Matrix factorization and the generic plane projection of curve a singularity *

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Abstract

We study the matrix factorizations defined by the generic plane projections of a curve singularity of $(\mathbb{C}^n,0)$. On the other hand, given a plane curve singularity $(Y,0)\subset(\mathbb{C}^2,0)$ we study the family of matrix factorizations defined by the space curve singularities $(X,0)\subset(\mathbb{C}^n,0)$ such that (Y,0) is the generic plane projection of (X,0).

1 Introduction

The description of the structure and the computation of the length of minimal resolutions of finitely generated modules are central problems in several areas of mathematics. Hilbert's syzygies theorem and the results of Auslander, Buchsbaum and Serre prove that regular rings are those for which all finitely generated modules have finite minimal resolutions. Koszul complexes and the theorems of Hilbert-Burch and Buschbaum-Eisenbud give explicit structures of the (finite) minimal resolutions of complete intersections, codimension two perfect ideals and codimension three Gorenstein ideals, respectively. In all these cases the information provided by the minimal resolution is finite and essentially unique.

On the other hand, infinite minimal resolutions naturally arise in algebra since we consider modules over non-regular rings. See the contribution of Avramov in [16] for a good introduction to infinite resolutions. Eisenbud proved in [8] that finitely generated modules over a complete intersections of codimension one, i.e. R = S/(F) with R a local regular ring, eventually becomes periodic of period two. The matrices A, B of the morphisms defining this periodic structure is a matrix factorization of F, that means AB = BA = FId. This construction was generalized to complete intersection rings by Eisenbud and Peeva in [9]. Although the minimal resolution is infinite in such cases, the

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information defining it remains finite. See [9, Chapter 1] for a introduction and survey to matrix factorizations, see also [10], [11] and the references therein.

The aim of this paper is to study matrix factorizations that naturally appear when we consider generic plane projections of an irreducible curve singularity of $(\mathbb{C}^n, 0)$. Given a curve singularity $(X, 0) \subset (\mathbb{C}^n, 0)$ the generic plane projections (Y, 0) are well defined up to equisingularity type, see Definition 3.8, inducing finite ring extensions $\pi^* : \mathcal{O}_{(Y,0)} \longrightarrow \mathcal{O}_{(X,0)}$. Since $\mathcal{O}_{(Y,0)} = \mathbb{C}\{x,y\}/(F)$ is a complete intersection ring, by [8] there exists a matrix factorization of F defined by a pair of square matrices A, B, with entries in $(x,y)\mathbb{C}\{x,y\}$, such that $AB = BA = F\mathrm{Id}$, inducing an exact sequence

$$0 \longrightarrow \mathbb{C}\{x,y\}^b \stackrel{A}{\longrightarrow} \mathbb{C}\{x,y\}^b \longrightarrow \mathcal{O}_{(X,0)} \longrightarrow 0,$$

and a minimal periodic free resolution of $\mathcal{O}_{(Y,0)}$ -modules

$$\cdots \longrightarrow \mathcal{O}^b_{(Y,0)} \xrightarrow{B} \mathcal{O}^b_{(Y,0)} \xrightarrow{A} \mathcal{O}^b_{(Y,0)} \xrightarrow{B} \mathcal{O}^b_{(Y,0)} \xrightarrow{A} \mathcal{O}^b_{(Y,0)} \longrightarrow \mathcal{O}_{(X,0)} \longrightarrow 0,$$

see Theorem 4.2. Moreover, we know that the family of generic plane projections is flat, see section 3. This flat family induces a global matrix factorization specializing to the above matrix factorization, Theorem 4.2.

In the section 5 we consider the family of matrix factorizations defined by the space curve singularities $(X,0) \subset (\mathbb{C}^n,0)$ with a given a generic plane projection $(Y,0) \subset (\mathbb{C}^2,0)$. In Proposition 5.7 we characterize the matrix factorizations of $F \in \mathbb{C}\{x,y\}$, equation defining (Y,0), induced by a space curve singularity $(X,0) \subset (\mathbb{C}^n,0)$. Equivalently, we characterize the dimension b square matrices A such that the cokernel of $\mathbb{C}\{x,y\}^b \xrightarrow{A} \mathbb{C}\{x,y\}^b$ is a curve singularity with generic plane projection (Y,0).

In Theorem 5.12 we establish a bijection between the orbits of the action of the general linear group in the family of matrix factorizations induced by curve singularities with generic plane projection (Y,0) and the clases of isomorphism of $\mathcal{O}_{(Y,0)}$ -algebras of the rings $\mathcal{O}_{(X,0)}$, where $(X,0) \subset (\mathbb{C}^n,0)$ is a curve singularity with $(Y,0) \subset (\mathbb{C}^2,0)$ as a generic plane projection.

The explicit computations of this paper are performed by using the computer algebra system Singular [7].

2 Preliminaries

Let (X,0) be a an analytic reduced curve singularity of $(\mathbb{C}^n,0)$, from now on curve singularity. The local ring $\mathcal{O}_{(X,0)}$ is a one-dimensional reduced local ring with maximal ideal $\mathbf{m}_{(X,0)}$. We write by $I_{(X,0)} \subset \mathcal{O}_{(\mathbb{C}^n,0)} = \mathbb{C}\{x_1,\ldots,x_n\}$ the radical ideal defining (X,0), i.e. $\mathcal{O}_{(X,0)} = \mathbb{C}\{x_1,\ldots,x_n\}/I_{(X,0)}$.

The Hilbert function of (X,0) is the numerical function $\operatorname{HF}_{(X,0)}:\mathbb{N} \longrightarrow \mathbb{N}$ such that $\operatorname{HF}_{(X,0)}(n) = \operatorname{Length}_{\mathcal{O}_{(X,0)}}(\mathbf{m}^n_{(X,0)}/\mathbf{m}^{n+1}_{(X,0)})$ for all $n \in \mathbb{N}$, The Hilbert-Samuel function of (X,0) is defined by $\operatorname{HF}^1_{(X,0)}(n) = \sum_{j=0}^n \operatorname{HF}_{(X,0)}(j) = \operatorname{Length}(\mathcal{O}_{(X,0)}/\mathbf{m}^{n+1}_{(X,0)}), n \in \mathbb{N}$. It

is well known that there exist integers $e_0(X,0), e_1(X,0) \in \mathbb{N}$ such that $HP^1_{(X,0)}(T) =$ $e_0(X,0)(T+1) - e_1(X,0)$ is the Hilbert-Samuel polynomial of (X,0), i.e. $\mathrm{HF}^1_{(X,0)}(n) =$ $HP_{(X,0)}^1(n)$ for $n \gg 0$. The integer $e_0(X,0)$ is the multiplicity of (X,0) and the embedding dimension of (X, 0) is $HF_{(X,0)}(1) = \dim_{\mathbb{C}}(\mathbf{m}_{(X,0)}/\mathbf{m}_{(X,0)}^2)$.

Let $\nu: \overline{X} = \mathbf{Spec}(\overline{\mathcal{O}_{(X,0)}}) \longrightarrow (X,0)$ be the normalization of (X,0), where $\overline{\mathcal{O}_{(X,0)}}$ is the integral closure of $\mathcal{O}_{(X,0)}$ on its full ring of fractions $\operatorname{tot}(\mathcal{O}_{(X,0)})$. The singularity order of (X,0) is $\delta(X,0) = \dim_{\mathbb{C}} \left(\overline{\mathcal{O}_{(X,0)}} / \mathcal{O}_{(X,0)} \right)$. We denote by $\mathcal{C}(X,0)$ the conductor of the finite extension $\nu^*: \mathcal{O}_{(X,0)} \hookrightarrow \overline{\mathcal{O}_{(X,0)}}$ and by c(X,0) the dimension of $\overline{\mathcal{O}_{(X,0)}}/\mathcal{C}(X,0)$. Let $\omega_{(X,0)} = \operatorname{Ext}_{\mathcal{O}_{(\mathbb{C}^n,0)}}^{n-1}(\mathcal{O}_{(X,0)},\Omega_{(\mathbb{C}^n,0)}^n)$ be the dualizing module of (X,0). We can consider the composition morphism of $\mathcal{O}_{(X,0)}$ -modules

$$\gamma_X: \Omega_{(X,0)} \longrightarrow \nu_* \Omega_{\overline{X}} \cong \nu_* \omega_{\overline{X}} \longrightarrow \omega_{(X,0)}.$$

Let $d: \mathcal{O}_{(X,0)} \longrightarrow \Omega_{(X,0)}$ the universal derivation. The Milnor number of (X,0) is, [3],

$$\mu(X,0) = \dim_{\mathbb{C}}(\omega_{(X,0)}/(\gamma_X \circ d)\mathcal{O}_{(X,0)})$$

Notice that (X,0) is non-singular iff $\mu(X,0)=0$ iff $\delta(X,0)=0$ iff c(X,0)=0.

In the following result we collect some basic results on $\mu(X,0)$ and other numerical invariants that we will use later on.

Proposition 2.1. Let (X,0) be a curve singularity of embedding dimension n. Then (i) $\mu(X,0) = 2\delta(X,0) - r(X,0) + 1$, where r(X,0) is the number of branches of (X,0). (ii) It holds

$$e_0(X,0) - 1 \le e_1(X,0) \le \delta(X,0) \le \mu(X,0)$$

and $e_1(X,0) \leq \binom{e_0(X,0)}{2} - \binom{n-1}{2}$. (iii) If X is singular then $\delta(X,0) + 1 \leq c(X,0) \leq 2\delta(X,0)$, and $c(X,0) = 2\delta(X,0)$ if and only if $\mathcal{O}_{(X,0)}$ is a Gorenstein ring.

Proof. (i) [3, Proposition 1.2.1]. (ii) [3, Proposition 1.2.4 (i)], [19], [14], [15]. (iii) [20, Proposition 7, pag. 80, and [1].

Given a local ring (A, \mathbf{m}) and a finitely generated A-module M we denote by $\beta_A(M)$ the minimal number of generators of M, i.e. $\beta_A(M) = \dim_{A/\mathbf{m}}(M/\mathbf{m}M)$.

3 On the generic projection of a curve singularity

Throughout this paper we only consider irreducible curve singularities.

Let (X,0) be a curve singularity of $(\mathbb{C}^n,0)$, i.e. (X,0) is an irreducible curve singularity of $(\mathbb{C}^n, 0)$. Then $\overline{\mathcal{O}_{(X,0)}} \cong \mathbb{C}\{t\}$ for some uniformation parameter $t \in \text{tot}(\mathcal{O}_{(X,0)})$ and the morphism $\nu^*: \mathcal{O}_{(X,0)} \hookrightarrow \overline{\mathcal{O}_{(X,0)}} \cong \mathbb{C}\{t\}$ can be described by a family of series $\nu^*(\overline{x_i}) = f_i(t) \in \mathbb{C}\{t\}, i = 1, \ldots, n$. We say that

$$(X,0): \begin{cases} x_1 = f_1(t) \\ \vdots \\ x_n = f_n(t) \end{cases}$$

is a parametrization of (X,0). Notice that the \mathbb{C} -algebra morphism defined by

$$\gamma: \mathbb{C}\{x_1, \dots, x_n\} \longrightarrow \mathbb{C}\{t\}$$

$$x_i \mapsto f_i(t)$$

has as kernel the ideal $I_{(X,0)}$ and induces the normalization map ν^* . The conductor of this ring extension is principal $\mathcal{C}(X,0)=(t^{c(X,0)})$. After a linear change of variables and choosing an appropriate uniformation parameter, i.e. an appropriate generator of the maximal ideal of $\mathbb{C}\{t\}$, we may assume that the parametrization of (X,0) is defined by: $f_1(t)=t^{e_0(X,0)}$ and $f_i(t)=\sum_{j\geq e_0(X,0)+1}b_{j,i}t^j,\ i=2,\ldots,n$, with $b_{j,i}\in\mathbb{C}$. We say that such a parametrization is presented in a standard form.

We denote by $\pi: Bl(X,0) \longrightarrow (X,0)$ the blowing-up of (X,0) on its closed point. The fiber of the closed point of (X,0) has a finite number of closed points: the so-called points of the first neighborhood of (X,0). We can iterate the process of blowing-up until we get the normalization of (X,0), see [6]. We denote by $\inf(X,0)$ the set of infinitely near points of (X,0). The curve singularity defined by an infinitely point p of (X,0) will be denote by (X,p). From [19] we get:

Proposition 3.1. Let (X,0) be a curve singularity, then

$$\delta(X,0) = \sum_{p \in \inf(X,0)} e_1(X,p)$$

See [18] for the corresponding result for non-irreducible curve singularities.

We denote by $\Gamma_{(X,0)} \subset \mathbb{N}$ the semigroup of values of (X,0): the set of integers $v_t(f) = ord_t(t)$ where $f \in \mathcal{O}_{(X,0)} \setminus \{0\}$. It is easy to see that $\delta(X,0) = \#(\mathbb{N} \setminus \Gamma_{(X,0)})$. The multiplicity of (X,0) is $e_0(X,0) = \min\{v_t(f); f \in \mathbf{m}_X \setminus \{0\}\}$.

Next, we recall the definition of the Whitney's $C_5(X,0)$ cone, see [23, Section 3] and [2, Chapter IV],

Definition 3.2. Let (X,0) be a curve singularity of $(\mathbb{C}^n,0)$ with a parametrization $x_i = f_i(t)$, i = 1, ..., n. The cone $C_5(X,0)$ is the set of secant vectors to (X,0): the set of vectors $w \in \mathbb{C}^n$ such that there exist two analytic functions $\alpha(u), \beta(u) \in \mathbb{C}\{u\}$ such that:

$$w = \lim_{u \to 0} (f_i(\alpha(u)) - f_i(\beta(u)))_{i=1,\dots,n}.$$

In [2, Proposition IV.1] the cone $C_5(X,0)$ is computed for a general reduced curve singularity. They show that such a cone is a finite union of dimension two planes containing the tangent lines of (X,0). Next, we will explicitly describe $C_5(X,0)$ for a irreducible curve singularity with a parametrization presented in a standard form.

Proposition 3.3. Let (X,0) be a curve singularity of $(\mathbb{C}^n,0)$ with a standard parametrization, $e = e_0(X,0)$,

$$\begin{cases} x_1 = t^e \\ x_i = \sum_{j>e+1} b_{j,i} t^j & i = 2, \dots, n \end{cases}$$

The cone $C_5(X,0)$ is the union of the dimension two planes generated by $(1,0,\ldots,0) \in \mathbb{C}^n$ and the vectors $(0,b_{k,2},\ldots,b_{k,n}) \in \mathbb{C}^n \setminus \{0\}$ for which there exist $\epsilon \in \mathbb{C}$ with $\epsilon^e = 1$ and k is the first integer $j \geq e+1$, if it exists, for which $(\epsilon^j - 1)b_{j,i} \neq 0$ for some integer $i \in \{2,\ldots,n\}$.

Proof. See the proof of [2, Proposition IV.1].

Remark 3.4. Notice that from the last result we get that the number of the irreducible components of $C_5(X,0)$ is at most $e_0(X,0)-1$.

Given a set of non-negative integers $2 \leq a_1 < \cdots < a_n$ such that $gcd(a_1, \cdots, a_n) = 1$, we consider the monomial curve singularity $(X,0) = M(a_1, \cdots, a_n)$ defined by the \mathbb{C} -algebra morphism $\gamma : \mathbb{C}\{x_1, \ldots, x_n\} \longrightarrow \mathbb{C}\{t\}$ with $\gamma(x_i) = t^{a_i}, i = 1, \ldots, n$, i.e. $I_{(M(a_1, \cdots, a_n), 0)} = \ker(\gamma)$. The induced monomorphism

$$\gamma: \mathcal{O}_{(X,0)} = \mathbb{C}\{x_1, \dots, x_n\}/I_{M(a_1, \dots, a_n)} \longrightarrow \mathbb{C}\{t\}$$

is the normalization map.

Next, we compute Whitney's C_5 cone of a monomial curve.

Proposition 3.5. Given a monomial curve singularity $(X,0) = M(n_1, n_2, n_3)$ of $(\mathbb{C}^3, 0)$.

- (i) If $gcd(n_1, n_2) = 1$ then $C_5(X, 0)$ is the plane generated by (1, 0, 0) and (0, 1, 0).
- (ii) If $gcd(n_1, n_2) = a > 1$ then $C_5(X, 0)$ is the union of the plane generated by (1, 0, 0) and (0, 1, 0) and the plane generated by (1, 0, 0) and (0, 0, 1).

Proof. If $gcd(n_1, n_2) = 1$ then from Proposition 3.3 we get that $C_5(X, 0)$ is the plane generated by (1, 0, 0) and (0, 1, 0). Assume now that $gcd(n_1, n_2) = a > 1$. Let ξ be a n_1 -th unit root. If $\xi^a \neq 1$ then we deduce that $C_5(X, 0)$ contains the plane generated by (1, 0, 0) and (0, 1, 0). Assume that $\xi^a = 1$; since $gcd(a, n_3) \neq 1$ from Proposition 3.3 we get that $C_5(X, 0)$ contains the plane generated by (1, 0, 0) and (0, 0, 1).

Example 3.6. Let us consider the monomial curve (X,0) of $(\mathbb{C}^3,0)$ defined by the standard parametrization $x_1 = t^4, x_2 = t^6, x_3 = t^7$. We know that the defining ideal of (X,0) is $I_{(X,0)} = (x_1^3 - x_2^2, x_3^2 - x_1^2x_2)$. We have $e_0(X,0) = 4$ and the 4-th unit roots are: $\{1,-1,i,-i\}$. Then $C_5(X,0)$ is the cone union of the two 2-dimensional planes generated by (1,0,0), defining the tangent line of (X,0), and (0,1,0), for $\epsilon = i,-i$, and the plane generated by (1,0,0) and (0,0,1), for $\epsilon = -1$.

We denote by $\mathcal{G}(n, n-2)$ the Grassmannian of the dimension n-2 linear varieties of $(\mathbb{C}^n, 0)$. For all $H \in \mathcal{G}(n, n-2)$ we denote by π_H the projection to $(\mathbb{C}^2, 0)$ parallel to H. We denote by $\mathcal{W}_{(X,0)}$ the open set of $H \in \mathcal{G}(n, n-2)$ such that $H \cap C_5(X, 0) = \{0\}$.

Next, we recall [2, Proposition IV.1] and we sketch the proof that we will use later on. For the theory of equisingularity for plane curve curves see [24], or [2] and [17].

Proposition 3.7. For all $H \in \mathcal{W}_{(X,0)}$ the image $\pi_H(X,0)$ is a germ of a reduced curve singularity with constant equisingularity type.

Next, we recall how this result was proved; see [2, Proposition IV.1] for the details. We denote by $W = W_{(X,0)}$ the Zariski open subset of \mathbb{C}^{2n} , with coordinates z_1, \ldots, z_{2n} , parameterizing the pairs of linear forms (L_1, L_2) with $L_1 = z_1x_1 + \cdots + z_nx_n$, $L_2 = z_{n+1}x_1 + \cdots + z_{2n}x_n$ such that the linear variety defined by them is of dimension n-2 and belonging to $W_{(X,0)}$. For each $L = (z_1, \ldots, z_{2n}) \in W$ we denote by

$$\pi_L: \quad \mathbb{C}^n \longrightarrow \mathbb{C}^2$$

$$(x_1, \dots, x_n) \mapsto (L_1, L_2)$$

the projection defined by the dimension n-2 linear variety $L_1=L_2=0$. The image of (X,0) is a plane curve singularity $\pi_L(X,0)\subset (\mathbb{C}^2,0)$. We denote by π the induced morphism

$$\pi: \quad \mathbb{C}^n \times W \quad \longrightarrow \quad \mathbb{C}^2 \times W$$
$$((x_1, \dots, x_n), L) \quad \mapsto \quad (\pi_L(x_1, \dots, x_n), L)$$

Given $L = (c_1, \ldots, c_{2n}) \in W$ the restriction of π to $(X, 0) \times W$ if a finite morphism in (0, L), see proof of [2, Proposition IV.1]. The image $\pi((X, 0) \times W, (0, L))$ is a hypersurface of $(\mathbb{C}^2, 0) \times (W, L)$ defined by a principal ideal $(F(x, y; z_1 - c_1, \ldots, z_{2n} - c_{2n})) \subset \mathbb{C}\{x, y; z_1 - c_1, \ldots, z_{2n} - c_{2n}\} \cong \mathcal{O}_{(\mathbb{C}^2, 0) \times (W, L)}$. Then we have a \mathbb{C} -algebra monomorphism

$$\mathcal{O}_{\pi((X,0)\times W,(0,L)),(0,L)} \cong \frac{\mathbb{C}\{x,y;z_1-c_1,\ldots,z_{2n}-c_{2n}\}}{(F(x,y;z_1-c_1,\ldots,z_{2n}-c_{2n}))} \xrightarrow{\pi^*} \mathcal{O}_{X\times W,(0,L)}$$

such that tensoring by $\cdot \otimes_{\mathbb{C}\{x,y;z_1-c_1,\dots,z_{2n}-c_{2n}\}} \frac{\mathbb{C}\{x,y;z_1-c_1,\dots,z_{2n}-c_{2n}\}}{(z_1-c_1,\dots,z_{2n}-c_{2n})}$ we get the morphism

$$\mathcal{O}_{\pi_L(X,0)} \xrightarrow{\pi_L^*} \mathcal{O}_{X,0},$$

morphism induced by the projection $(X,0) \xrightarrow{\pi_L} \pi_L(X,0)$. Moreover, the projection into the second component

$$\gamma: \pi((X,0) \times W, (0,L)) \longrightarrow (W,L)$$

defines a flat deformation of $\pi_L(X,0)$ whose fibers are curve singularities with constant Milnor number. This flat deformation γ admits a resolution in family, this means that if $\nu : \overline{\pi((X,0) \times W,(0,L))} \longrightarrow \pi((X,0) \times W,(0,L))$ is the normalization map then the composition

$$G = \gamma \circ \nu : \overline{\pi((X,0) \times W, (0,L))} \xrightarrow{\nu} \pi((X,0) \times W, (0,L)) \xrightarrow{\gamma} (W,L)$$

satisfies $G^{-1}(p) = \overline{\pi_p(X,0)}$ for a p belonging to a small neighbourhood of L in W. Given a parametrization

$$\begin{cases} x_1 = f_1(t) \\ \vdots \\ x_n = f_n(t) \end{cases}$$

of (X,0) the flat deformation γ admits a parametrization in family. This means that the normalization map $\overline{\pi_p(X,0)} \longrightarrow \pi_p(X,0)$ is defined, after a change of variables, by

$$\begin{cases} x = \sum_{j=1}^{n} (s_j + c_j) f_j(t) \\ y = \sum_{j=n+1}^{2n} (s_j + c_j) f_{j-n}(t) \end{cases}$$

for small s_1, \ldots, s_{2n} , [21], [22]. Furthermore, we have a commutative diagram where ν and ν_X are the corresponding normalization morphisms

$$\mathcal{O}_{(X,0)\times W,(0,L)} \xrightarrow{(\nu_X \times Id_W)^*} \overline{\mathcal{O}_{(X,0)\times W,(0,L)}} \cong \overline{\mathcal{O}_{\pi((X,0)\times W,(0,L)),(0,L)}}$$

$$\uparrow_{\pi^*} \xrightarrow{\nu^*}$$

$$\mathcal{O}_{\pi((X,0)\times W,(0,L)),(0,L)}$$

notice that ν_X^* is defined by the given parametrization of (X,0) and ν^* is defined by above parametrization in family.

Definition 3.8. A generic plane projection of a curve singularity (X,0) is $\pi_H(X,0)$ for any $H \in \mathcal{W}_{(X,0)}$. All these plane curve singularities share the same equisingularity type, Proposition 3.7. We denote by $\overline{\mu}(X,0)$ the Milnor number of a generic plane projection of (X,0).

Remark 3.9. A classical problem considered by F. Enriques and O. Chisini was to compare the equisingularity type of (X,0) and the equisingularity type of a generic plane projection $(\pi_L(X,0),0)$, see [25, page 12]. This problem was addressed in [5] and [4]. On the other hand, in [13] we gave a sharp upper bound of the singularity order of $(\pi_L(X,0),0)$, L generic, in terms of the singularity order of (X,0), we proved that

$$\delta(X,0) \le \delta(\pi_L(X,0),0) \le (e_0(X,0)-1)\delta(X,0) - \binom{e_0(X,0)-1}{2}.$$

Example 3.10. Let us consider Example 3.6. We take coordinates $z_1, z_2, z_3, z_4, z_5, z_6$ in \mathbb{C}^6 parameterizing the pairs of linear forms of $\mathbb{C}\{x_1, x_2, x_3\}$. In this case we have that the Zariski open set $W_{(X,0)} \subset \mathbb{C}^6$ is defined by the conditions $z_1z_6 - z_3z_4 \neq 0$ and $z_1z_5 - z_2z_4 \neq 0$. Next, we study $\pi_L(X,0)$ in a neighbourhood of $L = (1,0,0,0,1,1) \in W_{(X,0)}$. The parametrization in family is

$$\begin{cases} x = (1+s_1)t^4 + s_2t^6 + s_3t^7 \\ y = s_4t^4 + (1+s_5)t^6 + (1+s_6)t^7. \end{cases}$$

for small $s_1, \ldots, s_{2n} \in \mathbb{C}$. In order to get "small" equations we consider the sub-family defined by $s_1 = s_2 = s_3 = s_4 = s_5 = 0$

$$\begin{cases} x = t^4 \\ y = t^6 + (1 + s_6)t^7 \end{cases}$$

for a small s_6 . We can compute the equation F eliminating the variable t of the ideal $(x - t^4, y - t^6 - (1 + s_6)t^7), [7],$

$$F = y^4 - 2x^3y^2 + y^4s_6 + x^6 - 4x^5y - 2x^3y^2s_6 - x^7 + x^6s_6 - 12x^5ys_6 - 5x^7s_6 - 12x^5ys_6^2 - 10x^7s_6^2 - 4x^5ys_6^3 - 10x^7s_6^3 - 5x^7s_6^4 - x^7s_6^5.$$

Notice that, in particular, F(x, y, 0) = 0 is the equation of the ideal of the monomial curve singularity (X, 0). Hence we have a commutative diagram

$$\xrightarrow{ \begin{array}{c} \mathbb{C}\{x_1,\dots,x_n,s_6\} \\ I_{(X,0)}\mathbb{C}\{x_1,\dots,x_n,s_6\} \end{array}} \xrightarrow{ \begin{array}{c} \nu_X \times Id_W)^* \\ \uparrow \pi^* \end{array}} \mathbb{C}\{t,s_6\}$$

$$\xrightarrow{ \begin{array}{c} \mathbb{C}\{x,y,s_6\} \\ (F) \end{array}}$$

where $\nu_X^*(x_1) = t^4$, $\nu_X^*(x_2) = t^6$, $\nu_X^*(x_3) = t^7$, and $\nu^*(x) = t^4$ and $\nu^*(y) = t^6 + (1 + s_6)t^7$. Since the fibers of γ are plane curves we can compute their Milnor number as follows. Provided s_6 small the multiplicity sequence of the resolution process of the fiber $\gamma^{-1}(s_6)$ is $\{4, 2, 2, 1, \ldots\}$, so $\mu(\gamma^{-1}(s_6), 0) = 16$, Proposition 2.1 (iii) and Proposition 3.1. Hence $\overline{\mu}(X, 0) = 16$.

Example 3.11. In Proposition 3.7 we recall that all generic plane projections share the same equisingularity type. In the following example we show that the analytic type of the generic plane projections are not constant. We consider the example 4 of [26, Chapter V], $(X,0) = M(5,6,8,9) \subset (\mathbb{C}^4,0)$. By using Proposition 3.3 we get that $\mathcal{W}_{(X,0)}$ is the set of planes through the origin of \mathbb{C}^4 transversal to $x_2 = x_3 = 0$. Let us consider the plane $L(a_8,a_9)$ defined by the linear forms x_1 and $x_2 + a_8x_3 + a_9x_4$, with $a_8 \neq 0$ and $a_9 \neq 0$, we have that $L(a_8,a_9) \in \mathcal{W}_{(X,0)}$. The projection $\pi_{L(a_8,a_9)}(X,0)$ has the following parametrization

$$\begin{cases} x = t^5 \\ y = t^6 + a_8 t^8 + a_9 t^9. \end{cases}$$

From [26, Proposition 4.1, Chapter V] we have that two of such projections with respect to $L(a_8, a_9)$ and $L(b_8, b_9)$ are analytically isomorphic if and only if $a_8^3/a_9^2 = b_8^3/b_9^2$. Hence there are infinitely many analytic isomorphism classes of generic plane projections of (X, 0). The equisingularity type of these curve singularities is constant and it is defined by the set of characteristic exponents $\{6/5, 8/5, 9/5\}$, [24] or [2].

4 Matrix decomposition and the generic plane projection

In this section we study the family of matrix factorizations naturally appearing when we consider the generic plane projections of a curve singularity, [8], [9]. The key ingredient is that the generic plane projection define a flat family with constant equisingularity type, see proof of Proposition 3.7. We construct a "family" of matrix factorizations with closed fiber a matrix factorization of the initial germ of a curve singularity.

As a motivation we start describing a concrete example of matrix factorization. We construct a matrix factorization of the monomial curve singularity of Example 3.6.

Example 4.1. Let us consider the ring $\mathcal{O}_{(X,0)} = \mathbb{C}\{x_1, x_2, x_3\}/I_{(X,0)}$, Example 3.6. After the \mathbb{C} -algebra isomorphism of $\mathbb{C}\{x_1, x_2, x_3\}$

$$\phi: \begin{cases} \phi(x_1) = x \\ \phi(x_2) = y - z \\ \phi(x_3) = z \end{cases}$$

we get that $\mathcal{O}_{(X,0)} \cong M_X := \mathbb{C}\{x,y,z\}/J$ where J is the ideal generated by $x^3 - (y-z)^2$ and $z^2 - x^2(y-z)$. The $\mathbb{C}\{x,y\}$ -module M_X is generated by 1 and \overline{z} , so we have a complex of $\mathbb{C}\{x,y\}$ -modules

$$0 \longrightarrow \mathbb{C}\{x,y\}^2 \stackrel{d}{\longrightarrow} \mathbb{C}\{x,y\}^2 \stackrel{(1,\overline{z})}{\longrightarrow} M_X \longrightarrow 0$$

where

$$d = \begin{pmatrix} x^3 - y^2 - x^2y & x^4y + 2x^2y^2 \\ x^2 + 2y & x^3 - x^4 - 3x^2y - y^2 \end{pmatrix}.$$

With the help of Singular we can show that this complex is exact, [7]. Moreover, let us consider the morphism $h: \mathbb{C}\{x,y\}^2 \longrightarrow \mathbb{C}\{x,y\}^2$ defined by the adjoint matrix of d. Then we can check that

$$hd = dh = F \cdot \mathrm{Id}_{\mathbb{C}\{x,y\}^2}$$

where $F = y^4 - 2x^3y^2 + x^6 - 4x^5y - x^7$ is the equation of the plane projection

$$\pi_L(X,0): \left\{ \begin{array}{l} x = t^4 \\ y = t^6 + t^7 \end{array} \right.$$

with $L = (1, 0, 0, 0, 1, 1) \in W_{(X,0)}$. Hence the pair (d, h) is a matrix factorization of F, see [9, Section 1.2]. Then we have a minimal periodic two free resolution of the $\mathcal{O}_{\pi_L(X,0)} = \mathbb{C}\{x,y\}/(F)$ -module M_X

$$\cdots \longrightarrow \mathcal{O}^{2}_{(\pi_{L}(X,0),0)} \xrightarrow{h} \mathcal{O}^{2}_{(\pi_{L}(X,0),0)} \xrightarrow{d} \mathcal{O}^{2}_{(\pi_{L}(X,0),0)} \xrightarrow{h}$$

$$\xrightarrow{h} \mathcal{O}^{2}_{(\pi_{L}(X,0),0)} \xrightarrow{d} \mathcal{O}^{2}_{(\pi_{L}(X,0),0)} \xrightarrow{(1,\overline{z})} M_{X} \longrightarrow 0,$$

[9, Theorem 2.1.1].

In the next statement we use the notations of the proof of Proposition 3.7. For all $L \in W_{(X,0)}$ we consider $\mathcal{O}_{X\times W,(0,L)}$ as a $\mathcal{O}_{\pi(X\times W,(0,L)),(0,L)}$ -module via π^* , and $\mathcal{O}_{(X,0)}$ as $\mathcal{O}_{(\pi_L(X,0),0)}$ -module via π_L^* .

Theorem 4.2. Let (X,0) be a curve singularity of $(\mathbb{C}^n,0)$. Let b the minimal number of generators of $\mathcal{O}_{(X,0)}$ as $\mathcal{O}_{(\pi_L(X,0),0)}$ -module. Then for all $L=(c_1,\ldots,c_{2n})\in W_{(X,0)}$ there is $\widehat{F}=F(x,y;z_1-c_1,\ldots,z_{2n}-c_{2n})\in R=\mathbb{C}\{x,y;z_1-c_1,\ldots,z_{2n}-c_{2n}\}$ such that $\mathcal{O}=\mathcal{O}_{\pi((X,0)\times W,(0,L)),(0,L)}\cong \frac{R}{(\widehat{F})}$ and a matrix factorization of \widehat{F}

$$\widehat{d} \ \widehat{h} = \widehat{h} \ \widehat{d} = \widehat{F} \cdot \operatorname{Id}_{R^b}$$

with $\widehat{d}, \widehat{h} \in \operatorname{Mat}_{b \times b}(R)$ such that

$$0 \longrightarrow R^b \xrightarrow{\widehat{d}} R^b \longrightarrow \mathcal{O}_{(X,0) \times W,(0,L)} \longrightarrow 0$$

is exact. Moreover, there is a minimal free resolution of $\mathcal{O}_{(X,0)\times W,(0,L)}$ as \mathcal{O} -module

$$\cdots \longrightarrow \mathcal{O}^b \xrightarrow{\widehat{h}} \mathcal{O}^b \xrightarrow{\widehat{d}} \mathcal{O}^b \xrightarrow{\widehat{h}} \mathcal{O}^b \xrightarrow{\widehat{h}} \mathcal{O}^b \longrightarrow \mathcal{O}_{(X,0) \times W,(0,L)} \longrightarrow 0$$

such that tensoring by $\cdot \otimes_R \frac{R}{(z_1-c_1,...,z_{2n}-c_{2n})}$ we get a minimal free resolution of $\mathcal{O}_{(\pi_L(X,0),0)}$ -modules

$$\cdots \longrightarrow \mathcal{O}^{b}_{(\pi_{L}(X,0),0)} \xrightarrow{h} \mathcal{O}^{b}_{(\pi_{L}(X,0),0)} \xrightarrow{d} \mathcal{O}^{b}_{(\pi_{L}(X,0),0)} \xrightarrow{h}$$

$$\xrightarrow{h} \mathcal{O}^{b}_{(\pi_{L}(X,0),0)} \xrightarrow{d} \mathcal{O}^{b}_{(\pi_{L}(X,0),0)} \longrightarrow \mathcal{O}_{(X,0)} \longrightarrow 0$$

with h, d the induced morphisms by \widehat{d} , \widehat{h} . In particular, the pair d, h is a matrix factorization of F(x, y; 0, ..., 0), equation of $(\pi_L(X, 0), 0)$.

Proof. Let b the minimal number of generators of $\mathcal{O}_{(X,0)}$ as $\mathcal{O}_{(\pi_L(X,0),0)}$ -module. Then the minimal number of generators of $\mathcal{O}_{X\times W,(0,L)}$ as \mathcal{O} -module is b as well. Since \mathcal{O} is a quotient of R, the minimal number of generators of $\mathcal{O}_{(X,0)\times W,(0,L)}$ as R-module is b.

Since $\mathcal{O}_{(X,0)}$ is a finite and Cohen-Macaulay $\mathcal{O}_{(\pi_L(X,0),0)}$ -module of dimension one, $\mathcal{O}_{(X,0)}$ is a projective dimension one $\mathbb{C}\{x,y\}$ -module. On the other hand, since $\{z_1-c_1,\ldots,z_n-c_n\}$ is a regular sequence of R

$$\operatorname{pd}_{R}(\mathcal{O}_{(X,0)\times W,(0,L)}) = \operatorname{pd}_{\mathbb{C}\{x,y\}}(\mathcal{O}_{(X,0)}) = 1.$$

Assume that there exists a decomposition of \mathcal{O} -modules with $r \geq 1$

$$\mathcal{O}_{(X,0)\times W,(0,L)}\cong \mathcal{O}^r\oplus M,$$

tensoring by $\cdot \otimes_R \frac{R}{(z_1-c_1,\dots,z_{2n}-c_{2n})}$ we get the following decomposition of $\mathcal{O}_{(\pi_L(X,0),0)}$ -modules

$$\mathcal{O}_{(X,0)} \cong \mathcal{O}^r_{(\pi_L(X,0),0)} \oplus \overline{M}.$$

Since $\mathcal{O}_{(X,0)}$ is a finite length $\mathcal{O}_{(\pi_L(X,0),0)}$ -module we get a contradiction. Then $\mathcal{O}_{(X,0)\times W,(0,L)}$ is an \mathcal{O} -module with no free summands.

Recall that from the proof of Proposition 3.7 there exists $\widehat{F} = F(x, y; z_1 - c_1, \dots, z_{2n} - c_{2n}) \in R = \mathbb{C}\{x, y; z_1 - c_1, \dots, z_{2n} - c_{2n}\}$ such that $\mathcal{O} \cong \frac{R}{(\widehat{F})}$. Being $\mathcal{O}_{(X,0)\times W,(0,L)}$ a maximal Cohen-Macaulay $\mathcal{O}_{\pi((X,0)\times W,(0,L)),(0,L)}$ -module with no free summand, from [8, Theorem 6.1 (ii)] there exist a matrix factorization of \widehat{F}

$$\widehat{d} \ \widehat{h} = \widehat{h} \ \widehat{d} = \widehat{F} \cdot \operatorname{Id}_{R^b}$$

with $\widehat{d}, \widehat{h} \in \operatorname{Mat}_{b \times b}(R)$ defining a minimal free resolution of $\mathcal{O}_{(X,0) \times W,(0,L)}$ as \mathcal{O} -module

$$\cdots \longrightarrow \mathcal{O}^b \xrightarrow{\widehat{h}} \mathcal{O}^b \xrightarrow{\widehat{d}} \mathcal{O}^b \xrightarrow{\widehat{h}} \mathcal{O}^b \xrightarrow{\widehat{h}} \mathcal{O}^b \xrightarrow{\widehat{d}} \mathcal{O}^b \longrightarrow \mathcal{O}_{(X,0)\times W,(0,L)} \longrightarrow 0.$$

From [9, Theorem 2.1.1 (1)] we get a minimal resolution of R-modules

$$0 \longrightarrow R^b \xrightarrow{\widehat{d}} R^b \longrightarrow \mathcal{O}_{(X,0) \times W,(0,L)} \longrightarrow 0.$$

Tensoring this resolution by $\cdot \otimes_R \frac{R}{(z_1-c_1,...,z_{2n}-c_{2n})}$ we get a resolution

$$0 \longrightarrow \mathbb{C}\{x,y\}^b \stackrel{d}{\longrightarrow} \mathbb{C}\{x,y\}^b \longrightarrow \mathcal{O}_{(X,0)} \longrightarrow 0$$

with d the induced morphism. Notice that the matrix factorization of \hat{F} induces a matrix factorization of F(x, y; 0, ..., 0)

$$d h = h d = F \cdot \operatorname{Id}_{\mathbb{C}\{x,y\}^b}$$

with $d, h \in \operatorname{Mat}_{b \times b}(\mathbb{C}\{x, y\})$ the induced morphisms by \widehat{d}, \widehat{h} . From[9, Theorem 1.2.1 (2)] we get a minimal free resolution of $\mathcal{O}_{(\pi_L(X,0),0)}$ -modules

$$\cdots \longrightarrow \mathcal{O}^{b}_{(\pi_{L}(X,0),0)} \xrightarrow{h} \mathcal{O}^{b}_{(\pi_{L}(X,0),0)} \xrightarrow{d} \mathcal{O}^{b}_{(\pi_{L}(X,0),0)} \xrightarrow{h}$$

$$\xrightarrow{h} \mathcal{O}^{b}_{(\pi_{L}(X,0),0)} \xrightarrow{d} \mathcal{O}^{b}_{(\pi_{L}(X,0),0)} \longrightarrow \mathcal{O}_{(X,0)} \longrightarrow 0$$

In the next example we explicitly compute the resolutions and matrix factorizations appearing in the last result for the monomial curve singularity of Example 3.10.

Example 4.3. Let (X,0) be the curve singularity of Example 3.10. Let us consider the ring

$$\mathcal{O}_{((X,0)\times W,(0,L))} \cong \frac{R}{R I_{(X,0)}}$$

where $R = \mathbb{C}\{x_1, x_2, x_3, s\}$. After the \mathbb{C} -algebra isomorphism of R

$$\phi: \begin{cases} \phi(x_1) &= x \\ \phi(x_2) &= y - (1+s)z \\ \phi(x_3) &= z \\ \phi(s) &= s \end{cases}$$

we get that $\mathcal{O}_{((X,0)\times W,(0,L))}\cong M_X:=\mathbb{C}\{x,y,z,s\}/J$ where J is the ideal generated by $x^3-(y-(1+s)z)^2,z^2-x^2(y-(1+s)z)$. We have that $\mathcal{O}_{(\pi((X,0)\times W),(0,L)),(0,L)}\cong \frac{R}{(\widehat{F})}$, L=(1,0,0,0,1,1), with

$$\widehat{F} = y^4 - 2x^3y^2 + x^6 - 4x^5y - x^7 - s^4x^7 - 4s^3x^7 - 6s^2x^7 - 4s^2x^5y - 4sx^7 - 8sx^5y$$

The $\mathbb{C}\{x,y,s\}$ -module M_X is generated by $1,\overline{z}$ and we have a matrix factorization of \widehat{F} with associated matrices

$$\widehat{d} = \left(\begin{array}{ccc} x^3 - y^2 - (s+1)^2 x^2 y & x^4 y + 2 x^2 y^2 + s^3 x^4 y + 3 s^2 x^4 y + 3 s x^4 y + 2 s x^2 y^2 \\ s(s+1)^2 x^2 + (s+1)^2 x^2 + 2(s+1) y & x^3 - x^4 - 3 x^2 y - y^2 - s^4 x^4 - 4 s^3 x^4 - 6 s^2 x^4 - 3 s^2 x^2 y - 4 s x^4 - 6 s x^2 y \end{array}\right)$$

and $\hat{h} = \operatorname{Adj}(\hat{d})$. Notice that making s = 0 in \hat{d} , \hat{h} and \hat{F} we get d, h and F of Example 4.1.

In the following example, inspired in [12, Example 1], we show that there exists matrix factorizations (d,h) of a convergent series $F \in \mathbb{C}\{x,y\}$ such that $\mathrm{coker}(d)$ is not a \mathbb{C} -algebra. We will face this problem in the next section.

Example 4.4. Let (Y,0) be the monomial plane curve singularity with parametrization $x=t^3,y=t^4$, so $\mathcal{O}_{(Y,0)}=\mathbb{C}\{x,y\}/(x^4-y^3)\cong\mathbb{C}\{t^3,t^4\}$. We consider the $\mathcal{O}_{(Y,0)}$ -module $B=\mathcal{O}_{(Y,0)}+t\mathcal{O}_{(Y,0)}\subset\mathbb{C}\{t\}$. Notice that B is generated as \mathbb{C} -vector space by $1,t,t^3,t^4,t^5,\ldots$ and that B is not a ring. We have that the pair (d,h) is a matrix factorization of $F=x^4-y^3$ where

$$d = \left(\begin{array}{cc} y & -x^3 \\ -x & y^2 \end{array}\right)$$

and $h = \mathrm{Adj}(d)$, inducing a minimal resolution of B as $\mathbb{C}\{x,y\}$ -module:

$$0 \longrightarrow \mathbb{C}\{x,y\}^2 \stackrel{d}{\longrightarrow} \mathbb{C}\{x,y\}^2 \stackrel{(1,t)}{\longrightarrow} B \longrightarrow 0.$$

5 Matrix decompositions associated to a plane curve

In this section we study the family of curve singularities $(X,0) \subset (\mathbb{C}^n,0)$, and their associated matrix factorizations, sharing a given generic plane projection $(Y,0) \subset (\mathbb{C}^2,0)$.

Following the notations of the previous section, we know that there are matrix factorizations such that $\operatorname{coker}(d)$ is not a $\mathbb{C}\{x,y\}$ -algebra and, in particular, $\operatorname{coker}(d)$ is not the ring of regular functions of a curve singularity, Example 4.4. The first step of this section is to characterize the matrix factorizations coming from a curve singularity.

First, we give an upper bound of the first Betti number of matrix factorizations defined by a curve singularity. This result gives a partial answer to the question appearing in page 4 of the first paragraph of [9].

Lemma 5.1. Let (X,0) be a curve singularity of $(\mathbb{C}^n,0)$ and let (Y,0) be a generic plane projection of (X,0). Then

$$\beta_{\mathcal{O}(X,0)}(\mathcal{O}_{(X,0)}) \le e_0(X,0) - 1.$$

Proof. We may assume that (X,0) has, after a good election of the uniformation parameter, a standard parametrization

$$(X,0): \begin{cases} x_1 = t^e \\ x_2 = f_2(t) \\ \vdots \\ x_n = f_n(t) \end{cases}$$

with $e < v_t(f_2(t)) \le \cdots \le v_t(f_n(t))$ and $e = e_0(X, 0)$. The generic plane projection of (X, 0) from $L = (c_1, \ldots, c_{2n}) \in W_{(X,0)}$ has a parametrization

$$\begin{cases} x = c_1 t^e + \sum_{j=2}^n c_j f_j(t) \\ y = c_{n+1} t^e + \sum_{j=n+2}^{2n} c_j f_{j-n}(t) \end{cases}$$

see proof of Proposition 3.7.

Assume that $c_1 = c_{n+1} = 0$, then the (n-2)-plane H defined by L has equations $c_2x_2 + \cdots + c_nx_n = 0$, $c_{n+2}x_2 + \cdots + c_{2n}x_n = 0$. Notice that the tangent line of (X,0) is contained in H contradicting that $L \in W_{(X,0)}$, Proposition 3.3. Hence $c_1 \neq 0$ or $c_{n+1} \neq 0$, so $v_t(x) = e$ or $v_t(y) = e$. This implies that x, or y, is a superficial element of degree one of $\mathcal{O}_{(X,0)}$ and then $\mathcal{O}_{(X,0)}/a\mathcal{O}_{(X,0)} = e$, for a = x or a = y. Since x, y belong to a minimal system of generators of $\mathbf{m}_{(X,0)}$ and form a system of generators of $\mathbf{m}_{(X,0)}$, we have

$$\beta_{\mathcal{O}_{(Y,0)}}(\mathcal{O}_{(X,0)}) = \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{(X,0)}}{(x,y)\mathcal{O}_{(X,0)}} \right) < \dim_{\mathbb{C}} \left(\frac{\mathcal{O}_{(X,0)}}{a\mathcal{O}_{(X,0)}} \right) = e_0(X,0)$$

for a = x or a = y.

Next we characterize matrix factorizations (d, h) of a power series $F \in \mathbb{C}\{x, y\}$ for which coker (d) is a $\mathbb{C}\{x, y\}$ -algebra, see Example 4.4. We follow the ideas appearing in the introduction of [12].

Theorem 5.2. Let $(Y,0) \subset (\mathbb{C}^2,0)$ be an irreducible plane curve singularity defined by an equation $F \in \mathbb{C}\{x,y\}$. Let (d,h) be a matrix factorization of F inducing a free resolution of coker (d) as $\mathbb{C}\{x,y\}$ -modules

$$0 \longrightarrow \mathbb{C}\{x,y\}^b \stackrel{d}{\longrightarrow} \mathbb{C}\{x,y\}^b \longrightarrow \operatorname{coker}(d) \longrightarrow 0.$$

Then, coker (d) is a finitely generated faithful $\mathcal{O}_{(Y,0)}$ -module and the following conditions are equivalent:

(i) there is a finitely $\mathcal{O}_{(Y,0)}$ -algebra B, isomorphic to $\operatorname{coker}(d)$ as $\mathcal{O}_{(Y,0)}$ -module, and $\mathcal{O}_{(Y,0)}$ -algebra extensions

$$\mathcal{O}_{(Y,0)} \subset B \subset \overline{\mathcal{O}_{(Y,0)}} \cong \mathbb{C}\{t\}$$

(ii) there exists $\alpha \in \mathcal{O}_{(Y,0)} \setminus \{0\}$ and $e \in \operatorname{coker}(d) \setminus \{0\}$ such that

$$\alpha \operatorname{coker}(d) \subset e\mathcal{O}_{(Y,0)} \subset \operatorname{coker}(d)$$

and the natural morphism coker $(d) \to \operatorname{Ext}^1_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d)/e\mathcal{O}_{(Y,0)},\operatorname{coker}(d))$ induced by the exact sequence

$$0 \longrightarrow e\mathcal{O}_{(Y,0)} \longrightarrow \operatorname{coker}(d) \longrightarrow \operatorname{coker}(d)/e\mathcal{O}_{(Y,0)} \longrightarrow 0$$

is zero.

If these equivalent conditions hold then $B = e^{-1}\operatorname{coker}(d)$ is a finitely generated $\mathcal{O}_{(Y,0)}$ algebra local domain of dimension one. Moreover, there is a curve singularity $(X,0) \subset (\mathbb{C}^{b+2},0)$ such that $\mathcal{O}_{(X,0)} = B$.

Proof. We know that the matrix factorization induces a free periodic resolution of $\mathcal{O}_{(Y,0)}$ modules

$$\cdots \longrightarrow \mathcal{O}^b_{(Y,0)} \xrightarrow{h} \mathcal{O}^b_{(Y,0)} \xrightarrow{d} \mathcal{O}^b_{(Y,0)} \xrightarrow{\pi} \operatorname{coker}(d) \longrightarrow 0,$$

[9, Theorem 2.1.1]. Then coker (d) is a finitely generated $\mathcal{O}_{(Y,0)}$ -module.

Next, we prove that $\operatorname{coker}(d)$ is a faithful $\mathcal{O}_{(Y,0)}$ -module. Let $a \in \mathcal{O}_{(Y,0)}$ be a non-zero element. Assume that there is $g \in \operatorname{coker}(d) \setminus \{0\}$ such that ag = 0. Since we have a commutative diagram

$$\cdots \mathcal{O}_{(Y,0)}^{b} \xrightarrow{h} \mathcal{O}_{(Y,0)}^{b} \xrightarrow{d} \mathcal{O}_{(Y,0)}^{b} \xrightarrow{\pi} \operatorname{coker}(d) \longrightarrow 0$$

$$\downarrow^{a} \qquad \downarrow^{a} \qquad \downarrow^{a} \qquad \downarrow^{a}$$

$$\cdots \mathcal{O}_{(Y,0)}^{b} \xrightarrow{h} \mathcal{O}_{(Y,0)}^{b} \xrightarrow{d} \mathcal{O}_{(Y,0)}^{b} \xrightarrow{\pi} \operatorname{coker}(d) \longrightarrow 0$$

we get that there is $g' \in \mathcal{O}_{(Y,0)}^b$ such that $\pi(g') = g$. Hence $\pi(ag') = a\pi(g') = ag = 0$ and there is $g'' \in \mathcal{O}_{(Y,0)}^b$ with d(g'') = ag'. Moreover

$$0 = hd(g'') = h(ag') = ah(g'),$$

since $\mathcal{O}_{(Y,0)}^b$ is a domain and $a \neq 0$ we get h(g') = 0. Then there is $w \in \mathcal{O}_{(Y,0)}^b$ with g' = d(w), so

$$g = \pi(g') = \pi(d(w)) = 0.$$

Hence coker (d) is a faithful $\mathcal{O}_{(Y,0)}$ -module.

Let us assume (i). Then, identifying $\operatorname{coker}(d)$ with B, we have $\mathcal{O}_{(Y,0)}$ -algebra extensions

$$\mathcal{O}_{(Y,0)} \subset \operatorname{coker}(d) \subset \overline{\mathcal{O}_{(Y,0)}} \cong \mathbb{C}\{t\}.$$

We take e = 1 and α a generator of the conductor of the extension $\mathcal{O}_{(Y,0)} \subset \mathbb{C}\{t\}$. Applying the functor $\text{Hom}_{\mathcal{O}_{(Y,0)}}(\cdot, \text{coker}(d))$ to the exact sequence

$$0 \longrightarrow \mathcal{O}_{(Y,0)} \stackrel{i}{\longrightarrow} \operatorname{coker}(d) \stackrel{\pi}{\longrightarrow} \operatorname{coker}(d) / \mathcal{O}_{(Y,0)} \longrightarrow 0$$

we get the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d)/\mathcal{O}_{(Y,0)}, \operatorname{coker}(d)) \xrightarrow{\pi^*} \operatorname{Hom}_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d), \operatorname{coker}(d)) \xrightarrow{i^*} \\ \xrightarrow{i^*} \operatorname{Hom}_{\mathcal{O}_{(Y,0)}}(\mathcal{O}_{(Y,0)}, \operatorname{coker}(d)) \cong \operatorname{coker}(d) \xrightarrow{\phi} \operatorname{Ext}^1_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d)/\mathcal{O}_{(Y,0)}, \operatorname{coker}(d)).$$

Notice that $\phi = 0$ if and only if any $\mathcal{O}_{(Y,0)}$ —morphism $f : \mathcal{O}_{(Y,0)} \longrightarrow \operatorname{coker}(d)$ can be lifted to a $\mathcal{O}_{(Y,0)}$ —morphism $\tilde{f} : \operatorname{coker}(d) \longrightarrow \operatorname{coker}(d)$. We can define $\tilde{f}(q) = \alpha^{-1} f(\alpha q)$, for all $q \in \operatorname{coker}(d)$. Since $\alpha q \in \mathcal{O}_{(Y,0)}$

$$\tilde{f}(q) = \alpha^{-1} f(\alpha q) = \alpha^{-1} \alpha q f(1) = q f(1) \in \operatorname{coker}(d).$$

Being f a $\mathcal{O}_{(Y,0)}$ -linear map, \tilde{f} is $\mathcal{O}_{(Y,0)}$ -linear as well.

Assume now (ii). We have an exact sequence of $\mathcal{O}_{(Y,0)}$ -modules:

$$0 \longrightarrow e\mathcal{O}_{(Y,0)} \xrightarrow{i} \operatorname{coker}(d) \xrightarrow{\pi} \operatorname{coker}(d)/\mathcal{O}_{(Y,0)} \longrightarrow 0,$$

inducing the exact sequence

$$0 \longrightarrow \operatorname{Hom}_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d)/e\mathcal{O}_{(Y,0)},\operatorname{coker}(d)) \xrightarrow{\pi^*} \operatorname{Hom}_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d),\operatorname{coker}(d)) \xrightarrow{i^*} \operatorname{Hom}_{\mathcal{O}_{(Y,0)}}(e\mathcal{O}_{(Y,0)},\operatorname{coker}(d)) \cong \operatorname{coker}(d) \xrightarrow{\phi} \operatorname{Ext}^1_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d)/e\mathcal{O}_{(Y,0)},\operatorname{coker}(d)).$$

Since $\phi = 0$, any $\mathcal{O}_{(Y,0)}$ —morphism $f : e\mathcal{O}_{(Y,0)} \longrightarrow \operatorname{coker}(d)$ can be lifted to a $\mathcal{O}_{(Y,0)}$ -morphism $\tilde{f} : \operatorname{coker}(d) \longrightarrow \operatorname{coker}(d)$. On the other hand, since $\alpha \operatorname{coker}(d) \subset e\mathcal{O}_{(Y,0)}$ and $\alpha \in \mathcal{O}_{(Y,0)} \setminus \{0\}$ we get that $\operatorname{Hom}_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d)/e\mathcal{O}_{(Y,0)},\operatorname{coker}(d)) = 0$. Hence we have the isomorphism

$$\operatorname{Hom}_{\mathcal{O}(Y,0)}(\operatorname{coker}(d),\operatorname{coker}(d)) \stackrel{i^*}{\cong} \operatorname{Hom}_{\mathcal{O}(Y,0)}(e\mathcal{O}_{(Y,0)},\operatorname{coker}(d)).$$

Lemma 5.3. $\operatorname{coker}(d)$ admits a structure of $\mathcal{O}_{(Y,0)}$ -algebra with e as identity element.

Proof. Recall that, since $\operatorname{coker}(d)$ is a faithful $\mathcal{O}_{(Y,0)}$ -module, if $e\omega = e\omega'$ then $\omega = \omega'$. For all $q \in \operatorname{coker}(d)$ we consider the $\mathcal{O}_{(Y,0)}$ -morphism $f_q : e\mathcal{O}_{(Y,0)} \longrightarrow \operatorname{coker}(d)$ defined by $f_q(ew) = wq$, for all $w \in \mathcal{O}_{(Y,0)}$. Hence there exists a $\mathcal{O}_{(Y,0)}$ -morphism $\tilde{f}_q : \operatorname{coker}(d) \longrightarrow \operatorname{coker}(d)$ extending f_q . The structure of $\operatorname{coker}(d)$ as $\mathcal{O}_{(Y,0)}$ -algebra is defined by: for all $q_1, q_2 \in \operatorname{coker}(d)$,

$$q_1q_2 = \tilde{f}_{q_1}(q_2) = \tilde{f}_{q_2}(q_1).$$

If we write $\alpha q_1 = ea$, $a \in \mathcal{O}_{(Y,0)}$ then

$$\alpha^2 \tilde{f}_{q_2}(q_1) = \alpha \tilde{f}_{q_2}(\alpha q_1) = \alpha f_{q_2}(ea) = \alpha a q_2.$$

By symmetry, $\alpha q_2 = eb$, $b \in \mathcal{O}_{(Y,0)}$,

$$\alpha^2 \tilde{f}_{q_1}(q_2) = \alpha \tilde{f}_{q_1}(\alpha q_2) = \alpha f_{q_1}(eb) = \alpha b q_1.$$

Hence

$$\alpha^2 \tilde{f}_{q_2}(q_1) = eab = \alpha^2 \tilde{f}_{q_1}(q_2)$$

Since $\operatorname{coker}(d)$ is a faithful $\mathcal{O}_{(Y,0)}$ -module and $\alpha \neq 0$ we get $\tilde{f}_{q_2}(q_1) = \tilde{f}_{q_1}(q_2)$. We let to the reader the proof that the previous definition makes $\operatorname{coker}(d)$ a $\mathcal{O}_{(Y,0)}$ -algebra with e as identity element.

Notice that from the inclusion $\alpha \operatorname{coker}(d) \subset e\mathcal{O}_{(Y,0)}$ we deduce that

$$\operatorname{coker}(d) \subset e\alpha^{-1}\mathcal{O}_{(Y,0)} \subset eK(\mathcal{O}_{(Y,0)})$$

where $K(\mathcal{O}_{(Y,0)})$ is the ring of fractions of $\mathcal{O}_{(Y,0)}$. Then we define

$$B = e^{-1} \operatorname{coker}(d) \subset K(\mathcal{O}_{(Y,0)}).$$

Since e is the unit element of $\operatorname{coker}(d)$, B is an $\mathcal{O}_{(Y,0)}$ -algebra. On the other hand, since B is a finitely generated $\mathcal{O}_{(Y,0)}$ -module we get that $B \subset \overline{\mathcal{O}_{(Y,0)}} = \mathbb{C}\{t\}$.

Assume now that any of the two above equivalent conditions hold. We proved that B is a finitely generated sub- $\mathcal{O}_{(Y,0)}$ -algebra of $K(\mathcal{O}_{(Y,0)})$ so B is a domain and the extension $\mathcal{O}_{(Y,0)} \subset B$ is finite. Then B is a Cohen-Macaulay local ring of dimension one. Hence there exists a curve singularity $(X,0) \subset (\mathbb{C}^{b+2},0)$ such that $\mathcal{O}_{(X,0)} = B$.

Remark 5.4. Assume (ii) of the last result holds. Since $\operatorname{coker}(d) \cong B$ is a one-dimensional local domain then $\operatorname{Length}_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d)/e\mathcal{O}_{(Y,0)}) < +\infty$ for any $e \in \operatorname{coker}(d) \setminus \{0\}$, and then there exists $\alpha \in \mathcal{O}_{(Y,0)} \setminus \{0\}$ such that

$$\alpha \operatorname{coker}(d) \subset e\mathcal{O}_{(Y,0)} \subset \operatorname{coker}(d).$$

On the other hand, if $\operatorname{Length}_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d)/e\mathcal{O}_{(Y,0)}) < +\infty$ for an element $e \in \operatorname{coker}(d) \setminus \{0\}$ then there exists $\alpha \in \mathcal{O}_{(Y,0)} \setminus \{0\}$ satisfying the above two inclusions. The second condition of (ii) on the Ext^1 is easy computable.

Remark 5.5. With the notations of Example 4.4. We know that the $\mathcal{O}_{(Y,0)}$ -module $B = \mathcal{O}_{(Y,0)} + t\mathcal{O}_{(Y,0)} \subset \mathbb{C}\{t\}$ is not a ring. We can check this fact by using the last result. The second part of (ii) of the last theorem is equivalent to: any morphism $f: \mathcal{O}_{(Y,0)} \longrightarrow B$ can we lifted to a morphism $\tilde{f}: B \longrightarrow B$. If f is the natural inclusion then there is not a such \tilde{f} . In fact, we have tx - y = 0 in B, so $0 = \tilde{f}(tx - y) = \tilde{f}(t)x - f(y) = \tilde{f}(t)t^4 - t^3$. Hence $\tilde{f}(t)t - 1 = 0$, this is not possible in $\mathbb{C}\{t\}$.

Remark 5.6. Assume that we have the following $\mathcal{O}_{(Y,0)}$ -module extensions, where (Y,0) is a plane curve singularity,

$$\mathcal{O}_{(Y,0)} \stackrel{i}{\hookrightarrow} B \hookrightarrow \mathbb{C}\{t\} = \overline{\mathcal{O}_{(Y,0)}}.$$

Assume that B is an $\mathcal{O}_{(Y,0)}$ -algebra with i its the structural morphism. This structure is unique because is the induced by $B \subset B[c^{-1}] = \mathcal{O}_{(Y,0)}[c^{-1}]$ where c is a generator of the conductor of the ring extension $\mathcal{O}_{(Y,0)} \subset \mathbb{C}\{t\}$. Notice that this structure can be recovered as follows, see Lemma 5.3. Given $q \in B$ we consider the $\mathcal{O}_{(Y,0)}$ -linear map $f_q: \mathcal{O}_{(Y,0)} \longrightarrow B$ defined by $f_q(b) = bq$, for all $b \in \mathcal{O}_{(Y,0)}$. This map has a trivial lifting $\tilde{f}_q: B \longrightarrow B$ defined by $\tilde{f}_q(b) = bq$, for all $b \in B$. Then the product in B is

$$q_1q_2 = \tilde{f}_{q_1}(q_2) = \tilde{f}_{q_2}(q_1).$$

Notice that the lifting \tilde{f}_q is unique since the extension $\mathcal{O}_{(Y,0)} \subset \mathbb{C}\{t\}$ has a non-trivial conductor.

In the following result we refine Theorem 5.2 characterizing the matrix factorization defined by generic plane projections.

Proposition 5.7. Let $(Y,0) \subset (\mathbb{C}^2,0)$ be an irreducible plane curve singularity defined by an equation $F \in \mathbb{C}\{x,y\}$. Let (d,h) be a matrix factorization of F such that $\operatorname{coker}(d)$ is a $\mathcal{O}_{(Y,0)}$ -algebra satisfying the equivalent conditions of Theorem 5.2. Let $(X,0) \subset (\mathbb{C}^{2+b},0)$ be a curve singularity such that $\mathcal{O}_{(X,0)} \cong \operatorname{coker}(d)$ as \mathbb{C} -algebras with $b = \beta_{\mathcal{O}_{(Y,0)}}(\operatorname{coker}(d))$, Theorem 5.2. The following conditions are equivalent:

- (i) (Y,0) is the plane generic projection of (X,0),
- (ii) the cosets $\overline{x}, \overline{y} \in \mathcal{O}_{(X,0)}$ are \mathbb{C} -linear independent modulo $\mathbf{m}^2_{(X,0)}$ and $\mu(Y) = \overline{\mu}(X)$.

Proof. We write n=2+b. Assume that (Y,0) is the plane generic projection of (X,0). From Theorem 3.7 we get $\mu(Y) = \overline{\mu}(X)$. We know that the morphism

$$\pi^*: \mathcal{O}_{(Y,0)} \longrightarrow \mathcal{O}_{(X,0)} = \mathbb{C}\{x_1, \dots, x_n\}/I_X,$$

with $I_X \subset (x_1, \ldots, x_n)^2$, satisfies $\pi^*(\overline{x}) = z_1 \overline{x_1} + \cdots + z_n \overline{x_n}$ and $\pi^*(\overline{y}) = z_{n+1} \overline{x_1} + \cdots + z_{2n} \overline{x_n}$, for some $z_1, \ldots, z_{2n} \in \mathbb{C}$, and the linear variety $z_1 x_1 + \cdots + z_n x_n = z_{n+1} x_1 + \cdots + z_{2n} x_n = 0$

is of dimension n-2. Hence the cosets $\overline{x}, \overline{y} \in \mathcal{O}_{(X,0)}$ are \mathbb{C} -linear independent modulo $\mathbf{m}^2_{(X,0)}$ and we get (ii).

Assume now (ii). Notice that $\overline{x}, \overline{y} \in \mathcal{O}_{(X,0)}$ are \mathbb{C} -linear independent modulo $\mathbf{m}_{(X,0)}^2$ and $\mu(Y) = \overline{\mu}(X)$. Then, after a change of variables, we may assume that $\mathbf{m}_{(X,0)}/\mathbf{m}_{(X,0)}^2$ is generated by the cosets of x, y, x_3, \ldots, x_n , i.e. $\mathcal{O}_{(X,0)} \cong \mathbb{C}\{x, y, x_3, \ldots, x_n\}/I_X$ with $I_X \subset (x, y, x_3, \ldots, x_n)^2 \mathbb{C}\{x, y, x_3, \ldots, x_n\}$. Hence (Y, 0) is a linear projection of (X, 0) and since Y is $\mu(Y) = \overline{\mu}(X)$ we get that (Y, 0) is the plane generic projection of (X, 0), [3, Proposition IV.2 b].

Next we compare two curve singularities sharing the same generic plane projection.

Proposition 5.8. Let (Y,0) be a plane curve singularity. Let B_1 , B_2 be two $\mathcal{O}_{(Y,0)}$ -algebras

$$\mathcal{O}_{(Y,0)} \stackrel{i}{\hookrightarrow} B_i \hookrightarrow \mathbb{C}\{t\} = \overline{\mathcal{O}_{(Y,0)}}$$

i=1,2. A $\mathcal{O}_{(Y,0)}$ -linear map $\phi: B_1 \longrightarrow B_2$ is an isomorphism as $\mathcal{O}_{(Y,0)}$ -modules if and only if ϕ is an isomorphism of $\mathcal{O}_{(Y,0)}$ -algebras.

Proof. If ϕ is an isomorphism $\mathcal{O}_{(Y,0)}$ -algebras then it is an isomorphism of $\mathcal{O}_{(Y,0)}$ -modules. Assume now that ϕ is an isomorphism of $\mathcal{O}_{(Y,0)}$ -modules. I will use the notations of Remark 5.6. First we prove that for all $q_1, q_2 \in B_1$ it holds $\phi(q_1q_2) = \phi(q_1)\phi(q_2)$. Since

$$\phi(q_1q_2) = \phi(\widetilde{f_{q_1}}(q_2))$$
 and $\phi(q_1)\phi(q_2) = \widetilde{f_{\phi(q_1)}}(\phi(q_2))$

so we have to prove that

$$\phi(\widetilde{f_{q_1}}(q_2)) = \widetilde{f_{\phi(q_1)}}(\phi(q_2)).$$

We can take as lifting of $f_{\phi(q_1)}$

$$\widetilde{f_{\phi(q_1)}} = \phi \circ \widetilde{f_{q_1}} \circ \phi^{-1}.$$

Then

$$\widetilde{f_{\phi(q_1)}}(\phi(q_2)) = \phi \circ \widetilde{f_{q_1}} \circ \phi^{-1}\phi(q_2) = \phi \circ \widetilde{f_{q_1}}(q_2) = \phi(\widetilde{f_{q_1}}(q_2)).$$

Notice that $\phi \mid_{\mathcal{O}_{(Y,0)}} = Id_{\mathcal{O}_{(Y,0)}}$.

Definition 5.9. Given an irreducible series $F \in \mathbb{C}\{x,y\}$ and an integer $b \geq 2$ we denote by $\operatorname{MatRep}_{F,b}$ the set of pairs of dimension b square matrices (d,h) with entries in $(x,y)\mathbb{C}\{x,y\}$ such that

$$hd = dh = F \operatorname{Id}_{\mathbb{C}\{x,y\}^b},$$

and coker(d) satisfies the equivalent properties of Theorem 5.2 and Proposition 5.7, i.e. coker (d) is isomorphic to $\mathcal{O}_{(X,0)}$ as $\mathcal{O}_{(Y,0)}$ -modules, where (X,0) is a curve singularity with F as the equation defining a generic plane projection (Y,0) of (X,0).

We denote by $Gl_b(\mathbb{C}\{x,y\})$ the group of invertible $b \times b$ matrices with entries in $\mathbb{C}\{x,y\}$; we consider the following action in MatRep_{F,b}

$$\operatorname{MatRep}_{F,b} \times \operatorname{Gl}_b(\mathbb{C}\{x,y\})^2 \stackrel{\circ}{\longrightarrow} \operatorname{MatRep}_{F,b}$$
$$((d,h),(\phi,\psi)) \mapsto (\phi,\psi) \circ (d,h) = (\phi d\psi^{-1}, \psi h \phi^{-1})$$

Definition 5.10. Given a plane branch singularity $(Y,0) \subset (\mathbb{C}^2,0)$ we denote by $\mathcal{G}^n_{(Y,0),b}$, $n \geq 3$, the set of curve singularities $(X,0) \subset (\mathbb{C}^n,0)$ such that (Y,0) is a generic plane projection of (X,0) and $b = \beta_{\mathcal{O}_{(Y,0)}}(\mathcal{O}_{(X,0)})$. For each such a curve singularity (X,0) we choose a projection $\pi_L : (X,0) \longrightarrow (Y,0)$, $L \in W_{(X,0)}$, equivalently an immersion $\pi_L^* : \mathcal{O}_{(Y,0)} \longrightarrow \mathcal{O}_{(X,0)}$.

We consider in $\mathcal{G}^n_{(Y,0),b}$, $n \geq 3$ the following binary equivalence relation: given $(X_i,0) \in \mathcal{G}^n_{(Y,0),b}$, i=1,2, we write $(X_1,0) \sim (X_2,0)$ if and only if $\mathcal{O}_{(X_1,0)} \cong \mathcal{O}_{(X_2,0)}$ as $\mathcal{O}_{(Y,0)}$ -algebras, see Proposition 5.8.

Remark 5.11. Recall that two curve singularities $(X_i, 0) \subset (\mathbb{C}^n, 0)$, i = 1, 2, share the same generic plane projection if and only if $(X_1, 0)$ and $(X_2, 0)$ have isomorphic saturations, [2, Theorem VI.0.2].

Theorem 5.12. There is a bijective map

$$\mathcal{G}^n_{(Y,0),b}/\sim \stackrel{\epsilon}{\longrightarrow} \operatorname{MatRep}_{F,b}/Gl_b(\mathbb{C}\{x,y\})^2$$

$$(X,0) \mapsto \overline{(d,h)}$$

where (d,h) is a matrix factorization of F, where F is an equation defining (Y,0).

Proof. First we prove that the map ϵ is defined. We already proved in Theorem 4.2 that for all $(X,0) \in \mathcal{G}^n_{(Y,0),b}$ there exists a matrix factorization (d,h) of F inducing a minimal free resolution

$$0 \longrightarrow \mathbb{C}\{x,y\}^b \stackrel{d}{\longrightarrow} \mathbb{C}\{x,y\}^b \longrightarrow \mathcal{O}_{(X,0)} \longrightarrow 0$$

Given a second matrix factorization (d', h') of F we have a second minimal resolution

$$0 \longrightarrow \mathbb{C}\{x,y\}^b \stackrel{d'}{\longrightarrow} \mathbb{C}\{x,y\}^b \longrightarrow \mathcal{O}_{(X,0)} \longrightarrow 0$$

Both minimal resolution are isomorphic so there are $\phi, \psi \in Gl_b(\mathbb{C}\{x,y\})^2$ such that $\phi d = d'\psi$. Then $\phi d\psi^{-1} = d'$ and $\psi h\phi^{-1} = h'$, so $\overline{(d,h)} = \overline{(d',h')}$.

The map ϵ is exhaustive. Let $\overline{(d,h)}$ be an element of $\operatorname{MatRep}_{F,b}/Gl_b(\mathbb{C}\{x,y\})^2$. That means that $\operatorname{coker}(d) \cong \mathcal{O}_{(X,0)}$ as $\mathcal{O}_{(Y,0)}$ -modules, where (X,0) is a curve singularity with (Y,0) as a generic plane projection. Hence $\epsilon(X,0) = \overline{(d,h)}$.

The map ϵ is injective. Let $(X_1,0)$, $(X_2,0)$ be curve singularities of $\mathcal{G}^n_{(Y,0),b}$ such that $\epsilon(X_1,0) = \epsilon(X_2,0)$. This implies that $\mathcal{O}_{(X_1,0)} \cong \mathcal{O}_{(X_2,0)}$ as $\mathcal{O}_{(Y,0)}$ -modules. From Proposition 5.8 that isomorphism is of $\mathcal{O}_{(Y,0)}$ -algebras, so ϵ is injective.

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