

COMBINATORICS IN (2,1)-CATEGORIES

KRISTA ZEHR

ABSTRACT. Groupoid cardinality is an invariant of locally finite groupoids which has many of the properties of the cardinality of finite sets, but which takes values in all non-negative real numbers, and accounts for the morphisms of a groupoid. Several results on groupoid cardinality are proved, analogous to the relationship between cardinality of finite sets and i.e. injective or surjective functions. We also generalize to a broad class of (2,1)-categories a famous theorem of Lovász which characterizes the isomorphism type of relational structures by counting the number of homomorphisms into them.

1. Introduction

When studying mathematical objects defined as finite sets with extra structure, cardinality is a very powerful tool. While it is a complete invariant of finite sets up to bijection, this is not the main source of its power. For example, monomorphisms or epimorphisms enforce an order on the cardinalities of their source and target objects. Moreover, the cardinalities of the source and target of a morphism can enforce restrictions on the properties of that morphism. Categorical constructions such as products or coproducts reduce to algebraic equations involving the cardinalities of the component objects.

It was shown in [Lovász \[1967\]](#) that in a category of finite relational structures, it is enough to count the number of homomorphisms into an object from every other object to determine its isomorphism type. This result can be extended to many other locally finite categories. One way of understanding this result via the Yoneda embedding of a category $\mathcal{C} \rightarrow [\mathcal{C}^{op}, \mathbf{FinSet}]$. That is, given two representable functors h_A and h_B , and a *natural* isomorphism between them, the Yoneda lemma states that A and B must be isomorphic in \mathcal{C} . However, if we weaken the natural isomorphism to just a family of isomorphisms $\mathrm{Hom}(X, A) \cong \mathrm{Hom}(X, B)$ for each object X , then this is no longer true. Since cardinality is a complete invariant of sets up to isomorphism, Lovász’s result and its generalizations gives conditions where we can drop the naturality assumption and still conclude that $A \cong B$.

When discussing higher categories, the morphisms between any two objects no longer form a set, but a category, so an alternative invariant needs to be used. One invariant is the Euler characteristic of a category (c.f. [Leinster \[2006\]](#)). The Euler characteristic is an invariant up to equivalence (in fact, up to adjunction), and satisfies algebraic formula for products and coproducts identical to those for cardinality of finite sets. The downside

Thanks to Justin Curry and Marco Varisco for their feedback and for suggesting several of the questions investigated in this project.

2020 Mathematics Subject Classification: 18A99, 18A22, 18A25, 18A32.

Key words and phrases: groupoid cardinality, stuff types, (2,1)-categories, factorization systems.

© Krista Zehr, 2025. Permission to copy for private use granted.

is that it is hard to compute in general, and the categories that have well-defined Euler characteristic are not easy to classify. However, if the category in question is a groupoid, then the Euler characteristic is known as groupoid cardinality, and has a simple formula.

Groupoid cardinality essentially counts objects, but only up to isomorphism. So the cardinality of a discrete groupoid is equal to the number of objects, the cardinality of an equivalence relation is equal to the number of equivalence classes. Moreover, when the groupoid is not a preorder, we may have fractional cardinality. For instance, if G is a finite group, then the cardinality of the delooping groupoid BG is the reciprocal of the order of G . In this way, groupoid cardinality can be a useful invariant when we want to consider symmetries or equivalences between objects.

The goal of this paper is to further study the properties of groupoid cardinality analogous to the useful properties of set cardinality that were previously mentioned. In Section 3, we study the cardinalities of exponential objects in the category of groupoids \mathbf{Grpd} . These cardinalities are hard to compute in general, and have much more complex formulas than the simple exponential formula for the cardinality of a set of functions between finite sets. In Section 4, we relate properties of functors to groupoid cardinality.

Moving beyond the category of groupoids, we investigate other $(2,1)$ -categories. In such categories, the morphisms between objects form a groupoid, so we may calculate their groupoid cardinality. One example is the category of stuff types (c.f. [Morton \[2006\]](#)), a generalization of Joyal’s combinatorial species, which are themselves a sort of categorification of generating functions in combinatorics. Stuff types are simply functors from an arbitrary groupoid into the groupoid of finite sets and bijections. This picture can be generalized very broadly, replacing the groupoid of finite sets with any locally finite groupoid. One specific example mentioned in Section 3 is the groupoid of finite dimensional vector spaces over a finite field and the invertible linear maps between them. Section 5 discusses such $(2,1)$ -categories as well as a certain finiteness condition on the functors in them that is essential for combinatorial analysis.

The main theorem of this paper is a generalization of Lovász’s theorem to such categories of functors, and is proven in Section 6. Finally, in Section 7 we discuss possible future directions of this work to $(\infty,1)$ -categories, using the notion of homotopy cardinality of an ∞ -groupoid.

2. Background

2.0.1. GROUPOID CARDINALITY. Groupoid cardinality was first introduced in [Baez and Dolan \[2000\]](#) as a generalization of cardinality of finite sets which can take values of any non-negative real number.

Let \mathcal{G} be an essentially small groupoid. Denote the set of all isomorphism classes of \mathcal{G} by $\overline{\mathcal{G}}$. If x is an object of \mathcal{G} , then $[x]$ denotes the isomorphism class of x , and \mathcal{G}_x is the group of automorphisms of x . For the remainder of this paper, we will denote the cardinality of a finite set X by $\#X$, in order to avoid confusion with groupoid cardinality.

2.1. DEFINITION. Let \mathcal{G} be a locally finite, essentially small groupoid. Then the **groupoid cardinality** of \mathcal{G} is

$$|\mathcal{G}| = \sum_{[x] \in \overline{\mathcal{G}}} \frac{1}{\#\mathcal{G}_x}$$

We call such a groupoid **tame** if $|\mathcal{G}| < \infty$.

The following properties of groupoid cardinality are easy to verify.

2.2. PROPOSITION. Let \mathcal{G}, \mathcal{H} be tame groupoids.

- (a) If $\mathcal{G} \simeq \mathcal{H}$ then $|\mathcal{G}| \simeq |\mathcal{H}|$
- (b) $|\mathcal{G} \times \mathcal{H}| = |\mathcal{G}| |\mathcal{H}|$
- (c) $|\mathcal{G} \sqcup \mathcal{H}| = |\mathcal{G}| + |\mathcal{H}|$
- (d) $|\mathcal{G}| = 0$ if and only if \mathcal{G} is the empty groupoid.
- (e) If \mathcal{G} is finite, then $|\mathcal{G}|$ is rational.

2.3. PROPOSITION. Let G be a finite group, acting on a finite set X . Then the cardinality of the action groupoid $X//G$ is

$$|X//G| = \frac{\#X}{\#G}$$

Clearly, groupoid cardinality has similar formal properties to the cardinality of finite sets. Note that the category of functors between groupoids is again a groupoid, so it is natural to ask if the equation

$$|\mathcal{G}^{\mathcal{H}}| = |\mathcal{G}|^{|\mathcal{H}|}$$

holds for tame groupoids, as it does for the cardinality of finite sets. However, this fails miserably. In general the functor category between two tame groupoids is not even locally finite.

When \mathcal{H} is finite, the functor category is locally finite, however this equation still fails to hold. For example, consider the functors from the cyclic group C_2 to the discrete groupoid with 2 objects. This is obviously a finite groupoid. However, if the above equation were true, then the cardinality would be $\sqrt{2}$ which is not rational, contradicting Proposition 2.2(e). We will investigate cardinalities of functor groupoids in Section 3.

2.4. LOVÁSZ'S HOMOMORPHISM COUNTING LEMMA. In this section we will discuss the result in Lovász [1967] that distinguishes isomorphism types by counting homomorphisms into an object. The proof of this statement is elementary, and very simple to understand, so we will provide it in full here.

2.5. DEFINITION. A **finite relational structure** is a tuple (A, R_1, \dots, R_n) where A is a finite set and $R_i \subseteq A^{a_i}$ are relations on A with arity a_i (we assume n and the arities a_i to be fixed). A **homomorphism of relational structures** from (A, R) to (B, S) is a function $f : A \rightarrow B$ between the underlying sets such that $f(R_i) \subseteq S_i$ for all i . That is, f preserves the relations.

2.6. THEOREM. *Lovász [1967] For finite relational structures (A, R) , (B, S) if for every structure (C, T) , we have $\text{hom}((C, T), (A, R)) = \text{hom}((C, T), (B, S))$, then $(A, R) \cong (B, S)$. Where $\text{hom}((C, T), (A, R))$ is the number of homomorphisms from (C, T) to (A, R) .*

PROOF. In the first half of the proof, we prove by induction on the cardinality of C that if $\text{hom}((C, T), (A, R)) = \text{hom}((C, T), (B, S))$ for all (C, T) , then it is also true that $\text{inj}((C, T), (A, R)) = \text{inj}((C, T), (B, S))$, where $\text{inj}((C, T), (A, R))$ is the number of injective homomorphisms from (C, T) to (A, R) .

For the base case, if $C = \emptyset$, then $\text{inj}((C, T), (A, R)) = \text{inj}((C, T), (B, S)) = 1$ trivially.

For the inductive case, assume that $\text{inj}((C, T), (A, R)) = \text{inj}((C, T), (B, S))$ for all (C, T) such that $|C| < N$ for some $N > 0$. Consider that every homomorphism of relational structures $f : (C, T) \rightarrow (A, R)$ decomposes uniquely as $f = m \circ e$ where m is an injective homomorphism and e is a surjective homomorphism. (This is easy to check). Therefore, we have the equation

$$\text{hom}((C, T), (A, R)) = \sum_{\theta} \text{inj}((C/\theta, T), (A, R))$$

where the sum is over equivalence relations θ on C , interpreting C/θ as the image of a homomorphism in A . Now, fix (C, T) with $|C| = N$. Then we have

$$\begin{aligned} 0 &= \text{Hom}((C, T), (A, R)) - \text{Hom}((C, T), (B, S)) \\ &= \sum_{\theta} \text{inj}((C/\theta, T), (A, R)) - \text{inj}((C/\theta, T), (B, S)) \\ &= \text{inj}((C, T), (A, R)) - \text{inj}((C, T), (B, S)) + \sum_{\theta \text{ nontrivial}} \text{inj}((C/\theta, T), (A, R)) - \text{inj}((C/\theta, T), (B, S)) \end{aligned}$$

By our induction hypothesis, the terms in the sum on the right are all zero, therefore $\text{inj}((C, T), (A, R)) = \text{inj}((C, T), (B, S))$.

For the second half of the proof, we set $(C, T) = (A, R)$, so that $\text{inj}((A, R), (A, R)) = \text{inj}((A, R), (B, S)) \neq 0$. So there exists some injective homomorphism from (A, R) to (B, S) . Setting $(C, T) = (B, S)$, we can see that there also exists an injective homomorphism from (B, S) to (A, R) . Since these homomorphisms are functions of sets, we can see that they must be isomorphisms, which proves that $(A, R) \cong (B, S)$. ■

This theorem is very general, and yet the proof is very easy. It is easy to see how the basic structure of the proof could be generalized to other areas, and indeed, a more categorical description was developed in Lovász [1972] and Pultr [1973].

2.7. FACTORIZATION SYSTEMS IN $(2,1)$ -CATEGORIES. The proof of Theorem 2.6 relies on the existence of image factorizations. In this section we will review the theory of factorization systems in a 2-category. The following definition is from Kasangian and Vitale [2000].

2.8. DEFINITION. A **factorization system** in a (2,1)-category \mathcal{C} is a pair $(\mathcal{E}, \mathcal{M})$ of classes of 1-morphisms such that:

1. \mathcal{E} and \mathcal{M} contain all equivalences and are closed under composition with equivalences,
2. \mathcal{E} and \mathcal{M} are closed under 2-isomorphism classes of 1-morphisms,
3. for every 1-morphism $f : X \rightarrow Y$, there exist e in \mathcal{E} , m in \mathcal{M} and a 2-cell $\eta : me \Rightarrow f$.
4. Finally, \mathcal{E}, \mathcal{M} are required to satisfy an orthogonality or "fill-in" condition. Given any diagram:

$$\begin{array}{ccc} A & \xrightarrow{e} & B \\ u \downarrow & \swarrow \varphi & \downarrow v \\ C & \xrightarrow{m} & D \end{array}$$

with e in \mathcal{E} and m in \mathcal{M} , there exists $w : B \rightarrow C$ and 2-morphisms $\alpha : we \Rightarrow u$ and $\beta : mw \Rightarrow v$ such that $\varphi \circ (\beta * 1_e) = 1_m * \alpha$. Moreover, if (w', α', β') is another such fill-in, then then there exists a unique $\psi : w \Rightarrow w'$ such that $\alpha'(\psi * 1_e) = \alpha$ and $\beta'(1_m * \psi) = \beta$.

By Proposition 9.3 in Kasangian and Vitale [2000], we can consider \mathcal{E} and \mathcal{M} as subcategories of the underlying 1-category of \mathcal{C} . Furthermore, factorizations of morphisms are unique in the following sense:

2.9. PROPOSITION. If $(e : X \rightarrow U, m : U \rightarrow Y, \eta)$ and $(e' : X \rightarrow U', m' : U' \rightarrow Y, \eta')$ are two factorizations of f , then there exists an equivalence $w : U \rightarrow U'$ and 2-cells $\varphi : we \Rightarrow e', \psi : m'w \Rightarrow m$ such that $\eta'(1_{m'} * \varphi) = \eta(\psi * 1_e)$. Moreover w is unique up to a unique coherent 2-isomorphism.

The proof of this proposition is a straightforward exercise in using the orthogonality condition.

This definition of a factorization system is reminiscent of epi-mono factorization systems in ordinary 1-categories. Indeed, it makes sense to consider \mathcal{E} to be akin to quotient maps, and \mathcal{M} to be akin to subobject inclusions. Often, quotient maps and subobject inclusions have a variety of nice properties beyond being factors of morphisms, which motivates the following definition:

2.10. DEFINITION. [Dupont and Vitale, 2003, Definition 3.1] Let $f : C \rightarrow C'$ be a morphism in a 2-category \mathcal{C}

1. We say f is **fully faithful** if for all objects X of \mathcal{C} , the induced functor $f_* : \mathcal{C}(X, C) \rightarrow \mathcal{C}(X, C')$ is fully faithful.
2. We say f is **fully cofaithful** if for all objects X of \mathcal{C} , the induced functor $f^* : \mathcal{C}(C', X) \rightarrow \mathcal{C}(C, X)$ is fully faithful.

In section 4, we provide a number of results relating groupoid cardinality to properties of functors between tame groupoids (i.e. full, faithful, essentially surjective). Since Theorem 6.4 refers to more general (2,1)-categories than \mathbf{Grpd} , these properties of morphisms in a 2-category will allow us to use these results in this more general setting.

For small categories (and therefore, small groupoids) these definitions agree with well-understood properties of functors.

2.11. PROPOSITION. [Dupont and Vitale, 2003, Proposition 7.8] *Let $F : \mathcal{C} \rightarrow \mathcal{D}$ be a functor between small categories. Then*

1. *F is fully faithful in the sense of Definition 2.10 if and only if F is fully faithful as a functor.*
2. *F is fully cofaithful if F is essentially surjective and full as a functor.*

Factorizing morphisms into a composite of two morphisms, one a "quotient" and the other an "embedding", is a common and well-understood theme in mathematics. When moving to 2-categories, an extra layer of subtlety arises. Namely, there is more than one obvious choice of factorization system, leading to the following definition:

2.12. DEFINITION. A **ternary factorization system** is a pair of factorization systems $(\mathcal{E}, \widetilde{\mathcal{M}})$ and $(\widetilde{\mathcal{E}}, \mathcal{M})$ such that $\mathcal{E} \subseteq \widetilde{\mathcal{E}}$ and $\mathcal{M} \subseteq \widetilde{\mathcal{M}}$.

We call this a ternary factorization system because each morphism factors uniquely as three morphisms rather than two. Given a morphism φ , we first factor φ as $\tilde{m}e$ with e in \mathcal{E} and \tilde{m} in $\widetilde{\mathcal{M}}$, and then factor \tilde{m} as mf for m in \mathcal{M} and f in $\widetilde{\mathcal{E}}$. Alternatively, we may factor φ with $(\widetilde{\mathcal{E}}, \mathcal{M})$ first and then with $(\mathcal{E}, \widetilde{\mathcal{M}})$. The orthogonality condition ensures us that both of these factorizations are equivalent. Thus, we may consider a ternary factorization system to be a triple of subcategories $(\mathcal{E}, \mathcal{F}, \mathcal{M})$ with $\mathcal{F} = \widetilde{\mathcal{E}} \cap \widetilde{\mathcal{M}}$. With this definition, each morphism factors as a composition mfe with e in \mathcal{E} , f in \mathcal{F} , and m in \mathcal{M} .

Given such a triple, we can recover the original pair of ordinary factorization systems by defining $\widetilde{\mathcal{M}}$ to be the compositions of morphisms in \mathcal{F} followed by morphisms in \mathcal{M} , and likewise $\widetilde{\mathcal{E}}$ as compositions of morphisms in \mathcal{E} and \mathcal{F} . We will switch between both of these views of a ternary factorization system as convenient.

The 2-category \mathbf{Cat} of small categories and functors has a well-known ternary factorization system $(\mathcal{E}, \mathcal{F}, \mathcal{M})$ with \mathcal{E} consisting of all essentially surjective and full functors, \mathcal{F} consisting of all essentially surjective and faithful functors, and \mathcal{M} consisting of all fully-faithful functors. This factorization system restricts to small groupoids as well. That is to say, given a functor $F : \mathcal{G} \rightarrow \mathcal{H}$ between small groupoids, we can factor F in \mathbf{Cat} as

$$G \xrightarrow{F_2} \mathrm{im}_2 F \xrightarrow{F_1} \mathrm{im}_1 F \xrightarrow{F_0} \mathcal{H}$$

and the intermediate categories $\mathrm{im}_2 F$ and $\mathrm{im}_1 F$ will be groupoids. Since it will be useful later, we will give an explicit construction of this factorization and the intermediate categories for an arbitrary functor $F : \mathcal{G} \rightarrow \mathcal{H}$ between categories.

The 2-coimage of F is the functor $F_2 : \mathcal{G} \rightarrow \text{im}_2 F$. The objects of $\text{im}_2 F$ are the objects of \mathcal{G} and the morphisms are equivalence classes of morphisms in \mathcal{G} where $f \sim f'$ if and only if $F(f) = F(f')$. The functor F_2 acts as the identity on objects and sends morphisms to their corresponding equivalence class.

The 1-image of F is the functor $F_0 : \text{im}_1 F \rightarrow \mathcal{H}$. Where $\text{im}_1 F$ is the full subcategory of \mathcal{H} consisting of all objects Y such that $Y \cong F(X)$ for some object X of \mathcal{G} . The functor F_0 is the inclusion of this subcategory.

The functor $F_1 : \text{im}_2 F \rightarrow \text{im}_1 F$ sends each object X to $F(X)$ and each equivalence class of morphisms $[f]$ to $F(f)$.

3. Cardinalities of functor groupoids

Let \mathcal{H} be a finite groupoid and \mathcal{G} a locally-finite groupoid. Then the functor groupoid $\mathcal{G}^{\mathcal{H}}$ is locally finite and

$$|\mathcal{G}^{\mathcal{H}}| = \prod_{[y] \in \overline{\mathcal{H}}} \sum_{[x] \in \overline{\mathcal{G}}} \frac{\#\text{Hom}(\mathcal{H}_y, \mathcal{G}_x)}{\#\mathcal{G}_x} \quad (1)$$

This formula is not very useful for computing cardinalities, as it requires counting homomorphisms between arbitrary finite groups, which is a hard problem in general. However, if \mathcal{G} is the groupoid of finite sets and bijections, then Corollary 2.1.1 gives a nice formula for the cardinality. In the particular case where $\mathcal{H} \simeq BG$ for some finite group G , this groupoid is equivalent to the core of the category of finite G -Sets, which comes with a forgetful functor $U : \text{core}(G\text{-FinSet}) \rightarrow \text{FinSet}$, making it a stuff type (cf. Morton [2006]). We can then provide a nice formula for the generating function of U . Then the cardinality of the functor groupoid $\text{FinSet}^{\text{BG}} \cong \text{core}(G\text{-FinSet})$ is given by evaluating this generating function at 1.

3.1. THEOREM. *Let G be a finite group. Then the generating function of the stuff type $U : \text{core}(G\text{-FinSet}) \rightarrow \text{FinSet}$ is*

$$|U|(x) = \exp \left(\sum_{[H \leq G]} \frac{x^{[H:G]}}{\#C_{\text{Sym}(G/H)}(\text{im } \theta)} \right)$$

Where the sum is taken over subgroups $H \leq G$ up to conjugacy, and $\theta : G \rightarrow \text{Sym}(G/H)$ is the homomorphism defining the usual action of G on left cosets.

PROOF. First, for each $H \leq G$ we want to find the generating function Φ_H of finite G -sets where all orbits are isomorphic to G/H . Clearly, the cardinality of such G -sets must be a multiple of $[H : G]$, so the coefficients of Φ_H are supported only on $x^{[H:G]n}$. In order to compute these coefficients, we need to find the order of $\text{Aut}(G/H \times [n])$. Each such automorphism is described by the data of a permutation π of the n orbits, as well as an n -tuple (ϕ_i) of automorphisms of G/H . Since the automorphism group of G/H is $C_{\text{Sym}(G/H)}(\text{im } \theta)$, we have

$$\#\text{Aut}(G/H \times [n]) = n!(\#C_{\text{Sym}(G/H)}(\text{im } \theta))^n.$$

Therefore, the formula for the generating function Φ_H is

$$\begin{aligned}\Phi_H(x) &= \sum_{n=0}^{\infty} \frac{x^{[H:G]n}}{n!(\#C_{\text{Sym}(G/H)}(\text{im } \theta))^n} \\ &= \exp\left(\frac{x^{[H:G]}}{\#C_{\text{Sym}(G/H)}(\text{im } \theta)}\right)\end{aligned}$$

Now, in order to put a G -action on an arbitrary finite set, we can partition it into multiple disjoint (possibly empty) subsets - one for each conjugacy class $[H]$ of subgroups of G - and give each one the structure of a G -set where all orbits are isomorphic to G/H . This operation corresponds to taking the product of generating functions, so we have:

$$\begin{aligned}|U|(x) &= \prod_{[H \leq G]} \Phi_H(x) \\ &= \exp\left(\sum_{[H \leq G]} \frac{x^{[H:G]}}{\#C_{\text{Sym}(G/H)}(\text{im } \theta)}\right)\end{aligned}$$

■

3.2. COROLLARY. *For a finite groupoid \mathcal{G} , the cardinality of $\text{FinSet}_{\cong}^{\mathcal{G}}$ is*

$$\exp\left(\sum_{[x] \in \overline{\mathcal{G}}} \sum_{[H \leq \mathcal{G}_x]} \frac{1}{\#C_{\text{Sym}(\mathcal{G}_x/H)}(\text{im } \theta)}\right)$$

This result uses the so called convolution product or Cauchy product of stuff types, which relies only on the fact that FinSet has coproducts. Replacing FinSet with another locally finite category with coproducts, such as the category $\text{Vect}_{\text{f.d.}}(\mathbb{F}_q)$ of finite dimensional vector spaces over the field \mathbb{F}_q lets us prove a similar result:

Let G be a finite group and \mathbb{F}_q a finite field whose characteristic does not divide the order of G . The groupoid of all finite representations of G over \mathbb{F}_q has a functor $U : \text{core}(\text{Rep}_{\text{f.d.}}(G, \mathbb{F}_q)) \rightarrow \text{Vect}_{\text{f.d.}}(\mathbb{F}_q)$ sending each representation to its underlying vector space. We can define the generating function of U similarly by

$$|U|(x) = \sum_{n=0}^{\infty} |\mathcal{V}_n| x^n$$

where \mathcal{V}_n is the subgroupoid of all representations of dimension n .

3.3. THEOREM. *The generating function of $U : \text{core}(\text{Rep}_{\text{f.d.}}(G, \mathbb{F}_q)) \rightarrow \text{Vect}_{\text{f.d.}}(\mathbb{F}_q)$ is*

$$|U|(x) = \exp\left(\sum_V \frac{x^{\dim V}}{\#\text{Aut}(V)}\right)$$

where the sum is taken over irreducible representations of G over \mathbb{F}_q .

PROOF. For an irreducible representation V , we can form the generating function Φ_V for all representations of the form V^n , which equals

$$\begin{aligned}\Phi_V(x) &= \sum_{n=0}^{\infty} \frac{x^{n \dim V}}{n! \cdot (\#\text{Aut}(V))^n} \\ &= \exp\left(\frac{x^{\dim V}}{\#\text{Aut}(V)}\right)\end{aligned}$$

Then since the characteristic of \mathbb{F}_q does not divide the order of G , Maschke's theorem states that every representation is a direct sum of irreducibles, and thus

$$|U|(x) = \prod_V \Phi_V(x) = \exp\left(\sum_V \frac{x^{\dim V}}{\#\text{Aut}(V)}\right)$$

■

4. Morphisms between tame groupoids

In this section, we prove some facts relating properties of functors to the groupoid cardinalities of their source and target groupoids, analogous to the relationship between properties of functions and cardinality of sets.

4.1. PROPOSITION. *Let $\varphi : \mathcal{G} \rightarrow \mathcal{H}$ be a functor between tame groupoids.*

(a) *If φ is full, then $|\mathcal{G}| \leq |\mathcal{H}|$.*

(b) *If φ is essentially surjective and faithful, then $|\mathcal{G}| \geq |\mathcal{H}|$.*

PROOF. (a) Since φ is full, it induces an injective map on isomorphism classes. So, the formula for $|\mathcal{H}|$ gives

$$|\mathcal{H}| = \sum_{[x] \in \bar{\mathcal{G}}} \frac{1}{|\mathcal{H}_{\varphi(x)}|} + \sum_{[y] \in \bar{\mathcal{H}} \setminus \varphi(\bar{\mathcal{G}})} \frac{1}{|\mathcal{H}_y|}.$$

In addition, φ induces surjective maps $\mathcal{G}_x \rightarrow \mathcal{H}_{\varphi(x)}$, so

$$\frac{1}{|\mathcal{G}_x|} \leq \frac{1}{|\mathcal{H}_{\varphi(x)}|}$$

Therefore,

$$|\mathcal{G}| = \sum_{[x] \in \bar{\mathcal{G}}} \frac{1}{\#\mathcal{G}_x} \leq \sum_{[x] \in \bar{\mathcal{G}}} \frac{1}{\#\mathcal{H}_{\varphi(x)}} \leq \sum_{[x] \in \bar{\mathcal{G}}} \frac{1}{\#\mathcal{H}_{\varphi(x)}} + \sum_{[y] \in \bar{\mathcal{H}} \setminus \varphi(\bar{\mathcal{G}})} \frac{1}{\#\mathcal{H}_y} = |\mathcal{H}|.$$

(b) Since φ is essentially surjective, it induces a surjective map on isomorphism classes, so the formula for $|\mathcal{G}|$ gives

$$|\mathcal{G}| = \sum_{[y] \in \overline{\mathcal{H}}} \left(\sum_{[x] \in \varphi^{-1}([y])} \frac{1}{\#\mathcal{G}_x} \right)$$

Since φ is faithful, for each $[x]$ in $\overline{\mathcal{G}}$, φ induces an injective group homomorphism $\mathcal{G}_x \rightarrow \mathcal{H}_{\varphi(x)}$, so

$$\frac{1}{\#\mathcal{G}_x} \geq \frac{1}{\#\mathcal{H}_{\varphi(x)}}$$

Therefore,

$$|\mathcal{G}| = \sum_{[y] \in \overline{\mathcal{H}}} \left(\sum_{[x] \in \varphi^{-1}([y])} \frac{1}{\#\mathcal{G}_x} \right) \geq \sum_{[y] \in \overline{\mathcal{H}}} \frac{\#\varphi^{-1}([y])}{\#\mathcal{H}_y} \geq \sum_{[y] \in \overline{\mathcal{H}}} \frac{1}{\#\mathcal{H}_y} = |\mathcal{H}|.$$

■

4.2. THEOREM. Let \mathcal{G}, \mathcal{H} be tame groupoids such that $|\mathcal{G}| = |\mathcal{H}|$, and let $\varphi : \mathcal{G} \rightarrow \mathcal{H}$ be a functor that is either

- (a) essentially surjective and full,
- (b) essentially surjective and faithful, or
- (c) fully faithful.

Then φ is an equivalence.

PROOF.

- (a) Since φ is essentially surjective and full, it induces a bijection on isomorphism classes. So since $|\mathcal{G}| = |\mathcal{H}|$,

$$|\mathcal{G}| = \sum_{[x] \in \overline{\mathcal{G}}} \frac{1}{\#\mathcal{G}_x} = \sum_{[x] \in \overline{\mathcal{G}}} \frac{1}{\#\mathcal{H}_{\varphi(x)}} = |\mathcal{H}|$$

Additionally, since φ is full, it induces surjective group homomorphisms on all vertex groups, which means for all objects x of \mathcal{G} ,

$$\frac{1}{\#\mathcal{G}_x} \leq \frac{1}{\#\mathcal{H}_{\varphi(x)}}$$

Suppose for contradiction that this inequality is strict for some object x_0 of \mathcal{G} . So

$$0 < \frac{1}{\#\mathcal{H}_{\varphi(x_0)}} - \frac{1}{\#\mathcal{G}_{x_0}} = \sum_{\substack{[x] \in \overline{\mathcal{G}} \\ x \neq x_0}} \left(\frac{1}{\#\mathcal{G}_x} - \frac{1}{\#\mathcal{H}_{\varphi(x_0)}} \right)$$

However, this is a contradiction, since every term in the sum on the right must be non-positive. Therefore the induced map on each vertex group must be a bijection, making φ and equivalence.

(b) Since φ is essentially surjective, it induces a surjection on isomorphism classes. So,

$$|\mathcal{G}| = \sum_{[y] \in \overline{\mathcal{H}}} \sum_{\substack{[x] \in \overline{\mathcal{G}} \\ \varphi(x) \cong y}} \frac{1}{\#\mathcal{G}_x} = \sum_{[y] \in \overline{\mathcal{H}}} \frac{1}{\#\mathcal{H}_y} = |\mathcal{H}|.$$

Since φ is faithful, the induced maps on vertex groups are injective, so for $\varphi(x) \cong y$,

$$\frac{1}{\#\mathcal{H}_y} \leq \frac{1}{\#\mathcal{G}_x}, \text{ and}$$

$$\frac{1}{\#\mathcal{H}_y} \leq \sum_{\substack{[x] \in \overline{\mathcal{G}} \\ \varphi(x) \cong y}} \frac{1}{\#\mathcal{G}_x}.$$

Using the same argument as part (a), we see that the second inequality cannot be strict. Following from the first inequality, the sum on the right must contain only one term, which is equal to the left hand side. Thus the induced maps on vertex groups are all isomorphisms, and so φ is full, making it an equivalence.

(c) Since φ is fully faithful, we can treat \mathcal{G} as a full subgroupoid of \mathcal{H} , thus

$$|\mathcal{G}| = \sum_{[x] \in \overline{\mathcal{G}}} \frac{1}{\#\mathcal{G}_x} = \sum_{[x] \in \overline{\mathcal{G}}} \frac{1}{\#\mathcal{G}_x} + \sum_{[y] \in \overline{\mathcal{H}} \setminus \overline{\mathcal{G}}} \frac{1}{\#\mathcal{H}_y} = |\mathcal{H}|$$

Clearly, for these sums to be equal, $\overline{\mathcal{H}} \setminus \overline{\mathcal{G}} = \emptyset$, so φ is essentially surjective, and therefore an equivalence. ■

4.3. THEOREM. *Let \mathcal{G}, \mathcal{H} be tame groupoids, and let $\varphi : \mathcal{G} \rightarrow \mathcal{H}$ and $\psi : \mathcal{H} \rightarrow \mathcal{G}$ be functors.*

(a) *If φ and ψ are both full, then they are both equivalences.*

(b) *If φ and ψ are both essentially surjective and faithful, then they are both equivalences.*

PROOF. (a) Let x_0 be an object of \mathcal{G} , and denote $x_n = (\psi\varphi)^n(x_0)$. Since $\psi\varphi$ is full, we get a chain of surjective group homomorphisms on the vertex groups of the x_n :

$$\mathcal{G}_{x_0} \twoheadrightarrow \mathcal{G}_{x_1} \twoheadrightarrow \mathcal{G}_{x_2} \twoheadrightarrow \dots$$

Since all of these groups are finite, the chain eventually stabilizes. So for some M , and all $m \geq M$, $\mathcal{G}_{x_m} \cong \mathcal{G}_{x_{m+1}}$. If we assume that no two of the x_n are isomorphic in \mathcal{G} ,

then \mathcal{G} contains an infinite number of components with vertex groups all the same size, which contradicts the assumption that \mathcal{G} is tame. Thus, it must be the case that there is some n and m such that $x_{m+n} \cong x_m$. Then, since full functors induce injective functions on isomorphism classes, $x_n \cong x_0$. This implies that $\psi\varphi$ is essentially surjective, and it then follows that ψ is essentially surjective. Also, since \mathcal{G}_{x_0} and \mathcal{G}_{x_n} are isomorphic finite groups, it must be that each $\mathcal{G}_{x_k} \rightarrow \mathcal{G}_{x_{k+1}}$ is actually a bijection, so $\psi\varphi$ is faithful, and it then follows that φ is faithful. We can use the same procedure to show that φ is essentially surjective and ψ is faithful. Thus φ and ψ are full, faithful, and essentially surjective, and therefore are equivalences.

(b) Proposition 4.1 says that $|\mathcal{G}| \geq |\mathcal{H}|$ and $|\mathcal{H}| \geq |\mathcal{G}|$. Therefore $|\mathcal{G}| = |\mathcal{H}|$. Then Theorem 4.2 states that φ and ψ are equivalences. ■

5. Relatively-finite functors

A large class of (2,1)-categories with nice combinatorial properties arise as subcategories of the slice (2,1)-category $\mathbf{Grpd}/\mathcal{B}$ consisting of functors having a certain finiteness property. Before we discuss this property, it's worth discussing the category $\mathbf{Grpd}/\mathcal{B}$ in more detail.

The objects of $\mathbf{Grpd}/\mathcal{B}$ are pairs (\mathcal{G}, F) , where \mathcal{G} is a groupoid, and $F : \mathcal{G} \rightarrow \mathcal{B}$ is a functor. The 1-morphisms from (\mathcal{G}_1, F_1) to (\mathcal{G}_2, F_2) are pairs (Φ, η) , with Φ a functor and η a natural isomorphism forming a diagram in \mathbf{Grpd} :

$$\begin{array}{ccc} \mathcal{G}_1 & \xrightarrow{\Phi} & \mathcal{G}_2 \\ & \searrow F_1 \quad \nearrow F_2 & \\ & \mathcal{B} & \end{array}$$

η

Given 1-morphisms (Φ_1, η_1) and (Φ_2, η_2) from (\mathcal{G}, J) to (\mathcal{H}, K) are natural isomorphisms $\psi : \Phi_1 \Rightarrow \Phi_2$ such that $\eta_2 = K\psi \circ \eta_1$.

5.1. REMARK. Naturally since \mathcal{B} is a groupoid, the natural transformation in the definition of the 1-morphisms is necessarily an isomorphism. However, loosening this requirement (so that \mathcal{B} is a general category) does not change this definition. The slice 2-category \mathbf{Cat}/\mathcal{B} is also defined so that the natural transformations occurring in the definition of 1-morphisms be isomorphisms. If we allow $\eta : F_1 \Rightarrow F_2\Phi$ to not be an isomorphism, we obtain what are known as a lax morphism in \mathbf{Cat}/\mathcal{B} . There are interesting examples of lax morphisms arising in combinatorial contexts, for instance, when taking the derivative of a stuff type, but we will not consider them here.

5.2. DEFINITION. Let $F : \mathcal{G} \rightarrow \mathcal{B}$ be a functor. For any object y of \mathcal{B} , the **full inverse image** $F^{-1}(y)$ of F over y is the full subgroupoid of \mathcal{G} consisting of all objects x such that $F(x) \cong y$.

If we restrict F to $F^{-1}(y)$, we get a functor $F|_y : F^{-1}(y) \rightarrow \mathcal{B}$ called the **component of F at x** .

PROOF. First, to define $\hat{\Phi}$, recall that the objects of $\text{im}_2 F$ are the objects of \mathcal{G} , so $\hat{\Phi}(x) := \Phi(x)$. For an equivalence class of morphisms $[f]$, define $\hat{\Phi}([f]) := \Phi(f)$. To see that this is well defined, suppose we have $F(f) = F(f')$ for some distinct $f, f' : x \rightarrow y$. Since η is a natural isomorphism, we have

$$\begin{aligned}\Phi(f) &= \eta_y^{-1} \circ \Psi F(f) \circ \eta_x \\ &= \eta_y^{-1} \circ \Psi F(f') \circ \eta_x \\ &= \Phi(f')\end{aligned}$$

Then the components of η_2 are simply the identity morphisms of $\Phi(x)$ for each object x of \mathcal{G} .

Since $\text{im}_1 F$ is considered as a full subcategory of \mathcal{H} , simply define $\hat{\Psi}$ to be the restriction of Ψ to this subcategory. Then the components of η_0 are the identity morphisms of $\Psi(y)$ for each object y of $\text{im}_1 F$.

Finally, the component of η_1 corresponding to $x \in \text{Ob}(\text{im}_2 F) = \text{Ob}(\mathcal{G})$, must go from $\hat{\Phi}(x) = \Phi(x)$ to $\hat{\Psi}F_1(x) = \Psi F(x)$. The obvious choice is to take this component to be η_x . Then since $\hat{\Phi}([f]) = \Phi(f)$ and $\hat{\Psi}F_1([f]) = \Psi F(f)$, the fact that η is natural ensures that η_1 is natural. \blacksquare

It remains to show that $(\mathcal{E}, \widetilde{\mathcal{M}})$ and $(\widetilde{\mathcal{E}}, \mathcal{M})$ are factorization systems of Grpd/\mathcal{B} when extended this way. The first three axioms are obvious. The orthogonality axiom guarantees the existence of a functor acting as a fill-in, but we must still construct a natural transformation making that functor a morphism in Grpd/\mathcal{B} . The construction of this natural transformation is again a straightforward, yet tedious, use of the fill-in data from Grpd , and is left as an exercise to the reader.

So the standard ternary factorization system on groupoids extends nicely to Grpd/\mathcal{B} . Moreover, the nice properties of Definition 2.10 are also preserved.

5.10. PROPOSITION. *Let $(G, \varphi) : (X_1, F_1) \rightarrow (X_2, F_2)$ be a 1-morphism in Grpd/\mathcal{B} .*

- (a) *If G is fully faithful as a functor, then (G, φ) is fully faithful in the sense of Definition 2.10.*
- (b) *If G is essentially surjective and full, then (G, φ) is fully cofaithful.*

PROOF.

- (a) Fix an object (Y, H) in Grpd/\mathcal{B} . The induced functor is given by

$$\begin{aligned}(G, \varphi)_* : \text{Grpd}/\mathcal{B}((Y, H), (X_1, F_1)) &\rightarrow \text{Grpd}/\mathcal{B}((Y, H), (X_2, F_2)) \\ (\Phi, \eta) &\mapsto (G\Phi, \varphi\Phi \circ \eta) \\ (\nu : (\Phi_1, \eta_1) \Rightarrow (\Phi_2, \eta_2)) &\mapsto (G\nu : (G\Phi_1, \varphi\Phi_1 \circ \eta_1) \Rightarrow (G\Phi_2, \varphi\Phi_2 \circ \eta_2))\end{aligned}$$

Fix (Φ_1, η_1) and (Φ_2, η_2) from (Y, H) to (X_1, F_1) , which are sent to $(G\Phi_1, \varphi\Phi_1 \circ \eta_1)$ and $(G\Phi_2, \varphi\Phi_2 \circ \eta_2)$ by $(G, \varphi)_*$. Since G is fully faithful as a functor, Proposition 2.11

states that $G_* : \text{Grpd}(Y, X_1) \rightarrow \text{Grpd}(Y, X_2)$ is fully faithful as a functor. So given a 2-morphism $\mu : (G\Phi_1, \varphi\Phi_1 \circ \eta_1) \Rightarrow (G\Phi_2, \varphi\Phi_2 \circ \eta_2)$ in Grpd/\mathcal{B} , there is a unique natural transformation $\nu : \Phi_1 \Rightarrow \Phi_2$ such that $\mu = G\nu$. It remains to show that ν is a 2-morphism in Grpd/\mathcal{B} . That is, we want to show that $\eta_2 = F_1\nu \circ \eta_1$.

Since μ is a 2-morphism in Grpd/\mathcal{B} , we have the following identity:

$$\varphi\Phi_2 \circ \eta_2 = F_2\mu \circ \varphi\Phi_1 \circ \eta_1$$

Since all 2-morphisms are invertible, using cancellation, it is enough to show that

$$\varphi\Phi_2 \circ F_1\nu = F_2G\nu \circ \varphi\Phi_1$$

We will do this by comparing the components at each object in Y . So, fix an object a of Y , and consider the morphism ν_a . Since $\varphi : F_1 \Rightarrow F_2G$ is a natural transformation, we have the identity

$$\varphi_{\Phi_2(a)} \circ F_1(\nu_a) = F_2G(\nu_a) \circ \varphi_{\Phi_1(a)}$$

Then by definition of the whiskerings $\varphi\Phi_1$, $\varphi\Phi_2$, $F_1\nu$, and $F_2G\nu$, we have shown that ν is a 2-morphism in Grpd/\mathcal{B} from (Y, H) to (X_1, F_1) .

(b) Fix an object (Y, H) in Grpd/\mathcal{B} . The induced functor is given by

$$\begin{aligned} (G, \varphi)^* : \text{Grpd}/\mathcal{B}((X_2, F_2), (Y, H)) &\rightarrow \text{Grpd}/\mathcal{B}((X_1, F_1), (Y, H)) \\ (\Phi, \eta) &\mapsto (\Phi G, \eta G \circ \varphi) \\ (\nu : (\Phi_1, \eta_1) \Rightarrow (\Phi_2, \eta_2)) &\mapsto (\nu G : (\Phi_1 G, \eta_1 G \circ \varphi) \Rightarrow (\Phi_2 G, \eta_2 G \circ \varphi)) \end{aligned}$$

Fix (Φ_1, η_1) and (Φ_2, η_2) from (X_2, F_2) to (Y, H) , and let $\mu : (\Phi_1 G, \eta_1 G \circ \varphi) \Rightarrow (\Phi_2 G, \eta_2 G \circ \varphi)$ be a 2-morphism in Grpd/\mathcal{B} , so that

$$H\mu \circ \eta_1 G \circ \varphi = \eta_2 G \circ \varphi.$$

Then since φ is invertible, using right cancellation gives

$$H\mu \circ \eta_1 G = \eta_2 G$$

Proposition 2.11 states that there is a unique natural transformation $\nu : \Phi_1 \Rightarrow \Phi_2$ such that $\mu = \nu G$. Then,

$$(H\nu \circ \eta_1)G = \eta_2 G$$

Then using Proposition 2.11 again, we see that

$$H\nu \circ \eta_1 = \eta_2$$

Which proves that ν is a 2-morphism in Grpd/\mathcal{B} .

■

6. Combinatorial (2,1)-categories

In this section we prove the main theorem of this paper, which is a generalization of Theorem 2.6 to categories of relatively finite functors. The first half of the proof is similar to the proof of the original theorem. We will need the following definition and lemma:

6.1. DEFINITION. Let \mathcal{C} be a (2,1)-category with a factorization system $(\mathcal{E}, \mathcal{M})$. For any object X of \mathcal{C} , the \mathcal{E} -**quotients** of X are equivalence classes of morphisms $p : X \rightarrow Y$ in \mathcal{E} . Where two morphisms $p : X \rightarrow Y$ and $p' : X \rightarrow Y'$ are considered equivalent if there exists an equivalence $f : Y \rightarrow Y'$ and a 2-morphism $\varphi : fp \Rightarrow p'$.

6.2. LEMMA. Let \mathcal{C} be a (2,1)-category with a factorization system $(\mathcal{E}, \mathcal{M})$ such that all 1-morphisms in \mathcal{E} are fully cofaithful. Then for any two objects X, Y , there is an equivalence of groupoids

$$\mathcal{C}(X, Y) \simeq \coprod_{[X \rightarrow Z] \in \mathcal{E}} \mathcal{M}(Z, Y)$$

where the coproduct is taken over all \mathcal{E} -quotients of X .

PROOF. Choosing representatives for the \mathcal{E} -quotients of X , we can construct a functor from the coproduct on the right to $\mathcal{C}(X, Y)$ by pre-composing with the corresponding representatives. This functor is fully faithful when restricted to each component of the coproduct, and thus on the entire coproduct. It is obviously essentially surjective as a result of the factorization system. ■

This lemma is essential for the first part of the proof of the main theorem, as it gives us a formula for $\mathcal{C}(X, Y)$ in terms of a coproduct of $\mathcal{M}(Z, Y)$, which we can convert to a sum of groupoid cardinalities. The next lemma is used in the second part and will help us find equivalences in RelFin_{BG} .

6.3. LEMMA. Let H_1, H_2 and G be finite groups and $F_1 : BH_1 \rightarrow BG, F_2 : BH_2 \rightarrow BG$ be objects of RelFin_{BG} . Let $(\varphi, g) : F_1 \rightarrow F_2$ be a 1-morphism in RelFin_{BG} such that φ is an isomorphism of groups. Then (φ, g) is an equivalence in RelFin_{BG} .

PROOF. Treat $F_1 : H_1 \rightarrow G$ and $F_2 : H_2 \rightarrow G$ as group homomorphisms. Then since (φ, g) is a 1-morphism in RelFin_{BG} , we can treat g as an element of G such that for all $h \in H_1$, $F_1(h) = g^{-1}F_2(\varphi(h))g$. Then since φ is an isomorphism, for every $k \in H_2$, $F_2(k) = gF_1(\varphi^{-1}(k))g^{-1}$. Therefore (φ^{-1}, g^{-1}) is a strict inverse to (φ, g) . ■

6.4. THEOREM. Let \mathcal{B} be a locally finite groupoid and let $F : \mathcal{G} \rightarrow \mathcal{B}$ and $F' : \mathcal{G}' \rightarrow \mathcal{B}$ be functors in $\text{RelFin}_{\mathcal{B}}$ such that for all finite groups H and functors $S : BH \rightarrow \mathcal{B}$,

$$|\text{RelFin}_{\mathcal{B}}(S, F)| = |\text{RelFin}_{\mathcal{B}}(S, F')|.$$

Then F and F' are equivalent.

PROOF. First, note that any F in $\text{RelFin}_{\mathcal{B}}$ is completely determined up to equivalence by the components $F|_y$ for each isomorphism class $[y]$ of \mathcal{B} . Furthermore, if we have $S : BH \rightarrow \mathcal{B}$, then any morphism $(S, BH) \rightarrow (F, \mathcal{G})$ is completely determined by a morphism $(S, BH) \rightarrow (F|_{S*}, F^{-1}(S*))$, where $*$ is the unique object of BH . Therefore, it is enough to prove this theorem in the case where $\mathcal{B} = BG$ for some finite group G .

The proof has two steps: First, we use induction to show that for all $S : BH \rightarrow BG$,

$$|\widetilde{\mathcal{M}}(S, F)| = |\widetilde{\mathcal{M}}(S, F')| \quad (2)$$

Where $\widetilde{\mathcal{M}}$ consists of all morphisms whose underlying functors are faithful. Second, use equation 2 to show that F and F' are of the form $F \simeq U \sqcup V$ and $F' \simeq U \sqcup W$ with

$$|\widetilde{\mathcal{M}}(S, V)| = |\widetilde{\mathcal{M}}(S, W)|$$

Since \mathcal{G} and \mathcal{G}' are finite, we can then use induction to show that F and F' decompose as coproducts with equivalent components, and are therefore themselves equivalent.

For the first step, recall that the category RelFin_{BG} has the ternary factorization system $(\mathcal{E}, \mathcal{F}, \mathcal{M})$ as discussed in Section 5.8 (equivalently described by the pair of factorization systems $(\mathcal{E}, \widetilde{\mathcal{M}})$ and $(\widetilde{\mathcal{E}}, \mathcal{M})$). Furthermore, all morphisms in \mathcal{E} are fully-cofaithful. Therefore, we can use Lemma 6.2 with respect to the factorization system $(\mathcal{E}, \widetilde{\mathcal{M}})$. Applying this lemma to both sides the equation $|\text{RelFin}_{BG}(S, F)| = |\text{RelFin}_{BG}(S, F')|$ and rearranging terms gives

$$|\widetilde{\mathcal{M}}(S, F)| - |\widetilde{\mathcal{M}}(S, F')| = \sum_{\substack{[S \rightarrow T] \in \mathcal{E} \\ T \not\sim S}} |\widetilde{\mathcal{M}}(T, F)| - |\widetilde{\mathcal{M}}(T, F')|, \quad (3)$$

where the sum on the right is taken over \mathcal{E} -quotients of S that are not equivalent to S . Note that since S is of the form $BH \rightarrow BG$, for some finite group H , the \mathcal{E} -quotients of S are of the form $BH' \rightarrow BG$ for some quotient H' of H . So we can define a partial order on the equivalence classes of functors $BH \rightarrow BG$ for H a finite group as follows: Let $S : BH \rightarrow BG \leq S' : BH' \rightarrow BG$ if S is a \mathcal{E} -quotient of S' . It is easy to check that this is a partial order. Furthermore, since any finite group has only finitely many quotients, there can be no infinite descending chains, so this partial order is well-founded.

Now, fix $S : BH \rightarrow BG$, and assume for all quotient groups H' of H and functors $T : BH' \rightarrow BG$, that $|\widetilde{\mathcal{M}}(T, F)| = |\widetilde{\mathcal{M}}(T, F')|$. Then summands on the right side of equation 3 are all zero, so $|\widetilde{\mathcal{M}}(S, F)| = |\widetilde{\mathcal{M}}(S, F')|$. Then, using the principle of Noetherian induction, we can conclude that for any finite group H and functor $S : BH \rightarrow BG$,

$$|\widetilde{\mathcal{M}}(S, F)| = |\widetilde{\mathcal{M}}(S, F')|.$$

This concludes the first part of the proof.

For the second part, assume that \mathcal{G} and \mathcal{G}' are skeletal, and choose some isomorphism class BK_0 of \mathcal{G} so that $\mathcal{G} \simeq BK_0 \sqcup V_0$ and $F \simeq T_0 \sqcup F|_{V_0}$ for some functor $T_0 : BK_0 \rightarrow BG$. Clearly, the inclusion of BK_0 into \mathcal{G} is faithful, and so the morphism $T_0 \rightarrow F$ is in $\widetilde{\mathcal{M}}$. So

$$|\widetilde{\mathcal{M}}(T_0, F)| = |\widetilde{\mathcal{M}}(T_0, F')| \neq 0$$

and thus there exists a morphism $\varphi_0 : T_0 \rightarrow F'$ in $\widetilde{\mathcal{M}}$. The underlying functor of this morphism must send the unique object of BK_0 to some $BJ_0 \subseteq \mathcal{G}'$. We can then write $\mathcal{G}' \simeq BJ_0 \sqcup W_0$ and $F' \simeq R_0 \sqcup F'|_{W_0}$ for some functor $R_0 : BJ_0 \rightarrow BG$. Repeating the same procedure gives a morphism $\psi_0 : R_0 \rightarrow F$, also in $\widetilde{\mathcal{M}}$. Repeating this procedure gives a chain of morphisms in $\widetilde{\mathcal{M}}$:

$$T_0 \xrightarrow{\varphi_0} R_0 \xrightarrow{\psi_0} T_1 \xrightarrow{\varphi_1} R_1 \xrightarrow{\psi_1} T_2 \xrightarrow{\varphi_2} \dots$$

for $T_i : BK_i \rightarrow BG$, $R_i : BJ_i \rightarrow BG$ with $F \simeq T_i \sqcup V_i$ and $F' \simeq R_i \sqcup W_i$. Note that we can consider this chain as a chain of injective homomorphisms of finite groups:

$$K_0 \xrightarrow{\varphi_0} J_0 \xrightarrow{\psi_0} K_1 \xrightarrow{\varphi_1} J_1 \xrightarrow{\psi_1} K_2 \xrightarrow{\varphi_2} \dots$$

Since \mathcal{G} and \mathcal{G}' are finite, there are only finitely many such K_i, J_i up to isomorphism, so this chain must eventually stabilize, and thus there is some i for which $\varphi_i : K_i \rightarrow J_i$ is an isomorphism, and from Lemma 6.3 the morphism of relatively finite functors $\varphi_i : T_i \rightarrow R_i$ is an equivalence. Thus we have found a $U : BK \rightarrow BG$ such that $F \simeq U \sqcup V$ and $F' \simeq U \sqcup W$.

Now, for any $S : BH \rightarrow BG$, it is easy to see that

$$\begin{aligned} \widetilde{\mathcal{M}}(S, F) &\simeq \widetilde{\mathcal{M}}(S, U) \sqcup \widetilde{\mathcal{M}}(S, V) \text{ and} \\ \widetilde{\mathcal{M}}(S, F') &\simeq \widetilde{\mathcal{M}}(S, U) \sqcup \widetilde{\mathcal{M}}(S, W). \end{aligned}$$

So, from equation 2, we see that

$$\begin{aligned} |\widetilde{\mathcal{M}}(S, V)| &= |\widetilde{\mathcal{M}}(S, F)| - |\widetilde{\mathcal{M}}(S, U)| \\ &= |\widetilde{\mathcal{M}}(S, F')| - |\widetilde{\mathcal{M}}(S, U)| \\ &= |\widetilde{\mathcal{M}}(S, W)| \end{aligned}$$

We can then use induction to show that $V \simeq W$, and therefore $F \simeq F'$. ■

It's worth remarking that the overarching procedure of this proof differs slightly from that of Theorem 2.6. The first half is the same, where we prove by induction that $|\widetilde{\mathcal{M}}(S, F)| = |\widetilde{\mathcal{M}}(S, F')|$ for all S . The second half is quite different, however. We can show that there exist 1-morphisms $F \rightarrow F'$ and $F' \rightarrow F$, however, since the underlying functors are merely faithful, we cannot use Theorem 4.3 to show that $F \simeq F'$. Instead, we rely on relative finiteness to show that F and F' must have connected components that are equivalent, and then use induction to show that $F \simeq F'$.

7. Postnikov systems and homotopy cardinality

Groupoid cardinality is a special case of the homotopy cardinality of an ∞ -groupoid, for when the ∞ -groupoid is 1-truncated. If X is a connected ∞ groupoid with all homotopy groups finite, then the homotopy cardinality is

$$|X| = \prod_{k=1}^{\infty} (\#\pi_k(X))^{(-1)^k}$$

If X is not connected, then the homotopy cardinality is the sum of the cardinalities of each connected component. It is natural to ask which of the results discussed in this paper can be generalized to ∞ -groupoids.

The prevalence of factorization systems in the above results also leads us to the natural generalization of factorization systems to ∞ -categories. Given a morphism of ∞ -groupoids $f : X \rightarrow Y$, for each $n \geq 1$, there exists an ∞ -groupoid $\mathrm{im}_n f$ and morphisms $e_n : X \rightarrow \mathrm{im}_n f$ and $m_n : \mathrm{im}_n f \rightarrow Y$ such that the induced maps on homotpy groups satisfy:

- $e_n : \pi_{n-1}(X, x) \rightarrow \pi_{n-1}(\mathrm{im}_n f, e_n(x))$ is surjective.
- $e_n : \pi_k(X, x) \rightarrow \pi_k(\mathrm{im}_n f, e_n(x))$ is a bijection for $k < n - 1$.
- $m_n : \pi_{n-1}(\mathrm{im}_n f, x) \rightarrow \pi_{n-1}(Y, m_n(x))$ is injective.
- $m_n : \pi_k(\mathrm{im}_n f, x) \rightarrow \pi_k(Y, m_n(x))$ is a bijection for $k > n - 1$.

In other words, e_n is n -connected and m_n is n -truncated, so this factorization system is referred to as the $(n$ -connected, n -truncated) factorization system. Furthermore, these classes of morphisms satisfy a higher categorical version of the orthogonality condition in Definition 2.8.1 (see Example 5.2.8.16 and Proposition 5.2.8.11 in Lurie [2009]). It is easy to check that if we consider groupoids as ∞ -groupoids, the $(2$ -connected, 2 -truncated) factorization is exactly the (essentially surjective & full, faithful) factorization, and the $(1$ -connected, 1 -truncated) factorization is the (essentially surjective, fully-faithful) factorization system discussed in Section 2.7.

The orthogonality condition, along with the fact that n -connectivity implies $(n - 1)$ -connectivity, gives a morphism $\varphi_n : \mathrm{im}_n f \rightarrow \mathrm{im}_{n-1} f$ such that $e_{n-1} \simeq \varphi_n \circ e_n$ and $m_n \simeq m_{n-1} \circ \varphi_n$. The resulting tower of ∞ -groupoids is called the *relative Postnikov system of f* (c.f. [Paul G. Goerss, 1999, Definition VI.2.9]). This is the natural generalization of the ternary factorization system in Grpd to ∞ -groupoids. We thus get a commutative diagram:

$$\begin{array}{ccccccc}
 & & X & & & & \\
 & \swarrow e_{n+1} & \downarrow e_n & \searrow e_1 & & & \\
 \dots & \longrightarrow & \mathrm{im}_{n+1} f & \longrightarrow & \mathrm{im}_n f & \longrightarrow & \dots \longrightarrow \mathrm{im}_1 f \\
 & \searrow m_{n+1} & \downarrow m_n & \swarrow m_1 & & & \\
 & & Y & & & &
 \end{array}$$

The following result generalizes Proposition 4.1.

7.1. PROPOSITION. *Let $f : X \rightarrow Y$ be a morphism between tame ∞ -groupoids. Then*

$$\begin{aligned} |\mathrm{im}_n f| &\geq |\mathrm{im}_{n-1} f| \text{ if } n \text{ is even} \\ |\mathrm{im}_n f| &\leq |\mathrm{im}_{n-1} f| \text{ if } n \text{ is odd.} \end{aligned}$$

PROOF. In the case where $n = 2$, we have an surjective function $e_1 : \pi_0(X) \twoheadrightarrow \pi_0(\mathrm{im}_1 f)$, and bijections $\pi_k(\mathrm{im}_2 f, y) \cong \pi_k(Y, m_2(y))$ for $k > 1$. So,

$$\begin{aligned} |\mathrm{im}_2 f| &= \sum_{x \in \pi_0(X)} \left[\# \pi_1(\mathrm{im}_2 f, e_2(x))^{-1} \cdot \prod_{k=2}^{\infty} \# \pi_k(Y, f(x))^{(-1)^k} \right] \\ &= \sum_{u \in \pi_0(\mathrm{im}_1 f)} \left[\sum_{x \in e_1^{-1}(u)} \left(\# \pi_1(\mathrm{im}_2 f, e_2(x))^{-1} \cdot \prod_{k=2}^{\infty} \# \pi_k(Y, f(x))^{(-1)^k} \right) \right] \end{aligned}$$

We have bijections $\pi_k(\mathrm{im}_1 f, y) \cong \pi_k(Y, m_1(y))$ for $k > 0$, so

$$|\mathrm{im}_1 f| = \sum_{u \in \pi_0(\mathrm{im}_1 f)} \left[\# \pi_1(Y, m_1(u))^{-1} \cdot \prod_{k=2}^{\infty} \# \pi_k(Y, m_1(u))^{(-1)^k} \right]$$

So we have a correspondence of the terms in each sum with respect to $\pi_0(\mathrm{im}_1 f)$. When comparing the terms, it is clear that the terms in the formula for $|\mathrm{im}_2 f|$ will have at least as many summands as those for $|\mathrm{im}_1 f|$, which can have at most one term. Furthermore, we have an injective homomorphism $m_2 : \pi_1(\mathrm{im}_2 f, e_2(x)) \cong \pi_1(Y, f(x))$, so $\# \pi_1(\mathrm{im}_2 f, e_2(x))^{-1} \geq \# \pi_1(Y, f(x))^{-1}$. Therefore,

$$|\mathrm{im}_2 f| \geq |\mathrm{im}_1 f|$$

In the case where $n > 2$, we have $\pi_0(X) \cong \pi_0(\mathrm{im}_n f) \cong \pi_0(\mathrm{im}_{n-1} f)$, and bijections $\pi_k(\mathrm{im}_n f, x) \cong \pi_k(Y, m_n(x))$ for $k > n - 1$ and $\pi_k(X, x) \cong \pi_k(\mathrm{im}_n f, e_n(x))$ for $k < n - 1$. Therefore,

$$\begin{aligned} |\mathrm{im}_n f| &= \sum_{x \in \pi_0(X)} \left[\prod_{k=1}^{n-3} \# \pi_k(X, x)^{(-1)^k} \cdot \# \pi_{n-2}(X, x)^{(-1)^{n-2}} \right. \\ &\quad \left. \cdot \# \pi_{n-1}(\mathrm{im}_n f, e_n(x))^{(-1)^{n-1}} \cdot \prod_{k=n}^{\infty} \# \pi_k(Y, f(x))^{(-1)^k} \right] \end{aligned}$$

Similarly, in the formula for $|\mathrm{im}_{n-1} f|$, we have bijections $\pi_k(\mathrm{im}_{n-1} f, x) \cong \pi_k(Y, m_{n-1}(x))$ for $k > n - 2$ and $\pi_k(X, x) \cong \pi_k(\mathrm{im}_{n-1} f, e_{n-1}(x))$ for $k < n - 2$. Therefore,

$$\begin{aligned} |\mathrm{im}_{n-1} f| &= \sum_{x \in \pi_0(X)} \left[\prod_{k=1}^{n-3} \# \pi_k(X, x)^{(-1)^k} \cdot \# \pi_{n-2}(\mathrm{im}_{n-1} f, e_{n-1}(x))^{(-1)^{n-2}} \right. \\ &\quad \left. \cdot \# \pi_{n-1}(Y, f(x))^{(-1)^{n-1}} \cdot \prod_{k=n}^{\infty} \# \pi_k(Y, f(x))^{(-1)^k} \right] \end{aligned}$$

In addition, we have a surjective homomorphism $e_{n-1} : \pi_{n-2}(X, x) \twoheadrightarrow \pi_{n-2}(\mathrm{im}_{n-1}f, e_{n-1}(x))$ and an injective homomorphism $m_n : \pi_{n-1}(\mathrm{im}_nf, x) \hookrightarrow \pi_{n-1}(Y, m_n(x))$.

Thus, if n is even,

$$\begin{aligned} \# \pi_{n-1}(\mathrm{im}_nf, e_n(x))^{(-1)^{n-1}} \cdot \# \pi_{n-2}(X, x)^{(-1)^{n-2}} \\ \geq \# \pi_{n-1}(Y, f(x))^{(-1)^{n-1}} \cdot \# \pi_{n-2}(\mathrm{im}_{n-1}f, e_{n-1}(x))^{(-1)^{n-2}} \end{aligned}$$

and therefore $|\mathrm{im}_nf| \geq |\mathrm{im}_{n-1}f|$. Similarly, if n is odd, this inequality is reversed, and we have $|\mathrm{im}_nf| \leq |\mathrm{im}_{n-1}f|$. \blacksquare

7.2. COROLLARY. *If $f : X \rightarrow Y$ is a morphism of tame ∞ -groupoids that is $(n-1)$ -connected and n -truncated, then*

$$\begin{aligned} |X| &\geq |Y| \quad \text{if } n \text{ is even} \\ |X| &\leq |Y| \quad \text{if } n \text{ is odd.} \end{aligned}$$

PROOF. Since f is n -truncated,

$$X \xrightarrow{\mathrm{id}_X} X \xrightarrow{f} Y$$

is a valid factorization in the $(n$ -connected, n -truncated) factorization system, so by orthogonality, $X \simeq \mathrm{im}_nf$. Since f is $(n-1)$ -connected,

$$X \xrightarrow{f} Y \xrightarrow{\mathrm{id}_Y} Y$$

is a valid factorization in the $((n-1)$ -connected, $(n-1)$ -truncated) factorization system, so $\mathrm{im}_{n-1}f \simeq Y$. \blacksquare

Regarding Theorem 4.3, it is clear that if X and Y are tame ∞ -groupoids with only finitely many connected components and we have morphisms $X \rightarrow Y$ and $Y \rightarrow X$ which are $(n-1)$ -connected and n -truncated, then X and Y have all isomorphic homotopy groups. However, it remains open whether this would imply that the morphisms in question are equivalences.

The method used to prove Theorem 6.4 would likely not work when generalizing to ∞ -groupoids. First of all, the first step relies on the fact that groupoids are already 1-truncated, so we can begin by showing that the cardinality of the groupoids of faithful morphisms into F and F' agree. However, even if we assume that our objects are n -truncated, the method used to show that there exist equivalent connected components does not even generalize to 2-groupoids.

References

John C. Baez and James Dolan. From finite sets to feynman diagrams. 2000. URL <https://arxiv.org/abs/math/0004133>.

- M. Dupont and E.M. Vitale. Proper factorization systems in 2-categories. *Journal of Pure and Applied Algebra*, 179(1):65–86, 2003. doi: [https://doi.org/10.1016/S0022-4049\(02\)00244-X](https://doi.org/10.1016/S0022-4049(02)00244-X).
- S. Kasangian and E.M. Vitale. Factorization systems for symmetric cat-groups. *Theory and Applications of Categories*, 7:47–70, 2000. URL <https://api.semanticscholar.org/CorpusID:122521633>.
- Tom Leinster. The euler characteristic of a category. *Documenta Mathematica*, 2006.
- L. Lovász. Operations with structures. *Acta Mathematica Academiae Scientiarum Hungaricae*, 18:321–328, 09 1967. doi: <https://doi.org/10.1007/BF02280291>.
- L. Lovász. Direct product in locally finite categories. *Acta Scientiarum Mathematicarum*, 33, 01 1972.
- Jacob Lurie. *Higher Topos Theory*. Annals of Mathematics Studies. Princeton University Press, 2009.
- J. Morton. Categorified algebra and quantum mechanics. *Theory and Applications of Categories*, 16:785–854, 2006. URL <http://www.tac.mta.ca/tac/volumes/16/29/16-29abs.html>.
- John F. Jardine Paul G. Goerss. *Simplicial Homotopy Theory*. Progress in Mathematics. Birkhäuser Basel, 1999. doi: <https://doi.org/10.1007/978-3-0348-8707-6>.
- A. Pultr. Isomorphism types of objects in categories determined by numbers of morphisms. *Acta Scientiarum Mathematicarum*, 35:155–160, 1973.