WORD MAPS AND RANDOM WORDS

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ABSTRACT. We discuss some recent results by a number of authors regarding word maps on algebraic groups and finite simple groups, their mixing properties and the geometry of their fibers, emphasizing the role played by equidistribution results in finite fields via recent advances on character bounds and non-abelian arithmetic combinatorics. In particular, we discuss character varieties of random groups. In the last section, we give a new proof of a recent theorem of Hrushovski about the geometric irreducibility of the generic fibers of convolutions of dominant morphisms to simply connected algebraic groups.

These notes stem out of lectures given by the authors in Oxford, and by the first author in ICTS Bangalore, in spring 2024.

1. Introduction

A group G together with a word w in r letters (and their inverses) give rise to the associated word $map\ w_G: G^r \to G$ where an r-tuple of elements in G is sent to the value of the word evaluated at the tuple. When G is a field (K, +), this is nothing else than a linear form with integer coefficients in r variables. For non-abelian G, word maps are more subtle objects and a lot of effort has been devoted in the last decades to unravel some of their properties (see e.g. the surveys [Seg09, Sha13, GKP18], and the references therein). Here is a sample of questions that arise naturally in this context: is w_G surjective? if not, can every element of G be written as a product of a small number of w_G values? If G is finite, how close to uniformly distributed is $w_G(g_1, \ldots, g_r)$ where the g_i are chosen independently at random in G? What is the size or the dimension of a fiber $w_G^{-1}(g)$? These questions can be asked for finite groups, and in particular large finite simple groups, for compact Lie groups, or also for algebraic groups in arbitrary characteristic. Surveying the large body of works around these questions is out of the scope of this article. Rather, we propose here to present a brief introduction to these topics and the diverse methods they bring about through the lens of the following three concrete results:

- (1) The proof by Larsen, Shalev and Tiep [LST19] that every word map w_G on a large finite simple group has an L^{∞} -mixing time which is bounded by a number $t_{\infty}(w)$ depending only on w
- (2) The proof by Becker, Breuillard and Varjú [BBV] of a dimension formula for the fibers above the identity element of generic word maps,
- (3) A new analytic proof of a result of Hrushovski [Hru24], showing that the convolution of two dominant maps to a simply connected algebraic group has a geometrically irreducible generic fiber.

Given a map between two algebraic varieties, the Lang-Weil estimates, which we recall in Section 4, provide a dictionary between algebro-geometric notions (dominance, flatness, geometrically irreducible generic fibers, etc.) and analytic counting estimates in finite fields (size of the image, size of a fiber, approximate uniformity and boundedness of the pushforward of the uniform measure, etc.). They will be essential to the proofs. In fact, on the analytic side, a key role is played by equidistribution in finite fields. The large rank case in (1) combines recent advances by Larsen and Tiep [LT24]

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on sharp character bounds with an argument going back to [LS12] proving an upper bound on the size of fibers of the word map. The small rank case can be proved using the Lang-Weil estimates and using (3). We will give a proof of (3) that makes use of harmonic analysis on finite quasi-simple groups, as well as on bounds [Kow07] on exponential sums associated to arbitrary functions on an algebraic variety that generalize Deligne's celebrated exponential sums estimates [Del74], while Hrushovski's argument was purely model-theoretic. Regarding (2), essential to the proofs is the fact that random walks equidistribute very rapidly in the finite simple groups whose associated Cayley graphs are expander graphs. This expander property has been established in many instances in the last decade or so, such as in Bourgain–Gamburd–Sarnak [BGS10], often relying on methods from arithmetic combinatorics [BG08, BGT11, Bre15, PS16].

This article, which is based on lectures given in Bangalore and in Oxford in 2024, is mostly expository and we have put the emphasis on explaining some of the key ideas, sometimes working only on special illustrative cases, rather than presenting complete proofs. In this spirit, we have included various "exercises" along the way. It is organized as follows. In Section 2 we give further introductory remarks, recall some landmark results regarding word maps, set up our notation, and state (1). In Section 3 we sketch a proof of the upper bound for the size of fibers of w_G following an argument of Larsen and Shalev [LS12], and a proof of the high rank case of (1). In Section 4 we discuss the Lang-Weil estimates and make explicit the dictionary mentioned above. This is then utilized to prove the low rank case of (1). In Section 5 we gather general facts about representation and character varieties of finitely presented groups and discuss Gromov's random group model. In Section 6 we discuss the expander property for finite simple groups, and in Section 7 we sketch the proof of (2) and discuss the role of Chebotarev's density theorem. In the final section 8, we prove (3) and state further applications to algebro-geometric properties of word maps.

1.1. Conventions.

- We write \overline{K} for the algebraic closure of a field K, and \underline{G} for an algebraic K-group.
- Given a field extension $K \leq K'$, and a finite type K-scheme X, we denote by $X_{K'}$ the base change of X with respect to $\operatorname{Spec}(K') \to \operatorname{Spec}(K)$. Moreover, if $\varphi: X \to Y$ is a morphism of K-schemes, we denote by $\varphi_{K'}: X_{K'} \to Y_{K'}$ the corresponding base change to K'.
- We write \mathbb{A}_K^m for the *m*-dimensional affine space, as a *K*-scheme.
- Given an algebraic K-group \underline{G} and a subset S in $\underline{G}(K)$, we write $\overline{\langle S \rangle}^Z$ for the Zariski closure of the subgroup generated by S, which is an algebraic subgroup $\underline{H} \leq \underline{G}$.
- Let \mathcal{D} be a fixed set (possibly empty) of parameters (i.e. the given data).
 - Given functions $f, g: S \to \mathbb{R}$, possibly depending on \mathcal{D} , we write $f(s) \gg_{\mathcal{D}} g(s)$ (and also $g = O_{\mathcal{D}}(f)$) if $f(s) \geq C \cdot g(s)$ for some positive constant C depending on \mathcal{D} .
 - We write $O_{\mathcal{D}}(1)$ to indicate a constant depending only on the data \mathcal{D} . In particular, by O(1) we mean an absolute constant.
- We write \mathbb{N} for the set $\{1, 2, \dots\}$, and $\mathbb{Z}_{>0}$ for $\mathbb{N} \cup \{0\}$.

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2. Word maps on finite simple groups: probabilistic results

2.1. **Motivation.** In 1770, Lagrange proved the famous four squares theorem; every natural number can be represented as the sum of at most four square integers. Later that year, Waring considered the following generalization, which was confirmed by Hilbert 139 years later [Hil09]:

Problem 2.1 (Waring's problem, 1770). Can one find for every $k \in \mathbb{N}$, a number $t(k) \in \mathbb{N}$, such that every $n \in \mathbb{Z}_{\geq 0}$ can be represented as $n = \sum_{i=1}^{t(k)} x_i^k$ of t(k) for $x_1, \ldots, x_{t(k)} \in \mathbb{Z}_{\geq 0}$?

We now formulate Waring's problem in a slightly different language, using the following definition.

Definition 2.2 ([GH19, GH21]). Let $\varphi: X \to G$, $\psi: Y \to G$ be maps from sets X, Y to a (semi-)group (G, \cdot_G) . Define the *convolution* $\varphi * \psi: X \times Y \to G$ by

$$\varphi * \psi(x, y) = \varphi(x) \cdot_G \psi(y).$$

Furthermore, we denote by $\varphi^{*t}: X^t \to G$ the t-th convolution power.

In this language, Waring's problem can be stated as follows; let $\varphi_k : \mathbb{Z}_{\geq 0} \to (\mathbb{Z}_{\geq 0}, +)$ be the map $\varphi_k(x) = x^k$. Can one find $t(k) \in \mathbb{N}$ such that $\varphi_k^{*t(k)} : \mathbb{Z}_{\geq 0}^{t(k)} \to (\mathbb{Z}_{\geq 0}, +)$ is surjective?

More generally, analyzing the surjectivity of $\varphi^{*t}: X^t \to G$, varying over different semi-groups G and different maps $\varphi: X \to G$, gives rise to a family of problems called Waring-type problems. In addition, instead of just asking whether φ^{*t} is surjective, one can ask in how many ways one can write $g \in G$ as $g = \varphi^{*t}(x_1, \ldots, x_t)$? or in other words, estimate the size of the fiber $(\varphi^{*t})^{-1}(g)$. Such problems are called probabilistic Waring-type problems.

In the next lectures we will focus on the special case when φ is a word map and G is a simple group (finite or algebraic). This setting was extensively studied by Larsen, Liebeck, Shalev, Tiep, and many others. We refer to an excellent survey of Shalev [Sha13].

2.2. Waring type problems in the setting of word maps.

Definition 2.3. Let $w(x_1,...,x_r)$ be a word in a free group F_r (e.g. w=[x,y] or $w=x^{\ell}$). For any group G, we define a word map

$$w_G: G^r \to G, \quad by \ (g_1, ..., g_r) \mapsto w(g_1, ..., g_r), \ for \ g_1, ..., g_r \in G.$$

Note that $w_G * w_G = (w * w)_G$, where w * w denotes concatenation of words with different letters. For example, $w_{\text{com}} := [x, y] = xyx^{-1}y^{-1}$ induces the commutator map $w_{\text{com}}, G : G^2 \to G$, and $w_{\text{com}} * w_{\text{com}} := [x, y] \cdot [z, w]$.

Let us mention the state of the art for the Waring problem for word maps on simple algebraic groups and finite simple groups.

Theorem 2.4 (Borel [Bor83], and also Larsen [Lar04]). Let K be an algebraically closed field and let $1 \neq w \in F_r$. hen for every connected semisimple algebraic K-group, the map $w_{\underline{G}} : \underline{G}^r \to \underline{G}$ is dominant.

In particular, $(w*w)_{G(K)}: \underline{G}(K)^{2r} \to \underline{G}(K)$ is surjective.

Theorem 2.5 (Larsen–Shalev–Tiep, [LST11]). For every $1 \neq w \in F_r$, there exists $N(w) \in \mathbb{N}$ such that for every finite simple group G, with |G| > N(w), $(w * w)_G : G^{2r} \to G$ is surjective.

Certain words are in fact surjective over all finite simple groups, so that no convolutions are needed. A notable example is the following theorem, which answered a conjecture by Ore from the fifties.

Theorem 2.6 (The Ore conjecture 1951, [LOST10]). If G is a finite non-abelian simple group, then $w_{\text{com},G}: G^2 \to G$ is surjective.

Here are a few more examples in other settings:

- (1) Compact simple Lie groups [HLS15]: Let $1 \neq w \in F_r$ and let G be a compact connected simple Lie group of high rank $\mathrm{rk}(G) \gg_w 1$. Then $(w * w)_G : G^{2r} \to G$ is surjective.
- (2) Compact p-adic groups [AGKS13]: Let $1 \neq w \in F_r$. Then for every $n \geq 2$ and every $p \gg_n 1$, the map $(w^{*3})_{\mathrm{SL}_n(\mathbb{Z}_p)}$ is surjective. More generally, one can take $G = \underline{G}(\mathbb{Z}_p)$ for \underline{G} simply connected, simple algebraic \mathbb{Q} -group.
- (3) Arithmetic groups [AM19]: Let $1 \neq w \in F_r$. Then $w_{\mathrm{SL}_n(\mathbb{Z})}^{*87}$ is surjective for $n \gg_w 1$.
- (4) **Simple Lie algebras:** In [BGKP12], an analogue of Borel's theorem (Theorem 2.4) was shown for Lie algebra word maps on semisimple Lie algebras \mathfrak{g} under the additional assumption that the word map is not identically zero on \mathfrak{sl}_2 .
- 2.3. Probabilistic Waring type problems for word maps. In §2.2 we saw that word maps $w_G: G^r \to G$ in various settings, become surjective after taking very few self-convolutions. In other words, any $g \in G$ can be written as $w_G^{*t}(g_1, \ldots, g_{rt}) = g$ for some $g_1, \ldots, g_{rt} \in G$. We now consider the case when G is finite, and discuss in how many ways one can write $w_G^{*t}(g_1, \ldots, g_{rt}) = g$, or in other words: can one estimate the size of the fiber $(w_G^{*t})^{-1}(g)$? This boils down to analyzing the random walk induced by w on G. We start by introducing basic notions and results from the theory of random walks on finite groups.
- 2.3.1. Random walk on finite groups. Let G be a finite group, and denote by μ_G the uniform probability measure, which can be identified with the constant function $\frac{1}{|G|}$ on G. Let μ be a probability measure on G. It will later be convenient to write $\mu = f_{\mu}\mu_{G}$, where f_{μ} is the density of μ with respect to μ_{G} . If G is finite this simply means $f_{\mu}(g) = |G| \mu(g)$.

The measure μ induces a random walk on G as follows. In the first step, we choose a random element $h_1 \in G$, distributed according to μ . In the second step, choose a random element $h_2 \in G$, distributed according to μ , independently of step 1, and move to $h_1 \cdot h_2$. Continuing this way, choosing h_1, \ldots, h_t independently at random, the probability to reach $g \in G$ after t steps is given by

$$\mu^{*t}(g) = \mu * \cdots * \mu(g) := \sum_{h_1, \dots, h_t \in G \text{ s.t. } h_1 \cdot \dots \cdot h_t = g} \mu(h_1) \cdot \dots \cdot \mu(h_t),$$

and moreover,

$$f_{\mu^{*t}}(g) = f_{\mu} * \dots * f_{\mu}(g) = \frac{1}{|G|^{t-1}} \sum_{h_1, \dots, h_t \in G \text{ s.t. } h_1 \cdot \dots \cdot h_t = g} f_{\mu}(h_1) \cdot \dots \cdot f_{\mu}(h_t).$$

Denote by Irr(G) the set of irreducible characters of G. Recall that Irr(G) is an orthonormal basis for the space of conjugate invariant functions $\mathbb{C}[G]^G$, with respect to the inner product $\langle f_1, f_2 \rangle =$ $\frac{1}{|G|}\sum_{g\in G}f_1(g)\overline{f_2}(g)$. If μ is a conjugate invariant measure, we can write:

$$f_{\mu}(g) = \sum_{\rho \in Irr(G)} a_{\mu,\rho} \rho(g),$$

where $a_{\mu,\rho} := \sum_{g \in G} \overline{\rho(g)} \mu(g) = \langle f_{\mu}, \rho \rangle$ is the Fourier coefficient of μ at ρ .

Exercise 2.7. For every $\rho_1, \rho_2 \in \operatorname{Irr}(G)$, we have $\rho_1 * \rho_2 = \frac{\delta_{\rho_1, \rho_2}}{\rho_1(1)} \rho_1$.

By Exercise 2.7, we get:

$$f_{\mu}^{*t}(g) = \sum_{\rho \in Irr(G)} \frac{a_{\mu,\rho}^t}{\rho(1)^{t-1}} \rho(g) = 1 + \sum_{1 \neq \rho \in Irr(G)} \frac{a_{\mu,\rho}^t}{\rho(1)^{t-1}} \rho(g).$$

Definition 2.8. Let $1 \le q \le \infty$.

- (1) For every $f: G \to \mathbb{C}$, we set $||f||_q := \left(\frac{1}{|G|} \sum_{g \in G} |f(g)|^q\right)^{\frac{1}{q}}$. (2) Given a signed measure μ on G, we set $||\mu||_q := ||f_{\mu}||_q$. In particular, in this notation we have $\|\mu_G\|_q = 1$ for every $1 \le q \le \infty$.

Remark 2.9. The following two inequalities will be useful.

- (1) **Jensen's inequality**: for every $f: G \to \mathbb{C}$ and $1 \le q \le q' \le \infty$, we have $||f||_q \le ||f||_{q'}$.
- (2) Young's convolution inequality: given $f, h: G \to \mathbb{C}$, and given $1 \leq q, s, r \leq \infty$ with $\frac{1}{a} + \frac{1}{s} = 1 + \frac{1}{r}$, we have $||f * h||_r \le ||f||_q ||h||_s$.

Lemma 2.10. Let G be a finite group, and let μ be a conjugate invariant probability measure. Suppose that $1 \in \text{supp}\mu \not\subseteq N$ for every proper normal subgroup $N \triangleleft G$. Then there exists $0 < \alpha < 1$ such that for every $t \in \mathbb{N}$ and every $q \geq 1$,

$$\|\mu^{*t} - \mu_G\|_q \le \|\mu^{*t} - \mu_G\|_{\infty} \le |G| \cdot \alpha^t.$$

Proof. First note it is enough to show that $|a_{\mu,\rho}| < \rho(1)$ for all $1 \neq \rho \in Irr(G)$. Indeed, we then take $\alpha := \max_{1 \neq \rho \in Irr(G)} \frac{|a_{\mu,\rho}|}{\rho(1)}$, so that

$$\|\mu^{*t} - \mu_G\|_{\infty} = \left\| \sum_{1 \neq \rho \in \operatorname{Irr}(G)} \frac{a_{\mu,\rho}^t}{\rho(1)^{t-1}} \rho(g) \right\|_{\infty} \leq \sum_{1 \neq \rho \in \operatorname{Irr}(G)} \frac{|a_{\mu,\rho}^t|}{\rho(1)^t} \rho(1)^2 \leq \alpha^t \sum_{1 \neq \rho \in \operatorname{Irr}(G)} \rho(1)^2 \leq |G| \cdot \alpha^t.$$

Since $\operatorname{Supp}(\mu^{*t}) \subseteq \operatorname{Supp}(\mu^{*t+1})$, there exists $t_0 \in \mathbb{N}$ such that $S := \operatorname{Supp}(\mu^{*t_0}) = \operatorname{Supp}(\mu^{*(t_0+1)})$, so $S \cdot S = S$ and $S^{-1} \subseteq S^{|G|-1} = S$ and S is a normal subgroup, hence S = G. In particular, there exists $\delta > 0$ such that $f_{\mu^{*t_0}}(g) > \delta$ for every $g \in G$. Finally,

$$\frac{|a_{\mu,\rho}^{t_0}|}{\rho(1)^{t_0-1}} = |a_{\mu^{*t_0},\rho}| = \left| \frac{1}{|G|} \sum_{g \in G} \overline{\rho(g)} \cdot \delta + \frac{1}{|G|} \sum_{g \in G} \overline{\rho(g)} (f_{\mu^{*t_0}}(g) - \delta) \right| \\
\leq 0 + \rho(1) \frac{1}{|G|} \sum_{g \in G} (f_{\mu^{*t_0}}(g) - \delta) \leq \rho(1)(1 - \delta).$$

Definition 2.11. The minimal $t \in \mathbb{N}$ such that $\|\mu^{*t} - \mu_G\|_q < \frac{1}{2}$ is called the L^q -mixing time, and denoted $t_q(\mu)$, or t_q if μ is clear from the context.

Remark 2.12.

- (1) The choice of $\frac{1}{2}$ in Definition 2.11 is for definiteness; we could have taken any other numerical value smaller than 1.
- (2) Lemma 2.10 holds more generally for aperiodic (not necessarily conjugate invariant) measures. This follows e.g. from the Itô-Kawada equidistribution theorem [KI40] (see also [App14, Theorem 4.6.3]).
- (3) It follows from Remark 2.9(1) that $t_q(\mu) \le t_{q'}(\mu)$ if $1 \le q \le q'$.

Exercise 2.13. We have $\|\mu^{*t_q l} - \mu_G\|_q < 2^{-l}$ for any $l \in \mathbb{N}$ (see e.g. [LP17, Lemma 4.18]).

Example 2.14 (Bayer–Diaconis, [BD92]). Shuffling a deck of 52 playing cards can be seen as applying a random permutation of 52 elements, i.e. a probability measure on S_{52} . The randomness comes from the non-deterministic nature of shuffling performed by human beings. Repeating the same method of shuffling several times can be seen as applying a random walk on the symmetric group S_{52} , of the same type presented in §2.3.1. Bayer and Diaconis studied a common shuffling method called "riffle shuffle", which is used for example in many Casinos, and showed that it takes 7 to 8 riffle shuffles to mix a deck of 52 cards. More precisely, if μ is a probability measure on S_{52} , corresponding to a random riffle shuffle of 52 cards (based on the Gilbert–Shannon–Reeds model), then the L^1 -mixing time of μ , according to Definition 2.11, is 8 ($\|\mu^{*7} - \mu_{S_{52}}\|_1 \sim 0.67$ and $\|\mu^{*8} - \mu_{S_{52}}\|_1 \sim 0.33$). Here is a Numberphile video about this theorem.

2.3.2. Probabilistic Waring problem: finite simple groups.

Definition 2.15. Let $w \in F_r$ be a word and G be a group. We set $\tau_{w,G} := (w_G)_*(\mu_G^r)$ to be the corresponding word measure. Note that

$$\tau_{w,G}(g) = \frac{\left| w^{-1}(g) \right|}{\left| G \right|^r}.$$

Exercise 2.16. Show that $\tau_{w_1,G} * \tau_{w_2,G} = \tau_{w_1*w_2,G}$ and that $\tau_{w,G}^{*t}(g) = \frac{\left| (w^{*t})^{-1}(g) \right|}{|G|^{rt}}$.

We are interested in the family of random walks $\{\tau_{w,G}\}_{G \text{ f.s.g}}$ on the family of finite simple groups. We say that the family $\{\tau_{w,G}\}_{G \text{ f.s.g}}$ has a uniform L^q -mixing time of $t_q(w)$, if $t_q(w)$ is the minimal $t \in \mathbb{N}$ such that:

(2.1)
$$\lim_{|G| \to \infty \ G \text{ f.s.g}} \|\tau_{w,G}^{*t} - \mu_G\|_q = 0.$$

Since the family of finite simple groups is infinite, it is a priori not clear that $t_q(w)$ exists. A deep result of Larsen, Shalev and Tiep (Theorem 2.18 below) shows that this is indeed the case, in the strongest sense of $q = \infty$. Moreover, in the case that q = 1, it turns out that $t_1(w) \leq 2$ for every non-trivial word w.

Theorem 2.17 ([LST19, Theorem 1]). Let $1 \neq w \in F_r$ be a word. Then the family $\{\tau_{w,G}\}_{G \text{ f.s.g}}$ has a uniform L^1 -mixing time of $t_1(w) \leq 2$.

We do not prove Theorem 2.17 in these notes. However, the case of bounded rank groups of Lie type follows from a geometric statement (Theorem 4.18) that the convolution of any two non-trivial word maps has geometrically irreducible generic fiber. A generalization of this theorem is given in Section 8, where also the connection to L^1 -mixing time is discussed in details.

We now state the L^{∞} -result.

Theorem 2.18 ([LS12, LST19]). Let $1 \neq w \in F_r$.

- (1) There exists $\epsilon(w) > 0$ such that for every finite simple group G with $|G| \gg_w 1$, and every $g \in G$, one has $\tau_{w,G}(g) < |G|^{-\epsilon(w)}$ ([LS12]).
- (2) There exists $t_{\infty}(w) \in \mathbb{N}$ such that the family $\{\tau_{w,G}\}_{G \text{ f.s.g}}$ has a uniform L^{∞} -mixing time of $t_{\infty}(w)$ ([LST19]).

The exponents $\epsilon(w)^{-1}$ and $t_{\infty}(w)$ are both bounded from above by $C \cdot \ell(w)^4$, for a large absolute constant C, where $\ell(w)$ denotes the length of w.

Example 2.19. The following example shows that there is no uniform upper bound for $t_{\infty}(w)$ which is independent of w. Let $w_{(\ell)} = x^{\ell}$ be the power word, and let $G = \mathrm{SL}_n(\mathbb{F}_p)$. For simplicity, choose n divisible by ℓ . Choose a prime p such that \mathbb{F}_p contains a primitive ℓ -th root of unity ξ_{ℓ} (this happens if and only if $\ell|p-1$, and there are infinitely many such primes for each ℓ). Note that $(w_{(\ell)})_G^{-1}(e)$ contains the diagonal element g consisting of ℓ blocks of size n/ℓ , each is a scalar matrix $\xi_{\ell}^j \cdot I_{n/\ell}$ for $j = 0, ..., \ell - 1$. Since $(w_{\ell})_G^{-1}(e)$ is invariant under conjugation, it contains the conjugacy class g^G of g, so:

$$(2.2) \left| (w_{(\ell)})_G^{-1}(e) \right| \ge \left| g^G \right| = \frac{|G|}{|C_G(g)|} \ge \frac{|\operatorname{SL}_n(\mathbb{F}_p)|}{\left| \operatorname{SL}_n(\mathbb{F}_p) \cap \operatorname{GL}_{n/\ell}(\mathbb{F}_p)^{\ell} \right|}.$$

Arguing using the Lang-Weil estimates (see Theorem 4.2 below) and since $n^2 - 1 - (\frac{n^2}{\ell} - 1) > (n^2 - 1)(1 - \frac{1}{\ell})$, the RHS of (2.2) is larger than $|\operatorname{SL}_n(\mathbb{F}_p)|^{1 - \frac{1}{\ell}}$ for $p \gg_{n,\ell} 1$. **Exercise:** conclude that $\epsilon(w_{(\ell)}) \leq \ell^{-1}$ and $t_{\infty}(w_{(\ell)}) \geq \ell$ (note that $\operatorname{SL}_n(\mathbb{F}_p)$ is quasi-simple and not simple).

In the next section we discuss some key examples and applications, in particular the proof of the above theorems in the case when w is the commutator word, and then explain the main ideas of the proofs of the general case for finite groups of Lie type. The latter splits into the high rank case which is discussed in \S 3.3-3.4, and the low rank case, discussed in Section 4. The two cases require very different sets of ideas.

3. Commutator word, representation growth, and proof of the probabilistic results

3.1. Representation growth.

Definition 3.1. Let G be a compact group, and $r_n(G) := |\{\rho \in Irr(G) : \rho(1) = n\}|$. The representation zeta function of G is:

$$\zeta_G(s) := \sum_{n=1}^{\infty} r_n(G) n^{-s} = \sum_{\rho \in \operatorname{Irr}(G)} \rho(1)^{-s}, \text{ for } s \in \mathbb{C}.$$

The abscissa of convergence of $\zeta_G(s)$ is $\alpha(G) := \inf \{ s \in \mathbb{R}_{>0} : \zeta_G(s) < \infty \}$.

Theorem 3.2 (Larsen-Lubotzky, [LL08]). Let G be a compact, connected, simple Lie group. Then:

$$\alpha(G) = \frac{\operatorname{rk}(G_{\mathbb{C}})}{|\Sigma^{+}(G_{\mathbb{C}})|},$$

where $\Sigma^+(G_{\mathbb{C}})$ is the set of positive roots in the root system corresponding to $G_{\mathbb{C}}$.

Example 3.3. If $G = \mathrm{SU}_2$, then $r_n(G) = 1$ for each $n \in \mathbb{N}$ (the natural action on the space of homogeneous polynomials f(x,y) of degree n-1 with two variables is the unique n-dimensional irreducible representation of SU_2 , up to isomorphism). Hence $\zeta_G(s) := \sum_{n=1}^{\infty} n^{-s}$ is the Riemann zeta function, and $\alpha(G) = 1 = \frac{1}{1} = \frac{\mathrm{rk}(G_{\mathbb{C}})}{|\Sigma^+(G_{\mathbb{C}})|}$.

It is meaningless to discuss representation growth in a fixed finite group. However, one can ask about behavior in families of finite groups:

Theorem 3.4 (Liebeck–Shalev, [LS05a, Theorem 1.1], [LS05b, Theorem 1.1]).

- (1) For all s > 1, we have $\lim_{|G| \to \infty: G \text{ f.s.g}} \zeta_G(s) = 1$.
- (2) Let \underline{G} be a Chevalley group scheme (a \mathbb{Z} -model of a connected, simply connected, simple algebraic \mathbb{C} -group $\underline{G}_{\mathbb{C}}$). Then:

$$\lim_{q \to \infty} \zeta_{\underline{G}(\mathbb{F}_q)}(s) = 1 \text{ for every } s > \frac{\operatorname{rk}(\underline{G}_{\mathbb{C}})}{|\Sigma^+(\underline{G}_{\mathbb{C}})|}.$$

3.2. The commutator word and representation growth. Given a word $w \in F_r$ and a finite group G, recall that $\tau_{w,G} := (w_G)_*(\mu_{G^r}) = \frac{1}{|G|} \sum_{\rho \in \operatorname{Irr}(G)} a_{w,G,\rho} \cdot \rho$, where $a_{w,G,\rho} := a_{\tau_{w,G},\rho}$ is the Fourier coefficient of $\tau_{w,G}$ at $\rho \in \operatorname{Irr}(G)$.

Exercise 3.5. Since $\tau_{w,G} := (w_G)_*(\mu_G^r)$, we have:

$$a_{w,G,\overline{\rho}} = \sum_{g \in G} \rho(g) \cdot \tau_{w,G}(g) = \mathbb{E}_{(g_1,\dots,g_r) \in G^r}(\rho(w(g_1,\dots,g_r))).$$

Let $w_{\text{com}} := [x, y]$. A classical theorem of Frobenius describes $a_{w_{\text{com}},G,\rho}$.

Theorem 3.6 (Frobenius, 1896). For every finite group G,

$$a_{w_{\text{com}},G,\overline{\rho}} = \mathbb{E}_{(x,y)\in G^2}(\rho(xyx^{-1}y^{-1})) = \frac{1}{\rho(1)}.$$

Proof. Let $\rho \in Irr(G)$ be the character of an irreducible representation $\pi_{\rho}: G \longrightarrow GL(V_{\rho})$. Consider

$$T_y := \frac{1}{|G|} \sum_{x \in G} \pi_\rho(xyx^{-1}) \in \text{End}(V_\rho)$$

Note that for each $y, z \in G$:

$$\pi_{\rho}(z) \circ T_{y} = \frac{1}{|G|} \sum_{x \in G} \pi_{\rho}(zxyx^{-1}) \underset{x \mapsto z^{-1}x}{=} \frac{1}{|G|} \sum_{x \in G} \pi_{\rho}(xyx^{-1}z) = T_{y} \circ \pi_{\rho}(z).$$

Thus $T_y \in \text{End}(V_\rho)^G$, so by Schur's lemma, it is a scalar matrix $c_\rho \cdot I_{\rho(1)}$, where

$$c_{\rho}\rho(1) = \operatorname{tr}\left(\frac{1}{|G|} \sum_{x \in G} \pi_{\rho}(xyx^{-1})\right) = \frac{1}{|G|} \sum_{x \in G} \rho(xyx^{-1}) = \rho(y).$$

Finally,

$$\begin{split} \frac{1}{|G|^2} \sum_{x,y \in G} \rho(xyx^{-1}y^{-1}) &= \frac{1}{|G|} \sum_{y \in G} \left(\frac{1}{|G|} \sum_{x \in G} \rho(xyx^{-1}y^{-1}) \right) = \frac{1}{|G|} \sum_{y \in G} \left(\operatorname{tr} \frac{1}{|G|} \sum_{x \in G} \pi_{\rho}(xyx^{-1}y^{-1}) \right) \\ &= \frac{1}{|G|} \sum_{y \in G} \operatorname{tr} \left(T_y \circ \pi_{\rho}(y^{-1}) \right) = \frac{1}{|G|} \sum_{y \in G} \operatorname{tr} \left(\frac{\rho(y)}{\rho(1)} \cdot \pi_{\rho}(y^{-1}) \right) \\ &= \frac{1}{|\rho(1)|} \frac{1}{|G|} \sum_{y \in G} |\rho(y)|^2 = \frac{1}{|\rho(1)|}. \end{split}$$

Corollary 3.7. Let G be a finite group. Then:

$$\frac{\left|(w_{\text{com}}^{*t})^{-1}(g)\right|}{|G|^{2t}} = \tau_{w_{\text{com}},G}^{*t}(g) = \frac{1}{|G|} \sum_{\rho \in \text{Irr}(G)} \frac{\left(a_{w_{\text{com}},G,\rho}\right)^t}{\rho(1)^{t-1}} \rho(g) = \frac{1}{|G|} \sum_{\rho \in \text{Irr}(G)} \rho(1)^{1-2t} \rho(g).$$

Hence, $\zeta_G(2t-2) = |G| \tau_{w_{\text{com}},G}^{*t}(e)$.

Theorem 3.8 (Garion–Shalev, [GS09]). Over the family of finite simple groups G, with $|G| \gg_w 1$:

- (1) $t_1(w_{\text{com}}) = t_2(w_{\text{com}}) = 1$ (uniform L^1/L^2 -mixing time of 1).
- (2) $t_{\infty}(w_{\text{com}}) = 2$.

Proof. 1) By Corollary 3.7 we have:

$$\tau_{w_{\text{com}},G}(g) = \frac{1}{|G|} \sum_{\rho \in \text{Irr}(G)} a_{w_{\text{com}},G,\rho} \rho(g) = \frac{1}{|G|} \sum_{\rho \in \text{Irr}(G)} \frac{\rho(g)}{\rho(1)}.$$

Thus, by Jensen's inequality (Remark 2.9(1)) and Theorem 3.6,

$$\|\tau_{w_{\text{com}},G} - \mu_G\|_1^2 \le \|\tau_{w_{\text{com}},G} - \mu_G\|_2^2 = \sum_{1 \ne \rho \in \text{Irr}(G)} a_{w_{\text{com}},G,\rho}^2 = \sum_{1 \ne \rho \in \text{Irr}(G)} \rho(1)^{-2} = \zeta_G(2) - 1 \underset{|G| \to \infty}{\to} 0.$$

2) Note that $|G| \tau_{w_{\text{com}},G}(e) = |\text{Irr}(G)| = |\{\text{conj classes of } G\}|$. Hence $|G| \tau_{w_{\text{com}},G}(e) - 1 \to \infty$ and thus $t_{\infty}(w_{\text{com}}) > 1$. On the other hand, $t_{\infty}(w_{\text{com}}) = 2$ since by Young's convolution inequality (Remark 2.9(2)),

$$\|\tau_{w_{\text{com}},G}^{*2} - \mu_G\|_{\infty} = \|(\tau_{w_{\text{com}},G} - \mu_G)^{*2}\|_{\infty} \le \|\tau_{w_{\text{com}},G} - \mu_G\|_2^2 = \zeta_G(2) - 1 \underset{|G| \to \infty}{\longrightarrow} 0.$$

3.3. **Proof of Theorem 2.18.** The key ingredient in the proof of Theorem 2.18(2) is the following:

Theorem 3.9 (Larsen–Shalev–Tiep, [LST19]. See also [AG, Theorem 1.6]). Let $1 \neq w \in F_r$. Then there exists $\epsilon(w) > 0$ such that for every finite simple group G, with $|G| \gg_w 1$, and every $\rho \in \text{Irr}(G)$:

$$|a_{w,G,\rho}| \leq \rho(1)^{1-\epsilon(w)}$$
.

Theorem 3.9 implies Theorem 2.18(2). Let $t > 4\epsilon(w)^{-1}$. Then:

$$\begin{aligned} \|\tau_{w,G}^{*t} - \mu_G\|_{\infty} &= \max_{g \in G} |f_{w^{*t},G}(g) - 1| = \max_{g \in G} \left| \sum_{1 \neq \rho \in \operatorname{Irr}(G)} \frac{a_{w,G,\rho}^t}{\rho(1)^{t-1}} \rho(g) \right| \\ &\leq \sum_{1 \neq \rho \in \operatorname{Irr}(G)} \rho(1)^{1-t\epsilon(w)} \cdot \rho(1) \leq \sum_{1 \neq \rho \in \operatorname{Irr}(G)} \rho(1)^{-2} = \zeta_G(2) - 1 \to 0, \end{aligned}$$

where the last equality follows from Theorem 3.4.

We now sketch the main steps in the proof of Theorem 3.9 for groups of Lie type. We will not discuss the remaining case of the family of alternating groups. There are two main slogans:

Slogan #1: if $g \in G$ has small centralizer $C_G(g)$, then $|\rho(g)|$ is small.

Slogan #2: $C_G(w(g_1,...,g_r))$ is small with very high probability.

Slogan #1 is an important phenomenon in representation theory, which can already be seen from Schur's orthogonality.

Example 3.10. For every $\rho \in \text{Irr}(G)$, we have $\sum_{x \in G} |\rho(x)|^2 = |G|$ by Schur's orthogonality. In particular, $|\rho(g)|^2 |g^G| \leq |G|$, so:

$$|\rho(g)| \le \sqrt{|C_G(g)|}.$$

This is called the centralizer bound.

The character estimates required for mixing of word measures, or various other conjugate invariant measures, are of exponential form $|\rho(g)| \leq \rho(1)^{1-\epsilon}$. The current state of the art for (exponential) character estimates in finite simple groups of Lie type is the recent result of Larsen and Tiep:

Theorem 3.11 (Larsen–Tiep, [LT24]). There exists an absolute constant c > 0, such that for every finite simple group of Lie type G, every $1 \neq \rho \in Irr(G)$ and every $g \in G$:

(3.1)
$$|\rho(g)| < \rho(1)^{1 - c \frac{\log|g^G|}{\log|G|}}.$$

Remark 3.12. Theorem 2.18 was proved a few years before Theorem 3.11, using the characters estimates from [GLT20, GLT23], which are less general than (3.1), but strong enough for Theorem 2.18. In [GLT20, GLT23] a bound of the form (3.1) was given for all sufficiently regular elements, in the sense that $|C_G(g)| \leq |G|^{\delta}$ for some small $\delta > 0$. One of the key ideas in [LT24] to deal with elements of large centralizers, is to reduce to the setting of [GLT20, GLT23] using a mixing argument showing that sufficiently many self-convolutions of small conjugacy classes produces elements with small centralizers with high probability. We will not discuss the details of the proof of [LT24] or [GLT20, GLT23] in these notes.

For the next theorem, denote

(3.2)
$$T_{w,\delta,G} := \left\{ (g_1, ..., g_r) \in G^r : |C_G(w(g_1, ..., g_r))| > |G|^{\delta} \right\}.$$

Theorem 3.13 ([LST19]). Let $1 \neq w \in F_r$ and let $\delta > 0$. Then there exists $c'(\delta) > 0$ such that for every finite simple group of Lie type G, with $|G| \gg_w 1$, one has:

$$\mathbb{P}\left(T_{w,\delta,G}\right) < |G|^{-c'(\delta)/\ell(w)^2}.$$

Theorems 3.11 and 3.13 imply Theorem 3.9. Indeed, fixing (say) $\delta = \frac{1}{4}$, and by combining the two theorems, we get that for every $1 \neq \rho \in \text{Irr}(G)$, and every finite simple group G with $|G| \gg_w 1$,

$$|a_{w,G,\overline{\rho}}| = |\mathbb{E}(\rho(w(g_1,...,g_r)))| \le \mathbb{P}(T_{w,\delta,G}) \cdot \rho(1) + \rho(1)^{1-c(1-\delta)} \le |G|^{-c'(\delta)/\ell(w)^2} \cdot \rho(1) + \rho(1)^{1-c(1-\delta)} \le \rho(1)^{-2c'(\delta)/\ell(w)^2} \cdot \rho(1) + \rho(1)^{1-c(1-\delta)} \le 2\rho(1)^{1-2\epsilon(w)} < \rho(1)^{1-\epsilon(w)},$$

where the third inequality follows since $\rho(1) < |G|^{\frac{1}{2}}$, and the last inequality follows from the fact that $\lim_{|G| \to \infty} \min_{1 \le \rho \in \operatorname{Irr}(G)} \rho(1) = \infty$ (see [LS74]).

Remark 3.14. Note that Theorem 3.13, with $\delta = 0.99$ (for example) also implies Item (1) of Theorem 2.18 for finite groups of Lie type. Indeed, Theorem 3.13 implies that $\tau_{w,G}(g) < |G|^{-\epsilon'(w)}$ for every $g \in G$ with $|C_G(g)| > |G|^{\delta}$. To bound $\tau_{w,G}(g)$ for the other elements g, we can simply use the bound $\tau_{w,G}(g) = \frac{\tau_{w,G}(g^G)}{|g^G|} \le |g^G|^{-1}$.

3.4. Proof of probabilistic result (Theorem 3.13) in high rank. The proof of Theorem 3.13 in its full generality is rather complicated. In order to explain it we will make several simplifying assumptions. We consider the case that $G = GL_n(\mathbb{F}_p)$, that w has 2 letters, and $n > 32\ell(w)$, and prove the following Theorem 3.15, which is a slightly weaker version of Theorem 3.13. Still, its proof (which is based on Larsen–Shalev's proof of Proposition 3.3 in [LS12]) contains the main ideas used in the proof of the stronger Theorem 3.13. In §4 we will be able to prove the low rank case as well (i.e. $n \leq 32\ell(w)$).

Theorem 3.15. For every $1 \neq w \in F_2$, every $n \geq 32\ell(w)$ and for every prime p:

$$\tau_{w,\operatorname{GL}_n(\mathbb{F}_p)}(e) \le |\operatorname{GL}_n(\mathbb{F}_p)|^{-\frac{1}{11\ell(w)}}$$
.

Given a reduced word w(x,y), write $w = w_{\ell} \cdots w_1$, where each w_i is of the form x, x^{-1}, y or y^{-1} . We write $w^{(j)} := w_j \cdots w_2 w_1$, so $w^{(\ell)} = w$. Given $v_{1,0}, ..., v_{m,0} \in \mathbb{F}_p^n$, we denote $v_{i,j} := w^{(j)}.v_{i,0}$, for $j = 1, ..., \ell$. Each of the sequences $\{v_{i,0}, v_{i,1}, ..., v_{i,\ell}\}$ is what called a *trajectory* for w. For example, if $w = w_{\text{com}} = xyx^{-1}y^{-1}$, then

$$v_{i,1} = y^{-1} \cdot v_{i,0}$$
 and $v_{i,2} = x^{-1} \cdot v_{i,1}$ and $v_{i,3} = y \cdot v_{i,2}$ and $v_{i,4} = x \cdot v_{i,3}$.

We are going to conduct an "experiment" consisting of $m \sim \frac{n}{2\ell}$ small trials. We first choose two elements x, y in $GL_n(\mathbb{F}_p)$ independently at random. Then we do the following:

- In the first trial, we choose a random vector $v_{1,0} \in \mathbb{F}_p^n$, and then build its trajectory $\{v_{1,0}, v_{1,1}, ..., v_{1,\ell}\}$ according to w(x,y). At this point we check if $v_{1,\ell} = v_{1,0}$. If this is not the case, we stop the experiment. If it is the case, we continue to the next trial.
- In the second trial, we choose a random vector $v_{2,0} \in \mathbb{F}_p^n$ which does not belong to the span of all previous occurrences of the $v_{i',j'}$'s, i.e $v_{2,0} \notin \text{span}\{v_{1,0}, v_{1,1}, ..., v_{1,\ell-1}\}$. We again build its trajectory $\{v_{2,0}, v_{2,1}, ..., v_{2,\ell}\}$ according to w(x,y) and check if $v_{2,\ell} = v_{2,0}$. Again, if this is not the case, we stop the experiment, and if it is true, we continue to the next trial.
- We repeat the process $m \leq \frac{n}{2\ell}$ times.

Intuitively, it seems extremely unlikely that we will be able to complete the full experiment, and that a "miracle" is needed in order for this to happen. Let us define the "miracle set" $S_{w,m}$, which describes all tuples $(x, y, v_{1,0}, \ldots, v_{m,0})$ for which the experiment was successful. Let \prec denote the lexicographic order on pairs in $\{1, \ldots, m\} \times \{0, \ldots, \ell\}$, that is, $(a_1, a_2) \prec (b_1, b_2)$ if and only if $a_1 < b_1$, or, $a_1 = b_1$ and $a_2 < b_2$. Denote

$$Z_{i,j} := \text{Span} \{ v_{i',j'} \mid (i',j') \prec (i,j) \},$$

and

$$S_{w,m} := \{(x, y, v_{1,0}, \dots, v_{m,0}) \in GL_n(\mathbb{F}_p)^2 \times \mathbb{F}_p^{nm} : \forall i, v_{i,0} \notin Z_{i,0}, v_{i,\ell} = v_{i,0} \}.$$

The next proposition shows that $S_{w,m}$ is indeed a rare event (a "miracle").

Proposition 3.16. If $n \geq 2m\ell$, then $|S_{w,m}| < \ell^m p^{2n^2 + m^2\ell}$.

After we have shown that $S_{w,m}$ is a "miracle", we would like to show that $S_{w,m}$ must be large if $\left|w_{\mathrm{GL}_n(\mathbb{F}_p)}^{-1}(e)\right|$ is large. This will imply that $\left|w_{\mathrm{GL}_n(\mathbb{F}_p)}^{-1}(e)\right|$ must be small.

Exercise 3.17. For every prime p, we have $|GL_n(\mathbb{F}_p)| > 2^{-n}p^{n^2}$.

Proposition 3.16 implies Theorem 3.15. Indeed, for each $(x,y) \in w_{\mathrm{GL}_n(\mathbb{F}_p)}^{-1}(e)$ we always have $v_{i,\ell} = v_{i,0}$. Hence, there are at least $p^n(p^n - p^\ell)...(p^n - p^{\ell(m-1)}) \geq p^{nm}2^{-m}$ choices for $\{v_{i,0}\}_{i=1}^m$ such that $(x,y,v_1,\ldots,v_m) \in S_{w,m}$. Hence,

$$\left| w_{\mathrm{GL}_n(\mathbb{F}_p)}^{-1}(e) \right| p^{nm} 2^{-m} \le |S_{w,m}| < \ell^m p^{2n^2 + m^2 \ell}.$$

Let $m = \lfloor n/2\ell \rfloor$. Since $n \geq 32\ell$ we have, $\frac{n}{2\ell} - \frac{n}{32\ell} \leq \frac{n}{2\ell} - 1 \leq m \leq \frac{n}{2\ell}$. In particular,

$$\tau_{w,\operatorname{GL}_{n}(\mathbb{F}_{p})}(e) = \frac{\left|w_{\operatorname{GL}_{n}(\mathbb{F}_{p})}^{-1}(e)\right|}{\left|\operatorname{GL}_{n}(\mathbb{F}_{p})\right|^{2}} \leq \frac{2^{2n+m}}{p^{2n^{2}}} \cdot \ell^{m} p^{2n^{2}+m^{2}\ell-mn} \leq 2^{4n} p^{\frac{n^{2}}{4\ell} - \frac{n^{2}}{2\ell} + \frac{n^{2}}{32\ell}}.$$

$$\leq 2^{\frac{n^{2}}{8\ell}} p^{\frac{n^{2}}{4\ell} - \frac{n^{2}}{2\ell} + \frac{n^{2}}{32\ell}} \leq p^{\frac{n^{2}}{4\ell} - \frac{n^{2}}{2\ell} + \frac{n^{2}}{8\ell} + \frac{n^{2}}{32\ell}} \leq |\operatorname{GL}_{n}(\mathbb{F}_{p})|^{-\frac{1}{11\ell}}.$$

3.4.1. Proof of Proposition 3.16. We first analyze a special case.

Special (model) case: Let us bound the number the tuples $(x, y, v_{1,0}, \ldots, v_{m,0})$ in $S_{w,m}$ in which all $\{v_{i,j}\}_{i=1,\ldots,m,j=0,\ldots,3}$ are linearly independent. For simplicity of presentation, consider the commutator word $w = xyx^{-1}y^{-1}$ (the argument generalizes to any word).

We collect the information on $x, y, v_{i,j}$:

$$x(v_{i,2}) = v_{i,1} \text{ and } x(v_{i,3}) = v_{i,4}$$

$$(\star)$$

$$y(v_{i,1}) = v_{i,0} \text{ and } y(v_{i,2}) = v_{i,3}$$

Denote $V_x = \text{span} \{v_{i,2}, v_{i,3}\}_{i=1}^m$, $W_x = \text{span} \{v_{i,1}, v_{i,0}\}_{i=1}^m$, $V_y = \text{span} \{v_{i,1}, v_{i,2}\}_{i=1}^m$, $W_y = \text{span} \{v_{i,0}, v_{i,3}\}_{i=1}^m$. Note that each of these subspaces is 2m-dimensional. For every choice of linearly independent vectors $\{v_{i,j}\}_{i=1,\dots,m,j=0,\dots,3}$, the following holds:

$$(\star\star) x|_{V_x} = T_x \text{ and } y|_{V_y} = T_y,$$

for $T_x: V_x \to W_x$ and $T_y: V_y \to W_y$ determined by (\star) .

Exercise 3.18. The number of choices for $(x, y) \in GL_n(\mathbb{F}_p)^2$ satisfying $(\star\star)$ is $p^{4m(n-2m)} |GL_{n-2m}(\mathbb{F}_p)|^2 < p^{2n(n-2m)}$.

Since there are at most p^{4nm} options for $\{v_{i,j}\}_{i=1,\ldots,m}$, we get a total contribution of

$$p^{2n(n-2m)+4nm} \le p^{2n^2}.$$

General case: This is difficult to analyze directly. Instead, we relax the conditions in $S_{w,m}$, so that we count a larger set, but whose computation is similar to the special case above.

For each i, let b_i be the first index $j \ge 1$ such that $v_{i,j}$ is a linear combination of previous vectors $v_{i',j'}$ with $(i',j') \prec (i,j), j' \le b_{i'}$. We denote

$$R_{i,j} := \operatorname{Span} \{ v_{i',j'} \mid (i',j') \prec (i,j), j' \leq b_{i'} \},$$

and

$$S_{w,\{b_i\}_{i=1}^m} := \left\{ (x,y,v_{1,0},\ldots,v_{m,0}) \in G^2 \times \mathbb{F}_p^{nm} : \forall i,v_{i,0} \notin Z_{i,0},v_{i,b_i} \in R_{i,b_i} \text{ and } v_{i,j'} \notin R_{i,j'} \forall j' < b_i \right\}.$$

Note that $S_{w,m} \subseteq \bigcup_{\{b_i\}_{i=1}^m} S_{w,\{b_i\}_{i=1}^m}$. Hence, instead of bounding $S_{w,m}$ directly, we bound each individual $S_{w,\{b_i\}_{i=1}^m}$ and sum over all possible $\{b_i\}_{i=1}^m$. The advantage in working with $S_{w,\{b_i\}_{i=1}^m}$ is that we can provide estimates for $\left|S_{w,\{b_i\}_{i=1}^m}\right|$ which are very similar to the special case above. This is because we artificially stop each trial in the experiment at the moment any linear dependency occurs.

Hence, by conditioning on $\{v_{i,j}\}_{i\in[m],j< b_i}$, and by collecting all information on x,y as was done in (\star) and $(\star\star)$, we obtain the following upper bound:

$$|S_{w,m}| \le \sum_{\{b_i\}_{i=1}^m} \left| S_{w,\{b_i\}_{i=1}^m} \right| < \sum_{\{b_i\}_{i=1}^m} p^{m^2 \ell} p^{2n^2} \le \ell^m p^{2n^2 + m^2 \ell},$$

where:

- ℓ^m is an upper bound on the number of choices of $\{b_i\}_{i=1}^m$.
- $p^{m^2\ell}$ is an upper bound for the number of ways to write $v_{i,b_i} = \sum_{(i',j') \prec (i,b_i): j' < b_{i'}} a_{i,(i',j')} v_{i',j'}$, for $a_{i,(i',j')} \in \mathbb{F}_p$ and for each i.

Exercise 3.19. Using a similar computation as in the Special Case above, prove (3.3) to complete the proof of Proposition 3.16, by showing that for each $\{a_{i,(i',j')}\}_{i,i',j'}$, the number of tuples $(x, y, v_{1,0}, \ldots, v_{m,0}) \in S_{w,\{b_i\}_{i=1}^m}$ satisfying the condition $v_{i,b_i} = \sum_{(i',j') \prec (i,b_i):j' < b_{i'}} a_{i,(i',j')} v_{i',j'}$ is bounded by p^{2n^2} .

Remark 3.20. To go from Theorem 3.15 to the full generality of Theorem 3.13, one has to deal with a few more difficulties:

- (1) Estimating $\tau_{w,\operatorname{GL}_n(\mathbb{F}_p)}(g)$ for an arbitrary element g with $\left|C_{\operatorname{GL}_n(\mathbb{F}_p)}(g)\right| > \left|\operatorname{GL}_n(\mathbb{F}_p)\right|^{\delta}$.
- (2) Taking w to be a word with r letters instead of 2 letters.
- (3) Taking G to be any classical group of Lie type of rank $> C\ell(w)$, instead of $G = GL_n(\mathbb{F}_p)$.

Items (2) and (3) are mainly technical, and do not impose essential difficulties. To deal with Item (1), Larsen, Shalev and Tiep showed that if $g \in GL_n(\mathbb{F}_p)$ satisfies $|C_{GL_n(\mathbb{F}_p)}(g)| > p^{n^2\delta}$, then there exists a non-constant polynomial $Q(X) \in \mathbb{F}_p[X]$ such that dim $\ker Q(g) > \frac{1}{2}\delta n \deg Q$. One can then show that the condition that dim $\ker Q(w(g_1,...,g_r)) > \frac{1}{2}\delta n \deg Q$ is a rare event, by again analyzing $\Theta(n)$ trajectories of the word $w^{\deg Q}$. This reduces to a setting which is very similar to the one analyzed in the proof of Theorem 3.15.

4. Geometry of word maps on simple algebraic groups and interaction with probability.

In this section we make a connection between the geometry of word maps $w_{\underline{G}}: \underline{G}^r \to \underline{G}$ on simple algebraic groups (irreducible components and dimension of their fibers), and the probabilistic properties of the maps $w_{\underline{G}(\mathbb{F}_p)}: \underline{G}(\mathbb{F}_p)^r \to \underline{G}(\mathbb{F}_p)$.

Remark 4.1. This is a special case of a more general connection between the singularity properties of $w_{\underline{G}}$ to probabilistic properties of the maps $w_{\underline{G}(\mathbb{Z}_p)} : \underline{G}(\mathbb{Z}_p)^r \to \underline{G}(\mathbb{Z}_p)$. We will not discuss this here. However, this connection is studied thoroughly in [AA16, AA18, GH24, CGH23] and in [AGL, Section 5].

Our main tool will be the Lang-Weil estimates.

4.1. The Lang-Weil estimates and a geometric interpretation of L^{∞} -mixing time.

Theorem 4.2 (The Lang-Weil estimates, [LW54]). Let X be a finite type \mathbb{F}_q -scheme (e.g. defined by $f_1 = ... = f_r = 0$, for $f_1, ..., f_r \in \mathbb{F}_q[x_1, ..., x_n]$). Then:

$$\left| \frac{|X(\mathbb{F}_q)|}{q^{\dim X}} - C_X \right| \le Cq^{-1/2},$$

where C_X is the number of top-dimensional irreducible components of $X_{\overline{\mathbb{F}_q}}$, which are defined over \mathbb{F}_q , and C depends only on the complexity of X^1 (i.e. on r, n and the degrees of f_i).

Remark 4.3. In fact, uniform bounds can be given for the constant C above, which are polynomial in the degree $d := \max_{i=1,\dots,r} \deg f_i$. Namely (see [CM06, Theorem 7.5]) one can take:

$$C = O_n(d^{O_n(1)}).$$

This will be essential in §7 when we exploit uniform mixing bounds for word varieties whose complexity does not remain bounded.

Example 4.4.

(1) Consider $X = SL_2$. Then $\dim X_{\mathbb{F}_p} = 3$ and indeed

$$|\mathrm{SL}_2(\mathbb{F}_p)| = p(p^2 - 1) = p^3(1 - p^{-2}).$$

(2) Let $X = \{x^2 = -1\}$. Let p > 2. Then $\dim X_{\mathbb{F}_p} = 0$ and $X_{\overline{\mathbb{F}_p}}$ has 2 irreducible components $x = \pm i$. If $p = 1 \mod 4$, then $|X(\mathbb{F}_p)| = C_{X_{\mathbb{F}_p}} = 2$ and if $p \neq 1 \mod 4$ then $|X(\mathbb{F}_p)| = C_{X_{\mathbb{F}_p}} = 0$.

¹For the precise notion of complexity, see e.g. [GH19, Definition 7.7].

Definition 4.5. Let K be a field, and let $\varphi: X \to Y$ be a morphism of K-schemes. For every field extension $K \leq K'$ and $y \in Y(K')$, denote:

- $X_{y,\varphi}$ the K'-scheme $\{x \in X : \varphi(x) = y\}$, which is the fiber of φ above y.
- $\varphi^{-1}(y) := \{x \in X(K') : \varphi(x) = y\}$, the K' points of $X_{y,\varphi}$.

Definition 4.6. Let K be a field. A morphism $\varphi: X \longrightarrow Y$ between smooth, irreducible K-varieties is called:

- (1) Flat if for every $x \in X(\overline{K})$ we have $\dim X_{\varphi(x),\varphi} = \dim X \dim Y$.
- (2) Flat with geometrically irreducible fibers, or (FGI), if it is flat, and $X_{\varphi(x),\varphi}$ is irreducible (as a \overline{K} -scheme) for each $x \in X(\overline{K})$.

Remark 4.7. Definition 4.6(1) is not the standard definition of flatness, and it is a consequence of the Miracle Flatness Theorem [Har77, III, Exercise 10.9]. For the more general definition of flatness see e.g. [Har77, p.253-254].

Example 4.8. Let K be a field.

- The map $\varphi: \mathbb{A}^2_K \to \mathbb{A}^2_K$ $(x,y) \longmapsto (x,xy)$ is not flat since the fiber over $(0,0) \in \overline{K}^2$ is one dimensional although the "expected" dimension is 0.
- The map $\psi: \mathbb{A}^2_K \to \mathbb{A}_K \ (x,y) \mapsto x^2 + y^2$ is flat but not (FGI).

The following proposition is a direct consequence of the above definitions, the Lang-Weil estimates (Theorem 4.2), and the fact that the complexity of the fibers of φ is uniformly bounded.

Proposition 4.9. Fix a prime p. Let $\varphi: X \to Y$ be a morphism of smooth, geometrically irreducible \mathbb{F}_p -varieties.

(1) $\varphi: X \to Y$ is flat if and only if there exists C > 0 such that for every $q = p^r$ and every $y \in Y(\mathbb{F}_q)$:

$$\left| \frac{|\varphi^{-1}(y)|}{q^{(\dim X - \dim Y)}} - C_{X_{y,\varphi}} \right| < Cq^{-\frac{1}{2}},$$

or equivalently, by the Lang-Weil estimates,

$$\left| \frac{|\varphi^{-1}(y)|}{|X(\mathbb{F}_q)|} |Y(\mathbb{F}_q)| - C_{X_{y,\varphi}} \right| < Cq^{-\frac{1}{2}}.$$

(2) $\varphi: X \to Y$ is (FGI) if and only if there exists C > 0 such that for every $q = p^r$:

$$\left\|\varphi_*\mu_{X(\mathbb{F}_q)}-\mu_{Y(\mathbb{F}_q)}\right\|_{\infty}=\max_{y\in Y(\mathbb{F}_q)}\left|\frac{|\varphi^{-1}(y)|}{|X(\mathbb{F}_q)|}\left|Y(\mathbb{F}_q)\right|-1\right|< Cq^{-\frac{1}{2}}.$$

Moreover, C depends only on the complexity of the map φ .

Example 4.10. Back to Example 4.8, if $K = \mathbb{F}_q$:

- (1) For $(0,0) \in \mathbb{F}_q^2$, $|\varphi^{-1}(0,0)| = q$, where the expected size is $q^0 = 1$.
- (2) The map $\psi_{\mathbb{F}_q}$ is flat, so all non-empty fibers are of dimension 1. However, for $0 \in \mathbb{F}_q$, the scheme $X_{0,\psi} = \{x^2 + y^2 = 0\}$ has 2 irreducible components over $\overline{\mathbb{F}_q}$: $\{x iy = 0\}$ and $\{x + iy = 0\}$. Also:
 - (a) If $q = 1 \mod 4$, then $C_{X_{0,\psi}} = 2$ and $|\psi^{-1}(0)| = 2q 1$.
 - (b) If $q = 3 \mod 4$, then $C_{X_{0,\psi}} = 0$ and $|\psi^{-1}(0)| = 1$.

In either case, the equivalent conditions of Item (2) of Proposition 4.9 are not satisfied.

4.2. Completing the proof of Theorem 2.18 for groups of Lie type. We have seen so far a (sketch of) proof of Theorem 2.18 (and Theorem 3.9) for groups of Lie type in the case that $\operatorname{rk}(G) > C\ell(w)$, where the bottleneck was the probabilistic argument (Theorem 3.13). The low rank case is based on the following geometric statement, whose proof we discuss in the next subsection.

Theorem 4.11. Let $w \in F_r$ and let \underline{H} be a simple, simply connected algebraic \mathbb{F}_q -group. Then for every $t \ge \dim \underline{H} + 1$, the map $(w^{*t})_{\underline{H}} : \underline{H}^{rt} \to \underline{H}$ is (FGI).

We can now complete the proof of Theorem 2.18 for groups of Lie type.

Sketch of proof of Theorem 2.18 for groups of Lie type. For simplicity, we consider the case of untwisted Chevalley groups (the other case can be handled similarly, with a few more complications). So, suppose $G \simeq \underline{G}(\mathbb{F}_q)/Z(\underline{G}(\mathbb{F}_q))$, where \underline{G} is a Chevalley group scheme.

- (1) If $\operatorname{rk}(\underline{G}) > C'\ell(w)$ for some $C' \gg 1$, this case follows from the argument at the end of §3.3, using Theorem 3.13 which was proved in high rank in §3.4, and using Theorem 3.11.
- (2) If $\operatorname{rk}(\underline{G}) \leq C'\ell(w)$, we use Theorem 4.11 and Proposition 4.9 to deduce

(4.1)
$$\lim_{q \to \infty} \left\| \tau_{w,\underline{G}(\mathbb{F}_q)}^{*t} - \mu_{\underline{G}(\mathbb{F}_q)} \right\|_{\infty} = 0,$$

for $t \geq C''\ell(w)^2 \geq \dim \underline{G} + 1$. **Exercise:** (4.1) holds if we replace each $\underline{G}(\mathbb{F}_q)$ with G = $\underline{G}(\mathbb{F}_q)/Z(\underline{G}(\mathbb{F}_q)).$

Combining Items (1) and (2) proves Theorem 2.18(2).

Exercise 4.12. Show, using the Lang-Weil estimates, that Theorem 4.11 further implies Theorem 3.13 for untwisted Chevalley groups G of low rank (i.e. $\operatorname{rk}(G) \leq C'\ell(w)$).

By Theorem 2.18 and Proposition 4.9, we deduce:

Corollary 4.13 ([LST19, Theorem 5]). For every $1 \neq w \in F_r$, there exist $\epsilon(w) > 0$ and $t(w) \in \mathbb{N}$, such that for every simple, simply connected algebraic \mathbb{F}_q -group \underline{G} :

- (1) The fibers $(\underline{G}^r)_{g,w_{\underline{G}}}$ of $w_{\underline{G}}:\underline{G}^r\to\underline{G}$ are of codimension $\geq \epsilon(w)\dim\underline{G}$, for every $g\in\underline{G}(\overline{\mathbb{F}_q})$. (2) For every t>t(w), the map $w_{\underline{G}}^{*t}:\underline{G}^{rt}\to\underline{G}$ is (FGI).
- 4.3. Proof of the low rank geometric statement (Theorem 4.11). We use the notion of convolution in algebraic geometry, defined by the second author and Hendel in [GH21]. Let K be a field.

Definition 4.14 ([GH21, Definition 1.1]). Let $\varphi: X \to \underline{G}, \psi: Y \to \underline{G}$ be maps from K-varieties X, Y to an algebraic K-group \underline{G} . Define the convolution $\varphi * \psi : X \times Y \to \underline{G}$ by

$$\varphi * \psi(x, y) = \varphi(x) \cdot \psi(y).$$

We denote by $\varphi^{*t}: X^t \to \underline{G}$ the t-th convolution power.

As is well known in classical analysis, convolution of functions improves regularity. For example, if one convolves a k-differentiable function $f: \mathbb{R}^n \to \mathbb{R}$, with an l-differentiable function $h: \mathbb{R}^n \to \mathbb{R}$, the result f * h is k + l-differentiable. Another example is Young's convolution inequality (as in Remark 2.9).

A similar phenomenon takes place here; the convolution operation in Definition 4.14 improves singularity properties of morphisms. We shall see manifestations of this phenomenon below, and also in $\S 8$.

Fact 4.15 (cf. [GH21, Proposition 3.1]). Let X, Y be K-varieties. Let $\varphi : X \to \underline{G}$ be a flat morphism and $\psi : Y \to \underline{G}$ be any morphism. Then $\varphi * \psi : X \times Y \to \underline{G}$ is flat.

Proof. Consider the following diagram:

$$\begin{array}{ccc} X \times Y & \xrightarrow{\pi_X} & X \\ \downarrow \alpha & & \downarrow \varphi \\ \underline{G} \times Y & \xrightarrow{\beta} & \underline{G} \\ \downarrow \pi_{\underline{G}} & \\ \underline{G}, & \end{array}$$

where $\pi_X(x,y) = x$, $\pi_{\underline{G}}(g,y) = g$ are the natural projections, $\alpha(x,y) = (\varphi * \psi(x,y), y)$ and $\beta(g,y) = g \cdot \psi(y)^{-1}$. Note that the morphism $\alpha : X \times Y \to \underline{G} \times Y$ is a base change of $\varphi : X \to \underline{G}$, and that the morphism $\pi_{\underline{G}} : \underline{G} \times Y \to \underline{G}$ is flat, since $Y \to \operatorname{Spec} K$ is flat. Since $\varphi * \psi = \pi_{\underline{G}} \circ \alpha$ and since flatness is preserved under base change and compositions, the proposition follows.

Theorem 4.16 (Glazer-Hendel, [GH21, Thm B]). Let $\varphi: X \to \underline{G}$ be a dominant K-morphism from a smooth, geometrically irreducible K-variety X to an algebraic K-group \underline{G} . Then φ^{*t} is flat for all $t \geq \dim \underline{G}$.

Proof. Let $Z_1 \subseteq X$ be the non-flat locus of φ . Since φ is dominant, it is generically flat, hence $\dim Z_1 \leq \dim X - 1$. By Fact 4.15, the non-flat locus Z_t of φ^{*t} in X^t must be contained in $Z_1 \times ... \times Z_1$, and hence $\dim Z_t \leq t \dim X - t$.

Suppose toward contradiction that φ^{*t} is not flat, so that it has a fiber of dimension $> t \dim X - \dim \underline{G}$. Any top-dimensional irreducible component of this fiber is contained in Z_t , so we must have $t < \dim \underline{G}$, and a contradiction.

Remark 4.17. The smoothness hypothesis in Theorem 4.16 can be omitted, when one works with the more general definition of flatness (see Remark 4.7). The proof remains the same.

Theorem 4.18 ([LST19, Lemma 2.4]). Let \underline{G} be a simply connected semisimple K-algebraic group and let $w_1 \in F_{r_1}$ and $w_2 \in F_{r_2}$. Then there exists a Zariski open subset $U \subseteq \underline{G}$ such that the fiber $(\underline{\underline{G}}^{r_1+r_2})_{g,(w_1*w_2)_{\underline{G}}}$ is geometrically irreducible for every $g \in U(\overline{K})$ ("geometrically irreducible generic fiber").

Example 4.19 (Warning(!)). The assumption that \underline{G} is simply connected is crucial. The map $(w_{\text{com}}^{*t})_{\text{PGL}_n}$ does not have geometrically irreducible generic fibers, for every $t \in \mathbb{N}$.

In §8, we discuss a vast generalization of Theorem 4.18 (see Theorem 8.2).

Lemma 4.20. Let $f: Z \to Y$ be a continuous map of topological spaces, such that f is open and Y is irreducible. Suppose there exists a dense collection U of points $y \in Y$ s.t. $f^{-1}(y)$ is irreducible. Then Z is irreducible.

Proof. If Z is not irreducible, then $Z = Z_1 \cup Z_2$, with Z_1, Z_2 closed, proper subsets. For each $y \in U$, since $f^{-1}(y)$ is irreducible, either $f^{-1}(y) \subseteq Z_1$ or $f^{-1}(y) \subseteq Z_2$. Let $W_i := \{y \in Y : f^{-1}(y) \subseteq Z_i\}$. Then $W_i = Y \setminus f(Z_i^c)$ is closed, as f is open. Moreover, $W_i \neq Y$, since $Z_i \neq X$. Finally, $W_1 \cup W_2 \supseteq U$ and hence $W_1 \cup W_2 \supseteq \overline{U} = Y$, and we are done.

Proposition 4.21 ([GH24, Corollary 3.20]). Let X,Y be geometrically irreducible K-varieties, let $\varphi: X \to \underline{G}$ be a flat morphism with geometrically irreducible generic fiber, and let $\psi: Y \to \underline{G}$ be a dominant morphism. Then $\varphi * \psi: X \times Y \to \underline{G}$ is (FGI).

Proof. Let $Z := (X \times Y)_{g,\varphi * \psi}$ be a fiber of $\varphi * \psi$ over $g \in \underline{G}(\overline{K})$, and consider the map $f : Z(\overline{K}) \to Y(\overline{K})$, induced from the standard projection $X \times Y \to Y$. Then $Y(\overline{K})$ is irreducible and there exists $U \subseteq Y$ such that $\forall y \in U(\overline{K})$,

$$f^{-1}(y) = \left\{ (x,y) \in Z(\overline{K}) : \varphi * \psi(x,y) = g \right\} \simeq \left\{ x \in X(\overline{K}) : \varphi(x) = g\psi(y)^{-1} \right\},$$

is irreducible. By Lemma 4.20, Z is irreducible. By Fact 4.15, $\varphi * \psi$ is flat, hence it is (FGI). \Box

We are now ready to prove Theorem 4.11.

Proof of Theorem 4.11. Let $w \in F_r$ and let \underline{H} be a simple, simply connected algebraic \mathbb{F}_q -group. Since $\dim \underline{H} > 1$, by Theorems 4.16 and 4.18, the map $(w^{*t})_{\underline{H}} : \underline{H}^{rt} \to \underline{H}$ is flat, with geometrically irreducible generic fiber, for every $t \geq \dim \underline{H}$. By Proposition 4.21 and Theorem 2.4, $(w^{*(t+1)})_{\underline{H}} : \underline{H}^{r(t+1)} \to \underline{H}$ is (FGI) for every $t \geq \dim \underline{H}$, as required.

5. Representation varieties of random groups

In Sections 5 to 7, we change the setting, so that now we fix a connected semisimple algebraic group \underline{G} , and consider the word maps $w_{\underline{G}} : \underline{G}^r \to \underline{G}$, varying over all words $w \in F_r$ of a given length ℓ , as $\ell \to \infty$.

However, unlike Sections 2 to 4, instead of considering self-convolutions of word measures $\tau_{w,G}$, we consider self-convolutions of measures of the form $\mu_{\underline{A}} = \frac{1}{2r} \sum_{i=1}^{r} \left(\delta_{A_i} + \delta_{A_i^{-1}} \right)$, where $\underline{A} := (A_1, ..., A_r)$ is a generating tuple in $\underline{G}(\mathbb{F}_p)$. Applying ℓ self-convolutions of μ amounts to applying a **random** word of length ℓ on the tuple $(A_1, ..., A_r)$. Hence, good equidistribution of $\mu_{\underline{A}}^{*\ell}$ on $\underline{G}(\mathbb{F}_p)$ for all generating tuples \underline{A} , and for all primes, should say something about the geometry of **random** words.

In this section we consider all algebraic varieties and algebraic groups over the complex numbers. This is merely for simplicity of presentation. In sections 6 and 7, we discuss the same objects over other fields as well.

5.1. Representation varieties and universal theories.

Definition 5.1. We consider a presentation for an arbitrary finitely presented group as follows:

$$\Gamma_w := \langle x_1, ..., x_r : w_1(\underline{x}) = ... = w_k(\underline{x}) = 1 \rangle$$

where r is the number of generators and $\underline{w} = (w_1, ..., w_k)$, with $w_i \in F_r$, is the tuple of relators. We denote by

$$X_{\underline{w},\underline{G}} = \operatorname{Hom}(\Gamma_{\underline{w}},\underline{G}) = \left\{ (g_1,...,g_r) \in \underline{G}^r : w_1(\underline{g}) = ... = w_k(\underline{g}) = 1 \right\},\,$$

the **representation variety** of $\Gamma_{\underline{w}}$ in \underline{G} . This is a closed subscheme of \underline{G}^r .

If \underline{w} consists of a single word $w \in F_r$, then we write $X_{w,G}$. For example, if w = 1, then $X_{w,G} = \underline{G}^r$.

Remark 5.2. By Hilbert's Basis Theorem, for every r-generated group Γ (even infinitely presented), there exist finitely many $w_1, ..., w_k \in F_r$, such that

$$\operatorname{Hom}(\Gamma, \underline{G}) \simeq \operatorname{Hom}(\Gamma_w, \underline{G}) = X_{w,G}.$$

Clearly, there is no uniqueness in the tuple \underline{w} in the above remark as any other tuple that generates (as a normal subgroup) the same subgroup of F_r as \underline{w} would also work. Yet the relations $\underline{w} = 1$ in \underline{G} can imply other relations of the form v = 1, where v may not belong to the normal subgroup generated by \underline{w} . For instance, it is easily seen that the relation $ba^3b^{-1} = a^2$ in $\mathrm{SL}_2(\mathbb{C})$ implies

that the subgroup $\langle a,b\rangle \leq \operatorname{SL}_2(\mathbb{C})$ is metabelian. However the one-relator group $\langle a,b|ba^3b^{-1}=a^2\rangle$ is a non-solvable Baumslag-Solitar group. The main results of this section will give ample further evidence of this phenomenon. However, determining the exact set of relations entailed by a given one in \underline{G} is an intriguing problem that is completely open as far as we know even for $\underline{G} = \operatorname{SL}_2$.

Problem 5.3. Let $G = \mathrm{SL}_2(\mathbb{C})$. Given $\underline{w} \in F_r^k$, find all $\nu \in F_r$, such that

$$\forall g_1, \dots, g_r \in G: \underline{w}(g_1, \dots, g_r) = 1 \Longrightarrow \nu(g_1, \dots, g_r) = 1.$$

This problem can be paraphrased (and extended) in terms of first order logic by asking to determine the universal theory of $SL_2(\mathbb{C})$. In logic, a universal sentence in the language of groups is a formula with a single (universal) quantifier and no free variables. It is easy to see that it must be (the conjunction of) formulas of the following form, where $\underline{w} \in F_r^k$ and $\underline{\nu} \in F_r^m$:

$$\forall g_1, \dots, g_r \in G: \bigwedge_{1}^k w_i(g_1, \dots, g_r) = 1 \Longrightarrow \bigvee_{1}^m \nu_i(g_1, \dots, g_r) = 1,$$

and the set of all universal sentences that are true in G is called the *universal theory of G*. For example the universal theory of the free group can be described using the Makanin–Razborov algorithm and groups with the same universal theory as the free group have been given a geometric description as the so-called *limit groups* by Z. Sela [Sel01]. Hence, Problem 5.3 can be rephrased as:

Problem 5.4. Describe the universal theory of $SL_2(\mathbb{C})$.

The universal theory of $SL_2(\mathbb{C})$ is decidable, because of Hilbert's Nullstellensatz and the decidability of ideal membership (elimination theory, Groebner bases, etc.).

5.2. Character varieties. Note that \underline{G} acts on \underline{G}^r by conjugation $g.(g_1, ..., g_r) = (gg_1g^{-1}, ..., gg_rg^{-1})$, and that $X_{\underline{w},\underline{G}}$ is \underline{G} -invariant with respect to this action. Assume from now on that \underline{G} is reductive, we then define the **character variety** as the categorical quotient

$$\mathcal{X}_{\underline{w},\underline{G}} := X_{\underline{w},\underline{G}} /\!\!/ \underline{G}$$

in the sense of geometric invariant theory (GIT) (see e.g. [VP89, Section 4]). If $\mathbb{C}[X_{\underline{w},\underline{G}}]$ is the coordinate ring of $X_{\underline{w},\underline{G}}$, then $\mathcal{X}_{\underline{w},\underline{G}}$ is the affine scheme whose coordinate ring² is $\mathbb{C}[X_{\underline{w},\underline{G}}]^{\underline{G}}$.

Example 5.5 (see e.g. [CS83, ABL18], [Bow98, Section 4]). Let r=2 and $\underline{G}=\mathrm{SL}_2$. Suppose w=1. Then $\Gamma_w=F_2,\,X_{w,\mathrm{SL}_2}=\mathrm{SL}_2\times\mathrm{SL}_2$ and

$$\mathcal{X}_{w,\mathrm{SL}_2} = (\mathrm{SL}_2 \times \mathrm{SL}_2) /\!\!/ \mathrm{SL}_2 \simeq \mathbb{A}_{\mathbb{C}}^3.$$

The isomorphism is given by $(A, B) \longmapsto (\operatorname{tr}(A), \operatorname{tr}(B), \operatorname{tr}(AB))$. Indeed:

- For every $(x, y, z) \in \mathbb{C}^3$, we can find $(A, B) \in \mathrm{SL}_2(\mathbb{C})$ with $\mathrm{tr}(A) = x$, $\mathrm{tr}(B) = y$ and $\mathrm{tr}(AB) = z$.
- There is a Zariski open set $U \subseteq \mathbb{A}^3_{\mathbb{C}}$ (defined by $\Delta \neq 0$, see Example 5.23 below) where

$$(\operatorname{tr}(A),\operatorname{tr}(B),\operatorname{tr}(AB))=(\operatorname{tr}(A'),\operatorname{tr}(B'),\operatorname{tr}(A'B')) \Longleftrightarrow (A',B') \in (A,B)^{\operatorname{SL}_2}$$

These are called the *Fricke-Klein* coordinates on $\mathcal{X}_{1,\mathrm{SL}_2}$.

²A priori it is not clear that $\mathbb{C}[X_{\underline{w},\underline{G}}]^{\underline{G}}$ is a finitely generated \mathbb{C} -algebra. However, this is true whenever \underline{G} is reductive (Hilbert, [VP89]).

Example 5.6. When r=2 and $\underline{G}=\mathrm{SL}_3$, then X_{1,SL_3} is a (branched) 2-cover of $\mathbb{A}^8_{\mathbb{C}}$ via the map $(A,B)\longmapsto (\mathrm{tr}(A),\mathrm{tr}(B),\mathrm{tr}(AB),\mathrm{tr}(A^{-1}),\mathrm{tr}(B^{-1}),\mathrm{tr}(AB^{-1}),\mathrm{tr}(A^{-1}B),\mathrm{tr}(A^{-1}B^{-1}))$, see [Law07].

Note that in general, the map $\underline{G}(\mathbb{C})^r \to \operatorname{Hom}(\mathbb{C}[X_{\underline{w},\underline{G}}]^{\underline{G}},\mathbb{C})$ that sends (g_1,\ldots,g_r) to the evaluation map at this point descends to an onto map between \underline{G} -orbits closures of r-tuples and (closed) points of $X_{\underline{w},\underline{G}}/\!\!/\underline{G}$. Moreover, two orbit closures are mapped to the same point if and only if they intersect. We refer to [Sik12] for general background on character varieties.

Recall our notation: $\langle g_1, \ldots, g_r \rangle$ stands for the (abstract) subgroup generated by $g_1, \ldots, g_r \in \underline{G}(\mathbb{C})$ and $\overline{\langle g_1, \ldots, g_r \rangle}^Z$ for the Zariski-closure of this subgroup, which is a closed algebraic subgroup of \underline{G} . The following fact is essential:

Fact 5.7 ([PRR23, Theorem 2.50]). Let \underline{G} be a reductive group and suppose that $g_1, ..., g_r \in \underline{G}(\mathbb{C})$ Zariski-generates a reductive subgroup $\underline{H} \leq \underline{G}$ (i.e. $\overline{\langle g_1, ..., g_r \rangle}^Z = \underline{H}$). Then the \underline{G} -orbit $(g_1, ..., g_r)^{\underline{G}}$ of $(g_1, ..., g_r)$ is closed.

Corollary 5.8. If $\overline{\langle g_1,...,g_r\rangle}^Z = \underline{G}$ then $(g_1,...,g_r)^{\underline{G}}$ is closed.

Example 5.9. The element

$$u_t := \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$$

Zariski-generates (when $t \neq 0$) the maximal unipotent subgroup $\{u_t, t \in \mathbb{C}\}$, and indeed, $d_{\lambda}u_td_{\lambda}^{-1} = u_{\lambda^2 t}$, where d_{λ} is the diagonal matrix $diag(\lambda, \lambda^{-1})$. So the identity is contained in the closure of the $\mathrm{SL}_2(\mathbb{C})$ -orbit of u_t , but not in the orbit itself. For a similar reason it can be shown that the converse to Fact 5.7 holds (e.g. see [Sik12, Theorem 29]).

5.3. Random groups. Fix $r \geq 2$. We can define a random group on r generators,

$$\Gamma_w := \langle x_1, ..., x_r : w_1(\underline{x}) = ... = w_k(\underline{x}) = 1 \rangle$$

by choosing the relators \underline{w} at random among words of length ℓ . There are several probabilistic models for random words, among them:

- (1) Choosing reduced words of length ℓ : there are 2r options to choose the first letter in $\left\{x_i^{\pm 1}\right\}_{i=1}^r$ and then 2r-1 options for choosing each of the next $\ell-1$ letters, so a total of $2r \cdot (2r-1)^{\ell-1}$ possible reduced words of length ℓ .
- (2) Choosing reduced words of length in an interval $[c_1\ell, c_2\ell]$ for constants $c_1 < c_2$.
- (3) Choosing non-reduced words of length ℓ : there are $(2r)^{\ell}$ options.

It turns out that the analysis of Models (1) and (2) can be reduced to that of Model (3), so we will focus on the latter, which is more convenient as it allows to see random words as random walks on the free group. All three models usually behave similarly and give the same qualitative answers. Nevertheless, the Gromov density model is usually phrased using reduced words as in Model (1). It is as follows:

Definition 5.10 (Gromov density model, [Gro93], [Oll05]). A random group on r generators, at density $\delta \in [0,1]$ and length ℓ is the group $\Gamma_{\underline{w}}$, where $\underline{w} = (w_1, ..., w_k)$, for $k = \lfloor (2r-1)^{\delta \ell} \rfloor$, and where $w_1, ..., w_k$ are chosen independently uniformly at random among all reduced words of length ℓ .

We say that an event (or a sequence of events parameterized by ℓ) occurs **asymptotically almost** surely (or a.a.s.) if it occurs in probability $\geq 1 - o(1)$ as $\ell \to \infty$.

Theorem 5.11. Let $\delta \in [0,1]$ and $k = \lfloor (2r-1)^{\delta \ell} \rfloor$. The following hold a.a.s.:

- (1) [Gro93, 9.B]; If $\delta \in [0, \frac{1}{12})$. Then $\Gamma_{\underline{w}}$ has small cancellation $C'(\frac{1}{6})$. This means that no two relators (or their cyclic permutations) overlap on a segment of length $\geq \frac{\ell}{6}$.
- (2) [Gro93, 9.B],[Oll05]; If $\delta \in [0, \frac{1}{2})$, then $\Gamma_{\underline{w}}$ is infinite and Gromov-hyperbolic.
- (3) [DGP11]; If $\delta > 0$ then $\Gamma_{\underline{w}}$ does not act on a tree without a global fixed point.
- (4) [Gro93], [Z03], [KK13]; If $\delta > \frac{1}{3}$ then $\Gamma_{\underline{w}}$ has Kazhdan's property T.
- (5) [Gro93]; If $\delta > \frac{1}{2}$ then $\Gamma_{\underline{w}}$ is trivial if ℓ is odd and isomorphic to $\mathbb{Z}/2\mathbb{Z}$ if ℓ is even.

A good general reference for the Gromov model is Ollivier's monograph [Oll05]. In addition to the properties above, random groups at positive density do not have low dimensional representations:

Theorem 5.12 (Kozma–Lubotzky, [KL19]). If $\delta > 0$ and $d \in \mathbb{N}$ is fixed, then a.a.s., every $\varphi \in \text{Hom}(\Gamma_w, \text{GL}_d(\mathbb{C}))$ satisfies:

$$\begin{cases} \varphi \text{ is trivial} & \text{if } \ell \text{ is odd} \\ |\varphi(\Gamma_{\underline{w}})| \leq 2 & \text{if } \ell \text{ is even.} \end{cases}$$

Remark 5.13. In contrast to Kozma–Lubotzky's result, Ollivier–Wise [OW11] and Agol [Ago13] showed that every C'(1/6) group embeds in a right-angled Artin group (RAAG) and hence embeds in $GL_N(\mathbb{Z})$ for some N (see also [Ber15, Theorem 5.9]). Hence for a random group, the minimal such N must grow with the length ℓ of the relators.

5.3.1. Sketch of proof of Theorem 5.12. Note that if $w \neq 1$, then $X_{w,\mathrm{GL}_d(\mathbb{C})} \subseteq \mathrm{GL}_d(\mathbb{C})^r$ is a proper subvariety.

The idea of the proof is to take a random relator w_1 , and consider $X_{w_1,\mathrm{GL}_d(\mathbb{C})}$, and then consider another random relator w_2 and show that $X_{w_1,\mathrm{GL}_d(\mathbb{C})} \cap X_{w_2,\mathrm{GL}_d(\mathbb{C})}$ is of smaller dimension with high probability. Then we iterate until we are only left with points $\underline{a} \in \mathrm{GL}_d(\mathbb{C})^r$ with $|\langle a_1, ..., a_r \rangle| \leq 2$. The main tool to make this work is Bezout's theorem.

Definition 5.14. The **degree** of an irreducible subvariety $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ of dimension m, denoted $\deg(X)$, is the number of intersection points of X with an affine subspace H of codimension m, in general position. If X has irreducible components $X_1, ..., X_M$, then we define

$$\deg(X) := \sum_{i=1}^{M} \deg(X_i).$$

Note that in particular, the number of irreducible components of X is at most its degree.

From this definition, we see immediately that if X is a hypersurface defined by the vanishing of a polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ of degree d, then X has degree d.

Theorem 5.15 (Bezout's theorem). Let X,Y be closed subvarieties of $\mathbb{A}^n_{\mathbb{C}}$. Then

$$\deg(X \cap Y) \le \deg(X) \cdot \deg(Y).$$

For the proof, we refer the reader to [Ful84]) and [Dan94, Theorem II.3.2.2], which deal mostly with projective varieties and to [Sch81] for the affine case. See also [KL19] where the following is deduced:

Exercise 5.16. If $X = \{f_1 = ... = f_s = 0\} \subseteq \mathbb{A}^n_{\mathbb{C}}, f_i \in \mathbb{C}[x_1, ..., x_n], \text{ and denote } d := \max_i \deg(f_i).$ Then X has degree at most $d^{\min(s,n)}$.

For a similar reason, given a word w of length ℓ , the degree of the word variety $X_{w,\mathrm{GL}_d(\mathbb{C})}$ is at most $\ell^{O_{d,r}(1)}$.

A sketch of proof of Theorem 5.12:

(1) Let $U \subseteq \operatorname{GL}_d(\mathbb{C})^r$ be the open subset $\operatorname{GL}_d(\mathbb{C})^r \setminus \{(a, ..., a) : a^2 = I_n\}$. Pick $w_1 \in F_r$ of length ℓ at random, and decompose $X_{w_1, \operatorname{GL}_d(\mathbb{C})} \cap U$ into its irreducible components $X_{w_1, \operatorname{GL}_d(\mathbb{C})} \cap U = X_{w_1}^{(1)} \cup ... \cup X_{w_1}^{(k_1)}$.

Claim. For every $\underline{a} \in \mathrm{GL}_d(\mathbb{C})^r$, if $|\langle a_1, ..., a_r \rangle| > 2$, then

(5.1)
$$\Pr_{|w|=\ell}[w(\underline{a}) = 1] \le 1 - \frac{1}{2r}.$$

Proof. Let $\mu := \frac{1}{2r} \sum_{i=1}^r \left(\delta_{a_i} + \delta_{a_i^{-1}} \right)$. Since we assumed $\underline{a} \in U$, we must have $\max_{x \in \mathrm{GL}_d(\mathbb{C})} \mu(x) \le 1 - \frac{1}{2r}$. In particular,

$$\Pr_{|w|=\ell}[w(\underline{a}) = 1 \mid \text{given first } \ell - 1 \text{ letters}] \le 1 - \frac{1}{2r},$$

which implies the claim.

(2) For each $i = 1, ..., k_1$, we choose a tuple $\underline{a}^{(i)}$ belonging to $X_{w_1}^{(i)}$, and further choose $c \log(\ell)$ independent words $\underline{w_2} = w_2^{(1)}, ..., w_2^{(c \log(\ell))}$. By Item (1), for each fixed i, the probability that $\underline{w_2}$ will not break the irreducible component of $X_{w_1}^{(i)}$ into smaller components is bounded by:

$$\Pr\left[\underline{w_2}(\underline{a}^{(i)}) = 1\right] \le (1 - \frac{1}{2r})^{c \log(\ell)},$$

and hence,

$$\Pr\left[\dim X_{w_1,\operatorname{GL}_d(\mathbb{C})} \cap U > \dim X_{(w_1,\underline{w_2}),\operatorname{GL}_d(\mathbb{C})} \cap U\right] \ge \Pr\left[\forall i : \underline{w_2}(\underline{a}^{(i)}) \ne 1\right]$$

$$= 1 - \Pr\left[\exists i \in [k_1] : \underline{w_2}(\underline{a}^{(i)}) = 1\right] \ge 1 - k_1(1 - \frac{1}{2r})^{c\log(\ell)}.$$

By Bezout's theorem: $k_1 \leq \ell^{O_{r,d}(1)}$, and hence by taking c large enough,

$$\Pr\left[\dim X_{w_1,\operatorname{GL}_d(\mathbb{C})} \cap U > \dim X_{(w_1,\underline{w_2}),\operatorname{GL}_d(\mathbb{C})} \cap U\right] \ge 1 - \frac{1}{\ell^{O_{r,d}(1)}}.$$

We repeat this process $O_{r,d}(1)$ times, until we obtain an empty set.

(3) We have proved that if $\Gamma_{\underline{w}}$ is a random group with $\gg_{d,r} \log(\ell)$ random relators (and in particular, if $\Gamma_{\underline{w}}$ is a random group at density $\delta > 0$), then a.a.s., the variety $X_{\underline{w},\mathrm{GL}_d(\mathbb{C})}$ contains only elements \underline{a} of the form (a,...,a) with $a^2 = I$. I.e., $X_{\underline{w},\mathrm{GL}_d(\mathbb{C})}$ consists of homomorphisms $\varphi : \Gamma_{\underline{w}} \to \mathrm{GL}_d(\mathbb{C})$ sending $(x_1,...,x_r)$ to (a,...,a), so $|\varphi(\Gamma_{\underline{w}})| \leq 2$.

Remark 5.17. Note that if we take only $O(\log(\ell))$ random relations, then the error term in "a.a.s." obtained in this proof is only $> 1 - 1/\ell^{O(1)}$. To get an exponential error term, the argument requires at least a linear in ℓ number of independent relators (which is the case if we assume $\delta > 0$ of course).

5.4. **Principal part of character varieties.** We will be mostly interested in understanding the structure of the so-called *principal part* of the character variety (defined below), which is a Zariski-open set accounting for 'most' of the character variety.

We first prove the following fact:

Fact 5.18. Assume that \underline{G} is semisimple. The variety

$$(\underline{G}^r)^Z := \left\{ (g_1, ..., g_r) \in \underline{G}^r : \overline{\langle g_1, ..., g_r \rangle}^Z = \underline{G} \right\},$$

is an open subvariety of \underline{G}^r .

The assumption that \underline{G} is semisimple is essential. Note that it fails for a torus, since tuples made of finite order elements form a dense subset in this case.

Fact 5.18 for \underline{G} simple, follows from the following lemma, by observing that the condition that $\langle g_1, ..., g_r \rangle$ act irreducibly on \mathbb{C}^d is a Zariski open condition on $\underline{G}(\mathbb{C})^r$. (exercise: prove this).

Lemma 5.19. Given a simple algebraic group \underline{G} , there exist finite dimensional $\rho_1, \rho_2 \in \operatorname{Irr}(\underline{G}(\mathbb{C}))$ such that for every $g_1, ..., g_r \in \underline{G}(\mathbb{C})$:

$$\overline{\langle g_1,...,g_r\rangle}^Z = \underline{G}(\mathbb{C}) \Longleftrightarrow \rho_1, \rho_2|_{\langle g_1,...,g_r\rangle} \text{ are both irreducible.}$$

Proof. We take $\rho_1 = \operatorname{Ad} : \underline{G}(\mathbb{C}) \to \operatorname{GL}(\mathfrak{g})$, for $\mathfrak{g} = \operatorname{Lie}(\underline{G}(\mathbb{C}))$ and take ρ_2 to be any irreducible representation of large enough dimension.

The direction \Longrightarrow is easy, so it is left to prove \longleftarrow . Suppose that $\overline{\langle g_1,...,g_r\rangle}^Z = \underline{H}(\mathbb{C}) \lneq \underline{G}(\mathbb{C})$ for a proper algebraic subgroup \underline{H} . Then $\mathfrak{h} = \mathrm{Lie}(\underline{H}(\mathbb{C}))$ is a subrepresentation of \mathfrak{g} , which is contradiction to the irreducibility of ρ_1 , unless $\underline{H}(\mathbb{C})$ is finite. To deal with the case that $\underline{H}(\mathbb{C})$ is finite, we apply Jordan's theorem:

Theorem 5.20 (Jordan, 1878). There exists a function $f : \mathbb{N} \to \mathbb{N}$, such that if H is a finite subgroup of $GL_d(\mathbb{C})$, then it must contain a normal, abelian subgroup $H_0 \lhd H$ of index $|H : H_0| \leq f(d)$. In fact, Collins [Col07] showed that f(d) can be taken to be (d+1)! whenever $d \geq 71$.

See [Bre23] for a discussion of Jordan's theorem. Note that $\underline{G}(\mathbb{C}) \leq \operatorname{GL}_N(\mathbb{C})$ for some $N = N(\underline{G})$, and write $\rho_2 : \underline{G}(\mathbb{C}) \to \operatorname{GL}(V_{\rho_2})$. Applying Jordan's theorem to $H := \underline{H}(\mathbb{C})$, we can find $H_0 \lhd H$ abelian of index $|H : H_0| \leq f(N)$. Since $\rho_2(H)$ is finite it consists of semisimple elements, so that the elements in $\rho_2(H_0)$ are simultaneously diagonalizable. Hence $V_{\rho_2}|_{\rho_2(H_0)}$ is a direct sum of one dimensional representations. Since $|H/H_0| \leq f(N)$, taking dim $\rho_2 > f(N)$ guarantees that ρ_2 cannot be irreducible.

Exercise 5.21. Adapt the proof of Fact 5.18 to handle the case that \underline{G} is semisimple.

By Fact 5.18, we deduce that

$$\begin{split} X^Z_{\underline{w},\underline{G}} &:= \operatorname{Hom}(\Gamma_{\underline{w}},\underline{G}) \cap (\underline{G}^r)^Z \text{ is open in } X_{\underline{w},\underline{G}}, \\ \mathcal{X}^Z_{\underline{w},\underline{G}} &:= \operatorname{Hom}^Z(\Gamma_{\underline{w}},\underline{G}) /\!\!/ \underline{G} \text{ is open in } \mathcal{X}_{\underline{w},\underline{G}}. \end{split}$$

Moreover, by Fact 5.7, distinct \underline{G} -orbits in $X_{\underline{w},\underline{G}}^Z$ correspond to distinct points in $\mathcal{X}_{\underline{w},\underline{G}}^Z$.

Definition 5.22. We call $\mathcal{X}_{\underline{w},\underline{G}}^Z$ the **principal part** of $\mathcal{X}_{\underline{w},\underline{G}}$.

In the case of the free group on two generators and $\underline{G} = SL_2$, the principal part of the character variety is entirely described as follows:

Example 5.23 (exercise!). Let $\underline{G} = \operatorname{SL}_2$, r = 2. Recall that $\mathcal{X}_{1,\underline{G}}(\mathbb{C}) = \mathbb{C}^3$ with the parametrization $(a,b) \longmapsto (x,y,z) := (\operatorname{tr}(a),\operatorname{tr}(b),\operatorname{tr}(ab))$. The principal part $\mathcal{X}_{1,\underline{G}}^Z(\mathbb{C})$ of $\mathcal{X}_{1,\underline{G}}(\mathbb{C})$ is $\mathcal{X}_{1,\underline{G}}^Z(\mathbb{C}) = \mathbb{C}^3 \setminus \{F \cup \{\Delta = 0\}\}$, where

$$\triangle = x^2 + y^2 + z^2 - 4xyz - 4 = \operatorname{tr}(aba^{-1}b^{-1}) - 2,$$

and where F is a certain finite set $F \subseteq \left\{(x, y, z); x, y, z \in \{0, \pm 1, \pm \sqrt{2}, \frac{1 \pm \sqrt{5}}{2}\}\right\}$. Moreover,

$$\triangle = 0 \iff \langle a, b \rangle$$
 is not irreducible in \mathbb{C}^2 .

 $(x, y, z) \in F \iff \langle a, b \rangle$ is finite and irreducible on \mathbb{C}^2 .

Example 5.24. When w is arbitrary, the trace tr(w(A, B)) can easily be computed in these coordinates. Using the Cayley–Hamilton theorem to express A^2 as a linear combination of 1 and A and similarly for B^2 and expanding one obtains:

$$tr(w(A,B)) = P_w(x,y,z)$$

where $P_w \in \mathbb{Z}[x, y, z]$ is called the *word polynomial* associated to w in SL_2 . Little is known in general about these polynomials, see [Hor72]. But they provide explicit equations (exercise!) for the principal part of the character variety $\mathcal{X}_{w,SL_2}(\mathbb{C})$, namely:

(5.2)
$$\mathcal{X}_{w,\underline{\mathrm{SL}}_2}^Z(\mathbb{C}) = \left\{ (x,y,z) \in \mathbb{C}^3 \setminus F, \Delta \neq 0, P_w = 2, P_{aw} = x, P_{bw} = y \right\}.$$

6. The main theorem and uniform gap results in finite simple groups

Kesten [Kes59] gave a characterization of non-amenability of a group in terms of the probability of return to the identity of the simple random walk on one (or any) of its Cayley graphs: the group is non-amenable if and only if the probability of return decays exponential fast. In [Bre11], the first author proved a uniform version of Kesten's theorem for linear groups, which tells us the following:

Theorem 6.1 ([Bre11, Corollary 1.6]). Let \underline{G} be a semisimple algebraic group. There exists $c = c(r,\underline{G}) > 0$ such that for every $\ell \in \mathbb{N}$ and every $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ and $\ell \in \mathbb{N}$ are $\ell \in \mathbb{N}$ a

$$\Pr_{w:|w|=\ell} \left[w(\underline{x}) = 1 \right] < e^{-c\ell}.$$

We stress that the exponential rate c > 0 is independent of the generating set \underline{x} . The proof of uniformity relies of global estimates for Diophantine heights over $\overline{\mathbb{Q}}$. This is to be contrasted with the following other basic result of Kesten regarding the free group:

Theorem 6.2 (Kesten, [Kes59]). If $\langle x_1, ..., x_r \rangle = F_r$, then

$$\Pr_{w:|w|=\ell}[w(\underline{x})=1] \approx \left(\frac{\sqrt{2r-1}}{r}\right)^{\ell} = e^{-c_r\ell},$$

which is equal to the probability that a random (in the non-reduced model) word of length ℓ is trivial.

Theorem 6.1 and the proof of Theorem 5.12 imply the following:

Corollary 6.3. Let \underline{G} be a semisimple algebraic group, let $r, k \in \mathbb{N}$. There exists $c' = c'(r, k, \underline{G}) > 0$ such that if $\underline{w} = (w_1, ..., w_k)$, for random words $w_i \in F_r$ of length ℓ , and if $k > (r-1) \dim \underline{G}$, then $\mathcal{X}_{\underline{w},\underline{G}}^Z = \varnothing$ with probability $\geq 1 - e^{c'\ell}$.

In other words, it is enough to ask for $(r-1)\dim \underline{G} + 1$ random generators to be able to conclude that the random group $\Gamma_{\underline{w}}$ has no group homomorphism with a Zariski-dense image in \underline{G} . And this is achieved with an exponentially small probability of exception. In a similar vein:

Corollary 6.4. If $d \in \mathbb{N}$ and $k > (r-1)d^2$, then a.a.s., every $\varphi \in \text{Hom}(\Gamma_{\underline{w}}, \text{GL}_d(\mathbb{C}))$ has virtually solvable image.

³More generally, it is enough to demand that $\langle x_1,...,x_r \rangle$ is not virtually solvable.

Sketch of proof of Corollary 6.3. The proof is similar to the proof of Theorem 5.12, where one just replaces the upper bound (5.1) with the upper bound of Theorem 6.1, using the extra assumption that we only consider generating tuples $g \in (\underline{G}(\mathbb{C})^r)^Z$, respectively (for Corollary 6.4) tuples that generate a non-virtually solvable subgroup.

6.1. Statement of the main result. Let K be a field of characteristic 0. We note that if \underline{G} is defined over K (e.g. if $K = \mathbb{Q}$), then so are the character variety $\mathcal{X}_{\underline{w},\underline{G}}$ and its principal part $\mathcal{X}_{w,G}^Z$. Consequently the Galois group $Gal(\overline{K}|K)$ permutes its geometric irreducible components.

We are now ready to state the main theorem of [BBV]:

Theorem 6.5 (Becker–Breuillard–Varjú '24+, [BBV]). Let <u>G</u> be a semisimple algebraic K-group, and let $k, r \in \mathbb{N}$. Let $w_1, ..., w_k \in F_r$ be random, independent, non-reduced words of length ℓ , with $\underline{w} = (w_1, ..., w_k)$. Denote by $\delta := r - k$ the **defect** of $\Gamma_{\underline{w}}$. Then there exists $c = c(r, k, \underline{G}) > 0$ such that with probability $> 1 - e^{-c\ell}$:

- (i) If $\delta \geq 1$, then $\mathcal{X}_{\underline{w},\underline{G}}^Z \neq \varnothing$. (ii) If $\delta \leq 0$, then $\mathcal{X}_{\underline{w},\underline{G}}^Z = \varnothing$. (iii) If $\delta \geq 1$, then $\dim \mathcal{X}_{\underline{w},\underline{G}}^Z = (\delta 1) \dim \underline{G}$.
- (iv) If $\delta = 1$ and \underline{G} is simply connected, then $\mathcal{X}_{\underline{w},\underline{G}}^Z$ is finite and a single Galois orbit. (v) If $\delta \geq 2$ and \underline{G} is simply connected, then $\mathcal{X}_{\underline{w},\underline{G}}^Z$ is geometrically irreducible.

Remark 6.6.

- (1) All items but (i) are under the assumption of the generalized Riemann hypothesis (GRH). This assumption is used when applying an effective version of Chebotarev's theorem 7.7 with error term. At the expense of weakening the bound on the probability of exceptional words, this assumption can be relaxed by assuming only that Dedekind zeta functions of number fields have no zeroes in a certain small disc around s=1.
- (2) Items (iv) and (v) imply that the Galois action on $\mathcal{X}^Z_{\underline{w},\underline{G}}$ is transitive, when $\underline{G}(\mathbb{C})$ is simplyconnected.
- (3) Note that item (iii) says that $\mathcal{X}_{\underline{w},\underline{G}}^{Z}$ has the expected dimension (counting the number of degrees of freedom) with probability $> 1 - e^{-c\ell}$. But this is **not** the case for $\mathcal{X}_{\underline{w},\underline{G}}$ as the following example shows.

Example 6.7. Let $\underline{G} = \operatorname{SL}_2$, and let $\underline{B} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ be a Borel subgroup of \underline{G} . Since dim $\underline{B} = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\}$ 2, by Item (iii) of Theorem 6.5, for each r < 3k, with probability $> 1 - e^{-c\ell}$, a random tuple $\underline{w} = (w_1, ..., w_k)$ of words of length ℓ satisfies:

$$\dim \mathcal{X}_{w,G}^Z = 3(r-1-k) < 2r-3 = \dim \underline{B}^r - 3.$$

On the other hand, if $w_1, ..., w_k \in F_r'' := [[F_r, F_r], [F_r, F_r]]$ then $X_{w,G} = \underline{B}^r$, and hence

$$\dim \mathcal{X}_{w,G} \ge \dim X_{w,G} - 3 = \dim \underline{B}^r - 3 > \dim \mathcal{X}_{w,G}^Z$$

However F_r/F_r'' is solvable, hence amenable, and Kesten's theorem tells us that:

$$\Pr_{w \in F_r: |w| = \ell} \left[w \in F_r'' \right] \gtrsim e^{-o(\ell)}.$$

So it is **not true** that dim $\mathcal{X}_{\underline{w},\underline{G}} = \dim \mathcal{X}_{\underline{w},\underline{G}}^Z$, with probability $> 1 - e^{-c'\ell}$.

Remark 6.8. Nonetheless, one can further show that for $r \geq 3$ and k = 1 (i.e. Γ_w is a one-relator group), we have $\dim \mathcal{X}_{\underline{w},\underline{G}} = \dim \mathcal{X}_{\underline{w},\underline{G}}^Z = (r-2)\dim \underline{G}$. In other words, a word map $w_{\underline{G}} : \underline{G}^r \to \underline{G}$ is flat over $1 \in \underline{G}$ with probability $> 1 - e^{-c\ell}$.

Example 6.9. Here are several other examples for Theorem 6.5 in the case that $\underline{G} = \mathrm{SL}_2$, r = 2and k = 1, so $\Gamma_w = \langle a, b \mid w \rangle$:

- (1) $\mathcal{X}_{w,G}^Z = \emptyset$ whenever $w = ba^n b^{-1} a^{-m}$ and gcd(m,n) = 1 ($\Gamma_w = BS(m,n)$ is the **Baum**slag-Solitar group), and also whenever $w = baba^3b^2a^2$ or $w = bab^{-3}ab^2a$, by direct computation (using the formula (5.2) in Example 5.24).
- (2) If $w = a^2ba^{-2}b^{-2}$, then Γ_w is non-linear and residually finite [DS05], and $\mathcal{X}_{w,\underline{G}}^Z = \left\{ \left(\mp \sqrt{2}, -1, \pm \frac{1}{\sqrt{2}} \right) \right\}$ is zero-dimensional.
- (3) If $w = uau^{-1}b^{-1}$ for $u = ab^{-1}a^{-1}b$, then Γ_w is the fundamental group of the figure eight knot. Then

$$\mathcal{X}_{w,\underline{G}}^{Z} = \left\{ \begin{array}{c} z^2 + 2x^2 = 1 + z(1+x^2) \\ y = x \end{array} \right\} \text{ is of dimension } 1.$$

(4) If w = [a, v] for $v = [b, a]b^{-1}ab$, then Γ_w is the fundamental group of the complement of the Whitehead link, and $\mathcal{X}_{w,G}^{Z}$ is a hypersurface of dimension 2 given by

$$\mathcal{X}_{w,G}^{Z} = \left\{ x^2 z + y^2 z + z^3 = xy + 2z + xyz^2 \right\},\,$$

For Items (3) and (4) see [MR03, ABL18].

- (5) If $w = b^5 a b^{-1} a^{-1} b a^3$, one can check (using (5.2) and Mathematica), that $\mathcal{X}_{w,G}^Z$ consists of 10 points with algebraic coordinates permuted transitively by the Galois action.
- 6.2. Representations of surface groups. Let Σ_g be a closed orientable surface of genus g, and let $\Gamma_g = \pi_1(\Sigma_g)$ be its fundamental group. Then

$$\Gamma_g = \langle (a_1, b_1, ..., a_g, b_g) : \prod_{i=1}^g [a_i, b_i] = 1 \rangle = \Gamma_{w_{\text{com}}^{*g}}$$

Let $X_g := \operatorname{Hom}(\Gamma_g, \underline{G})$ and $\mathcal{X}_g = X_g /\!\!/ \underline{G}$ be the character variety.

Theorem 6.10 ([Li93], [Sim94, Section 11], [RBKC96], [LS05b, Corollary 1.11], [Sha09, Theorem 2.8]). If \underline{G} is a simply connected complex semisimple group. Then for $g \geq 2$:

- (1) \mathcal{X}_g and \mathcal{X}_g^Z have dimension $(2g-2)\dim \underline{G}$. (2) \mathcal{X}_g and \mathcal{X}_g^Z are irreducible.

Proof. Every semisimple complex Lie group admits a model defined over the rationals. So without loss of generality, we may assume that G is defined over $\mathbb Q$ and it then makes sense to consider its reductions modulo large primes. By the proof of Item (2) of Theorem 3.8 restricting to the family $\{\underline{G}(\mathbb{F}_p)\}_n$, and by Theorem 3.4(2), we have:

$$\left\|\tau_{w_{\text{com}},\underline{G}(\mathbb{F}_p)}^{*g} - \mu_{\underline{G}(\mathbb{F}_p)}\right\|_{\infty} \leq \left\|\tau_{w_{\text{com}},\underline{G}(\mathbb{F}_p)}^{*2} - \mu_{\underline{G}(\mathbb{F}_p)}\right\|_{\infty} \leq \left\|\tau_{w_{\text{com}},\underline{G}(\mathbb{F}_p)} - \mu_{\underline{G}(\mathbb{F}_p)}\right\|_{2}^{2} \leq \zeta_{\underline{G}(\mathbb{F}_p)}(2) - 1 \underset{p \to \infty}{\to} 0.$$

We therefore get:

$$\tau_{w_{\text{com}},\underline{G}(\mathbb{F}_p)}^{*g}(e) |\underline{G}(\mathbb{F}_p)| = \frac{|X_g(\mathbb{F}_p)|}{|\underline{G}(\mathbb{F}_p)|^{2g-1}} \underset{p \to \infty}{\to} 1.$$

Since $\underline{G}_{\mathbb{F}_p}$ is geometrically irreducible for $p \gg 1$, by the Lang-Weil estimates (Theorem 4.2), $|\underline{G}(\mathbb{F}_p)| = p^{\dim \underline{G}}(1 + O(p^{-1/2}))$. This implies that $|X_g(\mathbb{F}_p)| p^{-(2g-1)\dim \underline{G}} \xrightarrow[p \to \infty]{} 1$, and again by

Theorem 4.2 and by Chebotarev's density theorem (Corollary 7.4 bellow), this means that X_g is geometrically irreducible of dimension $(2g-1) \cdot \dim \underline{G}$.

6.3. Mixing results in finite groups. Let G be a finite group, with generating set $\underline{x} = (x_1, ..., x_r)$. We consider the random walk induced by the measure $\mu := \frac{1}{2r} \sum_{i=1}^r \left(\delta_{x_i} + \delta_{x_i^{-1}} \right)$, and would like to find the L^{∞} -mixing time (recall Definition 2.11), i.e. the minimal t_{∞} such that

$$\max_{g \in G} \left| \Pr_{w:|w|=t_{\infty}} \left[w(\underline{x}) = g \right] - \frac{1}{|G|} \right| = \max_{g \in G} \left| \mu^{*t_{\infty}}(g) - \frac{1}{|G|} \right| < \frac{1}{2|G|}.$$

We denote by $\triangle: L^2(G) \to L^2(G)$ the operator $\triangle(f)(g) := 2rf(g) - \sum_{i=1}^r \left(f(x_ig) + f(x_i^{-1}g)\right)$, also called the **combinatorial Laplacian**. It is a self-adjoint operator, with real eigenvalues $0 = \lambda_0 < \lambda_1 \le \lambda_2 ... \le \lambda_{|G|-1}$, where the constant functions are eigenvectors with eigenvalue $\lambda_0 = 0$. Write $f_0, f_1, ..., f_{|G|-1}$ for an orthonormal basis of eigenvectors, where f_i has eigenvalue λ_i . It is classical that a lower bound on the first eigenvalue λ_1 yields an upper bound on mixing time. Concretely:

Fact 6.11. We have

$$\max_{g \in G} \left| \Pr_{w:|w|=\ell} \left[w(\underline{x}) = g \right] - \frac{1}{|G|} \right| < e^{-\frac{\lambda_1 \ell}{2r}}.$$

Proof. Note that $\mu^{*\ell}(g) = (2r)^{-\ell}(2r - \Delta) \circ \dots \circ (2r - \Delta)(\delta_e)(g)$. Writing $\delta_e = \frac{1}{|G|} + \sum_{i=1}^{|G|-1} \overline{f_i(e)} f_i$, and since $(1+x) \leq e^x$ for all $x \in \mathbb{R}$:

$$\left|\mu^{*\ell}(g) - \frac{1}{|G|}\right|^2 = \left|\sum_{i=1}^{|G|-1} (1 - \frac{\lambda_i}{2r})^{\ell} \overline{f_i(e)} f_i(g)\right|^2 \le \sum_{i=1}^{|G|-1} (1 - \frac{\lambda_i}{2r})^{2\ell} |f_i(e)|^2 \cdot \sum_{i=1}^{|G|-1} |f_i(g)|^2 \le (1 - \frac{\lambda_1}{2r})^{2\ell} \le e^{-\frac{\lambda_1 \ell}{r}}.$$

In particular, the mixing time $t_{\infty}(\mu) \leq \frac{2r}{\lambda_1} \ln(2|G|)$.

The following fact is also classical, see e.g. [BT16, Section 5] or [LP16].

Fact 6.12. Let $\gamma = \operatorname{diam}_{\underline{x}}(G) = \inf \{ \ell : \forall g \in G : \exists w : |w| \leq \ell, \ w(\underline{x}) = g \}$. Then $\lambda_1 \geq \frac{1}{8\gamma^2}$.

Combining Facts 6.11 and 6.12 yields:

$$(6.1) t_{\infty}(\mu) \le 16r\gamma^2 \ln(2|G|).$$

We now recall the following key result, which is an essential step in establishing lower bounds on λ_1 for Cayley graphs following the so-called Bourgain–Gamburd machine [BG08].

Theorem 6.13 (Product Theorem [BGT11], [PS16], [Hru12]). Let \underline{G} be a simple algebraic group over \mathbb{F}_q . Let $A \subseteq \underline{G}(\mathbb{F}_q)$ be a generating set. Then there exists $\epsilon = \epsilon(\underline{G}) > 0$ s.t.:

$$|AAA| \ge \min \left\{ |\underline{G}(\mathbb{F}_q)|, |A|^{1+\epsilon} \right\}.$$

Corollary 6.14. Let \underline{G} be a simply connected, simple algebraic group over \mathbb{F}_p . For every generating set $\underline{x} = (x_1, ..., x_r)$ of $\underline{G}(\mathbb{F}_q)$, writing $\mu := \frac{1}{2r} \sum_{i=1}^r \left(\delta_{x_i} + \delta_{x_i^{-1}} \right)$ we have:

- (1) $\gamma \leq O_G(1) \left(\log |G(\mathbb{F}_q)| \right)^{O_{\underline{G}}(1)}$.
- $(2) \ t_{\infty}(\mu) \leq O_{r,\underline{G}}(1) \cdot (\log |\underline{G}(\mathbb{F}_q)|)^{O_{\underline{G}}(1)}.$

We note that a celebrated conjecture of Babai asserts that the constants $O_{\underline{G}}(1)$ in part (1) (diameter bound) of the above corollary ought to be a numerical (absolute) constant. In fact a stronger conjecture can be made in the case of groups of bounded rank:

Conjecture 6.15 (Conjecture 4.5 in [Bre14] for (1)). In Corollary 6.14 (1)+(2), the constant $O_{\underline{G}}(1)$ in the exponent of log can be taken to be equal to 1.

Conjecture 6.15 has now been proven for a large family of finite fields. Namely, every generating set $\underline{x} = (x_1, ..., x_r)$ of $\underline{G}(\mathbb{F}_p)$, for any prime p outside a certain "small" (and possibly empty) bad set Bad_{ϵ} of primes, will give rise to a Cayley graph with logarithmic diameter and mixing time. Explicitly:

Theorem 6.16 (Breuillard–Gamburd [BG10], Becker–Breuillard [BB]). Given $\epsilon > 0$, there exist $\delta_{\epsilon} > 0$ and a possibly empty exceptional set Bad_{\epsilon} of primes such that:

- (1) $|\operatorname{Bad}_{\epsilon} \cap [1, T)| \leq T^{\epsilon}$ for any T > 1.
- (2) Every Cayley graph of $\underline{G}(\mathbb{F}_p)$, for $p \notin \operatorname{Bad}_{\epsilon}$, has $\lambda_1 > \delta_{\epsilon}$.

In the proof of this theorem, the uniformity is deduced from a uniform spectral gap for infinite groups established in characteristic zero, that refines Theorem 6.1 above, and whose proof uses Diophantine heights over $\overline{\mathbb{Q}}$.

7. EFFECTIVE LANG-WEIL, EFFECTIVE CHEBOTAREV AND PROOF OF THE MAIN THEOREM

The proof of the main theorem of Section 6, Theorem 6.5, will proceed by reduction modulo large primes. To be able to lift information mod p to characteristic zero it is essential to have a good upper bound on the size of the primes of bad reduction. In the next subsection, we provide such bounds for general \mathbb{Z} -schemes; for detailed proofs, see [BB].

7.1. **Good reduction.** Let $X := \{f_1(x) = ... = f_s(x) = 0\}$, with $f_i \in \mathbb{Z}[x_1, ..., x_m]$, and $d = \max_i \deg(f_i)$. Set $\operatorname{height}(f_i) = \log \max |\operatorname{coeff}(f_i)|$ and

$$h_X := \max_i \text{ height}(f_i).$$

Lemma 7.1. Let Z be a geometrically irreducible component of $X_{\mathbb{Q}}$, and let K be the field of definition of Z (i.e. $K := \overline{\mathbb{Q}}^H$ where H is the stabilizer of Z under the action of $Gal(\overline{\mathbb{Q}}/\mathbb{Q})$ on the geometrically irreducible components of $X_{\mathbb{Q}}$). Then:

- $(1) [K:\mathbb{Q}] \le d^m.$
- (2) $\log(\Delta_K) \leq O_m(d^{O_m(1)}h_X)$, where Δ_K is the discriminant of the extension K/\mathbb{Q} .

Part (1) follows from Bezout's theorem 5.15, since the number of components of $X_{\mathbb{Q}}$ is bounded by its degree and part (2) follows from the Mahler bound for discriminants and from standard bounds on the discriminant of a compositum of number fields, see [BB].

We now examine the Galois action. Suppose L/\mathbb{Q} be a finite Galois extension, and let \mathcal{O}_L be the ring of integers of L. Let p be a prime number, and let \mathfrak{p} be a prime ideal of \mathcal{O}_L dividing $p\mathcal{O}_L$. Denote by $L_{\mathfrak{p}} := \mathcal{O}_L/\mathfrak{p}\mathcal{O}_L$ the residue field at \mathfrak{p} . The **decomposition group** of \mathfrak{p} over p is defined as:

$$D_{\mathfrak{p},p} := \{ \sigma \in \operatorname{Gal}(L/\mathbb{Q}) : \sigma(\mathfrak{p}) = \mathfrak{p} \}.$$

Denote by $a \mapsto \overline{a}$ the reduction map mod \mathfrak{p} (and similarly mod p). There is a unique map $\Psi_{\mathfrak{p},p}: D_{\mathfrak{p},p} \longrightarrow \operatorname{Gal}(L_{\mathfrak{p}}/\mathbb{F}_p) \ \sigma \mapsto \overline{\sigma}$ satisfying $\overline{\sigma(a)} = \overline{\sigma}(\overline{a})$, which is a surjective homomorphism. The kernel of $\Psi_{\mathfrak{p},p}$ is the **inertia subgroup**

$$I_{\mathfrak{p},p} := \{ \sigma \in \operatorname{Gal}(L/\mathbb{Q}) : \sigma(\alpha) = \alpha \mod \mathfrak{p} \text{ for all } \alpha \in \mathcal{O}_L \}.$$

Hence, $\Psi_{\mathfrak{p},p}$ induces an isomorphism $\Psi_{\mathfrak{p},p}:D_{\mathfrak{p},p}/I_{\mathfrak{p},p}\to \mathrm{Gal}(L_{\mathfrak{p}}/\mathbb{F}_p)$.

Now suppose that p is unramified in \mathcal{O}_L (or equivalently that $p \nmid \triangle_{L/\mathbb{Q}}$). In this case, $I_{\mathfrak{p},p}$ is trivial, and we have an isomorphism

$$\Psi_{\mathfrak{p},p}:D_{\mathfrak{p},p}\to\operatorname{Gal}(L_{\mathfrak{p}}/\mathbb{F}_p).$$

Let Frob $\in \operatorname{Gal}(L_{\mathfrak{p}}/\mathbb{F}_p)$ be the Frobenius automorphism $x \mapsto x^p$. Then we call the automorphism $\operatorname{Frob}_{\mathfrak{p},p} := \Psi_{\mathfrak{p},p}^{-1}(\operatorname{Frob}) \in \operatorname{Gal}(L/\mathbb{Q})$ the Frobenius automorphism of $D_{\mathfrak{p},p}$. Note that if $\mathfrak{p}' \in \mathcal{O}_L$ is another prime lying over p, there is an element $\sigma \in \operatorname{Gal}(L/\mathbb{Q})$ such that $\mathfrak{p}' = \sigma(\mathfrak{p})$ and then we get that $\operatorname{Frob}_{\mathfrak{p}',p} = \sigma \circ \operatorname{Frob}_{\mathfrak{p},p} \circ \sigma^{-1}$. We therefore define the **Frobenius symbol** Frob_p of p in L/\mathbb{Q} , to be the conjugacy class $\{\operatorname{Frob}_{\mathfrak{p},p}\}_{\mathfrak{p} \text{ lying over } p}$.

Lemma 7.2 ([BB]). In the setting of Lemma 7.1, there exists $\triangle \in \mathbb{N}$ with $\log \triangle = O_m(d^{O_m(1)}h_X)$ such that if $p \nmid \triangle$ is a prime, then the reduction mod p yields a 1-to-1 correspondence between

$$\left\{irreducible\ components\ of\ X_{\overline{\mathbb{Q}}}\right\} \Longleftrightarrow \left\{irreducible\ components\ of\ X_{\overline{\mathbb{F}_p}}\right\},$$

which preserves the dimension and the action of the Frobenius automorphism of $Gal(L/\mathbb{Q})$ and $\operatorname{Gal}(L_{\mathfrak{p}}/\mathbb{F}_p)$ respectively, where L is the compositum of the fields of definition of the irreducible components of $X_{\overline{\mathbb{Q}}}$.

The proof involves effective elimination theory and bounds on the degree and height of Groebner bases for ideals over \mathbb{Z} . We refer to [BB] for details. Below we will apply Lemma 7.2 to $X_{w,\underline{G}}$, with $d = O_G(\ell), h_X = O_G(\ell), \text{ and } \Delta = O_G(\ell^{O_{\underline{G}}(1)}).$

7.2. Chebotarev's density theorem. Chebotarev's density theorem states that as one vary over primes in large enough intervals $\left[\frac{1}{2}T,T\right]$, the map $p \longmapsto \operatorname{Frob}_p$ produces a uniformly random element of $Gal(L/\mathbb{Q})$. More precisely:

Theorem 7.3 (Chebotarev's Density Theorem 1926, [Tsc26]). Let L/\mathbb{Q} be a Galois extension, with $G = \operatorname{Gal}(L/\mathbb{Q})$. Let $f: G \to \mathbb{C}$ be a class function on G. Then if p is a random prime in the interval $[\frac{1}{2}T,T]$, we have:

$$\mathbb{E}_p(f(\operatorname{Frob}_p)) \xrightarrow[T \to \infty]{} \mathbb{E}_{g \in G}(f(g)).$$

Here the notation \mathbb{E}_p stands for the average over all primes p in the interval $\left[\frac{1}{2}T,T\right]$.

We will apply Theorem 7.3 in the setting where f(g) := fix(g) is the number of fixed points of the action of $Gal(L/\mathbb{Q})$ on the finite set Ω_X of top-dimensional geometrically irreducible components of $X_{\overline{\mathbb{Q}}}$. Recall that for $p > \triangle_{L/\mathbb{Q}}$, we have $\Omega_X = \Omega_{X,p}$ by Lemma 7.2 and thus $f(\operatorname{Frob}_p) = C_{X_{\mathbb{F}_p}}$ is the number of geometrically irreducible components of $X_{\mathbb{F}_p}$ that are defined over \mathbb{F}_p , so that

$$\mathbb{E}_p(C_{X_{\mathbb{F}_p}}) \xrightarrow[T \to \infty]{} \mathbb{E}_{g \in G}(\mathrm{fix}(g)).$$

Combining this with the Lang-Weil estimates (Theorem 4.2), we therefore deduce:

Corollary 7.4. Let X be a \mathbb{Q} -variety. Then:

- (1) dim $X \leq N \iff |X(\mathbb{F}_p)| \leq O(p^N)$ as $p \to \infty$.
- (2) $\limsup_{p\to\infty} \frac{|X(\mathbb{F}_p)|}{p^{\dim X}} = C_{X_{\overline{\mathbb{Q}}}} = |\Omega_X| = \#top\text{-}dim.$ geometrically irreducible components of X. (3) $\mathbb{E}_p\left(\frac{|X(\mathbb{F}_p)|}{p^{\dim X}}\right) \xrightarrow{T\to\infty} \text{TDIC}(X_{\mathbb{Q}}) = \#top\text{-}dim.$ \mathbb{Q} -irreducible components of X, or equivalently, the number of $\operatorname{Gal}(L/\mathbb{Q})$ -orbits on Ω_X .

We now state an effective version of Chebotarev's density theorem.

Theorem 7.5 (Effective Chebotarev, [LO77, Ser12], [Ser81, Eq. 34 on page 137]). Let L/\mathbb{Q} be a Galois extension. Then, under $(GRH)^4$, for every class function f on $Gal(L/\mathbb{Q})$,

$$|\mathbb{E}_p(f(\operatorname{Frob}_p) - \mathbb{E}_{g \in G}(f(g))| \le 40\mathbb{E}_{g \in G}(|f(g)|) \left(\log \triangle_{L/\mathbb{Q}} + [L:\mathbb{Q}]\right) \frac{\log(T)^2}{T^{1/2}}.$$

Remark 7.6. By Minkowski's bound [Lan70, V.4], $\triangle_{L/\mathbb{Q}} \ge \pi^{[L:\mathbb{Q}]}$ if $[L:\mathbb{Q}] \ge 3$. But $[L:\mathbb{Q}]$ could be very large! For example, in the case of $X_{w,\underline{G}}$, $[L:\mathbb{Q}]$ could be as large as ℓ ! (since $[L:\mathbb{Q}] = |\operatorname{Gal}(L/\mathbb{Q})|$, and $\operatorname{Gal}(L/\mathbb{Q})$ could be as large as the symmetric group on $\ell^{O(1)}$ elements), where on the other hand, T must be taken smaller than $e^{c'\ell}$ (as $\ell \ge c \log p$ is required for mixing time). This is **not good enough!**

By restriction to a special case, namely that f = fix(g) for the action of G on Ω_X , and relying on [GM19], we obtain, an effective version of Chebotarev, which is suitable for our purposes.

Theorem 7.7. Let L/\mathbb{Q} be a Galois field extension, with $G = \operatorname{Gal}(L/\mathbb{Q})$. Let Ω be a transitive G-set, let $\omega_0 \in \Omega$ and set $K := L^{\operatorname{stab}_G(w_0)}$. Then under (GRH),

$$|\mathbb{E}_p(\operatorname{fix}(\operatorname{Frob}_p) - \mathbb{E}_{g \in G}(\operatorname{fix}(g))| \le 40 \left(\log \triangle_{K/\mathbb{Q}} + [K : \mathbb{Q}]\right) \frac{\log(T)^2}{T^{1/2}}.$$

Remark 7.8. In contrast with Remark 7.6, now by Lemma 7.1 $[K:\mathbb{Q}] \leq O(\ell^{O(1)})$ and $\log(\triangle_{K/\mathbb{Q}}) \leq O(\ell^{O(1)}h) \leq O(\ell^{O(1)})$, while T is again around $e^{c'\ell}$.

We refer the reader to [BG] where Theorem 7.7 is discussed and generalized. It implies the following effective version of Corollary 7.4.

Corollary 7.9. Let $X = \{f_1(x) = ... = f_s(x) = 0\}$, with $f_i \in \mathbb{Z}[x_1, ..., x_m]$, $d = \max_i \deg(f_i)$, and $h_X := \max_i \operatorname{height}(f_i)$. Then

$$\left| \mathbb{E}_{p \in [\frac{1}{2}T,T]}(C_{X_{\mathbb{F}_p}}) - \text{TDIC}(X_{\mathbb{Q}}) \right| \le O_m(d^{O_m(1)}h_X) \frac{\log(T)^2}{T^{1/2}}.$$

7.3. Sketch of the proof of the main theorem. We now sketch the proof of Theorem 6.5, focusing only on the dimension formula, Item (iii). We will use the effective Lang-Weil estimates to find the right dimension of the character varieties together with the combination of the above effective Chebotarev theorem (which needs sufficiently large primes to be useful) and the uniform mixing on Cayley graphs of $\underline{G}(\mathbb{F}_p)$ (which requires the prime to be not too large compare to the length of the words). Fortunately the two requirements leave enough room to find plenty of primes to complete the argument. We now pass to the details.

Let $\delta = r - k \ge 1$. We would like to show that for every simply connected, simple algebraic group \underline{G} , we have $\dim X^Z_{\underline{w},\underline{G}} = \delta \dim \underline{G}$ for all \underline{w} but an exponentially small proportion of words of length ℓ . The main idea is to estimate

$$\left|X_{w,G}^{Z}(\mathbb{F}_p)\right| = \left|\left\{(x_1, ..., x_r) \in (\underline{G}^r)^{Z}(\mathbb{F}_p) : w_1(\underline{x}) = ... = w_k(\underline{x}) = 1\right\}\right|,$$

using a double counting argument, in which we average over all \underline{w} of length ℓ , and average over $(\epsilon\text{-good})$ primes p in $[\frac{1}{2}T,T]$, where $T\sim e^{c\ell}$ for small c>0.

Step 1 - reduction to a mixing statement:

⁴i.e. under the assumption that the non-trivial zeros of the Dedekind zeta function ζ_L are on the critical line.

Note that

$$(7.1) \qquad \mathbb{E}_{\underline{w} \in F_r^k : |w_i| = \ell} \left| X_{\underline{w}, \underline{G}}^Z(\mathbb{F}_p) \right| = \mathbb{E}_{\underline{w}} \left(\sum_{\underline{x} \in (\underline{G}^r)^Z(\mathbb{F}_p)} 1_{\underline{w}(\underline{x}) = 1} \right)$$

$$= \sum_{\underline{x} \in (\underline{G}^r)^Z(\mathbb{F}_p)} \Pr_{\underline{w}} \left[\underline{w}(\underline{x}) = 1 \right] = \sum_{\underline{x} \in (\underline{G}^r)^Z(\mathbb{F}_p)} \left(\Pr_{w \in F_r, |w| = \ell} \left[w(\underline{x}) = 1 \right] \right)^k,$$

where the last equality follows since $w_1, ..., w_k$ are independent (non-reduced) words.

Step 2 - generation mod p:

By Fact 6.11, if \underline{x} generates $\underline{G}(\mathbb{F}_p)$, then the probability distribution of $w(\underline{x})$ converges to the Haar measure on $\underline{G}(\mathbb{F}_p)$ as $\ell \to \infty$. However, a priori \underline{x} only lies in $(\underline{G}^r)^Z(\mathbb{F}_p)$. In his celebrated work, Nori [Nor87] classified arbitrary subgroups of $\mathrm{GL}_n(\mathbb{F}_p)$ for p large. His work was extended by Larsen–Pink to all finite fields [LP11]. These results imply the following:

Theorem 7.10 (Nori, Larsen–Pink, [Nor87, LP11]). Let \underline{G} be a simply connected, semisimple algebraic group over \mathbb{F}_p . Let $H \lneq \underline{G}(\mathbb{F}_p)$ be a proper subgroup. Then H is a subgroup of $\underline{H}(\mathbb{F}_p)$, for a proper algebraic subgroup $\underline{H} \leq \underline{G}$ of bounded complexity (hence degree).

The simply connectedness assumption is necessary only because otherwise $\underline{G}(\mathbb{F}_p)$ may have proper subgroups of bounded index (see also Lemma 8.9 below). Theorem 7.10 implies:

Corollary 7.11. If
$$p \gg_{\underline{G},r} 1$$
, then every $\underline{x} \in (\underline{G}^r)^Z(\mathbb{F}_p)$ generates $\underline{G}(\mathbb{F}_p)$.

Note that since we consider the asymptotics in $\ell \to \infty$, and take p in a window $[\frac{1}{2}T, T]$, where $T \sim e^{c\ell}$, we may safely assume that the condition $p \gg_{\underline{C},r} 1$ in Corollary 7.11 is satisfied.

Step 3 - uniform spectral gap and Lang-Weil estimates:

Thanks to Corollary 7.11, we may apply the uniform gap result (Theorem 6.16) to all $\underline{x} \in (\underline{G}^r)^Z(\mathbb{F}_p)$. Hence, for every $\epsilon > 0$, there exists $\delta_{\epsilon} > 0$, so that for every $p \notin \operatorname{Bad}_{\epsilon}$ and every $\underline{x} \in (\underline{G}^r)^Z(\mathbb{F}_p)$ we get from Fact 6.11,

(7.2)
$$\left| \Pr_{w:|w|=\ell} \left[w(\underline{x}) = 1 \right] - \frac{1}{|G(\mathbb{F}_n)|} \right| \le e^{-\frac{\delta_{\epsilon}\ell}{2r}}.$$

By (7.1) and (7.2), we deduce for $\ell \gg_{\epsilon,r,\underline{G}} \log(p)$ and $p \notin \operatorname{Bad}_{\epsilon}$:

$$\mathbb{E}_{\underline{w}\in F_r^k:|w_i|=\ell} \left| X_{\underline{w},\underline{G}}^Z(\mathbb{F}_p) \right| \leq \sum_{\underline{x}\in(\underline{G}^r)^Z(\mathbb{F}_p)} \left(|\underline{G}(\mathbb{F}_p)|^{-1} + e^{-\frac{\delta_{\ell}\ell}{2r}} \right)^k \\
= \sum_{\underline{x}\in(\underline{G}^r)^Z(\mathbb{F}_p)} |\underline{G}(\mathbb{F}_p)|^{-k} \left(1 + e^{-\frac{\delta_{\ell}\ell}{2r}} |\underline{G}(\mathbb{F}_p)| \right)^k \\
= \frac{\left| (\underline{G}^r)^Z(\mathbb{F}_p) \right|}{|\underline{G}(\mathbb{F}_p)|^k} + O(ke^{-\frac{\delta_{\ell}\ell}{2r}} |\underline{G}^{r+1}(\mathbb{F}_p)|) = p^{\dim \underline{G}(r-k)} (1 + O_{\underline{G},r}(p^{-1/2})) + o(1),$$

where o(1) goes to zero as $\ell \sim \frac{1}{c} \log(T)$ for $\frac{1}{c} \gg_{\epsilon,r,\underline{G}} 1$.

Step 4 - use of effective Chebotarev:

By Corollary 7.9, since $h_{X_{\underline{w},\underline{G}}} < \ell^{O_{\underline{G},r}(1)}$ and since $\operatorname{Bad}_{\epsilon} \cap [\frac{1}{2}T,T] \leq T^{\epsilon} \leq T^{\frac{1}{2}}$ for $\epsilon < \frac{1}{2}$, one has:

$$\Pr_{\underline{w}:|w_i|=\ell} \left[\dim X_{\underline{w},\underline{G}}^Z > \delta \dim \underline{G} \right] \leq \mathbb{E}_{\underline{w}} \left(1_{\dim X_{\underline{w},\underline{G}}^Z > \delta \dim \underline{G}} \cdot \mathrm{TDIC}(X_{\underline{w},\underline{G}}^Z) \right) \\
\leq \mathbb{E}_{\underline{w}} \left(1_{\dim X_{w,G}^Z > \delta \dim \underline{G}} \mathbb{E}_{p \in [\frac{1}{2}T,T] \setminus \mathrm{Bad}_{\epsilon}} C_{\mathbb{F}_p}(X_{\underline{w},\underline{G}}^Z) \right) + O_{\underline{G},r}(\ell^{O_{\underline{G},r}(1)}) \frac{(\log T)^2}{T^{1/2}}.$$

By the effective Lang-Weil estimates (Theorem 4.2 and Remark 4.3), we have:

$$C_{\mathbb{F}_p}(X_{\underline{w},\underline{G}}^Z) = \frac{\left|X_{\underline{w},\underline{G}}^Z(\mathbb{F}_p)\right|}{p^{\dim(X_{\underline{w},\underline{G}}^Z)\mathbb{F}_p}} + O_{\underline{G},r}(\frac{\ell^{O_{\underline{G},r}(1)}}{\sqrt{p}}),$$

and therefore,

$$\begin{split} &\Pr_{\underline{w} \in F_r^k: |w_i| = \ell} \left[\dim X_{\underline{w},\underline{G}}^Z > \delta \dim \underline{G} \right] \\ &\leq \mathbb{E}_{\underline{w}} \left(1_{\dim X_{\underline{w},\underline{G}}^Z > \delta \dim \underline{G}} \mathbb{E}_{p \in [\frac{1}{2}T,T] \backslash \operatorname{Bad}_{\epsilon}} \frac{\left| X_{\underline{w},\underline{G}}^Z(\mathbb{F}_p) \right|}{p^{\dim(X_{\underline{w},\underline{G}}^Z)\mathbb{F}_p}} \right) + O_{\underline{G},r}(\ell^{O_{\underline{G},r}(1)}) + O_{\underline{G},r}(\ell^{O_{\underline{G},r}(1)}) \frac{(\log T)^2}{T^{1/2}}. \end{split}$$

Note that if $\dim X_{\underline{w},\underline{G}}^Z > \delta \dim \underline{G}$ then $\dim (X_{\underline{w},\underline{G}}^Z)_{\mathbb{F}_p} \geq \dim (X_{\underline{w},\underline{G}}^Z) \geq \delta \dim \underline{G} + 1$. Combining with (7.3), and swapping the expectations $\mathbb{E}_{\underline{w}}$ and $\mathbb{E}_{p \in [\frac{1}{2}T,T] \setminus \mathrm{Bad}_{\epsilon}}$, we have:

$$\begin{split} &\Pr_{\underline{w} \in F_r^k: |w_i| = \ell} \left[\dim X_{\underline{w},\underline{G}}^Z > \delta \dim \underline{G} \right] \\ &\leq \mathbb{E}_{p \in [\frac{1}{2}T,T] \backslash \text{Bad}_{\epsilon}} \left(p^{-\delta \dim \underline{G} - 1} \mathbb{E}_{\underline{w}} \left(1_{\dim X_{\underline{w},\underline{G}}^Z > \delta \dim \underline{G}} \left| X_{\underline{w},\underline{G}}^Z (\mathbb{F}_p) \right| \right) \right) + O_{\underline{G},r}(\ell^{O_{\underline{G},r}(1)}) \frac{(\log T)^2}{T^{1/2}} \\ &\leq \mathbb{E}_{p \in [\frac{1}{2}T,T] \backslash \text{Bad}_{\epsilon}} \left(p^{-\delta \dim \underline{G} - 1} \mathbb{E}_{\underline{w}} \left| X_{\underline{w},\underline{G}}^Z (\mathbb{F}_p) \right| \right) + O_{\underline{G},r}(\ell^{O_{\underline{G},r}(1)}) \frac{(\log T)^2}{T^{1/2}} \\ &\leq \mathbb{E}_{p \in [\frac{1}{2}T,T] \backslash \text{Bad}_{\epsilon}} \left(\frac{1 + o(1)}{p} \right) + O_{\underline{G},r}(\ell^{O_{\underline{G},r}(1)}) \frac{(\log T)^2}{T^{1/2}} \\ &= O_{\underline{G},r}(\ell^{O_{\underline{G},r}(1)}) \left(\frac{(\log T)^2}{T^{1/2}} + \frac{1}{T} \right). \end{split}$$

This finishes the proof of Item (iii) of Theorem 6.5.

8. Generic fibers of convolutions of morphisms

In §4, we discussed a connection between the geometry of polynomial maps, and the probabilistic properties of pushforward measures by such maps (Proposition 4.9). By a variant of Proposition 4.9 (more precisely, by [GH24, Theorem 8.4]), if $\varphi: X \to Y$ is a morphism of smooth, geometrically irreducible \mathbb{Q} -varieties, then

(8.1)
$$\lim_{q=p^r\to\infty, p\gg_{\varphi}1} \|\varphi_*\mu_{X(\mathbb{F}_q)} - \mu_{Y(\mathbb{F}_q)}\|_{\infty} = 0 \iff \varphi \text{ is } (FGI).$$

In the special case that $Y = \underline{G}$ is an algebraic \mathbb{Q} -group, finding a uniform L^{∞} -mixing time t_{∞} for the family of measures $\{\varphi_*\mu_{X(\mathbb{F}_q)}\}_q$ as in (8.1) is equivalent to showing that $\varphi^{*t}: X^t \to \underline{G}$ is (FGI), for every $t \geq t_{\infty}$.

A weaker geometric condition than the (FGI) property is when φ is (FGI) over a Zariski open set $U \subseteq Y$, or equivalently, if the generic fiber of φ is geometrically irreducible. This property has an interesting probabilistic interpretation (see [LST19, Theorem 2] and also [GH24, Theorem 9.4])

$$\lim_{q=p^r\to\infty, p\gg_{\varphi}1}\left\|\varphi_*\mu_{X(\mathbb{F}_q)}-\mu_{Y(\mathbb{F}_q)}\right\|_1=0\iff \varphi \text{ has geometrically irreducible generic fiber}.$$

As mentioned at the end of §4, proving that φ^{*t} has geometrically irreducible generic fiber is a key step in showing that φ^{*t} becomes (FGI) after sufficiently many self convolutions (see Proposition 4.21).

In Theorem 4.18, we saw that the convolution of two word maps on a simply connected simple algebraic group has geometrically irreducible generic fiber, and we saw that simply connectedness is a crucial assumption (Example 4.19). In this section, we give an analytic proof of a generalization

of Theorem 4.18, which was recently proved by Hrushovski in [Hru24, Appendix C] using model theory.

The following definition plays a key role:

Definition 8.1. Let K be a field. A connected algebraic K-group \underline{G} is called *simply connected*, if it does not admit a non-trivial isogeny⁵ $\psi : \underline{H} \to \underline{G}$ from a connected algebraic K-group. We say that \underline{G} is absolutely simply connected if $\underline{G}_{\overline{K}}$ is simply connected.

Theorem 8.2 ([Hru24, Corollary C.8]). Let X and Y be geometrically irreducible \mathbb{Q} -varieties, let \underline{G} be an absolutely simply connected algebraic \mathbb{Q} -group and let $\varphi: X \to \underline{G}$ and $\psi: Y \to \underline{G}$ be dominant morphisms. Then

(8.2)
$$\lim_{q=p^r\to\infty, p\gg_{\varphi,\psi}1} \left\| (\varphi * \psi)_* \mu_{X(\mathbb{F}_q)\times Y(\mathbb{F}_q)} - \mu_{\underline{G}(\mathbb{F}_q)} \right\|_1 = 0.$$

In particular, the generic fiber of $\varphi * \psi : X \times Y \to \underline{G}$ is geometrically irreducible.

Note that Theorem 8.2 is an asymptotic statement about families of measures. In many cases, an asymptotic statement can be phrased as a non-asymptotic statement on a limit object. In this case, the limit object is the group $\underline{G}(\mathbb{F})$ where \mathbb{F} is a non-principal ultraproduct of finite fields. The group $\underline{G}(\mathbb{F})$ carries a natural probability measure $\mu_{\underline{G}(\mathbb{F})}$, which generalizes the family of counting measures $\{\mu_{\underline{G}(\mathbb{F}_q)}\}_q$. Similarly, one can define $\mu_{X(\mathbb{F})}$ and $\mu_{Y(\mathbb{F})}$. In this language, (8.2) is now equivalent to the non-asymptotic statement that the pushforward measure $(\varphi * \psi)_*\mu_{X(\mathbb{F})\times Y(\mathbb{F})}$ is equal to $\mu_{\underline{G}(\mathbb{F})}$. Hrushovski uses a very general result on convolutions of definable measures in a suitable model theoretic framework, and shows that such measures must be invariant with respect to a definable subgroup of $\underline{G}(\mathbb{F})$ of bounded index. To finish the proof he shows that if \underline{G} is absolutely simply connected, that $\underline{G}(\mathbb{F})$ has no definable subgroups of bounded index [Hru24, Corollary C.5], or equivalently, that for every $M \in \mathbb{N}$ there exists $p_0(M)$ such that for every $p > p_0(M)$ and $q = p^r$, the group $\underline{G}(\mathbb{F}_q)$ has no proper subgroups of index smaller than M.

We now give a purely analytic proof to Theorem 8.2. Some aspects of the proof share similar components to Hrushovski's proof, but the proof itself is different. We thank Udi for sharing his ideas and insights on this problem.

8.1. Analytic proof of Theorem 8.2. Let G be a compact group with Haar probability measure μ_G . Write Irr(G) for the set of irreducible characters of G. For each $\rho \in Irr(G)$ we write $\pi_{\rho} : G \to U(V_{\rho})$ for the irreducible representation with character ρ . Given a measure μ on G, we denote by

$$A_{\mu,\rho} := \widehat{\mu}(\pi_{\rho}) = \int_{G} \pi_{\rho}(g^{-1})\mu(g),$$

the (non-commutative) Fourier coefficient of μ at ρ . Note that by the Plancherel theorem for compact groups (see e.g. [App14, Theorem 2.3.1(2)]), if $\mu \in L^2(\mu_G)$ (i.e. $\mu = f\mu_G$, with $f \in L^2(G, \mu_G)$), then we have

(8.3)
$$\|\mu\|_{2}^{2} = \sum_{\rho \in Irr(G)} \rho(1) \|A_{\mu,\rho}\|_{HS}^{2},$$

where $||C||_{\text{HS}} : \sqrt{\text{tr}(CC^*)}$ denotes the Hilbert-Schmidt norm. Further denote by $||C||_{\text{op}}$ the operator norm on matrices. Similarly to Definition 2.8, if $\mu = f\mu_G$ for a measurable function f, for each $q \ge 1$, we define $||\mu||_q := \left(\int_G |f(g)|^q \mu_G(g)\right)^{\frac{1}{q}}$.

⁵An isogeny is a surjective homomorphism with finite kernel.

Definition 8.3. Let $\mathcal{G} = \{G_n\}_{n \in \mathbb{N}}$ be a family of compact groups. A family $\{\mu_n\}_{n \in \mathbb{N}}$ of probability measures on \mathcal{G} :

(1) Has uniform spectral decay, if

$$\lim_{n\to\infty} \sup_{1\neq\rho\in\operatorname{Irr}(G_n)} \|A_{\mu_n,\rho}\|_{\operatorname{op}} = 0.$$

(2) Is almost uniform in L^s for $1 \le s \le \infty$ if

$$\lim_{n\to\infty} \|\mu_n - \mu_{G_n}\|_s = 0.$$

Remark 8.4. By Remark 2.9(1), if $\{\mu_n\}_{n\in\mathbb{N}}$ is almost uniform in L^s , then it is almost uniform in $L^{s'}$ for every $1 \leq s' \leq s$.

Lemma 8.5. Let $\mathcal{G} = \{G_n\}_{n \in \mathbb{N}}$ be a family of compact groups. Let $\{\mu_n\}_{n \in \mathbb{N}}$ and $\{\gamma_n\}_{n \in \mathbb{N}}$ be families of probability measures on \mathcal{G} , where $\limsup_{n \to \infty} \|\mu_n\|_2 < C$ and $\{\gamma_n\}_{n \in \mathbb{N}}$ has uniform spectral decay. Then the family $\{\mu_n * \gamma_n\}_n$ is almost uniform in L^2 in \mathcal{G} .

Proof. For $n \gg 1$, we have:

(8.4)
$$\sum_{\rho \in Irr(G_n)} \rho(1) \|A_{\mu_n,\rho}\|_{HS}^2 = \|\mu_n\|_2^2 < C^2.$$

Since $A_{\mu_n * \gamma_n, \rho} = A_{\mu_n, \rho} \circ A_{\gamma_n, \rho}$, and since $||BD||_{HS} \le ||B||_{HS} ||D||_{op}$ for every $B, D \in \operatorname{Mat}_m(\mathbb{C})$, we have:

(8.5)
$$||A_{\mu_n * \gamma_n, \rho}||_{HS}^2 = ||A_{\mu_n, \rho} \circ A_{\gamma_n, \rho}||_{HS}^2 \le ||A_{\mu_n, \rho}||_{HS}^2 ||A_{\gamma_n, \rho}||_{op}^2.$$

By (8.4), (8.5) and since $\{\gamma_n\}_{n\in\mathbb{N}}$ has uniform spectral decay, we have:

$$\|\mu_n * \gamma_n - \mu_{G_n}\|_2 = \sum_{1 \neq \rho \in \operatorname{Irr}(G_n)} \rho(1) \|A_{\mu_n * \gamma_n, \rho}\|_{\operatorname{HS}}^2 \leq \sum_{1 \neq \rho \in \operatorname{Irr}(G_n)} \rho(1) \|A_{\mu_n, \rho}\|_{\operatorname{HS}}^2 \|A_{\gamma_n, \rho}\|_{\operatorname{op}}^2 \to 0,$$

as
$$n \to \infty$$
, as required.

We now turn to the proof of Theorem 8.2. We start with a reduction to the case of flat morphisms from smooth varieties.

Proposition 8.6. It is enough to prove Theorem 8.2 in the case that X, Y are smooth \mathbb{Q} -varieties and φ and ψ are flat \mathbb{Q} -morphisms.

Proof. To prove (8.2), we can throw away any constructible set $Z(\mathbb{F}_q) \subseteq X(\mathbb{F}_q)$ with $\operatorname{codim} Z \geq 1$, since by the Lang-Weil estimates (Theorem 4.2), the total contribution to (8.2) will go to 0 as $q \to \infty$. We may therefore assume X, Y are smooth \mathbb{Q} -varieties. By generic flatness, there exist Zariski open subsets $U \subseteq X$ and $V \subseteq Y$, such that $\varphi|_U : U \to \underline{G}$ and $\psi|_V : V \to \underline{G}$ are flat. Since X and Y are geometrically irreducible, $U \times V$ is Zariski dense in $X \times Y$, with complement of smaller codimension, so we may restrict to $U(\mathbb{F}_q) \times V(\mathbb{F}_q)$ for $q = p^r$ and $p \gg 1$. We may therefore assume that φ and ψ are flat morphisms, as required.

Notation. For each $M \in \mathbb{N}$, denote by \mathcal{P}_M the set of all prime powers $q = p^r$ for p > M.

With the help of Proposition 8.6, Theorem 8.2 follows from the following key proposition.

Proposition 8.7. Let X be a smooth, geometrically irreducible \mathbb{Q} -variety and $\varphi: X \to \underline{G}$ be a flat \mathbb{Q} -morphism to a connected, absolutely simply connected algebraic \mathbb{Q} -group. Then there exists $M \in \mathbb{N}$ such that the family $\{\varphi_*\mu_{X(\mathbb{F}_q)}\}_{q \in \mathcal{P}_M}$ of probability measures on $\{\underline{G}(\mathbb{F}_q)\}_{q \in \mathcal{P}_M}$ has uniform spectral decay.

Proof that Proposition 8.7 implies Theorem 8.2. By Proposition 8.6, we may assume that φ and ψ are flat morphisms. By Proposition 8.7, the family $\{\varphi_*\mu_{X(\mathbb{F}_q)}\}_{q\in\mathcal{P}_M}$ has uniform spectral decay. Since ψ is flat, by the Lang-Weil estimates (Proposition 4.9, Theorem 4.2, and more precisely [GH24, Theorem 8.4]), the density of $\psi_*\mu_{Y(\mathbb{F}_q)}$ with respect to $\mu_{\underline{C}(\mathbb{F}_q)}$ is bounded by a constant C>0 independent of q, for $q\in\mathcal{P}_M$, with M possibly larger. In particular, we have:

(8.6)
$$\sum_{\rho \in Irr(\underline{G}(\mathbb{F}_q))} \rho(1) \left\| A_{\psi_* \mu_{Y(\mathbb{F}_q)}, \rho} \right\|_{HS}^2 = \left\| \psi_* \mu_{Y(\mathbb{F}_q)} \right\|_2^2 \le \left\| \psi_* \mu_{Y(\mathbb{F}_q)} \right\|_{\infty}^2 \le C^2.$$

Theorem 8.2 now follows from (8.6) and Lemma 8.5.

Hence, it is left to prove Proposition 8.7. Let us first characterize simply connected algebraic groups.

Lemma 8.8. Let \underline{G} be a connected, absolutely simply connected algebraic \mathbb{Q} -group. Then $\underline{G} \simeq \underline{U} \rtimes \underline{L}$, where \underline{U} is the unipotent radical of \underline{G} and \underline{L} is an absolutely simply connected, semisimple algebraic \mathbb{Q} -group.

Proof. By Levi's theorem in characteristic 0 [Mos56], one has an isomorphism $\underline{G} \simeq \underline{U} \rtimes \underline{L}$, where \underline{L} is a connected reductive subgroup of \underline{G} . Since \underline{G} is absolutely simply connected, so is \underline{L} . Let $\underline{L}^{\mathrm{rad}}$ be the radical (also the center) of \underline{L} . Then the multiplication map $\underline{L}^{\mathrm{rad}} \times [\underline{L}, \underline{L}] \to \underline{L}$ is a central isogeny, and thus an isomorphism, by our assumption. Finally, note that $(\underline{L}^{\mathrm{rad}})_{\overline{\mathbb{Q}}}$ is a torus, which is never simply connected unless trivial, hence \underline{L} is perfect, and thus semisimple.

Lemma 8.9. Let \underline{G} be a connected, absolutely simply connected algebraic \mathbb{Q} -group. Then there exist $M \in \mathbb{N}$ and an absolute constant 0 < c < 1, such that for every $q \in \mathcal{P}_M$, any proper subgroup H of $\underline{G}(\mathbb{F}_q)$ is of index $\geq cq$.

Proof. Suppose that $H < \underline{G}(\mathbb{F}_q) \simeq \underline{U}(\mathbb{F}_q) \rtimes \underline{L}(\mathbb{F}_q)$ is a subgroup of index < q. The unipotent group \underline{U} is split over \mathbb{F}_q ([Bor91, Corollary 15.5(ii)]), and hence $H \cap \underline{U}(\mathbb{F}_q) = \underline{U}(\mathbb{F}_q)$. Moreover, $\underline{L}(\mathbb{F}_q)$ is a finite quasi-simple group of Lie type, whenever $q \in \mathcal{P}_M$ for $M \in \mathbb{N}$ (see e.g. [MT11, Theorem 24.17]). Hence, by [LS74] any $1 \neq \rho \in \operatorname{Irr}(\underline{L}(\mathbb{F}_q))$ satisfies $\rho(1) \geq cq$ for some absolute constant 0 < c < 1 (i.e. $\underline{L}(\mathbb{F}_q)$ is cq-quasi random). In particular, any proper subgroup $H' < \underline{L}(\mathbb{F}_q)$ is of index at least cq (as otherwise the space $\mathbb{C}[\underline{L}(\mathbb{F}_q)/H']$ is a representation of dimension < cq). We get that $H = \underline{G}(\mathbb{F}_q)$, unless it is of index $\geq cq$

Corollary 8.10. Let \underline{G} be a connected, absolutely simply connected algebraic \mathbb{Q} -group. Then there exist c > 0, and $M \in \mathbb{N}$ such that for every $q \in \mathcal{P}_M$, $\min_{\rho \in \operatorname{Irr}(G(\mathbb{F}_q)): \rho(1) > 1} \rho(1) \geq cq$.

Proof. Fix $1 < N \in \mathbb{N}$ and let $\rho \in \operatorname{Irr}(\underline{G}(\mathbb{F}_q))$ be a character of degree N. Let $\underline{L}, \underline{U}$ subgroups such that $\underline{G} \simeq \underline{U} \rtimes \underline{L}$ as in Levi's theorem. By the proof of Lemma 8.9, whenever $q \in \mathcal{P}_M$ for $M \in \mathbb{N}$, the restriction $\rho|_{\underline{L}(\mathbb{F}_q)}$ is either trivial or of dimension at least cq. Hence, it is left to deal with the case that $\rho|_{\underline{L}(\mathbb{F}_q)}$ is trivial. In this case, $\rho|_{\underline{U}(\mathbb{F}_q)}$ is irreducible and non-trivial. Since $\underline{U}(\mathbb{F}_q)$ is nilpotent, the character $\rho|_{\underline{U}(\mathbb{F}_q)}$ is monomial, and thus of degree at least q.

The final component we need for the proof of Proposition 8.7 are exponential character estimates for polynomials on varieties over finite fields.

Lemma 8.11 ([Kow07, Theorem 2]). Let X be a geometrically irreducible \mathbb{Q} -variety. Let $f: X \to \mathbb{A}^1_{\mathbb{O}}$ be a non-constant morphism. Then there exist constants C > 0 and $M \in \mathbb{N}$ such that for every

 $q \in \mathcal{P}_M$, and every non-trivial additive character $\psi : (\mathbb{F}_q, +) \to \mathbb{C}^{\times}$, we have:

$$\left| \frac{1}{|X(\mathbb{F}_q)|} \sum_{x \in X(\mathbb{F}_q)} \psi(f(x)) \right| \le Cq^{-1/2}.$$

Lemma 8.11 implies the following:

Corollary 8.12. Let X be a \mathbb{Q} -variety and $\varphi: X \to \mathbb{A}^m_{\mathbb{Q}}$ be a dominant morphism. Then there exist constants C > 0 and $M \in \mathbb{N}$ such that for every $q \in \mathcal{P}_M$ and every $1 \neq \Psi \in \operatorname{Irr}(\mathbb{F}_q^m)$, one has

$$\left| \frac{1}{|X(\mathbb{F}_q)|} \sum_{x \in X(\mathbb{F}_q)} \Psi(\varphi(x)) \right| \le Cq^{-1/2}.$$

Proof. Write $\varphi = (\varphi_1, ..., \varphi_m)$ for $\varphi_i : X \to \mathbb{A}^1_{\mathbb{Q}}$. Since φ is dominant, there are no complex numbers $c_1, ..., c_m$ such that $\sum_{i=1}^m c_i \varphi_i$ is equal to a constant. Hence, the same statement is true over $\overline{\mathbb{F}_p}$ for $p \gg 1$. Since any $\Psi(y) \in \operatorname{Irr}(\mathbb{F}_q^m)$ is of the form $\psi(a \cdot y)$ for $a \in \mathbb{F}_q^m$ and ψ a fixed non-trivial additive character of \mathbb{F}_q , we get $\Psi(\varphi(x)) = \psi(a \cdot \varphi(x))$, where $a \cdot \varphi(x) = (\sum_{i=1}^m a_i \varphi_i)(x)$ is a non-constant polynomial. The previous Lemma 8.11 implies the corollary.

We can now prove Proposition 8.7.

Proof of Proposition 8.7. Since $\varphi: X \to \underline{G}$ is flat, we can use (8.6) to deduce there exist C > 0 and $M \in \mathbb{N}$ such that for $q \in \mathcal{P}_M$:

$$\rho(1) \left\| A_{\varphi_* \mu_{X(\mathbb{F}_q)}, \rho} \right\|_{HS}^2 \le C^2$$
, for all $\rho \in \operatorname{Irr}(\underline{G}(\mathbb{F}_q))$.

If $\rho(1) > 1$ then by Corollary 8.10 with possibly enlarging M,

(8.7)
$$\left\| A_{\varphi_* \mu_{X(\mathbb{F}_q)}, \rho} \right\|_{\text{op}}^2 \le \left\| A_{\varphi_* \mu_{X(\mathbb{F}_q)}, \rho} \right\|_{\text{HS}}^2 \le \frac{C^2}{\rho(1)} \underset{q \to \infty}{\longrightarrow} 0.$$

It is left to consider the case when ρ is a one-dimensional character. In this case, ρ factors through $\underline{G}(\mathbb{F}_q)/[\underline{G}(\mathbb{F}_q),\underline{G}(\mathbb{F}_q)]$. Since \underline{L} is perfect, we get $[\underline{G},\underline{G}]\simeq\underline{L}\ltimes\underline{U}'$ for $\underline{U}':=\underline{U}\cap[\underline{G},\underline{G}]$ so $\underline{G}/[\underline{G},\underline{G}]$ is unipotent and $[\underline{G},\underline{G}]$ is absolutely simply connected. Note that for sufficiently large $t\in\mathbb{N}$, the map $w^{*t}_{\mathrm{com},\underline{G}}:\underline{G}^{2t}\to[\underline{G},\underline{G}]$ is flat (by Theorem 4.16) and the image of $w^{*t}_{\mathrm{com},\underline{G}(\mathbb{F}_q)}$ is contained in $[\underline{G}(\mathbb{F}_q),\underline{G}(\mathbb{F}_q)]$. By Proposition 4.9, the index $[\underline{G},\underline{G}](\mathbb{F}_q)/[\underline{G}(\mathbb{F}_q),\underline{G}(\mathbb{F}_q)]$ is bounded by a constant C independent of q, for $q\in\mathcal{P}_M$. By Lemma 8.9, $[\underline{G},\underline{G}](\mathbb{F}_q)=[\underline{G}(\mathbb{F}_q),\underline{G}(\mathbb{F}_q)]$. Hence, ρ factors through

$$\underline{G}(\mathbb{F}_q)/[\underline{G},\underline{G}](\mathbb{F}_q) \simeq (\underline{G}/[\underline{G},\underline{G}])(\mathbb{F}_q) \simeq (\underline{U}/\underline{U}')(\mathbb{F}_q),$$

where the first isomorphism follows by Lang's theorem [PR94, Theorem 6.1] and from the exact sequence of cohomologies [Spr98, Proposition 12.3.4]. Note that $\underline{U}/\underline{U}' \simeq \mathbb{A}_{\mathbb{Q}}^m$ and $(\underline{U}/\underline{U}')$ (\mathbb{F}_q) $\simeq \mathbb{F}_q^m$ for some $m \in \mathbb{N}$. Denote by $\Psi_{\rho} \in \operatorname{Irr}(\mathbb{F}_q^m) \simeq \operatorname{Irr}((\underline{U}/\underline{U}')(\mathbb{F}_q))$ the character corresponding to ρ and denote by $\widetilde{\varphi}: X \to \underline{G}/[\underline{G},\underline{G}] \simeq \mathbb{A}_{\mathbb{Q}}^m$ the composition of φ with the quotient map $\underline{G} \to \underline{G}/[\underline{G},\underline{G}]$. By Corollary 8.12, we have

$$\left| A_{\varphi_* \mu_{X(\mathbb{F}_q)}, \rho} \right| = \left| \sum_{g \in \underline{G}(\mathbb{F}_q)} \rho(g^{-1}) \varphi_* \mu_{X(\mathbb{F}_q)}(g) \right| = \left| \frac{1}{|X(\mathbb{F}_q)|} \sum_{x \in X(\mathbb{F}_q)} \overline{\rho}(\varphi(x)) \right|
= \left| \frac{1}{|X(\mathbb{F}_q)|} \sum_{x \in X(\mathbb{F}_q)} \Psi_{\overline{\rho}}(\widetilde{\varphi}(x)) \right| \le Cq^{-1/2},$$
(8.8)

for $q \in \mathcal{P}_M$. By (8.7) and (8.8), the family $\{\varphi_*\mu_{X(\mathbb{F}_q)}\}_{q:p\gg 1}$ has uniform spectral decay, as required.

8.2. Some consequences of Theorem 8.2. In [GH21, Thm B] and [GH19], the second author and Yotam Hendel showed that various singularities properties of morphisms are improved by taking sufficiently many self-convolutions. In particular, it was shown that any dominant morphism from a smooth, geometrically irreducible \mathbb{Q} -variety X to an algebraically \mathbb{Q} -group G becomes flat, with reduced fibers of rational singularities after sufficiently many self-convolutions. Combining Theorem 8.2, Proposition 4.21 and [GH21, Thm B], we strengthen these results in the setting when G is an absolutely simply connected algebraic \mathbb{Q} -group, by showing that convolved morphisms eventually have geometrically irreducible fibers as well.

Theorem 8.13. Let $\varphi: X \to \underline{G}$ be a dominant \mathbb{Q} -morphism from a smooth, geometrically irreducible \mathbb{Q} -variety X to a connected, absolutely simply connected algebraic \mathbb{Q} -group \underline{G} . Then:

- (1) φ^{*t} is flat with geometrically irreducible and reduced fibers (and in particular FGI) for all $t \ge \dim \underline{G} + 1$.
- (2) φ^{*t} is flat with geometrically irreducible and normal fibers for all $t \geq \dim \underline{G} + 2$.
- (3) $\exists N \in \mathbb{N}$ such that for every t > N, the map φ^{*t} is flat with geometrically irreducible fibers with rational singularities.

The condition of rational singularities in Item (3) is important because of the following proposition, which is a generalization of Proposition 4.9:

Proposition 8.14 ([CGH23, Theorems A and 4.7]). Let $\varphi: X \to Y$ be a dominant morphism of smooth, geometrically irreducible \mathbb{Q} -varieties. Then:

- (1) φ is flat with fibers of rational singularities (abbreviated "FRS") if and only if there exists C > 0 such that for every prime $p \gg 1$, the density of $\varphi_*\mu_{X(\mathbb{Z}_p)}$ with respect to $\mu_{Y(\mathbb{Z}_p)}$ is bounded by C, where $\mu_{X(\mathbb{Z}_p)}$ and $\mu_{Y(\mathbb{Z}_p)}$ denote the canonical measures on $X(\mathbb{Z}_p)$ and $Y(\mathbb{Z}_p)^7$.
- (2) φ is (FRS) and (FGI) if and only if for every prime $p \gg 1$,

$$\|\varphi_*\mu_{X(\mathbb{Z}_p)} - \mu_{Y(\mathbb{Z}_p)}\|_{\infty} < Cp^{-1/2},$$

We now turn to the second application which is about Lie algebra word maps.

Definition 8.15. A Lie algebra word $w(X_1, ..., X_r)$ is a \mathbb{Q} -linear combination of iterated commutators, or more precisely, an element in a free \mathbb{Q} -Lie algebra \mathcal{L}_r on a finite set $\{X_1, ..., X_r\}$. The space \mathcal{L}_r has a natural gradation, so one can define the degree of a word w as the maximal grade $d \in \mathbb{N}$ in which the image of w is non-trivial. For example, w = [X, Y] + 2[[X, Y], Y] is a Lie algebra word of degree 3. For each Lie algebra $\mathfrak{g}, w \in \mathcal{L}_r$ induces a Lie algebra word map $w_{\mathfrak{g}} : \mathfrak{g}^r \to \mathfrak{g}$.

In [GH24, Theorem A], the second author and Hendel have shown that non-zero word maps $w_{\mathfrak{g}}$: $\mathfrak{g}^r \to \mathfrak{g}$ on simple Lie algebra becomes (FRS) after a number t(w) of self-convolutions which is independent of \mathfrak{g} . Theorem 8.2, and Propositions 4.21 and 8.14 imply the following improvement of [GH24, Theorem A], which further shows that these maps have geometrically irreducible fibers:

⁶A Q-scheme of finite type X has rational singularities if it is normal and for every resolution of singularities $\pi: \widetilde{X} \to X$, one has $R^i\pi_*(O_{\widetilde{X}}) = 0$ for $i \geq 1$.

⁷See e.g. [CGH23, Lemma 4.2] for the definition of the canonical measure.

Corollary 8.16 (cf. [GH24, Theorem A]). Let $w \in \mathcal{L}_r$ be a Lie algebra word of degree d. Then there exists $0 < C < 10^6$, such that for every simple \mathbb{Q} -Lie algebra \mathfrak{g} , for which $0 \neq w_{\mathfrak{g}}$, one has:

- (1) If $t \geq Cd^3$ then $w_{\mathfrak{g}}^{*t}$ is (FGI).
- (2) If $t \ge Cd^4$ then $w_{\mathfrak{g}}^{*t}$ is (FRS) and (FGI).
- (3) There exists a constant $C' = C'(\mathfrak{g}, w)$, such that if $t \geq Cd^4$, one has

$$\left\| (w_{\mathfrak{g}}^{*t})_* \mu_{\mathfrak{g}(\mathbb{Z}_p)}^{rt} - \mu_{\mathfrak{g}(\mathbb{Z}_p)} \right\|_{\infty} < C' p^{-1/2},$$

where $\mu_{\mathfrak{g}(\mathbb{Z}_p)}$ is the Haar probability measure on $\mathfrak{g}(\mathbb{Z}_p)$. In particular, the L^{∞} -mixing time t_{∞} of the random walk on $\mathfrak{g}(\mathbb{Z}_p)$ induced by the word measure $(w_{\mathfrak{g}})_*\mu^r_{\mathfrak{g}(\mathbb{Z}_p)}$, is smaller than Cd^4 , whenever $p \gg_{w,\mathfrak{g}} 1$.

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