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GLOBAL WELL-POSEDNESS OF VLASOV-POISSON-BOLTZMANN EQUATIONS WITH NEUTRAL INITIAL DATA AND SMALL RELATIVE ENTROPY

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Abstract. The dynamics of dilute plasma particles such as electrons and ions can be modeled by the fundamental two species Vlasov-Poisson-Boltzmann equations, which describes mutual interactions of plasma particles through collisions in the self-induced electric field. In this paper, we are concerned with global well-posedness of mild solutions of the equations. We establish the global existence and uniqueness of mild solutions to the two species Vlasov-Poisson-Boltzmann equations in the torus for a class of initial data with bounded time-velocity weighted L^∞ norm under nearly neutral condition and some smallness condition on $L^1_xL^\infty_v$ norm as well as defect mass, energy and entropy so that the initial data allow large amplitude oscillations. Due to the nonlinear effect of electric field, we consider the problem in $W^{1,\infty}_{x,v}$ with large amplitude data, new difficulty arises when establishing globally uniform $W^{1,\infty}_{x,v}$ bound, which has been overcome based on nearly neutral condition, time-velocity weight function and a logarithmic estimate. Moreover, the large time behavior of solutions in $W^{1,\infty}_{x,v}$ norm with exponential decay rates of convergence is also obtained.

Contents

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1. Introduction

1.1. **The Vlasov-Poisson-Boltzmann equations.** The Vlasov-Poisson-Boltzmann (VPB) equations of two species can be used to model the time evolution of dilute charged particles (e.g., electrons and ions) in the absence of an external magnetic field. In general, the VPB of two species of particles in the torus take the form

$$\begin{cases} \partial_t F_+ + v\cdot\nabla_x F_+ - \dfrac{e_+}{m_+}\nabla_x\phi\cdot\nabla_v F_+ = Q(F_+, F_+ + F_-), \\ \partial_t F_- + v\cdot\nabla_x F_- + \dfrac{e_-}{m_-}\nabla_x\phi\cdot\nabla_v F_- = Q(F_-, F_+ + F_-), \\ -\Delta_x\phi = 4\pi\displaystyle\int_{\mathbb{R}^3}(e_+F_+ - e_-F_-)dv, \quad \int_{\mathbb{T}^3}\phi(t,x)dx = 0, \\ F_+(0,x,v) = F_{+,0}(x,v), \quad F_-(0,x,v) = F_{-,0}(x,v), \end{cases} \tag{1.1}$$

where $F_+ = F_+(t,x,v)$ and $F_- = F_-(t,x,v)$ are the spatially periodic number density functions for the ions (+) and electrons (-), respectively, at time $t\geq 0$, position $x = (x_1,x_2,x_3)\in[-\frac{1}{2},\frac{1}{2}]^3 = \mathbb{T}^3$, velocity $v = (v_1,v_2,v_3)\in\mathbb{R}^3$. $e_\pm$ and $m_\pm$ denote the magnitude of their charges and masses and $\phi(t,x)$ denotes the electric potential.

The charged particles are moving under the electric force and the inter-particle collisions along their trajectories. The differences between the equations of F_+ and F_- are that the directions of the electric force acting on ions and electrons are opposite and the collisions occur not only between the same kind of particles, but also between different types of particles.

The collision between particles is given by the standard Boltzmann collision operator $Q(G_1,G_2)$, where $G_1(v)$, $G_2(v)$ are two number density functions for two types of particles with masses m_i and diameters σ_i $(i = 1,2)$. By [13, p83 and p89], $Q(G_1,G_2)(v)$ is defined as

$$\begin{aligned} Q(G_1,G_2) &= \frac{1}{4}(\sigma_1+\sigma_2)^2\int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\left(G_1(v')\,G_2(u') - G_1(v)G_2(u)\right)\mathrm{d}\omega\mathrm{d}u \\ &:= Q_{\mathrm{gain}}(G_1,G_2) - Q_{\mathrm{loss}}(G_1,G_2), \end{aligned} \tag{1.2}$$

where the relationship between the post-collision velocity (v',u') of two particles with the pre-collision velocity (v,u) is given by

$$u' = u + \frac{2m_1}{m_1+m_2}[(v-u)\cdot\omega]\omega, \quad v' = v - \frac{2m_2}{m_1+m_2}[(v-u)\cdot\omega]\omega,$$

for $\omega\in\mathbb{S}^2$, which can be determined by conservation laws of momentum and energy

$$m_1v + m_2u = m_1v' + m_2u', \quad \frac{1}{2}m_1|v|^2 + \frac{1}{2}m_2|u|^2 = \frac{1}{2}m_1|v'|^2 + \frac{1}{2}m_2|u'|^2.$$

The Boltzmann collision kernel $B = B(v-u,\theta)$ in (1.1) depends only on $|v-u|$ and θ with $\cos\theta = (v-u)\cdot\omega/|v-u|$. Throughout this paper, we consider the hard potentials under the Grad's angular cut-off assumption, for instance,

$$B(v-u,\theta) = |v-u|^\gamma b(\theta),$$

with

$$0\leq\gamma\leq 1, \quad 0\leq b(\theta)\lesssim|\cos\theta|.$$

In this paper, we normalize all physical constants in the VPB (1.1) to be one. Then (1.1) can be written as

$$\begin{cases}\partial_t F_+ + v\cdot\nabla_x F_+ - \nabla_x\phi\cdot\nabla_v F_+ = Q(F_+,F_++F_-),\\ \partial_t F_- + v\cdot\nabla_x F_- + \nabla_x\phi\cdot\nabla_v F_- = Q(F_-,F_++F_-),\\ -\Delta_x\phi=\rho:=\displaystyle\int_{\mathbb{R}^3}(F_+-F_-)\mathrm{d}v,\quad \int_{\mathbb{T}^3}\phi(t,x)\mathrm{d}x=0,\\ F_+(0,x,v)=F_{+,0}(x,v),\quad F_-(0,x,v)=F_{-,0}(x,v),\end{cases}\tag{1.3}$$

where

$$\begin{aligned}Q(G_1,G_2)&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\left(G_1\left(v'\right)G_2\left(u'\right)-G_1(v)G_2(u)\right)\mathrm{d}\omega\mathrm{d}u\\ &:=Q_{\mathrm{gain}}(G_1,G_2)-Q_{\mathrm{loss}}(G_1,G_2),\end{aligned}\tag{1.4}$$

and

$$v+u=v'+u',\quad |v|^2+|u|^2=|v'|^2+|u'|^2.$$

1.2. **A brief history of VPB.** The mathematical studies on the VPB originated from the pure Boltzmann equation and Vlasov-Poisson system. There has been an enormous literature on the study of well-posedness of Boltzmann equation and Vlasov-Poisson equations; see [11, 31, 63, 67] and the references therein. In what follows we only mention some results which are related to our work.

For the Boltzmann equation, the local existence and uniqueness of general initial data in L^∞ framework was firstly investigated by Kaniel-Shinbrot [45] and the global existence was later obtained by Illner-Shinbrot [44] under additional smallness assumption on velocity weighted L^∞ norm. It is well known that for general initial data with finite mass, energy and entropy, the global existence of renormalized solutions was proved by Diperna-Lions [18] via the weak compactness method; however, the uniqueness of such solutions is unknown. On the other hand, the convergence of a class of large amplitude solutions toward the global Maxwellian with an explicit almost exponential rate in large time was also obtained by Desvillettes-Villani [17] conditionally under some assumptions on smoothness and polynomial moment bounds of the solutions. The result has been improved by Gualdani-Mischler-Mouhot [32] to derive a sharp exponential time rate. In the perturbation framework, due to the extensive study of the linearized operator (Grad [33], Ellis-Pinsky [29], and Baranger-Mouhot [4], for instance), the well-posedness theory of the Boltzmann equation is indeed well established in different kinds of settings since the pioneering work by Ukai [66]. For instance, the energy method in smooth Sobolev spaces was developed in Guo [37] and Liu-Yang-Yu [57]. Another $L^2\cap L^\infty$ approach was found by Guo [39, 40] even for treating the Boltzmann equation on a general bounded domain. Recently, Duan-Huang-Wang-Yang [26] developed a $L^\infty_x L^1_v\cap L^\infty_{x,v}$ approach to prove the global well-posedness and uniqueness of the Boltzmann equation for a class of initial data with large amplitude. For other interesting results, see [1, 6, 25, 28, 30, 34, 42, 46, 47, 48, 59, 60, 64, 65] and the references therein.

The well-posedness on the Vlasov-Poisson equations is well established. The local existence and uniqueness of general initial data in the framework of classical solutions was established by Kurth [49], Horst [43], and Batt [2]; Bardos-Degond established the first global classical solution under small initial data by introducing the free streaming condition. The development for Vlasov-Poisson system culminated in 1989 when independently and almost simultaneously two different proofs for global existence of classical solutions for general large data were given, one by Pfaffelmoser [62] and one by Lions-Perthame [55]. For other interesting results on Vlasov-Poisson system, see [9, 10, 14]; see also the books [31, 63] and the references therein.

As for the VPB, the global existence of the renormalized solutions with large initial data was constructed in [56] and this result was latter generalized to the case with physical boundary in [58].

In the perturbation framework of the VPB, Guo [35] proved the global smooth small-amplitude solutions near vacuum; it was shown in [36] that the VPB near Maxwellian admits a unique global classical solution in periodic box; see also [73, 74] in the whole space. Since then, the robust energy method was developed in [38] to deal with Vlasov-Maxwell-Boltzmann equations with the self-consistent electric and magnetic fields; see also[20, 22, 23, 73, 74, 76]. In particular, Duan-Yang-Zhao developed the time weighted energy method to study the VPB for general collision potentials [22, 23]; see also [70, 71, 72]. Moreover, the large-time behavior of global solutions is also extensively studied by using different approaches [21, 38, 50, 51, 52, 54, 69, 74, 75]. We would like to mention $L^2\cap L^\infty$ approach is also widely used for the VPB [7, 8, 41, 53] on hydrodynamic limit and bounded domain. For other interesting results, see [12, 15, 16, 19, 24, 68] and the references therein.

All of the aforementioned works in the perturbation framework for the VPB require that the initial data have a small amplitude around the global Maxwellian. The purpose of this paper is study the well-posedness of the VPB when initial data are allowed to have small relative entropy with large amplitude. Moreover, we also show that the solutions tend to the global Maxwellian with exponential convergence rates in $W^{1,\infty}_{x,v}$ norm.

1.3. **Notations.** Throughout this paper, we shall use the following conventions:

- C denotes a generic positive constant and $C_a, C_b, C_N\ldots$ denote the generic positive constants depending on $a,b,N,\ldots$, respectively, which may vary from line to line. And we use C_1, C_2, C_3, etc., denote fixed positive constants. $A\lesssim B$ means that there exists a constant $C>0$ such that $A\le CB$. $A\cong B$ means that both $A\lesssim B$ and $B\lesssim A$ hold.
- For a real-valued function $\mathrm{g}(t,x,v)$ defined on $\mathbb{R}_+\times\mathbb{T}^3\times\mathbb{R}^3$, the derivative of the i-th component x_i of the spatial variable $x=(x_1,x_2,x_3)$ is defined as $\partial_{x_i}\mathrm{g}$ or $\partial_i\mathrm{g}$. The derivative of the i-th component v_i of the spatial variable $v=(v_1,v_2,v_3)$ is defined as $\partial_{v_i}\mathrm{g}$ or $\partial^i\mathrm{g}$. $\nabla_x\mathrm{g}=\partial_x\mathrm{g}:=(\partial_1\mathrm{g},\partial_2\mathrm{g},\partial_3\mathrm{g})$ and $\nabla_v\mathrm{g}=\partial_v\mathrm{g}:=(\partial^1\mathrm{g},\partial^2\mathrm{g},\partial^3\mathrm{g})$ denote the gradient of g with respect to the spatial variable and velocity variable respectively.
- We use $[\cdot,\cdot]^{\mathrm{T}}$ to denote a column vector. For a vector-valued function $\mathbf{g}(t,x,v)=[g_+,g_-]^{\mathrm{T}}$ defined on $\mathbb{R}_+\times\mathbb{T}^3\times\mathbb{R}^3$, we define $\partial_{x,v}\mathbf{g}(t,x,v)=[\partial_{x,v}g_+,\partial_{x,v}g_-]^{\mathrm{T}}$.
- $\|\cdot\|_{L^2}$ denotes either the standard $L^2\left(\mathbb{T}^3_x\right)$-norm or $L^2\left(\mathbb{R}^3_v\right)$-norm or $L^2\left(\mathbb{T}^3_x\times\mathbb{R}^3_v\right)$-norm. Similarly, $\|\cdot\|_{L^\infty}$ denotes either the $L^\infty\left(\mathbb{R}^3_x\right)$-norm or $L^\infty\left(\mathbb{R}^3_v\right)$-norm or $L^\infty\left(\mathbb{T}^3_x\times\mathbb{R}^3_v\right)$-norm. We denote $\langle\cdot,\cdot\rangle$ as either the $L^2\left(\mathbb{T}^3_x\right)$ inner product or $L^2\left(\mathbb{R}^3_v\right)$ inner product or $L^2\left(\mathbb{T}^3_x\times\mathbb{R}^3_v\right)$ inner product. We denote $\|\cdot\|_\nu:=\|\sqrt{\nu}\cdot\|_{L^2}$.
- For the vector-valued function $\mathbf{g}$, we denote $\|\mathbf{g}\|_{L^2}:=(\|g_+\|^2_{L^2}+\|g_-\|^2_{L^2})^{\frac12}$, $\|\mathbf{g}\|_{L^\infty}:=\|g_+\|_{L^\infty}+\|g_-\|_{L^\infty}$, $\|\mathbf{g}\|_{L^1_xL^\infty_v}:=\|g_+\|_{L^1_xL^\infty_v}+\|g_-\|_{L^1_xL^\infty_v}$ and $\|\mathbf{g}\|_\nu:=(\|g_+\|^2_\nu+\|g_-\|^2_\nu)^{\frac12}$. Moreover, we can also define $\|\partial_{x,v}\mathbf{g}\|_{L^\infty}:=\|\partial_{x,v}g_+\|_{L^\infty}+\|\partial_{x,v}g_-\|_{L^\infty}$, where $\|\partial_{x,v}g_\pm\|_{L^\infty}:=\sum_{i=1}^{3}(\|\partial_{x_i}g_\pm\|_{L^\infty}+\|\partial_{v_i}g_\pm\|_{L^\infty})$. Similarly,for any vector-valued function $\mathbf{b}(x):=[b_1,b_1,b_3]^{\mathrm{T}}$, we can define the norm $\|\mathbf{b}\|_{L^2}:=\sum_{i=1}^{3}\|b_i\|_{L^2}$ where $b_i(x)$ $(i=1,2,3)$ is a real-valued function. For simplicity, we denote $\|(\mathit{\Xi},\mathit{\Pi})\|_{L^2}:=\|\mathit{\Xi}\|_{L^2}+\|\mathit{\Pi}\|_{L^2}$, where $\mathit{\Xi}$, $\mathit{\Pi}$ are real/vector-valued functions.

1.4. **Main results.** In this paper, we study the classical solutions of (1.3) around a normalized global Maxwellian:

$$\mu(v)=(2\pi)^{-\frac32}e^{-\frac{|v|^2}{2}}.$$

We define

$$f_+(t,x,v)=\frac{F_+(t,x,v)-\mu}{\sqrt{\mu}},\quad f_-(t,x,v)=\frac{F_-(t,x,v)-\mu}{\sqrt{\mu}}. \tag{1.5}$$

Throughout the paper, let's define

$$\mathbf{F}(t,x,v):=[F_+(t,x,v),F_-(t,x,v)]^{\mathrm{T}},\quad \mathbf{f}(t,x,v):=[f_+(t,x,v),f_-(t,x,v)]^{\mathrm{T}}.$$

For any given $\mathbf{g}=[g_+,g_-]^{\mathrm{T}}$, the linearized collision operator is defined as

$$\mathbf{Lg}:=[\mathrm{L}^+\mathbf{g},\mathrm{L}^-\mathbf{g}]^{\mathrm{T}}, \tag{1.6}$$

where

$$\mathrm{L}^{\pm}\mathbf{g}:=-\frac{1}{\sqrt{\mu}}\{2Q(\sqrt{\mu}g_{\pm},\mu)+Q(\mu,\sqrt{\mu}g_++\sqrt{\mu}g_-)\}.$$

We split $\mathbf{L}$ in a standard way: $\mathbf{Lg}=\nu(v)\mathbf{g}-\mathbf{Kg}$. The collision frequency is

$$\nu(v):=2\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\mu(u)\mathrm{d}\omega\mathrm{d}u\cong(1+|v|)^{\gamma}, \tag{1.7}$$

and $\mathbf{K}:=[\mathrm{K}^+,\mathrm{K}^-]^{\mathrm{T}}$ with $\mathrm{K}^{\pm}:=\mathrm{K}_2^{\pm}-\mathrm{K}_1^{\pm}$ which are defined as

$$\begin{aligned}\left(\mathrm{K}_1^+\mathbf{g}\right)(v)=\left(\mathrm{K}_1^-\mathbf{g}\right)(v)&:=\frac{1}{\sqrt{\mu}}Q_{\mathrm{loss}}(\mu,\sqrt{\mu}g_++\sqrt{\mu}g_-)\\&=\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(v)\mu(u)}(g_+(u)+g_-(u))\mathrm{d}\omega\mathrm{d}u,\end{aligned} \tag{1.8}$$

$$\begin{aligned}\left(\mathrm{K}_2^+\mathbf{g}\right)(v)&:=\frac{2}{\sqrt{\mu}}Q_{\mathrm{gain}}(\sqrt{\mu}g_+,\mu)+\frac{1}{\sqrt{\mu}}Q_{\mathrm{gain}}(\mu,\sqrt{\mu}g_++\sqrt{\mu}g_-)\\&=2\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)\mu(u')}g_+(v')\mathrm{d}\omega\mathrm{d}u\\&\quad+\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)\mu(v')}(g_+(u')+g_-(u'))\mathrm{d}\omega\mathrm{d}u,\end{aligned} \tag{1.9}$$

$$\begin{aligned}\left(\mathrm{K}_2^-\mathbf{g}\right)(v)&:=\frac{2}{\sqrt{\mu}}Q_{\mathrm{gain}}(\sqrt{\mu}g_-,\mu)+\frac{1}{\sqrt{\mu}}Q_{\mathrm{gain}}(\mu,\sqrt{\mu}g_++\sqrt{\mu}g_-)\\&=2\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)\mu(u')}g_-(v')\mathrm{d}\omega\mathrm{d}u\\&\quad+\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)\mu(v')}(g_+(u')+g_-(u'))\mathrm{d}\omega\mathrm{d}u.\end{aligned} \tag{1.10}$$

For $\mathbf{g}=[g_+,g_-]^{\mathrm{T}}$ and $\mathbf{f}=[f_+,f_-]^{\mathrm{T}}$, the nonlinear collision operator is defined as

$$\mathbf{\Gamma}(\mathbf{g},\mathbf{f}):=[\Gamma^+(\mathbf{g},\mathbf{f}),\Gamma^-(\mathbf{g},\mathbf{f})]^{\mathrm{T}} \tag{1.11}$$

with

$$\Gamma^{\pm}(\mathbf{g},\mathbf{f}):=\frac{1}{\sqrt{\mu}}\left\{Q(\sqrt{\mu}g_{\pm},\sqrt{\mu}(f_++f_-))\right\} \tag{1.12}$$

Thus the VPB (1.3) can be rewritten as

$$\begin{cases}(\partial_t+v\cdot\nabla_x-\nabla_x\phi\cdot\nabla_v)f_++\mathrm{L}^+\mathbf{f}=\Gamma^+(\mathbf{f},\mathbf{f})-\dfrac{1}{2}\nabla_x\phi\cdot vf_+-\nabla_x\phi\cdot v\sqrt{\mu},\\(\partial_t+v\cdot\nabla_x+\nabla_x\phi\cdot\nabla_v)f_-+\mathrm{L}^-\mathbf{f}=\Gamma^-(\mathbf{f},\mathbf{f})+\dfrac{1}{2}\nabla_x\phi\cdot vf_-+\nabla_x\phi\cdot v\sqrt{\mu},\\-\Delta_x\phi=\rho:=\displaystyle\int_{\mathbb{R}^3}\sqrt{\mu}(f_+-f_-)\mathrm{d}v,\quad\int_{\mathbb{T}^3}\phi(t,x)\mathrm{d}x=0,\\f_+(0,x,v)=f_{+,0}(x,v),\quad f_-(0,x,v)=f_{-,0}(x,v).\end{cases} \tag{1.13}$$

For later use, $(1.13)_{1,2}$ can be rewritten as

$$\partial_t \mathbf{f} + v \cdot \nabla_x \mathbf{f} - \mathbf{q}(\nabla_x \phi \cdot \nabla_v)\mathbf{f} + \mathbf{q}(\frac{v}{2} \cdot \nabla_x \phi)\mathbf{f} + \mathbf{L}\mathbf{f} = \mathbf{\Gamma}(\mathbf{f},\mathbf{f}) - \mathbf{q_1} v \cdot \nabla_x \phi \sqrt{\mu}, \tag{1.14}$$

where $\mathbf{q} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, and $\mathbf{q_1} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

It is well known that the the linearized collision operator $\mathbf{L}$ is non-negative. The null space $\mathcal{N}$ of $\mathbf{L}$ is the six dimensional space (Lemma 1)

$$\mathcal{N} = \operatorname{span}\left\{ \begin{bmatrix} \sqrt{\mu} \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \sqrt{\mu} \end{bmatrix}, \begin{bmatrix} \frac{v_i}{\sqrt{2}}\sqrt{\mu} \\ \frac{v_i}{\sqrt{2}}\sqrt{\mu} \end{bmatrix}, \begin{bmatrix} \frac{|v|^2-3}{2\sqrt{2}}\sqrt{\mu} \\ \frac{|v|^2-3}{2\sqrt{2}}\sqrt{\mu} \end{bmatrix} \right\}, \ (i=1,2,3).$$

For any fixed (t,x), and any function

$$\mathbf{g}(t,x,v) = \begin{bmatrix} g_+(t,x,v) \\ g_-(t,x,v) \end{bmatrix},$$

we define $\mathbf{P}$ as its $v-$ projection onto $L^2_v\left(\mathbb{R}^3\right)$, to the null space $\mathcal{N}$. We then decompose $\mathbf{g}(t,x,v)$ uniquely as

$$\mathbf{g}(t,x,v) = \{\mathbf{P}\mathbf{g}\}(t,x,v) + \{\mathbf{I}-\mathbf{P}\}\mathbf{g}(t,x,v),$$

where $\mathbf{P}\mathbf{g}$ is denoted by

$$\mathbf{P}\mathbf{g}(t,x,v) := \left\{ a_+(t,x) \begin{bmatrix} \sqrt{\mu} \\ 0 \end{bmatrix} + a_-(t,x) \begin{bmatrix} 0 \\ \sqrt{\mu} \end{bmatrix} + \mathbf{b}(t,x) \cdot \frac{v}{\sqrt{2}} \begin{bmatrix} \sqrt{\mu} \\ \sqrt{\mu} \end{bmatrix} + c(t,x) \frac{|v|^2-3}{2\sqrt{2}} \begin{bmatrix} \sqrt{\mu} \\ \sqrt{\mu} \end{bmatrix} \right\},$$

where $\mathbf{b}(t,x) = [b_1(t,x), b_2(t,x), b_3(t,x)]^{\mathrm{T}}$. Usually, we call $\mathbf{P}\mathbf{g}$ the hydrodynamic part of $\mathbf{g}$, and $\{\mathbf{I}-\mathbf{P}\}\mathbf{g}$ the microscopic part.

A direct calculation shows that the classical solutions $F_\pm(t,x,v)$ for VPB (1.3) satisfies the conservation laws of defect mass, momentum, and total energy:

$$\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \{F_\pm(t,x,v) - \mu(v)\} \mathrm{d}v\mathrm{d}x = \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \{F_{\pm,0}(t,x,v) - \mu(v)\} \mathrm{d}v\mathrm{d}x := M_{\pm,0}, \tag{1.15}$$

$$\begin{aligned} &\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} v\{(F_+(t,x,v)-\mu(v)) + (F_-(t,x,v)-\mu(v))\}\mathrm{d}v\mathrm{d}x \\ =&\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} v\{(F_{+,0}(t,x,v)-\mu(v)) + (F_{-,0}(t,x,v)-\mu(v))\}\mathrm{d}v\mathrm{d}x := \mathbf{J}_0, \end{aligned} \tag{1.16}$$

$$\begin{aligned} &\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |v|^2\{F_+(t,x,v) + F_-(t,x,v) - 2\mu\}\mathrm{d}v\mathrm{d}x + \int_{\mathbb{T}^3} |\nabla_x \phi(t,x)|^2 \mathrm{d}x \\ =&\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} |v|^2\{F_{+,0}(x,v) + F_{-,0}(x,v) - 2\mu\}\mathrm{d}v\mathrm{d}x + \int_{\mathbb{T}^3} |\nabla_x \phi_0|^2 \mathrm{d}x := E_0, \end{aligned} \tag{1.17}$$

as well as the inequality of defect entropy

$$\begin{aligned} &\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \{(F_+\ln F_+ + F_-\ln F_-)(t,x,v) - 2\mu(v)\ln\mu(v)\}\mathrm{d}v\mathrm{d}x \\ &\quad \le \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \{(F_{+,0}\ln F_{+,0} + F_{-,0}\ln F_{-,0})(x,v) - 2\mu(v)\ln\mu(v)\}\mathrm{d}v\mathrm{d}x. \end{aligned} \tag{1.18}$$

For later use, we define the relative entropy

$$\begin{aligned} \mathcal{E}(\mathbf{F}(t)) :=& \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \left\{F_+(t)\ln F_+(t) + F_-(t)\ln F_-(t) - 2\mu\ln\mu\right\}\mathrm{d}v\mathrm{d}x \\ &- \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} (1+\ln\mu)(F_+(t)+F_-(t)-2\mu)\mathrm{d}v\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3} |\nabla_x\phi(t)|^2\mathrm{d}x. \end{aligned} \tag{1.19}$$

Motivated by [22], we introduce a time-velocity weight function

$$w_\beta(t,v) := (1+|v|^2)^{\frac{\beta}{2}} e^{\frac{\sigma_0}{1+t}|v|^2}, \tag{1.20}$$

for some $\beta \geq 4$, $0 < \sigma_0 \leq \frac{1}{16}$. Throughout the paper, we denote

$$\tilde{h}_\pm(t,x,v) := w_{\beta_1}(t,v) f_\pm(t,x,v), \quad h_\pm(t,x,v) := w_\beta(t,v) f_\pm(t,x,v). \tag{1.21}$$

Similarly, we can define $\mathbf{h} := [h_+, h_-]^{\mathrm{T}}$ and $\tilde{\mathbf{h}} := [\tilde{h}_+, \tilde{h}_-]^{\mathrm{T}}$, then define $\partial_{x,v}\tilde{\mathbf{h}} := [\partial_{x,v}\tilde{h}_+, \partial_{x,v}\tilde{h}_-]^{\mathrm{T}}$.

The main result of this paper is stated as follows:

Theorem 1.1. *Assume $0 \leq \gamma \leq 1$, $4 \leq \beta_1 < \beta - 4$ and $(M_{\pm,0}, \mathbf{J}_0, E_0) = (0, \mathbf{0}, 0)$. For given $M_0 \geq 1$, suppose the initial data $F_{\pm,0}(x,v) = \mu(v) + \sqrt{\mu(v)} f_{\pm,0}(x,v) \geq 0$ satisfy $\|h_{\pm,0}\|_{L^\infty} \leq M_0$, $\|\partial_{x,v}\tilde{h}_{\pm,0}\|_{L^\infty} < +\infty$. Then there are small constant ε_1 depending on $\gamma, \beta, \beta_1, M_0$, and ε_0 sufficiently small depending on $\|\partial_{x,v}\tilde{h}_{\pm,0}\|_{L^\infty}$ such that if*

$$\|f_{+,0} - f_{-,0}\|_{L^\infty} \leq \varepsilon_0, \quad \mathcal{E}(\mathbf{F}_0) + \|\tilde{h}_{\pm,0}\|_{L^1_x L^\infty_v} \leq \varepsilon_1, \tag{1.22}$$

the VPB (1.3) has a unique global mild solution $F_\pm(t,x,v) = \mu(v) + \sqrt{\mu(v)} f_\pm(t,x,v) \geq 0$ satisfying the conservation laws of defect mass, momentum, energy (1.15)-(1.17) as well as the additional defect entropy inequality (1.18), and

$$\|h_\pm(t)\|_{L^\infty} \leq C \exp\{-\lambda_1 t\}, \tag{1.23}$$

$$\|f_+(t) - f_-(t)\|_{L^\infty} \leq C \min\left\{\varepsilon_0 \exp\{C(1 + \|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 t\}, e^{-\lambda_1 t}\right\}, \tag{1.24}$$

$$\|\partial_{x,v}\tilde{h}_\pm(t)\|_{L^\infty} \leq C(1 + \|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 \min\left\{1, \Big(1 + \ln(1 + \|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})\Big) e^{-\lambda_2 t}\right\}, \tag{1.25}$$

for constants $\lambda_1 > \lambda_2 > 0$, where C depends only on $\gamma, M_0, \beta_1, \beta$. Moreover, if initial data $f_{\pm,0}$ are continuous, then the solution $f_\pm(t,x,v)$ is continuous in $[0,\infty) \times \mathbb{T}^3 \times \mathbb{R}^3$.

Remark 1.2. *We point out that the initial data satisfying (1.22) are allowed to have large amplitude oscillations in spatial variable. For instance, one may take*

$$F_{+,0}(x,v) = \rho_0(x)\mu, \quad F_{-,0}(x,v) = (\rho_0(x) - \varepsilon_0)\mu,$$

where $\rho_0(x) \geq 0$ is bounded and ε_0 sufficiently small. It is straightforward to verify

$$\mathcal{E}(\mathbf{F}_0) + \|\tilde{h}_{\pm,0}\|_{L^1_x L^\infty_v} \lesssim \|\rho_0 \ln \rho_0 - \rho_0 + 1\|_{L^1_x} + \|\rho_0 - 1\|_{L^1_x}.$$

Even though $\|\rho_0 \ln \rho_0 - \rho_0 + 1\|_{L^1_x} + \|\rho_0 - 1\|_{L^1_x}$ is required to be small, initial data are still allowed to have large amplitude oscillations.

Remark 1.3. *By Gagliardo-Nirenberg's inequality, one has*

$$\|f_0\|_{L^\infty} \lesssim \|f_0\|_{L^2}^{\frac{1}{4}} \|\nabla_{x,v} f_0\|_{L^\infty}^{\frac{3}{4}} \text{ or } (\|\rho_0\|_{L^\infty} \lesssim \|\rho_0\|_{L^1}^{\frac{1}{4}} \|\nabla_x \rho_0\|_{L^\infty}^{\frac{3}{4}}),$$

and we need this quantity to be bounded, i.e., $\|f_0\|_{L^2}^{\frac{1}{4}} \|\nabla_{x,v} f_0\|_{L^\infty}^{\frac{3}{4}} \approx 1$, thus we must have

$$\|\nabla_{x,v} f_0\|_{L^\infty} \gtrsim \|f_0\|_{L^2}^{-\frac{1}{3}} \sim \varepsilon_1^{-\frac{1}{3}}, \text{ or } (\|\nabla_x \rho_0\|_{L^\infty} \gtrsim \|\rho_0\|_{L^1}^{-\frac{1}{3}}),$$

which shows that $\|\nabla_{x,v} f_0\|_{L^\infty}$ will be very large since $\|f_0\|_{L^2}$ is sufficiently small.

1.5. **Main difficulties and strategy of the proof.** We make some comments on the main ideas of the proof and explain the main difficulties and techniques of the present paper. For the global well-posedness of VPB with small relative entropy but large amplitude initial data, the difficulties arise from the nonlinearity of the characteristic flow of particles and nonlinearity of the collision operator. The characteristic flow is the solution of a Hamiltonian system, which leads us to solve VPB in $W^{1,\infty}_{x,v}$. The nonlinear effect of collision operator brings great challenges to the large initial data problem of Boltzmann equation. As explained previously, Diperna-Lions [18] established the global existence of renormalized solutions for general large initial data, but the uniqueness of the solution is still open. Recently, Duan-Huang-Wang-Yang [26] established the first unique global solution of the Boltzmann equation under the condition of small relative entropy. However, under the condition of small relative entropy, to include a large amplitude for the solution of VPB, the derivative must be sufficiently large, as explained in Remark 1.3, which makes the estimation of characteristics very difficult. In the present paper, we shall consider the case of VPB with small relative entropy under nearly neutral condition.

In the following, we briefly explain the key strategies of proof of present paper.

As far as we know, due to the appearance of the electric field $\nabla_x\phi(t,x)$, there is no existence result even for the local smooth solution of VPB with large initial data. Specifically, it is difficult to control $\nabla_x\phi\cdot vf_\pm$ in (1.13) because the sign is not clear. Inspired by [22, 23], we introduce the time-velocity weight $w_\beta(t,v)$, see (1.20), which helps us to control the electric field. As explained in Remark 1.3, $\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty}$ may be very large in our setting, then we have to establish the local existence theorem with the lifespan depending only on $\|\mathbf{h}_0\|_{L^\infty}$ but independent of $\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty}$, which is important to for us to derive the uniform estimates. Fortunately, using the classical potential analysis theory, we can obtain a logarithmic estimate on the derivative of electric field (see Lemma 2.4), which helps us to establish uniform estimates for $\|\mathbf{h}(t)\|_{L^\infty}$ and $\|\partial_{x,v}\tilde{\mathbf{h}}(t)\|_{L^\infty}$ during the time interval $[0,t_1]$ with t_1 depending only on $\|\mathbf{h}_0\|_{L^\infty}$, see Proposition 3.1 for details.

In order to extend the local solution into a global one, we need to establish some uniform *a priori* estimates. Under the *a priori* assumption (2.18), we can perform the $L^\infty_xL^1_v\cap L^\infty_{x,v}$ method to obtain global uniform estimate on $\|\mathbf{h}\|_{L^\infty}$, see Proposition 4.3. Specifically, one can first show that

$$\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}\lesssim\{\|\mathbf{h}_0\|_{L^\infty}+\|\mathbf{h}_0\|^2_{L^\infty}+\sqrt{\mathcal{E}(\mathbf{F}_0)}\}+\sup_{\substack{t_1\le s\le t\\ y\in\mathbb{T}^3}}\Big\{\|\mathbf{h}(s)\|^{\frac{3}{2}}_{L^\infty}\Big(\int_{\mathbb{R}^3_v}|\tilde{\mathbf{h}}(s,y,v)|\mathrm{d}v\Big)^{\frac{1}{2}}\Big\},$$

see Lemma 4.1. Motivated by [26], we can also prove

$$\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(t,x,v)|\mathrm{d}v\lesssim t_1^{-3}\|\tilde{\mathbf{h}}_0\|_{L^1_xL^\infty_v}+\sqrt{\mathcal{E}(\mathbf{F}_0)}+\text{smallness},\quad\forall t\ge t_1,$$

see Lemma 4.2, which yields that $\sup\limits_{x\in\mathbb{T}^3,t\ge t_1}\int_{\mathbb{R}^3_v}|\tilde{\mathbf{h}}(t,x,v)|\mathrm{d}v$ can be sufficiently small, see Remark 4.4.

Then, in order to close the *a priori* assumption (2.18), a crucial step is to prove that the hydrodynamic part $\mathbf{Pf}$ can be controlled by the microscopic part $\{\mathbf{I}-\mathbf{P}\}\mathbf{f}$, see Lemma 5.1 below. Motivated by [30, 7, 8], the key is to construct some test function. However, the periodic boundary conditions naturally require $\int_{\mathbb{T}^3}\bar{c}(t,x)\mathrm{d}x=0$ (the definition of $\bar{c}$ is shown in (5.2)), so we cannot handle the estimate of $\bar{c}(t,x)$ as in [30, 7, 8], because

$$-\Delta\varphi_c(t,x)=\bar{c}(t,x),\quad\int_{\mathbb{T}^3}\varphi_c(t,x)\mathrm{d}x=0$$

is ill-posed. To overcome the difficulty, by noting the conservation laws of defect total energy (1.17), we define a new function

$$\tilde{c}(t,x) := \bar{c}(t,x) + \frac{\sqrt{2}}{6} e^{-\lambda t} |\nabla_x \bar{\phi}(t,x)|^2,$$

which satisfies $\int_{\mathbb{T}^3} \tilde{c}(t,x)\mathrm{d}x = 0$ for all $t \geq 0$. Then we choose the test function for $\bar{c}$: $\Psi_c(t,x) = (|v|^2 - \beta_c)\sqrt{\mu} v \cdot \nabla_x \varphi_c(t,x)$ with

$$-\Delta \varphi_c(t,x) = \tilde{c}(t,x), \quad \int_{\mathbb{T}^3} \varphi_c(t,x)\mathrm{d}x = 0.$$

Thus we establish Lemma 5.1, and then obtain the L^2-L^∞ exponential decay of $\mathbf{f}$, see Proposition 5.2 and 5.3.

Next, since $\|\mathbf{h}(t)\|_{L^\infty}$ may not be small in local time, to close *a priori* assumption (2.18), then we impose the nearly neutral condition to ensure the smallness of $\|\nabla \phi(t)\|_{L^\infty}$ and $\|\nabla^2 \phi(t)\|_{L^\infty}$, which further yields that we have to establish estimates for $\|\partial_{x,v} \tilde{\mathbf{h}}(t)\|_{L^\infty}$. Then difficulty arises from the nonlinear term

$$\int_0^t e^{-\nu(v)(t-s)} \left| \left(\partial_{x,v}(w_{\beta_1} \Gamma^{\pm}(\mathbf{f},\mathbf{f}))\right)(s, X_{\pm}(s), V_{\pm}(s)) \right| \mathrm{d}s. \tag{1.26}$$

The general view is that this term is controlled by the integral

$$\int_0^t e^{-\nu(v)(t-s)} \nu(v) \|\mathbf{h}(s)\|_{L^\infty} \|\partial_{x,v} \tilde{\mathbf{h}}(s)\|_{L^\infty} \mathrm{d}s,$$

but it is hard to get global uniform estimate by using the Gronwall's inequality due to the factor $\nu(v)$. In the present paper, we control (1.26) by

$$\int_0^t \|\mathbf{h}(s)\|_{L^\infty} \|\partial_{x,v} \tilde{\mathbf{h}}(s)\|_{L^\infty} + e^{-\frac{\nu(v)}{2}(t-s)} \nu(v) \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} \|\mathbf{h}(s)\|_{L^\infty}^{\frac{1}{2}} \cdot \sup_{x \in \mathbb{T}^3} \left\{ \int_{\mathbb{R}^3_v} |\tilde{\mathbf{h}}(s,x,v)| \mathrm{d}v \right\}^{\frac{1}{2}} \mathrm{d}s,$$

which, together with (3.2), (3.3), the exponential decay of $\|\mathbf{h}(t)\|_{L^\infty}$ in Proposition 5.3, the smallness property of $\sup_{x \in \mathbb{T}^3} \int_{\mathbb{R}^3_v} |\tilde{\mathbf{h}}(s,x,v)| \mathrm{d}v$ in Remark 4.4, Gronwall's inequality, yields global uniform estimate (see (6.3) and (6.40)). Another difficult term is

$$\begin{aligned} &\int_0^t \int_0^{s-\frac{1}{N}} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau} e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \mathrm{d}s_1 ds \\ &\quad \times \iint_B \mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),u) \cdot \mathtt{k}^{(2)}_{w_{\beta_1}}(\hat{V}_+(s_1),u_1) \cdot \partial_x \tilde{h}_+(s_1, \hat{X}_+(s_1), u_1) du_1 du. \end{aligned}$$

Motivated by [41], we shall apply integration by parts, which reduce the control of above term as $\mathcal{E}(\mathbf{F}_0)$. Then we have to analyze the complex characteristics. In fact, Guo-Jang [41] deal with the characteristics in local time, while we need to control the characteristic flow global in time, which is more delicate, see Lemma 2.6, Corollary 2.7 and Lemma 2.8. With above analysis, we can obtain a globally uniform estimate of $\|\partial_{x,v}\tilde{\mathbf{h}}\|_{L^\infty}$, see section 6 for more details.

Finally, with above preparations, we can derive that

$$\begin{aligned} &\|\nabla \phi(t)\|_{L^\infty} \lesssim \min \left\{ \varepsilon_0 \exp\{C(1 + \|\partial_{x,v} \tilde{\mathbf{h}}_0\|_{L^\infty})^2 t\}, e^{-\lambda_1 t} \right\} \leq \delta^2 (1+t)^{-2}, \\ &\|\nabla^2 \phi(t)\|_{L^\infty} \lesssim \min \left\{ \varepsilon_0 \mathfrak{H}(t), e^{-\lambda_2 t} \Big(1 + 2 \ln\Big(\frac{1}{\delta}\Big) + \ln(1 + \|\partial_{x,v} \tilde{\mathbf{h}}_0\|_{L^\infty})\Big) \right\} \leq \delta^2 (1+t)^{-\frac{5}{2}}, \end{aligned}$$

for all $t \geq 0$. Indeed, for the case of finite time, we ensure (2.18) by choosing the neutral condition $\|f_{+,0} - f_{-,0}\|_{L^\infty} \leq \varepsilon_0$ to be small enough. For large time, (2.18) is proved based on the exponential decay of $\|\mathbf{h}(t)\|_{L^\infty}$, see Proposition 5.3. Thus we concludes the the *a priori* assumption (2.18), see section 7 for more details.

1.6. **Organization of the paper.** The paper is organized as follows. In Section 2, we list many useful lemmas and estimates for later use, and establish a new logarithmic estimate of the electric field. In Section 3, we establish the local existence theorem for VPB so that the lifespan of L^∞ solution depends only on $\|\mathbf{h}_0\|_{L^\infty}$ and is independent on $\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty}$. In Section 4, based $L^\infty_x L^1_v \cap L^\infty_{x,v}$ method with the time-velocity-exponential weight function, we obtain global uniform estimate of $\|\mathbf{h}\|_{L^\infty}$. In Section 5, using estimate the hydrodynamic part by microscopic part and standard L^2-L^∞ theory, we obtain the exponential decay rate of $\|\mathbf{h}\|_{L^\infty}$. In Section 6, we establish global uniform estimate of $\|\partial_{x,v}\tilde{\mathbf{h}}\|_{L^\infty}$. In Section 7, we prove the Theorem 1.1.

2. Preliminaries

We first present a useful result on the linear collision operator $\mathbf{L}$. We refer to [38, Lemma 1] for it proof.

Lemma 2.1 ([38])**.** *It holds that*

$$\langle \mathbf{Lg}, \mathbf{f}\rangle = \langle \mathbf{Lf}, \mathbf{g}\rangle, \quad \textit{and} \quad \langle \mathbf{Lg}, \mathbf{g}\rangle \geq 0.$$

And $\mathbf{Lg} = 0$ *if and only if*

$$\mathbf{g} = \mathbf{Pg}.$$

Moreover, there is $\lambda_0 > 0$ *such that*

$$\langle \mathbf{Lg}, \mathbf{g}\rangle \geq \lambda_0 \|\{\mathbf{I}-\mathbf{P}\}\mathbf{g}\|_\nu^2.$$

Lemma 2.2. *Let* $F_\pm(t,x,v)$ *satisfy* (1.15), (1.17) *and* (1.18), *then it holds that*

$$\begin{aligned}
&\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \frac{|F_+(t,x,v)-\mu(v)|^2}{4\mu(v)} I_{\{|F_+(t,x,v)-\mu(v)|\leq \mu(v)\}} \mathrm{d}v\ \mathrm{d}x \\
&\quad + \int_{\Omega}\int_{\mathbb{R}^3} \frac{1}{4}|F_+(t,x,v)-\mu(v)| I_{\{|F_+(t,x,v)-\mu(v)|\geq \mu(v)\}} \mathrm{d}v\ \mathrm{d}x \\
&\quad + \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \frac{|F_-(t,x,v)-\mu(v)|^2}{4\mu(v)} I_{\{|F_-(t,x,v)-\mu(v)|\leq \mu(v)\}} \mathrm{d}v\ \mathrm{d}x \\
&\quad + \int_{\Omega}\int_{\mathbb{R}^3} \frac{1}{4}|F_-(t,x,v)-\mu(v)| I_{\{|F_-(t,x,v)-\mu(v)|\geq \mu(v)\}} \mathrm{d}v\ \mathrm{d}x \\
&\quad + \frac{1}{2}\int_{\mathbb{T}^3} |\nabla_x \phi(t)|^2 dx \leq \mathcal{E}\left(\mathbf{F}_0\right).
\end{aligned}$$

Proof. By Taylor expansion, we have

$$F_\pm(t)\ln F_\pm(t) - \mu\ln\mu = (1+\ln\mu)[F_\pm(t)-\mu] + \frac{1}{2\tilde{F}_\pm}|F_\pm(t)-\mu|^2, \tag{2.1}$$

where $\tilde{F}_\pm$ is between $F_\pm(t)$ and μ. It follows from (1.19) and (2.1) that $\mathcal{E}(F(t)) \geq 0$ for any $t \geq 0$. Noting $1+\ln\mu = -\left(\frac{3}{2}\ln(2\pi)-1\right) - \frac{1}{2}|v|^2$, we have from (1.15) and (1.17) $-$ (1.19) that

$$\begin{aligned}
&\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \frac{1}{2\tilde{F}_+}|F_+(t)-\mu|^2 \mathrm{d}v\mathrm{d}x + \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} \frac{1}{2\tilde{F}_-}|F_-(t)-\mu|^2 \mathrm{d}v\mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}^3} |\nabla_x\phi(t)|^2 \mathrm{d}x \\
&= \int_{\mathbb{T}^3}\int_{\mathbb{R}^3} (F_+(t)\ln F_+(t) + F_-(t)\ln F_-(t) - 2\mu\ln\mu) \mathrm{d}v\mathrm{d}x \\
&\quad + \left(\frac{3}{2}\ln(2\pi)-1\right)\int_{\mathbb{T}^3}\int_{\mathbb{R}^3} (F_+(t)+F_-(t)-2\mu)\mathrm{d}v\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\Big\{\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}|v|^2(F_+(t)+F_-(t)-2\mu)\mathrm{d}v\mathrm{d}x+\int_{\mathbb{T}^3}|\nabla_x\phi(t)|^2\mathrm{d}x\Big\}\\
\le&\int_{\mathbb{T}^3}\int_{\mathbb{R}^3}(F_{+,0}\ln F_{+,0}+F_{-,0}\ln F_{-,0}-2\mu\ln\mu)\mathrm{d}v\mathrm{d}x+\Big(\frac{3}{2}\ln(2\pi)-1\Big)(M_{+,0}+M_{-,0})+\frac{1}{2}E_0\\
=&\mathcal{E}(\mathbf{F}_0).
\end{aligned}\tag{2.2}
$$

By similar arguments as in [26, Lemma 2.7], we have

$$
\frac{|F_\pm-\mu|}{\tilde{F}_\pm}\ge\frac{1}{2}\ \text{for}\ |F_\pm-\mu|\ge\mu,\tag{2.3}
$$

and

$$
\frac{1}{\tilde{F}_\pm}\ge\frac{1}{2\mu}\ \text{for}\ |F_\pm-\mu|\le\mu.\tag{2.4}
$$

Then Lemma 2.2 follows from (2.2)–(2.4). The proof is completed. □

Multiplying the first equation of $(1.13)_3$ by ϕ and using Poincaré's inequality, one has

$$
\|\nabla\phi\|_{L^2}^2\le C\|\rho\|_{L^2}\|\phi\|_{L^2}\le C\|\rho\|_{L^2}\|\nabla\phi\|_{L^2},
$$

which yield that

$$
\|\nabla\phi\|_{L^2}\le C\|\rho\|_{L^2}\le C\|f_+-f_-\|_{L^2}.\tag{2.5}
$$

It follows from $(1.13)_3$ and (2.5) that

$$
\|\nabla\phi_0\|_{L^2}\le C\|\rho_0\|_{L^2}\le C\Big\|\int_{\mathbb{R}^3}\sqrt{\mu}(f_{+,0}-f_{-,0})\mathrm{d}v\Big\|_{L^\infty}\ll 1.
$$

By the standard L^p-theory for elliptic equations, for $p\in[2,6]$, we have

$$
\begin{aligned}
\|\phi\|_{W^{2,p}}&\lesssim\|\phi\|_{L^p}+\|\rho\|_{L^p}\lesssim\|\phi\|_{H^1}+\|f_+-f_-\|_{L^\infty}\\
&\lesssim\|\nabla\phi\|_{L^2}+\|\mathbf{h}\|_{L^\infty}.
\end{aligned}\tag{2.6}
$$

By Gagliardo-Nirenberg's inequality, for $3<p\le 6$, it follows from (2.5) and (2.6) that

$$
\begin{aligned}
\|\nabla\phi\|_{L^\infty}&\lesssim\|\nabla\phi\|_{L^2}^{\theta}\cdot\|\nabla^2\phi\|_{L^p}^{1-\theta}\lesssim\|\nabla\phi\|_{L^2}^{\theta}(\|\nabla\phi\|_{L^2}+\|\mathbf{h}\|_{L^\infty})^{1-\theta}\\
&\lesssim\varepsilon\|\mathbf{h}\|_{L^\infty}+\frac{C}{\varepsilon}\|\nabla\phi\|_{L^2}.
\end{aligned}\tag{2.7}
$$

2.1. **The Poisson equation in the torus $\mathbb{T}^3$.** In this subsection, we list some useful results of the Poisson equation in $\mathbb{T}^3$.

We first introduce some precision concerning the expression of the field.

Lemma 2.3 ([5], [61])**.** *There exists a unique $\mathbb{T}^3$-periodic function $\varphi\in C^\infty(\mathbb{R}^3)$ such that*
(1) For every periodic $\rho\in C^1(\mathbb{T}^3)$ with $\int_{\mathbb{T}^3}\rho(x)dx=0$,

$$
\Phi(x):=\int_{\mathbb{T}^3}\varphi(x-y)\rho(y)dy=\int_{\mathbb{T}^3+x'}\varphi(x-y)\rho(y)dy,\quad x,x'\in\mathbb{R}^3.
$$

is the unique-periodic solution in $C^2(\mathbb{T}^3)$ of the problem:

$$
\Delta\Phi=\rho,\quad\int_{\mathbb{T}^3}\Phi(x)dx=0.\tag{2.8}
$$

(2) There exists a function $\varphi_0\in C^\infty(\mathbb{R}^3/\mathbb{Z}^3\cup\{(0,0,0)\})$ such that

$$
\varphi(x)=-\frac{1}{4\pi|x|}+\varphi_0(x),\ x\in\mathbb{R}^3/\mathbb{Z}^3.\tag{2.9}
$$

Based on Lemma 2.3, we can easily deduce the gradient and second derivative estimates.

Lemma 2.4. *Let Φ satisfy (2.8). For any $0 < d < R < \frac{1}{2}$, we have*

$$\|\nabla_x \Phi\|_{L^\infty} \le C\|\rho\|_{L^\infty}. \tag{2.10}$$

$$\|\nabla_x^2 \Phi\|_{L^\infty} \le C\Big(\|\rho\|_{L^\infty}(1 + \ln\frac{R}{d} + R^{-3}) + d\|\nabla\rho\|_{L^\infty}\Big). \tag{2.11}$$

Proof. According to the periodic condition, it is easy to have that

$$\Phi(x) = \int_{\mathbb{T}^3} \varphi(x-y)\rho(y)dy = \int_{\mathbb{T}^3} \varphi(y)\rho(x-y)dy. \tag{2.12}$$

which implies that

$$\partial_{x_i}\Phi(x) = \int_{\mathbb{T}^3} \partial_{x_i}\varphi(x-y)\rho(y)dy = \int_{\mathbb{T}^3} \partial_{y_i}\varphi(y)\rho(x-y)dy. \tag{2.13}$$

Then (2.10) follows from (2.9) and (2.13).

For $\nabla_x^2\Phi(x)$, we have from (2.13) that

$$\begin{aligned}
\partial_{x_i}\partial_{x_j}\Phi(x) &= \int_{\mathbb{T}^3} \partial_{y_i}\varphi(y)\partial_{x_j}\rho(x-y)dy = -\int_{\mathbb{T}^3} \partial_{y_i}\varphi(y)\partial_{y_j}\rho(x-y)dy \\
&= -\Big\{\int_{\mathbb{T}^3\cap\{|y|\le\varepsilon\}} + \int_{\mathbb{T}^3\cap\{|y|>\varepsilon\}}\Big\}\partial_{y_i}\varphi(y)\partial_{y_j}\rho(x-y)dy \\
&= -\int_{\mathbb{T}^3\cap\{|y|>\varepsilon\}} \partial_{y_i}\varphi(y)\partial_{y_j}\rho(x-y)dy + O(\varepsilon)\|\nabla\rho\|_{L^\infty} \\
&= \int_{\mathbb{T}^3\cap\{|y|>\varepsilon\}} \partial_{y_j}\partial_{y_i}\varphi(y)\rho(x-y)dy + I(\rho) + O(\varepsilon)\|\nabla\rho\|_{L^\infty},
\end{aligned} \tag{2.14}$$

where

$$I(\rho) := \lim_{\varepsilon\to 0}\int_{\mathbb{T}^3\cap\{|y|=\varepsilon\}} \partial_{y_i}\varphi(y)\rho(x-y)\overrightarrow{n_j}\mathrm{d}s_y \cong \|\rho\|_{L^\infty}.$$

Recalling the definition of φ in (2.9), we know that

$$\partial_{y_j}\partial_{y_i}\varphi(y) = \Big(\frac{3y_iy_j}{|y|^5} - \frac{\delta_{ij}}{|y|^3}\Big) + \partial_{y_j}\partial_{y_i}\varphi_0(y).$$

which, together with (2.14), yields that

$$\begin{aligned}
\partial_{x_i}\partial_{x_j}\Phi(x) =& I(\rho) + \int_{\mathbb{T}^3\cap\{|y|>\varepsilon\}} \partial_{y_j}\partial_{y_i}\varphi_0(y)\rho(x-y)\mathrm{d}y + O(\varepsilon)\|\nabla\rho\|_{L^\infty} \\
&+ \int_{\mathbb{T}^3\cap\{|y|>\varepsilon\}} \Big(\frac{3y_iy_j}{|y|^5} - \frac{\delta_{ij}}{|y|^3}\Big)\rho(x-y)dy := I.
\end{aligned} \tag{2.15}$$

A direct calculation shows that

$$\begin{aligned}
I &= \Big\{\int_{\mathbb{T}^3\cap\{\varepsilon<|y|\le d\}} + \int_{\mathbb{T}^3\cap\{|y|>d\}}\Big\}\Big(\frac{3y_iy_j}{|y|^5} - \frac{\delta_{ij}}{|y|^3}\Big)\rho(x-y)\mathrm{d}y \\
&= \int_{\mathbb{T}^3\cap\{\varepsilon<|y|\le d\}} \Big(\frac{3y_iy_j}{|y|^5} - \frac{\delta_{ij}}{|y|^3}\Big)(\rho(x-y) - \rho(x))\mathrm{d}y \\
&\quad + \Big\{\int_{\mathbb{T}^3\cap\{d<|y|\le R\}} + \int_{\mathbb{T}^3\cap\{|y|\ge R\}}\Big\}\Big(\frac{3y_iy_j}{|y|^5} - \frac{\delta_{ij}}{|y|^3}\Big)\rho(x-y)dy.
\end{aligned} \tag{2.16}$$

It follows from (2.15) and (2.16) that

$$\partial_{y_j}\partial_{y_i}\Phi(y) = \int_{\mathbb{T}^3\cap\{\varepsilon<|y|\le d\}} \Big(\frac{3y_iy_j}{|y|^5} - \frac{\delta_{ij}}{|y|^3}\Big)(\rho(x-y) - \rho(x))\mathrm{d}y$$

$$+\Big\{\int_{\mathbb{T}^3\cap\{d<|y|\le R\}}+\int_{\mathbb{T}^3\cap\{|y|\ge R\}}\Big\}\Big(\frac{3y_iy_j}{|y|^5}-\frac{\delta_{ij}}{|y|^3}\Big)\rho(x-y)dy$$
$$+\int_{\mathbb{T}^3}\partial_{y_i}\partial_{y_j}\varphi_0(y)\rho(x-y)dy+I(\rho)+O(\varepsilon)\|\nabla\rho\|_{L^\infty}. \tag{2.17}$$

Taking ε sufficiently small, then (2.11) follows from (2.17). Therefore the proof of Lemma 2.4 is completed. □

2.2. **Characteristics.** As in [3, 63], we study the characteristics under the following free streaming condition,

$$\|\nabla_x\phi(t)\|_{L^\infty}\le\delta(1+t)^{-2},\quad \|\nabla_x^2\phi(t)\|_{L^\infty}\le\delta(1+t)^{-\frac52}, \tag{2.18}$$

where δ is a sufficiently small positive constant determined later.

Under the condition of (2.18), we can define the following backward characteristics of the ions (+) and electrons (-):

$$\begin{cases}\dfrac{dX_\pm(\tau)}{\mathrm{d}\tau}=V_\pm(\tau),\\ \dfrac{dV_\pm(\tau)}{\mathrm{d}\tau}=\mp\nabla_x\phi(\tau,X_\pm(\tau)),\quad \tau\in[0,t],\\ X_\pm(t;t,x,v)=x,\ V_\pm(t;t,x,v)=v,\end{cases} \tag{2.19}$$

where we have used the simplified notations

$$X_\pm(\tau)=X_\pm(\tau;t,x,v),\quad V_\pm(\tau)=V_\pm(\tau;t,x,v).$$

Lemma 2.5. *Assume* (2.18). *For any* (t,x,v) *and* $0\le s_1\le s_2\le t<\infty$, *it holds*

$$|V_\pm(s_2)-V_\pm(s_1)|\le C\delta.$$

Proof. Integrating $(2.19)_2$ over $[s_1,s_2]$, we have from (2.18) that

$$|V_\pm(s_2)-V_\pm(s_1)|=\Big|\int_{s_1}^{s_2}\nabla_x\phi(\tau,X_+(\tau))\mathrm{d}\tau\Big|\le C\delta\int_0^\infty(1+\tau)^{-2}\mathrm{d}\tau\le C\delta.$$

Therefore the proof of Lemma 2.5 is completed. □

Lemma 2.6 ([3])**.** *With the assumption of* (2.18), *for any* $0\le s\le t<\infty$, *it holds that*

$$\Big|\frac{\partial X_\pm(s;t,x,v)}{\partial v}+(t-s)Id\Big|+\Big|\frac{\partial V_\pm(s;t,x,v)}{\partial v}-Id\Big|\le C\delta(t-s), \tag{2.20}$$
$$\Big|\frac{\partial X_\pm(s;t,x,v)}{\partial x}-Id\Big|+\Big|\frac{\partial V_\pm(s;t,x,v)}{\partial x}\Big|\le C\delta, \tag{2.21}$$

for some positive constant $C>0$.

Corollary 2.7. *For* δ *suitably small and* $0\le s\le t$, *it holds that*

$$\Big|\det\Big(\frac{\partial X_\pm(s)}{\partial v}\Big)\Big|\ge\frac12(t-s)^3.$$

Lemma 2.8. *With the assumption of* (2.18), *for any* $0\le s\le t<\infty$, *it holds that*

$$\sup_{0\le s\le t,x\in\mathbb{T}^3}\Big\{\int_{|v|\le CN}|\partial_{vv}X_\pm(s;t,x,v)|^2\mathrm{d}v\Big\}^{1/2}\le C_N(t-s)^{\frac52}\sup_{0\le s\le t}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}. \tag{2.22}$$

Proof. From (2.19) and a direct computation, for $0\le s\le t$, one has

$$\partial_{vv}^2\ddot{X}_\pm(\tau)=\pm\nabla_x^3\phi(\tau,X_\pm(\tau))(\partial_vX_\pm(\tau))^2\pm\nabla_x^2\phi(\tau,X_\pm(\tau))(\partial_{vv}^2X_\pm(\tau)), \tag{2.23}$$

with $\partial_{vv}^2 X_\pm(t) = \partial_{vv}^2 \dot{X}_\pm(t) = 0$. Integrating (2.23) on τ twice, then switching the order of integration, we obtain

$$\partial_{vv}^2 X_\pm(s) = \int_s^t \int_\tau^t \partial_{vv}^2 \ddot{X}_\pm(\tau_1)\mathrm{d}\tau_1\mathrm{d}\tau = \int_s^t \int_s^{\tau_1} \partial_{vv}^2 \ddot{X}_\pm(\tau_1)\mathrm{d}\tau\mathrm{d}\tau_1. \tag{2.24}$$

Recalling (2.18), it follows from (2.20), (2.23) and (2.24) that

$$\begin{aligned}
|\partial_{vv}^2 X_\pm(s)| \leq& \int_s^t \int_s^{\tau_1} |\nabla_x^3\phi(\tau_1, X_\pm(\tau_1))||\partial_v X_\pm(\tau_1)|^2\mathrm{d}\tau\mathrm{d}\tau_1 + \delta\int_s^t (1+\tau_1)^{-\frac{3}{2}}|\partial_{vv}^2 X_\pm(\tau_1)|\mathrm{d}\tau_1\\
\leq& C\int_s^t |\nabla_x^3\phi(\tau_1, X_\pm(\tau_1))|(\tau_1 - s)(t-\tau_1)^2\mathrm{d}\tau_1 + \delta\int_s^t (1+\tau_1)^{-\frac{3}{2}}|\partial_{vv}^2 X_\pm(\tau_1)|\mathrm{d}\tau_1\\
\leq& C(t-s)\int_s^t |\nabla_x^3\phi(\tau_1, X_\pm(\tau_1))|(t-\tau_1)^2\mathrm{d}\tau_1 + \delta\int_s^t (1+\tau_1)^{-\frac{3}{2}}|\partial_{vv}^2 X_\pm(\tau_1)|\mathrm{d}\tau_1.
\end{aligned}$$

Gronwall's inequality implies that for $0 \leq s \leq t$

$$|\partial_{vv}^2 X_\pm(s;t,x,v)| \leq C(t-s)\int_s^t |\nabla_x^3\phi(\tau, X_\pm(\tau;t,x,v))|(t-\tau_1)^2\mathrm{d}\tau.$$

We thus conclude that

$$\begin{aligned}
&\Big(\int_{|v|\leq CN} |\partial_{vv} X_\pm(s;t,x,v)|^2\mathrm{d}v\Big)^{1/2}\\
\leq& C(t-s)\Big\{\int_{|v|\leq CN}\Big(\int_s^t |\nabla_x^3\phi(\tau, X_\pm(\tau;t,x,v))|(t-\tau_1)^2\mathrm{d}\tau\Big)^2\mathrm{d}v\Big\}^{1/2}\\
\leq& C(t-s)\int_s^t (t-\tau_1)^2\Big(\int_{|v|\leq CN}|\nabla_x^3\phi(\tau, X_\pm(\tau;t,x,v))|^2\mathrm{d}v\Big)^{1/2}\mathrm{d}\tau.
\end{aligned} \tag{2.25}$$

On the other hand, making the change of variable $y = X_\pm(\tau;t,x,v)$, one has from Corollary 2.7 that

$$\begin{aligned}
\Big(\int_{|v|\leq CN}|\nabla_x^3\phi(\tau, X_\pm(\tau;t,x,v))|^2\mathrm{d}v\Big)^{\frac{1}{2}} \leq& \Big(\int_{|y-x|\leq CN(t-\tau)}|\nabla_x^3\phi(\tau,y)|^2|\det(\frac{\partial v}{\partial y})|\mathrm{d}y\Big)^{\frac{1}{2}}\\
\leq& C(t-\tau)^{-\frac{3}{2}}\Big(\int_{|y-x|\leq CN(t-\tau)}|\nabla_x^3\phi(\tau,y)|^2\mathrm{d}y\Big)^{\frac{1}{2}}\\
\leq& C_N\frac{\big((t-\tau)^3+1\big)^{\frac{1}{2}}}{(t-\tau)^{\frac{3}{2}}}\|\nabla_x^3\phi(\tau)\|_{L^2}\\
\leq& C_N(t-\tau)^{-\frac{3}{2}}\|\partial_x\tilde{\mathbf{h}}(\tau)\|_{L^\infty},
\end{aligned} \tag{2.26}$$

where we have used the the standard elliptic estimate $\|\nabla_x^3\phi(\tau)\|_{L^2} \leq C\|\nabla_x\rho(\tau)\|_{L^2} \leq C\|\partial_x\tilde{\mathbf{h}}(\tau)\|_{L^\infty}$ in the last inequality. Substituting (2.26) into (2.25), we complete the proof of Lemma 2.8 . □

2.3. **Some useful estimates.** Lastly, for convenient use later, we will provide some very important estimates regarding the collision operators.

For any scalar function $\mathrm{g}(v)$, we define the operator K in scalar form:

$$\mathrm{Kg} := \frac{1}{\sqrt{\mu}}\left\{Q_{\mathrm{gain}}(\sqrt{\mu}\mathrm{g}, \mu) + Q(\mu, \sqrt{\mu}\mathrm{g})\right\}.$$

The Kg can be further split into $\mathrm{Kg} = \mathrm{K}_2\mathrm{g} - \mathrm{K}_1\mathrm{g}$ with

$$(\mathrm{K}_1\mathrm{g})(v) := \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\sqrt{\mu(v)\mu(u)}\mathrm{g}(u)\mathrm{d}\omega\mathrm{d}u = \int_{\mathbb{R}^3}\mathrm{k}_1(v,\eta)\mathrm{g}(\eta)\,\mathrm{d}\eta, \tag{2.27}$$

$$
\begin{aligned}
(\mathrm{K}_2\mathrm{g})(v) :=& \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\sqrt{\mu(u)\mu(u')}\mathrm{g}(v')\mathrm{d}\omega\mathrm{d}u \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{S}^2} B(v-u,\theta)\sqrt{\mu(u)\mu(v')}\mathrm{g}(u')\mathrm{d}\omega\mathrm{d}u = \int_{\mathbb{R}^3}\mathrm{k}_2(v,\eta)\mathrm{g}(\eta)\,\mathrm{d}\eta, \qquad (2.28)
\end{aligned}
$$

where $\mathrm{k}_i(v,\eta)=\mathrm{k}_i(\eta,v)$ $(i=1,2)$ satisfies the following Grad's estimates [33]:

$$
0 \le \mathrm{k}_1(v,\eta) \le C|v-\eta|^{\gamma} e^{-\frac{1}{4}(|v|^2+|u|^2)}, \qquad (2.29)
$$

$$
0 \le \mathrm{k}_2(v,\eta) \le C\frac{1}{|v-\eta|} e^{-\frac{|v-\eta|^2}{8}} e^{-\frac{||v|^2-|\eta|^2|^2}{8|v-\eta|^2}}. \qquad (2.30)
$$

By comparing (1.8)$-$(1.10) with (2.27)$-$(2.28), for any $\mathbf{g}=[g_+,g_-]^{\mathrm{T}}$, we see

$$
\begin{aligned}
(\mathrm{K}^{\pm}\mathbf{g})(v) =& \int_{\mathbb{R}^3}\Big(\frac{3}{2}\mathrm{k}_2(v,\eta)-\mathrm{k}_1(v,\eta)\Big)g_{\pm}(\eta)\,\mathrm{d}\eta + \int_{\mathbb{R}^3}\Big(\frac{1}{2}\mathrm{k}_2(v,\eta)-\mathrm{k}_1(v,\eta)\Big)g_{\mp}(\eta)\,\mathrm{d}\eta \\
=:& \int_{\mathbb{R}^3}\mathtt{k}^{(2)}(v,\eta)g_{\pm}(\eta)\,\mathrm{d}\eta + \int_{\mathbb{R}^3}\mathtt{k}^{(1)}(v,\eta)g_{\mp}(\eta)\,\mathrm{d}\eta. \qquad (2.31)
\end{aligned}
$$

According to (2.29) $-$ (2.31), one has

Lemma 2.9. *There is $C>0$ such that*

$$
0 \le |\mathtt{k}^{(i)}(v,\eta)| \le C\Big(\frac{1}{|v-\eta|}+|v-\eta|^{\gamma}\Big)e^{-\frac{|v-\eta|^2}{8}} e^{-\frac{||v|^2-|\eta|^2|^2}{8|v-\eta|^2}}, \quad i=1,2, \qquad (2.32)
$$

for any $v,\eta\in\mathbb{R}^3$ with $v\neq\eta$.

It is easy to show that

$$
\partial_x(\mathrm{K}^{\pm}\mathbf{g}) = \int_{\mathbb{R}^3}\mathtt{k}^{(2)}(v,\eta)\partial_x g_{\pm}(\eta)\,\mathrm{d}\eta + \int_{\mathbb{R}^3}\mathtt{k}^{(1)}(v,\eta)\partial_x g_{\mp}(\eta)\,\mathrm{d}\eta, \qquad (2.33)
$$

$$
\begin{aligned}
\partial_v(\mathrm{K}^{\pm}\mathbf{g}) =& \int_{\mathbb{R}^3}\tilde{\mathtt{k}}^{(2)}(v,\eta)g_{\pm}(\eta)\,\mathrm{d}\eta + \int_{\mathbb{R}^3}\mathtt{k}^{(2)}(v,\eta)\partial_v g_{\pm}(\eta)\,\mathrm{d}\eta \\
&+ \int_{\mathbb{R}^3}\tilde{\mathtt{k}}^{(1)}(v,\eta)g_{\mp}(\eta)\,\mathrm{d}\eta + \int_{\mathbb{R}^3}\mathtt{k}^{(1)}(v,\eta)\partial_v g_{\mp}(\eta)\,\mathrm{d}\eta, \qquad (2.34)
\end{aligned}
$$

where $\tilde{\mathtt{k}}^{(i)}(v,\eta)$ $(i=1,2)$ satisfies for all $0<\varepsilon<1$

$$
0 \le |\tilde{\mathtt{k}}^{(i)}(v,\eta)| \le C\Big(\frac{1}{|v-\eta|}+|v-\eta|\Big)e^{-\frac{(1-\varepsilon)|v-\eta|^2}{8}} e^{-\frac{(1-\varepsilon)||v|^2-|\eta|^2|^2}{8|v-\eta|^2}}, \quad i=1,2. \qquad (2.35)
$$

Moreover, an argument similar to the one used in [39], for $\beta\ge 0$ and $\sigma_0<\frac{1}{16}$, it holds that

$$
\int_{\mathbb{R}^3}\Big|\mathtt{k}^{(i)}(v,\eta)\frac{w_\beta(t,v)}{w_\beta(t,\eta)}\Big|\mathrm{d}\eta + \int_{\mathbb{R}^3}\Big|\tilde{\mathtt{k}}^{(i)}(v,\eta)\frac{w_\beta(t,v)}{w_\beta(t,\eta)}\Big|\mathrm{d}\eta \le C(1+|v|)^{-1} \quad i=1,2. \qquad (2.36)
$$

For any scalar functions $\mathrm{g}(v)$ and $\mathrm{f}(v)$, we define the nonlinear operator Γ:

$$
\begin{aligned}
\Gamma(\mathrm{f},\mathrm{g}) := \frac{1}{\sqrt{\mu}}Q(\sqrt{\mu}\mathrm{f},\sqrt{\mu}\mathrm{g}) =& \frac{1}{\sqrt{\mu}}\left\{Q_{\mathrm{gain}}(\sqrt{\mu}\mathrm{f},\sqrt{\mu}\mathrm{g}) - Q_{\mathrm{loss}}(\sqrt{\mu}\mathrm{f},\sqrt{\mu}\mathrm{g})\right\} \\
=:& \Gamma_{\mathrm{gain}}(\mathrm{f},\mathrm{g}) - \Gamma_{\mathrm{loss}}(\mathrm{f},\mathrm{g}).
\end{aligned}
$$

The next lemma is very powerful and important. The reader can refer to [27] for a rigorous proof.

Lemma 2.10. *There is a generic constant $C>0$ such that*

$$\Big|w_\beta\Gamma_{gain}(\mathrm{f},\mathrm{g})\Big|+\Big|w_\beta\Gamma_{gain}(\mathrm{g},\mathrm{f})\Big|\leq\frac{C\|w_\beta\mathrm{f}\|_{L^\infty_v}}{1+|v|}\Big(\int_{\mathbb{R}^3}(1+|\eta|)^4|e^{\frac{\sigma_0}{1+t}|\eta|^2}\mathrm{g}(\eta)|^2\,\mathrm{d}\eta\Big)^{\frac12},\tag{2.37}$$

for all $v\in\mathbb{R}^3$. In particular, for $\beta\geq4$, one has

$$|w_\beta\Gamma_{gain}(\mathrm{f},\mathrm{g})|+|w_\beta\Gamma_{gain}(\mathrm{g},\mathrm{f})|\leq C\|w_\beta\mathrm{f}\|_{L^\infty}\|w_\beta\mathrm{g}\|_{L^\infty},\tag{2.38}$$

for all $v\in\mathbb{R}^3$.

Lemma 2.11. *There is a generic constant $C>0$ such that*

$$|w_\beta(t,v)\Gamma_{loss}(\mathrm{f},\mathrm{g})(v)|\leq C\nu(v)|w_\beta(t,v)\mathrm{f}(v)|\cdot\|w_\beta\mathrm{g}\|^{\frac12}_{L^\infty}\Big(\int_{\mathbb{R}^3_v}|\mathrm{g}(v)|\mathrm{d}v\Big)^{\frac12},\tag{2.39}$$

for all $v\in\mathbb{R}^3$. In particular, for $\beta\geq3$,

$$|w_\beta\Gamma_{loss}(\mathrm{f},\mathrm{g})|\leq C\nu(v)\|w_\beta\mathrm{f}\|_{L^\infty}\|w_\beta\mathrm{g}\|_{L^\infty},\tag{2.40}$$

for all $v\in\mathbb{R}^3$.

Proof. It can be calculated directly that

$$\begin{aligned}|w_\beta\Gamma_{\rm loss}(\mathrm{f},\mathrm{g})|&=\Big|\frac{w_\beta(t,v)}{\sqrt{\mu}(v)}Q_{\rm loss}(\sqrt{\mu}\mathrm{f},\sqrt{\mu}\mathrm{g})\Big|\\&=\Big|w_\beta(t,v)\mathrm{f}(v)\Big|\cdot\Big|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)}\mathrm{g}(u)\mathrm{d}\omega\mathrm{d}u\Big|\\&\leq C\nu(v)|w_\beta(t,v)\mathrm{f}(v)|\cdot\|w_\beta\mathrm{g}\|^{\frac12}_{L^\infty}\Big(\int_{\mathbb{R}^3_v}|\mathrm{g}(v)|\mathrm{d}v\Big)^{\frac12},\end{aligned}$$

which yields that (2.39) and (2.40) hold. This completes the proof. □

With the help of the preceding Lemmas 2.10 and 2.11, one can obtain that

Corollary 2.12. *There is a generic constant $C>0$ such that*

$$\Big|w_\beta(t,v)\Gamma^\pm(\mathbf{f},\mathbf{f})(v)\Big|\leq C\nu(v)\|w_\beta\mathbf{f}\|^{\frac32}_{L^\infty}\Big(\int_{\mathbb{R}^3}e^{\frac{\sigma_0}{1+t}|\eta|^2}|\mathbf{f}(\eta)|\,\mathrm{d}\eta\Big)^{\frac12},\tag{2.41}$$

for all $v\in\mathbb{R}^3$ and $\beta\geq4$.

Using Lemmas 2.10 and 2.11, we will prove the following two key lemmas, which play an important role in the subsequent proof.

Lemma 2.13. *There is a generic constant $C>0$ such that, for $4\leq\beta_1\leq\beta-1$,*

$$|\partial_x\left(w_{\beta_1}\Gamma_{gain}(\mathrm{f},\mathrm{g})\right)|+|\partial_x\left(w_{\beta_1}\Gamma_{gain}(\mathrm{g},\mathrm{f})\right)|\leq C\|w_{\beta_1}\partial_x\mathrm{f}\|_{L^\infty}\|w_{\beta_1}\mathrm{g}\|_{L^\infty},\tag{2.42}$$

$$\begin{aligned}|\partial_x\left(w_{\beta_1}\Gamma_{loss}(\mathrm{f},\mathrm{g})\right)|&\leq C\nu(v)\|w_{\beta_1}\partial_x\mathrm{f}\|_{L^\infty}\|w_{\beta_1}\mathrm{g}\|^{\frac12}_{L^\infty}\Big(\int_{\mathbb{R}^3_v}|\mathrm{g}(v)|\mathrm{d}v\Big)^{\frac12}\\&\quad+C\|w_\beta\mathrm{f}\|_{L^\infty}\|w_{\beta_1}\partial_x\mathrm{g}\|_{L^\infty}.\end{aligned}\tag{2.43}$$

Proof. Notice that

$$\partial_x\left(w_{\beta_1}\Gamma_{\rm gain}(\mathrm{f},\mathrm{g})\right)=w_{\beta_1}\Gamma_{\rm gain}(\partial_x\mathrm{f},\mathrm{g})+w_{\beta_1}\Gamma_{\rm gain}(\mathrm{f},\partial_x\mathrm{g}).\tag{2.44}$$

Thus, by the rotation and (2.37), (2.42) immediately holds.

For the loss term, it is obvious that

$$\partial_x\left(w_{\beta_1}\Gamma_{\rm loss}(\mathrm{f},\mathrm{g})\right)=w_{\beta_1}\Gamma_{\rm loss}(\partial_x\mathrm{f},\mathrm{g})+w_{\beta_1}\Gamma_{\rm gain}(\mathrm{f},\partial_x\mathrm{g}).\tag{2.45}$$

Use (2.39), one has

$$|w_{\beta_1}\Gamma_{\text{loss}}(\partial_x \mathrm{f},\mathrm{g})| \le \nu(v)\|w_{\beta_1}\partial_x \mathrm{f}\|_{L^\infty}\|w_{\beta_1}\mathrm{g}\|_{L^\infty}^{\frac12}\Big(\int_{\mathbb{R}^3_v}|\mathrm{g}(v)|\mathrm{d}v\Big)^{\frac12}, \tag{2.46}$$

and

$$\begin{aligned}\Big|w_{\beta_1}\Gamma_{\text{loss}}(\mathrm{f},\partial_x\mathrm{g})\Big| &\le \nu(v)|w_{\beta_1}(t,v)\mathrm{f}(v)|\cdot\Big|\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)}\partial_x\mathrm{g}(u)\mathrm{d}\omega\mathrm{d}u\Big|\\ &\le \|w_\beta \mathrm{f}\|_{L^\infty}\|w_{\beta_1}\partial_x\mathrm{g}\|_{L^\infty}.\end{aligned} \tag{2.47}$$

Thus we conclude (2.43) from (2.45) − (2.47). Therefore the proof of Lemma 2.13 is completed. □

Lemma 2.14. *There is a generic constant $C>0$ such that, for $4\le\beta_1\le\beta-2$,*

$$\begin{aligned}&|\partial_v(w_{\beta_1}\Gamma_{gain}(\mathrm{f},\mathrm{g}))| + |\partial_v(w_{\beta_1}\Gamma_{gain}(\mathrm{g},\mathrm{f}))| \le C\|\partial_v(w_{\beta_1}\mathrm{f})\|_{L^\infty}\|w_\beta\mathrm{g}\|_{L^\infty}\\ &+C\|\partial_v(w_{\beta_1}\mathrm{g})\|_{L^\infty}\|w_\beta\mathrm{f}\|_{L^\infty}+C\|w_\beta\mathrm{f}\|_{L^\infty}\|w_\beta\mathrm{g}\|_{L^\infty},\end{aligned} \tag{2.48}$$

$$\begin{aligned}&|\partial_v(w_{\beta_1}\Gamma_{loss}(\mathrm{f},\mathrm{g}))| \le C\nu(v)\|\partial_v(w_{\beta_1}\mathrm{f})\|_{L^\infty}\|w_{\beta_1}\mathrm{g}\|_{L^\infty}^{\frac12}\Big(\int_{\mathbb{R}^3_v}|\mathrm{g}(v)w_{\beta_1}(t,v)|\mathrm{d}v\Big)^{\frac12}\\ &+C\|\partial_v(w_{\beta_1}\mathrm{g})\|_{L^\infty}\|w_\beta\mathrm{f}\|_{L^\infty}+C\|w_\beta\mathrm{f}\|_{L^\infty}\|w_\beta\mathrm{g}\|_{L^\infty}.\end{aligned} \tag{2.49}$$

Proof. Noting $\partial_v(w_{\beta_1}\Gamma_{\text{gain}}(\mathrm{f},\mathrm{g})) = \partial_v w_{\beta_1}\Gamma_{\text{gain}}(\mathrm{f},\mathrm{g}) + w_{\beta_1}\partial_v(\Gamma_{\text{gain}}(\mathrm{f},\mathrm{g}))$, one has

$$|\partial_v(w_{\beta_1}\Gamma_{\text{gain}}(\mathrm{f},\mathrm{g}))| \le |\partial_v w_{\beta_1}\Gamma_{\text{gain}}(\mathrm{f},\mathrm{g})| + |w_{\beta_1}\partial_v(\Gamma_{\text{gain}}(\mathrm{f},\mathrm{g}))|. \tag{2.50}$$

Using the change of variables $u'=v+z_\perp$, $v'=v+z_{\shortparallel}$ with $z=u-v$, $z_{\shortparallel}=(z\cdot\omega)\omega$, $z_\perp=z-z_{\shortparallel}$, one get

$$\begin{aligned}\Gamma_{\text{gain}}(\mathrm{f},\mathrm{g}) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)\sqrt{\mu(u)}\mathrm{f}(v')\mathrm{g}(u')\mathrm{d}\omega\mathrm{d}u\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(z,\theta)\sqrt{\mu(v+z)}\mathrm{f}(v+z_{\shortparallel})\mathrm{g}(v+z_\perp)\mathrm{d}\omega\mathrm{d}z,\end{aligned}$$

which yields that

$$\begin{aligned}\partial_v(\Gamma_{\text{gain}}(\mathrm{f},\mathrm{g})) &= \int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(z,\theta)\partial_v\Big(\sqrt{\mu(v+z)}\Big)\mathrm{f}(v+z_{\shortparallel})\mathrm{g}(v+z_\perp)\mathrm{d}\omega\mathrm{d}z\\ &+\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(z,\theta)\sqrt{\mu(v+z)}\partial_v\mathrm{f}(v+z_{\shortparallel})\mathrm{g}(v+z_\perp)\mathrm{d}\omega\mathrm{d}z\\ &+\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(z,\theta)\sqrt{\mu(v+z)}\mathrm{f}(v+z_{\shortparallel})\partial_v\mathrm{g}(v+z_\perp)\mathrm{d}\omega\mathrm{d}z.\end{aligned} \tag{2.51}$$

It follows from (2.51) and (2.38) that for $4\le\beta_1\le\beta-2$,

$$\begin{aligned}|w_{\beta_1}\partial_v(\Gamma_{\text{gain}}(\mathrm{f},\mathrm{g}))| &\le C\left(\|w_{\beta_1}\partial_v\mathrm{f}\|_{L^\infty}\|w_\beta\mathrm{g}\|_{L^\infty}+\|w_{\beta_1}\partial_v g\|_{L^\infty}\|w_\beta\mathrm{f}\|_{L^\infty}+\|w_\beta\mathrm{f}\|_{L^\infty}\|w_\beta\mathrm{g}\|_{L^\infty}\right)\\ &\le C\left(\|\partial_v(w_{\beta_1}\mathrm{f})\|_{L^\infty}\|w_\beta\mathrm{g}\|_{L^\infty}+\|\partial_v(w_{\beta_1}\mathrm{g})\|_{L^\infty}\|w_\beta\mathrm{f}\|_{L^\infty}\right)\\ &\quad+C\|w_\beta\mathrm{f}\|_{L^\infty}\|w_\beta\mathrm{g}\|_{L^\infty},\end{aligned} \tag{2.52}$$

where we used the fact $\|w_{\beta_1}\partial_v\mathrm{f}\|_{L^\infty}\le C(\|\partial_v(w_{\beta_1}\mathrm{f})\|_{L^\infty}+\|w_\beta\mathrm{f}\|_{L^\infty})$. From (2.50), (2.52) and (2.38), we conclude (2.48).

It is clear that

$$|\partial_v(w_{\beta_1}\Gamma_{\text{loss}}(\mathrm{f},\mathrm{g}))| \le |\partial_v w_{\beta_1}\Gamma_{\text{loss}}(\mathrm{f},\mathrm{g})| + |w_{\beta_1}\partial_v(\Gamma_{\text{loss}}(\mathrm{f},\mathrm{g}))|. \tag{2.53}$$

Making a change of variables $z=u-v$ in (2.53), one has

$$\Gamma_{\text{loss}}(\mathrm{f},\mathrm{g}) = \mathrm{f}(v)\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(z,\theta)\sqrt{\mu(v+z)}\mathrm{g}(v+z)\mathrm{d}\omega\mathrm{d}z,$$

which yields that

$$\begin{aligned}\partial_v\left(\Gamma_{\text{loss}}(\mathrm{f},\mathrm{g})\right)=&\partial_v\mathrm{f}(v)\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(z,\theta)\sqrt{\mu(v+z)}\mathrm{g}(v+z)\mathrm{d}\omega\mathrm{d}z\\&+f(v)\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(z,\theta)\partial_v\left(\sqrt{\mu(v+z)}\right)\mathrm{g}(v+z)\mathrm{d}\omega\mathrm{d}z\\&+\mathrm{f}(v)\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(z,\theta)\sqrt{\mu(v+z)}\partial_v\mathrm{g}(v+z)\mathrm{d}\omega\mathrm{d}z.\end{aligned}\tag{2.54}$$

Using (2.54) and similar arguments as in (2.46) and (2.47), one can obtain that for $4\le\beta_1\le\beta-2$,

$$\begin{aligned}|w_{\beta_1}\partial_v\left(\Gamma_{\text{loss}}(\mathrm{f},\mathrm{g})\right)|\le&\nu(v)\|\partial_v(w_{\beta_1}\mathrm{f})\|_{L^\infty}\|w_{\beta_1}\mathrm{g}\|_{L^\infty}^{\frac12}\Big(\int_{\mathbb{R}^3_v}|\mathrm{g}(v)w_{\beta_1}(t,v)|\mathrm{d}v\Big)^{\frac12}\\&+\|\partial_v(w_{\beta_1}\mathrm{g})\|_{L^\infty}\|w_\beta\mathrm{f}\|_{L^\infty}+\|w_\beta\mathrm{f}\|_{L^\infty}\|w_\beta\mathrm{g}\|_{L^\infty}.\end{aligned}\tag{2.55}$$

We conclude (2.49) from (2.53), (2.55) and (2.39). Therefore the proof of Lemma 2.14 is complete. □

From Lemmas 2.13 and 2.14, we have following results.

Corollary 2.15. *There is a generic constant $C>0$ such that, for $4\le\beta_1\le\beta-2$,*

$$\begin{aligned}\left|\partial_x\left(w_{\beta_1}\Gamma^{\pm}(\mathbf{f},\mathbf{f})\right)\right|\le&\nu(v)\|w_{\beta_1}\partial_x\mathbf{f}\|_{L^\infty}\|w_{\beta_1}\mathbf{f}\|_{L^\infty}^{\frac12}\Big(\int_{\mathbb{R}^3_v}|\mathbf{f}(v)w_{\beta_1}(t,v)|\mathrm{d}v\Big)^{\frac12}\\&+\|w_\beta\mathbf{f}\|_{L^\infty}\|w_{\beta_1}\partial_x\mathbf{f}\|_{L^\infty},\end{aligned}\tag{2.56}$$

$$\begin{aligned}\left|\partial_v\left(w_{\beta_1}\Gamma^{\pm}(\mathbf{f},\mathbf{f})\right)\right|\le&\nu(v)\|\partial_v(w_{\beta_1}\mathbf{f})\|_{L^\infty}\|w_{\beta_1}\mathbf{f}\|_{L^\infty}^{\frac12}\Big(\int_{\mathbb{R}^3_v}|\mathbf{f}(v)w_{\beta_1}(t,v)|\mathrm{d}v\Big)^{\frac12}\\&+\|\partial_v(w_{\beta_1}\mathbf{f})\|_{L^\infty}\|w_\beta\mathbf{f}\|_{L^\infty}+\|w_\beta\mathbf{f}\|_{L^\infty}^2.\end{aligned}\tag{2.57}$$

3. Local-in-time existence

Due to the influence of electric field, it seems that there is no reference on the existence of strong solution of VPB for large data. To prove Theorem 1.1, we need to figure out more quantitative properties of the local existence about the lifespan of local L^∞ solution. More precisely, since the derivative of the initial data is very large in our own setting, (see Remark 1.3), we have to establish the local existence with the lifespan depending only on $\|\mathbf{h}_0\|_{L^\infty}$, but independent of $\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty}$. Then the key point is to overcome the difficulties arose the nonlinear terms. Fortunately, by utilizing the logarithmic estimate on derivative of electric field, (see Lemma 2.4), one can obtain the uniform estimate of derivative with lifespan of the local solution depending only on $\|\mathbf{h}_0\|_{L^\infty}$.

Proposition 3.1 (Local Existence)**.** *Let $\beta_1\ge 4$, $\beta>\beta_1+4$. Assuming $F_{\pm,0}(x,v)=\mu(v)+\sqrt{\mu(v)}f_{\pm,0}(x,v)\ge 0$, $\|h_{\pm,0}\|_{L^\infty}<\infty$ and $\|\partial_{x,v}\tilde{h}_{\pm,0}\|_{L^\infty}<\infty$, then there exists a positive time*

$$t_1:=\frac{1}{16C_1C_2C_3C_4(\|\mathbf{h}_0\|_{L^\infty}+1)^2}\tag{3.1}$$

such that the VPB (1.3) *has a unique solution $F(t,x,v)=\mu(v)+\sqrt{\mu(v)}f(t,x,v)\ge 0$ satisfying*

$$\sup_{0\le t\le t_1}\|\mathbf{h}(t)\|_{L^\infty}\le 2C_1(1+\|\mathbf{h}_0\|_{L^\infty}),\tag{3.2}$$

$$\sup_{0\le t\le t_1}\|\partial_{x,v}\tilde{\mathbf{h}}(t)\|_{L^\infty}\le C_5(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0)\|_{L^\infty})^2,\tag{3.3}$$

where the positive constant C,C_i $(i=1,2,3,4,5)$ depending only on β,β_1. Moreover, the conservation laws of defect mass, momentum, energy (1.15)-(1.17) *as well as the additional defect entropy* *inequality* (1.18) *hold. Finally, if initial data* $f_{\pm,0}$ *are continuous, then the solution* $f_{\pm}(t,x,v)$ *is continuous in* $[0,t_1]\times\mathbb{T}^3\times\mathbb{R}^3$.

Moreover, if $\|f_{+,0}-f_{-,0}\|_{L^\infty}\le\varepsilon_0$*, it holds that*

$$\sup_{0\le t\le t_1}\|f_{+,0}-f_{-,0}(t)\|_{L^\infty}\le C\varepsilon_0\exp\{(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2\}. \tag{3.4}$$

Proof. Since the proof is lengthy, we divide it into seven steps.

Step 1. To consider the local existence of solutions for the VPB (1.3), we start from the iteration that for $n=0,1,2,\cdots$,

$$\begin{cases}\{\partial_t+v\cdot\nabla_x-\nabla_x\phi^n\cdot\nabla_v\}F_+^{n+1}+\displaystyle\iint B(v-u,w)(F_+^n+F_-^n)(u)\mathrm{d}w\mathrm{d}u\cdot F_+^{n+1}\\ =Q_{gain}(F_+^n,F_+^n+F_-^n),\\ \{\partial_t+v\cdot\nabla_x+\nabla_x\phi^n\cdot\nabla_v\}F_-^{n+1}+\displaystyle\iint B(v-u,w)(F_+^n+F_-^n)(u)\mathrm{d}w\mathrm{d}u\cdot F_-^{n+1}\\ =Q_{gain}(F_-^n,F_+^n+F_-^n),\\ -\Delta_x\phi^n=\displaystyle\int_{\mathbb{R}^3}(F_+^n-F_-^n)\mathrm{d}v,\quad\int_{\mathbb{T}^3}\phi^n\mathrm{d}x=0,\\ F_\pm^{n+1}(t,x,v)\Big|_{t=0}=F_{\pm,0}(x,v)\ge0,\quad F_\pm^0(t,x,v)=\mu(v).\end{cases} \tag{3.5}$$

Denote

$$f_\pm^{n+1}=\frac{F_\pm^{n+1}-\mu}{\sqrt{\mu}}.$$

Then $(3.5)_1$ and $(3.5)_2$ can be written equivalently as

$$\begin{cases}\{\partial_t+v\cdot\nabla_x-\nabla_x\phi^n\cdot\nabla_v\}f_+^{n+1}+g^nf_+^{n+1}+\nabla_x\phi^n\cdot\dfrac{v}{2}f_+^{n+1}+v\cdot\nabla_x\phi^n\sqrt{\mu}\\ =\mathrm{K}^+\mathbf{f}^n+\Gamma_{\mathrm{gain}}^+(\mathbf{f}^n,\mathbf{f}^n),\\ \{\partial_t+v\cdot\nabla_x+\nabla_x\phi^n\cdot\nabla_v\}f_-^{n+1}+g^nf_-^{n+1}-\nabla_x\phi^n\cdot\dfrac{v}{2}f_-^{n+1}-v.\nabla_x\phi^n\sqrt{\mu}\\ =\mathrm{K}^-\mathbf{f}^n+\Gamma_{\mathrm{gain}}^-(\mathbf{f}^n,\mathbf{f}^n),\\ -\Delta_x\phi^n=\displaystyle\int_{\mathbb{R}^3}\sqrt{\mu}(f_+^n-f_-^n)\mathrm{d}v,\quad\int_{\mathbb{T}^3}\phi^n\mathrm{d}x=0,\\ f_\pm^{n+1}(0,x,v)=f_{\pm,0}(x,v),\quad f_\pm^0(0,x,v)=0,\end{cases} \tag{3.6}$$

with

$$\begin{aligned}g^n(t,x,v)&=\iint B(v-u,\omega)\left(F_+^n+F_-^n\right)(t,x,u)\mathrm{d}\omega\mathrm{d}u\\ &=\iint B(v-u,\omega)\left(2\mu(u)+\sqrt{\mu}(u)\big(f_+^n(t,x,u)+f_-^n(t,x,u)\big)\right)\mathrm{d}\omega\mathrm{d}u.\end{aligned} \tag{3.7}$$

Step 2. Next, we shall use the induction argument on $n=0,1,\cdots$ to prove that there exists a positive time $t_1>0$, independent of n, such that (3.5) (equivalently (3.6)), admits a unique mild solution on the time interval $[0,t_1]$, with

$$F_\pm^n(t,x,v)\ge0,\quad\|\mathbf{h}^n(t)\|_{L^\infty}\le2C_1(\|\mathbf{h}_0\|_{L^\infty}+1),\quad\|\partial_{x,v}\tilde{\mathbf{h}}^n(t)\|_{L^\infty}\le B(t),\quad\forall t\in[0,t_1], \tag{3.8}$$

where $B(t)$ is some continuous positive function.

Firstly, we consider the positivity of $F_\pm^{n+1}$. By induction on n, it follows from (3.5) and (3.7) that $F_\pm^{n+1}\ge0,\ n=0,1,\cdots$, if $F_\pm^n\ge0$.

Next, we consider the uniform estimate for the approximation sequence. And it is more convenient to use the equivalent form $f_{\pm}^{n+1}$. To control the bad term $\nabla_x\phi^n\cdot\frac{v}{2}f_{\pm}^{n+1}$, we need the time-dependent weight function $w_\beta(t,v)$ defined in (1.20), and denote $h_{\pm}^n(t,x,v):=w_\beta(t,v)f_{\pm}^n(t,x,v)$. It follows from (3.6) that

$$\left\{\partial_t+v\cdot\nabla_x\mp\nabla_x\phi^n\cdot\nabla_v+\tilde{g}_{\pm}^n\right\}h_{\pm}^{n+1}=\Big(\mp v\cdot\nabla_x\phi^n\sqrt{\mu}+\mathrm{K}^{\pm}\mathbf{f}^n+\Gamma_{\text{gain}}^{\pm}\left(\mathbf{f}^n,\mathbf{f}^n\right)\Big)\cdot w_\beta, \tag{3.9}$$

where

$$\tilde{g}_{\pm}^n(t,x,v)=\frac{\sigma_0}{(1+t)^2}|v|^2\pm\nabla_x\phi^n\cdot v\left\{\frac{1}{2}+\frac{\beta}{1+|v|^2}+\frac{2\sigma_0}{1+t}\right\}+g^n(t,x,v). \tag{3.10}$$

We define the backward characteristics $(X_{\pm}^n(\tau;t,x,v),\ V_{\pm}^n(\tau;t,x,v))$ passing though (t,x,v) such that

$$\begin{cases}\dfrac{dX_{\pm}^n(\tau;t,x,v)}{\mathrm{d}\tau}=V_{\pm}^n(\tau;t,x,v),\\ \dfrac{dV_{\pm}^n(\tau;t,x,v)}{\mathrm{d}\tau}=\mp\nabla_x\phi^n(\tau,X_{\pm}^n(\tau;t,x,v)),\\ X_{\pm}^n(t;t,x,v)=x,\quad V_{\pm}^n(t;t,x,v)=v.\end{cases} \tag{3.11}$$

For convenience, we abbreviate $(X_{\pm}^n(\tau),V_{\pm}^n(\tau)):=(X_{\pm}^n(\tau;t,x,v),V_{\pm}^n(\tau;t,x,v))$ and $\tilde{g}_{\pm}^n(s):=\tilde{g}_{\pm}^n(s,X_{\pm}^n(s),V_{\pm}^n(s))$. Then integrating along the characteristics, the mild solution of (3.9) can be represented as

$$\begin{aligned}h_{\pm}^{n+1}(t,x,v)=&\,h_{\pm,0}\left(X_{\pm}^n(0),V_{\pm}^n(0)\right)e^{-\int_0^t\tilde{g}_{\pm}^n(\tau)\mathrm{d}\tau}\\&\mp\int_0^t e^{-\int_s^t\tilde{g}_{\pm}^n(\tau)\mathrm{d}\tau}\left(v\cdot\nabla_x\phi^n\sqrt{\mu}w_\beta\right)(s,X_{\pm}^n(s),V_{\pm}^n(s))\mathrm{d}s\\&+\int_0^t e^{-\int_s^t\tilde{g}_{\pm}^n(\tau)\mathrm{d}\tau}\left(w_\beta\mathrm{K}^{\pm}\mathbf{f}^n\right)(s,X_{\pm}^n(s),V_{\pm}^n(s))\mathrm{d}s\\&+\int_0^t e^{-\int_s^t\tilde{g}_{\pm}^n(\tau)\mathrm{d}\tau}\Big(w_\beta\Gamma_{\text{gain}}^{\pm}\left(\mathbf{f}^n,\mathbf{f}^n\right)\Big)(s,X_{\pm}^n(s),V_{\pm}^n(s))\mathrm{d}s.\end{aligned} \tag{3.12}$$

For simplicity, we always take $t_1<1$ in the following proof. For convenience, we agree on $C_0=\frac{\sigma_0}{4}$. If $\sup\limits_{0\le s\le t_1}\|\mathbf{h}^n(s)\|_{L^\infty}\le 2C_1(\|\mathbf{h}_0\|_{L^\infty}+1)$, where $C_1\ge e$, a direct calculation shows that

$$\tilde{g}_{\pm}^n(t,x,v)\ge C_0|v|^2-2CC_1(\|\mathbf{h}_0\|_{L^\infty}+1)|v|\ge -C_2(\|\mathbf{h}_0\|_{L^\infty}+1)^2,$$

where $C_2=\frac{C^2C_1^2}{C_0}$. Taking $t_1\le\frac{1}{C_2(\|\mathbf{h}_0\|_{L^\infty}+1)^2}$, which yields that

$$\exp\Big\{-\int_0^t\tilde{g}_{\pm}^n(\tau)\mathrm{d}\tau\Big\}\le C_1. \tag{3.13}$$

For $0\le t\le t_1$, a direct calculation shows that

$$\|\mathbf{h}^{n+1}(t)\|_{L^\infty}\le C_1\|\mathbf{h}_0\|_{L^\infty}+C_3t_1\sup_{0\le s\le t_1}\|\mathbf{h}^n(s)\|_{L^\infty}+C_3t_1\sup_{0\le s\le t_1}\|\mathbf{h}^n(s)\|_{L^\infty}^2, \tag{3.14}$$

where $C_3>0$ is a specific constant. Taking $t_1\le\frac{1}{8C_1C_2C_3(\|\mathbf{h}_0\|_{L^\infty}+1)^2}$, we have

$$\sup_{0\le s\le t_1}\|\mathbf{h}^{n+1}(s)\|_{L^\infty}\le\frac{3}{2}C_1(\|\mathbf{h}_0\|_{L^\infty}+1). \tag{3.15}$$

Next we consider the estimate of $\partial_{x,v}\tilde{\mathbf{h}}^{n+1}(s)$. We denote

$$\tilde{h}_{\pm}^n(t,x,v):=f_{\pm}^n(t,x,v)w_{\beta_1}(t,v),\quad\text{with}\quad w_{\beta_1}(t,v)=(1+|v|^2)^{\frac{\beta_1}{2}}e^{\frac{\sigma_0}{1+t}|v|^2}. \tag{3.16}$$

It follows from (3.6) that

$$\Big\{\partial_t + v\cdot\nabla_x \mp \nabla_x\phi^n\cdot\nabla_v + \tilde{g}^n_{\pm,1}\Big\}\tilde{h}^{n+1}_{\pm} = \Big(\mp v\cdot\nabla_x\phi^n\sqrt{\mu} + \mathrm{K}^{\pm}\mathbf{f}^n + \Gamma^{\pm}_{\text{gain}}\left(\mathbf{f}^n,\mathbf{f}^n\right)\Big)\cdot w_{\beta_1}, \quad (3.17)$$

where $\tilde{g}^n_{\pm,1}(t,x,v) = \frac{\sigma_0}{(1+t)^2}|v|^2 \pm \Big\{\frac{1}{2} + \frac{\beta_1}{1+|v|^2} + \frac{2\sigma_0}{1+t}\Big\} v\cdot\nabla_x\phi^n + g^n$. Denote $\partial_i = \frac{\partial}{\partial x_i}$, then it follows from (3.16) that

$$\begin{aligned}&\left\{\partial_t + v\cdot\nabla_x \mp \nabla_x\phi^n\cdot\nabla_v\right\}\partial_i\tilde{h}^{n+1}_{\pm} + \tilde{g}^n_{\pm,1}\partial_i\tilde{h}^{n+1}_{\pm} = \pm\nabla_x\partial_i\phi^n\cdot\nabla_v\tilde{h}^{n+1}_{+} - \partial_i\tilde{g}^n_{\pm,1}\tilde{h}^{n+1}_{\pm}\\ &\mp v\cdot\nabla_x\partial_i\phi^n\sqrt{\mu}(v)w_{\beta_1}(t,v) + \partial_i\left(w_{\beta_1}\mathrm{K}^{\pm}\mathbf{f}^n\right) + \partial_i\Big(w_{\beta_1}\Gamma^{\pm}_{\text{gain}}\left(\mathbf{f}^n,\mathbf{f}^n\right)\Big). \qquad (3.18)\end{aligned}$$

Integrating (3.18) along the characteristics, we have

$$\begin{aligned}\partial_i\tilde{h}^{n+1}_{\pm}(t,x,v) =& \partial_i\tilde{h}_{\pm,0}(X_{\pm}(0),V_{\pm}(0))e^{-\int_0^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau}\\ &\pm\int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau}\Big(\nabla_x\partial_i\phi^n\cdot\nabla_v\tilde{h}^{n+1}_{\pm}\Big)(s,X^n_{\pm}(s),V^n_{\pm}(s))\mathrm{d}s\\ &-\int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau}\Big(\partial_i\tilde{g}^n_{\pm,1}\tilde{h}^{n+1}_{\pm}\Big)(s,X^n_{\pm}(s),V^n_{\pm}(s))\mathrm{d}s\\ &\mp\int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau}\left(v\cdot\nabla_x\partial_i\phi^n\sqrt{\mu}(v)w_{\beta_1}(t,v)\right)(s,X^n_{\pm}(s),V^n_{\pm}(s))\mathrm{d}s\\ &+\int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau}\big(\partial_i\big(w_{\beta_1}\mathrm{K}^{\pm}\mathbf{f}^n\big)\big)(s,X^n_{\pm}(s),V^n_{\pm}(s))\mathrm{d}s\\ &+\int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau}\Big(\partial_i\big(w_{\beta_1}\Gamma^{\pm}_{\text{gain}}\left(\mathbf{f}^n,\mathbf{f}^n\right)\big)\Big)(s,X^n_{\pm}(s),V^n_{\pm}(s))\mathrm{d}s\\ :=&\sum_{i=0}^{5} D_i. \qquad (3.19)\end{aligned}$$

Similar to (3.13), it holds for $t\in[0,t_1]$ that

$$\exp\Big\{-\int_0^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau\Big\} \le C_1. \quad (3.20)$$

For D_1, taking $d = \frac{1}{C(\|\nabla\rho\|_{L^\infty}+1)}$ and $R = \frac{1}{4}$ in (2.11), then one gets that

$$\|\nabla^2\phi\|_{L^\infty} \le C(1+\|\rho\|_{L^\infty})\left(1+\ln(\|\nabla\rho\|_{L^\infty}+1)\right). \quad (3.21)$$

which together with (3.21) yields that for $0\le t\le t_1$

$$\begin{aligned}D_1 &\le C\int_0^t (1+\|\mathbf{h}^n(s)\|_{L^\infty})\Big(1+\ln(1+\|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty})\Big)\|\partial_v\tilde{\mathbf{h}}^{n+1}(s)\|_{L^\infty}\mathrm{d}s\\ &\le C(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t\Big(1+\ln(1+\|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty})\Big)\|\partial_v\tilde{\mathbf{h}}^{n+1}(s)\|_{L^\infty}\mathrm{d}s \qquad (3.22)\end{aligned}$$

For D_2, taking $d = \frac{1}{8}$ and $R = \frac{1}{4}$ in (2.11), one has

$$\|\nabla^2\phi\|_{L^\infty} \le C(\|\rho\|_{L^\infty} + \|\nabla\rho\|_{L^\infty}). \quad (3.23)$$

Recalling the definition of $\tilde{g}^n_{\pm,1}(t,x,v)$ in (3.17), one has from (3.23) that

$$\begin{aligned}|\partial_i\tilde{g}^n_{\pm,1}| &\le C(1+|v|)(\|\nabla_x\partial_i\phi^n\|_{L^\infty} + \|\partial_x\tilde{\mathbf{h}}^n\|_{L^\infty})\\ &\le C(1+|v|)(\|\mathbf{h}^n(s)\|_{L^\infty} + \|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty}),\end{aligned}$$

which yields that for $0 \le t \le t_1$

$$\begin{aligned}D_2 &\le C\int_0^t \|\mathbf{h}^{n+1}(s)\|_{L^\infty}(\|\nabla_x\partial_i\phi^n(s)\|_{L^\infty}+\|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty})\mathrm{d}s\\ &\le C\int_0^t \|\mathbf{h}^{n+1}(s)\|_{L^\infty}(\|\mathbf{h}^n(s)\|_{L^\infty}+\|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty})\mathrm{d}s\\ &\le C+C(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t \|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\mathrm{d}s.\end{aligned}\tag{3.24}$$

Similarly, it follows from (3.23) that for $0 \le t \le t_1$

$$\begin{aligned}D_3 &\le C\int_0^t (\|\mathbf{h}^n(s)\|_{L^\infty}+\|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty})\mathrm{d}s\\ &\le C+C\int_0^t \|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\mathrm{d}s.\end{aligned}\tag{3.25}$$

From (2.33) and (2.36), it is clear that

$$\begin{aligned}D_4 &\le C\int_0^t \|\partial_i\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\int_{\mathbb{R}^3}\frac{w_{\beta_1}(s,V_\pm^n(s))}{w_{\beta_1}(s,u)}\big(|\mathrm{k}^+(V_\pm^n(s),u)|+|\mathrm{k}^-(V_\pm^n(s),u)|\big)\,\mathrm{d}u\mathrm{d}s\\ &\le C\int_0^t \|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\mathrm{d}s.\end{aligned}\tag{3.26}$$

For the last term on the RHS of (3.19), it follows from (2.42) that

$$\begin{aligned}D_5 &\le C\int_0^t \|\mathbf{h}^n(s)\|_{L^\infty}\|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\mathrm{d}s\\ &\le C(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t \|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\mathrm{d}s.\end{aligned}\tag{3.27}$$

Substituting (3.22) and (3.24) $-$ (3.27) into (3.19), one obtains that

$$\begin{aligned}\|\partial_x\tilde{h}^{n+1}\|_{L^\infty} &\le C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})+C(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t \|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\mathrm{d}s\\ &\quad+C(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t \Big(1+\ln(1+\|\partial_x\tilde{\mathbf{h}}^n(s)\|_{L^\infty})\Big)\|\partial_v\tilde{\mathbf{h}}^{n+1}(s)\|_{L^\infty}\mathrm{d}s.\end{aligned}\tag{3.28}$$

To close the estimate of $\|\partial_x\tilde{h}^{n+1}\|_{L^\infty}$, we still need to control $\|\partial_v\tilde{h}^{n+1}\|_{L^\infty}$. We denote $\partial^j=\partial_{v_j}$, it follows from (3.17) that

$$\begin{aligned}\{\partial_t+v\cdot\nabla_x\mp\nabla_x\phi^n\cdot\nabla_v\}\partial^j\tilde{h}_\pm^{n+1}+\tilde{g}_{\pm,1}^n\partial^j\tilde{h}_\pm^{n+1} &= -\partial_j\tilde{h}_\pm^{n+1}-\partial^j\tilde{g}_{\pm,1}^n\tilde{h}_\pm^{n+1}\mp\partial^j(\sqrt{\mu}w_{\beta_1}v)\cdot\nabla_x\phi^n\\ &\quad+\partial^j\Big(w_{\beta_1}\mathrm{K}^\pm\mathbf{f}^n\Big)+\partial^j\Big(w_{\beta_1}\Gamma_{\text{gain}}^\pm(\mathbf{f}^n,\mathbf{f}^n)\Big).\end{aligned}\tag{3.29}$$

Integrating (3.29) along the characteristics, we have

$$\begin{aligned}\partial^j\tilde{h}_\pm^{n+1}(t,x,v) =&\Big(\partial^j\tilde{h}_{\pm,0}\Big)(X_\pm^n(0),V_\pm^n(0))e^{-\int_0^t\tilde{g}_{\pm,1}^n(\tau)\mathrm{d}\tau}\\ &-\int_0^t e^{-\int_s^t\tilde{g}_{\pm,1}^n(\tau)\mathrm{d}\tau}\Big(\partial_j\tilde{h}_\pm^{n+1}\Big)(s,X_\pm^n(s),V_\pm^n(s))\mathrm{d}s\\ &-\int_0^t e^{-\int_s^t\tilde{g}_{\pm,1}^n(\tau)\mathrm{d}\tau}\Big(\partial^j\tilde{g}_{\pm,1}^n\cdot\tilde{h}_\pm^{n+1}\Big)(s,X_\pm^n(s)\cdot V_\pm^n(s))\mathrm{d}s\\ &\mp\int_0^t e^{-\int_s^t\tilde{g}_{\pm,1}^n(\tau)\mathrm{d}\tau}\Big(\partial^j\big(\sqrt{\mu}w_{\beta_1}v\big)\cdot\nabla_x\phi^n\Big)(s,X_\pm^n(s),V_\pm^n(s))\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau}\Big(\partial^j\big(w_{\beta_1}\mathrm{K}^{\pm}\mathbf{f}^n\big)\Big)(s,X^n_{\pm}(s),V^n_{\pm}(s))\mathrm{d}s\\
&+\int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau}\Big(\partial^j\big(w_{\beta_1}\Gamma^{\pm}_{\text{gain}}(\mathbf{f}^n,\mathbf{f}^n)\big)\Big)(s,X^n_{\pm}(s),V^n_{\pm}(s))\mathrm{d}s\\
:=&\sum_{i=0}^{5}E_i. \qquad (3.30)
\end{aligned}
$$

It follows from (3.20) that for $0\le t\le t_1$

$$
\begin{aligned}
\sum_{i=0,1,3}E_i&\le C(\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty}+1)+C\int_0^t\|\partial_x\tilde{\mathbf{h}}^{n+1}(s)\|_{L^\infty}\mathrm{d}s+C\int_0^t\|\mathbf{h}^n(s)\|_{L^\infty}\mathrm{d}s\\
&\le C(\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty}+1)+C\int_0^t\|\partial_x\tilde{\mathbf{h}}^{n+1}(s)\|_{L^\infty}\mathrm{d}s. \qquad (3.31)
\end{aligned}
$$

For E_2, we note that

$$
|(\partial_{v_j}\tilde{g}^n_{\pm,1})(t,x,v)|\le C(1+|v|+\|\nabla_x\phi^n\|_{L^\infty}+\|\mathbf{h}^n\|_{L^\infty}),
$$

which yields that for $0\le t\le t_1$

$$
\begin{aligned}
E_2\le&C\int_0^t\left(1+\|\nabla_x\phi^n(s)\|_{L^\infty}+\|\mathbf{h}^n(s)\|_{L^\infty}\right)\|\mathbf{h}^{n+1}(s)\|_{L^\infty}\mathrm{d}s\\
\le&C\int_0^t(1+\|\mathbf{h}^n(s)\|_{L^\infty})\|\mathbf{h}^{n+1}(s)\|_{L^\infty}ds\le C. \qquad (3.32)
\end{aligned}
$$

For E_4, noting $\partial^j w_{\beta_1}\le Cw_\beta$. From (2.34) and (2.36), it is obvious that

$$
\begin{aligned}
|\partial^j(w_{\beta_1}\mathrm{K}^{\pm}\mathbf{f}^n)|\le&|\partial^j(w_{\beta_1})\mathrm{K}^{\pm}\mathbf{f}^n|+|w_{\beta_1}\partial^j(\mathrm{K}^{\pm}\mathbf{f}^n)|\\
\le&C(\|w_{\beta_1}\mathbf{f}^n\|_{L^\infty}+\|w_{\beta_1}\partial^j\mathbf{f}^n\|_{L^\infty}+\|w_\beta\mathbf{f}^n\|_{L^\infty})\\
\le&C(\|\mathbf{h}^n\|_{L^\infty}+\|\partial^j\tilde{\mathbf{h}}^n\|_{L^\infty}),
\end{aligned}
$$

which yields that for $0\le t\le t_1$

$$
E_4\le C\int_0^t(\|\mathbf{h}^n(s)\|_{L^\infty}+\|\partial_v\tilde{\mathbf{h}}^n(s)\|_{L^\infty})\mathrm{d}s\le C+C\int_0^t\|\partial_v\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\mathrm{d}s. \qquad (3.33)
$$

For E_5, we have from (2.48) that

$$
\begin{aligned}
E_5&\le C\int_0^t(\|\mathbf{h}^n(s)\|^2_{L^\infty}+\|\mathbf{h}^n(s)\|_{L^\infty}\|\partial_v\tilde{\mathbf{h}}^n(s)\|_{L^\infty})\mathrm{d}s\\
&\le C+C(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t\|\partial_v\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\mathrm{d}s. \qquad (3.34)
\end{aligned}
$$

Combining $(3.30)-(3.34)$, one has

$$
\begin{aligned}
\|\partial_v\tilde{\mathbf{h}}^{n+1}(t)\|_{L^\infty}\le&C(\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty}+1)+C(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t\|\partial_v\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\mathrm{d}s\\
&+\int_0^t\|\partial_x\tilde{\mathbf{h}}^{n+1}(s)\|_{L^\infty}\mathrm{d}s. \qquad (3.35)
\end{aligned}
$$

It follows from (3.28) and (3.35) that

$$
\|\partial_{x,v}\tilde{\mathbf{h}}^{n+1}(t)\|_{L^\infty}\le C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})+C(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t\|\partial_{x,v}\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\mathrm{d}s
$$

$$+ C(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t \Big(1+\ln\|\partial_{x,v}\tilde{\mathbf{h}}^n(s)\|_{L^\infty}\Big)\|\partial_{x,v}\tilde{\mathbf{h}}^{n+1}(s)\|_{L^\infty}\mathrm{d}s. \tag{3.36}$$

We make the *a priori* assumption $\|\partial_{x,v}\tilde{\mathbf{h}}^n(t)\|_{L^\infty}\le B(t)$. Then (3.36) yields that

$$\begin{aligned}\|\partial_{x,v}\tilde{\mathbf{h}}^{n+1}(t)\|_{L^\infty} \le{}& C_4(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty}) + C_4(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t B(s)\mathrm{d}s\\ &+ C_4(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t (1+\ln B(s))\,\|\partial_{x,v}\tilde{\mathbf{h}}^{n+1}(s)\|_{L^\infty}\mathrm{d}s,\end{aligned} \tag{3.37}$$

where C_4 is a specific constant.

We are now in a position to determine $B(t)$. In fact, we take $B(t)$ as the unique solution of the following integral equation:

$$\begin{aligned}B(t) ={}& C_4(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty}) + C_4(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t B(s)\mathrm{d}s\\ &+ C_4(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t (1+\ln(1+B(s)))\,B(s)\mathrm{d}s.\end{aligned} \tag{3.38}$$

After simple manipulation, (3.38) implies that

$$B(t)\le e^{-1}\cdot\Big\{\Big(C_4(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})+1\Big)\cdot e\Big\}^{\exp\{2C_4(1+\|\mathbf{h}_0\|_{L^\infty})t\}}-1.$$

Moreover, combining (3.37) and (3.38), Gronwall's inequality yields that $\|\partial_{x,v}\tilde{\mathbf{h}}^{n+1}(t)\|_{L^\infty}\le B(t)$.

Without loss of generality, taking $t_1=\frac{1}{16C_1C_2C_3C_4(\|\mathbf{h}_0\|_{L^\infty}+1)^2}$, for $0\le t\le t_1$ one has

$$B(t)\le C_5(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2.$$

By induction on n, we can prove that if

$$\|\partial_{x,v}\tilde{\mathbf{h}}^n(t)\|_{L^\infty}\le B(t)\le C_5(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2, \tag{3.39}$$

then it follows from (3.39) that $0\le t\le t_1$

$$\|\partial_{x,v}\tilde{\mathbf{h}}^{n+1}(t)\|_{L^\infty}\le B(t)\le C_5(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2, \tag{3.40}$$

for all $n\ge 0$.

Step 3. Now we prove that $\tilde{h}^{n+1},\ n=0,1,2,\cdots$ is a Cauchy sequence. It follows from (3.17) that

$$\begin{aligned}&\{\partial_t+v\cdot\nabla_x-\nabla_x\phi^n\cdot\nabla_v\}(\tilde{h}_+^{n+1}-\tilde{h}_+^n)+\tilde{g}_{+,1}^n(\tilde{h}_+^{n+1}-\tilde{h}_+^n)\\ ={}&\nabla_x(\phi^n-\phi^{n-1})\nabla_v\tilde{h}_+^n-(\tilde{g}_{+,1}^n-\tilde{g}_{+,1}^{n-1})\tilde{h}_+^n-v\cdot\nabla_x(\phi^n-\phi^{n-1})\sqrt{\mu}(v)w_{\beta_1}(t,v)\\ &+w_{\beta_1}\Big(\mathrm{K}^+\mathbf{f}^n-\mathrm{K}^+\mathbf{f}^{n-1}\Big)+w_{\beta_1}\Big(\Gamma^+_{\mathrm{gain}}(\mathbf{f}^n,\mathbf{f}^n)-\Gamma^+_{\mathrm{gain}}(\mathbf{f}^{n-1},\mathbf{f}^{n-1})\Big),\end{aligned} \tag{3.41}$$

and

$$\begin{aligned}&\{\partial_t+v\cdot\nabla_x+\nabla_x\phi^n\cdot\nabla_v\}(\tilde{h}_-^{n+1}-\tilde{h}_-^n)+\tilde{g}_{-,1}^n(\tilde{h}_-^{n+1}-\tilde{h}_-^n)\\ ={}&-\nabla_x(\phi^n-\phi^{n-1})\nabla_v\tilde{h}_-^n-(\tilde{g}_{-,1}^n-\tilde{g}_{-,1}^{n-1})\tilde{h}_-^n+v\cdot\nabla_x(\phi^n-\phi^{n-1})\sqrt{\mu}(v)w_{\beta_1}(t,v)\\ &+w_{\beta_1}\left(\mathrm{K}^-\mathbf{f}^n-\mathrm{K}^-\mathbf{f}^{n-1}\right)+w_{\beta_1}\Big(\Gamma^-_{\mathrm{gain}}(\mathbf{f}^n,\mathbf{f}^n)-\Gamma^-_{\mathrm{gain}}(\mathbf{f}^{n-1},\mathbf{f}^{n-1})\Big).\end{aligned} \tag{3.42}$$

Then integrating (3.41) and (3.42) along the characteristics, one has

$$\Big(\tilde{h}_\pm^{n+1}-\tilde{h}_\pm^n\Big)(t,x,v)$$

$$
\begin{aligned}
&= \pm \int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau} \Big(\nabla_x(\phi^n - \phi^{n-1}) \cdot \nabla_v \tilde{h}^{n+1}_{\pm}\Big)(s, X^n_{\pm}(s), V^n_{\pm}(s))\mathrm{d}s \\
&\quad - \int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau} \Big((\tilde{g}^n_{\pm,1} - \tilde{g}^{n-1}_{\pm,1}) \cdot \tilde{h}^n_{\pm}\Big)(s, X^n_{\pm}(s), V^n_{\pm}(s))\mathrm{d}s \\
&\quad \mp \int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau} \Big(v \cdot \nabla_x(\phi^n - \phi^{n-1})\sqrt{\mu}(v) w_{\beta_1}(t,v)\Big)(s, X^n_{\pm}(s), V^n_{\pm}(s))\mathrm{d}s \\
&\quad + \int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau} \Big(w_{\beta_1}(\mathrm{K}^{\pm}\mathbf{f}^n - \mathrm{K}^{\pm}\mathbf{f}^{n-1})\Big)(s, X^n_{\pm}(s), V^n_{\pm}(s))\mathrm{d}s \\
&\quad + \int_0^t e^{-\int_s^t \tilde{g}^n_{\pm,1}(\tau)\mathrm{d}\tau} \Big(w_{\beta_1}\big(\Gamma^{\pm}_{\text{gain}}(\mathbf{f}^n, \mathbf{f}^n) - \Gamma^{\pm}_{\text{gain}}(\mathbf{f}^{n-1}, \mathbf{f}^{n-1})\big)\Big)(s, X^n_{\pm}(s), V^n_{\pm}(s))\mathrm{d}s \\
&=: \sum_{i=1}^5 J_i.
\end{aligned}
\tag{3.43}
$$

Utilizing (3.40) and (2.10), one gets that

$$
J_1 \le C(1 + \|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 \int_0^t \|(\tilde{\mathbf{h}}^n - \tilde{\mathbf{h}}^{n-1})(s)\|_{L^\infty}\mathrm{d}s. \tag{3.44}
$$

For J_2, we note from (3.16) and (3.7) that

$$
\Big|(\tilde{g}^n_{\pm,1} - \tilde{g}^{n-1}_{\pm,1})(t,x,v)\Big| \le C(1+|v|)\|(\tilde{\mathbf{h}}^n - \tilde{\mathbf{h}}^{n-1})(t)\|_{L^\infty},
$$

which combined with (3.8) yield that

$$
J_2 + J_3 \le C(1 + \|\mathbf{h}_0\|_{L^\infty}) \int_0^t \|(\tilde{\mathbf{h}}^n - \tilde{\mathbf{h}}^{n-1})(s)\|_{L^\infty}\mathrm{d}s. \tag{3.45}
$$

For J_4, It is easy from (2.31) and (2.36) to see that

$$
\begin{aligned}
\Big|w_{\beta_1}(\mathrm{K}^{\pm}\mathbf{f}^n - \mathrm{K}^{\pm}\mathbf{f}^{n-1})\Big| =& \Big|w_{\beta_1}\int_{\mathbb{R}^3} \mathtt{k}^{(2)}(v,u)(f^n_{\pm} - f^{n-1}_{\pm})(u)\,\mathrm{d}u\Big| \\
&+ \Big|w_{\beta_1}\int_{\mathbb{R}^3} \mathtt{k}^{(1)}(v,u)(f^n_{\mp} - f^{n-1}_{\mp})(u)\,\mathrm{d}u\Big| \\
\le& C\|\tilde{\mathbf{h}}^n - \tilde{\mathbf{h}}^{n-1}\|_{L^\infty},
\end{aligned}
$$

which implies that

$$
J_4 \le C\int_0^t \|(\tilde{\mathbf{h}}^n - \tilde{\mathbf{h}}^{n-1})(s)\|_{L^\infty}\mathrm{d}s. \tag{3.46}
$$

For J_5, we have from (1.12) and (2.38) that

$$
w_{\beta_1}\Big|\Gamma^{\pm}_{\text{gain}}(\mathbf{f}^n, \mathbf{f}^n) - \Gamma^{\pm}_{\text{gain}}(\mathbf{f}^{n-1}, \mathbf{f}^{n-1})\Big| \le C(\|\tilde{\mathbf{h}}^n\|_{L^\infty} + \|\tilde{\mathbf{h}}^{n-1}\|_{L^\infty})\|\tilde{\mathbf{h}}^n - \tilde{\mathbf{h}}^{n-1})\|_{L^\infty},
$$

which with (3.8) yield that

$$
J_5 \le C(1 + \|\mathbf{h}_0\|_{L^\infty}) \int_0^t \|(\tilde{\mathbf{h}}^n - \tilde{\mathbf{h}}^{n-1})(s)\|_{L^\infty}\mathrm{d}s. \tag{3.47}
$$

Substituting (3.44)−(3.47) into (3.43), one obtains that for $0 \le t \le t_1$,

$$
\|\tilde{\mathbf{h}}^{n+1} - \tilde{\mathbf{h}}^n(t)\|_{L^\infty} \le C(1 + \|\mathbf{h}_0\|_{L^\infty} + \|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 \int_0^t \|(\tilde{\mathbf{h}}^n - \tilde{\mathbf{h}}^{n-1})(s)\|_{L^\infty}\mathrm{d}s,
$$

and by induction on n

$$\sup_{0\le s\le t_1}\|\tilde{\mathbf{h}}^{n+1}-\tilde{\mathbf{h}}^{n}(s)\|_{L^\infty}\le\frac{C^n(1+\|\mathbf{h}_0\|_{L^\infty}+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^{2n}}{n!}t_1^n\le\frac{C^n(1+\|\mathbf{h}_0\|_{L^\infty}+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^{2n}}{n!},$$

which yields immediately that $\tilde{h}_{\pm}^{n+1},\ n=0,1,2,\cdots$ is a Cauchy sequence. Therefore, there exists a limit $f_{\pm}$ such that

$$\sup_{0\le s\le t_1}\|w_{\beta_1}(\mathbf{f}^n-\mathbf{f})(s)\|_{L^\infty_{x,v}}\to 0\text{ as }n\to+\infty.$$

It is clear to know that $f_{\pm}$ is a mild solution of (1.13), thus $F_{\pm}=\mu(v)+\sqrt{\mu(v)}f_{\pm}\ge 0$ is a mild solution of VPB (1.3). It follows from (3.15), (3.39) and (3.40) that

$$\begin{aligned}&\sup_{0\le s\le t_1}\|\mathbf{h}(s)\|_{L^\infty}\le 2C_1(1+\|\mathbf{h}_0\|_{L^\infty}),\\&\sup_{0\le s\le t_1}\|\partial_{x,v}\tilde{\mathbf{h}}(s)\|_{L^\infty}\le C_5(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2.\end{aligned}$$

Step 4. Now we consider the uniqueness. Let $\mathbf{g}(t,x,v)$ be another solution of (1.13) with

$$\begin{aligned}&\sup_{0\le t\le t_1}\|w_\beta\mathbf{g}(t)\|_{L^\infty}\le 2C_1(1+\|\mathbf{h}_0\|_{L^\infty}),\\&\sup_{0\le t\le t_1}\|\partial_{x,v}(w_{\beta_1}\mathbf{g})\|_{L^\infty})\le C_5(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2.\end{aligned}$$

By similar arguments as in (3.41) $-$ (3.47), it is direct to obtain that

$$\|w_{\beta_1}(\mathbf{f}-\mathbf{g})(t)\|_{L^\infty}\le C(1+\|\mathbf{h}_0\|_{L^\infty}+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2\int_0^t\|w_{\beta_1}(\mathbf{f}-\mathbf{g})(s)\|_{L^\infty}\mathrm{d}s,$$

which, together with the Gronwall's inequality, yields the uniqueness, i.e., $\mathbf{f}=\mathbf{g}$.

Step 5. Taking the limit $n\to+\infty$ in (3.5), and then multiplying $(3.5)_1$ by $1,v,|v|^2$ and $\ln F_+$, and multiplying $(3.5)_2$ by $1,v,|v|^2$ and $\ln F_-$, integrating by parts, together with $(3.5)_3$, one can obtain (1.15)$-$(1.17) and (1.18).

Step 6. If $F_{\pm,0}$ (or equivalent $f_{\pm,0}$) is continuous, it is direct to check that $F_{\pm}^{n+1}(t,x,v)$ (or equivalent $f_{\pm}^{n+1}(t,x,v)$) is continuous in $[0,\infty)\times\mathbb{T}^3\times\mathbb{R}^3$. The continuous of $\mathbf{f}(t,x,v)$ is an immediate consequence of $\sup\limits_{0\le s\le t_1}\|(\mathbf{f}^{n+1}-\mathbf{f})(s)\|_{L^\infty}\to 0$ as $n\to+\infty$.

Step 7. Let $\|f_{+,0}-f_{-,0}\|_\infty\le\varepsilon_0$. It follows from (3.6) that

$$\begin{aligned}&\{\partial_t+v\cdot\nabla_x\}(f_+-f_-)+g(t,x,v)(f_+-f_-)\\=&\nabla_x\phi\cdot\nabla_v(f_++f_-)-\nabla_x\phi\cdot\frac{v}{2}(f_++f_-)-2v\cdot\nabla_x\phi\sqrt{\mu}+\mathrm{K}^+\mathbf{f}-\mathrm{K}^-\mathbf{f}\\&+\Gamma^+_{\text{gain}}(\mathbf{f},\mathbf{f})-\Gamma^-_{\text{gain}}(\mathbf{f},\mathbf{f}),\end{aligned}\tag{3.48}$$

where

$$g(t,x,v)=\iint B(v-u,\omega)\left(F_++F_-\right)(t,x,u)\mathrm{d}\omega\mathrm{d}u\ge 0.$$

From (3.48), it is direct to have that

$$\begin{aligned}(f_+-f_-)(t,x,v)=&\left(f_{+,0}-f_{-,0}\right)(x-tv,v)e^{-\int_0^t g(\tau,\mathfrak{X}(\tau),v)\mathrm{d}\tau}\\&+\int_0^t e^{-\int_s^t g(\tau,\mathfrak{X}(\tau),v)\mathrm{d}\tau}\Big(\nabla_x\phi\cdot\nabla_v(f_++f_-)\Big)(s,\mathfrak{X}(\tau),v)\mathrm{d}s\\&-\int_0^t e^{-\int_s^t g(\tau,\mathfrak{X}(\tau),v)\mathrm{d}\tau}\Big(\nabla_x\phi\cdot\frac{v}{2}(f_++f_-)\Big)(s,\mathfrak{X}(\tau),v)\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&-2\int_0^t e^{-\int_s^t g(\tau,\mathfrak{X}(\tau),v)\mathrm{d}\tau}\Big(v\cdot\nabla_x\phi\sqrt{\mu}\Big)(s,\mathfrak{X}(\tau),v)\mathrm{d}s\\
&+\int_0^t e^{-\int_s^t g(\tau,\mathfrak{X}(\tau),v)\mathrm{d}\tau}\Big(\mathrm{K}^+\mathbf{f}-\mathrm{K}^-\mathbf{f}\Big)(s,\mathfrak{X}(\tau),v)\mathrm{d}s\\
&+\int_0^t e^{-\int_s^t g(\tau,\mathfrak{X}(\tau),v)\mathrm{d}\tau}\Big(\Gamma^+_{\text{gain}}(\mathbf{f},\mathbf{f})-\Gamma^-_{\text{gain}}(\mathbf{f},\mathbf{f})\Big)(s,\mathfrak{X}(\tau),v)\mathrm{d}s\\
:=&\sum_{i=0}^5 H_i,
\end{aligned}
\tag{3.49}
$$

where $\mathfrak{X}(s):=x-v(t-s)$. Noting (3.2), (3.3), we have

$$
\sum_{i=0}^3 H_i\le\|f_{+,0}-f_{-,0}\|_{L^\infty}+C(1+\|\mathbf{h}_0\|_{L^\infty}+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2\int_0^t\|(f_+-f_-)(s)\|_{L^\infty}\mathrm{d}s. \tag{3.50}
$$

For H_4, by a rotation, it follows from (2.28) that

$$
\begin{aligned}
\mathrm{K}^+\mathbf{f}-\mathrm{K}^-\mathbf{f}=&2\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}B(v-u,\theta)(f_+-f_-)(v')\sqrt{\mu(u)}\sqrt{\mu(u')}dud\omega\\
=&\int_{\mathbb{R}^3}\mathrm{k}_2(v,u)(f_+-f_-)(u)\,\mathrm{d}\eta,
\end{aligned}
$$

which combined with (2.30) and (2.36) yield that

$$
|\mathrm{K}^+\mathbf{f}-\mathrm{K}^-\mathbf{f}|\le C\|f_+-f_-\|_{L^\infty}. \tag{3.51}
$$

Then it follows from (3.51) that

$$
H_4\le C\int_0^t\|(f_+-f_-)(s)\|_{L^\infty}\mathrm{d}s. \tag{3.52}
$$

For H_5, by a rotation, it is noted that

$$
\begin{aligned}
\Gamma^+_{\text{gain}}(\mathbf{f},\mathbf{f})-\Gamma^-_{\text{gain}}(\mathbf{f},\mathbf{f})=&\frac{1}{\sqrt{\mu}}Q_{\text{gain}}\left(\sqrt{\mu}(f_+-f_-),\sqrt{\mu}(f_++f_-)\right)\\
=&\iint B(v-u,\theta)\sqrt{\mu(u)}(f_+-f_-)(v')(f_++f_-)(u')\mathrm{d}\omega\mathrm{d}u\\
=&\iint B(v-u,\theta)\sqrt{\mu(u)}(f_+-f_-)(u')(f_++f_-)(v')\mathrm{d}\omega\mathrm{d}u=:I_A.
\end{aligned}
$$

To I_A, as in [31], we use the change of variables $u'=v+z_\perp$, $v'=v+z_{\shortparallel}$ and $\eta=v+z_{\shortparallel}$ with $z=u-v$, $z_{\shortparallel}=(z\cdot\omega)\omega$, $z_\perp=z-z_{\shortparallel}$. Moreover,

$$
|B(v-u,\theta)|\le C(|z_{\shortparallel}|^2+|z_\perp|^2)^{\frac{\gamma-1}{2}}|z_{\shortparallel}|,
$$

and

$$
\mathrm{d}\omega\mathrm{d}u=\mathrm{d}z_{\shortparallel}\mathrm{d}z_\perp\mathrm{d}\omega=2|z_{\shortparallel}|^2\mathrm{d}|z_{\shortparallel}|\mathrm{d}\omega\frac{\mathrm{d}z_\perp}{|z_{\shortparallel}|^2}=\frac{2}{|z_{\shortparallel}|^2}\mathrm{d}z_{\shortparallel}\mathrm{d}z_\perp=\frac{2}{|\eta-v|^2}\mathrm{d}\eta\mathrm{d}z_\perp.
$$

Then one has

$$
\begin{aligned}
I_A\le&\|(f_+-f_-)\|_{L^\infty}\int_\eta\int_{z_\perp}\frac{1}{|\eta-v|}|(z_{\shortparallel}|^2+|z_\perp|^2)^{\frac{\gamma-1}{2}}e^{-\frac{|\eta+z_\perp|^2}{4}}|\mathbf{f}(\eta)|\mathrm{d}\eta\mathrm{d}z_\perp\\
\le&C\|f_+-f_-\|_{L^\infty}\|\mathbf{h}\|_{L^\infty}.
\end{aligned}
\tag{3.53}
$$

which yields that

$$H_5 \leq C(1+\|\mathbf{h}_0\|_{L^\infty})\int_0^t \|(f_+-f_-)(s)\|_{L^\infty}\mathrm{d}s. \tag{3.54}$$

Submitting (3.50), (3.52) and (3.54) into (3.49), one has

$$\|(f_+-f_-(t)\|_{L^\infty} \leq \varepsilon_0 \exp\left\{C(1+\|\mathbf{h}_0\|_{L^\infty}+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 t\right\}.$$

Then for $0 \leq t \leq t_1$,

$$\|(f_+-f_-(t)\|_{L^\infty} \leq C\varepsilon_0 \exp\left\{C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 t\right\} \leq C\varepsilon_0 \exp\left\{C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2\right\}. \tag{3.55}$$

Therefore the proof of Proposition 3.1 is completed. $\square$

4. Uniform L^∞-Estimate

From now on, we make the *a priori* assumption (2.18), i.e.,

$$\|\nabla_x\phi(t)\|_{L^\infty} \leq \delta(1+t)^{-2}, \quad \|\nabla_x^2\phi(t)\|_{L^\infty} \leq \delta(1+t)^{-\frac{5}{2}},$$

where δ is a small positive constant to be determined later.

4.1. L^∞ *a priori* **estimate.** Recall the weight function $w_\beta(t,v)$ in (1.20). It follows from $(1.13)_{1,2}$ that

$$(\partial_t + v\cdot\nabla_x - \nabla_x\phi\cdot\nabla_v)\, h_+ + \tilde{\nu}_+ h_+ = w_\beta \mathrm{K}^+\mathbf{f} + w_\beta\Gamma^+(\mathbf{f},\mathbf{f}) - \nabla_x\phi\cdot v w_\beta\sqrt{\mu}, \tag{4.1}$$

$$(\partial_t + v\cdot\nabla_x + \nabla_x\phi\cdot\nabla_v)\, h_- + \tilde{\nu}_- h_- = w_\beta \mathrm{K}^-\mathbf{f} + w_\beta\Gamma^-(\mathbf{f},\mathbf{f}) + \nabla_x\phi\cdot v w_\beta\sqrt{\mu}, \tag{4.2}$$

where

$$\tilde{\nu}_\pm(t,x,v) := \frac{\sigma_0}{(1+t)^2}|v|^2 + \nu(v) \pm \nabla_x\phi\cdot v\Big(\frac{1}{2}+\frac{\beta}{1+|v|^2}+\frac{2\sigma_0}{1+t}\Big). \tag{4.3}$$

For $\delta \ll 1$, it follows from (2.18) and (4.3) that

$$\begin{aligned}\tilde{\nu}_\pm(t,x,v) &\geq \frac{1}{(1+t)^2}\left(\sigma_0|v|^2 - C\delta|v|\right) + \nu(v)\\ &= \frac{\sigma_0}{(1+t)^2}\Big(|v|-\frac{C\delta}{2\sigma_0}\Big)^2 - \frac{C^2\delta^2}{4\sigma_0(1+t)^2} + \nu(v) \geq \frac{1}{2}\nu(v) \geq \tilde{\nu}_0 > 0,\end{aligned} \tag{4.4}$$

for a positive constant $\tilde{\nu}_0$.

Recall the backward characteristics $(X_\pm(\tau), V_\pm(\tau)) = (X_\pm(\tau;t,x,v), V_\pm(\tau;t,x,v))$ in (2.19), the mild formulations of (4.1), (4.2) take the form

$$\begin{aligned}h_\pm(t,x,v) &= (h_{\pm,0})\,(X_\pm(0),V_\pm(0))e^{-\int_0^t\tilde{\nu}_\pm(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\\ &\quad+\int_0^t e^{-\int_s^t\tilde{\nu}_\pm(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\Big(w_\beta\mathrm{K}^\pm\mathbf{f}\Big)(s,X_\pm(s),V_\pm(s))\mathrm{d}s\\ &\quad\mp\int_0^t e^{-\int_s^t\tilde{\nu}_\pm(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\Big(\nabla_x\phi\cdot v w_\beta\sqrt{\mu}\Big)(s,X_\pm(s),V_\pm(s))\mathrm{d}s\\ &\quad+\int_0^t e^{-\int_s^t\tilde{\nu}_\pm(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\Big(w_\beta\Gamma^\pm(\mathbf{f},\mathbf{f})\Big)(s,X_\pm(s),V_\pm(s))\mathrm{d}s.\end{aligned} \tag{4.5}$$

Lemma 4.1. *Under the condition* (2.18), *it holds that for* $\beta \geq 4$ *and* $\beta_1 \geq 0$,

$$\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty} \leq C_6\{\|\mathbf{h}_0\|_{L^\infty}+\|\mathbf{h}_0\|_{L^\infty}^2+\sqrt{\mathcal{E}(\mathbf{F}_0)}\}$$

$$+ C_6 \sup_{\substack{t_1 \le s \le t \\ y \in \mathbb{T}^3}} \Big\{ \|\mathbf{h}(s)\|_{L^\infty}^{\frac{3}{2}} \Big(\int_{\mathbb{R}^3} |\tilde{\mathbf{h}}(s,y,u)| \mathrm{d}u \Big)^{\frac{1}{2}} \Big\}, \tag{4.6}$$

where the positive constant $C_6 \ge 1$ depends only on γ, β, Here $t_1 > 0$ is the lifespan defined in (3.1).

Proof. From (2.41) and (4.5), it is easy to have that

$$\begin{aligned}
|h_\pm(t,x,v)| \le{}& \|\mathbf{h}_0\|_{L^\infty} e^{-\tilde{\nu}_0 t} + \int_0^t e^{-\tilde{\nu}_0(t-s)} |(w_\beta \mathrm{K}^\pm \mathbf{f})(s, X_\pm(s), V_\pm(s))| \mathrm{d}s \\
&+ C \int_0^t e^{-\tilde{\nu}_0(t-s)} \mu^{\frac{1}{4}}(V_\pm(s)) |\nabla_x \phi(s, X_\pm(s), V_\pm(s))| \mathrm{d}s \\
&+ C \int_0^t e^{-\int_s^t \tilde{\nu}_\pm(\tau)\mathrm{d}\tau} \cdot \nu(V_\pm(s)) \|\mathbf{h}(s)\|_{L^\infty}^{\frac{3}{2}} \Big(\int_{\mathbb{R}^3} |\tilde{\mathbf{h}}(s, X_\pm(s), u)| \mathrm{d}u \Big)^{\frac{1}{2}} \mathrm{d}s.
\end{aligned} \tag{4.7}$$

It follows from (2.7) that

$$\begin{aligned}
&\int_0^t e^{-\tilde{\nu}_0(t-s)} \mu^{\frac{1}{4}}(V_\pm(s)) |\nabla_x \phi(s, X_\pm(s))| \mathrm{d}s \\
&\le \frac{C}{N} \int_0^t e^{-\tilde{\nu}_0(t-s)} \|\mathbf{h}(s)\|_{L^\infty} \mathrm{d}s + C_N \int_0^t e^{-\tilde{\nu}_0(t-s)} \|\nabla_x \phi(s)\|_{L^2} \mathrm{d}s \\
&\le \frac{C}{N} \sup_{0 \le s \le t} \|\mathbf{h}(s)\|_{L^\infty} + C_N \sup_{0 \le s \le t} \|\nabla_x \phi(s)\|_{L^2}.
\end{aligned} \tag{4.8}$$

Substituting (4.8) into (4.7), one has

$$\begin{aligned}
|h_\pm(t,x,v)| \le{}& \|\mathbf{h}_0\|_{L^\infty} e^{-\tilde{\nu}_0 t} + \frac{C}{N} \sup_{0 \le s \le t} \|\mathbf{h}(s)\|_{L^\infty} + C_N \sup_{0 \le s \le t} \|\nabla_x \phi(s)\|_{L^2} \\
&+ C \sup_{\substack{0 \le s \le t \\ y \in \mathbb{T}^3}} \Big\{ \|\mathbf{h}(s)\|_{L^\infty}^{\frac{3}{2}} \cdot \Big(\int_{\mathbb{R}^3} |\tilde{\mathbf{h}}(s,y,u)| \mathrm{d}u \Big)^{\frac{1}{2}} \Big\} \\
&+ C \int_0^t e^{-\tilde{\nu}_0(t-s)} \mathrm{d}s \int_{\mathbb{R}^3} |\mathtt{k}_{w_\beta}^{(2)}(V_\pm(s), u) h_\pm(s, X_\pm(s), u)| \mathrm{d}u \\
&+ C \int_0^t e^{-\tilde{\nu}_0(t-s)} \mathrm{d}s \int_{\mathbb{R}^3} |\mathtt{k}_{w_\beta}^{(1)}(V_\pm(s), u) h_\mp(s, X_\pm(s), u)| du,
\end{aligned} \tag{4.9}$$

where $\mathtt{k}_{w_\beta}^{(i)}(v,u) = \mathtt{k}^{(i)}(v,u) \cdot \frac{w_\beta(t,v)}{w_\beta(t,u)}$ $(i = 1, 2)$. We only consider $|h_+(t,x,v)|$ because $|h_-(t,x,v)|$ can be dealt similarly. For simplicity of presentation, we denote

$$(\hat{X}_\pm(\tau), \hat{V}_\pm(\tau)) := (X_\pm(\tau; s, X_+(s), u), V_\pm(\tau; s, X_+(s), u)). \tag{4.10}$$

Using (4.9) again, one has

$$\begin{aligned}
|h_+(t,x,v)| \le{}& C\|\mathbf{h}_0\|_{L^\infty} + \frac{C}{N} \sup_{0 \le s \le t} \|\mathbf{h}(s)\|_{L^\infty} + C_N \sup_{0 \le s \le t} \|\nabla_x \phi(s)\|_{L^2} \\
&+ C \sup_{\substack{0 \le s \le t \\ y \in \mathbb{T}^3}} \Big\{ \|\mathbf{h}(s)\|_{L^\infty}^{\frac{3}{2}} \cdot \Big(\int_{\mathbb{R}^3} |\tilde{\mathbf{h}}(s,y,u)| \mathrm{d}u \Big)^{\frac{1}{2}} \Big\} + \sum_{i=1}^4 I_i,
\end{aligned} \tag{4.11}$$

where,

$$I_1 := \int_0^t \int_0^s e^{-\tilde{\nu}_0(t-s)} e^{-\tilde{\nu}_0(s-\tau)} \mathrm{d}\tau \mathrm{d}s$$

$$\begin{aligned}
&\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\mathtt{k}^{(2)}_{w_\beta}(V_+(s),u)\mathtt{k}^{(2)}_{w_\beta}(\hat V_+(\tau),u')||h_+(\tau,\hat X_+(\tau),u')|\mathrm{d}u'du;\\
I_2:=&\int_0^t\int_0^s e^{-\tilde\nu_0(t-s)}e^{-\tilde\nu_0(s-\tau)}\mathrm{d}\tau\mathrm{d}s\\
&\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\mathtt{k}^{(2)}_{w_\beta}(V_+(s),u)\mathtt{k}^{(1)}_{w_\beta}(\hat V_+(\tau),u')||h_-(\tau,\hat X_+(\tau),u')|\mathrm{d}u'du;\\
I_3:=&\int_0^t\int_0^s e^{-\tilde\nu_0(t-s)}e^{-\tilde\nu_0(s-\tau)}\mathrm{d}\tau\mathrm{d}s\\
&\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\mathtt{k}^{(1)}_{w_\beta}(V_+(s),u)\mathtt{k}^{(2)}_{w_\beta}(\hat V_-(\tau),u')||h_-(\tau,\hat X_-(\tau),u')|\mathrm{d}u'du;\\
I_4:=&\int_0^t\int_0^s e^{-\tilde\nu_0(t-s)}e^{-\tilde\nu_0(s-\tau)}\mathrm{d}\tau\mathrm{d}s\\
&\times\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|\mathtt{k}^{(1)}_{w_\beta}(V_+(s),u)\mathtt{k}^{(1)}_{w_\beta}(\hat V_-(\tau),u')||h_+(\tau,\hat X_-(\tau),u')|\mathrm{d}u'du.
\end{aligned}$$

We fix $N>0$ large enough so that for any $0\le\tau\le s\le t$

$$\sup_{0\le s\le t}|V_\pm(s)-v|\le C\delta\le\frac{N}{8},\qquad \sup_{0\le\tau\le s\le t}|\hat V_\pm(\tau)-u|\le C\delta\le\frac{N}{8}. \tag{4.12}$$

We first discuss I_1 of (4.11) and split it into three cases.
Case 1: $|v|\ge N$. In this case, we have $|V_+(s)|\ge\frac{N}{2}$. It is noted that

$$\int_{\mathbb{R}^3}|\mathtt{k}^{(i)}_{w_\beta}(V_+(s),u)|\mathrm{d}u\le\frac{C}{1+|V_+(s)|},\quad i=1,2, \tag{4.13}$$

which yields that

$$I_1\le\frac{C}{N}\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}. \tag{4.14}$$

Case 2: For either $|v|\le N, |u|\ge 2N$ or $|u|\le 2N, |u'|\ge 3N$, we get either

$$|V_+(s)-u|\ge|u-v|-|V_+(s)-v|\ge|u|-|v|-|V_+(s)-v|\ge\frac{N}{2},$$

or

$$|\hat V_+(\tau)-u'|\ge|u'-u|-|\hat V_+(\tau)-u|\ge|u'|-|u|-|\hat V_+(\tau)-u|\ge\frac{N}{2},$$

for N suitably large, $s,\tau\in[0,t]$. Then either one of the following is valid: for $i=1,2$,

$$\begin{aligned}
|\mathtt{k}^{(i)}_{w_\beta}(V_+(s),u)|&\le e^{-\frac{N^2}{64}}|\mathtt{k}^{(i)}_{w_\beta}(V_+(s),u)|e^{\frac{|V_+(s)-u|^2}{16}},\quad |V_+(s)-u|\ge\frac{N}{2},\\
|\mathtt{k}^{(i)}_{w_\beta}(\hat V_+(\tau),u')|&\le e^{-\frac{N^2}{64}}|\mathtt{k}^{(i)}_{w_\beta}(\hat V_+(\tau),u')|e^{\frac{|\hat V_+(\tau)-u'|^2}{16}},\quad |\hat V_+(\tau)-u'|\ge\frac{N}{2}.
\end{aligned} \tag{4.15}$$

We also note for $i=1,2$,

$$\begin{aligned}
\int_{\mathbb{R}^3}|\mathtt{k}^{(i)}_{w_\beta}(V_+(s),u)|e^{\frac{|V_+(s)-u|^2}{16}}du&\le\frac{C}{1+|V_+(s)|},\\
\int_{\mathbb{R}^3}|\mathtt{k}^{(i)}_{w_\beta}(\hat V_+(\tau),u')|e^{\frac{|\hat V_+(\tau)-u'|^2}{16}}\mathrm{d}u'&\le\frac{C}{1+|\hat V_+(\tau)|}.
\end{aligned} \tag{4.16}$$

Using (4.15)–(4.16), one has

$$\begin{aligned}&\int_0^t e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s\int_0^s e^{-\tilde{\nu}_0(s-\tau)}\mathrm{d}\tau\Big\{\int_{|u|\geq 2N}\int_{\mathbb{R}^3}+\int_{|u|\leq 2N}\int_{|u'|\geq 3N}\Big\}|h_+(\tau,\hat{X}_+(\tau),u')|\\&\quad\times|\mathtt{k}^{(2)}_{w_\beta}(V_+(s),u)\mathtt{k}^{(2)}_{w_\beta}(\hat{V}_+(\tau),u')|\mathrm{d}u'\mathrm{d}u\\&\leq Ce^{-\frac{N^2}{64}}\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}.\end{aligned}\tag{4.17}$$

Case 3: $|v|\leq N,|u|\leq 2N,|u'|\leq 3N,s-\frac{1}{N}\leq\tau\leq s$. In this case, it is easy to see that

$$\begin{aligned}&\int_0^t e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s\int_{s-\frac{1}{N}}^s e^{-\tilde{\nu}_0(s-\tau)}\mathrm{d}\tau\int_{|u|\leq 2N}\int_{|u'|\leq 3N}|h_+(\tau,\hat{X}_+(\tau),u')|\\&\quad\times|\mathtt{k}^{(2)}_{w_\beta}(V_+(s),u)\mathtt{k}^{(2)}_{w_\beta}(\hat{V}_+(\tau),u')|\mathrm{d}u'\mathrm{d}u\\&\leq\frac{C}{N}\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}.\end{aligned}\tag{4.18}$$

Case 4: $|v|\leq N,|u|\leq 2N,|u'|\leq 3N,0\leq\tau\leq s-\frac{1}{N}$. A direct calculation shows that

$$\begin{aligned}&\int_0^t e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s\int_0^{s-\frac{1}{N}} e^{-\tilde{\nu}_0(s-\tau)}\mathrm{d}\tau\int_{|u|\leq 2N}\int_{|u'|\leq 3N}|h_+(\tau,\hat{X}_+(\tau),u')|\\&\quad\times|\mathtt{k}^{(2)}_{w_\beta}(V_+(s),u)\mathtt{k}^{(2)}_{w_\beta}(\hat{V}_+(\tau),u')|\mathrm{d}u'\mathrm{d}u\\&\leq\int_0^t e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s\int_0^{s-\frac{1}{N}} e^{-\tilde{\nu}_0(s-\tau)}\mathrm{d}\tau\Big(\int_{|u|\leq 2N}\int_{|u'|\leq 3N}|h_+(\tau,\hat{X}_+(\tau),u')|^2\mathrm{d}u'\mathrm{d}u\Big)^{\frac{1}{2}}\\&\leq C_N\int_0^t e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s\int_0^{s-\frac{1}{N}} e^{-\tilde{\nu}_0(s-\tau)}\mathrm{d}\tau\Big(\int_{|u|\leq 2N}\int_{|u'|\leq 3N}|f_+(\tau,\hat{X}_+(\tau),u')|^2\mathrm{d}u'\mathrm{d}u\Big)^{\frac{1}{2}}.\end{aligned}\tag{4.19}$$

Let $y=\hat{X}_+(\tau)\equiv X_+(\tau;s,X_+(s),u)$, then it follows from Corollary 2.7 that

$$\Big|\frac{dy}{du}\Big|\geq\frac{1}{2}(s-\tau)^3,\quad|y-X_+(s)|\leq C(s-\tau)N.\tag{4.20}$$

Using Lemma 2.2, we have

$$\begin{aligned}&\int_{|u|\leq 2N}\int_{|u'|\leq 3N}|f_+(\tau,\hat{X}_+(\tau),u')|^2\mathrm{d}u'\mathrm{d}u\\&=\int_{|u|\leq 2N}\int_{|u'|\leq 3N}|f_+(\tau,\hat{X}_+(\tau),u')|^2\mathbf{1}_{\{|F_+(\tau,\hat{X}_+(\tau),u')-\mu(u')|\geq\mu(u')\}}\mathrm{d}u'\mathrm{d}u\\&\quad+\int_{|u|\leq 2N}\int_{|u'|\leq 3N}|f_+(\tau,\hat{X}_+(\tau),u')|^2\mathbf{1}_{\{|F_+(\tau,\hat{X}_+(\tau),u')-\mu(u')|<\mu(u')\}}\mathrm{d}u'\mathrm{d}u\\&\leq C_N\|h_+(\tau)\|_{L^\infty}\int_{|u|\leq 2N}\int_{|u'|\leq 3N}|F_+(\tau,y,u')-\mu(u')|\mathbf{1}_{\{|F_+(\tau,y,u')-\mu(u')|\geq\mu(u')\}}\mathrm{d}u'\mathrm{d}u\\&\quad+C_N\int_{|u|\leq 2N}\int_{|u'|\leq 3N}\frac{|F_+(\tau,y,u')-\mu(u')|^2}{\mu(u')}\mathbf{1}_{\{|F_+(\tau,y,u')-\mu(u')|<\mu(u')\}}\mathrm{d}u'\mathrm{d}u\\&\leq C_N\frac{1+(s-\tau)^3}{(s-\tau)^3}\|h_+(\tau)\|_{L^\infty}\int_{\mathbb{T}^3}\int_{|u'|\leq 3N}|F_+(\tau,y,u')-\mu(u')|\mathbf{1}_{\{|F_+(\tau,y,u')-\mu(u')|\geq\mu(u')\}}\mathrm{d}u'\mathrm{d}y\\&\quad+C_N\frac{1+(s-\tau)^3}{(s-\tau)^3}\int_{\mathbb{T}^3}\int_{|u'|\leq 3N}\frac{|F_+(\tau,y,u')-\mu(u')|^2}{\mu(u')}\mathbf{1}_{\{|F_+(\tau,y,u')-\mu(u')|<\mu(u')\}}\mathrm{d}u'\mathrm{d}y\end{aligned}$$

$$\leq C_N N^3\Big(\|h_+(\tau)\|_{L^\infty}\mathcal{E}(\mathbf{F}_0)+\mathcal{E}(\mathbf{F}_0)\Big), \tag{4.21}$$

which yields that

$$\Big(\int_{|u|\leq 2N}\int_{|u'|\leq 3N}|f_+(\tau,\hat{X}_+(\tau),u')|^2\mathrm{d}u'\mathrm{d}u\Big)^{\frac{1}{2}}\leq C_N\Big(\|h_+(\tau)\|_{L^\infty}^{\frac{1}{2}}\sqrt{\mathcal{E}(\mathbf{F}_0)}+\sqrt{\mathcal{E}(\mathbf{F}_0)}\Big). \tag{4.22}$$

Combining (4.19) and (4.22), we have

$$\begin{aligned}
&\int_0^t e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s\int_0^{s-\frac{1}{N}}e^{-\tilde{\nu}_0(s-\tau)}\mathrm{d}\tau\\
&\qquad\times\int_{|u|\leq 2N}\int_{|u'|\leq 3N}|\mathbf{k}_{w_\beta}^{(2)}(V_+(s),u)\mathbf{k}_{w_\beta}^{(2)}(\hat{V}_+(\tau),u')|\cdot|h_+(\tau,\hat{X}_+(\tau),u')|\mathrm{d}u'\mathrm{d}u\\
\leq&\frac{1}{N}\|h_+(\tau)\|_{L^\infty}+C_N\Big(\mathcal{E}(\mathbf{F}_0)+\sqrt{\mathcal{E}(\mathbf{F}_0)}\Big).
\end{aligned}\tag{4.23}$$

Then it is a consequence of (4.14), (4.17)-(4.18) and (4.23) that

$$I_1\leq\frac{C}{N}\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}+C_N\sqrt{\mathcal{E}(\mathbf{F}_0)}. \tag{4.24}$$

An argument similar to the one used in (4.24) shows that

$$\sum_{i=1}^4 I_i\leq\frac{C}{N}\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}+C_N\sqrt{\mathcal{E}(\mathbf{F}_0)}. \tag{4.25}$$

Combining (4.11) and (4.25), one has

$$\begin{aligned}
|h_+(t,x,v)|\leq& C\|\mathbf{h}_0\|_{L^\infty}+\frac{C}{N}\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}+C_N\sqrt{\mathcal{E}(\mathbf{F}_0)}\\
&+C\sup_{\substack{0\leq s\leq t\\ y\in\mathbb{T}^3}}\Big\{\|\mathbf{h}(s)\|_{L^\infty}^{\frac{3}{2}}\cdot\Big(\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(s,y,u)|\mathrm{d}u\Big)^{\frac{1}{2}}\Big\}.
\end{aligned}\tag{4.26}$$

For $h_-(t,x,v)$, similar as the proof of (4.26), one gets

$$\begin{aligned}
|h_-(t,x,v)|\leq& C\|\mathbf{h}_0\|_{L^\infty}+\frac{C}{N}\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}+C_N\sqrt{\mathcal{E}(\mathbf{F}_0)}\\
&+C\sup_{\substack{0\leq s\leq t\\ y\in\mathbb{T}^3}}\Big\{\|\mathbf{h}(s)\|_{L^\infty}^{\frac{3}{2}}\cdot\Big(\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(s,y,u)|\mathrm{d}u\Big)^{\frac{1}{2}}\Big\}.
\end{aligned}\tag{4.27}$$

Choosing N sufficiently large in (4.26) and (4.27) so that $\frac{C}{N}\leq\frac{1}{2}$, one obtains that for $\beta\geq 4$ and $\beta_1\geq 0$,

$$\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}\leq C\Big(\|\mathbf{h}_0\|_{L^\infty}+\sqrt{\mathcal{E}(\mathbf{F}_0)}\Big)+C\sup_{\substack{0\leq s\leq t\\ y\in\mathbb{T}^3}}\Big\{\|\mathbf{h}(s)\|_{L^\infty}^{\frac{3}{2}}\cdot\Big(\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(s,y,u)|\mathrm{d}u\Big)^{\frac{1}{2}}\Big\}. \tag{4.28}$$

Using Proposition 3.1, one has that for $\beta\geq 4$ and $\beta_1\geq 0$,

$$\sup_{\substack{0\leq s\leq t_1\\ y\in\mathbb{T}^3}}\Big\{\|\mathbf{h}(s)\|_{L^\infty}^{\frac{3}{2}}\cdot\Big(\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(s,y,u)|\mathrm{d}u\Big)^{\frac{1}{2}}\Big\}\leq C\sup_{0\leq s\leq t_1}\|\mathbf{h}(s)\|_{L^\infty}^2\leq C\|\mathbf{h}_0\|_{L^\infty}^2,$$

which, together with (4.28), yields (4.6). Therefore the proof of Lemma 4.1 is completed. $\square$

4.2. $L^\infty_x L^1_v$**-estimate.** In this part, we mainly focus on the estimate of

$$\int_{\mathbb{R}^3} |\tilde{\mathbf{h}}(t,x,v)|\mathrm{d}v.$$

Motivated by [26], if $\mathcal{E}(\mathbf{F}_0)+\|\tilde{\mathbf{h}}_0\|_{L^1_xL^\infty_v}$ is small, due to the hyperbolicity of the VPB system, one should be able to show that $\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(t,x,v)|\mathrm{d}v$ is small for $t\geq t_1$, even though it could be initially large, i.e., $\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}_0(x,v)|\mathrm{d}v$ is large. Indeed, we have the following lemma which plays a key role in this paper.

Lemma 4.2. *Under the condition* (2.18)*, it holds for* $0\leq\beta_1<\beta-4$*, and* $t\geq t_1$*, that*

$$\begin{aligned}\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(t,x,v)|\mathrm{d}v\leq\, & C\Big(\|\mathbf{h}_0\|_{L^\infty}^{\frac{3}{\beta-\beta_1}}\|\tilde{\mathbf{h}}_0\|_{L^1_xL^\infty_v}^{1-\frac{3}{\beta-\beta_1}}+(\|\mathbf{h}_0\|_{L^\infty}+1)^6\|\tilde{\mathbf{h}}_0\|_{L^1_xL^\infty_v}\Big)+C_{N,\lambda}\sqrt{\mathcal{E}(\mathbf{F}_0)}\\ &+C(\lambda+\frac{1}{N^{\beta-\beta_1-4}})(\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}+\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|^2_{L^\infty}),\end{aligned}$$

where $\lambda>0$ *and* $N\geq1$ *will be chosen later.*

Proof. 1. From $(1.13)_1$, one has

$$(\partial_t+v\cdot\nabla_x\mp\nabla_x\phi\cdot\nabla_v)\tilde{h}_\pm+\tilde{\nu}_{\pm,1}\tilde{h}_\pm=w_{\beta_1}\mathrm{K}^\pm\mathbf{f}+w_{\beta_1}\Gamma^\pm(\mathbf{f},\mathbf{f})\mp\nabla_x\phi\cdot vw_{\beta_1}\sqrt{\mu}\tag{4.29}$$

where

$$\tilde{\nu}_{\pm,1}(t,x,v):=\frac{\sigma_0}{(1+t)^2}|v|^2+\nu(v)\pm\nabla_x\phi\cdot v\Big(\frac{1}{2}+\frac{\beta_1}{1+|v|^2}+\frac{2\sigma_0}{1+t}\Big).\tag{4.30}$$

Similar as (4.4), it holds that

$$|\tilde{\nu}_{+,1}(t,x,v)|\geq\frac{1}{2}\nu(v)\geq\tilde{\nu}_0.\tag{4.31}$$

It suffices to prove the lemma for $\tilde{h}_+(t,x,v)$, since $\tilde{h}_-(t,x,v)$ can be obtained similarly.

The mild solution of (4.29) takes the form

$$\begin{aligned}\tilde{h}_+(t,x,v)=\,&\tilde{h}_{+,0}(X_\pm(0),V_\pm(0))e^{-\int_0^t\tilde{\nu}_{+,1}(\tau,X_+(\tau),V_+(\tau))\mathrm{d}\tau}\\ &+\int_0^t e^{-\int_s^t\tilde{\nu}_{+,1}(\tau,X_+(\tau),V_+(\tau))\mathrm{d}\tau}\Big(w_{\beta_1}\mathrm{K}^+\mathbf{f}\Big)(s,X_\pm(s),V_\pm(s))\mathrm{d}s\\ &-\int_0^t e^{-\int_s^t\tilde{\nu}_{+,1}(\tau,X_+(\tau),V_+(\tau))\mathrm{d}\tau}\Big(\nabla_x\phi\cdot vw_{\beta_1}\sqrt{\mu}\Big)(s,X_+(s),V_+(s))\mathrm{d}s\\ &+\int_0^t e^{-\int_s^t\tilde{\nu}_{+,1}(\tau,X_+(\tau),V_+(\tau))\mathrm{d}\tau}\Big(w_{\beta_1}\Gamma^+(\mathbf{f},\mathbf{f})\Big)(s,X_+(s),V_+(s))\mathrm{d}s,\end{aligned}\tag{4.32}$$

where we have denoted $\tilde{\nu}_{+,1}(\tau):=\tilde{\nu}_{+,1}(\tau,X_+(\tau),V_+(\tau))$. Similar to (4.9), we denote $\mathtt{k}^{(i)}_{w_{\beta_1}}(v,u)=\mathtt{k}^{(i)}(v,u)\cdot\frac{w_{\beta_1}(t,v)}{w_{\beta_1}(t,u)}$ $(i=1,2)$. Using (4.31)-(4.32), one has

$$\begin{aligned}\int_{\mathbb{R}^3}|\tilde{h}_+(t,x,v)|\mathrm{d}v\leq\,&e^{-\tilde{\nu}_0t}\int_{\mathbb{R}^3}|\tilde{h}_{+,0}(X_+(0),V_+(0))|\mathrm{d}v\\ &+\int_0^t\int_{\mathbb{R}^3}e^{-\int_s^t\tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int_{\mathbb{R}^3}|\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),v')\tilde{h}_+(s,X_+(s),v')|\mathrm{d}v'\mathrm{d}v\mathrm{d}s\\ &+\int_0^t\int_{\mathbb{R}^3}e^{-\int_s^t\tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int_{\mathbb{R}^3}|\mathtt{k}^{(1)}_{w_{\beta_1}}(V_+(s),v')\tilde{h}_-(s,X_+(s),v')|\mathrm{d}v'\mathrm{d}v\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_{\mathbb{R}^3} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\Big|\Big(w_{\beta_1}\Gamma^+(\mathbf{f},\mathbf{f})\Big)(s,X_+(s),V_+(s))\Big|\mathrm{d}v\mathrm{d}s\\
&+\int_0^t\int_{\mathbb{R}^3} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\nabla_x\phi(s,X_+(s))\cdot V_+(s)w_{\beta_1}(s,V_+(s))\sqrt{\mu(V_+(s))}\mathrm{d}v\mathrm{d}s\\
&:= H_1+H_2^+ + H_2^- + H_3 + H_4.
\end{aligned}\tag{4.33}
$$

2. It is noted that

$$
\begin{aligned}
H_2^+ \leq& \int_{t-\lambda}^t\int_{\mathbb{R}^3} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int_{\mathbb{R}^3}|\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),v')\tilde{h}_+(s,X_+(s),v')|\mathrm{d}v'\mathrm{d}v\mathrm{d}s\\
&+\int_0^{t-\lambda}\int_{\mathbb{R}^3} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int_{\mathbb{R}^3}|\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),v')\tilde{h}_+(s,X_+(s),v')|\mathrm{d}v'\mathrm{d}v\mathrm{d}s\\
:=& H_{21}^+ + H_{22}^+.
\end{aligned}\tag{4.34}
$$

Using the fact that $1+|v|\cong 1+|V_\pm(s)|$ for any $0\leq s\leq t$, we have

$$
\begin{aligned}
H_{21}^+ \leq& C\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}\int_{t-\lambda}^t\int_{\mathbb{R}^3} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}(1+|V_+(s)|^2)^{-\frac{\beta-\beta_1}{2}}\mathrm{d}v\mathrm{d}s\\
\leq& C\lambda\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}.
\end{aligned}\tag{4.35}
$$

To estimate H_{22}^+, we notice that

$$
\begin{aligned}
H_{22}^+ =& \int_0^{t-\lambda}\int_{|v|\geq N} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int_{\mathbb{R}^3}|\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),v')\tilde{h}_+(s,X_+(s),v')|\mathrm{d}v'\mathrm{d}v\mathrm{d}s\\
&+\int_0^{t-\lambda}\int_{|v|\leq N} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int_{|v'|\geq 2N}|\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),v')\tilde{h}_+(s,X_+(s),v')|\mathrm{d}v'\mathrm{d}v\mathrm{d}s\\
&+\int_0^{t-\lambda}\int_{|v|\leq N} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int_{|v'|\leq 2N}|\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),v')\tilde{h}_+(s,X_+(s),v')|\mathrm{d}v'\mathrm{d}v\mathrm{d}s\\
:=&H_{221}^+ + H_{222}^+ + H_{223}^+.
\end{aligned}\tag{4.36}
$$

It is straightforward to show that

$$
\begin{aligned}
H_{221}^+ \leq& \sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}\int_0^{t-\lambda}\int_{|v|\geq N} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}(1+|V_+(s)|)^{-(\beta-\beta_1)-1}\mathrm{d}v\mathrm{d}s\\
\leq& C\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}\int_0^{t-\lambda}\int_{|v|\geq N} e^{-\tilde{\nu}_0(t-s)}(1+|v|)^{-(\beta-\beta_1)-1}\mathrm{d}v\mathrm{d}s\\
\leq& \frac{C}{N^{\beta-\beta_1-2}}\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}.
\end{aligned}\tag{4.37}
$$

For H_{222}^+, it holds that $|V_+(s)-v'|\geq|v'-v|-|V_+(s)-v|\geq|v'|-|v|-|V_+(s)-v|\geq\frac{N}{2}$. Then we have

$$
\begin{aligned}
H_{222}^+ \leq& e^{-\frac{N^2}{64}}\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}\int_0^{t-\lambda}\int_{|v|\leq N} e^{-\tilde{\nu}_0(t-s)}(1+|V_+(s)|)^{-(\beta-\beta_1)}\mathrm{d}v\mathrm{d}s\\
&\times\int_{|v'|\geq 2N}|\mathtt{k}^{(2)}_{w_\beta}(V_+(s),v')|e^{\frac{|V_+(s)-v'|^2}{16}}\mathrm{d}v'\\
\leq& Ce^{-\frac{N^2}{64}}\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}\int_0^{t-\lambda}\int_{|v|\leq N} e^{-\tilde{\nu}_0(t-s)}(1+|V_+(s)|)^{-(\beta-\beta_1)}\mathrm{d}v\mathrm{d}s
\end{aligned}
$$

$$\leq Ce^{-\frac{N^2}{64}} \sup_{0\leq s\leq t} \|\mathbf{h}(s)\|_{L^\infty}. \tag{4.38}$$

For H_{223}^+, similar argument as in (4.21), we have

$$\begin{aligned}
H_{223}^+ \leq & C_N \int_0^{t-\lambda} e^{-\tilde{\nu}_0(t-s)} \mathrm{d}s \Big(\int_{|v|\leq N} \int_{|v'|\leq 2N} |f_+(s, X_+(s), v')|^2 \mathrm{d}v' \mathrm{d}v \Big)^{\frac{1}{2}} \\
\leq & C_{N,\lambda} \Big(\|h_+(\tau)\|_{L^\infty}^{\frac{1}{2}} \sqrt{\mathcal{E}(\mathbf{F}_0)} + \sqrt{\mathcal{E}(\mathbf{F}_0)} \Big) \\
\leq & \lambda \sup_{0\leq s\leq t} \|\mathbf{h}(s)\|_{L^\infty} + C_{N,\lambda} \sqrt{\mathcal{E}(\mathbf{F}_0)}.
\end{aligned} \tag{4.39}$$

Combining (4.34)-(4.39), we obtain

$$H_2^+ \leq C(\lambda + \frac{1}{N}) \sup_{0\leq s\leq t} \|\mathbf{h}(s)\|_{L^\infty} + C_{N,\lambda} \sqrt{\mathcal{E}(\mathbf{F}_0)}. \tag{4.40}$$

Similar as (4.40), one has

$$H_2^- \leq C(\lambda + \frac{1}{N}) \sup_{0\leq s\leq t} \|\mathbf{h}(s)\|_{L^\infty} + C_{N,\lambda} \sqrt{\mathcal{E}(\mathbf{F}_0)}. \tag{4.41}$$

3. For H_3, we note

$$\begin{aligned}
H_3 \leq & C \int_0^t e^{-\tilde{\nu}_0(t-s)} \int_{\mathbb{R}^3} w_{\beta_1}(s, V_+(s)) \int_{\mathbb{R}^3\times\mathbb{S}^2} B(V_+(s) - u, \theta) e^{-\frac{|u|^2}{4}} \\
& \times |f_+(s, X_+(s), V_+(s))| \left(|f_+(s, X_+(s), u)| + |f_-(s, X_+(s), u)| \right) \mathrm{d}\omega \mathrm{d}u \mathrm{d}v \mathrm{d}s \\
& + C \int_0^t e^{-\tilde{\nu}_0(t-s)} \int_{\mathbb{R}^3} w_{\beta_1}(s, V_+(s)) \int_{\mathbb{R}^3\times\mathbb{S}^2} B(V_+(s) - u, \theta) e^{-\frac{|u|^2}{4}} \\
& \times |f_+(s, X_+(s), V_+'(s))| \big(|f_+(s, X_+(s), u')| + |f_-(s, X_+(s), u')| \big) \mathrm{d}\omega \mathrm{d}u \mathrm{d}v \mathrm{d}s \\
:= & H_{31} + H_{32}.
\end{aligned} \tag{4.42}$$

For H_{31}, it is clear to see that

$$\begin{aligned}
H_{31} \leq & C \int_{t-\lambda}^t e^{-\tilde{\nu}_0(t-s)} \mathrm{d}s \int_{\mathbb{R}^3} w_{\beta_1}(s, V_+(s)) \int_{\mathbb{R}^3} |V_+(s) - u|^\gamma e^{-\frac{|u|^2}{4}} \\
& \times |f_+(s, X_+(s), V_+(s))| \left(|f_+(s, X_+(s), u)| + |f_-(s, X_+(s), u)| \right) \mathrm{d}u \mathrm{d}v \\
& + C \int_0^{t-\lambda} e^{-\tilde{\nu}_0(t-s)} \mathrm{d}s \Big\{ \int_{|v|\geq N} \int_{\mathbb{R}^3} + \int_{|v|\leq N} \int_{|u|\geq N} + \int_{|v|\leq N} \int_{|u|\leq N} \Big\} |V_+(s) - u|^\gamma e^{-\frac{|u|^2}{4}} \\
& \times w_{\beta_1}(s, V_+(s)) |f_+(s, X_+(s), V_+(s))| \left(|f_+(s, X_+(s), u)| + |f_-(s, X_+(s), u)| \right) \mathrm{d}u \mathrm{d}v \\
:= & H_{311} + H_{312} + H_{313} + H_{314}.
\end{aligned} \tag{4.43}$$

Noting $\beta - \beta_1 > 4$, one has

$$\begin{aligned}
H_{311} \leq & C \int_{t-\lambda}^t e^{-\tilde{\nu}_0(t-s)} \mathrm{d}s \int_{\mathbb{R}^3} |\tilde{\mathbf{h}}(s, X_+(s), V_+(s))| \mathrm{d}v \int_{\mathbb{R}^3\times\mathbb{S}^2} |(V_+(s) - u)|^\gamma e^{-\frac{|u|^2}{4}} \mathrm{d}\omega \mathrm{d}u \|\mathbf{h}(s)\|_{L^\infty} \\
\leq & C \|\mathbf{h}(s)\|_{L^\infty}^2 \int_{t-\lambda}^t e^{-\tilde{\nu}_0(t-s)} \mathrm{d}s \int_{\mathbb{R}^3} \nu(V_+(s)) (1 + |V_+(s)|)^{-(\beta-\beta_1)} \mathrm{d}v \\
\leq & C\lambda \sup_{0\leq s\leq t} \|\mathbf{h}(s)\|_{L^\infty}^2,
\end{aligned} \tag{4.44}$$

and

$$
\begin{aligned}
H_{312} &\le C\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2\int_0^{t-\lambda}e^{-\tilde\nu_0(t-s)}\mathrm{d}s\int_{|v|\ge N}\nu(V_+(s))(1+|V_+(s)|)^{-(\beta-\beta_1)}\mathrm{d}v\\
&\le \frac{C}{N^{\beta-\beta_1-4}}\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2.
\end{aligned}\tag{4.45}
$$

For H_{313}, it is direct to have

$$
\begin{aligned}
H_{313} &\le Ce^{-\frac{N^2}{8}}\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2\int_0^{t-\lambda}e^{-\tilde\nu_0(t-s)}\mathrm{d}s\int_{|v|\le N}\nu(V_+(s))(1+|V_+(s)|)^{-(\beta-\beta_1)}\mathrm{d}v\\
&\le Ce^{-\frac{N^2}{8}}\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2.
\end{aligned}\tag{4.46}
$$

For H_{314}, similar argument as in (4.21), we have

$$
\begin{aligned}
H_{314} &\le C\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}\int_0^{t-\lambda}e^{-\tilde\nu_0(t-s)}\mathrm{d}s\\
&\quad\times\int_{|v|\le N}\int_{|u|\le N}|V_+(s)-u|^\gamma(1+|V_+(s)|)^{-(\beta-\beta_1)}e^{-\frac{|u|^2}{4}}|\mathbf{f}(s,X_+(s),u)|\mathrm{d}u\mathrm{d}v\\
&\le C\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}\int_0^{t-\lambda}e^{-\tilde\nu_0(t-s)}\mathrm{d}s\Big(\int_{|v|\le N}\int_{|u|\le N}|\mathbf{f}(s,X_+(s),u)|^2\mathrm{d}u\mathrm{d}v\Big)^{\frac12}\\
&\quad\times\Big(\int_{|v|\le N}\int_{|u|\le N}|V_+(s)-u|^{2\gamma}(1+|V_+(s)|)^{-2(\beta-\beta_1)}e^{-\frac{|u|^2}{2}}\mathrm{d}u\mathrm{d}v\Big)^{\frac12}\\
&\le C\lambda\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2+C_{N,\lambda}\mathcal{E}(\mathbf{F}_0).
\end{aligned}\tag{4.47}
$$

Combining (4.43) − (4.47), we obtain

$$
H_{31}\le C(\lambda+\frac{1}{N^{\beta-\beta_1-4}})\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2+C_{N,\lambda}\mathcal{E}(\mathbf{F}_0).\tag{4.48}
$$

For H_{32}, using the fact $w_{\beta_1}(s,V_+(s))\le w_{\beta_1}(s,V_+'(s))w_{\beta_1}(s,u')$, here $V_+'(s)$ represents post-collision velocity with respect to pre-collision velocity $V_+(s)$, one has

$$
\begin{aligned}
H_{32} &\le C\int_{t-\lambda}^t e^{-\tilde\nu_0(t-s)}\mathrm{d}s\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}|V_+(s)-u|^\gamma e^{-\frac{|u|^2}{4}}|\tilde h_+(s,X_+(s),V_+'(s))||\tilde{\mathbf{h}}(s,X_+(s),u')|\mathrm{d}u\mathrm{d}v\\
&\quad+C\int_0^{t-\lambda}e^{-\tilde\nu_0(t-s)}\mathrm{d}s\Big\{\int_{|v|\ge N}\int_{\mathbb{R}^3}\int_{\mathbb{S}^2}+\int_{|v|\le N}\int_{|u|\ge N}\int_{\mathbb{S}^2}+\int_{|v|\le N}\int_{|u|\le N}\int_{\mathbb{S}^2}\Big\}e^{-\frac{|u|^2}{4}}\\
&\quad\times B(V_+(s)-u,\theta)|\tilde h_+(s,X_+(s),V_+'(s))||\tilde{\mathbf{h}}(s,X_+(s),u')|\mathrm{d}\omega\mathrm{d}u\mathrm{d}v\\
&:=H_{321}+H_{322}+H_{323}+H_{324}.
\end{aligned}\tag{4.49}
$$

Using the fact $(1+|V_+(s)|^2)^{\frac{\beta-\beta_1}{2}}\le(1+|u'|^2)^{\frac{\beta-\beta_1}{2}}(1+|V_+'(s)|^2)^{\frac{\beta-\beta_1}{2}}$, one has

$$
\begin{aligned}
H_{321} &\le \int_{t-\lambda}^t e^{-\tilde\nu_0(t-s)}\mathrm{d}s\int_{\mathbb{R}^3}(1+|V_+(s)|^2)^{-\frac{\beta-\beta_1}{2}}\nu(V_+(s))\mathrm{d}v\|\mathbf{h}(s)\|_{L^\infty}^2\\
&\le C\lambda\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2.
\end{aligned}\tag{4.50}
$$

Similar as (4.45) − (4.46), it holds that

$$
H_{322}+H_{323}\le\frac{C}{N^{(\beta-\beta_1)-4}}\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2.\tag{4.51}
$$

Noting

$$\begin{cases} V_+'(s) = V_+(s) + \{(u - V_+(s)) \cdot \omega\} \cdot \omega, \\ u' = u - \{(u - V_+(s)) \cdot \omega\} \cdot \omega, \end{cases} \tag{4.52}$$

we make a change of variables

$$z = u - V_+(s), \quad z_{\shortparallel} = (z \cdot \omega)\omega, \quad z_\perp = z - z_{\shortparallel}, \quad \eta = V_+(s) + z_{\shortparallel}. \tag{4.53}$$

Owing to

$$|B(V_+(s) - u, \theta)| \le C|V_+(s) - u|^\gamma |\cos\theta| \le C(|z_{\shortparallel}|^2 + |z_\perp|^2)^{\frac{\gamma-1}{2}} |z_{\shortparallel}|,$$

and

$$\mathrm{d}\omega\mathrm{d}u = \mathrm{d}z_{\shortparallel}\mathrm{d}z_\perp\mathrm{d}\omega = 2|z_{\shortparallel}|^2 d|z_{\shortparallel}|\mathrm{d}\omega\frac{\mathrm{d}z_\perp}{|z_{\shortparallel}|^2} = \frac{2}{|z_{\shortparallel}|^2}\mathrm{d}z_{\shortparallel}\mathrm{d}z_\perp = \frac{2}{|\eta - V_+(s)|^2}\mathrm{d}\eta\mathrm{d}z_\perp,$$

then one has

$$\begin{aligned} H_{324} &\le C\int_0^{t-\lambda} e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s \int_{|v|\le N}\int_{|u|\le N}\int_{\mathbb{S}^2} B(V_+(s) - u, \theta)(1 + |V_+(s)|^2)^{-\frac{\beta-\beta_1}{2}} \\ &\quad \times e^{-\frac{|u|^2}{4}} |h_+(s, X_+(s), V_+'(s))| \cdot \|\mathbf{h}(s)\|_{L^\infty}\mathrm{d}\omega\mathrm{d}u\mathrm{d}v \\ &\le C\int_0^{t-\lambda} e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s \int_{|v|\le N}\int_{|z|\le 3N}\int_{\mathbb{S}^2} e^{-\frac{|z_{\shortparallel}+z_\perp+V_+(s)|^2}{4}}(|z_{\shortparallel}|^2 + |z_\perp|^2)^{\frac{\gamma-1}{2}}|z_{\shortparallel}| \\ &\quad \times (1 + |V_+(s)|^2)^{-\frac{\beta-\beta_1}{2}} |h_+(s, X_+(s), V_+(s) + z_{\shortparallel})|\mathrm{d}v d|z_{\shortparallel}|\mathrm{d}z_\perp\mathrm{d}\omega\|\mathbf{h}(s)\|_{L^\infty} \\ &\le C\int_0^{t-\lambda} e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s \int_{|v|\le N}\int_{|\eta|\le 5N}\int_{z_\perp} e^{-\frac{|\eta+z_\perp|^2}{4}}(|\eta - V_+(s)|^2 + |z_\perp|^2)^{\frac{\gamma-1}{2}} \\ &\quad \times |\eta - V_+(s)|^{-1}(1 + |V_+(s)|^2)^{-\frac{\beta-\beta_1}{2}} |h_+(s, X_+(s), \eta)|\mathrm{d}v\mathrm{d}z_\perp\mathrm{d}\eta\|\mathbf{h}(s)\|_{L^\infty} \\ &\le C\int_0^{t-\lambda} e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s \int_{|v|\le N}\int_{|\eta|\le 5N} |\eta - V_+(s)|^{-1}(1 + |V_+(s)|^2)^{-\frac{\beta-\beta_1}{2}} \\ &\quad \times |h_+(s, X_+(s), \eta)|\mathrm{d}v\mathrm{d}\eta\|\mathbf{h}(s)\|_{L^\infty} \\ &\le C_N\int_0^{t-\lambda} e^{-\tilde{\nu}_0(t-s)}\mathrm{d}s\|\mathbf{h}(s)\|_{L^\infty}\Big(\int_{|v|\le N}\int_{|\eta|\le 5N} |\mathbf{f}(s, X_+(s), \eta)|^2\mathrm{d}\eta\mathrm{d}v\Big)^{\frac{1}{2}} \\ &\le C\lambda\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2 + C_{N,\lambda}\mathcal{E}(\mathbf{F}_0). \end{aligned} \tag{4.54}$$

Hence we have from (4.49)–(4.54) that

$$H_{32} \le C(\lambda + \frac{1}{N^{\beta-\beta_1-4}})\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2 + C_{N,\lambda}\mathcal{E}(\mathbf{F}_0). \tag{4.55}$$

Combining (4.42), (4.48) and (4.55), we have

$$H_3 \le C(\lambda + \frac{1}{N^{\beta-\beta_1-4}})\sup_{0\le s\le t}\|\mathbf{h}(s)\|_{L^\infty}^2 + C_{N,\lambda}\mathcal{E}(\mathbf{F}_0). \tag{4.56}$$

4. For H_4, using (2.7), one gets that

$$\begin{aligned} H_4 &\le \int_0^t e^{-\tilde{\nu}_0(t-s)}\mu^{\frac{1}{4}}(V_\pm(s))\|\nabla_x\phi(s)\|_{L^\infty}\mathrm{d}v\mathrm{d}s \\ &\le C\lambda\int_0^t e^{-\tilde{\nu}_0(t-s)}\|\mathbf{h}(s)\|_{L^\infty}\mathrm{d}s + \frac{C}{\lambda}\int_0^t e^{-\tilde{\nu}_0(t-s)}\|\nabla_x\phi(s)\|_{L^2}\mathrm{d}s \end{aligned}$$

$$\leq C\lambda \sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty} + \frac{C}{\lambda}\sqrt{\mathcal{E}(\mathbf{F}_0)}. \tag{4.57}$$

5. It remains to deal with the initial term. For $t \geq t_1 > 0$, it holds that

$$\begin{aligned}
&\int_{\mathbb{R}^3} |\tilde{h}_{+,0}(X_+(0), V_+(0))|\mathrm{d}v \leq \Big\{\int_{|v|\geq \tilde{N}} + \int_{|v|\leq \tilde{N}}\Big\} |\tilde{h}_{+,0}(X_+(0), V_+(0))|\mathrm{d}v\\
\leq& C\|\mathbf{h}_0\|_{L^\infty}\int_{|v|\geq \tilde{N}} (1+V_+(0))^{-(\beta-\beta_1)}dv + \int_{|v|\leq \tilde{N}} |\tilde{h}_{+,0}(X_+(0), V_+(0))|\mathrm{d}v\\
\leq& C\|\mathbf{h}_0\|_{L^\infty}\int_{|v|\geq \tilde{N}} (1+|v|^2)^{-\frac{\beta-\beta_1}{2}}dv + \frac{C}{t^3}\int_{|y-x|\leq C\tilde{N}t} |\tilde{h}_{+,0}(y, V_+(0))|\mathrm{d}y\\
\leq& C\|\mathbf{h}_0\|_{L^\infty}\tilde{N}^{3-(\beta-\beta_1)} + C\frac{1+\tilde{N}^3t_1^3}{t_1^3}\|\tilde{h}_{+,0}\|_{L^1_xL^\infty_v}\\
\leq& C\|\mathbf{h}_0\|_{L^\infty}^{\frac{3}{\beta-\beta_1}}\|\tilde{h}_{+,0}\|_{L^1_xL^\infty_v}^{1-\frac{3}{\beta-\beta_1}} + C(\|\mathbf{h}_0\|_{L^\infty}+1)^6\|\tilde{h}_{+,0}\|_{L^1_xL^\infty_v},
\end{aligned} \tag{4.58}$$

where we have chosen $\tilde{N} = \|\mathbf{h}_0\|_{L^\infty}^{\frac{1}{\beta-\beta_1}}\|\tilde{h}_{+,0}\|_{L^1_xL^\infty_v}^{-\frac{1}{\beta-\beta_1}}$ and $t_1 := \frac{1}{16C_1C_2C_3C_4(\|\mathbf{h}_0\|_\infty+1)^2}$.

6. Inserting (4.40), (4.56), (4.57) and (4.58) into (4.33) shows

$$\begin{aligned}
\int_{\mathbb{R}^3} |\tilde{h}_+(t,x,v)|\mathrm{d}v \leq& C\Big(\|\mathbf{h}_0\|_{L^\infty}^{\frac{3}{\beta-\beta_1}}\|\tilde{h}_{+,0}\|_{L^1_xL^\infty_v}^{1-\frac{3}{\beta-\beta_1}} + (\|\mathbf{h}_0\|_{L^\infty}+1)^6\|\tilde{h}_{+,0}\|_{L^1_xL^\infty_v}\Big) + C_{N,\lambda}\sqrt{\mathcal{E}(\mathbf{F}_0)}\\
&+ C(\lambda + \frac{1}{N^{\beta-\beta_1-4}})\Big(\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty} + \sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}^2\Big).
\end{aligned} \tag{4.59}$$

For $\int_{\mathbb{R}^3} |\tilde{h}_-(t,x,v)|\mathrm{d}v$, similarly, one has

$$\begin{aligned}
\int_{\mathbb{R}^3} |\tilde{h}_-(t,x,v)|\mathrm{d}v \leq& C\Big(\|\mathbf{h}_0\|_{L^\infty}^{\frac{3}{\beta-\beta_1}}\|\tilde{h}_{-,0}\|_{L^1_xL^\infty_v}^{1-\frac{3}{\beta-\beta_1}} + (\|\mathbf{h}_0\|_{L^\infty}+1)^6\|\tilde{h}_{-,0}\|_{L^1_xL^\infty_v}\Big) + C_{N,\lambda}\sqrt{\mathcal{E}(\mathbf{F}_0)}\\
&+ C(\lambda + \frac{1}{N^{\beta-\beta_1-4}})\Big(\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty} + \sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty}^2\Big).
\end{aligned} \tag{4.60}$$

Therefore we complete the proof of Lemma 4.2 from (4.59) and (4.60). □

4.3. **Uniform L^∞- bound.** With the help of Lemmas 4.1 and 4.2, one can obtain the following uniform L^∞-bound.

Proposition 4.3. *Assume* (2.18)*, let $0 \leq \beta_1 < \beta - 4$, there exists small $\varepsilon_1 > 0$ such that if $\mathcal{E}(\mathbf{F}_0) + \|\tilde{\mathbf{h}}_0\|_{L^1_xL^\infty_v} \leq \varepsilon_1$, it holds*

$$\sup_{0\leq s\leq t}\|\mathbf{h}(s)\|_{L^\infty} \leq CM_0^2,$$

where the positive constant $C \geq 1$ depends only on γ, β and β_1.

Proof. In terms of Lemma 4.1, we make the *a priori* assumption

$$\|\mathbf{h}(t)\|_{L^\infty} \leq 2A_0 := 2C_6\Big\{2M_0^2 + \sqrt{\mathcal{E}(\mathbf{F}_0)}\Big\}, \tag{4.61}$$

where the positive constant $C_6 \geq 1$ is defined in Lemma 4.1. Then it follows from Lemma 4.1 and the *a priori* assumption (4.61) that

$$\|\mathbf{h}(t)\|_{L^\infty} \leq A_0 + C_6(2A_0)^{\frac{3}{2}} \cdot \sup_{\substack{t_1\leq s\leq t\\ y\in\mathbb{T}^3}}\Big(\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(s,y,u)|\mathrm{d}u\Big)^{\frac{1}{2}}. \tag{4.62}$$

To estimate the second term on the RHS of (4.61), it follows from (4.61) and Lemma 4.2 that

$$\sup_{\substack{t_1\le s\le t\\ y\in\mathbb{T}^3}}\Big(\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(s,y,u)|\mathrm{d}u\Big)\le C\Big(M_0^{\frac{3}{\beta-\beta_1}}\|\tilde{\mathbf{h}}_0\|_{L^1_xL^\infty_v}^{1-\frac{3}{\beta-\beta_1}}+M_0^6\|\tilde{\mathbf{h}}_0\|_{L^1_xL^\infty_v}\Big)+C_{N,\lambda}\sqrt{\mathcal{E}(\mathbf{F}_0)}$$
$$+C\Big(\lambda+\frac{1}{N^{\beta-\beta_1-4}}\Big)\Big(2A_0+(2A_0)^2\Big). \tag{4.63}$$

Note $0\le\beta_1<\beta-4$. One can firstly choose λ sufficiently small, then $N\ge1$ large enough, and finally let $\mathcal{E}(\mathbf{F}_0)+\|\tilde{\mathbf{h}}_0\|_{L^1_xL^\infty_v}\le\varepsilon_1$ with ε_1 suitably small depending only on β, β_1, γ and M_0 such that

$$C_6(2A_0)^{\frac{3}{2}}\cdot\sup_{\substack{t_1\le s\le t\\ y\in\mathbb{T}^3}}\Big(\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(s,y,u)|\mathrm{d}u\Big)^{\frac{1}{2}}\le\frac{3}{4}A_0,$$

which, together with (4.62), yields immediately that

$$\|\mathbf{h}(t)\|_{L^\infty}\le\frac{7}{4}A_0,$$

for all $t\ge0$. Hence we have closed the *a priori* assumption (4.61). Therefore the proof of Proposition 4.3 is completed. □

Remark 4.4. *Assume* (2.18)*, Proposition 4.3 and Lemma 4.2 yield immediately that*

$$\sup_{\substack{t_1\le s\le t\\ y\in\mathbb{T}^3}}\Big(\int_{\mathbb{R}^3}|\tilde{\mathbf{h}}(s,y,u)|\mathrm{d}u\Big)\ll1,$$

for $\mathcal{E}(\mathbf{F}_0)+\|\tilde{\mathbf{h}}_0\|_{L^1_xL^\infty_v}\le\varepsilon_1$ *with* ε_1 *sufficiently small.*

5. Exponential decay of L^∞

In this section, the aim is to establish the exponential time of $\|\mathbf{h}(t)\|_{L^\infty}$.

5.1. **Estimate on the hydrodynamic part.** Denote $\bar{\mathbf{f}}(t,x,v):=e^{\lambda t}\mathbf{f}(t,x,v)$, where $\lambda>0$ is a suitably small constant determined later. Then it follows from (1.14) that

$$\begin{cases}\partial_t\bar{\mathbf{f}}+v\cdot\nabla_x\bar{\mathbf{f}}-e^{-\lambda t}\mathbf{q}(\nabla_x\bar\phi\cdot\nabla_v)\bar{\mathbf{f}}+\mathbf{L}\mathbf{y}=e^{-\lambda t}\mathbf{\Gamma}(\bar{\mathbf{f}},\bar{\mathbf{f}})-e^{-\lambda t}(\nabla_x\bar\phi\cdot\dfrac{v}{2})\mathbf{q}\bar{\mathbf{f}}\\ \quad-\nabla_x\bar\phi\cdot v\sqrt{\mu}\mathbf{q_1}+\lambda\bar{\mathbf{f}},\\ -\Delta_x\bar\phi=\displaystyle\int\sqrt{\mu}\,(\bar f_+-\bar f_-)\,dv,\quad\int_{\mathbb{T}^3}\bar\phi\;\mathrm{d}x=0,\end{cases}\tag{5.1}$$

where $\bar{\mathbf{f}}=\begin{bmatrix}\bar f_+\\ \bar f_-\end{bmatrix}$, $\mathbf{q}=\begin{bmatrix}1&0\\0&-1\end{bmatrix}$, $\mathbf{q_1}=\begin{bmatrix}1\\-1\end{bmatrix}$.

Define the hydrodynamic part $\mathbf{P}\bar{\mathbf{f}}$ of $\bar{\mathbf{f}}$ as

$$\mathbf{P}\bar{\mathbf{f}}:=\left\{\bar a_+(t,x)\begin{bmatrix}\sqrt{\mu}\\0\end{bmatrix}+\bar a_-(t,x)\begin{bmatrix}0\\\sqrt{\mu}\end{bmatrix}+\bar{\mathbf{b}}(t,x)\cdot\frac{v}{\sqrt2}\begin{bmatrix}\sqrt{\mu}\\\sqrt{\mu}\end{bmatrix}+\bar c(t,x)\frac{|v|^2-3}{2\sqrt2}\begin{bmatrix}\sqrt{\mu}\\\sqrt{\mu}\end{bmatrix}\right\},\tag{5.2}$$

where $\bar{\mathbf{b}}=[\bar b_1,\bar b_2,\bar b_3]^{\mathrm{T}}$.

Lemma 5.1. *Under the condition* (2.18)*, one has that, for* $1\le s\le t$*,*

$$\int_s^t\|\mathbf{P}\bar{\mathbf{f}}\|_{L^2}^2\mathrm{d}\tau+\int_s^t\|\nabla_x\bar\phi\|_{L^2}^2\mathrm{d}\tau\le G(s)-G(t)+C_7\int_s^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\mathrm{d}\tau,\tag{5.3}$$

where $|G(s)|\le C\|\bar{\mathbf{f}}(s)\|_{L^2}^2$.

Proof. From the Green's identity, a mild solution $\bar{\mathbf{f}}$ of (5.1) satisfies

$$\begin{aligned}
&\left\langle \bar{\mathbf{f}}(t), \mathbf{\Psi}(t)\right\rangle - \left\langle \bar{\mathbf{f}}(s), \mathbf{\Psi}(s)\right\rangle - \int_s^t \underbrace{\left\langle \bar{\mathbf{f}}, \partial_t \mathbf{\Psi}\right\rangle}_{I} + \underbrace{\left\langle \mathbf{P}\bar{\mathbf{f}}, v\cdot\nabla_x \mathbf{\Psi}\right\rangle}_{II} \mathrm{d}\tau \\
&- \int_s^t \underbrace{\langle \{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}, v\cdot\nabla_x\mathbf{\Psi}\rangle - \langle \mathbf{\Psi}, \mathbf{L}\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\rangle + e^{-\lambda\tau}\langle \mathbf{\Psi}, \mathbf{\Gamma}(\bar{\mathbf{f}},\bar{\mathbf{f}})\rangle}_{III} \mathrm{d}\tau \\
=& -\int_s^t \underbrace{e^{-\lambda\tau}\langle \mathbf{q}\sqrt{\mu}\bar{\mathbf{f}}, \nabla_x\bar{\phi}\cdot\nabla_v(\frac{\mathbf{\Psi}}{\sqrt{\mu}})\rangle}_{IV} + \underbrace{\langle \mathbf{q_1}\nabla_x\bar{\phi}\cdot v\sqrt{\mu}, \mathbf{\Psi}\rangle}_{V} - \underbrace{\lambda\langle \bar{\mathbf{f}}, \mathbf{\Psi}\rangle}_{VI} \mathrm{d}\tau.
\end{aligned} \tag{5.4}$$

Recall the definition of $\bar{\mathbf{f}}$, it is clear from (1.15) − (1.17) that

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \bar{f}_\pm(t,x,v)\sqrt{\mu(v)}\mathrm{d}x\mathrm{d}v = 0, \tag{5.5}$$

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(\bar{f}_+ + \bar{f}_-\right)(t,x,v)v\sqrt{\mu(v)}\mathrm{d}x\mathrm{d}v = \mathbf{0}, \tag{5.6}$$

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \left(\bar{f}_+ + \bar{f}_-\right)(t,x,v)|v|^2\sqrt{\mu(v)}\mathrm{d}x\mathrm{d}v + \int_{\mathbb{T}^3} e^{-\lambda t}|\nabla_x\bar{\phi}(t,x)|^2\mathrm{d}x = 0, \tag{5.7}$$

which implies that for $t \geq 0$

$$\begin{cases} \displaystyle\int_{\mathbb{T}^3} \bar{a}_\pm(t,x)\mathrm{d}x = \int_{\mathbb{T}^3} \bar{b}_i(t,x)\mathrm{d}x = 0,\ i = 1,2,3, \\ \displaystyle\int_{\mathbb{T}^3} \left(3\sqrt{2}\bar{c}(t,x) + e^{-\lambda t}|\nabla_x\bar{\phi}(t,x)|^2\right)\mathrm{d}x = 0. \end{cases} \tag{5.8}$$

However, due to $\int_{\mathbb{T}^3}\bar{c}(t,x)\mathrm{d}x \not\equiv 0$, another question arises: the Poisson equation $-\Delta\varphi_c(t,x) = \bar{c}(t,x)$, with $\int_{\mathbb{T}^3}\varphi_c\mathrm{d}x = 0$ is ill-posed, so we cannot use it to estimate $\bar{c}$ as in [30, 7]. To solve this problem, we define a new function

$$\tilde{c}(t,x) := \bar{c}(t,x) + \frac{\sqrt{2}}{6}e^{-\lambda t}|\nabla_x\bar{\phi}(t,x)|^2. \tag{5.9}$$

Thus it follows from (5.8) that

$$\int_{\mathbb{T}^3}\tilde{c}(t,x)\mathrm{d}x = 0. \tag{5.10}$$

We define $\varphi_a^\pm(t,x)$, $\varphi_{b_i}(t,x)$ and $\varphi_c(t,x)$ as the solutions of the following Poisson equations, respectively.

$$-\Delta\varphi_a^\pm(t,x) = \bar{a}_\pm(t,x), \quad \int_{\mathbb{T}^3}\varphi_{\bar{a}_\pm}(t,x)\mathrm{d}x = 0, \tag{5.11}$$

$$-\Delta\varphi_{b_i}(t,x) = \bar{b}_i(t,x), \quad \int_{\mathbb{T}^3}\varphi_{b_i}(t,x)\mathrm{d}x = 0, \tag{5.12}$$

$$-\Delta\varphi_c(t,x) = \tilde{c}(t,x), \quad \int_{\mathbb{T}^3}\varphi_c(t,x)\mathrm{d}x = 0. \tag{5.13}$$

It follows from the standard elliptic estimate that

$$\|\varphi_a^\pm\|_{H^2} \leq C\|\bar{a}_\pm\|_{L^2}, \quad \|\varphi_{b_i}\|_{H^2} \leq C\|\bar{b}_i\|_{L^2}, \quad \|\varphi_c\|_{H^2} \leq C\|\tilde{c}\|_{L^2}. \tag{5.14}$$

For the sake of later use, we define $\bar{\mathbf{a}} = \begin{bmatrix}\bar{a}_+ \\ \bar{a}_-\end{bmatrix}$.

Now we divide the proof into four steps.

Step 1. Estimate on $\nabla_x\partial_t\varphi_a^\pm$. Choosing the test function $\mathbf{\Psi} = \begin{bmatrix}\chi_+(x)\sqrt{\mu}\\ 0\end{bmatrix}$, with $\chi_+(x)$ depending only on x and substituting it into (5.4) (with time integration over $[t, t+\varepsilon]$), then we obtain

$$\int_{\mathbb{T}^3}[\bar{a}_+(t+\varepsilon)-\bar{a}_+(t)]\chi_+(x)\mathrm{d}x = \frac{1}{\sqrt{2}}\int_t^{t+\varepsilon}\int_{\mathbb{T}^3}\bar{\mathbf{b}}\cdot\nabla_x\chi_+\mathrm{d}x\mathrm{d}\tau + \lambda\int_t^{t+\varepsilon}\int_{\mathbb{T}^3}\bar{a}_+\chi_+\mathrm{d}x\mathrm{d}\tau,$$

where we have used that I, III, IV and V vanish. Taking the difference quotient, we have that for almost $t>0$,

$$\int_{\mathbb{T}^3}\partial_t\bar{a}_+(t)\chi_+(x)\mathrm{d}x = \frac{1}{\sqrt{2}}\int_{\mathbb{T}^3}\bar{\mathbf{b}}\cdot\nabla_x\chi_+\mathrm{d}x + \lambda\int_{\mathbb{T}^3}\bar{a}_+\chi_+\mathrm{d}x. \tag{5.15}$$

Choosing the test function $\mathbf{\Psi} = \begin{bmatrix}0\\ \chi_-(x)\sqrt{\mu}\end{bmatrix}$ in (5.4), with $\chi_-(x)$ depending only on x. Similarly, it holds

$$\int_{\mathbb{T}^3}\partial_t\bar{a}_-(t)\chi_-(x)\mathrm{d}x = \frac{1}{\sqrt{2}}\int_{\mathbb{T}^3}\bar{\mathbf{b}}\cdot\nabla_x\chi_-\mathrm{d}x + \lambda\int_{\mathbb{T}^3}\bar{a}_-\chi_-\mathrm{d}x. \tag{5.16}$$

It follows from (5.15) and (5.16) that

$$\left|\int_{\mathbb{T}^3}\partial_t\bar{a}_\pm(t)\chi_\pm\mathrm{d}x\right| \lesssim \{\|\bar{\mathbf{b}}(t)\|_{L^2}+\lambda\|\bar{a}_\pm(t)\|_{L^2}\}\cdot\|\chi_\pm\|_{H^1}.$$

Hence, for almost $t\geq 1$, one has

$$\|\partial_t\bar{a}_\pm(t)\|_{(H^1)^*} \lesssim \|\bar{\mathbf{b}}(t)\|_{L^2}+\lambda\|\bar{a}_\pm(t)\|_{L^2},$$

where $(H^1)^* = \big(H^1(\mathbb{T}^3)\big)^*$ is the dual space of $H^1(\mathbb{T}^3)$ with respect to the dual pair $\langle A, B\rangle = \int_{\mathbb{T}^3}A(x)B(x)dx$, for $A\in H^1$ and $B\in\big(H^1\big)^*$.

Since $\partial_t\bar{a}_\pm(t)\in\big(H^1\big)^*$, by the standard elliptic theory, we can solve $-\Delta\Phi_a^\pm = \partial_t\bar{a}_\pm(t)$, $\int_{\mathbb{T}^3}\Phi_a^\pm dx = 0$. Noting $\Phi_a^\pm = -\Delta^{-1}\partial_t\bar{a}_\pm = \partial_t\varphi_a^\pm$ with $\varphi_a^\pm$ defined in (5.11), thus we have

$$\begin{aligned}\left\|\nabla_x\partial_t\varphi_a^\pm\right\|_{L^2} &\cong \left\|\Delta^{-1}\partial_t\bar{a}_\pm(t)\right\|_{H^1} = \left\|\Phi_a^\pm\right\|_{H^1} \lesssim \|\partial_t\bar{a}_\pm(t)\|_{(H^1)^*}\\ &\leq C\Big(\|\bar{\mathbf{b}}(t)\|_{L^2}+\lambda\|\bar{a}_\pm(t)\|_{L^2}\Big).\end{aligned} \tag{5.17}$$

Step 2. Estimate on $\nabla_x\partial_t\varphi_c$. Choosing the test function $\mathbf{\Psi} = \mathbf{\Psi}_c^t := \begin{bmatrix}\frac{|v|^2-3}{2\sqrt{2}}\sqrt{\mu}\chi_c(x)\\ \frac{|v|^2-3}{2\sqrt{2}}\sqrt{\mu}\chi_c(x)\end{bmatrix}$ in (5.4), one has

$$\begin{aligned}\frac{3}{2}\int_{\mathbb{T}^3}[\bar{c}(t+\varepsilon)-\bar{c}(t)]\chi_c(x)\mathrm{d}x &= \int_t^{t+\varepsilon}\int_{\mathbb{T}^3}\bar{\mathbf{b}}\cdot\nabla_x\chi_c\mathrm{d}x\mathrm{d}\tau + \frac{3\lambda}{2}\int_t^{t+\varepsilon}\int_{\mathbb{T}^3}\bar{c}\chi_c\mathrm{d}x\mathrm{d}\tau\\ &\quad+\int_t^{t+\varepsilon}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\cdot(v\cdot\nabla_x)\mathbf{\Psi}_c^t\mathrm{d}v\mathrm{d}x\mathrm{d}\tau\\ &\quad+\int_t^{t+\varepsilon}e^{-\lambda\tau}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\mathbf{q}\sqrt{\mu}\bar{\mathbf{f}}\cdot(\nabla_x\bar{\phi}\cdot\nabla_v)\left(\frac{\mathbf{\Psi}_c^t}{\sqrt{\mu}}\right)\mathrm{d}v\mathrm{d}x\mathrm{d}\tau,\end{aligned}$$

where we have used that $\mathbf{L}\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}$, $\mathbf{\Gamma}(\bar{\mathbf{f}},\bar{\mathbf{f}})$ and $\nabla_x\bar{\phi}\cdot v\sqrt{\mu}\mathbf{q_1}$, integrated against $\mathbf{\Psi}_c^t$ are zero. Taking the difference quotient, we obtain for almost $t\geq 1$,

$$\begin{aligned}\frac{3}{2}\int_{\mathbb{T}^3}\partial_t\bar{c}(t)\chi_c(x)\mathrm{d}x &= \int_{\mathbb{T}^3}\bar{\mathbf{b}}\cdot\nabla_x\chi_c\mathrm{d}x + \frac{3\lambda}{2}\int_{\mathbb{T}^3}\bar{c}(t)\chi_c(x)\mathrm{d}x\\ &\quad+\iint_{\mathbb{T}^3\times\mathbb{R}^3}\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\cdot(v\cdot\nabla_x)\mathbf{\Psi}_c^t\mathrm{d}v\mathrm{d}x\end{aligned}$$

$$+ e^{-\lambda t}\iint_{\mathbb{T}^3\times\mathbb{R}^3} \mathbf{q}\sqrt{\mu}\bar{\mathbf{f}}\cdot(\nabla_x\bar{\phi}\cdot\nabla_v)\left(\frac{\mathbf{\Psi}_c^t}{\sqrt{\mu}}\right)\mathrm{d}v\mathrm{d}x. \tag{5.18}$$

For fixed $t\geq 1$, we choose $\chi_c(x)=\Phi_c(t,x)$ with $-\Delta\Phi_c=\partial_t\tilde{c}(t)$, $\int_{\mathbb{T}^3}\Phi_c dx=0$. It is clear that $\Phi_c=-\Delta^{-1}\partial_t\tilde{c}(t)=\partial_t\varphi_c$, where φ_c is defined in (5.13). Using the fact that

$$\partial_t\bar{c}=\partial_t\tilde{c}+\frac{\sqrt{2}}{6}\lambda e^{-\lambda t}|\nabla_x\bar{\phi}|^2-\frac{\sqrt{2}}{3}e^{-\lambda t}\nabla_x\bar{\phi}\cdot\nabla_x\partial_t\bar{\phi}, \tag{5.19}$$

we have

$$\begin{aligned}\int_{\mathbb{T}^3}\partial_t\bar{c}\partial_t\varphi_c\mathrm{d}x=&\int_{\mathbb{T}^3}\partial_t\tilde{c}\partial_t\varphi_c\mathrm{d}x+\frac{\sqrt{2}}{6}\lambda\int_{\mathbb{T}^3}e^{-\lambda t}|\nabla_x\bar{\phi}|^2\partial_t\varphi_c\mathrm{d}x\\&-\frac{\sqrt{2}}{3}\int_{\mathbb{T}^3}e^{-\lambda t}(\nabla_x\bar{\phi}\cdot\nabla_x\partial_t\bar{\phi})\partial_t\varphi_c\mathrm{d}x\\=&\|\nabla\Delta^{-1}\partial_t\tilde{c}(t)\|_{L^2}^2+\frac{\sqrt{2}}{6}\lambda\int_{\mathbb{T}^3}e^{-\lambda t}|\nabla_x\bar{\phi}|^2\partial_t\varphi_c\mathrm{d}x\\&-\frac{\sqrt{2}}{3}\int_{\mathbb{T}^3}e^{-\lambda t}\nabla_x\bar{\phi}\cdot\nabla_x\partial_t\bar{\phi}\partial_t\varphi_c\mathrm{d}x.\end{aligned} \tag{5.20}$$

For the second term on RHS of (5.20), it follows from (2.18), (1.14) and (5.1) that

$$\|e^{-\lambda t}\nabla_x\bar{\phi}(t)\|_{L^\infty}\leq\delta(1+t)^{-2}, \tag{5.21}$$

which yields that

$$\begin{aligned}\frac{\sqrt{2}}{6}\lambda\Big|\int_{\mathbb{T}^3}e^{-\lambda t}|\nabla_x\bar{\phi}|^2\partial_t\varphi_c\mathrm{d}x\Big|&\lesssim\delta\|\nabla_x\bar{\phi}(t)\|_{L^2}\|\partial_t\varphi_c(t)\|_{L^2}\\&\lesssim\delta\left(\|\nabla_x\bar{\phi}(t)\|_{L^2}^2+\|\nabla\Delta^{-1}\partial_t\tilde{c}(t)\|_{L^2}^2\right),\end{aligned} \tag{5.22}$$

where we have used Poincaré's inequality

$$\|\partial_t\varphi_c(t)\|_{L^2}\lesssim\|\nabla_x\partial_t\varphi_c(t)\|_{L^2}=\|\nabla\Delta^{-1}\partial_t\tilde{c}(t)\|_{L^2}. \tag{5.23}$$

For the last term on RHS of (5.20), it follows from (5.21) that

$$\begin{aligned}\frac{\sqrt{2}}{3}\Big|\int_{\mathbb{T}^3}e^{-\lambda t}\nabla_x\bar{\phi}\cdot\nabla_x\partial_t\bar{\phi}\partial_t\varphi_c\mathrm{d}x\Big|\lesssim&\delta\|\nabla_x\partial_t\bar{\phi}(t)\|_{L^2}\|\partial_t\varphi_c(t)\|_{L^2}\\\lesssim&\delta\left(\|\nabla\Delta^{-1}\partial_t\bar{\mathbf{a}}(t)\|_{L^2}^2+\|\nabla\Delta^{-1}\partial_t\tilde{c}(t)\|_{L^2}^2\right),\end{aligned} \tag{5.24}$$

where we used $\|\nabla_x\partial_t\bar{\phi}(t)\|_{L^2}^2\lesssim\left(\|\nabla\Delta^{-1}\partial_t\bar{a}_+(t)\|_{L^2}^2+\|\nabla\Delta^{-1}\partial_t\bar{a}_-(t)\|_{L^2}^2\right)=:\|\nabla\Delta^{-1}\partial_t\bar{\mathbf{a}}(t)\|_{L^2}^2$.

Substituting (5.24) and (5.22) into (5.20), one obtains that

$$\frac{3}{2}\int_{\mathbb{T}^3}\partial_t\bar{c}\partial_t\varphi_c\mathrm{d}x\geq\Big(\frac{3}{2}-2\delta\Big)\|\nabla\Delta^{-1}\partial_t\tilde{c}(t)\|_{L^2}^2-\delta\left(\|\nabla\Delta^{-1}\partial_t\bar{\mathbf{a}}(t)\|_{L^2}^2+\|\nabla_x\bar{\phi}(t)\|_{L^2}^2\right). \tag{5.25}$$

Next, we deal with the terms on RHS of (5.18) with $\chi_c=\Phi_c=\partial_t\varphi_c$.

For the first term on RHS of (5.18), it is clear that

$$\int_{\mathbb{T}^3}\bar{\mathbf{b}}\cdot\nabla_x\Phi_c\mathrm{d}x\leq\frac{m}{2}\|\nabla\Delta^{-1}\partial_t\tilde{c}(\tau)\|_{L^2}^2+\frac{1}{2m}\|\bar{\mathbf{b}}(\tau)\|_{L^2}^2, \tag{5.26}$$

for some small constant $m>0$.

For the second term on RHS of (5.18), a direct calculation shows that

$$\Big|\iint_{\mathbb{T}^3\times\mathbb{R}^3}\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\cdot(v\cdot\nabla_x)\mathbf{\Psi}_c^t\mathrm{d}v\mathrm{d}x\Big|\leq\frac{m}{2}\|\nabla\Delta^{-1}\partial_t\tilde{c}(\tau)\|_2^2+\frac{C}{2m}\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2. \tag{5.27}$$

For the third term on RHS of (5.18), according to Remark 4.4, there is a small constant $\kappa>0$ such that

$$\sup_{\substack{t\geq 1\\ y\in\mathbb{T}^3}}\left\{\int_{\mathbb{R}^3_v}|\mathbf{f}(t,x,v)|\mathrm{d}v\right\}\leq\kappa. \tag{5.28}$$

which, together with (5.23), yields that

$$\begin{aligned}&\left|e^{-\lambda t}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\mathbf{q}\sqrt{\mu}\bar{\mathbf{f}}\cdot(\nabla_x\bar{\phi}\cdot\nabla_v)(\frac{\boldsymbol{\Psi}_c^t}{\sqrt{\mu}})\mathrm{d}v\mathrm{d}x\right|\\ \lesssim&\sup_{x\in\mathbb{T}^3}\left\{\int_{\mathbb{R}^3_v}|\mathbf{f}(t,x,v)|\mathrm{d}v\right\}\int_{\mathbb{T}^3}|\nabla_x\bar{\phi}||\nabla\Delta^{-1}\partial_t\tilde{c}|\mathrm{d}x\\ \lesssim&\kappa\left(\|\nabla\Delta^{-1}\partial_t\tilde{c}\|_{L^2}^2+\|\nabla_x\bar{\phi}\|_{L^2}^2\right).\end{aligned} \tag{5.29}$$

For the last term on RHS of (5.18), using (5.23), one has

$$\frac{3\lambda}{2}\left|\int_{\mathbb{T}^3}\bar{c}(t)\partial_t\varphi_c\mathrm{d}x\right|\lesssim\lambda\left(\|\nabla\Delta^{-1}\partial_t\tilde{c}\|_{L^2}^2+\|\bar{c}\|_{L^2}^2\right). \tag{5.30}$$

It follows from (5.9) and (5.21) that

$$\|\bar{c}(t)\|_{L^2}^2\lesssim\|\tilde{c}(t)\|_{L^2}^2+\delta^2\|\nabla_x\bar{\phi}(t)\|_{L^2}^2, \tag{5.31}$$

which, together with (5.30), yields that

$$\frac{3\lambda}{2}\left|\int_{\mathbb{T}^3}\bar{c}(t)\partial_t\varphi_c\mathrm{d}x\right|\lesssim\lambda\left(\|\nabla\Delta^{-1}\partial_t\tilde{c}\|_{L^2}^2+\|\tilde{c}(t)\|_{L^2}^2+\delta^2\|\nabla_x\bar{\phi}(t)\|_{L^2}^2\right). \tag{5.32}$$

Taking $m=\frac{1}{2}$, and let δ, κ and λ suitably small such that $2\delta+\kappa+\lambda<\frac{1}{2}$, then substituting (5.25), (5.26) $-$ (5.29) and (5.32) into (5.18), we obtain that for $t\geq 1$,

$$\begin{aligned}\|\nabla_x\partial_t\varphi_c(t)\|_{L^2}^2=\|\nabla\Delta^{-1}\partial_t\tilde{c}(t)\|_{L^2}^2\leq&C\left(\|\bar{\mathbf{b}}(t)\|_{L^2}^2+\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}(t)\|_{\nu}^2\right)+\lambda\|\tilde{c}(t)\|_{L^2}^2\\&+\delta\|\bar{\mathbf{a}}(t)\|_{L^2}^2+(\kappa+\delta)\|\nabla_x\bar{\phi}(t)\|_{L^2}^2.\end{aligned} \tag{5.33}$$

Step 3. Estimate on $\nabla_x\partial_t\varphi_{b_i}$. Choosing the test function $\boldsymbol{\Psi}=\boldsymbol{\Psi}_{b_i}^t:=\begin{bmatrix}\frac{v_i}{\sqrt{2}}\sqrt{\mu}\chi_{b_i}(x)\\ \frac{v_i}{\sqrt{2}}\sqrt{\mu}\chi_{b_i}(x)\end{bmatrix}$ with $\chi_{b_i}(x)$ depending only on x, then we obtain from (5.4) that

$$\begin{aligned}\int_{\mathbb{T}^3}[\bar{b}_i(t+\varepsilon)-\bar{b}_i(t)]\chi_{b_i}(x)\mathrm{d}x=&\int_t^{t+\varepsilon}\int_{\mathbb{T}^3}\left(\frac{1}{\sqrt{2}}\bar{a}_+\partial_i\chi_{b_i}+\frac{1}{\sqrt{2}}\bar{a}_-\partial_i\chi_{b_i}+\bar{c}\partial_i\chi_{b_i}+\lambda\bar{b}_i\chi_{b_i}\right)\mathrm{d}x\mathrm{d}\tau\\&+\int_t^{t+\varepsilon}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\cdot(v\cdot\nabla_x)\boldsymbol{\Psi}_{b_i}^t\mathrm{d}v\mathrm{d}x\mathrm{d}\tau\\&+\int_t^{t+\varepsilon}e^{-\lambda\tau}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\mathbf{q}\sqrt{\mu}\bar{\mathbf{f}}\cdot(\nabla_x\bar{\phi}\cdot\nabla_v)\left(\frac{\boldsymbol{\Psi}_{b_i}^t}{\sqrt{\mu}}\right)\mathrm{d}v\mathrm{d}x\mathrm{d}\tau.\end{aligned}$$

Taking the difference quotient, we obtain

$$\begin{aligned}\int_{\mathbb{T}^3}\partial_t\bar{b}_i(t)\chi_{b_i}(x)\mathrm{d}x=&\int_{\mathbb{T}^3}\left(\frac{1}{\sqrt{2}}\bar{a}_+\partial_i\chi_{b_i}+\frac{1}{\sqrt{2}}\bar{a}_-\partial_i\chi_{b_i}+\bar{c}\partial_i\chi_{b_i}+\lambda\bar{b}_i\chi_{b_i}\right)\mathrm{d}x\\&+\iint_{\mathbb{T}^3\times\mathbb{R}^3}\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\cdot(v\cdot\nabla_x)\boldsymbol{\Psi}_{b_i}^t\mathrm{d}v\mathrm{d}x\\&+e^{-\lambda t}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\mathbf{q}\sqrt{\mu}\bar{\mathbf{f}}\cdot(\nabla_x\bar{\phi}\cdot\nabla_v)\left(\frac{\boldsymbol{\Psi}_{b_i}^t}{\sqrt{\mu}}\right)\mathrm{d}v\mathrm{d}x,\ \text{for } t\geq 1.\end{aligned} \tag{5.34}$$

According to Remark 4.4, it follows that

$$\left| e^{-\lambda t} \iint_{\mathbb{T}^3\times\mathbb{R}^3} \mathbf{q}\sqrt{\mu}\bar{\mathbf{f}} \cdot (\nabla_x\bar{\phi}\cdot\nabla_v)\left(\frac{\mathbf{\Psi}^t_{b_i}}{\sqrt{\mu}}\right) \mathrm{d}v\mathrm{d}x \right| \lesssim \kappa\|\nabla_x\bar{\phi}\|_{L^2}\cdot\|\chi_{b_i}\|_{H^1}, \tag{5.35}$$

which, with (5.34), yields that

$$\begin{aligned}\|\partial_t\bar{b}_i(t)\|_{(H^1)^*} \lesssim& \|\bar{\mathbf{a}}(t)\|_{L^2} + \|\bar{c}(t)\|_{L^2} + \lambda\|\bar{b}_i(t)\|_{L^2} + \kappa\|\nabla_x\bar{\phi}\|_{L^2} + \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}(t)\|_\nu \\ \lesssim& \|\bar{\mathbf{a}}(t)\|_{L^2} + \|\tilde{c}(t)\|_{L^2} + \lambda\|\bar{b}_i(t)\|_{L^2} + (\kappa+\delta)\|\nabla_x\bar{\phi}\|_{L^2} + \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}(t)\|_\nu.\end{aligned}$$

For fixed $t\geq 1$, we choose $\chi_{b_i}=\Phi_{b_i}$ with $-\Delta\Phi_{b_i}=\partial_t\bar{b}_i(t)$, $\int_{\mathbb{T}^3}\Phi_{b_i}dx=0$. It is clear that $\Phi_{b_i}=-\Delta^{-1}\partial_t\bar{b}_i=\partial_t\varphi_{b_i}$, where φ_{b_i} is defined in (5.12). By similar arguments as in *Step 1*, we have

$$\begin{aligned}\|\nabla_x\partial_t\varphi_{b_i}(t)\|_{L^2} &\cong \left\|\Delta^{-1}\partial_t\bar{b}_i(t)\right\|_{H^1} = \|\Phi_{b_i}(t)\|_{H^1} \lesssim \left\|\partial_t\bar{b}_i(t)\right\|_{(H^1)^*} \\ &\lesssim \|(\bar{\mathbf{a}},\tilde{c})(t)\|_{L^2} + \lambda\|\bar{b}_i(t)\|_{L^2} + (\kappa+\delta)\|\nabla_x\bar{\phi}(t)\|_{L^2} + \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}(t)\|_\nu.\end{aligned} \tag{5.36}$$

Step 4. Estimate on $\bar{\mathbf{a}}$, $\bar{\mathbf{b}}$, $\bar{c}$.
Step 4.1. Estimate on $\tilde{c}$. Motivated by [30], we choose the test function

$$\mathbf{\Psi} = \begin{bmatrix}(|v|^2-\beta_c)\sqrt{\mu}v\cdot\nabla_x\varphi_c \\ (|v|^2-\beta_c)\sqrt{\mu}v\cdot\nabla_x\varphi_c\end{bmatrix},$$

where φ_c is the one defined in (5.13), and $\beta_c=5$ so that $\int(|v|^2-\beta_c)v_i^2\mu(v)dv=0$. It is easy to verify that V in (5.4) vanishes, due to the orthogonality.

For I, due to oddness in v and the choice of β_c,

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3} \mathbf{P}\bar{\mathbf{f}}\cdot\begin{bmatrix}(|v|^2-\beta_c)\sqrt{\mu}v\cdot\nabla_x\partial_t\varphi_c \\ (|v|^2-\beta_c)\sqrt{\mu}v\cdot\nabla_x\partial_t\varphi_c\end{bmatrix}\mathrm{d}v\mathrm{d}x = 0.$$

Thus using (5.33), we have

$$\begin{aligned} I &\lesssim m_c\int_s^t\|\nabla\Delta^{-1}\partial_t\tilde{c}(\tau)\|_{L^2}^2\mathrm{d}\tau + \frac{1}{m_c}\int_s^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}(\tau)\|_\nu^2\mathrm{d}\tau \\ &\lesssim m_c\int_s^t\left(\delta\|\bar{\mathbf{a}}(\tau)\|_{L^2}^2 + \lambda\|\tilde{c}(\tau)\|_{L^2}^2 + (\kappa+\delta)\|\nabla_x\bar{\phi}(\tau)\|_{L^2}^2 + \|\bar{\mathbf{b}}(\tau)\|_{L^2}^2\right)\mathrm{d}\tau \\ &\quad + \frac{1}{m_c}\int_s^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}(\tau)\|_\nu^2\mathrm{d}\tau. \end{aligned} \tag{5.37}$$

For II, the $\bar{a}_\pm$, $\bar{\mathbf{b}}$ contributions vanish owing to oddness in v and the choice of β_c, thus one has from (5.9) that

$$\begin{aligned} -II &= -5\sqrt{2}\int_s^t\int_{\mathbb{T}^3}\bar{c}(\tau,x)\Delta\varphi_c\mathrm{d}x\mathrm{d}\tau \\ &= -5\sqrt{2}\int_s^t\int_{\mathbb{T}^3}\tilde{c}(\tau,x)\Delta\varphi_c\mathrm{d}x\mathrm{d}\tau - \frac{5}{3}\int_s^t\int_{\mathbb{T}^3}e^{-\lambda t}|\nabla_x\bar{\phi}|^2\tilde{c}\mathrm{d}x\mathrm{d}\tau \\ &\geq 5\sqrt{2}\int_s^t\|\tilde{c}\|_{L^2}^2\mathrm{d}\tau - \delta\int_s^t\|\nabla_x\bar{\phi}\|_{L^2}\|\tilde{c}\|_{L^2}\mathrm{d}\tau \\ &\geq (5\sqrt{2}-\delta)\int_s^t\|\tilde{c}\|_{L^2}^2\mathrm{d}\tau - \delta\int_s^t\|\nabla_x\bar{\phi}\|_{L^2}^2\mathrm{d}\tau. \end{aligned} \tag{5.38}$$

For IV, it is apparent from (5.28) that

$$\begin{aligned} IV \lesssim & \sup_{\substack{t\geq 1\\ y\in\mathbb{T}^3}}\left\{\int_{\mathbb{R}^3_v}|\mathbf{f}(t,x,v)|\mathrm{d}v\right\}\int_s^t \|\nabla_x\bar{\phi}\|_{L^2}\|\nabla_x\varphi_c\|_{L^2}\mathrm{d}\tau \\ \lesssim & \ \kappa\int_s^t(\|\nabla_x\bar{\phi}\|_{L^2}^2+\|\tilde{c}\|_{L^2}^2)\mathrm{d}\tau. \end{aligned} \tag{5.39}$$

For VI, due to oddness in v, one has

$$VI \lesssim \lambda\int_s^t\left(\|\tilde{c}\|_{L^2}^2+\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\right)\mathrm{d}\tau. \tag{5.40}$$

It remains to control III. Using (5.28) and Proposition 4.3, a routine computation yields that

$$\begin{aligned} III \lesssim & \ m_c\int_s^t\|\tilde{c}\|_{L^2}^2\mathrm{d}\tau+\frac{1}{m_c}\int_s^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\mathrm{d}\tau \\ & +\sup_{\substack{t\geq 1\\ y\in\mathbb{T}^3}}\left\{\int_{\mathbb{R}^3_v}|\mathbf{f}(t,x,v)|\mathrm{d}v\|w_\beta\mathbf{f}\|_{L^\infty}\right\}^{\frac{1}{2}}\int_s^t\|\bar{\mathbf{f}}\|_{L^2}^2\mathrm{d}\tau \\ \lesssim & \ m_c\int_s^t\|\tilde{c}\|_{L^2}^2\mathrm{d}\tau+\frac{1}{m_c}\int_s^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\mathrm{d}\tau+\kappa^{\frac{1}{2}}\int_s^t\|\bar{\mathbf{f}}\|_{L^2}^2\mathrm{d}\tau. \end{aligned} \tag{5.41}$$

Plugging (5.37) $-$ (5.41) into (5.4), and taking $\delta+\lambda+\kappa+m_c<1$, one can get

$$\begin{aligned} \int_s^t\|\tilde{c}(\tau)\|_{L^2}^2\mathrm{d}\tau \lesssim & \frac{1}{m_c}\int_s^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}(\tau)\|_\nu^2\mathrm{d}\tau+(\delta+\kappa)\int_s^t(\|\bar{\mathbf{a}}(\tau)\|_{L^2}^2+\|\nabla_x\bar{\phi}(\tau)\|_{L^2}^2)\mathrm{d}\tau \\ & +m_c\int_s^t\|\bar{\mathbf{b}}(\tau)\|_{L^2}^2\mathrm{d}\tau+\kappa^{\frac{1}{2}}\int_s^t\|\bar{\mathbf{f}}(\tau)\|_{L^2}^2\mathrm{d}\tau+G(s)-G(t), \end{aligned} \tag{5.42}$$

where $G(s):=\iint_{\mathbb{T}^3\times\mathbb{R}^3}\bar{\mathbf{f}}(s,x,v)\cdot\mathbf{\Psi}(s,x,v)\mathrm{d}v\mathrm{d}x$, and $|G(s)|\leq C\|\bar{\mathbf{f}}(s)\|_{L^2}^2$.
Step 4.2. Estimate on $\bar{\mathbf{a}}$. We choose a test function

$$\mathbf{\Psi}=\mathbf{\Psi}_a:=\begin{bmatrix}-(|v|^2-\beta_a)\sqrt{\mu}v\cdot\nabla_x\varphi_a^+\\ -(|v|^2-\beta_a)\sqrt{\mu}v\cdot\nabla_x\varphi_a^-\end{bmatrix},$$

where $\varphi_a^\pm(t,x)$ are the ones defined in (5.11), and $\beta_a=10$ so that $\int(|v|^2-\beta_a)\left(\frac{|v|^2-3}{2\sqrt{2}}\right)v_i^2\mu(v)dv=0$.

For II, in view of oddness in v and the choice of β_a, the $\bar{\mathbf{b}}$, $\bar{c}$ contributions vanish, one has

$$\begin{aligned} II =& 5\int_s^t\int_{\mathbb{T}^3}\bar{a}_+\Delta\varphi_a^+\mathrm{d}x\mathrm{d}\tau+5\int_s^t\int_{\mathbb{T}^3}\bar{a}_-\Delta\varphi_a^-\mathrm{d}x\mathrm{d}\tau \\ =& -5\int_s^t\|\bar{\mathbf{a}}(\tau)\|_{L^2}^2\mathrm{d}\tau, \end{aligned} \tag{5.43}$$

where we have used the fact $\int(|v|^2-10)v_i^2\mu(v)\mathrm{d}v=-5$.

For V, noting $(5.1)_2$ and (5.11), we have $\bar{\phi}=\varphi_a^+-\varphi_a^-$. Then one obtains that

$$\begin{aligned} V =& \sum_{i,j}\int_s^t\int\partial_i\varphi_a^+\partial_j\bar{\phi}\cdot\int(|v|^2-10)v_iv_j\mu\mathrm{d}v \\ & -\sum_{i,j}\int_s^t\int\partial_i\varphi_a^-\partial_j\bar{\phi}\cdot\int(|v|^2-10)v_iv_j\mu\mathrm{d}v \end{aligned}$$

$$= -5\int_s^t\int(\nabla_x\varphi_a^+ - \nabla_x\varphi_a^-)\cdot\nabla_x\bar{\phi}\mathrm{d}x\mathrm{d}\tau = -5\int_s^t\|\nabla_x\bar{\phi}\|_{L^2}^2\mathrm{d}\tau. \tag{5.44}$$

For I, due to the oddness in v, it follows from (5.2) and (5.17) that

$$\begin{aligned}
I =& \int_s^t\langle\mathbf{P}\bar{\mathbf{f}},\partial_t\mathbf{\Psi}_a\rangle\mathrm{d}\tau + \int_s^t\langle\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}},\partial_t\mathbf{\Psi}_a\rangle\mathrm{d}\tau\\
=&\frac{5}{\sqrt{2}}\int_s^t\int(\bar{\mathbf{b}}\cdot\nabla_x\partial_t\varphi_a^+ + \bar{\mathbf{b}}\cdot\nabla_x\partial_t\varphi_a^-)\mathrm{d}x\mathrm{d}\tau + \int_s^t\langle\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}},\partial_t\mathbf{\Psi}_a\rangle\mathrm{d}\tau\\
\lesssim&\int_s^t\left(\|\nabla\Delta^{-1}\partial_t\bar{\mathbf{a}}\|_{L^2}^2 + \|\bar{\mathbf{b}}\|_{L^2}^2 + \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\right)\mathrm{d}\tau\\
\lesssim&\lambda\int_s^t\|\bar{\mathbf{a}}\|_{L^2}^2\mathrm{d}\tau + \int_s^t\left(\|\bar{\mathbf{b}}\|_{L^2}^2 + \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\right)\mathrm{d}\tau.
\end{aligned} \tag{5.45}$$

Using the same procedure as in *Step 4.1* for III, IV and VI, one can get

$$\begin{aligned}
III + IV + VI \le&(\kappa + m_a + \lambda)\int_s^t\left(\|\nabla_x\bar{\phi}\|_{L^2}^2 + \|\bar{\mathbf{a}}\|_{L^2}^2\right)\mathrm{d}\tau + C\int_s^t\|\bar{\mathbf{b}}\|_{L^2}^2\mathrm{d}\tau\\
&+\frac{C}{m_a}\int_s^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\mathrm{d}\tau + \kappa^{\frac{1}{2}}\int_s^t\|\bar{\mathbf{f}}\|_{L^2}^2\mathrm{d}\tau.
\end{aligned} \tag{5.46}$$

Substituting (5.43) $-$ (5.46) into (5.4), and noting the smallness of m_a,κ,λ, we obtain that

$$\begin{aligned}
\int_s^t\left(\|\bar{\mathbf{a}}\|_{L^2}^2 + \|\nabla_x\bar{\phi}\|_{L^2}^2\right)\mathrm{d}\tau \le& C\int_s^t\left(\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2 + \|\bar{\mathbf{b}}\|_{L^2}^2\right)\mathrm{d}\tau + \kappa^{\frac{1}{2}}\int_s^t\|\bar{\mathbf{f}}\|_{L^2}^2\mathrm{d}\tau\\
&+G(s) - G(t).
\end{aligned} \tag{5.47}$$

Step 4.3. Estimate on $\bar{\mathbf{b}}$. To close the estimate, we need to divide the proof into two cases.
Case 1: For fixed i,j, we choose the test function in (5.4)

$$\mathbf{\Psi} = \mathbf{\Psi}_{b,1}^{i,j} := \begin{bmatrix}(v_i^2-\beta_b)\sqrt{\mu}\partial_j\varphi_{b_j}\\(v_i^2-\beta_b)\sqrt{\mu}\partial_j\varphi_{b_j}\end{bmatrix},\ i,j = 1,2,3,$$

where φ_{b_j} is the one defined in (5.12) and $\beta_b = 1$ so that $\int(v_i^2-\beta_b)\mu(v)dv = 0$. It is clear that V in (5.4) is zero, due to the orthogonality.

For II, the $\bar{a}$, $\bar{c}$ contributions vanish due to oddness in v, one has

$$II = 2\sqrt{2}\int_s^t\int_{\mathbb{T}^3}\bar{b}_i\cdot\partial_i\partial_j\varphi_{b_j}\mathrm{d}x\mathrm{d}\tau = -2\sqrt{2}\int_s^t\int_{\mathbb{T}^3}\bar{b}_i\cdot(\partial_i\partial_j\Delta^{-1}\bar{b}_j)\mathrm{d}x\mathrm{d}\tau. \tag{5.48}$$

For I, due to oddness in v and the choice of β_b, it follows from (5.2), (5.36) and (5.31) that

$$\begin{aligned}
I =&\sqrt{2}\int_s^t\int\bar{c}\cdot\partial_j\partial_t\varphi_{b_j}\mathrm{d}x\mathrm{d}\tau + \int_s^t\left\langle\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}},\partial_t\mathbf{\Psi}_{b,1}^{i,j}\right\rangle\mathrm{d}\tau\\
\lesssim& m_b\int_s^t\|\nabla_x\partial_t\varphi_{b_j}(\tau)\|_{L^2}^2\mathrm{d}\tau + \frac{1}{m_b}\int_s^t\left(\|\bar{c}(\tau)\|_{L^2}^2 + \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\right)\mathrm{d}\tau\\
\lesssim&\lambda\int_s^t\|\bar{\mathbf{b}}\|_{L^2}^2\mathrm{d}\tau + (m_b + \frac{\delta}{m_b})\int_s^t\left(\|\bar{\mathbf{a}}\|_{L^2}^2 + \|\nabla_x\bar{\phi}\|_{L^2}^2\right)\mathrm{d}\tau\\
&+\frac{1}{m_b}\int_s^t\left(\|\tilde{c}\|_{L^2}^2 + \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\right)\mathrm{d}\tau,
\end{aligned} \tag{5.49}$$

where m_b is a small constant chosen later.

For VI, it is obvious from (5.31) that

$$VI \leq \lambda \int_s^t \|\bar{\mathbf{b}}\|_{L^2}^2 \mathrm{d}\tau + \delta \int_s^t \|\nabla_x \bar{\phi}\|_{L^2}^2 \mathrm{d}\tau + \lambda \int_s^t (\|\tilde{c}\|_{L^2}^2 + \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2) \mathrm{d}\tau, \tag{5.50}$$

where we have used (5.14).

For III, IV, by similar arguments as in *Step 4.1*, one can obtain

$$III + IV \leq (m_b+\kappa) \int_s^t (\|\nabla_x \bar{\phi}\|_{L^2}^2 + \|\bar{\mathbf{b}}\|_{L^2}^2) \mathrm{d}\tau + \frac{C}{m_b} \int_s^t \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2 \mathrm{d}\tau + \kappa^{\frac{1}{2}} \int_s^t \|\bar{\mathbf{f}}\|_{L^2}^2 \mathrm{d}\tau. \tag{5.51}$$

Plugging (5.48) $-$ (5.51) into (5.4), we obtain

$$\begin{aligned}
&\Big| \int_s^t \int \bar{b}_i \cdot (\partial_i \partial_j \Delta^{-1} \bar{b}_j) \mathrm{d}x \mathrm{d}\tau \Big| \\
&\leq (\lambda+\kappa+m_b) \int_s^t \|\bar{\mathbf{b}}\|_{L^2}^2 \mathrm{d}\tau + (m_b + \frac{\delta}{m_b}) \int_s^t \left(\|\bar{\mathbf{a}}\|_{L^2}^2 + \|\nabla_x \bar{\phi}\|_{L^2}^2 \right) \mathrm{d}\tau \\
&\quad + \frac{C}{m_b} \int_s^t \left(\|\tilde{c}\|_{L^2}^2 + \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2 \right) \mathrm{d}\tau + \kappa^{\frac{1}{2}} \int_s^t \|\bar{\mathbf{f}}\|_{L^2} \mathrm{d}\tau + G(s) - G(t).
\end{aligned} \tag{5.52}$$

Case 2: We choose the test function in (5.4)

$$\mathbf{\Psi} = \mathbf{\Psi}_{b,2}^{i,j} := \begin{bmatrix} |v|^2 v_i v_j \sqrt{\mu} \partial_j \varphi_{b_i} \\ |v|^2 v_i v_j \sqrt{\mu} \partial_j \varphi_{b_i} \end{bmatrix}, \ i \neq j.$$

Due to the oddness in v, it is clear $V = 0$.

For II, the $\bar{a}$, $\bar{c}$ contributions vanish due to oddness in v, one has

$$II = -7\sqrt{2} \int_s^t \int_{\mathbb{T}^3} \bar{b}_i \cdot (\partial_j \partial_j \Delta^{-1} \bar{b}_i) + \bar{b}_j \cdot (\partial_i \partial_j \Delta^{-1} \bar{b}_i) \mathrm{d}x \mathrm{d}\tau. \tag{5.53}$$

For I, due to oddness in v, it follows from (5.36) that

$$\begin{aligned}
I &= \int_s^t \left\langle \{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}, \partial_t \mathbf{\Psi}_{\mathbf{b},2}^{i,j} \right\rangle \mathrm{d}\tau \\
&\lesssim m_b \int_s^t \|\nabla_x \partial_t \varphi_{b_i}(\tau)\|_{L^2}^2 \mathrm{d}\tau + \frac{1}{m_b} \int_s^t \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2 \mathrm{d}\tau \\
&\lesssim m_b \int_s^t \left(\|\bar{\mathbf{a}}\|_{L^2}^2 + \|\nabla_x \bar{\phi}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2 \right) \mathrm{d}\tau + \lambda \int_s^t \|\bar{\mathbf{b}}\|_{L^2}^2 \mathrm{d}\tau + \frac{1}{m_b} \int_s^t \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2 \mathrm{d}\tau.
\end{aligned} \tag{5.54}$$

Moreover, it is not hard to show that III, IV and VI are bounded by

$$m_b \int_s^t \|\bar{\mathbf{b}}\|_{L^2}^2 \mathrm{d}\tau + \kappa \int_s^t \|\nabla_x \bar{\phi}\|_{L^2}^2 \mathrm{d}\tau + \frac{C}{m_b} \int_s^t \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2 \mathrm{d}\tau + \kappa^{\frac{1}{2}} \int_s^t \|\bar{\mathbf{f}}\|_{L^2}^2 \mathrm{d}\tau. \tag{5.55}$$

Combining (5.53) $-$ (5.55), one has

$$\begin{aligned}
&\Big| \int_s^t \int_{\mathbb{T}^3} \bar{b}_i \cdot (\partial_j \partial_j \Delta^{-1} \bar{b}_i) + \bar{b}_j \cdot (\partial_i \partial_j \Delta^{-1} b_i) \mathrm{d}x \mathrm{d}\tau \Big| \\
&\leq G(s) - G(t) + C(m_b+\lambda) \int_s^t \|\bar{\mathbf{b}}\|_{L^2}^2 \mathrm{d}\tau + m_b \int_s^t (\|\bar{\mathbf{a}}\|_{L^2}^2 + \|\nabla_x \bar{\phi}\|_{L^2}^2 + \|\tilde{c}\|_{L^2}^2) \mathrm{d}\tau \\
&\quad + \frac{C}{m_b} \int_s^t \|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2 \mathrm{d}\tau + \kappa^{\frac{1}{2}} \int_s^t \|\bar{\mathbf{f}}\|_{L^2} \mathrm{d}\tau.
\end{aligned} \tag{5.56}$$

Then it is follows (5.52) and (5.56) that

$$\int_s^t \|\bar{\mathbf{b}}(\tau)\|_{L^2}^2 \mathrm{d}\tau \leq G(s) - G(t) + (m_b + \frac{\delta}{m_b}) \int_s^t \left(\|\bar{\mathbf{a}}\|_{L^2}^2 + \|\nabla_x \bar{\phi}\|_{L^2}^2 \right) \mathrm{d}\tau$$

$$+\frac{C}{m_b}\int_s^t \left(\|\tilde{c}\|_{L^2}^2+\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\right)\mathrm{d}\tau+\kappa^{\frac{1}{2}}\int_s^t\|\bar{\mathbf{f}}\|_{L^2}\mathrm{d}\tau. \tag{5.57}$$

Using (5.42), (5.47) and (5.57), taking $m_b=\sqrt{m_c}$ suitably small, and then choose δ and κ sufficiently small, we have

$$\int_s^t\|\left(\bar{\mathbf{a}},\bar{\mathbf{b}},\tilde{c}\right)(\tau)\|_{L^2}^2+\|\nabla_x\bar{\phi}(\tau)\|_{L^2}^2\mathrm{d}\tau\leq G(s)-G(t)+C\int_s^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\mathrm{d}\tau, \tag{5.58}$$

which, with (5.31), yields that

$$\int_s^t\|\bar{c}(\tau)\|_{L^2}^2\mathrm{d}\tau\leq G(s)-G(t)+C\int_s^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\mathrm{d}\tau. \tag{5.59}$$

The lemma 5.1 is now a direct consequence of (5.58) $-$ (5.59). The proof is completed. □

5.2. **Exponential decay of L^2.** Next, we establish the L^2 decay estimate for $\mathbf{f}$, which is crucial step to get the exponential decay in L^∞ norm.

Proposition 5.2. *Assume* (2.18)*, there exists* $0<\lambda\ll 1$ *such that*

$$\|\mathbf{f}(t)\|_{L^2}\leq Ce^{-\frac{\lambda}{2}t}.$$

where the positive constant $C\geq 1$ *depends only on* M_0*.*

Proof. Multiplying (5.1) by $\bar{\mathbf{f}}$ and integrating over $\mathbb{T}^3\times\mathbb{R}^3$, one obtains that

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\bar{\mathbf{f}}(t)\|_{L^2}^2+\iint\nabla_x\bar{\phi}\cdot v\sqrt{\mu}(\bar{f}_+-\bar{f}_-)\mathrm{d}v\mathrm{d}x+\lambda_0\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2\\
\leq&e^{-\lambda t}\iint\nabla_x\bar{\phi}\cdot\frac{v}{2}(|\bar{f}_-|^2-|\bar{f}_+|^2)\mathrm{d}v\mathrm{d}x+e^{-\lambda t}\iint\mathbf{\Gamma}(\bar{\mathbf{f}},\bar{\mathbf{f}})\cdot\bar{\mathbf{f}}\mathrm{d}v\mathrm{d}x\\
&+\lambda\iint(|\bar{f}_+|^2+|\bar{f}_-|^2)\mathrm{d}v\mathrm{d}x,
\end{aligned} \tag{5.60}$$

where $\|\bar{\mathbf{f}}\|_{L^2}^2:=\|(\bar{f}_+,\bar{f}_-)\|_{L^2}^2$. Recalling (1.3) and (5.1), it follows from integration by parts that

$$\iint_{\mathbb{T}^3\times\mathbb{R}^3}\nabla_x\bar{\phi}\cdot v\sqrt{\mu}(\bar{f}_+-\bar{f}_-)\mathrm{d}v\mathrm{d}x=\frac{d}{dt}\Big(\frac{1}{2}\int|\nabla_x\bar{\phi}|^2dx\Big)-\lambda\int|\nabla_x\bar{\phi}|^2dx. \tag{5.61}$$

It is easy to show that the first term on RHS of (5.60) can be bounded by

$$\sup_{x\in\mathbb{T}^3}\Big\{\int_{\mathbb{R}^3_v}|\mathbf{f}(t,x,v)|\mathrm{d}v\cdot\|w_\beta\mathbf{f}\|_{L^\infty}\Big\}^{\frac{1}{2}}\cdot\|\bar{\mathbf{f}}\|_{L^2}^2. \tag{5.62}$$

A simple manipulation shows that

$$\begin{aligned}
&\Big|e^{-\lambda t}\iint_{\mathbb{T}^3\times\mathbb{R}^3}\mathbf{\Gamma}(\bar{\mathbf{f}},\bar{\mathbf{f}})\cdot\bar{\mathbf{f}}\mathrm{d}v\mathrm{d}x\Big|\\
\leq&\sup_{x\in\mathbb{T}^3}\Big\{\int_{\mathbb{R}^3_v}|\mathbf{f}(t,x,v)|\mathrm{d}v\cdot\|w_\beta\mathbf{f}\|_{L^\infty}\Big\}^{\frac{1}{2}}\|\bar{\mathbf{f}}\|_{L^2}\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu\\
\leq&\frac{\lambda_0}{2}\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2+\frac{1}{2\lambda_0}\sup_{x\in\mathbb{T}^3}\Big\{\int_{\mathbb{R}^3_v}|\mathbf{f}(t,x,v)|\mathrm{d}v\cdot\|w_\beta\mathbf{f}\|_{L^\infty}\Big\}\cdot\|\bar{\mathbf{f}}\|_{L^2}^2.
\end{aligned} \tag{5.63}$$

Substituting (5.61)-(5.63) into (5.60), one has that

$$\frac{1}{2}\frac{d}{dt}\left(\|\bar{\mathbf{f}}(t)\|_{L^2}^2+\|\nabla_x\bar{\phi}(t)\|_{L^2}^2\right)-\lambda\left(\|\bar{\mathbf{f}}(t)\|_{L^2}^2+\|\nabla_x\bar{\phi}(t)\|_{L^2}^2\right)+\frac{\lambda_0}{2}\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|_\nu^2$$

$$\leq\Big(\frac{1}{2\lambda_0}\sup_{x\in\mathbb{T}^3}\Big\{\int_{\mathbb{R}^3_v}|\mathbf{f}(t,x,v)|\mathrm{d}v\cdot\|w_\beta\mathbf{f}\|_{L^\infty}\Big\}+\sup_{x\in\mathbb{T}^3}\Big\{\int_{\mathbb{R}^3_v}|\mathbf{f}(t,x,v)|\mathrm{d}v\|w_\beta\mathbf{f}\|_{L^\infty}\Big\}^{\frac{1}{2}}\Big)\cdot\|\bar{\mathbf{f}}\|^2_{L^2}. \quad (5.64)$$

For $t\leq 1$, it follows from (5.64) and Proposition 4.3 that

$$\begin{aligned}&\left(\|\bar{\mathbf{f}}(t)\|^2_{L^2}+\|\nabla_x\bar{\phi}\|^2_{L^2}\right)+\frac{\lambda_0}{2}\int_0^t\|(\mathbf{I}-\mathbf{P})\bar{\mathbf{f}}(\tau)\|^2_\nu\mathrm{d}\tau\\&\leq\left(\|\mathbf{f}_0\|^2_{L^2}+\|\nabla_x\phi_0\|^2_{L^2}\right)\exp\left\{\frac{1}{\lambda_0}\left(\|w_\beta\mathbf{f}\|^2_{L^\infty}+1\right)\right\}\\&\leq\left(\|\mathbf{f}_0\|^2_{L^2}+\|\nabla_x\phi_0\|^2_{L^2}\right)\exp\Big\{C(1+M_0)^4\Big\},\quad 0\leq t\leq 1.\end{aligned}\quad (5.65)$$

For $t\geq 1$, integrating (5.64) over time on $(1,t)$, and noting Remark 4.4, one has

$$\begin{aligned}&\frac{1}{2}\left(\|\bar{\mathbf{f}}(t)\|^2_{L^2}+\|\nabla_x\bar{\phi}(t)\|^2_{L^2}\right)+\frac{\lambda_0}{2}\int_1^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|^2_\nu\\\leq&\frac{1}{2}\left(\|\bar{\mathbf{f}}(1)\|^2_{L^2}+\|\nabla_x\bar{\phi}(1)\|^2_{L^2}\right)+(\kappa+\lambda)\int_1^t\left(\|\bar{\mathbf{f}}(\tau)\|^2_{L^2}+\|\nabla_x\bar{\phi}(\tau)\|^2_{L^2}\right)\mathrm{d}\tau.\end{aligned}\quad (5.66)$$

Substituting (5.3) into (5.66)

$$\begin{aligned}&\Big(\frac{1}{2}-\kappa-\lambda\Big)\left(\|\bar{\mathbf{f}}(t)\|^2_{L^2}+\|\nabla_x\bar{\phi}(t)\|^2_{L^2}\right)+\Big(\frac{\lambda_0}{2}-(\kappa+\lambda)(C_7+1)\Big)\int_1^t\|\{\mathbf{I}-\mathbf{P}\}\bar{\mathbf{f}}\|^2_\nu\\\leq&C\left(\|\mathbf{f}(1)\|^2_{L^2}+\|\nabla_x\bar{\phi}(1)\|^2_{L^2}\right).\end{aligned}\quad (5.67)$$

Taking $\lambda\leq\frac{\lambda_0}{4(C_7+1)}$, and noting κ is sufficiently small, one has

$$\|\bar{\mathbf{f}}(t)\|^2_{L^2}+\|\nabla_x\bar{\phi}(t)\|^2_{L^2}\leq C\left(\|\bar{\mathbf{f}}(1)\|^2_{L^2}+\|\nabla_x\bar{\phi}(1)\|^2_{L^2}\right),$$

which yields that

$$\|\mathbf{f}(t)\|^2_{L^2}\leq Ce^{-\lambda t},$$

where C depend on M_0 and λ_0. Therefore the proof of Proposition 5.2 is completed. □

We now present the main conclusions of this section.

Proposition 5.3. *Assume* (2.18), *there is* $0<\lambda_1\leq\min\{\frac{\lambda}{2},\frac{\tilde{\nu}_0}{2}\}$ *such that*

$$\|\mathbf{h}(t)\|_{L^\infty}\leq Ce^{-\lambda_1 t}, \quad (5.68)$$

where the positive constant $C\geq 1$ *depends only on* M_0,β_1,β.

Proof. According to the standard L^2-L^∞ method of the Boltzmann equation, one can easily infer (5.68). Here we omit the details for brevity of presentations. The reader can refer to [39, 41, 26] for more details. Therefore the proof of Proposition 5.3 is completed. □

6. Uniform $W^{1,\infty}$-Bound

In order to close the free streaming condition (2.18), the remaining task is to perform the $W^{1,\infty}$ estimate of $\mathbf{f}$. Recall $\tilde{h}_\pm$ in (1.21), which satisfy (4.29) − (4.31), one can establish the following two key lemmas.

Lemma 6.1. *Assume* (2.18), *let* $4\leq\beta_1<\beta-4$, *there exits constant* C, *depending on* β, β_1 *and* M_0, *such that*

$$\|\partial_x\tilde{\mathbf{h}}(t)\|_{L^\infty}\leq C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2+\delta\sup_{0\leq s\leq t}\|\partial_v\tilde{\mathbf{h}}(s)\|_{L^\infty}.$$

Lemma 6.2. *Assume* (2.18)*, let* $4 \le \beta_1 < \beta - 4$*, there exists constant* C*, depending on* β*,* β_1 *and* M_0*, such that*

$$\|\partial_v \tilde{\mathbf{h}}(t)\|_{L^\infty} \le C(1 + \|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 + \sup_{0\le s\le t}\|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty}.$$

With the help of the two lemmas we can establish the following proposition:

Proposition 6.3. *Assume* (2.18)*, let* $4 \le \beta_1 < \beta - 4$*, there exists constant* C*, depending on* β*,* β_1 *and* M_0*, such that*

$$\|\partial_{x,v}\tilde{\mathbf{h}}(t)\|_{L^\infty} \le C(1 + \|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2.$$

Proof. Proposition 6.3 follows from Lemmas 6.1 and 6.2. Therefore the proof of Proposition 6.3 is completed. □

In the following two subsections, we prove the above two lemmas.

6.1. **Proof of Lemma 6.1.** Recall $\partial_i = \frac{\partial}{\partial x_i}$, applying ∂_i to (4.29), we get

$$\begin{aligned}&(\partial_t + v\cdot\nabla_x \mp \nabla_x\phi\cdot\nabla_v)\partial_i\tilde{h}_\pm + \tilde{\nu}_{\pm,1}\partial_i\tilde{h}_\pm\\ =&\pm\nabla_x\partial_i\phi\cdot\nabla_v\tilde{h}_\pm - \partial_i\tilde{\nu}_{\pm,1}\tilde{h}_\pm \mp \nabla_x\partial_i\phi\cdot v w_{\beta_1}\sqrt{\mu} + \partial_i(w_{\beta_1}\Gamma^\pm(\mathbf{f},\mathbf{f})) + \partial_i(w_{\beta_1}\mathrm{K}^\pm\mathbf{f}).\end{aligned}\tag{6.1}$$

Integrating (6.1) along the characteristic lines, one obtains

$$\begin{aligned}\partial_i\tilde{h}_\pm(t,x,v) =& \partial_i\tilde{h}_{\pm0}(X_\pm(0),V_\pm(0))e^{-\int_0^t\tilde{\nu}_{\pm,1}(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\\ &\pm\int_0^t e^{-\int_s^t\tilde{\nu}_{\pm,1}(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\left(\nabla_x\partial_i\phi\cdot\nabla_v\tilde{h}_\pm\right)(s,X_\pm(s),V_\pm(s))\mathrm{d}s\\ &-\int_0^t e^{-\int_s^t\tilde{\nu}_{\pm,1}(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\left(\partial_i\tilde{\nu}_{\pm,1}\tilde{h}_\pm\right)(s,X_\pm(s),V_\pm(s))\mathrm{d}s\\ &\mp\int_0^t e^{-\int_s^t\tilde{\nu}_{\pm,1}(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\left(\nabla_x\partial_i\phi\cdot v w_{\beta_1}\sqrt{\mu}\right)(s,X_\pm(s),V_\pm(s))\mathrm{d}s\\ &+\int_0^t e^{-\int_s^t\tilde{\nu}_{\pm,1}(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\left(\partial_i(w_{\beta_1}\Gamma^\pm(\mathbf{f},\mathbf{f}))\right)(s,X_\pm(s),V_\pm(s))\mathrm{d}s\\ &+\int_0^t e^{-\int_s^t\tilde{\nu}_{\pm,1}(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\left(\partial_i(w_{\beta_1}\mathrm{K}^\pm\mathbf{f})\right)(s,X_\pm(s),V_\pm(s))\mathrm{d}s = \sum_{i=0}^5 H_i.\end{aligned}\tag{6.2}$$

For the nonlinear collision term H_4, it follows from (2.56), (3.3), (5.68), Proposition 4.3 and Remark 4.4 that

$$\begin{aligned}&\int_0^t e^{-\int_s^t\tilde{\nu}_{\pm,1}(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\left|\left(\partial_i(w_{\beta_1}\Gamma^\pm(\mathbf{f},\mathbf{f}))\right)(s,X_\pm(s),V_\pm(s))\right|\mathrm{d}s\\ \le&\Big\{\int_0^{t_1}+\int_{t_1}^t\Big\}e^{-\frac{\nu(v)}{2}(t-s)}\nu(v)\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}\|\mathbf{h}(s)\|_{L^\infty}^{\frac12}\cdot\sup_{x\in\mathbb{T}^3}\Big\{\int_{\mathbb{R}^3_v}|\tilde{\mathbf{h}}(s,x,v)|\mathrm{d}v\Big\}^{\frac12}\mathrm{d}s\\ &+\int_0^t e^{-\tilde{\nu}_0(t-s)}\|\mathbf{h}(s)\|_{L^\infty}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}\mathrm{d}s\\ \le&C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0)\|_{L^\infty})^2+\int_0^t e^{-\tilde{\nu}_0(t-s)}e^{-\lambda_1 s}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}\mathrm{d}s\\ &+\int_{t_1}^t e^{-\frac{\nu(v)}{2}(t-s)}\nu(v)\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}\|\mathbf{h}(s)\|_{L^\infty}^{\frac12}\cdot\sup_{x\in\mathbb{T}^3}\Big\{\int_{\mathbb{R}^3_v}|\tilde{\mathbf{h}}(s,x,v)|\mathrm{d}v\Big\}^{\frac12}\mathrm{d}s\\ \le&C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0)\|_{L^\infty})^2+\kappa^{\frac12}\sup_{0\le s\le t}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}+\int_0^t e^{-\lambda_1 s}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}\mathrm{d}s.\end{aligned}\tag{6.3}$$

Next we estimate the remaining terms on RHS of (6.2) as follows.

It follows from (4.30) and (2.18) that

$$|\partial_i \tilde{\nu}_{\pm,1}(t,x,v)| \leq C\delta(1+t)^{-\frac{5}{2}}(1+|v|). \tag{6.4}$$

Then plugging (2.18), (2.33), (4.31), (6.3) $-$ (6.4) and Proposition 4.3 into (6.2), one has

$$\begin{aligned}
|\partial_x \tilde{h}_{\pm}(t,x,v)| \leq{}& C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0)\|_{L^\infty})^2 + \delta \sup_{0\leq s\leq t}\|\partial_v \tilde{\mathbf{h}}(s)\|_{L^\infty} \\
&+ \kappa^{\frac{1}{2}} \sup_{0\leq s\leq t}\|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} + \int_0^t e^{-\lambda_1 s}\|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty}\mathrm{d}s \\
&+ \Big|\int_0^t e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau)\mathrm{d}\tau}\int_{\mathbb{R}^3} \mathtt{k}^{(2)}_{w_{\beta_1}}(V_{\pm}(s),u)\partial_x \tilde{h}_{\pm}(s,X_{\pm}(s),u)\mathrm{d}u\Big| \\
&+ \Big|\int_0^t e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau)\mathrm{d}\tau}\int_{\mathbb{R}^3} \mathtt{k}^{(1)}_{w_{\beta_1}}(V_{\pm}(s),u)\partial_x \tilde{h}_{\mp}(s,X_{\pm}(s),u)du\Big|.
\end{aligned} \tag{6.5}$$

We only deal with the case for $|\partial_i \tilde{h}_{+}(t,x,v)|$, because $|\partial_i \tilde{h}_{-}(t,x,v)|$ can be controlled in the same way. We denote $\hat{\nu}_{\pm,1}(\tau_1) := \tilde{\nu}_{\pm,1}(\tau_1,\hat{X}(\tau_1),\hat{V}(\tau_1))$ for simplicity of presentation, where $\hat{X}(\tau_1),\hat{V}(\tau_1)$ has been defined in (4.10). Indeed, using (6.5) again for $\partial_x \tilde{h}_{\pm}(s,X_{\pm}(s),u)$, one has

$$\begin{aligned}
|\partial_x \tilde{h}_{+}(t,x,v)| \leq{}& C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0)\|_{L^\infty})^2 + \delta \sup_{0\leq s\leq t}\|\partial_v \tilde{\mathbf{h}}(s)\|_{L^\infty} + \kappa^{\frac{1}{2}} \sup_{0\leq s\leq t}\|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} \\
&+ \int_0^t e^{-\lambda_1 s}\|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty}\mathrm{d}s + \sum_{i=1}^4 I_i,
\end{aligned} \tag{6.6}$$

where,

$$\begin{aligned}
I_1 :=& \int_0^t e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\mathrm{d}s \int_0^s e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1}\mathrm{d}s_1 \\
&\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathtt{k}^{(2)}_{w_{\beta_1}}(V_{+}(s),u)\cdot \mathtt{k}^{(2)}_{w_{\beta_1}}(\hat{V}_{+}(s_1),u_1)\cdot \partial_x \tilde{h}_{+}(s_1,\hat{X}_{+}(s_1),u_1)du_1\mathrm{d}u, \\
I_2 :=& \int_0^t e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\mathrm{d}s \int_0^s e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1}\mathrm{d}s_1 \\
&\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathtt{k}^{(2)}_{w_{\beta_1}}(V_{+}(s),u)\cdot \mathtt{k}^{(1)}_{w_{\beta_1}}(\hat{V}_{+}(s_1),u_1)\cdot \partial_x \tilde{h}_{-}(s_1,\hat{X}_{+}(s_1),u_1)du_1\mathrm{d}u, \\
I_3 :=& \int_0^t e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\mathrm{d}s \int_0^s e^{-\int_{s_1}^s \hat{\nu}_{-,1}(\tau_1)\mathrm{d}\tau_1}\mathrm{d}s_1 \\
&\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathtt{k}^{(1)}_{w_{\beta_1}}(V_{+}(s),u)\cdot \mathtt{k}^{(2)}_{w_{\beta_1}}(\hat{V}_{-}(s_1),u_1)\cdot \partial_x \tilde{h}_{-}(s_1,\hat{X}_{-}(s_1),u_1)du_1\mathrm{d}u, \\
I_4 :=& \int_0^t e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\mathrm{d}s \int_0^s e^{-\int_{s_1}^s \hat{\nu}_{-,1}(\tau_1)\mathrm{d}\tau_1}\mathrm{d}s_1 \\
&\times \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \mathtt{k}^{(1)}_{w_{\beta_1}}(V_{+}(s),u)\cdot \mathtt{k}^{(1)}_{w_{\beta_1}}(\hat{V}_{-}(s_1),u_1)\cdot \partial_x \tilde{h}_{+}(s_1,\hat{X}_{-}(s_1),u_1)du_1\mathrm{d}u.
\end{aligned}$$

We discuss I_1 of (6.6) and split it into four cases.

Case 1: $|v| \geq N$. By similar argument as in (4.14),

$$I_1 \leq \frac{C}{N}\sup_{0\leq s\leq t}\|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty}. \tag{6.7}$$

Case 2: For either $|v| \le N, |u| \ge 2N$ or $|u| \le 2N, |u_1| \ge 3N$. By similar argument as in (4.17),

$$I_1 \le Ce^{-\frac{N^2}{64}} \sup_{0\le s\le t} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty}. \tag{6.8}$$

Case 3: $|v| \le N, |u| \le 2N, |u_1| \le 3N, s - \frac{1}{N} \le s_1 \le s$. By similar argument as in (4.18),

$$I_1 \le \frac{C}{N} \sup_{0\le s\le t} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty}. \tag{6.9}$$

Case 4: $|v| \le N, |u| \le 2N, |u_1| \le 3N, 0 \le s_1 \le s - \frac{1}{N}$.

$$\begin{aligned} I_1 \cong &\int_0^t \int_0^{s-\frac{1}{N}} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau} e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \mathrm{d}s_1 ds \\ &\times \iint_B \mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),u) \cdot \mathtt{k}^{(2)}_{w_{\beta_1}}(\hat{V}_+(s_1),u_1) \cdot \partial_x \tilde{h}_+(s_1, \hat{X}_+(s_1), u_1) du_1 du, \end{aligned} \tag{6.10}$$

where $B = \{(u,u_1): \ |u| \le 2N, \ |u_1| \le 3N\}$. From (2.29) $-$ (2.31), $\mathtt{k}^{(2)}_{w_{\beta_1}}(v,u)$ has integrable singularity of $\frac{1}{|v-u|}$, we can choose $\mathtt{k}^{(2)}_N(v,u)$ smooth with compact support such that

$$\sup_{|v|\le 3N} \int_{|u|\le 3N} |\mathtt{k}^{(2)}_N(v,u) - \mathtt{k}^{(2)}_{w_{\beta_1}}(v,u)| \mathrm{d}u \le \frac{1}{N}, \quad |\mathtt{k}^{(2)}_N(v,u)| + |\partial_{v,u}\mathtt{k}^{(2)}_N(v,u)| \le C_N.$$

Noting that

$$\begin{aligned} &\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),u) \cdot \mathtt{k}^{(2)}_{w_{\beta_1}}(\hat{V}_+(s_1),u_1) = \Big(\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),u) - \mathtt{k}^{(2)}_N(V_+(s),u)\Big) \cdot \mathtt{k}^{(2)}_{w_{\beta_1}}(\hat{V}_+(s_1),u_1) \\ &+ \Big(\mathtt{k}^{(2)}_{w_{\beta_1}}(\hat{V}_+(s_1),u_1) - \mathtt{k}^{(2)}_N(\hat{V}_+(s_1),u_1)\Big) \cdot \mathtt{k}^{(2)}_N(V_+(s),u) + \mathtt{k}^{(2)}_N(V_+(s),u) \cdot \mathtt{k}^{(2)}_N(\hat{V}_+(s_1),u_1), \end{aligned}$$

we can estimate I_1 by

$$\begin{aligned} I_1 \le &\frac{C}{N} \sup_{0\le s\le t} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} + \int_0^t \int_0^{s-\frac{1}{N}} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau} e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \mathrm{d}s_1 ds \\ &\times \iint_B \mathtt{k}^{(2)}_N(V_+(s),u)\mathtt{k}^{(2)}_N(\hat{V}_+(s_1),u_1) \cdot \partial_x \tilde{h}_+(s_1, \hat{X}_+(s_1), u_1) du_1 du =: I_1^*. \end{aligned} \tag{6.11}$$

Consider a change of variable:

$$y = \hat{X}_+(s_1) = X_+(s_1; s, X_+(s; t,x,v), u) \tag{6.12}$$

such that

$$\begin{aligned} I_1^* \cong &\int_0^t \int_0^{s-\frac{1}{N}} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau} e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \mathrm{d}s_1 ds \\ &\times \iint_{\hat{B}} \mathtt{k}^{(2)}_N(V_+(s),u)\mathtt{k}^{(2)}_N(\hat{V}_+(s_1),u_1) \cdot \partial_y \tilde{h}_+(s_1, y, u_1) |det(\frac{\partial u}{\partial y})| du_1 dy, \end{aligned} \tag{6.13}$$

where $\hat{B}$ is the image of B under the transformation $(u,u_1) \to (y,u_1)$. In addition, it is clear that $\hat{B} \subseteq \{(y,u_1): \ |y - X_+(s)| \le C(s-s_1)N, \ |u_1| \le 3N\}$. After integrating by parts, one has

$$\begin{aligned} I_1^* = &\int_0^t \int_0^{s-\frac{1}{N}} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau} \partial_y\{e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1}\} \mathrm{d}s_1 ds \\ &\times \iint_{\hat{B}} \mathtt{k}^{(2)}_N(V_+(s),u)\mathtt{k}^{(2)}_N(\hat{V}_+(s_1),u_1) \cdot \tilde{h}_+(s_1, y, u_1) |det(\frac{\partial u}{\partial y})| du_1 dy \\ &+ \int_0^t \int_0^{s-\frac{1}{N}} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau} e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \mathrm{d}s_1 ds \end{aligned}$$

$$
\begin{aligned}
&\times \iint_{\hat{B}} \partial_y \{\mathtt{k}_N^{(2)}(V_+(s),u)\mathtt{k}_N^{(2)}(\hat{V}_+(s_1),u_1)\} \cdot \tilde{h}_+(s_1,y,u_1)|det(\frac{\partial u}{\partial y})|du_1 dy \\
&+ \int_0^t \int_0^{s-\frac{1}{N}} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau} e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \mathrm{d}s_1 ds \\
&\times \iint_{\hat{B}} \mathtt{k}_N^{(2)}(V_+(s),u)\mathtt{k}_N^{(2)}(\hat{V}_+(s_1),u_1) \cdot \tilde{h}_+(s_1,y,u_1)\partial_y\{|det(\frac{\partial u}{\partial y})|\}du_1 dy \\
&+ \int_0^t \int_0^{s-\frac{1}{N}} e^{-\int_s^t \tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau} e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \mathrm{d}s_1 ds \\
&\times \iint_{\partial_y \hat{B}} \mathtt{k}_N^{(2)}(V_+(s),u)\mathtt{k}_N^{(2)}(\hat{V}_+(s_1),u_1) \cdot \tilde{h}_+(s_1,y,u_1)|det(\frac{\partial u}{\partial y})|du_1 dS_y \\
=:\ & I_{11}^* + I_{12}^* + I_{13}^* + I_B.
\end{aligned}
\tag{6.14}
$$

For the boundary term I_B, it follows from Corollary 2.7 and Proposition 4.3 that

$$
I_B \le C_N \sup_{0\le s\le t} \|h(s)\|_{L^\infty} \le C_N. \tag{6.15}
$$

To estimate I_{11}^*, we notice that

$$
\partial_y \left(e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \right) = -e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \cdot \int_{s_1}^s \partial_y \left(\hat{\nu}_{+,1}(\tau_1) \right) \mathrm{d}\tau_1, \tag{6.16}
$$

and

$$
\partial_y \left(\tilde{\nu}_{+,1}(\tau_1, \hat{X}_+(\tau_1), \hat{V}_+(\tau_1)) \right) = \partial_x \hat{\nu}_{+,1}(\tau_1) \cdot \frac{\partial \hat{X}_+(\tau_1)}{\partial u} \cdot \frac{\partial u}{\partial y} + \partial_v \hat{\nu}_{+,1}(\tau_1) \cdot \frac{\partial \hat{V}_+(\tau_1)}{\partial u} \cdot \frac{\partial u}{\partial y}. \tag{6.17}
$$

It is clear that

$$
|\partial_x \hat{\nu}_{+,1}(\tau_1)| + |\partial_v \hat{\nu}_{+,1}(\tau_1)| \le C(1+|\hat{V}_+(\tau_1)|) \le C(1+|u|) \le C_N. \tag{6.18}
$$

It follows from (2.20) that

$$
\Big|\frac{\partial \hat{X}_+(\tau_1)}{\partial u}\Big| \le C(s-s_1), \quad \Big|\frac{\partial \hat{V}_+(\tau_1)}{\partial u}\Big| \le C\delta(s-s_1)+1, \quad \Big|\frac{\partial u}{\partial y}\Big| \le C(s-s_1)^{-1}. \tag{6.19}
$$

Combining (6.17) $-$ (6.19), one gets

$$
\Big|\partial_y \left(\tilde{\nu}_{+,1}(\tau_1, \hat{X}_+(\tau_1), \hat{V}_+(\tau_1)) \right)\Big| \le C_N \left(1+(s-s_1)^{-1}\right), \tag{6.20}
$$

Which, together with (6.16), yields

$$
\Big|\partial_y \left(e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \right)\Big| \le C_N e^{-\int_{s_1}^s \hat{\nu}_{+,1}(\tau_1)\mathrm{d}\tau_1} \left(1+(s-s_1)\right). \tag{6.21}
$$

Substituting (6.21) into (6.14), following the same argument in L^∞ bound, one can deduce that

$$
\begin{aligned}
I_{11}^* &\le C_N \int_0^t \int_0^{s-\frac{1}{N}} e^{-\tilde{\nu}_0(t-s)} e^{-\tilde{\nu}_0(s-s_1)}(1+(s-s_1))\mathrm{d}s_1\mathrm{d}s \Big(\iint_{\hat{B}} |f_+(s_1,y,u_1)|^2 dy du_1 \Big)^{\frac{1}{2}} \\
&\le C_N \Big(\|h_+(\tau)\|_{L^\infty}^{\frac{1}{2}} \sqrt{\mathcal{E}(\mathbf{F}_0)} + \sqrt{\mathcal{E}(\mathbf{F}_0)} \Big).
\end{aligned}
\tag{6.22}
$$

For I_{12}^*, a direct calculation shows that

$$
\begin{aligned}
\partial_y \left(\mathtt{k}_N^{(2)}(V_+(s),u) \cdot \mathtt{k}_N^{(2)}(\hat{V}_+(s_1),u_1) \right) =& (\partial_u \mathtt{k}_N^{(2)})(V_+(s),u) \cdot \frac{\partial u}{\partial y} \cdot \mathtt{k}_N^{(2)}(\hat{V}_+(s_1),u_1) + \mathtt{k}_N^{(2)}(V_+(s),u) \\
& \times (\partial_v \mathtt{k}_N^{(2)})(\hat{V}_+(s_1),u_1) \cdot \frac{\partial \hat{V}_+(s_1)}{\partial u} \cdot \frac{\partial u}{\partial y},
\end{aligned}
\tag{6.23}
$$

which, together with (2.20), yields that

$$\left|\partial_y\left(\mathtt{k}_N^{(2)}(V_+(s),u)\mathtt{k}_N^{(2)}(\hat{V}_+(s_1),u_1)\right)\right| \le C_N\left(1+(s-s_1)^{-1}\right). \tag{6.24}$$

Combining (6.24) and (6.14), by similar arguments as in (6.22), one can obtain that

$$I_{12}^* \le C_N\Big(\|h_+(\tau)\|_{L^\infty}^{\frac12}\sqrt{\mathcal{E}(\mathbf{F}_0)}+\sqrt{\mathcal{E}(\mathbf{F}_0)}\Big). \tag{6.25}$$

For I_{13}^*, we denote $z = X_+(s;t,x,v)$. Recalling $y = \hat{X}(s_1) = X(s_1;s,X_+(s;t,x,v),u) = X(s_1;s,z,u)$, then

$$\begin{aligned} I_{13}^* \le{}& C_N\int_0^t\int_0^{s-\frac1N} e^{-\tilde{\nu}_0(t-s)}e^{-\tilde{\nu}_0(s-s_1)}\mathrm{d}s_1 ds \\ &\times\Big(\iint_{\hat{B}}|f_+(s_1,y,u_1)|^2dydu_1\Big)^{\frac12}\cdot\Big(\iint_{\hat{B}}|\partial_y\Big(|\det(\frac{\partial u}{\partial y})|\Big)|^2dydu_1\Big)^{\frac12}. \end{aligned} \tag{6.26}$$

A direct calculation shows that

$$\partial_y\Big(\det\Big(\frac{\partial u}{\partial y}\Big)\Big) = \partial_y\Big(\frac{1}{\det\Big(\frac{\partial y}{\partial u}\Big)}\Big) = -\frac{1}{\Big(\det\Big(\frac{\partial y}{\partial u}\Big)\Big)^2}\partial_u\Big(\det\Big(\frac{\partial y}{\partial u}\Big)\Big)\cdot\frac{\partial u}{\partial y}, \tag{6.27}$$

and

$$\left|\partial_u\left(\det\left(\frac{\partial y}{\partial u}\right)\right)\right| \le \left|\frac{\partial\hat{X}_+(s_1)}{\partial u}\right|^2\left|\partial_{uu}^2\hat{X}_+(s_1)\right| \le C(s-s_1)^2\left|\partial_{uu}^2X(s_1;s,z,u)\right|. \tag{6.28}$$

Then it follows from (2.20), (6.27) $-$ (6.28) that

$$\Big|\partial_y\Big(\det\Big(\frac{\partial u}{\partial y}\Big)\Big)\Big| \le C(s-s_1)\frac{1}{\Big(\det\Big(\frac{\partial y}{\partial u}\Big)\Big)^2}\Big|\partial_{uu}^2X(s_1;s,z,u)\Big|. \tag{6.29}$$

Substitute (6.29) into (6.26), one has

$$\begin{aligned} &\Big(\iint_{\hat{B}}|\partial_y\Big(|det(\frac{\partial u}{\partial y})|\Big)|^2\mathrm{d}y\mathrm{d}u_1\Big)^{\frac12} \\ \le{}& C(s-s_1)\Big(\iint_{\hat{B}}\frac{1}{\Big(\det\Big(\frac{\partial y}{\partial u}\Big)\Big)^4}|\partial_{uu}^2X(s_1;s,z,u)|^2\mathrm{d}y\mathrm{d}u_1\Big)^{\frac12} \\ \le{}& C(s-s_1)\Big(\iint_{B}\frac{1}{\Big(\det\Big(\frac{\partial y}{\partial u}\Big)\Big)^4}|\partial_{uu}^2X(s_1;s,z,u)|^2|\det(\frac{\partial y}{\partial u})|\mathrm{d}u\mathrm{d}u_1\Big)^{\frac12} \\ \le{}& C_N(s-s_1)^{-\frac72}\Big(\int_{|u|\le 2N}|\partial_{uu}^2X(s_1;s,z,u)|^2\mathrm{d}u\Big)^{\frac12}, \end{aligned} \tag{6.30}$$

where we have used Corollary 2.7. Noting Lemma 2.8, we have from (6.30) that

$$\Big(\iint_{\hat{B}}|\partial_y\Big(|det(\frac{\partial u}{\partial y})|\Big)|^2\mathrm{d}y\mathrm{d}u_1\Big)^{\frac12} \le C_N\sup_{0\le s\le t}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}. \tag{6.31}$$

Following the same argument as in (4.21), together with (6.31) and (6.26), one has

$$\begin{aligned} I_{13}^* &\le C_N\Big(\sup_{0\le s\le t}\|h(s)\|_{L^\infty}^{\frac12}\sqrt{\mathcal{E}(\mathbf{F}_0)}+\sqrt{\mathcal{E}(\mathbf{F}_0)}\Big)\sup_{0\le s\le t}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty} \\ &\le \frac1N\sup_{0\le s\le t}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}, \end{aligned} \tag{6.32}$$

where $\mathcal{E}(\mathbf{F}_0) \le \varepsilon_1$ with ε_1 sufficiently small. Then it follows from (6.15), (6.22), (6.25) and (6.32) that

$$I_1^* \le \frac{C}{N} \sup_{0\le s\le t} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} + C_N. \tag{6.33}$$

Combining (6.7) − (6.9), (6.11) and (6.33), we have

$$I_1 \le \frac{C}{N} \sup_{0\le s\le t} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} + C_N. \tag{6.34}$$

By similar arguments as in (6.7)-(6.34), one has for $i = 2, 3, 4, 5$

$$I_i \le \frac{C}{N} \sup_{0\le s\le t} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} + C_N. \tag{6.35}$$

Plugging (6.34) and (6.35) into (6.6), one obtains that

$$\begin{aligned}\|\partial_x \tilde{h}_+(t)\|_{L^\infty} \le C(1 + \|\partial_{x,v}\tilde{\mathbf{h}}_0)\|_{L^\infty})^2 + C_N + (\frac{C}{N} + \kappa^{\frac{1}{2}}) \sup_{0\le s\le t} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} \\ + \delta \sup_{0\le s\le t} \|\partial_v \tilde{\mathbf{h}}(s)\|_{L^\infty} + \int_0^t e^{-\lambda_1 s} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} \mathrm{d}s.\end{aligned} \tag{6.36}$$

Similarly, $h_-(t, x, v)$ has the same estimate as (6.36). Thus,

$$\begin{aligned}\|\partial_x \tilde{\mathbf{h}}(t)\|_{L^\infty} \le C(1 + \|\partial_{x,v}\tilde{\mathbf{h}}_0)\|_{L^\infty})^2 + C_N + (\frac{C}{N} + \kappa^{\frac{1}{2}}) \sup_{0\le s\le t} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} \\ + \delta \sup_{0\le s\le t} \|\partial_v \tilde{\mathbf{h}}(s)\|_{L^\infty} + \int_0^t e^{-\lambda_1 s} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} \mathrm{d}s.\end{aligned} \tag{6.37}$$

Then Lemma 6.1 follows from Gronwall's inequality and $N \gg 1$, $\kappa \ll 1$. Therefore the proof of Lemma 6.1 is completed.

□

6.2. **Proof of Lemma 6.2.** Recall $\partial^j = \frac{\partial}{\partial v_j}$, applying ∂^j to (4.29), we obtain

$$\begin{aligned}&(\partial_t + v\cdot\nabla_x \mp \nabla_x\phi\cdot\nabla_v)\partial^j \tilde{h}_\pm + \tilde{\nu}_{\pm,1}\partial^j \tilde{h}_\pm \\ &= -\partial_j \tilde{h}_\pm - \partial^j \tilde{\nu}_{\pm,1}\tilde{h}_\pm \mp \nabla_x\phi\cdot\partial^j(v w_{\beta_1}\sqrt{\mu}) + \partial^j(w_{\beta_1}\Gamma^\pm(\mathbf{f},\mathbf{f})) + \partial^j(w_{\beta_1}K_\pm f).\end{aligned} \tag{6.38}$$

Integrating (6.38) along the characteristic, one has

$$\begin{aligned}\partial^j \tilde{h}_\pm(t,x,v) = {} & \partial^j \tilde{h}_{\pm 0}(X_\pm(0), V_\pm(0)) e^{-\int_0^t \tilde{\nu}_{\pm,1}(\tau, X_\pm(\tau), V_\pm(\tau))\mathrm{d}\tau} \\ & - \int_0^t e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau, X_\pm(\tau), V_\pm(\tau))\mathrm{d}\tau} \left(\partial_j \tilde{h}_\pm\right)(s, X_\pm(s), V_\pm(s))\mathrm{d}s \\ & - \int_0^t e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau, X_\pm(\tau), V_\pm(\tau))\mathrm{d}\tau} \left(\partial^j \tilde{\nu}_{\pm,1}\tilde{h}_\pm\right)(s, X_\pm(s), V_\pm(s))\mathrm{d}s \\ & \mp \int_0^t e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau, X_\pm(\tau), V_\pm(\tau))\mathrm{d}\tau} \left(\nabla_x\phi\cdot\partial^j(v w_{\beta_1}\sqrt{\mu})\right)(s, X_\pm(s), V_\pm(s))\mathrm{d}s \\ & + \int_0^t e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau, X_\pm(\tau), V_\pm(\tau))\mathrm{d}\tau} \left(\partial^j(w_{\beta_1}\Gamma^\pm(\mathbf{f},\mathbf{f}))\right)(s, X_\pm(s), V_\pm(s))\mathrm{d}s \\ & + \int_0^t e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau, X_\pm(\tau), V_\pm(\tau))\mathrm{d}\tau} \left(\partial^j(w_{\beta_1}\mathrm{K}^\pm\mathbf{f})\right)(s, X_\pm(s), V_\pm(s))\mathrm{d}s = \sum_{i=0}^5 J_i.\end{aligned} \tag{6.39}$$

For the nonlinear collision term J_4, using (2.57) and similar arguments as in (6.3), one gets that

$$\begin{aligned}&\int_0^t e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau,X_\pm(\tau),V_\pm(\tau))\mathrm{d}\tau}\left|\left(\partial^j(w_{\beta_1}\Gamma^\pm(\mathbf{f},\mathbf{f}))\right)(s,X_\pm(s),V_\pm(s))\right|\mathrm{d}s\\ \le& C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0)\|_{L^\infty})^2+\kappa^{\frac{1}{2}}\sup_{0\le s\le t}\|\partial_v\tilde{\mathbf{h}}(s)\|_{L^\infty}+\int_0^t e^{-\lambda_1 s}\|\partial_v\tilde{\mathbf{h}}(s)\|_{L^\infty}\mathrm{d}s. \end{aligned}\tag{6.40}$$

Next, we deal with the remaining terms on RHS of (6.39). It is easy to see that

$$|\partial_{v_j}\tilde{\nu}_{\pm,1}(t,x,v)|\le C(1+|v|),\tag{6.41}$$

which, together with (5.68) and (6.40), yields that

$$\begin{aligned}\sum_{i=0}^4 J_i\le& C(\|\partial_v\tilde{\mathbf{h}}_0\|_{L^\infty}+1)^2+\sup_s\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}+\kappa^{\frac{1}{2}}\sup_{0\le s\le t}\|\partial_v\tilde{\mathbf{h}}(s)\|_{L^\infty}\\ &+\int_0^t e^{-\lambda_1 s}\|\partial_v\tilde{\mathbf{h}}(s)\|_{L^\infty}\mathrm{d}s.\end{aligned}\tag{6.42}$$

For J_5, we only consider $\mathrm{K}^+\mathbf{f}$, because $\mathrm{K}^-\mathbf{f}$ is similar. Using (2.34), it is easy to have

$$\begin{aligned}J_5=&\int_0^t e^{-\int_s^t\tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\left(\partial_{v_j}w_{\beta_1}\cdot\mathrm{K}^+\mathbf{f}\right)(s,X_+(s),V_+(s))\mathrm{d}s\\ &+\int_0^t e^{-\int_s^t\tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int\tilde{\mathtt{k}}^{(2)}_{w_{\beta_1}}(V_+(s),u)\tilde{h}_+(s,X_+(s),u)du\mathrm{d}s\\ &+\int_0^t e^{-\int_s^t\tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int\tilde{\mathtt{k}}^{(1)}_{w_{\beta_1}}(V_+(s),u)\tilde{h}_-(s,X_+(s),u)du\mathrm{d}s\\ &+\int_0^t e^{-\int_s^t\tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),u)\cdot(w_{\beta_1}\partial_{v_j}f_+)(s,X_+(s),u)du\mathrm{d}s\\ &+\int_0^t e^{-\int_s^t\tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int\mathtt{k}^{(1)}_{w_{\beta_1}}(V_+(s),u)\cdot(w_{\beta_1}\partial_{v_j}f_-)(s,X_+(s),u)du\mathrm{d}s\\ =:&J_{51}+J_{52}^++J_{52}^-+J_{53}^++J_{53}^-.\end{aligned}\tag{6.43}$$

It follows from (2.36) that

$$J_{51}+J_{52}^++J_{52}^-\le\int_0^t e^{-\nu_0(t-s)}\|\mathbf{h}(s)\|_{L^\infty}\mathrm{d}s\le C.\tag{6.44}$$

We only need to deal with J_{53}^+ since J_{53}^- can be dealt in the same way. It is clear that

$$\begin{aligned}J_{53}^+\le&\int_0^t e^{-\tilde{\nu_0}(t-s)}\int\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),u)\cdot\left((\partial_{v_j}w_{\beta_1})f_+\right)(s,X_+(s),u)du\mathrm{d}s\\ &+\int_0^t e^{-\int_s^t\tilde{\nu}_{+,1}(\tau)\mathrm{d}\tau}\int\mathtt{k}^{(2)}_{w_{\beta_1}}(V_+(s),u)\cdot\partial_{v_j}\tilde{h}_+(s,X_+(s),u)\mathrm{d}u\mathrm{d}s\\ \le& C+\bar{J}_{53}^+.\end{aligned}\tag{6.45}$$

To estimate $\bar{J}_{53}^+$, performing the same procedure as in (6.5) $-$ (6.11), one has

$$\begin{aligned}\bar{J}_{53}^+\le& C(\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty}+1)^2+\sup_s\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}+\left(\frac{C}{N}+\kappa^{\frac{1}{2}}\right)\sup_{0\le s\le t}\|\partial_v\tilde{\mathbf{h}}(s)\|_{L^\infty}\\ &+\int_0^t e^{-\lambda_1 s}\|\partial_v\tilde{\mathbf{h}}(s)\|_{L^\infty}\mathrm{d}s+(**),\end{aligned}\tag{6.46}$$

where

$$
\begin{aligned}
(**) \cong & \int_0^t \int_0^{s-\frac{1}{N}} e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau)\mathrm{d}\tau} e^{-\int_{s_1}^s \hat{\nu}_{\pm,1}(\tau_1))\mathrm{d}\tau_1} \mathrm{d}s_1 \mathrm{d}s \\
& \times \iint_B \mathtt{k}_N^{(2)}(V_+(s),u) \cdot \mathtt{k}_N^{(2)}(\hat{V}_+(s_1),u_1) \partial_{v_j} \tilde{h}_+(s_1,\hat{X}_+(s_1),u_1)\mathrm{d}u_1\mathrm{d}u \\
= & \int_0^t \int_0^{s-\frac{1}{N}} e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau)\mathrm{d}\tau} e^{-\int_{s_1}^s \hat{\nu}_{\pm,1}(\tau_1))\mathrm{d}\tau_1} \mathrm{d}s_1 \mathrm{d}s \\
& \times \iint_B \mathtt{k}_N^{(2)}(V_+(s),u) \cdot \mathtt{k}_N^{(2)}(\hat{V}_+(s_1),u_1) \partial_{u_{1,j}} \tilde{h}_+(s_1,\hat{X}_+(s_1),u_1)\mathrm{d}u_1\mathrm{d}u,
\end{aligned}
$$

here $u_{1,j}$ is the j-th component of u_1. Denote $y = X_+(s_1, s, X_+(s,t,x,v),u)$, after integrating by parts, and noting the boundary term contribution can be controlled by $C_N \sup_{0\le s\le t} \|\mathbf{h}(s)\|_{L^\infty}$, we have

$$
\begin{aligned}
(**) \le & \int_0^t \int_0^{s-\frac{1}{N}} \int_{\hat{B}} e^{-\int_s^t \tilde{\nu}_{\pm,1}(\tau)\mathrm{d}\tau} e^{-\int_{s_1}^s \hat{\nu}_{\pm,1}(\tau_1)\mathrm{d}\tau_1} \mathtt{k}_N^{(2)}(V_+(s),u) \partial_{u_{1,j}} \mathtt{k}_N^{(2)}(\hat{V}_+(s_1),u_1) \\
& \times \tilde{h}_+(s_1,y,u_1) \left| \det\left(\frac{\partial u}{\partial y}\right) \right| dy du_1 + C_N \sup_{0\le s\le t} \|\mathbf{h}(s)\|_{L^\infty} \\
\le & C_N \Big(\sup_{0\le s\le t} \|\mathbf{h}(s)\|_{L^\infty}^{1/2} \sqrt{\mathcal{E}(\mathbf{F}_0)} + \sqrt{\mathcal{E}(\mathbf{F}_0)} \Big) + C_N \le C_N.
\end{aligned}
\tag{6.47}
$$

It is a consequence of (6.39), (6.42) and (6.43)–(6.47) that

$$
\begin{aligned}
\|\partial_v \tilde{\mathbf{h}}(t)\|_{L^\infty} \le & C(1+\|\partial_v \tilde{\mathbf{h}}_0\|_{L^\infty})^2 + C_N + \sup_{0\le s\le t} \|\partial_x \tilde{\mathbf{h}}(s)\|_{L^\infty} + \left(\frac{1}{N} + \kappa^{\frac{1}{2}}\right) \sup_{0\le s\le t} \|\partial_v \tilde{\mathbf{h}}(s)\|_{L^\infty} \\
& + \int_0^t e^{-\lambda_1 s} \|\partial_v \tilde{\mathbf{h}}(s)\|_{L^\infty} \mathrm{d}s.
\end{aligned}
$$

Taking $N \gg 1$, and noting $\kappa \ll 1$, Lemma 6.2 follows from Gronwall's inequality. Therefore the proof of Lemma 6.2 is completed. □

7. Proof of Theorem 1.1

Step 1. Noting Propositions 5.3 and 6.3, we only need to close (1.25) − (1.24) and the *a priori* assumption (2.18).

It follows from (3.49) − (3.54) that for any $t \ge 0$,

$$
\begin{aligned}
\|(f_+ - f_-)(t)\|_{L^\infty} \le & (1 + \sup_{0\le s\le t} \|\mathbf{h}(s)\|_{L^\infty} + \sup_{0\le s\le t} \|\partial_v \tilde{\mathbf{h}}(s)\|_{L^\infty}) \int_0^t \|(f_+ - f_-)(s)\|_{L^\infty} \mathrm{d}s \\
& + \|(f_+ - f_-)(0)\|_{L^\infty}.
\end{aligned}
\tag{7.1}
$$

Then it is a consequence of Propositions 4.3 and 6.3 that for $t \ge 0$

$$
\|(f_+ - f_-)(t)\|_{L^\infty} \le \varepsilon_0 + C(1 + \|\partial_{x,v} \tilde{\mathbf{h}}_0\|_{L^\infty})^2 \int_0^t \|(f_+ - f_-)(s)\|_{L^\infty} \mathrm{d}s, \tag{7.2}
$$

which, together with Gronwall's inequality, yields that

$$
\|(f_+ - f_-)(t)\|_{L^\infty} \le \varepsilon_0 \exp\left\{ C(1 + \|\partial_{x,v} \tilde{\mathbf{h}}_0\|_{L^\infty})^2 t \right\}, \tag{7.3}
$$

where the positive constant $C > 0$ depends on $\|\mathbf{h}_0\|_{L^\infty}$. Combining (2.10) and (7.3), one has

$$
\|\nabla_x \phi(t)\|_{L^\infty} \le C\|(f_+ - f_-)(t)\|_{L^\infty} \le C\varepsilon_0 \exp\left\{ C(1 + \|\partial_{x,v} \tilde{\mathbf{h}}_0\|_{L^\infty})^2 t \right\}. \tag{7.4}
$$

On the other hand, it is follows from (5.68) that

$$\|\nabla_x\phi(t)\|_{L^\infty} \le C\|\mathbf{h}(t)\|_{L^\infty} \le Ce^{-\lambda_1 t}. \tag{7.5}$$

Thus we have from (7.4) − (7.5) that

$$\|\nabla_x\phi(t)\|_{L^\infty} \lesssim \min\Big\{\varepsilon_0 \exp\{C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 t\}, e^{-\lambda_1 t}\Big\}. \tag{7.6}$$

Taking $d=\delta^2 e^{-\lambda_1 t}(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^{-2}$ and $R=\frac14$ in Lemma 2.4, and using Proposition 6.3, one has

$$\begin{aligned}\|\nabla_x^2\phi(t)\|_{L^\infty} &\le C\|(f_+-f_-)(t)\|_{L^\infty}\Big(1+\ln\frac{R}{d}+R^{-3}\Big)+\|\partial_x\tilde{\mathbf{h}}(t)\|_{L^\infty}d\\ &\lesssim \delta^2 e^{-\lambda_1 t}+\|(f_+-f_-)(t)\|_{L^\infty}\Big(1+2\ln\Big(\frac{1}{\delta}\Big)+\lambda_1 t+\ln(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})\Big),\end{aligned} \tag{7.7}$$

which, together with (7.3), yields that

$$\|\nabla_x^2\phi(t)\|_{L^\infty} \lesssim \delta^2 e^{-\lambda_1 t}+\varepsilon_0\mathfrak{H}(t), \tag{7.8}$$

where $\mathfrak{H}(t):=\Big(1+2\ln\big(\frac{1}{\delta}\big)+\lambda_1 t+\ln(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})\Big)\exp\Big\{C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 t\Big\}$.

On the other side, it follows from (7.7) and Proposition 5.3 that

$$\|\nabla_x^2\phi(t)\|_{L^\infty} \lesssim \delta^2 e^{-\lambda_1 t}+e^{-\lambda_2 t}\Big(1+2\ln\Big(\frac{1}{\delta}\Big)+\ln(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})\Big),\ \text{where } 0<\lambda_2\le\frac{\lambda_1}{2}. \tag{7.9}$$

Noting (7.5) − (7.9), we choose ε_0 sufficiently small such that

$$\min\Big\{\varepsilon_0\exp\{C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 t\}, e^{-\lambda_1 t}\Big\} \le \delta^2(1+t)^{-2}, \tag{7.10}$$

and

$$\min\Big\{\varepsilon_0\mathfrak{H}(t), e^{-\lambda_2 t}\Big(1+2\ln\Big(\frac{1}{\delta}\Big)+\ln(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})\Big)\Big\} \le \delta^2(1+t)^{-\frac52}. \tag{7.11}$$

Thus we conclude the *a priori* assumption (2.18) from (7.6) − (7.11).

Step 2. Recall (6.2), Proposition 6.3 and (7.9), it is easy to have

$$\begin{aligned}H_1+H_3 \le& C\int_0^t e^{-\tilde{\nu}_0(t-s)}\|\nabla_x^2\phi(s)\|_{L^\infty}(1+\|\partial_v\tilde{\mathbf{h}}(s)\|_{L^\infty})\mathrm{d}s\\ \le& C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2\Big(\ln\Big(\frac{1}{\delta}\Big)+\ln(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})\Big)\int_0^t e^{-\tilde{\nu}_0(t-s)}e^{-\lambda_2 s}\mathrm{d}s\\ \le& C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2\Big(\ln\Big(\frac{1}{\delta}\Big)+\ln(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})\Big)e^{-\lambda_2 t}.\end{aligned} \tag{7.12}$$

In addition, it is not difficult to verify

$$H_2+H_4 \le C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2 e^{-\lambda_1 t}. \tag{7.13}$$

It remains to H_5. We shall adopt the same procedure as in (6.5) − (6.11). Indeed, in view of $\int_0^t e^{-\tilde{\nu}_0(t-s)}e^{-\lambda_2 s}\mathrm{d}s \le Ce^{-\lambda_2 t}$, all of the above cases in Section 6.1 except the last one can be bounded by

$$\frac{C}{N}e^{-\lambda_2 t}\sup_{0\le s\le t}\{e^{\lambda_2 s}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}\}. \tag{7.14}$$

For the last case, similarly as for treating (6.10)–(6.32), it follows from integration by part that that

$$I_1^* \le \frac{C}{N}e^{-\lambda_2 t}\sup_{0\le s\le t}\{e^{\lambda_2 s}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}\}+C_N e^{-\lambda_1 t}$$

$$
\begin{aligned}
&+C_N\int_0^t\int_0^{s-\frac{1}{N}} e^{-\tilde{\nu}_0(t-s)}e^{-\tilde{\nu}_0(s-s_1)}\|\mathbf{f}(s_1)\|_{L^2}^{\frac{1}{2}}\mathrm{d}s_1\mathrm{d}s\cdot\Big(\iint_{\hat{B}}|f_+(s_1,y,u_1)|^2dydu_1\Big)^{\frac{1}{4}}\\
&+C_N\int_0^t\int_0^{s-\frac{1}{N}} e^{-\tilde{\nu}_0(t-s)}e^{-\tilde{\nu}_0(s-s_1)}\|\mathbf{f}(s_1)\|_{L^2}^{\frac{1}{2}}\mathrm{d}s_1ds\cdot\Big(\iint_{\hat{B}}|f_+(s_1,y,u_1)|^2dydu_1\Big)^{\frac{1}{4}}\\
&\quad\times\Big(\iint_{\hat{B}}|\partial_y\Big(|\det(\frac{\partial u}{\partial y})|\Big)|^2dydu_1\Big)^{\frac{1}{2}}\\
\leq&\frac{C}{N}e^{-\lambda_2 t}\sup_{0\leq s\leq t}\{e^{\lambda_2 s}\|\partial_x\tilde{\mathbf{h}}(s)\|_{L^\infty}\}+C_Ne^{-\lambda_1 t}+C_N\mathcal{E}^{\frac{1}{4}}(\mathbf{F}_0)(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2e^{-\frac{\lambda}{4}t}. \qquad (7.15)
\end{aligned}
$$

Noting $\lambda_2\leq\frac{\lambda_1}{2}\leq\frac{\lambda}{4}$, using (7.12) $-$ (7.15), one can choose $N\geq 1$ large enough, then let ε_1 suitably small, thus we obtain

$$
\|\partial_x\tilde{\mathbf{h}}(t)\|_{L^\infty}\leq C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2\Big(\ln\Big(\frac{1}{\delta}\Big)+\ln(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})\Big)e^{-\lambda_2 t}. \qquad (7.16)
$$

Similarly,

$$
\|\partial_v\tilde{\mathbf{h}}(t)\|_{L^\infty}\leq C(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})^2\Big(\ln\Big(\frac{1}{\delta}\Big)+\ln(1+\|\partial_{x,v}\tilde{\mathbf{h}}_0\|_{L^\infty})\Big)e^{-\lambda_2 t}. \qquad (7.17)
$$

Combining (7.16), (7.17) and Proposition 6.3, we conclude (1.25). Therefore the proof of Theorem1.1 is completed. □

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