# THE LAMM-RIVIÈRE SYSTEM II: ENERGY IDENTITY

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ABSTRACT. In this paper, we establish an angular energy quantization for the following fourth order inhomogeneous Lamm-Rivière system

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \operatorname{div}(w \nabla u) + W \cdot \nabla u + f$$

in dimension four, with an inhomogeneous term  $f \in L \log L$ .

**Keywords:** Energy identity, blowup analysis, Lamm-Rivière system, biharmonic map, Lorentz space

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## 1. Introduction and main results

Let  $B_1 \subset \mathbb{R}^4$  be the unit ball and (N,h) be a l-dimensional, smooth, compact Riemannian manifold without boundary, that is isometrically embedded into an Euclidean space  $\mathbb{R}^n$  of dimension n. Consider the following second order energy functionals for maps  $u \in W^{2,2}(B_1, N)$ :

$$E_{ext}(u) = \int_{B_1} |\Delta u|^2 dx$$
 and  $E_{int}(u) = \int_{B_1} |(\Delta u)^T|^2 dx$ ,

where  $(\Delta u)^T$  is the orthogonal projection of  $\Delta u$  into the tangent space of N at point u,  $T_uN$ . Critical points of  $E_{ext}$  (or  $E_{int}$ ) are called extrinsic (or intrinsic, respectively) biharmonic maps. Note that the notion of extrinsic biharmonic maps may depend on the isometric embedding of (N, h) into  $\mathbb{R}^n$ , and biharmonic maps are natural higher order extensions of harmonic maps.

The Euler-Lagrange equations for critical points of  $E_{ext}$  and  $E_{int}$  are fourth order nonlinear elliptic systems with supercritical nonlinearities. For instance, an extrinsic biharmonic map  $u \in W^{2,2}(B_1, N)$  is a weak solution of

$$\Delta^2 u - \Delta \left( B(u)(\nabla u, \nabla u) \right) - 2\nabla \cdot \langle \Delta u, \nabla P(u) \rangle + \langle \Delta (P(u)), \Delta u \rangle = 0,$$

where  $B(\cdot)(\cdot,\cdot)$  is the second fundamental form of  $N\hookrightarrow\mathbb{R}^n$ , and P(y) is the orthogonal projection of  $\mathbb{R}^n$  into the tangent space  $T_yN$  for  $y\in N$ . These equations are critical in dimension four. Chang-Wang-Yang [2] initiated the study of regularity theory of extrinsic biharmonic maps from a m-dimensional domain  $\Omega\subset\mathbb{R}^m$  to an Euclidean sphere and established their smoothness when m=4. Shortly after, Wang developed a regularity theory of both extrinsic and intrinsic biharmonic maps into any smooth compact Riemannian manifold in a series of papers [28, 27, 26] via the method of Coulomb moving frames. Motivated by the celebrated result of Rivière [19], Lamm and Rivière [11] introduced a class of fourth order critical elliptic systems (see (1.1) below with f=0), including both extrinsic and intrinsic biharmonic maps, and proved the continuity of weak solutions; see also Scheven [22] for regularity results in supercritical dimensions. Built upon the

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techniques by [11], Guo and Xiang [7] derived the Hölder continuity of weak solutions, while Guo, Xiang and Zheng [8] established the sharp  $L^p$ -regularity theory, for the fourth order Lamm-Rivière system; see also [25, 24, 6] for alternative approaches on the regularity theory of biharmonic maps and related fourth order elliptic systems.

In this paper, we will apply the regularity theory by [8] for the inhomogeneous Lamm-Rivière system to study energy quantization of weak solutions. More precisely, we consider weak solutions  $u \in W^{2,2}(B_1, \mathbb{R}^n)$  of the inhomogeneous Lamm-Rivière system:

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \operatorname{div}(w \nabla u) + W \cdot \nabla u + f, \tag{1.1}$$

where  $V \in W^{1,2}(B_1, M_n \otimes \wedge^1 \mathbb{R}^4), w \in L^2(B_1, M_n), \text{ and } W \in W^{-1,2}(B_1, M_n \otimes \wedge^1 \mathbb{R}^4) \text{ is of the form}$ 

$$W = \nabla \omega + F$$

with  $\omega \in L^2(B_1, so_n)$ ,  $F \in L^{\frac{4}{3}, 1}(B_1, M_n \otimes \wedge^1 \mathbb{R}^4)$ . The drift term  $f \in L^p(B_1, \mathbb{R}^n)$  for some p > 1, or  $f \in L \log L(B_1, \mathbb{R}^n)$ .

Due to the scaling invariance property of (1.1) in dimension 4, weakly convergent sequences of solutions may exhibit the loss of compactness around finitely many points, similar to that of harmonic maps from Riemannian surfaces. Sacks and Uhlenbeck [20] was the first to study this type of concentration-compactness phenomena for harmonic maps in dimension two. Sacks and Uhlenbeck discovered that the loss of compactness arises from the creation of bubbles, which are nontrivial harmonic maps from  $\mathbb{S}^2$  to (N,h), near each energy concentration point. Subsequently, Parker [17], Ding-Tian [5], Qing-Tian [18], and Lin-Wang [14] established the energy identity that accounts for the loss of energy by the sum of energies of finitely many bubbles. For biharmonic maps or approximated biharmonic maps from a bounded domain  $\Omega \subset \mathbb{R}^4$  to (N,h), Wang-Zheng [29, 30], Hornung-Moser [10] and Laurain-Rivière [12] have independently established the energy identity, see Liu-Yin [15] for an alternate proof and also related results by Chen-Zhu [3, 4] in which the definition domain (M, g) is a 4-dimensional, compact Riemannian manifold without boundary. We would like to remark that the authors in [12] actually proved the energy identity for solutions to the Lamm-Rivière sysetm (1.1) under certain growth conditions on the coefficient functions V, w, W (see [12, Equation (2.7)]), which were satisfied by biharmonic maps. Furthermore, they made an expectation in [12, Theorem 5.1] that a similar angular energy quantization result should hold for the homogeneous Lamm-Rivière system (that is, (1.1) with f = 0). Motived by this expectation, we will establish the angular energy identity for solutions of the general fourth order linear system (1.1) with  $f \in L \log L$ .

Our main result reads as follows.

**Theorem 1.1.** Let  $\{u_k\} \subset W^{2,2}(B_1,\mathbb{R}^n)$  be a sequence of weak solutions of

$$\Delta^2 u_k = \Delta(V_k \cdot \nabla u_k) + \operatorname{div}(w_k \nabla u_k) + (\nabla \omega_k + F_k) \cdot \nabla u_k + f_k, \tag{1.2}$$

with

$$V_k \in W^{1,2}(B_1, M_n \otimes \wedge^1 \mathbb{R}^4), \quad w_k \in L^2(B_1, M_n), \quad \omega_k \in L^2(B_1, so_n),$$
  
 $F_k \in L^{\frac{4}{3}, 1}(B_1, M_n \otimes \wedge^1 \mathbb{R}^4), \quad f_k \in L \log L(B_1, \mathbb{R}^n).$ 

Assume that there exists a constant  $\Lambda > 0$  such that for all  $k \in \mathbb{N}$ ,

$$||u_k||_{W^{2,2}(B_1)} + ||V_k||_{W^{1,2}(B_1)} + ||w_k||_{L^2(B_1)} + ||\omega_k||_{L^2(B_1)} + ||F_k||_{L^{\frac{4}{3},1}(B_1)} + ||f_k||_{L\log L(B_1)} \le \Lambda. \quad (1.3)$$

Then there exists a subsequence, still denoted by  $u_k, V_k, w_k, \omega_k, F_k$  and  $f_k$ , such that  $u_k \rightharpoonup u_\infty$  weakly in  $W^{2,2}(B_1)$ ,  $V_k \rightharpoonup V_\infty$  in  $W^{1,2}(B_1)$ ,  $w_k \rightharpoonup w_\infty$  in  $L^2(B_1)$ ,  $\omega_k \rightharpoonup \omega_\infty$  in  $L^2(B_1)$ ,  $F_k \rightharpoonup F_\infty$ 

in  $L^{\frac{4}{3},1}(B_1)$ , and  $f_k \rightharpoonup f_\infty \in L \log L(B_1)$  in the distributional sense. And  $u_\infty$  is a weak solution of  $\Delta^2 u_\infty = \Delta(V_\infty \cdot \nabla u_\infty) + \operatorname{div}(w_\infty \nabla u_\infty) + (\nabla \omega_\infty + F_\infty) \cdot \nabla u_\infty + f_\infty.$ 

Moreover, there exists  $l \in \mathbb{N}^*$  and

(i) a family of solutions  $\{\theta^i\}_{i=1}^l \subset W^{2,2}(\mathbb{R}^4,\mathbb{R}^n)$  to the system:

$$\Delta^2 \theta^i = \Delta (V_{\infty}^i \cdot \nabla \theta^i) + \operatorname{div}(w_{\infty}^i \nabla \theta^i) + (\nabla \omega_{\infty}^i + F_{\infty}^i) \cdot \nabla \theta^i \quad in \ \mathbb{R}^4,$$

where

$$V_{\infty}^{i} \in W^{1,2}(\mathbb{R}^{4}, M_{n} \otimes \wedge^{1}\mathbb{R}^{4}), \quad w_{\infty}^{i} \in L^{2}(\mathbb{R}^{4}, M_{n}),$$
  
 $\omega_{\infty}^{i} \in L^{2}(\mathbb{R}^{4}, so_{n}) \quad and \quad F_{\infty}^{i} \in L^{\frac{4}{3}, 1}(\mathbb{R}^{4}, M_{n} \otimes \wedge^{1}\mathbb{R}^{4});$ 

- (ii) a family of convergent points  $\{a_k^1, \dots, a_k^l\} \subset B_1$ , with  $a_k^i \xrightarrow{k \to \infty} a^i \in B_1$ ;
- (iii) a family of sequences of positive real numbers  $\{\lambda_k^1, \cdots, \lambda_k^l\}$ , with  $\lambda_k^i \xrightarrow{k \to \infty} 0$ , such that up to a subsequence,

$$u_k \to u_\infty$$
 in  $W_{\text{loc}}^{2,2}(B_1 \setminus \{a^1, \cdots, a^l\})$ 

and

$$\left\|\left\langle \nabla^2 \left(u_k - u_\infty - \sum_{i=1}^l \theta_k^i\right), X_k \right\rangle \right\|_{L^2_{\text{loc}}(B_1)} + \left\|\left\langle \nabla \left(u_k - u_\infty - \sum_{i=1}^l \theta_k^i\right), X_k \right\rangle \right\|_{L^4_{\text{loc}}(B_1)} \to 0,$$
 where  $\theta_k^i = \theta^i((\cdot - a_k^i)/\lambda_k^i)$  and  $X_k = \nabla^\perp d_k$  with  $d_k = \min_{1 \le i < l} (\lambda_k^i + d(a_k^i, \cdot)).$ 

As a direct consequence of Theorem 1.1, we have the following angular energy identity when the drift term  $f \in L^p$ , for p > 1.

Corollary 1.2. Let  $\{u_k\} \subset W^{2,2}(B_1,\mathbb{R}^n)$  be a sequence of weak solutions of

$$\Delta^2 u_k = \Delta(V_k \cdot \nabla u_k) + \operatorname{div}(w_k \nabla u_k) + (\nabla \omega_k + F_k) \cdot \nabla u_k + f_k, \tag{1.4}$$

with

$$V_k \in W^{1,2}(B_1, M_n \otimes \wedge^1 \mathbb{R}^4), \quad w_k \in L^2(B_1, M_n), \quad \omega_k \in L^2(B_1, so_n),$$
  
 $F_k \in L^{\frac{4}{3},1}(B_1, M_n \otimes \wedge^1 \mathbb{R}^4), \quad f_k \in L^p(B_1, \mathbb{R}^n) \text{ for } p > 1.$ 

Assume that there exists a constant  $\Lambda > 0$  such that for every  $k \in \mathbb{N}$ ,

$$||u_k||_{W^{2,2}(B_1)} + ||V_k||_{W^{1,2}(B_1)} + ||w_k||_{L^2(B_1)} + ||\omega_k||_{L^2(B_1)} + ||F_k||_{L^{\frac{4}{3},1}(B_1)} + ||f_k||_{L^p(B_1)} \le \Lambda. \quad (1.5)$$

Then there exists a subsequence, still denoted by  $u_k, V_k, w_k, \omega_k, F_k$  and  $f_k$ , such that  $u_k \rightharpoonup u_\infty$  in  $W^{2,2}(B_1)$ ,  $V_k \rightharpoonup V_\infty$  in  $W^{1,2}(B_1)$ ,  $w_k \rightharpoonup w_\infty$  in  $L^2(B_1)$ ,  $\omega_k \rightharpoonup \omega_\infty$  in  $L^2(B_1)$ ,  $F_k \rightharpoonup F_\infty$  in  $L^{\frac{4}{3},1}(B_1)$  and  $f_k \rightharpoonup f_\infty$  in  $L^p(B_1)$ . And  $u_\infty$  is a weak solution of

$$\Delta^2 u_{\infty} = \Delta (V_{\infty} \cdot \nabla u_{\infty}) + \operatorname{div}(w_{\infty} \nabla u_{\infty}) + (\nabla \omega_{\infty} + F_{\infty}) \cdot \nabla u_{\infty} + f_{\infty}.$$

Moreover, there exists  $l \in \mathbb{N}^*$  and

(i) a family of solutions  $\{\theta^1, \dots, \theta^l\}$  to the system of the form

$$\Delta^2 \theta^i = \Delta (V^i_\infty \cdot \nabla \theta^i) + \operatorname{div}(w^i_\infty \nabla \theta^i) + (\nabla \omega^i_\infty + F^i_\infty) \cdot \nabla \theta^i \quad \text{in } \mathbb{R}^4,$$

where

$$V_{\infty}^{i} \in W^{1,2}(\mathbb{R}^{4}, M_{n} \otimes \wedge^{1}\mathbb{R}^{4}), \quad w_{\infty}^{i} \in L^{2}(\mathbb{R}^{4}, M_{n}),$$
  
$$\omega_{\infty}^{i} \in L^{2}(\mathbb{R}^{4}, so_{n}) \quad and \quad F_{\infty}^{i} \in L^{\frac{4}{3}, 1}(\mathbb{R}^{4}, M_{n} \otimes \wedge^{1}\mathbb{R}^{4});$$

(ii) a family of convergent points  $\{a_k^1, \cdots, a_k^l\} \subset B_1$ , with  $a_k^i \xrightarrow{k \to \infty} a^i \in B_1$ ;

(iii) a family of sequences of positive real numbers  $\{\lambda_k^1, \dots, \lambda_k^l\}$ , with  $\lambda_k^i \xrightarrow{k \to \infty} 0$ , such that up to a subsequence,

$$u_k \to u_\infty$$
 in  $W_{\mathrm{loc}}^{2,2}(B_1 \setminus \{a^1, \cdots, a^l\})$ 

and

$$\begin{split} \left\| \left\langle \nabla^2 \left( u_k - u_\infty - \sum_{i=1}^l \theta_k^i \right), X_k \right\rangle \right\|_{L^2_{\text{loc}}(B_1)} + \left\| \left\langle \nabla \left( u_k - u_\infty - \sum_{i=1}^l \theta_k^i \right), X_k \right\rangle \right\|_{L^4_{\text{loc}}(B_1)} \to 0, \\ where \ \theta_k^i = \theta^i ((\cdot - a_k^i)/\lambda_k^i) \ \ and \ X_k = \nabla^\perp d_k \ \ with \ d_k = \min_{1 \le i \le l} (\lambda_k^i + d(a_k^i, \cdot)). \end{split}$$

We remark that, by combining Corollary 1.2 with the Pohozaev identity for biharmonic maps, one can obtain routinely the energy identity for both extrinsic and intrinsic biharmonic maps, see, e.g., [29, 12].

The ideas of proof of Theorem 1.1 can be outlined as follows.

- Employ the  $\epsilon$ -regularity theorem, the energy gap theorem, the weak compactness theorem from [8] to derive  $L^{4,2}$  (and  $L^{2,1}$ ) estimates of the angular part of first (and second) order derivatives. In this step, we also prove a removable isolated singularity theorem.
- Prove no concentration of angular hessian energy in the neck region by a Lorentz duality argument similar to Laurain and Rivière [13, 12]. In this step, we also establish the well-known bubble tree decomposition.

The paper is organized as follows. In Section 2, we recall the necessary function spaces and regularity results for solutions of the inhomogeneous Lamm-Rivière system. In Section 3, we derive  $L^{4,2}$  (and  $L^{2,1}$ ) estimates of the angular part of first (and second) order derivatives. In Section 4, we prove Corollary 1.2 by establishing no concentration of angular hessian energy in the neck region by a Lorentz duality argument. In Section 5, we prove our main theorem, Theorem 1.1.

#### 2. Preliminaries and auxiliary results

## 2.1. Function spaces.

In this subsection, we recall the definitions and basic properties of Lorentz spaces. For more details, see for instance the monograph [1]. Throughout this subsection, we assume that  $\Omega \subset \mathbb{R}^n$  is a bounded smooth domain.

Given a measurable function  $f: \Omega \to \mathbb{R}$ , denote by  $\delta_f(t) = \mathcal{L}^n(\{x \in \Omega : |f(x)| > t\})$  its distributional function and by  $f^*(t) = \inf\{s > 0 : \delta_f(s) \le t\}$ ,  $t \ge 0$ , the nonincreasing rearrangement of |f|. Define

$$f^{**}(t) \equiv \frac{1}{t} \int_0^t f^*(s) ds, \quad t > 0.$$

For  $p \in (1, +\infty)$ ,  $q \in [1, +\infty]$ , the Lorentz space  $L^{p,q}(\Omega)$  consists of measurable functions with finite  $L^{p,q}(\Omega)$ -norm given by

$$||f||_{L^{p,q}(\Omega)} \equiv \begin{cases} \left[ \int_0^{+\infty} (t^{1/p} f^{**}(t))^q \frac{dt}{t} \right]^{1/q}, & 1 \le q < +\infty, \\ \sup_{t>0} t^{1/p} f^{**}(t), & q = \infty. \end{cases}$$

**Proposition 2.1** ([16, Theorem 3.4 and 3.5]). Let  $1 < p_1, p_2 < +\infty$  and  $1 \le q_1, q_2 \le +\infty$  be such that

$$\frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \le 1$$
 and  $\frac{1}{q} = \frac{1}{q_1} + \frac{1}{q_2} \le 1$ .

Then for all  $g \in L^{p_1,q_1}(\Omega)$  and  $h \in L^{p_2,q_2}(\Omega)$ , we have  $gh \in L^{p,q}(\Omega)$  with

$$||gh||_{L^{p,q}(\Omega)} \le ||g||_{L^{p_1,q_1}(\Omega)} ||h||_{L^{p_2,q_2}(\Omega)}.$$

We recall that  $L \log L(\Omega)$  is defined by

$$L\log L(\Omega) := \{ f : \Omega \to \mathbb{R} \mid ||f||_{L\log L(\Omega)} < \infty \},$$

where 
$$||f||_{L \log L(\Omega)} = \int_0^\infty f^*(t) \log(2 + \frac{1}{t}) dt$$
.

**Lemma 2.2** ([21, Lemma 2.1]). For any function  $f \in L \log L(B_r)$  with radius  $r \in (0, \frac{1}{2})$ , there exists a constant C > 0 which is independent of r, such that

$$||f||_{L^1(B_r)} \le C \left[ \log \left( \frac{1}{r} \right) \right]^{-1} ||f||_{L \log L(B_r)}.$$

# 2.2. Auxiliary regularity and compactness results.

In this subsection, we first recall the  $\epsilon$ -regularity theorem and the energy gap theorem, which shall be used in our later proofs. Theorem 2.3 and 2.4 below can be founded in the literatures, see for example [8, Theorem 1.2, Theorem 1.6 and Corollary 1.5].

**Theorem 2.3** ( $\varepsilon$ -regularity). Suppose  $u \in W^{2,2}(B_1, \mathbb{R}^n)$  is a weak solution of (1.1) in  $B_1$ .

(1) Let  $f \in L \log L(B_1)$ . There exist  $\varepsilon = \varepsilon(n) > 0$  and C = C(n) > 0 such that if

$$\|V\|_{W^{1,2}(B_1)} + \|w\|_{L^2(B_1)} + \|\omega\|_{L^2(B_1)} + \|F\|_{L^{\frac{4}{3},1}(B_1)} < \varepsilon,$$

then  $u \in W^{2,2,1}(B_{\frac{1}{2}}, \mathbb{R}^n)$ , and

$$||u||_{W^{2,2,1}(B_{1/2})} \le C(||f||_{L \log L(B_1)} + ||u||_{L^1(B_1)}).$$

(2) Let  $f \in L^p(B_1)$  for p > 1. There exist  $\varepsilon = \varepsilon(p,n) > 0$  and C = C(p,n) > 0 such that if

$$\|V\|_{W^{1,2}(B_1)} + \|w\|_{L^2(B_1)} + \|\omega\|_{L^2(B_1)} + \|F\|_{L^{\frac{4}{3},1}(B_1)} < \varepsilon,$$

then  $u \in W^{2,q}(B_{\frac{1}{2}}, \mathbb{R}^n)$ , and

$$||u||_{W^{2,q}(B_{1/2})} \le C(||f||_{L^p(B_1)} + ||u||_{L^1(B_1)}),$$

where  $q = \frac{2p}{2-n}$  if p < 2, and q can be any positive number if  $p \ge 2$ .

**Theorem 2.4** (Energy gap). Let  $u \in W^{2,2}(\mathbb{R}^4, \mathbb{R}^n)$  be a weak solution of (1.1) in  $\mathbb{R}^4$  with  $f \equiv 0$ . There exists  $\varepsilon = \varepsilon(n) > 0$  such that if

$$||V||_{W^{1,2}(\mathbb{R}^4)} + ||w||_{L^2(\mathbb{R}^4)} + ||\omega||_{L^2(\mathbb{R}^4)} + ||F||_{L^{\frac{4}{3},1}(\mathbb{R}^4)} < \varepsilon,$$

then  $u \equiv 0$  in  $\mathbb{R}^4$ .

Next, we prove a removable isolated singularity theorem for solutions of (1.1).

**Theorem 2.5** (Removable singularity). Let  $u \in W^{2,2}(B_1, \mathbb{R}^n)$  be a weak solution to the inhomogeneous system (1.1) in  $B_1 \setminus \{0\}$ . Then u is a solution of (1.1) on the whole  $B_1$ .

*Proof.* We need to show that for all  $\varphi \in W_0^{2,2} \cap L^{\infty}(B_1, \mathbb{R}^n)$ ,

$$\int_{B_1} \Delta u^i \Delta \varphi^i = \int_{B_1} \nabla (V_{ij}^{\alpha} \partial_{\alpha} u^j) \cdot \nabla \varphi^i + \omega_{ij} \partial_{\alpha} u^j \partial_{\alpha} \varphi^i + (W_{ij}^{\alpha} \partial_{\alpha} u^j + f) \varphi^i. \tag{2.1}$$

Since (2.1) already holds for all  $\Phi \in W_0^{2,2} \cap L^{\infty}(B_1 \setminus \{0\}, \mathbb{R}^n)$ , we may use a standard approximation argument to prove the claim. For each  $R \in (0,1)$ , we claim that there exists  $\tau_R \in W^{2,2} \cap L^{\infty}([0,1])$  such that

- $\tau_R = 0$  on  $[0, R^2]$  and  $\tau_R = 1$  on [R, 1];
- $0 \le \tau_R \le 1$  on [0, 1];
- $||D^2\tau_R(|x|)||_{L^2(B_1)} + ||D\tau_R(|x|)||_{L^{4,2}(B_1)} \to 0 \text{ as } R \to 0.$

Indeed, one can check that the function  $\hat{\tau}_R$  defined below satisfies the above properties:

• 
$$\hat{\tau}_R = 0$$
 on  $[0, R^2]$ ,  $\hat{\tau}_R(t) = \frac{\log \log(et/R^2)}{\log \log(e/R)}$  on  $[R^2, R]$  and  $\hat{\tau}_R = 1$  on  $[R, 1]$ .

Now the mapping  $\Phi_R \colon B_1 \to \mathbb{R}^n$  defined by

$$\Phi_R(x) := \tau_R(|x|)\varphi(x)$$
 for all  $x \in B_1$ 

belongs to  $W_0^{2,2} \cap L^{\infty}(B_1 \setminus \{0\}, \mathbb{R}^n)$  and thus it satisfies (2.1).

It is easy to see that

$$\int_{B_1} \Delta u^i \Delta \Phi_R^i \to \int_{B_1} \Delta u^i \Delta \varphi^i \quad \text{as } R \to 0.$$

Indeed, this convergence follows from Hölder's inequality and

$$\partial_{\beta}\partial_{\beta}(\tau_R(|x|)\varphi^i) = (D^2\tau_R(|x|))(\partial_{\beta}(|x|))^2\varphi^i + A + \tau_R(|x|)\partial_{\beta}\partial_{\beta}\varphi^i,$$

where

$$A = D\tau_R(|x|)\partial_\beta\partial_\beta(|x|)\varphi^i + 2D\tau_R(|x|)\partial_\beta(|x|)\partial_\beta\varphi^i,$$

and

- $D^2\tau_R(|x|) \to 0$  in  $L^2(B_1)$  and  $D\tau_R(|x|) \to 0$  in  $L^{4,2}(B_1)$  as  $R \to 0$ ;
- $\partial_{\beta}(|x|) \in L^{\infty}(B_1)$  and  $\partial_{\beta}\partial_{\beta}(|x|) \in L^{4,\infty}(B_1)$ ;
- $\tau_R(|x|) \to 1$  for almost every  $x \in B_1$  as  $R \to 0$ .

Similarly, we can prove that

$$\int_{B_1} \nabla (V_{ij}^{\alpha} \partial_{\alpha} u^j) \cdot \nabla \Phi_R^i \to \int_{B_1} \nabla (V_{ij}^{\alpha} \partial_{\alpha} u^j) \cdot \nabla \varphi^i \quad \text{as } R \to 0$$

$$\int_{B_1} \omega_{ij} \partial_{\alpha} u^j \partial_{\alpha} \Phi_R^i \to \int_{B_1} \omega_{ij} \partial_{\alpha} u^j \partial_{\alpha} \varphi^i \quad \text{as } R \to 0$$

$$\int_{B_1} (W_{ij}^{\alpha} \partial_{\alpha} u^j + f) \Phi_R^i \to \int_{B_1} (W_{ij}^{\alpha} \partial_{\alpha} u^j + f) \varphi^i \quad \text{as } R \to 0.$$

For simplicity, we only verify the first convergence. Note that

$$\nabla (V_{ij}^{\alpha} \partial_{\alpha} u^{j}) \cdot \nabla \Phi_{R}^{i} = \nabla V_{ij}^{\alpha} \partial_{\alpha} u^{j} \cdot \nabla \Phi_{R}^{i} + V_{ij}^{\alpha} \partial_{\alpha} \nabla u^{j} \cdot \nabla \Phi_{R}^{i}.$$

For the first term in the right hand side, applying Hölder's inequality together with the fact that

$$||D\tau_R(|x|)||_{L^4(B_1)} \to 0,$$

we infer that

$$\int_{B_1} \nabla V_{ij}^{\alpha} \partial_{\alpha} u^j \cdot \nabla \Phi_R^i = \int_{B_1} \nabla V_{ij}^{\alpha} \partial_{\alpha} u^j \cdot (\nabla \varphi^i \tau_R(|x|) + D\tau_R(|x|) \frac{x}{|x|} \varphi^i)$$

$$\to \int_{B_1} \nabla V_{ij}^{\alpha} \partial_{\alpha} u^j \cdot \nabla \varphi^i \quad \text{as } R \to 0.$$

Similarly, for the second term in the right hand side, we have

$$\int_{B_1} V_{ij}^{\alpha} \partial_{\alpha} \nabla u^j \cdot \nabla \Phi_R^i = \int_{B_1} V_{ij}^{\alpha} \partial_{\alpha} \nabla u^j \cdot (\nabla \varphi^i \tau_R(|x|) + D\tau_R(|x|) \frac{x}{|x|} \varphi^i)$$

$$\to \int_{B_1} V_{ij}^{\alpha} \partial_{\alpha} \nabla u^j \cdot \nabla \varphi^i \quad \text{as } R \to 0.$$

This completes the proof.

The following weak compactness result is well-known, see for example [7, Theorem 1.3] and [8, Theorem 1.7]. Using Theorem 2.5, we sketch a simple alternate proof below.

**Theorem 2.6** (Weak compactness). Let  $\{u_k\} \subset W^{2,2}(B_1, \mathbb{R}^n)$  be a sequence of weak solutions of (1.1) in  $B_1$  with  $f_k \in L \log L(B_1)$ . Suppose

$$u_k \rightharpoonup u \text{ in } W^{2,2}(B_1), \quad V_k \rightharpoonup V \text{ in } W^{1,2}(B_1), \quad w_k \rightharpoonup w \text{ in } L^2(B_1),$$
  
 $\omega_k \rightharpoonup \omega \text{ in } L^2(B_1), \quad F_k \rightharpoonup F \text{ in } L^{\frac{4}{3},1}(B_1),$   
 $f_k \rightharpoonup f \in L \log L(B_1) \text{ in the distributional sense.}$ 

Then

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \operatorname{div}(w \nabla u) + (\nabla \omega + F) \cdot \nabla u + f.$$

*Proof.* Since  $||V_k||_{W^{1,2}(B_1)}$ ,  $||w_k||_{L^2(B_1)}$ ,  $||\omega_k||_{L^2(B_1)}$ , and  $||F_k||_{L^{\frac{4}{3}}(B_1)}$  are uniformly bounded, there is a finite set  $\{a^i\}_{i=1}^N \subset B_1$  of  $\varepsilon$ -concentration points, with  $\varepsilon > 0$  given by Theorem 2.3, namely,

$$\lim_{r \to 0} \lim_{k \to \infty} (\|V_k\|_{W^{1,2}(B_r(a^i))} + \|w_k\|_{L^2(B_r(a^i))} + \|\omega_k\|_{L^2(B_r(a^i))} + \|F_k\|_{L^{\frac{4}{3},1}(B_r(a^i))}) \ge \varepsilon.$$

Applying Theorem 2.3, we have that  $u_k \to u$  strongly in  $W^{2,2}_{\text{loc}}(B_1 \setminus \{a^1, \cdots, a^N\})$ . If  $\{a^1, \cdots, a^N\} = \emptyset$ , then  $u_k \to u$  in  $W^{2,2}_{\text{loc}}(B_1)$  and the conclusion follows. If  $\{a^1, \cdots, a^N\} \neq \emptyset$ , then we have

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \operatorname{div}(w \nabla u) + (\nabla \omega + F) \cdot \nabla u + f \quad \text{in } B_1 \setminus \{a^1, \dots, a^N\}.$$

By Theorem 2.5, u solves the above system in  $B_1$ . The proof is complete.

## 3. Estimate of the tangential derivative in annular region

In this section, we will provide the  $L^{4,2}$  and  $L^{2,1}$  estimates for tangential derivatives in an annular region. For this purpose, we first need to present two Lemmas about harmonic maps in the annular region under appropriate boundary conditions.

The first Lemma was given in [12, Lemma 6.1]. For the convenience of readers, we provide the proof (with more details) here.

**Lemma 3.1.** Let  $0 < r < \frac{1}{8}$  and  $u \in W^{2,2}(B_1 \backslash B_r)$  be a harmonic function such that

$$\int_{\partial B_1} u \, d\sigma = 0 \quad and \quad \int_{\partial B_r} u \, d\sigma = 0. \tag{3.1}$$

Then there exist a positive constant C, independent of r and u, such that

$$||u||_{L^{2,1}(B_{\frac{1}{2}}\setminus B_{2r})} \le C||u||_{L^{2}(B_{1}\setminus B_{r})}$$

and

$$\|\nabla^T \nabla u\|_{L^{2,1}(B_{\frac{1}{2}} \setminus B_{2r})} \le C \|\nabla^T \nabla u\|_{L^2(B_1 \setminus B_r)},$$

where  $\nabla^T u = \nabla u - \frac{\partial u}{\partial r} \frac{\partial}{\partial r}$ .

*Proof.* Since u is a harmonic function in the four-dimensional domain, it can be decomposed with respect to the spherical harmonics as follows:

$$u = a_0 + b_0 r^{-2} + \sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} (d_k^l r^l + d_k^{-l} r^{-l-2}) \phi_k^l,$$
(3.2)

where  $(\phi_k^l)_{l,k}$  is the orthonormal basis of  $L^2(\mathbb{S}^3)$  given by eigenfunctions of the Laplacian on  $\mathbb{S}^3$ , i.e.,  $\Delta \phi_k^l = -l(l+2)\phi_k^l$  on  $\mathbb{S}^3$  for  $1 \leq k \leq N_l = (l+1)^2$ . Using the boundary condition (3.1), we have

$$\int_{\partial B_1} u(1,\theta) d\sigma = \int_{\partial B_1} (a_0 + b_0 + \sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} (d_k^l + d_k^{-l}) \phi_k^l) d\mathcal{H}^3(\theta) = 0,$$

and

$$\int_{\partial B_r} u(r,\theta) d\sigma = \int_{\partial B_r} (a_0 + b_0 r^{-2} + \sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} (d_k^l r^l + d_k^{-l} r^{-l-2}) \phi_k^l) d\mathcal{H}^3(\theta) = 0.$$

From  $\int_{\partial B_1} \phi_k^l d\mathcal{H}^3(\theta) = \int_{\partial B_r} \phi_k^l d\mathcal{H}^3(\theta) = 0$  for  $1 \leq k \leq N_l$  and all  $l \geq 1$ , we deduce that  $a_0 = b_0 = 0$ . Hence, we have

$$u = \sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} (d_k^l r^l + d_k^{-l} r^{-l-2}) \phi_k^l.$$

Applying regularity theory for elliptic equations, we have

$$\|\phi_k^l\|_{L^{\infty}} \le C(l(l+2))^2,$$

where C is a positive constant independent of l.

Denote by  $f_j(x) = |x|^j$  for integers j. Using the fact from [13, Page 8],

$$||f_j||_{L^{2,1}(\Omega)} \sim 4 \int_0^{+\infty} |\{x \in \Omega \mid |f_j(x)| \ge \lambda\}|^{\frac{1}{2}} d\lambda,$$

we can directly calculate

$$||f_j||_{L^{2,1}(B_{\frac{1}{2}}\backslash B_{2r})} \le \begin{cases} c(2r)^{j+2} & j \le -3, \\ \left(\frac{1}{2}\right)^{\frac{3j}{4}+1} & j \ge 0, \end{cases}$$

where c is a constant independent of j and r.

Applying the Cauchy-Schwarz inequality, we then have

$$\begin{split} \|u\|_{L^{2,1}(B_{\frac{1}{2}}\backslash B_{2r})} &\leq C \sum_{l=1}^{+\infty} \sum_{k=1}^{N_l} \left( d_k^l \left( \frac{1}{2} \right)^{\frac{3l}{4}+1} + d_k^{-l}(2r)^{-l} \right) (l(l+2))^2 \\ &\leq C \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{(l+1)^2} (d_k^l)^2 \frac{1}{2l+4} \right)^{\frac{1}{2}} \times \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{(l+1)^2} 4(2l+4)(l(l+2))^4 \left( \frac{1}{2} \right)^{\frac{3l}{2}+2} \right)^{\frac{1}{2}} \\ &+ C \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{(l+1)^2} (d_k^{-l})^2 \frac{r^{-2l}}{8l} \right)^{\frac{1}{2}} \times \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{(l+1)^2} 8l(l(l+2))^4 \left( \frac{1}{4} \right)^l \right)^{\frac{1}{2}} \\ &:= C \left( \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{(l+1)^2} (d_k^l)^2 \frac{1}{2l+4} \right)^{\frac{1}{2}} \times A + \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{(l+1)^2} (d_k^{-l})^2 \frac{r^{-2l}}{8l} \right)^{\frac{1}{2}} \times B \right). \end{split}$$

Since A and B are convergent series, we have

$$||u||_{L^{2,1}(B_{\frac{1}{2}}\backslash B_{2r})} \le C \left( \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{(l+1)^2} (d_k^l)^2 \frac{1}{2l+4} \right)^{\frac{1}{2}} + \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{(l+1)^2} (d_k^{-l})^2 \frac{r^{-2l}}{8l} \right)^{\frac{1}{2}} \right).$$

Moreover, it holds

$$||u||_{L^{2}(B_{1}\backslash B_{r})} = \left|\left|\sum_{l=1}^{+\infty} \sum_{k=1}^{N_{l}} (d_{k}^{l}|x|^{l} + d_{k}^{-l}|x|^{-l-2})\phi_{k}^{l}\right|\right|_{L^{2}(B_{1}\backslash B_{r})}$$

$$\geq \left(\int_{B_{1}\backslash B_{r}} \sum_{l=1}^{+\infty} \sum_{k=1}^{N_{l}} (d_{k}^{l}|x|^{l} + d_{k}^{-l}|x|^{-l-2})^{2} |\phi_{k}^{l}|^{2} dx\right)^{\frac{1}{2}}$$

$$\geq \left[ \int_{r}^{1} \int_{\partial B_{1}} \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{N_{l}} (d_{k}^{l})^{2} \rho^{2l} + (d_{k}^{-l})^{2} \rho^{-2l-4} \right) \rho^{3} |\phi_{k}^{l}|^{2} d\mathcal{H}^{3}(\theta) d\rho \right]^{\frac{1}{2}}$$

$$= \left( \int_{r}^{1} \sum_{l=1}^{+\infty} \sum_{k=1}^{N_{l}} \left( (d_{k}^{l})^{2} \rho^{2l+3} + (d_{k}^{-l})^{2} \rho^{-2l-1} \right) d\rho \right)^{\frac{1}{2}}$$

$$= \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{N_{l}} \frac{(d_{k}^{l})^{2}}{2l+4} (1 - r^{2l+4}) + \sum_{l=1}^{+\infty} \sum_{k=1}^{N_{l}} \frac{(d_{k}^{-l})^{2}}{2l} (r^{-2l} - 1) \right)^{\frac{1}{2}}$$

$$\geq C \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{N_{l}} \frac{(d_{k}^{l})^{2}}{2l+4} \right)^{\frac{1}{2}} + \left( \sum_{l=1}^{+\infty} \sum_{k=1}^{N_{l}} \frac{(d_{k}^{-l})^{2}}{2l} r^{-2l} \right)^{\frac{1}{2}}$$

$$\geq C \|u\|_{L^{2,1}(B_{\frac{1}{2}} \setminus B_{2r})},$$

where C > 0 is constant independent of r.

Similarly, one can prove the second inequality. The proof is now complete.

For our proofs, we will also need the following technical Lemma.

**Lemma 3.2.** Let  $n \ge 2$  and  $0 < r_0 < 1$ . Suppose u, v are two harmonic functions in  $B_1 \backslash B_{r_0}$ , and v = v(r) is radial symmetric. Assume that

$$\int_{\partial B_1} (u - v) d\sigma = 0 \quad and \quad \int_{\partial B_{r_0}} (u - v) d\sigma = 0.$$
 (3.3)

Then for any  $p \in [1, \infty)$ , we have

$$||v||_{L^p(B_1\setminus B_{r_0})} \le ||u||_{L^p(B_1\setminus B_{r_0})}.$$

In particular, we also have

$$||v||_{L^{2,1}(B_1 \setminus B_{r_0})} \le C||u||_{L^{2,1}(B_1 \setminus B_{r_0})},$$

where C is an absolute constant.

Proof. Firstly, define

$$\phi(r) := \int_{\partial B_1} (u - v)(r, \theta) \, d\sigma(\theta).$$

Since u-v is a harmonic function, direct computation shows that

$$\phi''(r) + \frac{n-1}{r}\phi'(r) = 0,$$
 for all  $r \in (r_0, 1)$ . (3.4)

Indeed, we have

$$\phi'(r) = \int_{\partial B_r} \frac{\partial}{\partial r} \left[ (u - v)(r, \omega) \right] d\sigma(\omega)$$

and

$$\phi''(r) = \int_{\partial B_1} \frac{\partial^2}{\partial r^2} [(u - v)(r, \omega)] d\sigma(\omega) = -\int_{\partial B_1} \frac{n - 1}{r} \frac{\partial}{\partial r} [(u - v)(r, \omega)] d\sigma(\omega),$$

where in the last equality, we used the expansion of Laplace operator in n-dimensional spherical coordinates.

Next, solving the equation (3.4), we obtain that

$$\phi(r) = \begin{cases} a + b \log r, & n = 2, \\ a + br^{2-n}, & n \ge 3. \end{cases}$$

Using the boundary condition (3.3), we deduce that a = b = 0. Hence, for any  $r_0 \le r \le 1$ , we have  $\phi(r) \equiv 0$ .

Finally, this together with Hölder's inequality yields

$$\int_{B_1 \setminus B_{r_0}} |v|^p dx = \int_{r_0}^1 r^{n-1} \int_{\partial B_1} |v(r)|^p d\sigma(\theta) dr = \int_{r_0}^1 r^{n-1} \omega_{n-1}^{-(p-1)} \left| \int_{\partial B_1} v(r) d\sigma(\theta) \right|^p dr 
= \int_{r_0}^1 r^{n-1} \omega_{n-1}^{-(p-1)} \left| \int_{\partial B_1} u(r,\theta) d\sigma(\theta) \right|^p dr 
\leq \int_{r_0}^1 r^{n-1} \omega_{n-1}^{-(p-1)} \int_{\partial B_1} |u(r,\theta)|^p d\sigma(\theta) \cdot \omega_{n-1}^{p(1-\frac{1}{p})} dr 
= \int_{r_0}^1 r^{n-1} \int_{\partial B_1} |u(r,\theta)|^p d\sigma(\theta) dr = \int_{B_1 \setminus B_{r_0}} |u|^p dx.$$

The proof is thus complete.

Now, we derive the  $L^{4,2}$  and  $L^{2,1}$  estimates of the angular part of derivatives of u.

**Theorem 3.3.** There exist constants  $\varepsilon, C = C(n) > 0$  such that for all  $0 < r < \frac{1}{8}$ , all  $f \in L^p(B_1 \backslash B_r, \mathbb{R}^n)$  with p > 1 and all  $u \in W^{2,2}(B_1 \backslash B_r, \mathbb{R}^n)$  satisfying

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \operatorname{div}(w \nabla u) + (\nabla \omega + F) \cdot \nabla u + f \quad in \ B_1 \backslash B_r$$

and

$$||V||_{W^{1,2}(B_1 \setminus B_r)} + ||w||_{L^2(B_1 \setminus B_r)} + ||\omega||_{L^2(B_1 \setminus B_r)} + ||F||_{L^{\frac{4}{3},1}(B_1 \setminus B_r)} \le \varepsilon,$$

then there holds

$$\|\nabla^{T}\nabla u\|_{L^{2,1}(B_{\frac{1}{4}}\backslash B_{4r})} + \|\nabla^{T}u\|_{L^{4,2}(B_{\frac{1}{4}}\backslash B_{4r})}$$

$$\leq C\left(\|\nabla^{2}u\|_{L^{2}(B_{1}\backslash B_{r})} + \|\nabla u\|_{L^{4}(B_{1}\backslash B_{r})} + \|f\|_{L^{p}(B_{1}\backslash B_{r})}\right).$$

Proof. First, by the Sobolev embedding theorem and interpolation inequalities, we have

$$\|\nabla^T u\|_{L^{4,2}(B_{\frac{1}{4}} \setminus B_{4r})} \le \|\nabla u\|_{L^{4,2}(B_{\frac{1}{4}} \setminus B_{4r})} \le C(\|\nabla^2 u\|_{L^2(B_1 \setminus B_r)} + \|\nabla u\|_{L^4(B_1 \setminus B_r)}).$$

Next, using Whitney's extension theorem [1, 23], we infer that there exist

$$\tilde{V} \in W^{1,2}(B_1, M_n \otimes \wedge^1 \mathbb{R}^4), \quad \tilde{w} \in L^2(B_1, M_n),$$
  
 $\tilde{\omega} \in L^2(B_1, so_n), \quad \tilde{F} \in L^{\frac{4}{3}, 1}(B_1, M_n \otimes \wedge^1 \mathbb{R}^4)$ 

such that  $\tilde{V} = V$ ,  $\tilde{w} = w$ ,  $\tilde{\omega} = \omega$ ,  $\tilde{F} = F$  on  $B_1 \backslash B_r$ , and

$$\|\tilde{V}\|_{W^{1,2}(B_1)} + \|\tilde{w}\|_{L^2(B_1)} + \|\tilde{\omega}\|_{L^2(B_1)} + \|\tilde{F}\|_{L^{\frac{4}{3},1}(B_1)} < 2\varepsilon.$$

By [9, Theorem 1.1] (see also [11, Theorem 1.4]), we know that for  $0 < \varepsilon < \frac{1}{2}$  sufficiently small, there exist

$$A \in L^{\infty} \cap W^{2,2}(B_1, GL_n)$$
 and  $B \in W^{1,\frac{4}{3}}(B_1)$ 

such that

$$\operatorname{dist}(A, SO_n) + \|A\|_{W^{2,2}} + \|B\|_{W^{1,\frac{4}{3}}} \le C(\|\tilde{V}\|_{W^{1,2}} + \|\tilde{w}\|_{L^2} + \|\tilde{\omega}\|_{L^2} + \|\tilde{F}\|_{L^{\frac{4}{3},1}})$$

and

$$\nabla \Delta A + \Delta A \tilde{V} - \nabla A \tilde{w} + A(\nabla \tilde{\omega} + \tilde{F}) = \text{curl } B.$$

Whitney's extension theorem also implies that there exists an extension  $\tilde{u} \in W^{2,2}(B_1)$  of u such that

$$\|\nabla^2 \tilde{u}\|_{L^2(B_1)} + \|\nabla \tilde{u}\|_{L^4(B_1)} \le 2 (\|\nabla^2 u\|_{L^2(B_1 \setminus B_r)} + \|\nabla u\|_{L^4(B_1 \setminus B_r)}).$$

An easy computation shows that  $\tilde{u}$  satisfies

$$\Delta(A\Delta \tilde{u}) = \operatorname{div}(K) + Af$$
 in  $B_1 \backslash B_r$ ,

where

$$K = 2\nabla A\Delta \tilde{u} - \Delta A\nabla \tilde{u} + A\tilde{w}\nabla \tilde{u} - \nabla A(\tilde{V}\cdot\nabla \tilde{u}) + A\nabla(\tilde{V}\cdot\nabla \tilde{u}) + B\cdot\nabla \tilde{u}.$$

By the Sobolev embedding  $W^{1,\frac{4}{3}} \hookrightarrow L^2$  and  $W^{1,2} \hookrightarrow L^{4,2}$ , we have  $K \in L^{\frac{4}{3},1}(B_1)$  with

$$||K||_{L^{\frac{4}{3},1}(B_1)} \le C(||\nabla^2 u||_{L^2(B_1 \setminus B_r)} + ||\nabla u||_{L^4(B_1 \setminus B_r)}).$$

Next, we let  $\tilde{f} \in L^p(B_1)$  be an extension of Af such that

$$\|\tilde{f}\|_{L^p(B_1)} \le 2\|Af\|_{L^p(B_1 \setminus B_r)}.$$

Let  $D \in W_0^{1,\frac{4}{3}}(B_1)$  solve the equation

$$\Delta D = \operatorname{div}(K) + \tilde{f} \quad \text{in } B_1.$$

Applying the Calderon-Zygmund theory and the Sobolev embeddings  $W^{1,(\frac{4}{3},1)} \hookrightarrow L^{2,1}$  and  $W^{2,p} \hookrightarrow L^{\frac{2p}{2-p},p} \hookrightarrow L^{2,1}$  (only if p>1), we obtain the following estimate

$$||D||_{L^{2,1}(B_1)} \le C(||K||_{L^{\frac{4}{3},1}(B_1)} + ||f||_{L^p(B_1 \setminus B_r)}). \tag{3.5}$$

Finally, select  $a, b \in \mathbb{R}^n$  such that

$$\begin{cases} \int_{\partial B_1} \left( D - A\Delta \tilde{u} + a + \frac{b}{|x|^2} \right) d\sigma = 0 \\ \int_{\partial B_r} \left( D - A\Delta \tilde{u} + a + \frac{b}{|x|^2} \right) d\sigma = 0. \end{cases}$$

Since  $D - A\Delta \tilde{u}$  and  $a + \frac{b}{|x|^2}$  are two harmonic functions in  $B_1 \backslash B_r$  with the same boundary values, Lemma 3.1 and Lemma 3.2 imply that there is a positive constant C independent of r such that

$$\left\| D - A\Delta \tilde{u} + a + \frac{b}{|x|^2} \right\|_{L^{2,1}(B_{\frac{1}{2}} \backslash B_{2r})} \leq C \left\| D - A\Delta \tilde{u} + a + \frac{b}{|x|^2} \right\|_{L^2(B_1 \backslash B_r)} 
\leq C \left\| D - A\Delta \tilde{u} \right\|_{L^2(B_1 \backslash B_r)} 
\leq C \left( \left\| \nabla^2 u \right\|_{L^2(B_1 \backslash B_r)} + \left\| K \right\|_{L^{\frac{4}{3},1}(B_1 \backslash B_r)} + \left\| f \right\|_{L^p(B_1 \backslash B_r)} \right).$$
(3.6)

Let  $h = D - A\Delta \tilde{u} + a + \frac{b}{|x|^2}$ . Then we have

$$\operatorname{div}(A\nabla \tilde{u}) = \nabla A\nabla \tilde{u} + D + a + \frac{b}{|x|^2} - h =: a + \frac{b}{|x|^2} + E \quad \text{in } B_1 \backslash B_r.$$

It follows from (3.5), (3.6) and Hölder's inequality that

$$||E||_{L^{2,1}(B_{\frac{1}{2}}\backslash B_{2r})} \leq ||\nabla A||_{L^{4,2}(B_{\frac{1}{2}}\backslash B_{2r})} ||\nabla^{2}\tilde{u}||_{L^{2,2}(B_{\frac{1}{2}}\backslash B_{2r})} + ||D||_{L^{2,1}(B_{\frac{1}{2}}\backslash B_{2r})} + ||h||_{L^{2,1}(B_{\frac{1}{2}}\backslash B_{2r})}$$

$$\leq C(||\nabla^{2}u||_{L^{2}(B_{1}\backslash B_{r})} + ||\nabla u||_{L^{4}(B_{1}\backslash B_{r})} + ||f||_{L^{p}(B_{1}\backslash B_{r})}).$$

By Hodge decomposition, we have

$$Ad\tilde{u} = d\alpha + d^*\beta,\tag{3.7}$$

where  $\alpha \in W_0^{1,2}(B_{\frac{1}{2}})$  and  $\beta \in W^{1,2}(B_{\frac{1}{2}})$ . Applying d and  $d^*$  on both side of (3.7), we obtain

$$\Delta \alpha = a + \frac{b}{|x|^2} + E \quad \text{in } B_{\frac{1}{2}} \backslash B_{2r}$$

and

$$\Delta \beta = dA \wedge d\tilde{u}$$
 in  $B_{\frac{1}{2}}$ .

Now, we extend E by  $\tilde{E} \in W^{1,2}(B_{\frac{1}{\alpha}})$  such that

$$\|\tilde{E}\|_{L^{2,1}(B_{\frac{1}{2}})} \le 2\|E\|_{L^{2,1}(B_{\frac{1}{2}}\setminus B_{2r})}.$$

Similarly, we may extend  $t:=a+\frac{b}{|x|^2}$  to  $\tilde{t}\in L^{2,1}(B_{\frac{1}{2}})$  such that

$$\|\tilde{t}\|_{L^{2,1}(B_{\frac{1}{2}})} \le 2\|t\|_{L^{2,1}(B_{\frac{1}{2}}\setminus B_{2r})}.$$

Let  $\tilde{\alpha} \in W_0^{1,2}(B_{\frac{1}{2}})$  solve the equation

$$\Delta \tilde{\alpha} = \tilde{E} + \tilde{t}$$
 in  $B_{\frac{1}{2}}$ .

Since  $\alpha - \tilde{\alpha}$  is harmonic in  $B_{\frac{1}{2}} \setminus B_{2r}$ , we may argue similarly as in (3.6) to obtain

$$\|\nabla^{T}\nabla(\alpha - \tilde{\alpha})\|_{L^{2,1}(B_{\frac{1}{4}}\setminus B_{4r})} \leq C\|\nabla^{2}(\alpha - \tilde{\alpha})\|_{L^{2}(B_{\frac{1}{2}}\setminus B_{2r})}$$

$$\leq C(\|\nabla^{2}\tilde{\alpha}\|_{L^{2}(B_{\frac{1}{2}}\setminus B_{2r})} + \|\nabla^{2}\alpha\|_{L^{2}(B_{\frac{1}{2}})})$$

$$\leq C(\|E\|_{L^{2}(B_{\frac{1}{2}}\setminus B_{2r})} + \|t\|_{L^{2}(B_{1}\setminus B_{r})} + \|\Delta\alpha\|_{L^{2}(B_{\frac{1}{2}})})$$

$$\leq C(\|E\|_{L^{2,1}(B_{\frac{1}{2}}\setminus B_{2r})} + \|D - A\Delta\tilde{u}\|_{L^{2}(B_{1}\setminus B_{r})} + \|\nabla A\nabla\tilde{u}\|_{L^{2}(B_{\frac{1}{2}})} + \|A\nabla^{2}\tilde{u}\|_{L^{2}(B_{\frac{1}{2}})})$$

$$\leq C(\|\nabla^{2}u\|_{L^{2}(B_{1}\setminus B_{r})} + \|\nabla u\|_{L^{4}(B_{1}\setminus B_{r})} + \|f\|_{L^{p}(B_{1}\setminus B_{r})}),$$
(3.8)

where C is a positive constant independent of r.

It follows from the standard  $L^p$ -theory, Hölder's inequality and the Sobolev embedding that

$$\|\nabla^{2}\beta\|_{L^{2,1}(B_{\frac{1}{4}})} \leq C\|dA \wedge d\tilde{u}\|_{L^{2,1}(B_{\frac{1}{2}})} \leq C\|\nabla\tilde{u}\|_{L^{4,2}(B_{\frac{1}{2}})}$$

$$\leq C(\|\nabla^{2}u\|_{L^{2}(B_{1}\backslash B_{r})} + \|\nabla u\|_{L^{4}(B_{1}\backslash B_{r})}).$$
(3.9)

Applying the standard estimate for harmonic functions and Lemma 3.2, we obtain

$$\left\| a + \frac{b}{|x|^2} \right\|_{L^{2,1}(B_{\frac{1}{2}} \setminus B_{2r})} \le C \left\| a + \frac{b}{|x|^2} \right\|_{L^2(B_1 \setminus B_r)} \le C \|D - A\Delta \tilde{u}\|_{L^2(B_1 \setminus B_r)}$$

$$\le C \left( \|\nabla^2 u\|_{L^2(B_1 \setminus B_r)} + \|\nabla u\|_{L^4(B_1 \setminus B_r)} + \|f\|_{L^p(B_1 \setminus B_r)} \right).$$
(3.10)

Notice that

$$\|\nabla^T \nabla u\|_{L^{2,1}} \le C(\|\nabla^T (A\nabla u)\|_{L^{2,1}} + \|\nabla^T A\nabla u\|_{L^{2,1}}).$$

Combining this with (3.7), (3.8), (3.9) and (3.10), we conclude that

$$\begin{split} &\|\nabla^T \nabla u\|_{L^{2,1}(B_{\frac{1}{4}} \backslash B_{4r})} \leq C(\|\nabla^T \nabla \alpha + \nabla^T \nabla^{\perp} \beta\|_{L^{2,1}(B_{\frac{1}{4}} \backslash B_{4r})} + \|\nabla^T A \nabla u\|_{L^{2,1}(B_{\frac{1}{4}} \backslash B_{4r})}) \\ &\leq C(\|\nabla^T \nabla (\alpha - \tilde{\alpha})\|_{L^{2,1}(B_{\frac{1}{4}} \backslash B_{4r})} + \|\nabla^2 \tilde{\alpha}\|_{L^{2,1}(B_{\frac{1}{4}} \backslash B_{4r})} + \|\nabla^2 \beta\|_{L^{2,1}(B_{\frac{1}{4}})} + \|\nabla u\|_{L^{4,2}(B_{\frac{1}{4}} \backslash B_{4r})}) \\ &\leq C(\|E\|_{L^{2,1}(B_{\frac{1}{2}} \backslash B_{2r})} + \|t\|_{L^{2,1}(B_{\frac{1}{2}} \backslash B_{2r})} + \|\nabla^2 u\|_{L^{2}(B_{1} \backslash B_{r})} + \|\nabla u\|_{L^{4}(B_{1} \backslash B_{r})} + \|f\|_{L^{p}(B_{1} \backslash B_{r})}) \\ &\leq C(\|\nabla^2 u\|_{L^{2}(B_{1} \backslash B_{r})} + \|\nabla u\|_{L^{4}(B_{1} \backslash B_{r})} + \|f\|_{L^{p}(B_{1} \backslash B_{r})}). \end{split}$$

Hence, the proof is complete.

# 4. Proof of Corollary 1.2

In this section, we will prove Corollary 1.2. In the first step, we prove  $L^{2,\infty}$  and  $L^{4,\infty}$  estimates for  $\nabla^2 u$  and  $\nabla u$ , respectively.

**Lemma 4.1.** There exists  $\delta > 0$  such that for all  $r_k$ ,  $R_k > 0$  with  $2r_k < R_k$  and  $\lim_{k \to \infty} R_k = 0$ , all  $f_k \in L^p(B_{R_k} \setminus B_{r_k}, \mathbb{R}^n)$  with uniformly bounded norm in  $L^p(p > 1)$  and all  $u_k \in W^{2,2}(B_{R_k} \setminus B_{r_k}, \mathbb{R}^n)$ 

satisfying

$$\Delta^2 u_k = \Delta(V_k \cdot \nabla u_k) + \operatorname{div}(w_k \nabla u_k) + (\nabla \omega_k + F_k) \cdot \nabla u_k + f_k \quad \text{in } B_{R_k} \setminus B_{r_k}$$

and

$$\sup_{k} \sup_{r_{k} < \rho < \frac{R_{k}}{2}} \left( \|V_{k}\|_{W^{1,2}(B_{2\rho} \setminus B_{\rho})} + \|w_{k}\|_{L^{2}(B_{2\rho} \setminus B_{\rho})} + \|\omega_{k}\|_{L^{2}(B_{2\rho} \setminus B_{\rho})} + \|F_{k}\|_{L^{\frac{4}{3},1}(B_{2\rho} \setminus B_{\rho})} \right) \leq \delta,$$

then there exists C > 0, independent of  $u_k, r_k$  and  $R_k$ , such that

$$\begin{split} &\|\nabla^2 u_k\|_{L^{2,\infty}(B_{R_k/2}\setminus B_{2r_k})} + \|\nabla u_k\|_{L^{4,\infty}(B_{R_k/2}\setminus B_{2r_k})} \\ \leq &C \sup_{r_k < \rho < \frac{R_k}{r_k}} \left( \|\nabla^2 u_k\|_{L^2(B_{2\rho}\setminus B_\rho)} + \|\nabla u_k\|_{L^4(B_{2\rho}\setminus B_\rho)} \right) + CR_k^{\frac{4(p-1)}{p}}. \end{split}$$

*Proof.* Choose  $\delta \leq \frac{\varepsilon}{4}$  and set

$$M := \sup_{r_k < \rho < \frac{R_k}{2}} \left( \|\nabla^2 u_k\|_{L^2(B_{2\rho} \setminus B_\rho)} + \|\nabla u_k\|_{L^4(B_{2\rho} \setminus B_\rho)} \right).$$

Then for all  $2r_k \leq \rho \leq \frac{R_k}{4}$ , Theorem 2.3 implies that there exist q > 2 and a constant C independent of  $u_k$  such that

$$\rho^{2-\frac{4}{q}} \|\nabla^2 u_k\|_{L^q(B_{2\rho}\setminus B_\rho)} + \rho^{1-\frac{2}{q}} \|\nabla u_k\|_{L^{2q}(B_{2\rho}\setminus B_\rho)}$$

$$\leq C(M + (R_k/2)^{\frac{4(p-1)}{p}} \|f_k\|_{L^p(B_{R_k}\setminus B_{r_k})}) =: CM_1.$$

Next, we shall estimate the level set  $A(\lambda) \equiv \{x \in B_{R_k/2} \setminus B_{2r_k} : |\nabla^2 u_k|(x) > \lambda\}$  for any  $\lambda > 0$ . Write  $\hat{f}_k = |\nabla^2 u_k| \chi_{B_{R_k/2} \setminus B_{2r_k}}$ . Then  $A(\lambda) = \{x \in \mathbb{R}^4 : \hat{f}_k(x) > \lambda\}$ . Direct computation shows that for  $\rho \in [2r_k, R_k/4]$ , it holds

$$\int_{B_{2\rho}\setminus B_{\rho}} \hat{f}_k(x)^q dx \le C^q M_1^q \rho^{4-2q}.$$

Fix  $\lambda > 0$ . Then for any  $j \in \mathbb{Z}$ , we have

$$\lambda^2 |\{x \in B_{2^{j+1}\rho} \setminus B_{2^j\rho} : \hat{f}_k(x) > \lambda\}| \leq \lambda^{2-q} \int_{B_{2^{j+1}\rho} \setminus B_{2^j\rho}} \hat{f}_k(x)^q \, dx \leq C^q \lambda^{2-q} M_1^q (2^j \rho)^{4-2q}.$$

Choosing  $\rho = \frac{1}{\sqrt{\lambda}}$ , we obtain

$$\lambda^2 |\{x \in \mathbb{R}^4 \backslash B_{2^j/\sqrt{\lambda}} : \hat{f}_k(x) > \lambda\}| \le C \sum_{i > j} M_1^q 2^{i(4-2q)} = C M_1^q 2^{j(4-2q)}.$$

On the other hand, note that

$$\lambda^{2} |\{x \in B_{2^{j}/\sqrt{\lambda}} : \hat{f}_{k}(x) > \lambda\}| \leq \lambda^{2} |B_{2^{j}/\sqrt{\lambda}}| = \frac{\pi^{2}}{2} 2^{4j}$$

and thus for any  $j \in \mathbb{Z}$  we have

$$\lambda^2 A(\lambda) \le C(2^{j(4-2q)} M_1^q + 2^{4j}).$$

Selecting j such that  $2^{4j} \approx M_1^q$ , we obtain that

$$\|\nabla^2 u_k\|_{L^{2,\infty}(B_{R_k/2}\setminus B_{2r_k})} \le CM_1^{\frac{q}{2}} \le CM_1.$$

Similarly, we can prove the estimate for  $\|\nabla u_k\|_{L^{4,\infty}(B_{R_k/2}\setminus B_{2r_k})}$ . This completes the proof.

Combining Theorem 3.3 and Lemma 4.1 yields the estimate of the angular energy in the neck region.

**Theorem 4.2.** There exist constants  $\delta, C > 0$  such that for all  $r_k, R_k > 0$  with  $2r_k < R_k$  and  $\lim_{k \to \infty} R_k = 0$ , all  $f_k \in L^p(B_{R_k} \setminus B_{r_k}, \mathbb{R}^n)$  with p > 1 and all  $u_k \in W^{2,2}(B_{R_k} \setminus B_{r_k}, \mathbb{R}^n)$  satisfying

$$\Delta^2 u_k = \Delta(V_k \cdot \nabla u_k) + \operatorname{div}(w_k \nabla u_k) + (\nabla \omega_k + F_k) \cdot \nabla u_k + f_k \quad \text{in } B_{R_k} \setminus B_{r_k}$$

and

$$\sup_{k} \sup_{r_{k} < \rho < \frac{R_{k}}{2}} \left( \|V_{k}\|_{W^{1,2}(B_{2\rho} \setminus B_{\rho})} + \|w_{k}\|_{L^{2}(B_{2\rho} \setminus B_{\rho})} + \|\omega_{k}\|_{L^{2}(B_{2\rho} \setminus B_{\rho})} + \|F_{k}\|_{L^{\frac{4}{3},1}(B_{2\rho} \setminus B_{\rho})} \right) \le \delta,$$

there holds

$$\|\nabla^{T}(\nabla u_{k})\|_{L^{2}(B_{R_{k}/2}\setminus B_{2r_{k}})} + \|\nabla^{T}u_{k}\|_{L^{4}(B_{R_{k}/2}\setminus B_{2r_{k}})}$$

$$\leq C \left( \sup_{r_{k}<\rho<\frac{R_{k}}{2}} \left( \|\nabla^{2}u_{k}\|_{L^{2}(B_{2\rho}\setminus B_{\rho})} + \|\nabla u_{k}\|_{L^{4}(B_{2\rho}\setminus B_{\rho})} \right) + R_{k}^{\frac{4(p-1)}{p}} \right)$$

$$\cdot \left( \|\nabla^{2}u_{k}\|_{L^{2}(B_{R_{k}}\setminus B_{r_{k}})} + \|\nabla u_{k}\|_{L^{4}(B_{R_{k}}\setminus B_{r_{k}})} + \|f_{k}\|_{L^{p}(B_{R_{k}}\setminus B_{r_{k}})} \right).$$

Next, we prove the *bubble-tree* decomposition.

**Proposition 4.3** (Basic properties of blow-up). Under the same notations as in the Corollary 1.2, there hold

(1) For each  $i \neq j$ ,

$$\lim_{k\to\infty}(\frac{\lambda_k^j}{\lambda_k^i}+\frac{\lambda_k^i}{\lambda_k^j}+\frac{|a_k^i-a_k^j|}{\lambda_k^i+\lambda_k^j})=\infty.$$

(2) For each i, there exists a family of neck domains  $N_k^i = B_{\mu_k^i}(a_k^i) \setminus B_{\lambda_k^i}(a_k^i)$  with  $\lim_{k \to \infty} (\mu_k^i/\lambda_k^i) = \infty$ , such that

$$\|V_k\|_{W^{1,2}(N_k^i)} + \|w_k\|_{L^2(N_k^i)} + \|\omega_k\|_{L^2(N_k^i)} + \|F_k\|_{L^{\frac{4}{3},1}(N_i^i)} \leq \min\left\{\delta, \frac{\varepsilon}{2}\right\},$$

where  $\varepsilon$  and  $\delta$  are the constants given by Theorem 2.3 and Lemma 4.1, respectively.

(3) For any neck region  $N_k^i$ , it holds that for each  $\varepsilon > 0$ , there exists  $\lambda(\varepsilon) > 1$  such that for all  $\lambda > \lambda(\varepsilon)$  and for all  $k \gg 1$ ,

$$\sup_{\lambda \lambda_k^i \le \rho \le \frac{\mu_k^i}{2\lambda}} \int_{B_{2\rho}(a_k^i) \setminus B_{\rho}(a_k^i)} (|\nabla^2 u_k|^2 + |\nabla u_k|^4) \, dx \le \varepsilon,$$

which is equivalent to

$$\lim_{\lambda \to \infty} \left( \lim_{k \to \infty} \sup_{\lambda \lambda_k^i \le \rho \le \frac{\mu_k^i}{2\lambda}} \int_{B_{2\rho}(a_k^i) \setminus B_{\rho}(a_k^i)} (|\nabla^2 u_k|^2 + |\nabla u_k|^4) \, dx \right) = 0.$$

*Proof.* The proof consists of four steps.

Step 1: Find energy concentration points. Let  $\varepsilon > 0$  be the constant given by Theorem 2.3 and let  $\delta$  be given by Lemma 4.1. By the boundedness assumption, there exists at most finitely many points  $\{a^1, \dots, a^l\} \subset B_1$  such that

$$\lim_{r\to 0} \lim_{k\to \infty} \left( \|V_k\|_{W^{1,2}(B(a^i,r))} + \|w_k\|_{L^2(B(a^i,r))} + \|\omega_k\|_{L^2(B(a^i,r))} + \|F_k\|_{L^{\frac{4}{3},1}(B(a^i,r))} \right) \ge \varepsilon.$$

Since  $V_k \rightharpoonup V_\infty$  in  $W^{1,2}(B_1)$ ,  $w_k \rightharpoonup w_\infty$  in  $L^2(B_1)$ ,  $\omega_k \rightharpoonup \omega_\infty$  in  $L^2(B_1)$ ,  $F_k \rightharpoonup F_\infty$  in  $L^{\frac{4}{3},1}(B_1)$ ,  $f_k \rightharpoonup f_\infty$  in  $L^p(B_1)$  and  $u_k \rightharpoonup u_\infty$  in  $W^{2,2}(B_1)$ , we infer from Theorem 2.6 that

$$\Delta^2 u_{\infty} = \Delta (V_{\infty} \cdot \nabla u_{\infty}) + \operatorname{div}(w_{\infty} \nabla u_{\infty}) + (\nabla \omega_{\infty} + F_{\infty}) \cdot \nabla u_{\infty} + f_{\infty}.$$

Furthermore, Theorem 2.3 implies that  $u_k \to u_\infty$  in  $W_{loc}^{2,2}(B_1 \setminus \{a^1, \dots, a^l\})$ .

Step 2: The first time blow-up analysis. For simplicity, we may assume that l=1 and  $a^1=0$ . For r>0, define a center of mass on B(0,r) by

$$a_k = \frac{\|zV_k(z)\|_{W^{1,2}(B(0,r))} + \|zw_k(z)\|_{L^2(B(0,r))} + \|z\omega_k(z)\|_{L^2(B(0,r))} + \|zF_k(z)\|_{L^{4/3,1}(B(0,r))}}{\|V_k\|_{W^{1,2}(B(0,r))} + \|w_k\|_{L^2(B(0,r))} + \|\omega_k\|_{L^2(B(0,r))} + \|F_k\|_{L^{4/3,1}(B(0,r))}}.$$

Observe that  $|a_k| \leq r$ . Choose  $\lambda_k > 0$  so that

$$||V_k||_{W^{1,2}(B(a_k,r)\setminus B(a_k,\lambda_k))} + ||w_k||_{L^2(B(a_k,r)\setminus B(a_k,\lambda_k))} + ||\omega_k||_{L^2(B(a_k,r)\setminus B(a_k,\lambda_k))} + ||F_k||_{L^{\frac{4}{3},1}(B(a_k,r)\setminus B(a_k,\lambda_k))} = \min\left\{\delta, \frac{\varepsilon}{2}\right\}.$$
(4.1)

If  $\lambda_k \neq o_k(1)$ , we restart the process with r replaced by  $\liminf_k \lambda_k/2$  until  $\lambda_k \to 0$ . In other word, we find an infinitesimal ball  $B(a_k, \lambda_k)$  which concentrates most of the energy of  $\{V_k, w_k, \omega_k, F_k\}$ . But the point here is that outside the infinitesimal ball there is also a small but fixed amount of energy for all  $\{V_k, w_k, \omega_k, F_k\}$ . This will imply that the following iteration process stops after finitely many times. The region  $N_k := B_r(a_k) \setminus B_{\lambda_k}(a_k)$  is the first sequence of necks.

Now we start to blow up  $u_k$  and the coefficient functions  $\{V_k, w_k, \omega_k, F_k, f_k\}$  on  $B(a_k, r)$  as follows. Define

$$\tilde{u}_k(x) := u_k(a_k + \lambda_k x), \quad \tilde{V}_k(x) := \lambda_k V_k(a_k + \lambda_k x),$$

$$\tilde{w}_k(x) := \lambda_k^2 w_k(a_k + \lambda_k x), \quad \tilde{\omega}_k(x) := \lambda_k^2 \omega_k(a_k + \lambda_k x),$$

$$\tilde{F}_k(x) := \lambda_k^3 F_k(a_k + \lambda_k x), \quad \tilde{f}_k(x) := \lambda_k^4 f_k(a_k + \lambda_k x).$$

Then we have

$$\Delta^2 \tilde{u}_k = \Delta(\tilde{V}_k \cdot \nabla \tilde{u}_k) + \operatorname{div}(\tilde{w}_k \nabla \tilde{u}_k) + (\nabla \tilde{\omega}_k + \tilde{F}_k) \cdot \nabla \tilde{u}_k + \tilde{f}_k \quad \text{in} \quad B(0, \frac{r}{\lambda_k}).$$

Note that  $B(0, \frac{r}{\lambda_k}) \xrightarrow{k \to \infty} \mathbb{R}^4$ . Thanks to the conformal invariance, we know that

$$\begin{split} \|\nabla \tilde{u}_k\|_{L^4(B(0,\frac{r}{\lambda_k}))} &= \|\nabla u_k\|_{L^4(B(a_k,r))}, \quad \|\nabla^2 \tilde{u}_k\|_{L^2(B(0,\frac{r}{\lambda_k}))} = \|\nabla^2 u_k\|_{L^2(B(a_k,r))}, \\ \|\nabla \tilde{V}_k\|_{L^2(B(0,\frac{r}{\lambda_k}))} &= \|\nabla V_k\|_{L^2(B(a_k,r))}, \quad \|\tilde{w}_k\|_{L^2(B(0,\frac{r}{\lambda_k}))} = \|w_k\|_{L^2(B(a_k,r))}, \\ \|\tilde{\omega}_k\|_{L^2(B(0,\frac{r}{\lambda_k}))} &= \|\omega_k\|_{L^2(B(a_k,r))}, \quad \|\tilde{F}_k\|_{L^{\frac{4}{3},1}(B(0,\frac{r}{\lambda_k}))} = \|F_k\|_{L^{\frac{4}{3},1}(B(a_k,r))}, \\ \|\tilde{f}_k\|_{L^p(B(0,\frac{r}{\lambda_k}))} &= \lambda_k^{4(1-\frac{1}{p})} \|f_k\|_{L^p(B(a_k,r))}. \end{split}$$

Thus, up to a subsequence, we may assume that  $\tilde{V}_k \rightharpoonup \tilde{V}_\infty$  in  $W^{1,2}_{\rm loc}(\mathbb{R}^4)$ ,  $\tilde{w}_k \rightharpoonup \tilde{w}_\infty$  in  $L^2_{\rm loc}(\mathbb{R}^4)$ ,  $\tilde{\omega}_k \rightharpoonup \tilde{\omega}_\infty$  in  $L^2_{\rm loc}(\mathbb{R}^4)$ ,  $\tilde{F}_k \rightharpoonup \tilde{F}_\infty$  in  $L^{\frac{4}{3},1}_{\rm loc}(\mathbb{R}^4)$ ,  $\tilde{u}_k \rightharpoonup \tilde{u}_\infty$  in  $W^{2,2}_{\rm loc}(\mathbb{R}^4)$  and  $\tilde{f}_k \rightharpoonup 0$  in  $L^p_{\rm loc}(\mathbb{R}^4)$ . By Theorem 2.6,  $\tilde{u}_\infty$  is a bubble, i.e.,

$$\Delta^2 \tilde{u}_{\infty} = \Delta (\tilde{V}_{\infty} \cdot \nabla \tilde{u}_{\infty}) + \operatorname{div}(\tilde{w}_{\infty} \nabla \tilde{u}_{\infty}) + (\nabla \tilde{\omega}_{\infty} + \tilde{F}_{\infty}) \cdot \nabla \tilde{u}_{\infty}, \text{ in } \mathbb{R}^4.$$

Moreover, we have

$$\tilde{u}_k \to \tilde{u}_{\infty}$$
 in  $W_{\text{loc}}^{2,2}(\mathbb{R}^4 \setminus \{b^1, \cdots, b^m\}),$ 

where  $\{b^1, \dots, b^m\}$  is the set of possible concentration points of  $\{\tilde{V}_k, \tilde{w}_k, \tilde{\omega}_k, \tilde{F}_k\}$ , that is,

$$\lim_{r\to 0}\lim_{k\to \infty}\left(\|\tilde{V}_k\|_{W^{1,2}(B(b^i,r))}+\|\tilde{w}_k\|_{L^2(B(b^i,r))}+\|\tilde{\omega}_k\|_{L^2(B(b^i,r))}+\|\tilde{F}_k\|_{L^{\frac{4}{3},1}(B(b^i,r))}\right)\geq \varepsilon, 1\leq i\leq m.$$

If no such  $b^i$  exists, the blow-up process stops and we obtain  $\tilde{u}_k \to \tilde{u}_\infty$  in  $W^{2,q}_{loc}(\mathbb{R}^4)$ . Otherwise,  $\{b^1, \dots, b^m\}$  is nonempty. By direct calculations, (4.1) implies

$$\begin{split} &\|\nabla \tilde{V}_k\|_{L^2(B_{r/\lambda_k} \setminus B_1)} + \|\tilde{w}_k\|_{L^2(B_{r/\lambda_k} \setminus B_1))} + \|\tilde{\omega}_k\|_{L^2(B_{r/\lambda_k} \setminus B_1)} + \|\tilde{F}_k\|_{L^{\frac{4}{3},1}(B_{r/\lambda_k} \setminus B_1)} \\ &= \|\nabla V_k\|_{L^2(N_k)} + \|w_k\|_{L^2(N_k)} + \|\omega_k\|_{L^2(N_k)} + \|F_k\|_{L^{\frac{4}{3},1}(N_k)} = \min\left\{\delta, \frac{\varepsilon}{2}\right\}. \end{split}$$

Thus, we have  $\{b^1, \dots, b^m\} \subseteq B_1$ . Furthermore, the above equality means that in each blow-up process  $\{\tilde{V}_k, \tilde{w}_k, \tilde{\omega}_k, \tilde{F}_k\}$  takes away a fixed amount of energy.

**Step 3: Iteration.** At each  $a^i$ ,  $i=1,\dots,l$ , we can repeat the above process. For instance, at point  $a^i$ , we find the center of mass  $\bar{a}^i_k$ , and choose  $\bar{\lambda}^i_k \to 0$ ,  $r^i > 0$  such that

$$\begin{split} &\|\tilde{V}_{k}\|_{W^{1,2}(B(\bar{a}_{k}^{i},r^{i})\backslash B(\bar{a}_{k}^{i},\bar{\lambda}_{k}^{i}))} + \|\tilde{w}_{k}\|_{L^{2}(B(\bar{a}_{k}^{i},r^{i})\backslash B(\bar{a}_{k}^{i},\bar{\lambda}_{k}^{i}))} \\ &+ \|\tilde{\omega}_{k}\|_{L^{2}(B(\bar{a}_{k}^{i},r^{i})\backslash B(\bar{a}_{k}^{i},\bar{\lambda}_{k}^{i}))} + \|\tilde{F}_{k}\|_{L^{\frac{4}{3},1}(B(\bar{a}_{k}^{i},r^{i})\backslash B(\bar{a}_{k}^{i},\bar{\lambda}_{k}^{i}))} = \min\left\{\delta,\frac{\varepsilon}{2}\right\}. \end{split}$$

Hence, we may perform the blow up process again to get

$$\bar{u}_{k}^{i}(x) := \tilde{u}_{k}(\bar{a}_{k}^{i} + \bar{\lambda}_{k}^{i}x), \quad \bar{V}_{k}^{i}(x) := \bar{\lambda}_{k}^{i}\tilde{V}_{k}(\bar{a}_{k}^{i} + \bar{\lambda}_{k}^{i}x), 
\bar{w}_{k}^{i}(x) := (\bar{\lambda}_{k}^{i})^{2}\tilde{w}_{k}(\bar{a}_{k}^{i} + \bar{\lambda}_{k}^{i}x), \quad \bar{\omega}_{k}^{i}(x) := (\bar{\lambda}_{k}^{i})^{2}\tilde{\omega}_{k}(\bar{a}_{k}^{i} + \bar{\lambda}_{k}^{i}x), 
\bar{F}_{k}^{i}(x) := (\bar{\lambda}_{k}^{i})^{3}\tilde{F}_{k}(\bar{a}_{k}^{i} + \bar{\lambda}_{k}^{i}x), \quad \bar{f}_{k}^{i}(x) := (\bar{\lambda}_{k}^{i})^{4}\tilde{f}_{k}(\bar{a}_{k}^{i} + \bar{\lambda}_{k}^{i}x).$$

We can relabel the sequence  $\{\bar{u}_k^i, \bar{V}_k^i, \bar{w}_k^i, \bar{\omega}_k^i, \bar{F}_k^i, \bar{f}_k^i\}$  in terms of  $\{u_k, V_k, w_k, \omega_k, F_k, f_k\}$ . For simplicity, we write

$$a_k^0 = a_k, \quad \lambda_k^0 = \lambda_k, \quad a_k^i = a_k^0 + \lambda_k^0 \cdot \bar{a}_k^i \quad \text{and} \quad \lambda_k^i = \lambda_k^0 \cdot \bar{\lambda}_k^i.$$

Then, we have

$$\bar{u}_{k}^{i}(x) := u_{k}(a_{k}^{i} + \lambda_{k}^{i}x), \quad \bar{V}_{k}^{i}(x) := \lambda_{k}^{i}V_{k}(a_{k}^{i} + \lambda_{k}^{i}x),$$

$$\bar{w}_{k}^{i}(x) := (\lambda_{k}^{i})^{2}w_{k}(a_{k}^{i} + \lambda_{k}^{i}x), \quad \bar{\omega}_{k}^{i}(x) := (\lambda_{k}^{i})^{2}\omega_{k}(a_{k}^{i} + \lambda_{k}^{i}x),$$

$$\bar{F}_{k}^{i}(x) := (\lambda_{k}^{i})^{3}F_{k}(a_{k}^{i} + \lambda_{k}^{i}x), \quad \bar{f}_{k}^{i}(x) := (\lambda_{k}^{i})^{4}f_{k}(a_{k}^{i} + \lambda_{k}^{i}x)$$

and

$$\begin{split} & \|\nabla V_k\|_{L^2(B(a_k^i, r^i\lambda_k^0) \setminus B(a_k^i, \lambda_k^i))} + \|w_k\|_{L^2(B(a_k^i, r^i\lambda_k^0) \setminus B(a_k^i, \lambda_k^i))} \\ & + \|\omega_k\|_{L^2(B(a_k^i, r^i\lambda_k^0) \setminus B(a_k^i, \lambda_k^i))} + \|F_k\|_{L^{\frac{4}{3}, 1}(B(a_k^i, r^i\lambda_k^0) \setminus B(a_k^i, \lambda_k^i))} \\ & = \|\nabla \tilde{V}_k\|_{L^2(B(\bar{a}_k^i, r^i) \setminus B(\bar{a}_k^i, \bar{\lambda}_k^i))} + \|\tilde{w}_k\|_{L^2(B(\bar{a}_k^i, r^i) \setminus B(\bar{a}_k^i, \bar{\lambda}_k^i))} \\ & + \|\tilde{\omega}_k\|_{L^2(B(\bar{a}_k^i, r^i) \setminus B(\bar{a}_k^i, \bar{\lambda}_k^i))} + \|\tilde{F}_k\|_{L^{\frac{4}{3}, 1}(B(\bar{a}_k^i, r^i) \setminus B(\bar{a}_k^i, \bar{\lambda}_k^i))} = \min \left\{\delta, \frac{\varepsilon}{2}\right\}. \end{split}$$

To unify the notation, for each  $i = 0, 1, \dots, l$ , we set

$$\begin{split} u_k^i(x) &:= u_k (a_k^i + \lambda_k^i x), \quad V_k^i(x) := \lambda_k^i V_k (a_k^i + \lambda_k^i x), \\ w_k^i(x) &:= (\lambda_k^i)^2 w_k (a_k^i + \lambda_k^i x), \quad \omega_k^i(x) := (\lambda_k^i)^2 \omega_k (a_k^i + \lambda_k^i x), \\ F_k^i(x) &:= (\lambda_k^i)^3 F_k (a_k^i + \lambda_k^i x), \quad f_k^i(x) := (\lambda_k^i)^4 f_k (a_k^i + \lambda_k^i x). \end{split}$$

Then for any  $i \geq 1$ , it holds

$$\lim_{k \to \infty} \frac{\lambda_k^0}{\lambda_k^i} = \infty \quad \text{and} \quad \frac{|a_k^0 - a_k^i|}{\lambda_k^0} \le C.$$

For any  $i, j \geq 1$  and  $i \neq j$ , we have

$$\lim_{k\to\infty}\frac{|a_k^i-a_k^j|}{\lambda_k^i+\lambda_k^j}=\lim_{k\to\infty}\frac{|\bar{a}_k^i-\bar{a}_k^j|}{\bar{\lambda}_k^i+\bar{\lambda}_k^j}=\infty.$$

This proves the first assertion (1).

Denote by  $N_k^i := B(a_k^i, r^i \lambda_k^0) \setminus B(a_k^i, \lambda_k^i)$  the neck region with  $\frac{r^i \lambda_k^0}{\lambda_k^i} \xrightarrow{k \to \infty} \infty$ . Then

$$\|V_k\|_{W^{1,2}(N_k^i)} + \|w_k\|_{L^2(N_k^i)} + \|\omega_k\|_{L^2(N_k^i)} + \|F_k\|_{L^{\frac{4}{3},1}(N_k^i)} = \min\left\{\delta, \frac{\varepsilon}{2}\right\}.$$

This proves the second assertion (2).

We may repeat the above process. Since each time the energy of the coefficient functions  $\{V_k, w_k, \omega_k, F_k\}$  decreases at least  $\min\{\delta, \frac{\varepsilon}{2}\}$ , which is independent of k, the process will stop after finitely many times.

Step 4: Proof of assertion (3). We argue by contradiction. Suppose there is a neck domain  $N_k^i = B_{\mu_k^i}(a_k^i) \setminus B_{\lambda_k^i}(a_k^i)$  satisfying assertion (2), and there exists  $\varepsilon_1 > 0$  such that for all  $r_k > 0$ , there exist  $\rho_k$  such that  $B_{2\rho_k}(a_k^i) \setminus B_{\rho_k}(a_k^i) \subseteq N_k^i(r_k)$  and

$$\int_{B_{2\rho_k}(a_k^i)\backslash B_{\rho_k}(a_k^i)} (|\nabla^2 u_k|^2 + |\nabla u_k|^4) \, dx \ge \varepsilon_1,$$

where  $N_k^i(\lambda) = B(a_k^i, \frac{\mu_k^i}{\lambda}) \setminus B(a_k^i, \lambda \lambda_k^i)$ . Choose  $r_k = k$  such that  $B_{2\rho_k}(a_k^i) \setminus B_{\rho_k}(a_k^i) \subseteq N_k^i(k)$ , then  $k\lambda_k^i \le \rho_k \le 2\rho_k \le \frac{\mu_k^i}{k}$ , which implies that  $\frac{\mu_k^i}{\rho_k} \ge 2k$  and  $\frac{\lambda_k^i}{\rho_k} \le \frac{1}{k}$ , and so  $\lim_{k \to \infty} \frac{\mu_k^i}{\rho_k} = \infty$ ,  $\lim_{k \to \infty} \frac{\lambda_k^i}{\rho_k} = 0$ .

Now, for  $x \in B(0, \frac{\mu_k^i}{\rho_k})$ , we blow up  $\{u_k, V_k, w_k, \omega_k, F_k, f_k\}$  on  $B(a_k^i, \mu_k^i)$  by setting

$$\hat{u}_k(x) := u_k(a_k^i + \rho_k x), \quad \hat{V}_k(x) := \rho_k V_k(a_k^i + \rho_k x),$$

$$\hat{w}_k(x) := \rho_k^2 w_k(a_k^i + \rho_k x), \quad \hat{\omega}_k(x) := \rho_k^2 \omega_k(a_k^i + \rho_k x).$$

$$\hat{F}_k(x) := \rho_k^3 F_k(a_k^i + \rho_k x), \quad \hat{f}_k(x) := \rho_k^4 f_k(a_k^i + \rho_k x).$$

Then it follows that

$$\Delta^2 \hat{u}_k = \Delta(\hat{V}_k \cdot \nabla \hat{u}_k) + \operatorname{div}(\hat{w}_k \nabla \hat{u}_k) + (\nabla \hat{\omega}_k + \hat{F}_k) \cdot \nabla \hat{u}_k + \hat{f}_k \quad \text{in } B(0, \frac{\mu_k^i}{\sigma_k}).$$

Similar to **Step 2**, we may assume that  $\hat{V}_k \rightharpoonup \hat{V}_\infty$  in  $W^{1,2}_{loc}(\mathbb{R}^4)$ ,  $\hat{w}_k \rightharpoonup \hat{w}_\infty$  in  $L^2_{loc}(\mathbb{R}^4)$ ,  $\hat{\omega}_k \rightharpoonup \hat{\omega}_\infty$  weakly in  $L^2_{loc}(\mathbb{R}^4)$ ,  $\hat{F}_k \rightharpoonup \hat{F}_\infty$  in  $L^{\frac{4}{3},1}_{loc}(\mathbb{R}^4)$ ,  $\hat{u}_k \rightharpoonup \hat{u}_\infty$  in  $W^{2,2}_{loc}(\mathbb{R}^4)$  and  $\hat{f}_k \rightharpoonup 0$  in  $L^p_{loc}(\mathbb{R}^4)$ . By Theorem 2.6,  $\hat{u}_\infty$  is a bubble:

$$\Delta^2 \hat{u}_{\infty} = \Delta (\hat{V}_{\infty} \cdot \nabla \hat{u}_{\infty}) + \operatorname{div}(\hat{w}_{\infty} \nabla \hat{u}_{\infty}) + (\nabla \hat{\omega}_{\infty} + \hat{F}_{\infty}) \cdot \nabla \hat{u}_{\infty} \quad \text{in } \mathbb{R}^4.$$

Now we claim that  $\hat{u}_{\infty}$  is a non-trivial bubble. Indeed, by assertion (2), we have

$$\|\nabla V_k\|_{L^2(\hat{N}_k^i)} + \|\hat{w}_k\|_{L^2(\hat{N}_k^i)} + \|\hat{\omega}_k\|_{L^2(\hat{N}_k^i)} + \|\bar{F}_k\|_{L^{\frac{4}{3},1}(\hat{N}_k^i)}$$

$$= \|\nabla V_k\|_{L^2(N_k^i)} + \|w_k\|_{L^2(N_k^i)} + \|\omega_k\|_{L^2(N_k^i)} + \|F_k\|_{L^{\frac{4}{3},1}(N_k^i)} = \min\left\{\delta, \frac{\varepsilon}{2}\right\},$$

where

$$\hat{N}_k^i \equiv \frac{N_k^i - a_k^i}{\rho_k} = \frac{B_{\mu_k^i}(a_k^i) \backslash B_{\lambda_k^i}(a_k^i) - a_k^i}{\rho_k} = B(0, \frac{\mu_k^i}{\rho_k}) \backslash B(0, \frac{\lambda_k^i}{\rho_k}) \to \mathbb{R}^4 \backslash \{0\}.$$

By Theorem 2.3,  $\hat{u}_k \to \hat{u}_\infty$  in  $W_{loc}^{2,2}(\mathbb{R}^4 \setminus \{0\})$ . Furthermore, note that

$$\int_{B_2(0)\backslash B_1(0)} |\nabla^2 \hat{u}_k|^2 + |\nabla \hat{u}_k|^4 \, dx = \int_{B_2\rho_k(a_k^i)\backslash B_{\rho_k}(a_k^i)} |\nabla^2 u_k|^2 + |\nabla u_k|^4 \, dx \ge \varepsilon_1,$$

and thus we infer from the strong convergence  $\hat{u}_k \to \hat{u}_{\infty}$  in  $W^{2,2}_{loc}(\mathbb{R}^4 \setminus \{0\})$  that

$$\int_{B_2(0)\backslash B_1(0)} |\nabla^2 \hat{u}_{\infty}|^2 + |\nabla \hat{u}_{\infty}|^4 dx = \lim_{k \to \infty} \int_{B_2(0)\backslash B_1(0)} |\nabla^2 \hat{u}_k|^2 + |\nabla \hat{u}_k|^4 dx \ge \varepsilon_1.$$

This shows  $\hat{u}_{\infty}$  is non-trivial in  $\mathbb{R}^4 \setminus \{0\}$ . Due to Theorem 2.5, it has to be a non-trivial solution in  $\mathbb{R}^4$ . As a consequence, we deduce from Theorem 2.4 that

$$\|\hat{V}_{\infty}\|_{W^{1,2}(\mathbb{R}^4)} + \|\hat{w}_{\infty}\|_{L^2(\mathbb{R}^4)} + \|\hat{\omega}_{\infty}\|_{L^2(\mathbb{R}^4)} + \|\hat{F}_{\infty}\|_{L^{\frac{4}{3},1}(\mathbb{R}^4)} \ge \varepsilon.$$

On the other hand, by the lower semicontinuity, we have

$$\begin{split} &\|\hat{V}_{\infty}\|_{W^{1,2}(K)} + \|\hat{w}_{\infty}\|_{L^{2}(K)} + \|\hat{\omega}_{\infty}\|_{L^{2}(K)} + \|\hat{F}_{\infty}\|_{L^{\frac{4}{3},1}(K)} \\ \leq & \liminf_{k \to \infty} \left( \|\hat{V}_{k}\|_{W^{1,2}(\hat{N}_{k}^{i})} + \|\hat{w}_{k}\|_{L^{2}(\hat{N}_{k}^{i})} + \|\hat{\omega}_{k}\|_{L^{2}(\hat{N}_{k}^{i})} + \|\hat{F}_{k}\|_{L^{\frac{4}{3},1}(\hat{N}_{k}^{i})} \right) \leq \frac{\varepsilon}{2} \end{split}$$

for any compact set  $K \subset \mathbb{R}^4 \setminus \{0\}$ . This implies that

$$\|\hat{V}_{\infty}\|_{W^{1,2}(\mathbb{R}^4)} + \|\hat{w}_{\infty}\|_{L^2(\mathbb{R}^4)} + \|\hat{\omega}_{\infty}\|_{L^2(\mathbb{R}^4)} + \|\hat{F}_{\infty}\|_{L^{\frac{4}{3},1}(\mathbb{R}^4)} \le \frac{\varepsilon}{2},$$

which is a contradiction. This proves the assertion (3). Hence, the proof of Proposition 4.3 is complete.  $\Box$ 

Proof of Corollary 1.2. Let  $N_k^i = B(a_k^i, \mu_k^i) \backslash B(a_k^i, \lambda_k^i)$  be a neck domain given by Proposition 4.3 and  $N_k^i(\lambda) = B(a_k^i, \frac{\mu_k^i}{\lambda}) \backslash B(a_k^i, \lambda \lambda_k^i)$ . Since  $\|h\|_{L^2}^2 \leq \|h\|_{L^{2,1}} \|h\|_{L^{2,\infty}}$  and  $\|h\|_{L^4}^2 \leq \|h\|_{L^{4,2}} \|h\|_{L^{4,\infty}}$ , we obtain from Theorem 4.2 and Proposition 4.3 that

$$\lim_{\lambda \to \infty} \lim_{k \to \infty} \left( \|\nabla^T (\nabla u_k)\|_{L^2(N_k^i(\lambda))} + \|\nabla^T u_k\|_{L^4(N_k^i(\lambda))} \right) = 0.$$

This implies Corollary 1.2.

### 5. Proof of Theorem 1.1

In this section, we will prove Theorem 1.1. Since the proof is similar to that of Corollary 1.2, we only point out the main differences.

As in the proof of Corollary 1.2, we first prove  $L^{2,1}$  and  $L^{4,2}$  estimates for the tangential derivatives in the annular region.

**Theorem 5.1.** There exist constants  $\varepsilon, C = C(n) > 0$  such that for all  $0 < r < \frac{1}{8}$ , all  $f \in L \log L(B_1 \backslash B_r, \mathbb{R}^n)$  and all  $u \in W^{2,2}(B_1 \backslash B_r, \mathbb{R}^n)$  satisfying

$$\Delta^2 u = \Delta(V \cdot \nabla u) + \operatorname{div}(w \nabla u) + (\nabla \omega + F) \cdot \nabla u + f \quad in \ B_1 \backslash B_r$$

and

$$||V||_{W^{1,2}(B_1 \backslash B_r)} + ||w||_{L^2(B_1 \backslash B_r)} + ||\omega||_{L^2(B_1 \backslash B_r)} + ||F||_{L^{\frac{4}{3},1}(B_1 \backslash B_r)} < \varepsilon,$$

then there holds

$$\begin{split} \|\nabla^T \nabla u\|_{L^{2,1}(B_{\frac{1}{4}} \setminus B_{4r})} + \|\nabla^T u\|_{L^{4,2}(B_{\frac{1}{4}} \setminus B_{4r})} \\ & \leq C \left( \|\nabla^2 u\|_{L^2(B_1 \setminus B_r)} + \|\nabla u\|_{L^4(B_1 \setminus B_r)} + \|f\|_{L \log L(B_1 \setminus B_r)} \right). \end{split}$$

*Proof.* The proof is quite similar to that of Theorem 3.3. The only difference lies in the estimates involving f. More precisely, we extend Af by  $\tilde{f} \in L \log L(B_1)$  such that

$$\|\tilde{f}\|_{L\log L(B_1)} \le 2\|Af\|_{L\log L(B_1\setminus B_r)}$$

Let  $D \in W_0^{1,\frac{4}{3}}(B_1)$  solve the equation

$$\Delta D = \operatorname{div}(K) + \tilde{f} \quad \text{in } B_1.$$

Then we may apply the Calderón-Zygmund theory with  $\tilde{f} \in L \log L$  as in [8, Proof of Theorem 1.6] and the Sobolev embeddings  $W^{1,(\frac{4}{3},1)} \hookrightarrow L^{2,1}$  and  $W^{2,1} \hookrightarrow L^{2,1}$  to obtain

$$\|D\|_{L^{2,1}(B_1)} \leq C(\|K\|_{L^{\frac{4}{3},1}(B_1)} + \|f\|_{L\log L(B_1\backslash B_r)}).$$

With the above estimate at hand, all the other parts of proof works without any changes. Thus the proof is complete.  $\Box$ 

Next, we prove the *bubble-tree* decomposition in the setting of  $L \log L$ .

**Proposition 5.2** (Basic properties of blow-up). Using the same notations as in the Theorem 1.1, there hold

(1) For each  $i \neq j$ ,

$$\lim_{k \to \infty} \left( \frac{\lambda_k^j}{\lambda_k^i} + \frac{\lambda_k^i}{\lambda_k^j} + \frac{|a_k^i - a_k^j|}{\lambda_k^i + \lambda_k^j} \right) = \infty.$$

(2) For each i, there exists a family of neck domains  $N_k^i = B_{\mu_k^i}(a_k^i) \setminus B_{\lambda_k^i}(a_k^i)$  with  $\lim_{k \to \infty} (\mu_k^i / \lambda_k^i) = \infty$ , such that

$$||V_k||_{W^{1,2}(N_k^i)} + ||w_k||_{L^2(N_k^i)} + ||\omega_k||_{L^2(N_k^i)} + ||F_k||_{L^{\frac{4}{3},1}(N_k^i)} = \min\left\{\delta, \frac{\varepsilon}{2}\right\},\,$$

where  $\varepsilon$  and  $\delta$  are the constants given by Theorem 2.3 and Lemma 5.3, respectively.

(3) For any neck region  $N_k^i$ , it holds that for each  $\varepsilon > 0$ , there exists  $\lambda(\varepsilon) > 1$  such that for all  $\lambda > \lambda(\varepsilon)$  and for all  $k \gg 1$ ,

$$\sup_{\lambda \lambda_k^i \le \rho \le \frac{\mu_k^i}{2\lambda}} \int_{B_{2\rho}(a_k^i) \setminus B_{\rho}(a_k^i)} (|\nabla^2 u_k|^2 + |\nabla u_k|^4) \, dx \le \varepsilon,$$

which is equivalent to

$$\lim_{\lambda \to \infty} \left( \lim_{k \to \infty} \sup_{\lambda \lambda_k^i \le \rho \le \frac{\mu_k^i}{2\lambda}} \int_{B_{2\rho}(a_k^i) \setminus B_{\rho}(a_k^i)} (|\nabla^2 u_k|^2 + |\nabla u_k|^4) \, dx \right) = 0.$$

*Proof.* We will follow the proof of Proposition 4.3. The only difference lies in **Step 2**. In **Step 2**, we have by [8, Section 2.3] that

$$\|\hat{f}_k\|_{L\log L(B(0,\frac{r}{\lambda_k}))} \le C \|f_k\|_{L\log L(B(a_k,r))}.$$

Then  $\|\tilde{f}_k\|_{L\log L}$  is uniformly bounded in  $B(0,\frac{r}{\lambda_k})\to\mathbb{R}^4$  and  $\tilde{f}_k\rightharpoonup\tilde{f}$  in a distributional sense, for some  $\tilde{f}\in L\log L(\mathbb{R}^4)$ . On the other hand, by Lemma 2.2, we deduce that  $\|\tilde{f}_k\|_{L^1_{loc}(\mathbb{R}^4)}\to 0$  as  $k\to\infty$ . In fact, for any compact subset U, we have

$$\|\tilde{f}_{k}\|_{L^{1}(U)} = \lambda_{k}^{4} \int_{U} |f_{k}(a_{k} + \lambda_{k}x)| dx = \int_{a_{k} + \lambda_{k}U} |f_{k}(y)| dy$$

$$\leq C \left[ \log \left( \frac{1}{\lambda_{k}} \right) \right]^{-1} \|f_{k}\|_{L \log L(a_{k} + \lambda_{k}U)}.$$

Thus,  $\tilde{f} = 0$ . Hence, by Theorem 2.6,  $\tilde{u}_{\infty}$  is a bubble:

$$\Delta^2 \tilde{u}_{\infty} = \Delta (\tilde{V}_{\infty} \cdot \nabla \tilde{u}_{\infty}) + \operatorname{div}(\tilde{w}_{\infty} \nabla \tilde{u}_{\infty}) + (\nabla \tilde{\omega}_{\infty} + \tilde{F}_{\infty}) \cdot \nabla \tilde{u}_{\infty}.$$

Moreover, we have

$$\tilde{u}_k \to \tilde{u}_{\infty}$$
 in  $W_{\text{loc}}^{2,2}(\mathbb{R}^4 \setminus \{b^1, \cdots, b^m\}),$ 

where  $\{b^1, \dots, b^m\}$  are possible concentration points of  $\{\tilde{V}_k, \tilde{w}_k, \tilde{\omega}_k, \tilde{F}_k\}$ .

Since the arguments for all other steps are essentially the same, we omit them and hence this completes the proof of proposition.

Next, we will establish the following decay estimates, analogous to Lemma 4.1, via a proof similar to [8, Theorem 1.6].

**Lemma 5.3.** There exists  $\delta > 0$  such that for all  $r_k, R_k > 0$  with  $2r_k < R_k$  and  $\lim_{k \to \infty} R_k = 0$ , all  $f_k \in L \log L(B_{R_k} \setminus B_{r_k}, \mathbb{R}^n)$  with uniformly bounded norm and all  $u_k \in W^{2,2}(B_{R_k} \setminus B_{r_k}, \mathbb{R}^n)$  satisfying

$$\Delta^2 u_k = \Delta(V_k \cdot \nabla u_k) + \operatorname{div}(w_k \nabla u_k) + (\nabla \omega_k + F_k) \cdot \nabla u_k + f_k \quad \text{in } B_{R_k} \setminus B_{r_k}$$

and

$$\sup_{k} \sup_{r_k < \rho < \frac{R_k}{2}} \left( \|V_k\|_{W^{1,2}(B_{2\rho} \setminus B_\rho)} + \|w_k\|_{L^2(B_{2\rho} \setminus B_\rho)} + \|\omega_k\|_{L^2(B_{2\rho} \setminus B_\rho)} + \|F_k\|_{L^{\frac{4}{3},1}(B_{2\rho} \setminus B_\rho)} \right) \le \delta,$$

there exists C > 0 independent of  $u_k, r_k$  and  $R_k$  such that

$$\|\nabla^{2} u_{k}\|_{L^{2,\infty}(B_{R_{k}} \setminus B_{2r_{k}})} + \|\nabla u_{k}\|_{L^{4,\infty}(B_{R_{k}} \setminus B_{2r_{k}})}$$

$$\leq C \sup_{r_{k} < \rho < \frac{R_{k}}{2}} \left( \|\nabla^{2} u_{k}\|_{L^{2}(B_{2\rho} \setminus B_{\rho})} + \|\nabla u_{k}\|_{L^{4}(B_{2\rho} \setminus B_{\rho})} \right) + C \left( \log \frac{1}{R_{k}} \right)^{-1}.$$

*Proof.* For the  $L^{2,\infty}$  estimate of  $\nabla^2 u$ , we note

$$\lambda^{2} |\{x \in B_{R_{k}} \setminus B_{2r_{k}} : |\Delta u_{k}(x)| > \lambda\}| = \sum_{j=1}^{N-1} \lambda^{2} |\{x \in B_{2^{j+1}r_{k}} \setminus B_{2^{j}r_{k}} : |\Delta u_{k}(x)| > \lambda\}|,$$

where  $N = N(r_k, R_k)$ .

Fix  $\rho \in (r_k, \frac{R_k}{2})$ . Similar to the proof in beginning part of Theorem 3.3, we know that for  $0 < \delta < \frac{1}{2}$  sufficiently small, there exist

$$A_k \in L^{\infty} \cap W^{2,2}(B_{2\rho}, GL_n)$$
 and  $B_k \in W^{1,\frac{4}{3}}(B_{2\rho})$ 

such that

$$\operatorname{dist}(A_k, SO_n) + \|A_k\|_{W^{2,2}} + \|B_k\|_{W^{1,\frac{4}{3}}} \le C(\|V_k\|_{W^{1,2}} + \|w_k\|_{L^2} + \|\omega_k\|_{L^2} + \|F_k\|_{L^{\frac{4}{3},1}})$$

and

$$\Delta(A_k \Delta u_k) = \operatorname{div}(K_k) + A_k f_k \quad \text{in } B_{2\rho} \backslash B_{\rho}, \tag{5.1}$$

where  $K_k = 2\nabla A_k \Delta u_k - \Delta A_k \nabla u_k + A_k w_k \nabla u_k - \nabla A_k (V_k \cdot \nabla u_k) + A_k \nabla (V_k \cdot \nabla u_k) + B_k \cdot \nabla u_k$ .

Next extend all the functions from  $B_{2\rho}$  to the whole space  $\mathbb{R}^4$  in such a way that their norms in  $\mathbb{R}^4$  are bounded by a constant multiple of the corresponding norms in  $B_{2\rho}$ . With no confusion of notations, we use the same symbols for all the extended functions.

Applying the  $L^{\infty}$  boundedness of  $A^{-1}$  and the classical Calderon-Zygmund theory, we have

$$\sum_{j=1}^{N-1} \lambda^{2} \left| \left\{ x \in B_{2^{j+1}r_{k}} \backslash B_{2^{j}r_{k}} : |\Delta u_{k}| > \lambda \right\} \right|$$

$$\leq \sum_{j=1}^{N-1} \lambda^{2} \left| \left\{ x \in B_{2^{j+1}r_{k}} \backslash B_{2^{j}r_{k}} : |A_{k}\Delta u_{k}| > \frac{\lambda}{\|A_{k}^{-1}\|_{L^{\infty}}} \right\} \right|$$

$$\leq \sum_{j=1}^{N-1} \lambda^{2} \left| \left\{ x \in B_{2^{j+1}r_{k}} \backslash B_{2^{j}r_{k}} : |I_{2}(\operatorname{div}K_{k})| > C_{1}\lambda \right\} \right|$$

$$+ \sum_{i=1}^{N-1} \lambda^2 \left| \left\{ x \in B_{2^{j+1}r_k} \backslash B_{2^j r_k} : |I_2(A_k f_k)| > C_1 \lambda \right\} \right| := I + II.$$

Now, we estimate the two terms separately. By Adams [1],  $I_1: L^{\frac{4}{3}}(\mathbb{R}^4) \to L^2(\mathbb{R}^4)$  is a bounded linear operator, we have

$$I \leq \sum_{j=1}^{N-1} \lambda^{2} |\{x \in B_{2^{j+1}r_{k}} \setminus B_{2^{j}r_{k}} : |I_{1}(K_{k})| > C_{1}\lambda\}| \leq \sum_{j=1}^{N-1} \lambda^{2} \int_{B_{2^{j+1}r_{k}} \setminus B_{2^{j}r_{k}}} \lambda^{-2} |I_{1}(K_{k})|^{2}$$

$$\leq \|I_{1}(K_{k})\|_{L^{2}(\mathbb{R}^{4})}^{2} \leq C \|K_{k}\|_{L^{\frac{4}{3}}(\mathbb{R}^{4})}^{2} \leq C \left(\sum_{j=1}^{N-1} \|K_{k}\|_{L^{\frac{4}{3}}(B_{2^{j+1}r_{k}} \setminus B_{2^{j}r_{k}})}\right)^{2}.$$

We can estimate the first term of  $K_k$  as follows. By the Hölder inequality, we have

$$\sum_{j=1}^{N-1} \|\nabla A_k \Delta u_k\|_{L^{\frac{4}{3}}(B_{2^{j+1}r_k} \setminus B_{2^j r_k})} \leq \sum_{j=1}^{N-1} \|\nabla A_k\|_{L^4(B_{2^{j+1}r_k} \setminus B_{2^j r_k})} \|\Delta u_k\|_{L^2(B_{2^{j+1}r_k} \setminus B_{2^j r_k})} \\
\leq C \sup_{r_k < \rho < \frac{R_k}{2}} \|\nabla^2 u_k\|_{L^2(B_{2\rho} \setminus B_{\rho})}.$$

The remaining terms of  $K_k$  can be estimated similarly. Hence, we conclude that

$$I \le C \left( \sup_{r_k < \rho < \frac{R_k}{2}} \left( \|\nabla^2 u_k\|_{L^2(B_{2\rho} \setminus B_{\rho})} + \|\nabla u_k\|_{L^4(B_{2\rho} \setminus B_{\rho})} \right) \right)^2.$$

For the second term, again by Adams [1],  $I_2: L^1(\mathbb{R}^4) \to L^{2,\infty}(\mathbb{R}^4)$  is a bounded linear operator. Thus we have

$$II \leq \|I_2(A_k f_k)\|_{L^{2,\infty}(\mathbb{R}^4)}^2 \leq C \|A_k f_k\|_{L^1(\mathbb{R}^4)}^2 \leq C \|f_k\|_{L^1(B_{R_k})}^2$$

$$\stackrel{\text{Lemma 2.2}}{\leq} C \left(\log \frac{1}{R_k}\right)^{-2} \|f_k\|_{L \log L(B_{R_k})}^2 \leq C \left(\log \frac{1}{R_k}\right)^{-2}.$$

Combining the two inequalities above, we obtain

$$\|\Delta u_k\|_{L^{2,\infty}(B_{R_k}\backslash B_{2r_k})} \leq C \sup_{r_k < \rho < \frac{R_k}{2}} \left( \|\nabla^2 u_k\|_{L^2(B_{2\rho}\backslash B_\rho)} + \|\nabla u_k\|_{L^4(B_{2\rho}\backslash B_\rho)} \right) + C \left(\log \frac{1}{R_k} \right)^{-1}.$$

For the  $L^{4,\infty}$ -estimate on  $|\nabla u|$ , we use (5.1) to rewrite our system as

$$\Delta \operatorname{div}(A_k \nabla u_k) = \operatorname{div} \hat{K}_k + A_k f_k \quad \text{in } B_{2\rho} \backslash B_{\rho},$$

where  $\hat{K}_k = K_k + \nabla^2 A_k \cdot \nabla u_k + \nabla A_k \cdot \nabla^2 u_k$ .

Similar to the estimate for  $\|\Delta u_k\|_{L^{2,\infty}(B_{R_k}\setminus B_{2r_k})}$ , we only need to apply the boundedness of Riesz operators  $I_2: L^{\frac{4}{3}}(\mathbb{R}^4) \to L^4(\mathbb{R}^4)$  and  $I_3: L^1(\mathbb{R}^4) \to L^{4,\infty}(\mathbb{R}^4)$ . Hence, we have

$$\|\nabla u_k\|_{L^{4,\infty}(B_{R_k}\setminus B_{2r_k})} \le C \sup_{r_k < \rho < \frac{R_k}{2}} \left( \|\nabla^2 u_k\|_{L^2(B_{2\rho}\setminus B_\rho)} + \|\nabla u_k\|_{L^4(B_{2\rho}\setminus B_\rho)} \right) + C \log^{-1} \frac{1}{R_k}.$$

Thus the proof is complete.

Combining Theorem 5.1 and Lemma 5.3 yields the estimate of the angular energy in the neck region.

**Theorem 5.4.** There exist constants  $\delta, C > 0$  such that for all  $r_k, R_k > 0$  with  $2r_k < R_k$  and  $\lim_{k \to \infty} R_k = 0$ , all  $f_k \in L \log L(B_{R_k} \setminus B_{r_k}, \mathbb{R}^n)$  with uniformly bounded norm and all  $u_k \in R_k$ 

 $W^{2,2}(B_{R_k}\backslash B_{r_k},\mathbb{R}^n)$  satisfying

$$\Delta^2 u_k = \Delta(V_k \cdot \nabla u_k) + \operatorname{div}(w_k \nabla u_k) + (\nabla \omega_k + F_k) \cdot \nabla u_k + f_k \quad in \ B_{R_k} \setminus B_{r_k}$$

and

$$\sup_{k} \sup_{r_{k} < \rho < \frac{R_{k}}{2}} \left( \|V_{k}\|_{W^{1,2}(B_{2\rho} \setminus B_{\rho})} + \|w_{k}\|_{L^{2}(B_{2\rho} \setminus B_{\rho})} + \|\omega_{k}\|_{L^{2}(B_{2\rho} \setminus B_{\rho})} + \|F_{k}\|_{L^{\frac{4}{3},1}(B_{2\rho} \setminus B_{\rho})} \right) \leq \delta,$$

there holds

$$\|\nabla^{T}(\nabla u_{k})\|_{L^{2}(B_{R_{k}/2}\setminus B_{2r_{k}})} + \|\nabla^{T}u_{k}\|_{L^{4}(B_{R_{k}/2}\setminus B_{2r_{k}})}$$

$$\leq C \left( \sup_{r_{k}<\rho<\frac{R_{k}}{2}} \left( \|\nabla^{2}u_{k}\|_{L^{2}(B_{2\rho}\setminus B_{\rho})} + \|\nabla u_{k}\|_{L^{4}(B_{2\rho}\setminus B_{\rho})} \right) + \log^{-1}\frac{1}{R_{k}} \right)$$

$$\cdot \left( \|\nabla^{2}u_{k}\|_{L^{2}(B_{R_{k}}\setminus B_{r_{k}})} + \|\nabla u_{k}\|_{L^{4}(B_{R_{k}}\setminus B_{r_{k}})} + \|f_{k}\|_{L\log L(B_{R_{k}}\setminus B_{r_{k}})} \right).$$

Proof of Theorem 1.1. Let  $N_k^i = B(a_k^i, \mu_k^i) \backslash B(a_k^i, \lambda_k^i)$  be a neck domain given as in Proposition 5.2 and  $N_k^i(\lambda) = B(a_k^i, \frac{\mu_k^i}{\lambda}) \backslash B(a_k^i, \lambda \lambda_k^i)$ . Since  $\|h\|_{L^2}^2 \leq \|h\|_{L^{2,1}} \|h\|_{L^{2,\infty}}$  and  $\|h\|_{L^4}^2 \leq \|h\|_{L^{4,2}} \|h\|_{L^{4,\infty}}$ , we obtain from Theorem 5.4 and Proposition 5.2 that

$$\lim_{\lambda \to \infty} \lim_{k \to \infty} \left( \|\nabla^T (\nabla u_k)\|_{L^2(N_k^i(\lambda))} + \|\nabla^T u_k\|_{L^4(N_k^i(\lambda))} \right) = 0.$$

This proves Theorem 1.1.

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