pi_x^{-1}\, a) = \kappa\,(x \vee y, \phi_y^{x\vee y}(\pi_y^{-1}\, a))$$

applying def. 4.17 to both sides,

$$a = \pi_{x\vee y}(\phi_y^{x\vee y}(\pi_y^{-1}\, a))$$

$$\pi_{x\vee y}^{-1}\, a = \phi_y^{x\vee y}(\pi_y^{-1}\, a) \tag{b}$$

And thus, (c) $a \in \Lambda_{x\vee y}$.

By hypothesis,

$$a \leq_x b$$

by def. 4.22,

$$\pi_x^{-1}\, a \leq \pi_x^{-1}\, b$$

by (a) above, $\pi_x^{-1}\, a \in F_x^{x\vee y}$, so $\pi_x^{-1}\, b \in F_x^{x\vee y}$, and since $\phi_x^{x\vee y}$ is an isomorphism on $F_x^{x\vee y}$,

$$\phi_x^{x\vee y}(\pi_x^{-1}\, a) \leq \phi_x^{x\vee y}(\pi_x^{-1}\, b) \tag{d}$$

since $\phi_{x\vee y}^{x\vee y}$ is the identity,

$$\phi_x^{x\vee y}(\pi_x^{-1}\, b) = \phi_{x\vee y}^{x\vee y}(\phi_x^{x\vee y}(\pi_x^{-1}\, b))$$

by def. 4.11,

$$(x, \pi_x^{-1} b) \sim (x \vee y, \phi_x^{x\vee y}(\pi_x^{-1}\, b))$$

by def. 4.16,

$$\kappa\,(x, \pi_x^{-1} b) = \kappa\,(x \vee y, \phi_x^{x\vee y}(\pi_x^{-1}\, b))$$

applying def. 4.17 to both sides,

$$b = \pi_{x\vee y}(\phi_x^{x\vee y}(\pi_x^{-1}\, b))$$

$$\pi_{x\vee y}^{-1}\, b = \phi_x^{x\vee y}(\pi_x^{-1}\, b) \tag{e}$$

And thus, (f) $b \in \Lambda_{x\vee y}$.

Substituting (b) and (d) into (e),

$$\pi_{x\vee y}^{-1}\, a \leq \pi_{x\vee y}^{-1}\, b \tag{g}$$

by def. 4.22, (c), (f), and (g) together show

$$a \leq_{x\vee y} b$$

□

Lemma 4.25. *For $a, b \in L$ and $x, y \in S$, if $a \leq_y b$ and $b \in \Lambda_x$, then $a \leq_{x\wedge y} b$ (which implies $a, b \in \Lambda_{x\wedge y}$).*

Proof.
This is proved dually to lem. 4.24. □

Lemma 4.26. *For $a, b \in L$ and $x \leq y$ in S with $a, b \in \Lambda_x \cap \Lambda_y$, $a \leq_x b$ iff $a \leq_y b$.*

Proof.
Regarding ⇒: By hypothesis $a \leq_x b$ and $a \in \Lambda_y$, so by lem. 4.24, $a \leq_{x\vee y} b$, which is equivalent to $a \leq_y b$.
Regarding ⇐: This is proved dually to the ⇒ case. □

Definition 4.27. *We define an* ascending sequence of length n *to be a pair $((x_i)_{1\leq i\leq n}, (a_i)_{0\leq i\leq n})$ where:*

1. *for all $1 \leq i \leq n$, $x_i \in S$ and for all $0 \leq i \leq n$, $a_i \in L$,*
2. *$x_1 \leq x_2 \leq \cdots \leq x_n$, and*
3. *$a_0 \leq_{x_1} a_1 \leq_{x_2} \cdots \leq_{x_n} a_n$.*

We call the $x_\bullet$ blocks *of the ascending sequence (since they are elements of S which index the blocks $L_\bullet$), and we call the $a_\bullet$ the* elements *of the sequence.*

Note that by def. 4.22, in any ascending sequence $(x_\bullet, a_\bullet)$ of length n, for all $1 \leq i \leq n$, $a_{i-1}, a_i \in \Lambda_{x_i}$. Thus, for all $1 \leq i \leq n-1$, $a_i \in \Lambda_{x_i} \cap \Lambda_{x_{i+1}}$.

Lemma 4.28. *Given an ascending sequence $(x_\bullet, a_\bullet)$ of length n and an $x \in S$ where $a_0 \in \Lambda_x$, there exists an ascending sequence $(x'_\bullet, a'_\bullet)$ of length $n+1$ with $x'_0 = x$, $x'_{n+1} = x_n \vee x$, $a'_0 = a_0$, and $a'_{n+1} = a_n$.*

Proof.

We define:

$$x'_i = \begin{cases} x & \text{if } i = 1 \\ x_{i-1} \vee x & \text{if } 1 < i \leq n+1 \end{cases}$$

$$a'_i = \begin{cases} a_0 & \text{if } i = 0 \\ a_{i-1} & \text{if } 1 \leq i \leq n+1 \end{cases}$$

Properties (1) and (2) of an ascending sequence for $(x'_\bullet, a'_\bullet)$ are trivial; what remains is to show is property (3), that for all $1 \leq i \leq n+1$, $a'_{i-1} \leq_{x'_i} a'_i$. We prove this by induction on i from 1 to $n+1$.

For $i = 1$, the requirement $a'_{i-1} \leq_{x'_i} a'_i$ is equivalent to $a_0 \leq_x a_0$, which is implied by the hypothesis $a_0 \in \Lambda_x$.

For $i > 1$, by hypothesis, $a_{i-2} \leq_{x_{i-1}} a_{i-1}$, which is equivalent to $a'_{i-1} \leq_{x_{i-1}} a'_i$. By the induction hypothesis of case $i-1$, $a'_{i-2} \leq_{x'_{i-1}} a'_{i-1}$, which implies $a'_{i-1} \in \Lambda_{x'_{i-1}}$. Applying lem. 4.24 gives $a'_{i-1} \leq_{x_{i-1} \vee x'_{i-1}} a'_i$. We then compute $x_{i-1} \vee x'_{i-1} = x_{i-1} \vee x_{i-2} \vee x = x_{i-1} \vee x = x'_i$, which shows that $a'_{i-1} \leq_{x'_i} a'_i$. □

Definition 4.29. [Herr1973a, sec. 2][Herr1973a-en, sec. 2] *For $a, b \in L$, we define $a \leq b$ iff there exists an ascending sequence $(x_\bullet, a_\bullet)$ of some length n with $a = a_0$ and $b = a_n$.*

Lemma 4.30. [Herr1973a, (10)][Herr1973a-en, (10)] *If $a \leq b$ in L and if $a \in \Lambda_x$ for a specified $x \in S$, there is an ascending sequence $(y_\bullet, c_\bullet)$ of some length n with $y_1 = x$, $c_0 = a$, and $c_n = b$ (which also shows $a \leq b$).*

Proof.

By def. 4.29, there exists an ascending sequence $((z_i)_{1 \leq i \leq m}, (d_i)_{0 \leq i \leq m})$ of some length m with $d_0 = a$ and $d_m = b$. Apply lem. 4.28 to $(z_\bullet, d_\bullet)$ and x to produce an ascending sequence $(y_\bullet, c_\bullet)$ of length n; hence $n = m+1$, $y_1 = x$, $y_n = z_m \vee x$, $c_0 = d_0 = a$, and $c_n = d_m = b$. □

Lemma 4.31. *(part of* [Herr1973a, (14)][Herr1973a-en, (14)]*) If $a \leq b$ in L and if $a \in \Lambda_x$ for a specified $x \in S$, there is an ascending sequence $(y_\bullet, c_\bullet)$ of some length n with $y_1 \geq x$, $c_0 = a$, $c_n = b$, and the elements $y_\bullet$ are strictly increasing: $y_i <_{c_{i+1}} y_{i+1}$.*

Proof.

By lem. 4.30, there exists an ascending sequence $(y_\bullet, c_\bullet)$ of length n that satisfies all of the requirements except that there may be adjacent pairs of elements that are are not strictly increasing, that is, not $<_{c_i}$.

We can meet that requirement by repeatedly deleting the second element of any adjacent pair of equal elements. Since the sequence has a finite length, this process eventually terminates. (Note that if $a = b$, the sequence may be reduced to length zero, which is valid and satisfies the requirements.)

We remove adjacent equal elements as follows: Let $y_i \not<_{c_{i+1}} y_{i+1}$ for some $0 \leq i < n$. Since $y_i \leq_{c_{i+1}} y_{i+1}$; y_i and y_{i+1} must have the same images under $\pi^{-1}_{c_{i+1}}$, and since $\pi^{-1}_{c_{i+1}}$ is bijective, they must be equal as elements of L. Then

$$y_0 \leq_{c_1} \cdots \leq_{c_i} y_i = y_{i+1} \leq_{c_{i+2}} y_{i+2} \leq_{c_{i+3}} \cdots \leq_{c_n} y_n$$
$$c_1 \leq \cdots \leq c_i \leq c_{i+1} \leq c_{i+2} \leq \cdots \leq c_n$$

We create a new ascending sequence:

$$y_0 \leq_{c_1} \cdots \leq_{c_i} y_i \leq_{c_{i+2}} y_{i+2} \leq_{c_{i+3}} \cdots \leq_{c_n} y_n$$
$$c_1 \leq \cdots \leq c_i \leq c_{i+2} \leq \cdots \leq c_n$$

This a valid ascending sequence because the only new condition on its validity is $y_i \leq_{c_{i+2}}$, which follows because $y_i = y_{i+1}$.

Note that if $i = 0$, then the first element of the sequence remains unchanged as a, but the first block is removed and replaced with y_1, which is $\geq x$ but may be $> x$. If $i = n-1$, the last element of the sequence is replaced y_{n-1} which is equal to the former first element y_n and the last block is removed and replaced with c_{n-1}. □

Lemma 4.32. *The relation $\leq$ on L is reflexive.*

Proof.

The ascending sequence $((), (a))$ (of length one) shows $a \leq a$. □

Lemma 4.33. *The relation $\leq$ on L is antisymmetric.*

Proof.

Assume $a, b \in L$ with $a \leq b$ and $b \leq a$. By def. 4.29, there are ascending sequences $((x_i)_{1\leq i\leq n}, (d_i)_{0\leq i\leq n})$ and $((y_i)_{1\leq i\leq m}, (e_i)_{0\leq i\leq m})$ with $d_0 = a$, $d_n = b$, $e_0 = b$, $e_m = a$, and $b = d_n \in \Lambda_{x_n}$. Applying lem. 4.28 to $(y_\bullet, e_\bullet)$ and x_n produces an ascending sequence $((y'_i)_{1\leq i\leq m+1}, (e'_i)_{0\leq i\leq m+1})$ with $y'_0 = x_n$, $y'_{m+1} = y_m \vee x_n$, $e'_0 = e_0 = b$, and $e'_{m+1} = e_m = a$. Concatenate $(x_\bullet, d_\bullet)$ and $(y'_\bullet, e'_\bullet)$, combining $d_n = b = e'_0$, to produce the ascending sequence $((z_i)_{1\leq i\leq n+m+1}, (f_i)_{0\leq i\leq n+m+1})$ (of length $n + m + 1$):

$$z_i = \begin{cases} x_i & \text{if } 1 \leq i \leq n \\ y'_{i-n} & \text{if } n < i \leq n+m+1 \end{cases}$$

$$f_i = \begin{cases} d_i & \text{if } 0 \leq i \leq n \\ e'_{i-n} & \text{if } n < i \leq n+m+1 \end{cases}$$

Note that $z_{n+m+1} \geq z_i$ for all $1 \leq i \leq n+m+1$ and $a = e_m = e'_{m+1} = f_{n+m+1} \in \Lambda_{z_{n+m+1}}$.

Now apply lem. 4.28 again to $(z_\bullet, f_\bullet)$ and z_{n+m+1} produce an ascending sequence $((z'_i)_{1\leq i\leq n+m+2}, (f'_i)_{0\leq i\leq n+m+2})$ of length $n+m+2$, with $z'_1 = z_{n+m+1}$, $z'_{n+m+2} = z_{n+m+1} \vee z_{n+m+1} = z_{n+m+1}$, $f'_0 = f_0 = d_0 = a$, $f'_{n+1} = f_n = d_n = b$, and $f'_{n+m+2} = f_{n+m+1} = e'_{m+1} = e_m = a$.

Since the $z'_\bullet$ are a weakly ascending sequence in S and the first and last elements are $= z_{n+m+1}$, all of the $z'_\bullet$ are $= z_{n+m+1}$. This implies that all of the $\pi^{-1}_{z_{n+m+1}} f'_\bullet$ exist, and they are a weakly ascending sequence in $L_{z_{n+m+1}}$. Since $f'_0 = a = f'_{n+m+2}$, the first and last values of $\pi^{-1}_{z_{n+m+1}} f'_\bullet$ are equal, and hence all $\pi^{-1}_{z_{n+m+1}} f'_\bullet = \pi^{-1}_{z_{n+m+1}} a$. That means that $\pi^{-1}_{z_{n+m+1}} b = \pi^{-1}_{z_{n+m+1}} f'_{n+1} = \pi^{-1}_{z_{n+m+1}} a$. By lem. 4.19, $\pi^{-1}_{z_{n+m+1}}$ is bijective, so $b = a$. □

Lemma 4.34. *The relation $\leq$ on L is transitive.*

Proof.

Assume $a, b, c \in L$ with $a \leq b$ and $b \leq c$. By def. 4.29, there are ascending sequences $((x_i)_{1\leq i\leq n}, (d_i)_{0\leq i\leq n})$ and $((y_i)_{1\leq i\leq m}, (e_i)_{0\leq i\leq m})$ with $d_0 = a$, $d_n = b$, $e_0 = b$, $e_m = c$, and $b = d_n \in \Lambda_{x_n}$. Applying lem. 4.28 to $(y_\bullet, e_\bullet)$ and x_n produces an ascending sequence $((y'_i)_{1\leq i\leq m+1}, (e'_i)_{0\leq i\leq m+1})$ with $y'_0 = x_n$, $y'_{m+1} = y_m \vee x_n$, $e'_0 = b$, and $e'_{m+1} = c$. Concatenate $(x_\bullet, d_\bullet)$ and $(y'_\bullet, e'_\bullet)$, combining $d_n = b = e'_0$, to produce the ascending sequence $((z_i)_{1\leq i\leq n+m+1}, (f_i)_{0\leq i\leq n+m+1})$ (of length $n + m + 1$):

$$z_i = \begin{cases} x_i & \text{if } 1 \leq i \leq n \\ y'_{i-n} & \text{if } n < i \leq n+m+1 \end{cases}$$

$$f_i = \begin{cases} d_i & \text{if } 0 \leq i \leq n \\ e'_{i-n} & \text{if } n < i \leq n+m+1 \end{cases}$$

Since $f_0 = d_0 = a$ and $f_{n+m+1} = e'_{m+1} = e_m = c$, this ascending sequence shows that $a \leq c$. □

Lemma 4.35. *$\leq$ on L is a partial ordering.*

Proof.

This is proved by lem. 4.32, 4.33, and 4.34. □

Lemma 4.36. [Herr1973a, (11)][Herr1973a-en, (11)] *If $a \leq c \leq b$ in L and for some $x \in S$, $a, b \in \Lambda_x$, then $c \in \Lambda_x$ and $a \leq_x c \leq_x b$.*

Proof.

As in previous proofs, by def. 4.29, there exist ascending sequences from a to c and from c to b. Using lem. 4.30, these sequences can be concatenated to produce one sequence with starting element a, ending element b, and with c as an element. We apply lem. 4.30 to the concatenated sequence and x to produce such a sequence with starting block x.

Since the starting block of the final sequence is x, every block is $\geq x$. The ending element is b and $b \in \Lambda_x$. Applying lem. 4.25 recursively from the end of the sequence to the beginning proves that every element $\in \Lambda_x$ and every consecutive pair of elements is $\leq_x$. That shows that $c \in \Lambda_x$ and by lem. 4.23, $a \leq_x c \leq_x b$. □

Lemma 4.37. *If $a, b \in L_x$, then $a \leq b$ in L_x iff $\pi_x a \leq \pi_x b$ in L.*

Proof.

Regarding $\Rightarrow$: If $a \leq b$ in L_x, then $((x), (\pi_x(a), \pi_x(b)))$ is an ascending sequence (of length one) which shows that $\pi_x\, a \leq \pi_x\, b$.

Regarding $\Leftarrow$: If $\pi_x\, a \leq \pi_x\, b$, since $\pi_x(a), \pi_x(b) \in \Lambda_x$, by lem. 4.36, $\pi_x\, a \leq_x \pi_x\, b$, which is equivalent to $a \leq b$ in L_x. □

4.4. The Π intervals of S.

Definition 4.38. *For any $a \in L$, we define $\Pi_a = \{x \in S \,|\, a \in \Lambda_x\}$, the set of $x \in S$ for which L_x has an element that maps to a via π_x.*

Lemma 4.39. *For any $x \in S$, x is the maximum element of Π_{0_x} and the minimum element of Π_{1_x}.*

Proof.

Suppose $z \in \Pi_{0_x}$ and so $0_x \in \Lambda_z$. Then there exists $a \in L_z$ such that $(z, a) \sim (x, \hat{0}_{L_x})$, requiring $\phi_z^{x\vee z}\, a = \phi_x^{x \vee z}\, \hat{0}_{L_x}$. Thus $\hat{0}_x \in F_x^{x\vee z}$. But since $\hat{0}_{L_x}$ is the minimum of L_x, by (MC8), it can only be in $F_x^{x\vee z}$ if $x \vee z = x$. Thus $z \leq x$, showing x is the maximum of Π_{0_x}.

Dually, we prove x is the minimum element of Π_{1_x}. □

Lemma 4.40. *For any $a \in L$, Π_a is a finite interval in S.*

Proof.

Given $x, x' \in \Pi_a$, there exist $b \in L_x$ and $b' \in L_{x'}$ such that $a = \kappa\,(x, b) = \kappa\,(x', b')$. By def. 4.11, that implies $(x, b) \sim (x', b')$ and so $x \,\gamma\, x'$, $b \in F_x^{x\vee x'}$, $b' \in F_{x'}^{x\vee x'}$, and $\phi_x^{x\vee x'}\, b = \phi_{x'}^{x\vee x'}\, b'$. Defining $x'' = x \vee x'$ and $b'' = \phi_x^{x\vee x'}$, we can assemble that $x \,\gamma\, x''$ and $(x, b) \sim (x'', b'')$, and so $\kappa\,(x'', b'') = a$, showing $x'' \in \Pi_a$. This shows that Π_a is closed under joins.

Dually, using lem. 4.12 instead of def. 4.11, we prove that Π_a is closed under meets.

Given $x, x'' \in \Pi_a$ and $x \leq x' \leq x''$, there exist $b \in L_x$ and $b'' \in L_{x''}$ such that $a = \kappa\,(x, b) = \kappa\,(x'', b)$. By def. 4.11, $x \,\gamma\, x''$, so by lem. 2.7(1) $x \,\gamma\, x'$ and by lem. 4.5, $b \in F_x^{x''}$, and so $b \in F_x^{x'}$. Defining $b' = \phi_x^{x''}\, b$, $\kappa\,(x', b') = a$ and so $x' \in \Pi_a$. Thus Π_a is convex.

Assume that there is no maximum element of Π_a. Then there is an infinite ascending chain in Π_a, $x_0 < x_1 < x_x < \cdots$. For each $y \in \Pi_a$, define τ_y to be $\rho_{L_y}(\pi_y^{-1}\, a)$, the rank of $\pi_y^{-1}\, a$ in L_y, which since L_y is a finite modular lattice, is well-defined and $\in \mathbb{N}$.

τ is a strictly decreasing map: Let $x < y$ in Π_a. Since every $\phi_z^{w\,-1}$ is an isomorphism on its domain and image, which are convex in L_z and L_w, respectively, $\phi_z^{w\,-1}$ preserves relative ranks.

$$\begin{aligned}
\tau_y &= \rho_{L_y}(\pi_y^{-1}\, a) \\
&= \rho_{L_y}(\pi_y^{-1}\, a) - \rho_{L_y}(\hat{0}_{L_y}) \\
&= \rho_{L_x}(\phi_x^{y\,-1}(\pi_y^{-1}\, a)) - \rho_{L_x}(\phi_x^{y\,-1}(\hat{0}_{L_y})) \\
&= \rho_{L_x}(\pi_x^{-1}\, a) - \rho_{L_x}(\phi_x^{y\,-1}(\hat{0}_{L_y})) \\
&= \tau_x - \rho_{L_x}(\phi_x^{y\,-1}(\hat{0}_{L_y}))
\end{aligned}$$

By lem. 4.4 we know that $\phi_x^{y\,-1}(\hat{0}_{L_y}) = \hat{0}_x^y > \hat{0}_{L_x}$, so $\rho_{L_x}(\phi_x^{y\,-1}(\hat{0}_{L_y})) > 0$, thus showing $\tau_y < \tau_x$.

This implies that $\tau_{x_0} > \tau_{x_1} > \cdots$, which is impossible as they are all $\in \mathbb{N}$, and so there is no infinite ascending chain in Π_a. Since we have shown that Π_a is closed under joins, it then must have a maximum element.

Dually, we show that Π_a has minimum element.

Together, these facts show that Π_a is an interval in S, and because S is locally finite (by (MC1)), Π_a is finite. □

4.5. The relations of the Λ intervals of L.

Lemma 4.41. *For any $x, y \in S$, $\Lambda_x \cap \Lambda_y = \Lambda_{x\wedge y} \cap \Lambda_{x\vee y}$.*

Proof.

Regarding $\subset$: Assume $c \in \Lambda_x \cap \Lambda_y$. Then there exists $a \in L_x$ such that $\pi_x\, a = c$ and there exists $b \in L_y$ such that $\pi_y\, b = c$. This implies $(x, a) \sim (y, b)$, $\kappa\,(x, a) = \kappa\,(y, b) = c$, $\phi_x^{x\vee y}\, a = \phi_y^{x\vee y}\, b$, and $\pi_{x\vee y}(\phi_x^{x\vee y}\, a) = \pi_{x\vee y}(\phi_y^{x\vee y}\, b) = c$. Thus, $c \in \Lambda_{x\vee y}$. Dually, we prove $c \in \Lambda_{x\wedge y}$.

Regarding $\supset$: Assume $f \in \Lambda_{x\wedge y} \cap \Lambda_{x\vee y}$. Then there exists $d \in L_{x\wedge y}$ such that $\pi_{x\wedge y}\, d = f$ and there exists $e \in L_{x\vee y}$ such that $\pi_{x\vee y}\, e = f$. This implies $(x \wedge y, d) \sim (x \vee y, e)$, $\phi^{x\vee y}_{x\wedge y}\, d = e$, and so $\phi^{x\vee y}_{x}(\phi^{x}_{x\wedge y}\, d) = e$. Thus, $\kappa\,(x, \phi^{x}_{x\wedge y}\, d) = f$, showing $f \in \Lambda_x$. Similarly, we prove $f \in \Lambda_y$. □

Lemma 4.42. *For $x, y \in S$, then $\Lambda_x \cap \Lambda_y \neq \emptyset$ iff $x\ \gamma\ y$.*

Proof.
Regarding $\Rightarrow$: Let $e \in \Lambda_x \cap \Lambda_y$. Then there exists $a \in L_x$ such that $\pi_x\, a = e$ and $b \in L_y$ such that $\pi_y\, b = e$. Thus $(x, a) \sim (y, b)$, and by def. 4.11, $x\ \gamma\ y$.

Regarding $\Leftarrow$: Since $x\ \gamma\ y$, by lem. 4.6(1), $I^{x\vee y}_x \cap I^{x\vee y}_y = I^{x\vee y}_{x\wedge y} \neq \emptyset$. Let $a \in I^{x\vee y}_{x\wedge y}$, so that $a \in I^{x\vee y}_x$ and $a \in I^{x\vee y}_y$, and define $a_x = {\phi^{x\vee y}_x}^{-1}\, a$ and $a_y = {\phi^{x\vee y}_y}^{-1}\, a$. Then $(x, a_x) \sim (y, a_y)$ so that $\pi_x\, a_x = \pi_y\, a_y$. Since $\pi_x\, a_x \in \Lambda_x$ and $\pi_y\, a_y \in \Lambda_y$, $\Lambda_x \cap \Lambda_y \neq \emptyset$. □

Lemma 4.43. [Herr1973a, (1)][Herr1973a-en, (1)] *If $x \leq_\gamma y$, then*
1. *$\Lambda_x \cap \Lambda_y = \pi_x\, F^y_x = \pi_y\, I^y_x$ and*
2. *$\Lambda_x \cap \Lambda_y$ is a filter of Λ_x and an ideal of Λ_y.*

Proof.
Regarding (1) $\subset$: Assume $e \in \Lambda_x \cap \Lambda_y$. Then there exists $a \in L_x$ such that $\pi_x\, a = e$ and $b \in L_y$ such that $\pi_y\, b = e$. Thus $(x, a) \sim (y, b)$, and by lem. 4.13, $\phi^y_x\, a = b$, ${\phi^y_x}^{-1}\, b = a$, $a \in F^y_x$, and $b \in I^y_x$, showing $e \in \pi_x\, F^y_x$ and $e \in \pi_y\, I^y_x$.

Regarding (1) $\supset$: Assume $e \in \pi_x\, F^y_x$. Since $F^y_x \subset L_x$, $e \in \Lambda_x$. There exists $a \in F^y_x$ such that $\pi_x\, a = e$. Define $b = \phi^y_x\, a$, so $b \in L_y$. By lem. 4.13, $(x, a) \sim (y, b)$, so $\pi_y\, b = e$ and $e \in \Lambda_y$. Dually, we prove that $e \in \pi_y\, I^y_x$ implies $e \in \Lambda_x$.

Regarding (2): This statement follows trivially from (1) because the $\pi_\bullet$ are poset isomorphisms by lem. 4.37. □

Lemma 4.44. *(part of* [Herr1973a, Satz 2.1][Herr1973a-en, Th. 2.1]*) Λ_x is an interval in L.*

Proof.
First, we show that if $a, b \in \Lambda_x$, then there is an upper bound $c \geq a, b$ in Λ_x. Define $c = \pi_x(\pi_x^{-1}\, a \vee \pi_x^{-1}\, b)$. Clearly $c \in \Lambda_x$. By def. 4.22, $c \geq_x a, b$ and by lem. 4.37, $c \geq a, b$.

Dually, we show that if $a, b \in \Lambda_x$, then there is a lower bound $c \leq a, b$ in Λ_x.

Since Λ_x (the image of L_x under π_x) is finite and has upper and lower bounds, it must have maximum and minimum elements. Because lem. 4.36 shows that Λ_x is convex in L, Λ_x is an interval in L. □

Note that the following lemma shows that for any x, Λ_x (under the partial ordering inherited from L) is a lattice, but not that L as a whole is a lattice (under its partial ordering).

Lemma 4.45. *(part of* [Herr1973a, Satz 2.1][Herr1973a-en, Th. 2.1]*) π_x is a lattice isomorphism from L_x to Λ_x.*

Proof.
Because L_x is a lattice ((MC2)), π_x is a partial order isomorphism from L_x to Λ_x (lem. 4.37), and Λ_x is an interval in L (lem. 4.44), Λ_x is a lattice (under the partial order of L) and π_x is a lattice isomorphism from L_x to Λ_x. □

4.6. The sum is locally finite and has finite covers.

Theorem 4.46. *L is locally finite.*

Proof.
Given a particular $a \leq b$ in L, we will establish a finite bound on the number of c with $a \leq c \leq b$. As in previous proofs, by def. 4.29, there exist ascending sequences from a to c and from c to b. Using lem. 4.30, these sequences can be concatenated to produce one sequence with starting element a, ending element b, and with c as an element. Define x_a to be the first block of the sequence, x_b to be the last block of the sequence, and, where c is the i-th element of the sequence, x_c is either the i-th or $(i+1)$-st block of the sequence (whichever exists). Of necessity, $a \in \Lambda_{x_a}$, $b \in \Lambda_{x_b}$, and $c \in \Lambda_{x_c}$.

Thus, for any such c, there exists one and possibly more combinations of x_a, x_b, x_c, and c that satisfy:
1. $x_a \in \Pi_a$,
2. $x_b \in \Pi_b$,
3. $x_a \leq x_c \leq x_b$, and

(4) $c \in \Lambda_{x_c}$.

But clearly, for any combination that satisfies these requirements, there is at most one c for which $a \leq c \leq b$.

Given any particular a and b, by lem. 4.40 the number of possible x_a and x_b are finite. Given any particular x_a and x_b, by (MC1), the number of possible x_c are finite. Given any particular x_c, since Λ_{x_c} is finite, the number of possible c are finite. All together, this means that the number of possible combinations are finite, and so the number of possible c are finite. □

Remark 4.47. It is an open question whether (MC8) is required to prove L is locally finite. ((MC8) is used in proving lem. 4.4, which is used in proving lem. 4.40, that each $\Pi_\bullet$ is finite.)

Lemma 4.48. *(part of* [Herr1973a, Satz 2.1][Herr1973a-en, Th. 2.1]*) If $a \lessdot b$ in L, then there exists $x \in S$ for which $a, b \in \Lambda_x$ and $\pi_x^{-1} a \lessdot \pi_x^{-1} b$. If $a \lessdot b$ in L_x for some $x \in S$, then $\pi_x a \lessdot \pi_x b$.*

Proof.

Regarding the first statement:

If $a \lessdot b$ in L then there exists an ascending sequence $(x_\bullet, c_\bullet)$ of length n with the first element $= a$ and the last element $= b$. All of the elements in the sequence must be either a or b, because if there was an element c in the sequence that is $\neq a, b$, the sequence could be cut into two parts, with the first one showing that $a \leq c$ and the second one showing that $c \leq b$, which would contradict that $a \lessdot b$.

Because the first element of the sequence is a and the last element is b, there is some $0 \leq j < n$ for which $c_j = a$ and $c_{j+1} = b$, and so $a \leq_{x_{j+1}} b$. Define $x = x_{j+1}$. By def. 4.22 and the fact that $a \neq b$, $\pi_x^{-1} a < \pi_x^{-1} b$. By lem. 4.37 and 4.44, if there was some $c \in L_x$ such that $\pi_x^{-1} a < c < \pi_x^{-1} b$, we would have $a < \pi_x c < b$, which would contradict that $a \lessdot b$. Thus $\pi_x^{-1} a \lessdot \pi_x^{-1} b$.

Regarding the second statement: If $a \lessdot b$ in L_x for some $x \in S$ then because π_x is an isomorphism into an interval of L, $\pi_x a \lessdot \pi_x b$. □

Theorem 4.49. *L has finite covers.*

Proof.

Select an $x \in L$. For any y for which $x \lessdot y$, by lem. 4.48, there is a $z \in S$ for which $\pi_z^{-1} x \lessdot \pi_z^{-1} y$. Thus, for any such y there is one or more combinations that satisfy:

(1) $z \in \Pi_x$ and
(2) $a \in L_z$,

and for any such combination, $y = \pi_x a$. But by lem. 4.40, for any fixed x, the number of such z's is finite, and because each L_z is finite, the number of such a's for any particular z is finite. Thus there are a finite number of combinations that satisfy the constraints, and so there are a finite of upper covers of x.

We prove there are a finite number of lower covers of x dually. □

Remark 4.50. (MC8) is required to prove L has finite covers: Let S be $\mathbb{N}$ ordered by $\leq$, and L_i be B_i, the Boolean algebra with i atoms. Let ϕ_i^j be the injection from B_i into B_j generated by mapping the i atoms of B_i to the first i atoms of B_j. Then the $\hat{0}$ of sum L is covered by an infinite number of atoms.

4.7. The sum is a lattice.

Lemma 4.51. [Herr1973a, (12)][Herr1973a-en, (12)] *For all $x, y \in S$:*

(1) *$x < y$ iff $0_x < 0_y$ iff $1_x < 1_y$,*
(2) *$x \leq y$ iff $0_x \leq 0_y$ iff $1_x \leq 1_y$, and*
(3) *$x \mid\mid y$ iff $0_x \mid\mid 0_y$ iff $1_x \mid\mid 1_y$.*[8]

Proof.

Regarding $x < y$ implies $0_x < 0_y$: Assume $x \lessdot y$. Then by (MC8), $x \;\gamma\; y$, and by lem. 4.4, $\hat{0}_{L_x} < 0_x^y$. Since π_x is an isomorphism, $0_x = \pi_x \hat{0}_{L_x} < \pi_x 0_x^y = \pi_y \hat{0}_{L_y} = 0_y$. For general $x < y$, induction on a saturated chain from x to y shows $0_x < 0_y$.

Regarding $x \leq y$ implies $0_x \leq 0_y$: This follows directly from the preceding statement.

[8]We use $x \mid\mid y$ to denote that x and y are incomparable, that is, $x \not\leq y$ and $y \not\leq x$.

Regarding $0_x \leq 0_y$ implies $x \leq y$: By def. 4.29 and lem. 4.30, we can construct an ascending sequence whose first element is 0_x, last element is 0_y, and first block is x. Define z to be the last block. Necessarily $x \leq z$ and $0_y \in \Lambda_z$. Thus, $z \in \Pi_{0_y}$, and by lem. 4.39, $z \leq y$, which implies $x \leq y$.

Regarding $0_x < 0_y$ implies $x < y$: This follows directly from the preceding statement.

Regarding $x \,||\, y$ iff $0_x \,||\, 0_y$: This follows directly from the preceding statements.

Dually, we prove: $x < y$ implies $1_x < 1_y$, $x \leq y$ implies $1_x \leq 1_y$, $1_x \leq 1_y$ implies $x \leq y$, $1_x < 1_y$ implies $x < y$, and $x \,||\, y$ iff $1_x \,||\, 1_y$. ☐

Remark 4.52. Note that if $x \,||\, y$ but $x \,\gamma\, y$, then $0_x \,||\, 0_y$ and $1_x \,||\, 1_y$, but since lem. 4.42 ensures there exists an $a \in \Lambda_x \cap \Lambda_y$, $0_x, 0_y \leq a \leq 1_x, 1_y$.

Lemma 4.53. *If $x \neq y$ in S, then $\Lambda_x \not\subset \Lambda_y$ and $\Lambda_y \not\subset \Lambda_x$.*

Proof.

If $\Lambda_x \subset \Lambda_y$, then $0_x \geq 0_y$ and $1_x \leq 1_y$. But the first condition implies by lem. 4.51 that $x \geq y$ and the second implies $x \leq y$. Together, they imply $x = y$.

Similarly, $\Lambda_y \subset \Lambda_x$ implies $x = y$. ☐

Definition 4.54. *We define $\sup_L(a_1, \ldots, a_n)$ to be the supremum in L of the elements (in L) or subsets (of L) $a_1, \ldots, a_n$, when it exists. We define $\inf_L(a_1, \ldots, a_n)$ to be the infimum in L of the elements (in L) or subsets (of L) $a_1, \ldots, a_n$, when it exists.*

Lemma 4.55. [Herr1973a, (13)][Herr1973a-en, (13)] $\sup_L(0_x, 0_y)$ *exists and* $\sup_L(0_x, 0_y) = 0_{x \vee y}$.

Proof.

By lem. 4.51, $0_{x\vee y}$ is an upper bound of 0_x and 0_y. To show that it is the least upper bound, assume $e \geq 0_x, 0_y$ in L. By lem. 4.30, there exists an ascending sequence $(z_\bullet, a_\bullet)$ with first element 0_x, last element e, and first block x. Define z^+ to be its last block, so $z^+ \geq x$. Similarly, there exists an ascending sequence $(w_\bullet, b_\bullet)$ with first element 0_y, last element e, and first block y. Define w^+ to be its last block, so $w^+ \geq y$.

Thus, $e \in \Lambda_{z^+} \cap \Lambda_{w^+}$, and by lem. 4.41, $e \in \Lambda_{z^+ \vee w^+}$, so $e \geq 0_{z^+\vee w^+}$. Since $z^+ \vee w^+ \geq x \vee y$, by lem. 4.51, $e \geq 0_{z^+\vee w^+} \geq 0_{x\vee y}$. Since we have shown that any upper bound of 0_x and 0_y is $\geq 0_{x\vee y}$, $\geq 0_{x\vee y}$ is the least upper bound. ☐

Definition 4.56. *For every $v \in S$, we define on L the partial binary operations $\vee_v$ and $\wedge_v$: $a \vee_v b = \pi_v(\pi_v^{-1}\, a \vee \pi_v^{-1}\, b)$ and $a \wedge_v b = \pi_v(\pi_v^{-1}\, a \wedge \pi_v^{-1}\, b)$ when $a, b \in \Lambda_v$, that is, the images of $\vee$ and $\wedge$ on L_v under the map π_v.*

Lemma 4.57. [Herr1973a, (8)][Herr1973a-en, (8)] *Suppose $\Lambda_x \cap \Lambda_y \neq \emptyset$. Then $\Lambda_x \cap \Lambda_y$ is an interval in Λ_x and in Λ_y. The operations $\wedge_x$ and $\wedge_y$ are closed on and agree on $\Lambda_x \cap \Lambda_y$. The operations $\vee_x$ and $\vee_y$ are also closed on and agree on $\Lambda_x \cap \Lambda_y$.*

Proof.

Lem. 4.45 shows that Λ_x and Λ_y are intervals in L, and since they intersect, $\Lambda_x \cap \Lambda_y$ is an interval in Λ_x and in Λ_y. Define $S_x = \pi_x^{-1}(\Lambda_x \cap \Lambda_y)$ and $S_y = \pi_y^{-1}(\Lambda_x \cap \Lambda_y)$. Define $\rho_x = \pi_x|_{S_x}$ and $\rho_y = \pi_y|_{S_y}$. By lem. 4.45, ρ_x is a lattice isomorphism from S_x to $\Lambda_x \cap \Lambda_y$, ρ_y is a lattice isomorphism from S_y to $\Lambda_x \cap \Lambda_y$, and ${\rho_y}^{-1} \circ \rho_x$ is a lattice isomorphism from S_x to S_y. Thus the operations $\wedge_x$ and $\wedge_y$ are closed on and agree on $\Lambda_x \cap \Lambda_y$, and the operations $\vee_x$ and $\vee_y$ are closed on and agree on $\Lambda_x \cap \Lambda_y$. ☐

Lemma 4.58. [Herr1973a, (14)][Herr1973a-en, (14)] *For $x \in S$ and $a, b \in \Lambda_x$, then* $\sup_L(a, b)$ *exists and* $= a \vee_x b$.

Proof.

Clearly, $a \vee_x b \geq a, b$, so the lemma is implied by the statement:

$$\text{If } e \in L,\ x \in S,\ a, b \in \Lambda_x, \text{ and with } e \geq a, b, \text{ then } e \geq a \vee_x b. \tag{a}$$

Consider any particular e. Define w to be the maximum element of Π_e, which exists by lem. 4.40, so $e \in \Lambda_w$. Since $a \leq e$ and $a \in \Lambda_x$, there is an ascending sequence with first block x and a last block that is $\in \Pi_e$, so $x \leq w$. For any given e, we prove (a) by induction on $x \in S$ descending from w. By th. 4.46, the induction will reach all $x \leq w$.

The base case of induction (a) is $x = w$. Then $a, b \in \Lambda_w$ and $e \geq a, b$. By lem. 4.37, $a, b \leq_x e$, so $a \vee_x b \leq_x e$, and so $e \geq a \vee_x b$.

For $x < w$, we have the induction hypothesis:

$$\text{For all } z \text{ with } w \geq z > x \text{ and all } g, h \in \Lambda_z \text{ where } e \geq g, h, \text{ we have } e \geq g \vee_z h. \tag{b}$$

From this we must prove: For all $a, b \in \Lambda_x$ with $e \geq a, b$; $e \geq a \vee_x b$. This can in turn be proven inductively by (a, b) descending in $\Lambda_x \times \Lambda_x$. The induction start, $a = b = 1_x$, is trivial since $e \geq a = a \vee_x b$.

So let $a, b \in \Lambda_x$ with the induction hypothesis:

$$\text{For all } c, d \in \Lambda_x \text{ with } e \geq c,\ e \geq d,\ c \geq a,\ d \geq b, \text{ and either } c > a \text{ or } d > b; \text{ we have } e \geq c \vee_x d. \tag{c}$$

From this we must prove: $e \geq a \vee_x b$.

By lem. 4.31, there exists an ascending sequence $(x_\bullet, a_\bullet)$ with first element a, last element e, $x_1 \geq x$, last block $\leq w$, and the elements strictly increasing, so $a_1 > a$. Similarly, there exists an ascending sequence $(b_\bullet, y_\bullet)$ with first element b, last element e, $y_1 \geq x$, last block $\leq w$, and the elements strictly increasing, so $b_1 > b$.

If $x_1 = x$, then from (c), $a_1 \vee_x b \leq e$. Then $a \vee_x b \leq a_1 \vee_x b \leq e$, so $a \vee_x b \leq e$. Analogously, if $y_1 = x$, then $a \vee_x b \leq e$.

The remaining case is $x_1, y_1 > x$. Since $x \leq x_1 \wedge y_1 \leq x_1$ and $a \in \Lambda_x, \Lambda_{x_1}$, from lem. 4.41, it follows that $a \in \Lambda_{x_1 \wedge y_1}$. Similarly, $b \in \Lambda_{x_1 \wedge y_1}$.

In the case $x_1 \wedge y_1 > x$, we already know $x_1 \wedge y_1 \leq w$ and we can therefore use (b) to conclude that $a \vee_{x_1 \wedge y_1} b \leq e$. Since $a, b \in \Lambda_x \cap \Lambda_{x_1 \wedge y_1}$, by lem. 4.43, $a \vee_x b \in \Lambda_x \cap \Lambda_{x_1 \wedge y_1}$. By lem. 4.57, $a \vee_x b = a \vee_{x_1 \wedge y_1} b \leq e$.

The remaining case is reduced to $x_1, y_1 > x$ and $x_1 \wedge y_1 = x$. Since $a \in \Lambda_x \cap \Lambda_{x_1}$ and $b \in \Lambda_x \cap \Lambda_{y_1}$, lem. 4.42 ensures that $0_{x_1}, 0_{y_1} \in \Lambda_x$. Let $c = 0_{x_1} \vee_x 0_{y_1}$. By lem. 4.43, $c \in \Lambda_{x_1}$ and $c \in \Lambda_{y_1}$, so by lem. 4.41, $c \in \Lambda_{x_1} \cap \Lambda_{y_1} = \Lambda_x \cap \Lambda_{x_1 \vee y_1}$. Hence $c \geq 0_{x_1 \vee y_1}$. On the other hand, since $\Lambda_x \cap \Lambda_{x_1 \vee y_1} \neq \emptyset$, $0_{x_1 \vee y_1} \in \Lambda_x$, and by lem. 4.43, $0_{x_1}, 0_{y_1} \leq_x 0_{x_1 \vee y_1}$. Assembling these, $0_{x_1} \vee_x 0_{y_1} = 0_{x_1 \vee y_1} = c$.

By lem. 4.55, $c = 0_{x_1 \vee y_1} = \sup_L(0_{x_1}, 0_{y_1})$. By lem. 4.57 and (b), $a \vee_x c = a \vee_{x_1} c \leq e$. Similarly we show $b \vee_x c = b \vee_{y_1} c \leq e$.

If $a \not\geq c$, then also $a \not\geq_x c$ and therefore $a \vee_x c >_x a$. From that, (c) guarantees $(a \vee_x c) \vee_x (b \vee_x c) \leq e$, from which immediately follows $a \vee_x b \leq e$. We similarly handle when $b \not\geq c$.

So ultimately the case $a \geq c$ and $b \geq c$ remains. By lem. 4.51, $c \leq a \leq 1_x \leq 1_{x_1} \leq 1_{x_1 \vee y_1}$ and similarly for b, so lem. 4.36 shows a and b are elements of $\Lambda_{x_1 \vee y_1}$. Since $x_1 \vee y_1 > x$, we can conclude with (b) that $a \vee_{x_1 \vee y_1} b \leq e$. However, $a \vee_{x_1 \vee y_1} b = a \vee_x b$ by lem. 4.57, so $a \vee_x b \leq e$. □

Lemma 4.59. [Herr1973a, (15)][Herr1973a-en, (15)] *If $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$ is a saturated chain in S with $n \geq 0$, $a \in \Lambda_x$, and $b \in \Lambda_y$, then $\sup_L(a, b)$ exists and*

$$\sup_L(a, b) = ((\cdots((a \vee_{x_0} 0_{x_1}) \vee_{x_1} 0_{x_2}) \vee_{x_2} \cdots) \vee_{x_{n-1}} 0_{x_n}) \vee_y b.$$

Proof.

The proof proceeds inductively on n. The case $n = 0$ is immediate from lem. 4.58.

So let $n \geq 1$. Set $c_0 = a \in \Lambda_{x_0}$. For $1 \leq i \leq n$, we inductively define $c_i = c_{i-1} \vee_{x_{i-1}} 0_{x_i}$ and inductively prove $c_i \in \Lambda_{x_i}$ using (MC8), (MC3), and $0_{x_{i+1}} \in \Lambda_{x_i}$. By the induction on n, $c_{n-1} = \sup_L(a, 0_{x_{n-1}})$. Because of lem. 4.58,

$$c_n = c_{n-1} \vee_{x_{n-1}} 0_{x_{n-1}} = \sup_L(c_{n-1}, 0_{x_{n-1}}) = \sup_L(a, 0_{x_{n-1}}, 0_{x_n}) = \sup_L(a, 0_{x_n}).$$

We've shown $c_n \in \Lambda_{x_n}$, so $c_n \vee_{x_n} b$ is defined and by lem. 4.58, $c_n \vee_{x_n} b = \sup_L(c_n, b)$. Therefore $c_n \vee_{x_n} b = \sup_L(a, 0_{x_n}, b) = \sup_L(a, b)$. □

Lemma 4.60. [Herr1973a, (16)][Herr1973a-en, (16)] *If $a \in \Lambda_x$ and $b \in \Lambda_y$, then $\sup_L(a, b)$ exists and* $= \sup_L(a, 0_{x \vee y}) \vee_{x \vee y} \sup_L(b, 0_{x \vee y})$.

Proof.

By lem. 4.59, $\sup_L(a, 0_{x \vee y}), \sup_L(b, 0_{x \vee y}) \in \Lambda_{x \vee y}$. Further, by lem. 4.58 and 4.55,

$$\sup_L(a, 0_{x \vee y}) \vee_{x \vee y} \sup_L(b, 0_{x \vee y}) = \sup_L(a, b, 0_{x \vee y}) = \sup_L(a, b, 0_x, 0_y) = \sup_L(a, b).$$

□

Dually, we prove:

Lemma 4.61. [Herr1973a, (13*)][Herr1973a-en, (13*)] *$\inf_L(1_x, 1_y)$ exists and $\inf_L(1_x, 1_y) = 1_{x \wedge y}$.*

Lemma 4.62. [Herr1973a, (14*)][Herr1973a-en, (14*)] *For $x \in S$ and $a, b \in \Lambda_x$, then $\inf_L(a, b)$ exists and $= a \wedge_x b$.*

Lemma 4.63. [Herr1973a, (15*)][Herr1973a-en, (15*)] *If $x = x_0 \gtrdot x_1 \gtrdot \cdots \gtrdot x_n = y$ is a saturated chain in S with $n \geq 0$, $a \in \Lambda_x$, and $b \in \Lambda_y$, then $\inf_L(a, b)$ exists and*

$$\inf_L(a, b) = ((\cdots((a \wedge_{x_0} 0_{x_1}) \wedge_{x_1} 0_{x_2}) \wedge_{x_2} \cdots) \wedge_{x_{n-1}} 0_{x_n}) \wedge_y b.$$

Lemma 4.64. [Herr1973a, (16*)][Herr1973a-en, (16*)] *If $a \in \Lambda_x$ and $b \in \Lambda_y$, then $\inf_L(a, b)$ exists and $= \inf_L(a, 0_{x\wedge y}) \wedge_{x \wedge y} \sup_L(b, 0_{x\wedge y})$.*

Definition 4.65. *For $x, y \in L$, define $x \vee y = \sup_L(x, y)$ and $x \wedge y = \inf_L(x, y)$.*

Theorem 4.66. *(part of* [Herr1973a, Satz 2.1][Herr1973a-en, Th. 2.1]*) L (with the order relation $\leq$) is a lattice with $\vee$ and $\wedge$ as the lattice operations.*

Proof.
This is immediate from lem. 4.60 and 4.64. □

Remark 4.67. It is an open question whether the proof that L is a lattice depends on monotony, property (MC8). Intuitively, it seems like it should not. However the proof of th. 4.66 depends on lem. 4.60, which depends on lem. 4.58, which depends on lem. 4.40, which depends on lem. 4.4, which depends on monotony.

Lemma 4.68. [Herr1973a, Zusatz 2.1][Herr1973a-en, Add. 2.1] *The map $x \mapsto 0_x$ is an injective* sup*-homomorphism of S into L. The map $x \mapsto 1_x$ is an injective* inf*-homomorphism of S into L.*

Proof.
This is direct from lem. 4.51, 4.55, and 4.61. □

4.8. The sum is a modular lattice.

Theorem 4.69. [Herr1973a, Satz 3.2][Herr1973a-en, Th. 3.2] *The sum L of a m.c. system is a modular lattice.*

Proof.
Th. 4.66 has proven that L is a lattice. To prove it is modular: Assume $a \lessdot b, c$ in L and $b \neq c$. Then by lem. 4.48 there exists $x \in S$ such that $a, b \in \Lambda_x$. Similarly there exists $y \in S$ such that $a, c \in \Lambda_y$. Since $a \in \Lambda_x \cap \Lambda_y$, by lem. 4.41, $a \in \Lambda_{x\vee y}$, and by lem. 4.43, $b, c \in \Lambda_{x\vee y}$. By (MC2), $\Lambda_{x\vee y}$ is modular, so there exists $d \in \Lambda_{x\vee y}$ such that $b, c \lessdot d$. Thus, L is upper semimodular.
Dually, we prove L is lower semimodular.
Now assume $a, b, c \in L$. Define $N = [a \wedge b \wedge c, a \vee b \vee c]$, which is a sublattice of L. By the above, N is upper and lower semimodular. By th. 4.46, N is finite, and so N is modular. Thus, a, b, and c satisfy the modular identity in N and thus in L. This proves that L is modular. □

4.9. Existence of $\hat{0}$.

Theorem 4.70. *The sum lattice L has a minimum element iff the skeleton lattice S has a minimum element.*

Proof.
Regarding $\Rightarrow$: Assume L has a minimum element $\hat{0}_L$. There exists $x \in S$ and $a \in L_x$ such that $\pi_x\, a = \hat{0}_L$. If S does not have a minimum element, there exists $y \in S$ such that $y < x$. By lem. 4.51 and 4.45 $0_y < 0_x \leq \hat{0}_L$, contradicting that $\hat{0}_L$ is the minimum element of L.
Regarding $\Leftarrow$: Assume that S has a minimum element $\hat{0}_S$. If L does not have a minimum element, then $0_{\hat{0}_S}$ is not the minimum element of L and there is an $a \in L$ such that $a < 0_{\hat{0}_S}$. By lem. 4.30, there is an ascending sequence with first element a and last element $0_{\hat{0}_S}$ Define x to be the first block of the sequence and y to be the last block, with $x \leq y$ in S. Then $a \in \Lambda_x$ and $0_{\hat{0}_S} \in \Lambda_y$, so $0_{\hat{0}_S} \in \Pi_y$ and by lem. 4.39, $y \leq \hat{0}_S$, requiring $y = \hat{0}_S$. That implies $x = \hat{0}_S$ and so $a \geq 0_{\hat{0}_S}$, contradicting that $a < 0_{\hat{0}_S}$. □

4.10. The adjoint maps Φ and Ψ. The family of partial functions $\phi_\bullet^\bullet$ and their inverses $\phi_\bullet^{\bullet\,-1}$ defined for pairs of indexes x and y with $x \leq_\gamma y$ can be extended to a family of total functions defined for all pairs $x \leq y$ in the manner of [DayHerr1988a, Def. 4.4]:

Definition 4.71. *For $x \leq y$ in S, we define:*

$$\Phi_x^y\, a = \begin{cases} \phi_x^y(a \vee 0_x^y) & \text{if } x \mathrel{\gamma} y \\ \hat{0}_{L_y} & \text{if } x \not\mathrel{\gamma} y \end{cases}$$

$$\Psi_x^y\, a = \begin{cases} \phi_x^{y\,-1}(a \wedge 1_x^y) & \text{if } x \mathrel{\gamma} y \\ \hat{1}_{L_x} & \text{if } x \not\mathrel{\gamma} y \end{cases}$$

Lemma 4.72. *For $x \leq y$ in S, Φ_x^y is a lattice homomorphism from L_x to L_y and Ψ_x^y is a lattice homomorphism from L_y to L_x.*

Theorem 4.73. *For $x \leq y$ in S, Φ_x^y and Ψ_x^y are an adjoint pair of homomorphisms between L_x and L_y.*[WikiGal]

Proof.

To show that Φ_x^y and Ψ_x^y are an adjoint pair of functions, we must show that for all $a \in L_x$ and $b \in L_y$,

$$\Phi_x^y(a) \leq b \text{ iff } a \leq \Psi_x^y(b).$$

If $x \not\mathrel{\gamma} y$, then for any $a \in L_x$ and $b \in L_y$, $\Phi_x^y(a) = \hat{0}_{L_y} \leq b$ and $a \leq \hat{1}_{L_x} = \Psi_x^y(b)$, so their equivalence is trivial.

If $x \mathrel{\gamma} y$, then the following statements are all equivalent:

$$\Phi_x^y(a) \leq b$$

by def. 4.71,

$$\phi_x^y(a \vee 0_x^y) \leq b$$

since 1_x^y is the maximum of I_x^y, the range of ϕ_x^y, this is equivalent to

$$\phi_x^y(a \vee 0_x^y) \leq b \wedge 1_x^y$$

because $\phi_x^{y\,-1}$ is an isomorphism,

$$a \vee 0_x^y \leq \phi_x^{y\,-1}(b \wedge 1_x^y)$$

since 0_x^y is the minimum of F_x^y, the range of $\phi_x^{y\,-1}$,

$$a \leq \phi_x^{y\,-1}(b \wedge 1_x^y)$$

by def. 4.71,

$$a \leq \Psi_x^y(b)$$

□

Remark 4.74. Φ_x^y and Ψ_x^y are characterized by more than that they are adjoints. Inter alia, the image of Φ_x^y is an ideal of L_y and the image of Ψ_x^y is a filter of L_x.

Lemma 4.75. *For $x \in S$, Φ_x^x and Ψ_x^x are both the identity map on L_x.*

Lemma 4.76. *For $x \leq y$ in S, $\Phi_x^y\, \hat{0}_{L_x} = \hat{0}_{L_y}$ and $\Psi_x^y\, \hat{1}_{L_y} = \hat{1}_{L_x}$.*

$\Phi_\bullet^\bullet$ and $\Psi_\bullet^\bullet$ extend the corresponding $\phi_\bullet^\bullet$ and $\phi_\bullet^{\bullet\,-1}$:

Lemma 4.77. *For $x \leq_\gamma y$ in S, $\Phi_x^y|_{F_x^y} = \phi_x^y$ and $\Psi_x^y|_{I_x^y} = \phi_x^{y\,-1}$.*

Lemma 4.78. *For $x \leq y \leq z$ in S, $\Phi_x^z = \Phi_y^z \circ \Phi_x^y$ and $\Psi_x^z = \Psi_y^x \circ \Psi_z^y$.*

Proof.

The proof of $\Phi_x^z = \Phi_y^z \circ \Phi_x^y$ has several cases. Assume $a \in L_x$.

If $x \not\mathrel{\gamma} y$ (and consequently $x \not\mathrel{\gamma} z$): $\Phi_x^z\, a = \hat{0}_{L_z} = \Phi_y^z\, \hat{0}_{L_y} = \Phi_y^z(\Phi_x^y\, a)$.

If $y \not\mathrel{\gamma} z$ (and consequently $x \not\mathrel{\gamma} z$): $\Phi_x^z\, a = \hat{0}_{L_z} = \Phi_y^z(\Phi_x^y\, a)$.

If $y \mathrel{\gamma} x, z$ but $x \mathrel{\not\gamma} z$:

$$\begin{aligned}\Phi^z_x\, a &= \hat{0}_{L_z}\\ &= \phi^z_y(0^z_y)\\ &= \Phi^z_y(0^z_y)\end{aligned}$$

since $x \mathrel{\not\gamma} z$, by (MC5), I^y_x, the range of ϕ^y_x, is disjoint from F^z_y, so $\phi^y_x\, a \leq 1^y_x < 0^z_y$,

$$\begin{aligned}&= \Phi^z_y(\phi^y_x\, a \vee 0^z_y)\\ &= \Phi^z_y(\Phi^y_x\, a)\end{aligned}$$

If $x \mathrel{\gamma} z$ (and consequently $y \mathrel{\gamma} x, z$):

$$\Phi^z_x\, a = \phi^z_x(0^z_x \vee a)$$

by (MC6),

$$= \phi^z_y(\phi^y_x(0^z_x \vee a))$$

because $0^z_x \geq 0^y_x$,

$$= \phi^z_y(\phi^y_x(0^z_x \vee 0^y_x \vee a))$$

because ϕ^y_x is an isomorphism,

$$= \phi^z_y(\phi^y_x(0^z_x) \vee \phi^y_x(0^y_x \vee a))$$

because $\phi^y_x(0^z_x) = 0^z_y$,

$$\begin{aligned}&= \phi^z_y(0^z_y \vee \phi^y_x(0^y_x \vee a))\\ &= \Phi^z_y(\Phi^y_x\, a)\end{aligned}$$

We prove $\Psi^z_x = \Psi^x_y \circ \Psi^y_z$ dually. □

The adjoint functions allow us to restate lem. 4.59, 4.60, 4.63, and 4.64 in this form:

Lemma 4.79. *If $x \leq y$ in S, $a \in \Lambda_x$, and $b \in \Lambda_y$, then*

$$a \vee b = \pi_y(\Phi^y_x(\pi_x^{-1}\, a) \vee \pi_y^{-1}\, b).[9]$$

Or equivalently: If $c \in L_x$, and $d \in L_y$, then

$$\pi_x\, c \vee \pi_y\, d = \pi_y(\Phi^y_x\, c \vee d).$$

Proof.
Regarding the first statement:
Choose a saturated chain in S from x to y, $x = x_0 \lessdot x_1 \lessdot \cdots \lessdot x_n = y$ for some $n \geq 0$. By lem. 4.59,

$$a \vee b = ((\cdots((a \vee_{x_0} 0_{x_1}) \vee_{x_1} 0_{x_2}) \vee_{x_2} \cdots) \vee_{x_{n-1}} 0_{x_n}) \vee_y b. \tag{a}$$

We prove the lemma by induction on n.
If $n = 0$, then $x = x_0 = x_n = y$, and (a) reduces to

$$a \vee b = a \vee_y b$$

by def. 4.56 and $x = y$,

$$= \pi_y(\pi_x^{-1}\, a \vee \pi_y^{-1}\, b)$$

by lem. 4.75, Φ^y_x is the identity,

$$= \pi_y(\Phi^y_x(\pi_x^{-1}\, a) \vee_y \pi_y^{-1}\, b)$$

If $n > 0$, showing more terms of (a),

$$a \vee b = (((\cdots((a \vee_{x_0} 0_{x_1}) \vee_{x_1} 0_{x_2}) \vee_{x_2} \cdots) \vee_{x_{n-2}} 0_{x_{n-1}}) \vee_{x_{n-1}} 0_{x_n}) \vee_y b$$

[9]Note that the first $\vee$ is in L and the second is in L_y.

since $0_{x_{n-1}}$ is the identity of $\vee_{x_{n-1}}$,

$$= ((((\cdots((a \vee_{x_0} 0_{x_1}) \vee_{x_1} 0_{x_2}) \vee_{x_2} \cdots) \vee_{x_{n-2}} 0_{x_{n-1}}) \vee_{x_{n-1}} 0_{x_{n-1}}) \vee_{x_{n-1}} 0_{x_n}) \vee_y b$$

using lem. 4.59 on $a \vee 0_{x_{n-1}}$,

$$= ((a \vee 0_{x_{n-1}}) \vee_{x_{n-1}} 0_y) \vee_y b$$

defining $z = x_{n-1}$ and using $y = x_n$,

$$= ((a \vee 0_z) \vee_z 0_y) \vee_y b$$

using the induction hypothesis on $a \vee 0_z$, since $0_z \in \Lambda_z = \Lambda_{x_{n-1}}$,

$$= (\pi_z(\Phi_x^z(\pi_x^{-1}\, a) \vee \pi_z^{-1}\, 0_z) \vee_z 0_y) \vee_y b$$

using def. 4.56 on $\vee_z$,

$$= \pi_z((\Phi_x^z(\pi_x^{-1}\, a) \vee \pi_z^{-1}\, 0_z) \vee \pi_z^{-1}\, 0_y) \vee_y b$$

since $\pi_z^{-1}\, 0_z = \hat{0}_{L_z}$ and $\Phi_x^z(\pi_x^{-1}\, a) \in L_z$,

$$= \pi_z(\Phi_x^z(\pi_x^{-1}\, a) \vee \pi_z^{-1}\, 0_y) \vee_y b$$

since $\pi_z^{-1}\, 0_y = 0_z^y$,

$$= \pi_z(\Phi_x^z(\pi_x^{-1}\, a) \vee 0_z^y) \vee_y b$$

since $\Phi_x^z(\pi_x^{-1}\, a) \vee 0_z^y \in F_z^y$ and $\pi_z = \pi_y \circ \phi_z^y$ on F_z^y,

$$= \pi_y(\phi_z^y(\Phi_x^z(\pi_x^{-1}\, a) \vee 0_z^y)) \vee_y b$$

by def. 4.71,

$$= \pi_y(\Phi_z^y(\Phi_x^z(\pi_x^{-1}\, a))) \vee_y b$$

by lem. 4.78,

$$= \pi_y(\Phi_x^y(\pi_x^{-1}\, a)) \vee_y b$$

using def. 4.56 on $\vee_y$,

$$= \pi_y(\Phi_x^y(\pi_x^{-1}\, a) \vee \pi_y^{-1}\, b)$$

Regarding the second statement: This follows from the first statement by setting $a = \pi_x\, c$ and $b = \pi_y\, d$. □

Lemma 4.80. *If $a \in \Lambda_x$, and $b \in \Lambda_y$, then*

$$a \vee b = \pi_{x\vee y}(\Phi_x^{x\vee y}(\pi_x^{-1}\, a) \vee \Phi_y^{x\vee y}(\pi_y^{-1}\, b)).$$ [10]

Or equivalently: If $c \in L_x$, and $d \in L_y$, then

$$\pi_x\, c \vee \pi_y\, d = \pi_{x\vee y}(\Phi_x^{x\vee y}\, c \vee \Phi_x^{x\vee y}\, d).$$

Proof.

Regarding the first statement: By lem. 4.60,

$$a \vee b = (a \vee 0_{x\vee y}) \vee_{x\vee y} (b \vee 0_{x\vee y})$$

applying lem. 4.79 to both terms,

$$= \pi_{x\vee y}(\Phi_x^{x\vee y}(\pi_x^{-1}\, a) \vee \pi_{x\vee y}^{-1}\, 0_{x\vee y}) \vee_{x\vee y} \pi_{x\vee y}(\Phi_y^{x\vee y}(\pi_y^{-1}\, b) \vee \pi_{x\vee y}^{-1}\, 0_{x\vee y})$$

since $\pi_{x\vee y}^{-1}\, 0_{x\vee y} = \hat{0}_{L_{x\vee y}}$ and the ranges of $\Phi_x^{x\vee y}$ and $\Phi_y^{x\vee y}$ are $\subset L_{x\vee y}$,

$$= \pi_{x\vee y}(\Phi_x^{x\vee y}(\pi_x^{-1}\, a)) \vee_{x\vee y} \pi_{x\vee y}(\Phi_y^{x\vee y}(\pi_y^{-1}\, b))$$

using def. 4.56 on $\vee_{x\vee y}$,

[10] Note that the first $\vee$ is in L and the second $\vee$ is in $L_{x\vee y}$.

$$= \pi_{x\vee y}(\Phi_x^{x\vee y}(\pi_x^{-1}\, a) \vee \Phi_y^{x\vee y}(\pi_y^{-1}\, b))$$

Regarding the second statement: This follows from the first statement by setting $a = \pi_x\, c$ and $b = \pi_x\, d$. □

Dually, we prove:

Lemma 4.81. *If $x \leq y$ in S, $a \in \Lambda_x$, and $b \in \Lambda_y$, then*

$$a \wedge b = \pi_x(\pi_x^{-1}\, a \wedge \Psi_x^y(\pi_y^{-1}\, b)).$$

Or equivalently: If $c \in L_x$, and $d \in L_y$, then

$$\pi_x\, c \wedge \pi_y\, d = \pi_x(c \wedge \Psi_x^y\, d).$$

Lemma 4.82. *If $a \in \Lambda_x$, and $b \in \Lambda_y$, then*

$$a \wedge b = \pi_{x\wedge y}(\Psi_{x\wedge y}^x(\pi_x^{-1}\, a) \wedge \Psi_{x\wedge y}^y(\pi_y^{-1}\, b)).$$

Or equivalently: If $c \in L_x$, and $d \in L_y$, then

$$\pi_x\, c \wedge \pi_y\, d = \pi_{x\wedge y}(\Psi_{x\wedge y}^x\, c \wedge \Psi_{x\wedge y}^x\, d).$$

4.11. Gluing is natural. Gluing is "natural" in that it is compatible with isomorphism of m.c. systems (see def. 3.5) and isomorphism of lattices.

Theorem 4.83. *Gluing is compatible with isomorphism of m.c. systems and isomorphism of lattices: If two m.c. systems are isomorphic, their sum lattices are isomorphic.*

Proof.[11]
Assume we have two m.c. systems, $\mathcal{C}$ (comprised of skeleton S, overlap tolerance γ, blocks $L_\bullet$, connections $\phi_\bullet^\bullet$, connection sources $F_\bullet^\bullet$, and connection targets $I_\bullet^\bullet$) and $\mathcal{C}'$ (comprised of skeleton S', overlap tolerance γ', blocks $L'_\bullet$, connections $\phi_\bullet^{\bullet\prime}$, connection sources $F_\bullet^{\bullet\prime}$, and connection targets $I_\bullet^{\bullet\prime}$). Thus for $\mathcal{C}$ we have derived objects M, $\sim$, κ, $\pi_\bullet$, and its sum L, and for $\mathcal{C}'$ derived objects M', $\sim'$, κ', $\pi'_\bullet$ and its sum L'.

Assume we have an isomorphism χ from $\mathcal{C}$ to $\mathcal{C}'$. Thus, χ_S is a lattice isomorphism from S to S' and $(\chi_{Bx})_{x\in S}$ is a family of lattice isomorphisms, each from L_x to $L'_{\chi_S(x)}$. Then immediately from the definitions we have:

1. From def. 4.11, for $x \in S$ and $a \in \bigcup_{w\in S} L_w$, $(x, a) \in M$ iff $(\chi_S\, x, \chi_{Bx}\, a) \in M'$.
2. From def. 4.11, for $(x, a), (y, b) \in M$, $(x, a) \sim (y, b)$ iff $(\chi_S\, x, \chi_{Bx}\, a) \sim' (\chi_S\, y, \chi_{By}\, b)$.
3. From def. 4.16, for $(x, a), (y, b) \in M$, $\kappa\,(x, a) = \kappa\,(y, b)$ iff $\kappa'\,(\chi_S\, x, \chi_{Bx}\, a) = \kappa'\,(\chi_S\, y, \chi_{By}, b)$.
4. From def. 4.17, for $(x, a), (y, b) \in M$, $\pi_x\, a = \pi_y\, b$ iff $\pi'_{\chi_S\, x}\, \chi_{Bx}\, a = \pi'_{\chi_S\, y}\, \chi_{By}\, b$.

For $(x, a), (y, b) \in M$, we define $\chi_L(\kappa((x, a))) = \kappa'((\chi_S\, x, \chi_{Bx}\, a))$ from L to L'. Because of (3) and since by def. 4.15, L is M modulo $\sim$ and L' is M' modulo $\sim'$, χ_L is well-defined as a function from L to L': $\kappa'((\chi_S\, x, \chi_{Bx}\, a))$ is the same for all (x, a) that are $\sim$. Since χ_S and the $\chi_{B\bullet}$ are bijections, $\chi_L(\kappa((\chi_S^{-1}\, x', \chi_{B\chi_S^{-1}(x)}^{-1}\, a'))) = \kappa'((x', a'))$ for all $(x', a') \in M$, showing that χ_L is bijective.

Then it is straightforward that for any $a, b \in L$, there exists an ascending sequence $(x_\bullet, a_\bullet)$ with $a = a_0$ and $b = a_n$ iff there exists an ascending sequence $(x'_\bullet, a'_\bullet)$ with $\chi_L\, a = a'_0$ and $\chi_L\, b = a'_n$. From this, it follows from def. 4.29 that for any $a, b \in L$, $a \leq b$ iff $\chi_L\, a \leq \chi_L\, b$, which shows that L and L' are isomorphic as posets and consequently isomorphic as lattices. □

5. Dissection

The dissection of a lattice into maximal complemented blocks was introduced in Herrmann [Herr1973a, sec. 6] [Herr1973a-en, sec. 6]. We mostly use Haiman's notation [Haim1991a, sec. 1].

In this section, let L be a modular, l.f.f.c. lattice.

[11] It seems like there ought to be a metatheorem that takes note of how the sum of a m.c. system is defined and immediately concludes that the process is natural. But I do not know of one.

5.1. Basic properties of dissection.

Definition 5.1. [Herr1973a, sec. 6][Herr1973a-en, sec. 6] *For $x \in L$:*

$$x^* = \begin{cases} \hat{1}_L & \text{if } \hat{1}_L \text{ exists and } a = \hat{1}_L \\ \bigvee_{y, x \lessdot y} y & \text{otherwise} \end{cases}$$

$$x_* = \begin{cases} \hat{0}_L & \text{if } \hat{0}_L \text{ exists and } a = \hat{0}_L \\ \bigwedge_{y, y \lessdot x} y & \text{otherwise} \end{cases}$$

Lemma 5.2. *x^* and x_* are well-defined.*

Proof.
That x^* and x_* are well-defined follows from L having finite covers. □

Lemma 5.3. [Herr1973a, Lem. 6.1][Herr1973a-en, Lem. 6.1] *For all $a, b \in L$:*

(a) If $a \leq b$, then $a^* \leq b^*$.
(a^δ) If $a \leq b$, then $a_* \leq b_*$.
(b) $a \leq {a_*}^* \leq a^*$.
(b^δ) $a_* \leq {a^*}_* \leq a$.
(c) ${{a^*}_*}^* = a^*$.
(c^δ) ${{a_*}^*}_* = a_*$.
(d) If $a = {a^*}_*$ and $b = {b^*}_*$, then $a \vee b = {(a \vee b)^*}_*$.
(d^δ) If $a = {a_*}^*$ and $b = {b_*}^*$, then $a \wedge b = {(a \wedge b)_*}^*$.
(e) $a_* \vee b_* = (a \vee b)_*$.
(e^δ) $a^* \wedge b^* = (a \wedge b)^*$.

Proof.
Items (a) through (e^δ) are proved in [Herr1973a, Lem. 6.1][Herr1973a-en, Lem. 6.1]. □

Remark 5.4. Although $\bullet^*$ and $\bullet_*$ resemble a Galois connection between L and itself, in general they are not:[WikiGal] Define $L = \{0,1,2\} \times \{0,1\}$, the product of a 2-chain and a 1-chain. Define $x = (0,1)$ and $y = (2,1)$. Then $x_* = (0,0)$, $x^* = (1,1)$, $y_* = (1,0)$, and $y^* = (2,1)$. Thus $x_* = (1,1) \leq (2,2) = y$ but $x = (0,1) \not\leq (1,0) = y_*$, showing the two operations are not adjoints.

Lemma 5.5. [Herr1973a, sec. 6][Herr1973a-en, sec. 6] *For any x, the intervals $[x, x^*]$ and $[x_*, x]$ are finite, modular, and complemented.*

Proof.
These intervals are modular because L is modular. They are finite because L is locally finite. For $[x, x^*]$, the maximum of the interval is a join of atoms and for $[x_*, x]$, the minimum of the interval is a meet of coatoms, so in either case th. 2.9 can be applied to show that the interval is complemented. □

Definition 5.6. *A* block *of L is a maximal (under inclusion) complemented interval of L. The* dissection skeleton *S of L is the set of all blocks of L. Given a block $B \in S$, 0_B is the minimum element of the block and 1_B is the maximum element of the block.*

Lemma 5.7. *If two blocks are distinct, their minimum elements are distinct and their maximum elements are distinct.*

Proof.
Let B and C be two blocks with $0_B = 0_C$. Since B is complemented, by th. 2.9, 1_B is the join of atoms of B, and so $1_B \leq {0_B}^*$ Since B is a maximal complemented interval and $[0_B, {0_B}^*]$ is a complemented interval, $1_B = {0_B}^*$.
Similarly we show $1_C = {0_C}^*$. But since $0_B = 0_C$, this implies $1_B = 1_C$ and thus B and C are identical.
Dually, we show that $1_B = 1_C$ implies B and C are identical. □

Lemma 5.8. [Herr1973a, Satz 6.2 and 6.4][Herr1973a-en, Th. 6.2 and 6.4]

(1) *$[x, y]$ is a block iff $x^* = y$ and $y_* = x$.*
(2) *The set of minimum elements of blocks is a lattice under the join operation $x \vee y$ and the meet operation ${(x \wedge y)^*}_*$. Consequently, this set is a sub-*sup*-semilattice (and thus sub-poset) of L.*
(3) *The set of maximum elements of blocks is a lattice under the join operation ${(x \vee y)_*}^*$ and the meet operation $x \wedge y$. Consequently, this set is a sub-*inf*-semilattice (and thus sub-poset) of L.*

(4) *The set of minimum elements and the set of maximum elements are isomorphic lattices under the mutually inverse isomorphisms $x \mapsto x^*$ and $x \mapsto x_*$.*

Proof.

These are demonstrated in [Herr1973a, Satz 6.2, Lem. 6.3, and Satz 6.4][Herr1973a-en, Th. 6.2, Lem. 6.3, and Th. 6.4] ☐

Definition 5.9. *We define $\leq$, $\vee$, and $\wedge$ on blocks in S as the corresponding operations on the minimum elements of the blocks, which by lem. 5.8(4) are identical to the corresponding operations on the maximum elements of the blocks.*

Lemma 5.10. *S is a lattice under the operations defined in def. 5.9.*

Definition 5.11. *The dissection tolerance*[12] *γ on S is the relationship between blocks B and C, $B \mathrel{\gamma} C$ iff $B \cap C \neq \emptyset$.*

Lemma 5.12. *The dissection tolerance γ of L is a tolerance on S.*

Proof.

The dissection tolerance is clearly reflexive and symmetric. It remains to be shown that it is compatible with the meet and join operations on S.

Let there be four blocks P, Q, R, and S for which $P \mathrel{\gamma} Q$ and $R \mathrel{\gamma} S$. Then there exist $x, y \in L$ for which $x \in [0_P, 1_P] \cap [0_Q, 1_Q]$ and $y \in [0_R, 1_R] \cap [0_S, 1_S]$.

Then $0_P \leq x \leq 1_P$ and $0_R \leq y \leq 1_R$, so $0_P \vee 0_R \leq x \vee y \leq 1_P \vee 1_R \leq {(1_P \vee 1_R)_*}^* = 1_{P \vee R}$ by lem. 5.3(b) and 5.8(3). Thus $x \vee y \in [0_{P\vee R}, 1_{P\vee R}]$.

Similarly $0_Q \leq x \leq 1_Q$ and $0_S \leq y \leq 1_S$, so $0_Q \vee 0_S \leq x \vee y \leq 1_Q \vee 1_S \leq {(1_Q \vee 1_S)_*}^* = 1_{Q\vee S}$. Thus $x \vee y \in [0_{Q\vee S}, 1_{Q\vee S}]$.

This shows that $P \vee R$ and $Q \vee S$ both contain $x \vee y$ and so $P \vee R \mathrel{\gamma} Q \vee S$.

Dually, we prove that $P \wedge R \mathrel{\gamma} Q \wedge S$. These show that γ is compatible with meet and join. ☐

Definition 5.13. *For any blocks B, C with $B \leq C$ and $B \cap C \neq \emptyset$, define $F_B^C = I_B^C = B \cap C$. Define ϕ_B^C as the identity map from F_B^C to I_B^C.*

Lemma 5.14. *For any blocks B, C with $B \leq C$ and $B \cap C \neq \emptyset$, $F_B^C = I_B^C = [0_C, 1_B]$, which is a filter of B and an ideal of C.*

Proof.

Given that $B \leq C$, we know $0_B \leq 0_C$ and $1_B \leq 1_C$. That implies

$$\begin{aligned} B \cap C &= \{x \in L \mid 0_B \leq x \leq 1_B \text{ and } 0_C \leq x \leq 1_C\} \\ &= \{x \in L \mid 0_C \leq x \leq 1_B\} \\ &= [0_C, 1_B] \end{aligned}$$

☐

Lemma 5.15. *For any block B, $F_B^B = I_B^B = B$ and ϕ_B^B is the identity map from B to itself.*

Lemma 5.16. *For blocks $B \lessdot C$ in S, $B \cap C \neq \emptyset$, $B \not\subset C$, and $C \not\subset B$.*

Proof.

By lem. 5.7, $0_B < 0_C$ and $1_B < 1_C$. Thus $0_B \in B$ but $0_B \notin C$, showing $B \not\subset C$. And $1_C \in C$ but $1_C \notin B$, showing $C \not\subset B$.

Let x be a minimal element in $[0_B, 0_C]$ for which ${x^*}_* > 0_B$. Necessarily ${x^*}_* = x$ as otherwise we would have ${x^*}_* < x$, which would allow $y = {x^*}_*$ to be a smaller element with ${y^*}_* = {{{x^*}_*}^*}_* = {x^*}_* > 0_B$.

Let X be the block $[x, x^*]$. By construction, $B < X \leq C$ and since $B \lessdot C$, $C = X$. Since $x > 0_B$, choose y such that $0_B \leq y \lessdot x$. $y^* = {{y^*}_*}^* = {0_B}^* = 1_B$, so every upward cover of y is $\leq 1_B$. This implies $x \in B$. Since $x \in X = C$, $x \in B \cap C$ and $B \cap C \neq \emptyset$. ☐

Lemma 5.17. *S is a l.f.f.c. lattice.*

---

[12][DayHerr1988a] calls the dissection tolerance the "glue tolerance".

Proof.

Because S is a subposet of L and L is locally finite, S is locally finite.

Given any $B \in S$, for each $C \gtrdot B$ in S, $B \cap C \neq \emptyset$ by lem. 5.16. By lem. 5.14, $0_C \in B$. Since every distinct C has a distinct 0_C (by lem. 5.7) and B is finite, there are a finite number of distinct blocks $C \gtrdot B$. Thus, S has finite upward covers.

Dually, we prove S has finite downward covers. ☐

Lemma 5.18. *For blocks X, Y, and Z with $X \leq Z \leq Y$, $X \cap Y \subset Z$ and $X \cap Y = (X \cap Z) \cap (Z \cap Y)$.*

Proof.

$$
\begin{aligned}
0_Z \leq 0_Y &\text{ and } 1_X \leq 1_Z \\
0_Z \leq 0_X \vee 0_Y &\text{ and } 1_X \wedge 1_Y \leq 1_Z \\
[0_X \vee 0_Y, 1_X \wedge 1_Y] &\subset [0_Z, 1_Z] \\
[0_X, 1_X] \cap [0_Y, 1_Y] &\subset [0_Z, 1_Z] \\
X \cap Y &\subset Z \\
X \cap Y &= X \cap Z \cap Y \\
X \cap Y &= (X \cap Z) \cap (Z \cap Y)
\end{aligned}
$$

☐

Lemma 5.19. *For blocks X and Y, $X \cap Y = (X \wedge Y) \cap (X \vee Y)$ where $X \vee Y$ and $X \wedge Y$ are the join and meet operations in S.*

Proof.

$$
\begin{aligned}
X \cap Y &= [0_X, 1_X] \cap [0_Y, 1_Y] \\
&= [0_X \vee 0_Y, 1_X \wedge 1_Y]
\end{aligned}
$$

by lem. 5.8(2)(3),

$$= [0_{X \vee Y}, 1_{X \wedge Y}]$$

since $X \wedge Y \leq X \vee Y$ and def. 5.9 imply $0_{X \wedge Y} \leq 0_{X \vee Y}$ and $1_{X \wedge Y} \leq 1_{X \vee Y}$,

$$
\begin{aligned}
&= [0_{X \wedge Y} \vee 0_{X \vee Y}, 1_{X \wedge Y} \wedge 1_{X \vee Y}] \\
&= [0_{X \wedge Y}, 1_{X \wedge Y}] \cap [0_{X \vee Y}, 1_{X \vee Y}] \\
&= (X \wedge Y) \cap (X \vee Y)
\end{aligned}
$$

☐

Lemma 5.20. *For blocks X and Y,*

(1) $X \cap Y \subset X \vee Y$,
(2) $X \cap Y \subset X \wedge Y$,
(3) $X \cap Y \cap (X \vee Y) = (X \wedge Y) \cap (X \vee Y)$, *and*
(4) $X \cap Y \cap (X \wedge Y) = (X \vee Y) \cap (X \wedge Y)$,[13]

where $X \vee Y$ and $X \wedge Y$ are the join and meet operations in S.

Proof.

These are all immediate from lem. 5.19. ☐

Theorem 5.21. *For any modular l.f.f.c. lattice L,*

(1) *the skeleton lattice S (of blocks),*
(2) *the dissection tolerance γ,*
(3) *the blocks S, indexed by themselves (as elements of S), and*

[13]Statement (4) is dual statement (3), but that is hard to see from the formulas: Duality on L interchanges meet ($\wedge$) and join ($\vee$) as expected, but it leaves the intersection of blocks ($\cap$) unchanged.

(4) *the connections* $\phi_\bullet^\bullet$

form a m.c. system.

Proof.

We prove the axioms:

Regarding (MC1): By lem. 5.17, S is a l.f.f.c. lattice.

Regarding (MC2): By lem. 5.5, each block in S is finite, modular, and complemented.

Regarding (MC3): By def. 5.13, each ϕ_B^C is the identity map on the lattice $F_B^C = I_B^C$, and so is a lattice isomorphism. By lem. 5.14, each F_B^C is a filter of B and and I_B^C is an ideal of C.

Regarding (MC4): This is proved as lem. 5.15.

Regarding (MC5): If $B \le C \le D$ in S and $I_B^C \cap F_C^D \ne \emptyset$, then by def. 5.13 and lem. 5.18, $\emptyset \ne (B \cap C) \cap (C \cap D) = B \cap D$, implying $B \;\gamma\; D$.

Regarding (MC6): This follows from lem. 5.18 and that all of the $\phi_\bullet^\bullet$ are identity maps.

Regarding (MC7): This is proved as lem. 5.20.

Regarding (MC8): This is proved as lem. 5.16. ☐

Definition 5.22. *We define the m.c. system (S, γ, S, ϕ) described in th. 5.21 as the* dissection *of L.*

Theorem 5.23. *The lattice L has a minimum element iff its skeleton lattice S has a minimum element.*

Proof.

Regarding $\Rightarrow$: Assume L has a minimum element $\hat{0}_L$. Then $X = [\hat{0}_L, (\hat{0}_L)^*]$ is a block with minimum element $\hat{0}_L$. By lem. 5.8(2), the fact that $\hat{0}_L$ is the minimum of L implies X is $\le$ all blocks in S.

Regarding $\Leftarrow$: Assume S has a minimum element $\hat{0}_S$. Every element c of L is a member of the block $[c^*{}_*, c^*]$, By lem. 5.8(2), the fact that $\hat{0}_S$ is the minimum of S implies that $0_{\hat{0}_S} \le c^*{}_*$. Thus $0_{\hat{0}_S} \le c^*{}_* \le$c, showing that $\hat{0}_L = 0_{\hat{0}_S}$. ☐

5.2. Blocks containing an element.

Definition 5.24. *For $x \in L$, we define $\nabla_x = [x_*, x_*{}^*]$ and $\Delta_x = [x^*{}_*, x^*]$.*

Lemma 5.25. *For $x \in L$, ∇_x and Δ_x are blocks $\in S$, $x \in \nabla_x$ and $x \in \Delta_x$.*

Proof.

By lem. 5.3(c)(c$^\delta$) and lem. 5.8(10), $[x^*{}_*, x^*]$ and $[x_*, x_*{}^*]$ are both blocks. By lem. 5.3(b)(b$^\delta$), $x \in [x^*{}_*, x^*]$ and $x \in [x_*, x_*{}^*]$. ☐

Definition 5.26. Γ_x *for $x \in L$ is defined as the set of blocks containing x.*

Note that Γ_x in a dissection is the parallel of Π_x in a gluing.

Lemma 5.27. *For $x \in L$, Γ_x is convex.*

Proof.

Suppose $B, D \in \Gamma_x$ and $C \in S$ with $B \le C \le D$. Then $0_B \le x \le 1_B$, $0_D \le x \le 1_D$, $0_B \le 0_C \le 0_D$ and $1_B \le 1_C \le 1_D$. From these follows $0_C \le 0_D \le x \le 1_B \le 1_C$, so $C \in \Gamma_x$. ☐

Lemma 5.28. [Haim1991a, sec. 1] ∇_x *is the lowest block in* Γ_x *and* Δ_x *is the highest block in* Γ_x.[14]

Proof.

By lem. 5.25, both ∇_x and Δ_x are $\in \Gamma_x$.

Let $B \in \Gamma_x$ and so $x \in B$, $x \le 1_B$, so $x_* \le (1_B)_* = 0_B$. Thus $\nabla_x = [x_*, x_*{}^*] \le B$ by def. 5.9. Similarly, $x \ge 0_B$, so $x^* \ge (0_B)^* = 1_B$. Thus $\Delta_x = [x^*{}_*, x^*] \ge B$ by def. 5.9. ☐

Lemma 5.29. *For $x \in L$, Γ_x is the interval $[\nabla_x, \Delta_x]$ of S, which is finite.*

Proof.

By lem. 5.28, Γ_x has maximum element Δ_x and minimum element ∇_x and by lem. 5.27, Γ_x is convex. Thus Γ_x is the interval $[\nabla_x, \Delta_x]$ of S. By lem. 5.17, S is locally finite, so Γ_x is finite. ☐

[14] Thus there is this mnemonic for $\nabla_\bullet$ and $\Delta_\bullet$: ∇_x points *downward* and is the *lowest* block containing x, and conversely $\Delta_\bullet$ points *upward* and is the *highest* block containing x.

5.3. Dissection is natural. Dissection is "natural" in that it is compatible with isomorphism of lattices and isomorphism of m.c. systems (see def. 3.5).

Theorem 5.30. *Dissection is compatible with isomorphism of lattices and isomorphism of m.c. systems: If two lattices are isomorphic, their dissections are isomorphic.*

Proof.

Assume we have two lattices, L and L', and an isomorphism χ from L to L'.

Let $\mathcal{C}$ be the dissection of L and be comprised of skeleton S, overlap tolerance γ, blocks $L_\bullet$, connections $\phi_\bullet^\bullet$, connection sources $F_\bullet^\bullet$, and connection targets $I_\bullet^\bullet$. Let $\mathcal{C}'$ be the dissection of L' and be comprised of skeleton S', overlap tolerance γ', blocks $L'_\bullet$, connections $\phi_\bullet^{\bullet\prime}$, connection sources $F_\bullet^{\bullet\prime}$, and connection targets $I_\bullet^{\bullet\prime}$. *** [Thus for $\mathcal{C}$ we have derived objects M, $\sim$, κ, $\pi_\bullet$, and its sum L, and for $\mathcal{C}'$ derived objects M', $\sim'$, κ', $\pi'_\bullet$ and its sum L'.]

Define χ_S from S to S' that maps a block $x = [0_x, 1_x] \in S$ with $0_x, 1_x \in L$ into $[\chi\, 0_x, \chi\, 1_x] \in S'$. Since χ is a lattice isomorphism, $[\chi\, 0_x, \chi\, 1_x]$ is a block of L'. Conversely, $\chi_S^{-1}\, x' = \chi_S^{-1}\, [0_{x'}, 1_{x'}] = [\chi^{-1}\, 0_{x'}, \chi^{-1}\, 1_{x'}]$ so χ_S^{-1} maps the blocks of L' to blocks of L, showing that χ_S is a bijection. Since χ is a lattice isomorphism, def. 5.9 shows that χ_S is a lattice isomorphism from S to S'.

For any $x \in S$, define χ_{Bx} on L_x by $\chi_{Bx}\, a = \chi\, a$ for all $a \in L_x$. Then given $a \in L_x$,

$$a \in [0_x, 1_x]$$

because χ is an isomorphism,

$$\chi\, a \in [\chi\, 0_x, \chi\, 1_x]$$

by the definition of χ_S,

$$\chi\, a \in \chi_S\, [0_x, 1_x] = \chi_S\, x$$

by the definition of χ_{Bx},

$$\chi_{Bx}\, a \in \chi_S\, x$$

since the blocks in S' are their own indexes (th. 5.21),

$$\chi_{Bx}\, a \in L'_{\chi_S\, x}$$

showing that χ_{Bx} maps L_x to $L'_{\chi_S\, x}$.

Similarly, $\chi_{Bx}^{-1}\, a' = \chi^{-1}\, a'$ and χ^{-1} maps $L'_{\chi_S\, x}$ to L_x. Those facts, combined with the fact that χ is a lattice isomorphism, shows that χ_{Bx} is a lattice isomorphism from L_x to $L'_{\chi_S\, x}$.

Since γ on S and γ' on S' are defined as non-null intersection of blocks (def. 5.11) and χ_S on S is defined as applying the bijection χ on the elements of its argument, for $x, y \in S$, $x\ \gamma\ y$ iff $\chi_S(x)\ \gamma'\ \chi_S(y)$.

Given $x \leq_\gamma y \in S$, by lem. 5.14, $F_x^y = [0_y, 1_x]$, so

$$\chi_{Bx}(F_x^y) = \chi_{Bx}([0_y, 1_x])$$

by the definition of χ_{Bx},

$$= \chi([0_y, 1_x])$$

applying χ to each member of $[0_y, 1_x]$,

$$= [\chi\, 0_y, \chi\, 1_x]$$

by the definition of χ_S,

$$= [0_{\chi_S\, y}, 1_{\chi_S\, x}]$$

by lem. 5.14,

$$= F_{\chi_S\, x}^{\chi_S\, y\,\prime}$$

Thus showing that $F_{\chi_S(x)}^{\chi_S(y)\prime} = \chi_{Bx}(F_x^y)$.

Similarly, we show that $I_{\chi_S(x)}^{\chi_S(y)\prime} = \chi_{By}(I_x^y)$.

Since the connections $\phi_\bullet^\bullet$ in dissections are identity maps (def. 5.13), for all $a \in F_x^y$, $\chi_{By}(\phi_x^y(a)) = \chi_{By}(a) = \phi_{\chi_S(x)}^{\chi_S(y)'}(\chi_{Bx}(a))$, or equivalently, $\chi_{By} \circ \phi_x^y = \phi_{\chi_S(x)}^{\chi_S(y)'} \circ \chi_{Bx}|_{F_x^y}$.

These facts together show that χ_S and $\chi_{B\bullet}$ comprise a m.c. system isomorphism (def. 3.5 from $\mathcal{C}$ to $\mathcal{C}'$. □

6. Dissection of a gluing

In this section, we prove that if $\mathcal{C}'$ is the dissection of the sum lattice L of a m.c. system $\mathcal{C}$, then $\mathcal{C}'$ is isomorphic to $\mathcal{C}$.

We choose $\mathcal{C}$ to be an arbitrary m.c. system. We define L to be the sum (def. 4.15) of $\mathcal{C}$. We define $\mathcal{C}'$ to be the dissection (def. 5.22) of L.

From $\mathcal{C}$, we define its components: the skeleton S, the overlap tolerance γ, and the blocks $L_\bullet$, From $\mathcal{C}$, we define its derived objects: the sum L, the canonical projections (def. 4.17) $\pi_\bullet$, the blocks $\Lambda_\bullet$ (def. 4.18) of L, and the elements $0_\bullet$ and $1_\bullet$ (def. 4.18) in L.

From L, we define its derived objects: the set of blocks (def. 5.6) S', the dissection tolerance (def. 5.11) γ', and the dissection $\mathcal{C}'$.

Definition 6.1. *We define χ_S: for any $x \in S$, $\chi_S(x)$ is the interval $\Lambda_x = [0_x, 1_x]$ of L.*

Lemma 6.2. *For each $x \in S$, Λ_x is a block of L.*

Proof.

$\hat{1}_{L_x}$ is the join of the atoms of L_x. Λ_x is isomorphic to L_x and the atoms of L_x map via π_x to elements that cover 0_x. Thus, ${0_x}^* \geq 1_x$.

Suppose there is an element $a \in L$ covering 0_x that is not in Λ_x. Then by lem. 4.48 there is a $y \in S$ such that 0_x and a are in Λ_y. Then $0_x \in \Lambda_y$, so $0_x \geq 0_y$. By lem. 4.51, $x \geq y$. But if $x = y$, then $a \in \Lambda_x$, which is contrary to hypothesis, so we must have $x > y$. By lem. 4.51, $1_x \geq 1_y$. Then, $0_x \leq a \leq 1_y \leq 1_x$, and since $\Lambda_x = [0_x, 1_x]$, $a \in \Lambda_x$, contrary to hypothesis. Thus all $a \gtrdot 0_x$ are $\in \Lambda_x$ and ${0_x}^* \leq 1_x$.

So we have shown $(0_x)^* = 1_x$. Dually, we show $(1_x)_* = 0_x$. Thus by lem. 5.8(1), $\Lambda_x = [0_x, 1_x]$ is a block. □

Lemma 6.3. *Every block of L is a Λ_x for some $x \in S$.*

Proof.

Let $[a, b]$ be a block of L. Assuming L is not the trivial lattice, $a \neq b$. By lem. 5.8(1), $a^* = b$ and $b_* = a$.

Let $c_1, c_2, \ldots, c_n$ be the atoms of $[a, b]$, that is, the elements of $[a, b]$ that cover a. (This is a finite set because L has finite covers.) For each c_i, by lem. 4.48, there exists $x_i \in S$ such that $a, c_i \in \Lambda_{x_i}$. Define $x = \bigvee_i x_i$. By lem. 4.41, $a \in \Lambda_x$. By lem. 4.43, $\Lambda_{x_i} \cap \Lambda_x$ is a filter of Λ_x, and so $c_i \in \Lambda_x$. Because Λ_x is a sublattice of L, $\bigvee_i c_i \in \Lambda_x$, so $b = a^* = \bigvee_i c_i \in \Lambda_x$.

This shows that $[a, b] \subset \Lambda_x$. By lem. 6.2, Λ_x is a block, and since all blocks are maximal under containment, $[a, b] = \Lambda_x$. □

Lemma 6.4. *χ_S is a bijection from S to S'.*

Proof.

By lem. 6.2, χ_S maps S to S'. By lem. 4.53, χ_S is one-to-one. By lem. 6.3, χ_S is onto. Thus χ_S is a bijection from S to S'. □

Lemma 6.5. *For $x, y \in S$, $x \mathrel{\gamma} y$ iff $\chi_S(x) \mathrel{\gamma'} \chi_S(y)$.*

Proof.

Regarding $\Rightarrow$: Suppose $x \mathrel{\gamma} y$. By lem. 4.42, $\Lambda_x \cap \Lambda_y \neq \emptyset$. Thus by def. 6.1, $\chi_S(x) \cap \chi_S(y) \neq \emptyset$ and by def. 5.11, $\chi_S(x) \mathrel{\gamma'} \chi_S(y)$.

Regarding $\Leftarrow$: Suppose $\chi_S(x) \mathrel{\gamma'} \chi_S(y)$. By def. 5.11 and 6.1, $\Lambda_x \cap \Lambda_y \neq \emptyset$. By lem. 4.42, $x \mathrel{\gamma} y$. □

Definition 6.6. *For $x \in S$, we define $\chi_{Bx} : L_x \to \Lambda_x : y \mapsto \pi_x\, y$, which is a lattice isomorphism from L_x to Λ_x.*

Theorem 6.7. *The dissection $\mathcal{C}'$ of the sum L of the m.c. system $\mathcal{C}$ is isomorphic to $\mathcal{C}$.*

Proof.

We will prove that χ_S and $(\chi_{Bx})_{x \in S}$ as defined by def. 6.1 and 6.6 are a m.c. system isomorphism (def. 3.5) from $\mathcal{C}$ to $\mathcal{C}'$. The needed characteristics of a m.c. system isomorphism are proved as follows:

(1) This is proven by lem. 6.5.
(2) This is proven by def. 5.11, def. 5.13, and lem. 4.43.
(3) This is proven by def. 6.6 and lem. 4.43.

□

7. Gluing of a dissection

In this section, we prove that if L' is the sum of the dissection $\mathcal{C}$ of a modular, l.f.f.c. lattice L, then L' is isomorphic to L.

We choose L to be an arbitrary modular, l.f.f.c. lattice. We define $\mathcal{C}$ to be the m.c. system which is the dissection (def. 5.22) of L. We define L' to be the sum (def. 4.15) of $\mathcal{C}$.

From the dissection of L, we define its derived objects: the set of blocks (def. 5.6) S, the operations $\bullet_*$, $\bullet^*$ (def. 5.1), the blocks $\Delta_\bullet$, $\nabla_\bullet$ (def. 5.24), and the sets of blocks $\Gamma_\bullet$ (def. 5.26).

From $\mathcal{C}$, we define its components: the family of blocks $L'_\bullet$, which is the family of blocks (def. 5.13) of L indexed by themselves (as elements of S); and the connections $\phi_\bullet^{\bullet\prime}$ and their sources and targets $F_\bullet^{\bullet\prime}$, $I_\bullet^{\bullet\prime}$. From $\mathcal{C}$, we define its derived objects: M', $\sim'$, (def. 4.11), κ' (def. 4.16), the set of ascending sequences (def. 4.27), and the sum L'.

Definition 7.1. *We define* $\chi : L \to L' : a \mapsto \kappa'(\nabla_a, a)$.

That is, χ takes an element $a \in L$, derives the block of the dissection $\nabla_a \in S$, then maps the element/block pair through the gluing using κ' to an element of L'.

Lemma 7.2. χ *is well-defined.*

Proof.
Choose $a \in L$. By lem. 5.28, ∇_a is a block, and is the lowest block in Γ_a, which is the set of blocks containing a. Thus, $a \in \nabla_a$, and so κ' maps (∇_a, a) into an element of the sum L'. □

Lemma 7.3. *Given any* $x, y \in S$, $p \in L'_x$, *and* $q \in L'_y$, *then* $\kappa'(x,p) = \kappa'(y,q)$ *(in* L'*) iff* $p = q$ *(in* L*).*

Proof.
First we note that if $p = q$ is $\in L'_x, L'_y$, then $p = q \in L'_{x \vee y}$ by lem. 5.20(1). That shows that $p \in F_x^{x\vee y\prime} = L'_x \cap L'_{x\vee y}$ and $q \in F_y^{x\vee y\prime} = L'_y \cap L'_{x\vee y}$ by def. 5.13.

The following statements are all equivalent:

$$\kappa'(x,p) = \kappa'(y,q)$$

by def. 4.16,

$$(x,p) \sim' (y,q)$$

by def. 4.11,

$$\phi_x^{x\vee y\prime}\, p = \phi_y^{x\vee y\prime}\, q \text{ and } p \in F_x^{x\vee y\prime} \text{ and } q \in F_y^{x\vee y\prime}$$

by def. 5.13, the $\phi_\bullet^{\bullet\prime}$ are identity maps; and in the reverse direction, by the above derivation that $p = q$ implies $p \in F_x^{x\vee y\prime} = L'_x \cap L'_{x\vee y}$ and $q \in F_y^{x\vee y\prime} = L'_y \cap L'_{x\vee y}$,

$$p = q$$

□

This shows that our choice of ∇_a as the block in def. 7.1 was arbitrary; any other block in Γ_a (i.e., any other block containing a) could be used in the definition without changing $\chi\, a$.

Lemma 7.4. χ *is a bijection from* L *to* L'.

Proof.
Regarding that χ is one-to-one: Assume that $a, a' \in L$ and $\chi\, a = \chi\, a'$. Then $\kappa'(\nabla_a, a) = \kappa'(\nabla_{a'}, a')$. By the definition of κ', that requires that $(\nabla_a, a) \sim' (\nabla_{a'}, a')$ which implies that $\phi_{\nabla_a}^{\nabla_a \vee \nabla_{a'}\prime}\, a = \phi_{\nabla_{a'}}^{\nabla_a \vee \nabla_{a'}\prime}\, a'$. But since the $\phi_\bullet^{\bullet\prime}$ are identity maps, that means that $a = a'$.

Regarding that χ is onto: Choose any $a' \in L'$. Since L' is the set of values of κ', there exists a $B \in S$ and an $a \in B$ such that $\kappa'(B, a) = a'$. By def. 5.6, B is a block of L and $a \in L$, so $B \in \Gamma_a$. Then by lem. 7.3, $a' = \kappa'(B, a) = \chi\, a$.

Thus, χ is a bijection from L to L'. □

Theorem 7.5. *The sum L' of the m.c. system $\mathcal{C}$ which is the dissection of a modular, l.f.f.c. lattice L is is isomorphic to L.*

Proof.

We will show that χ is a lattice isomorphism from L to L'. In lem. 7.4 we have already shown that χ is a bijection, so what remains is to show that for any $a, b \in L$, $a \le b$ in L iff $\chi\, a \le \chi\, b$ in L'.

Regarding $\Rightarrow$:

Assume $a \le b$ in L. Choose a saturated chain in L from a to b:

$$a = c_0 \lessdot c_1 \lessdot c_2 \lessdot \cdots \lessdot c_{n-1} \lessdot c_n = b. \tag{a}$$

For $1 \le i \le n$, define $d_i = \nabla_{c_i} \in S$. Since the $L'_\bullet$ are themselves the blocks of L, which are the members of S, each d_i is also one of the $L'_\bullet$, specifically $d_i \in L'_{d_i}$.

Since $c_{i-1} \le c_i$ in L, by lem. 5.3(a$^\delta$), $(c_{i-1})_* \le (c_i)_*$, then by def. 5.9, $[(c_{i-1})_*, {(c_{i-1})_*}^*] \le [(c_i)_*, {(c_i)_*}^*]$ in S, and then by def. 5.24, $d_{i-1} = \nabla_{c_{i-1}} \le \nabla_{c_i} = d_i$ in S. Thus, in S,

$$d_1 \le d_2 \le \cdots \le d_{n-1} \le d_n. \tag{b}$$

That $c_i \in d_i = \nabla_{c_i}$ is immediate. Thus $(d_i, c_i) \in M'$ and $\chi\, c_i = \kappa'(d_i, c_i)$ exists in L'.

By def. 5.1, $c_{i-1} \lessdot c_i$ implies $(c_i)_* \le c_{i-1}$. Then by lem. 5.3(b), $c_{i-1} < c_i \le {(c_i)_*}^*$. These show $c_{i-1} \in [(c_i)_*, {(c_i)_*}^*] = \nabla_{c_i} = d_i$. Thus $(d_i, c_{i-1}) \in M'$ and $\kappa'(d_i, c_{i-1})$ exists in L'. By lem. 7.3, $\kappa'(d_i, c_{i-1}) = \kappa'(d_{i-1}, c_{i-1})$.

Because $c_{i-1}, c_i \in d_i$ and $c_{i-1} \le c_i$ (in the order of L'_{d_i}, which is inherited from the order of L), by def. 4.22, $\chi\, c_{i-1} = \kappa'(d_{i-1}, c_{i-1}) = \kappa'(d_i, c_{i-1}) \le_{d_i} \kappa'(d_i, c_i) = \chi\, c_i$ in L'.

Thus, in L',

$$\chi(a) = \chi(c_0) \le_{d_1} \chi(c_1) \le_{d_2} \chi(c_2) \le_{d_3} \cdots \le_{d_{n-1}} \chi(c_{n-1}) \le_{d_n} \chi(c_n) = \chi(b). \tag{c}$$

By def. 4.27, (c) and (b) show that $(d_\bullet, \chi\, c_\bullet)$ is an ascending sequence (in $\mathcal{C}$), and so by def. 4.29, $\chi\, a \le \chi\, b$.

Regarding $\Leftarrow$:

Assume $a, b \in L$ and $\chi\, a \le \chi\, b$ in L'. Then by def. 4.29, there is an ascending sequence (in $\mathcal{C}$) $(z_\bullet, e_\bullet)$ so that in L'

$$\chi(a) = e_0 \le_{z_1} e_1 \le_{z_2} e_2 \le_{z_3} \cdots \le_{z_{n-1}} e_{n-1} \le_{z_n} e_n = \chi(b). \tag{d}$$

and in S

$$z_1 \le z_2 \le z_3 \le \cdots \le z_{n-1} \le z_n. \tag{e}$$

Define $z_0 = \nabla_a$ so $a = e_0 \in L'_{z_0}$, and since $a = e_0 \in L'_{z_1}$ from (d), $z_0 \le L'_{z_1}$. For $0 \le i \le n$, define $f_i = \pi_{z_i}^{-1}\, e_i \in z_i$.

For $1 \le i \le n$, from (d) and the definition of z_0, $e_{i-1} \in L'_{z_{i-1}}$ and $e_{i-1} \in L'_{z_i}$. By def. 4.16, $\kappa'(z_{i-1}, \pi_{z_{i-1}}^{-1}\, e_{i-1}) = e_{i-1} = \kappa'(z_i, \pi_{z_i}^{-1}\, e_{i-1})$. By lem. 7.3, $\pi_{z_{i-1}}^{-1}\, e_{i-1} = \pi_{z_i}^{-1}\, e_{i-1}$, showing $f_{i-1} = \pi_{z_i}^{-1}\, e_{i-1}$.

For $1 \le i \le n$, from (d), $e_{i-1} \le_{z_i} e_i$, which is, by def. 4.22, $\pi_{z_i}^{-1}\, e_{i-1} \le \pi_{z_i}^{-1}\, e_i$, which by the above shows $f_{i-1} \le f_i$ in L'_{z_i}. Since L'_{z_i} inherits its order from L, $f_{i-1} \le f_i$ in L. Thus in L

$$f_0 \le f_1 \le f_2 \le \cdots \le f_{n-1} \le f_n. \tag{f}$$

Then $\kappa'(z_0, f_0) = \kappa'(z_0, \pi_{z_0}^{-1}\, e_0) = e_0 = \chi\, a = \kappa'(\nabla_a, a)$, so by lem. 7.3, $f_0 = a$. By lem. 7.3, $\kappa'(z_n, f_n) = e_n = \chi\, b$ implies $f_n = b$ in L. These facts, with (f), show $a \le b$ in L. □

8. Conclusion

Theorem 8.1. *The two mappings (1) gluing a m.c. system to create a modular, l.f.f.c. lattice and (2) dissecting a modular, l.f.f.c. lattice to create a m.c. system are mutually inverse mappings between the category of isomorphism classes of m.c. systems and the category of isomorphism classes of modular, l.f.f.c. lattices. Thus, the two mappings are both bijective. The the skeleton of the m.c. system has a minimum element iff the corresponding lattice has a minimum element.*

Proof.

That the two processes map between isomorphism classes of modular, l.f.f.c. lattices and isomorphism classes of m.c. systems is proven in th. 4.83 and 5.30.

That the two processes are inverses is proven in th. 6.7 and 7.5.

That the lattice has a minimum element iff the skeleton has a minimum element is proven in th. 4.70. □

This bijection allows us to informally conflate a lattice and its corresponding m.c. system. This allows us to consider the elements of the lattice to be simultaneously elements of the blocks of the m.c. system, to conflate the blocks $L_\bullet$ of the m.c. system with the blocks of the lattice, etc.[15] This resembles the approach in [Herr1973a, sec. 1] [Herr1973a-en, sec. 1].

$$\begin{aligned}
L_x &\longleftrightarrow \Lambda_x \\
x\ \gamma\ y &\longleftrightarrow \Lambda_x \cap \Lambda_y \neq \emptyset \\
F_x^y &\longleftrightarrow \Lambda_x \cap \Lambda_y \\
I_x^y &\longleftrightarrow \Lambda_x \cap \Lambda_y \\
\phi_x^y &\longleftrightarrow\ : \Lambda_x \cap \Lambda_y \to \Lambda_x \cap \Lambda_y : a \mapsto a \\
\pi_x &\longleftrightarrow\ : \Lambda_x \to \Lambda_x : a \mapsto a \\
\kappa\,(x, a) &\longleftrightarrow a \\
0_{L_x} &\longleftrightarrow 0_x \\
0_x^y &\longleftrightarrow 0_y \\
1_{L_x} &\longleftrightarrow 1_x \\
I_x^y &\longleftrightarrow 1_x \\
\Phi_x^y &\longleftrightarrow\ : \Lambda_x \to \Lambda_y : a \mapsto a \vee 0_y \\
\Psi_x^y &\longleftrightarrow\ : \Lambda_y \to \Lambda_x : a \mapsto a \wedge 1_x
\end{aligned}$$

This equivalence is a base for the further explication of the structure of modular, l.f.f.c. lattices.

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