ON EULER-SOMBOR ENERGY OF GRAPHS

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ABSTRACT. In 2024, Gutman et al. [6] defined a new molecular descriptor called as The Euler-Sombor (ES) index of graph. By using this index we define the Euler-Sombor (ES) matrix of a graph G whoes $(i,j)^{th}$ entry is $\sqrt{d_i^2 + d_j^2 + d_i.d_j}$ if vertex v_i is adjacent to vertex v_j, otherwise 0. The ES eigenvalues of the graph G are the eigenvalues of its ES matrix ,$ES(G)$. In this paper we discus ES eigenvalues ν_i and energy of some classes of graphs.

1. INTRODUCTION

The graphs considered here all are undirected,finite and simple. Let $G(V, E)$ be a graph with vertex set $V(G) = \{v_1, v_2, \ldots, v_n\}$ and edge set $E(G) = \{e_1, ..., e_m\}$. We denote d_i the degree of vertex v_i in G. Let K_n, C_n and S_n denote a complete, cycle and star graph resp. on n vertices also the complete bipartite graph on $m+n$ vertices is denoted by $K_{m,n}$. In chemical graph theory and mathematical chemistry, a molecular graph is a visual representation of a chemical compound where the vertices/nodes represent the atoms, and the edges signify the chemical bonds between them. This graph-based approach provides a structural formula of the compound.

A topological index known as a connectivity index, is a numerical value derived from the molecular graph. It captures various aspects of the graph's topology, which helps in characterizing the molecule's structure. The toplogical indices are invariant under graph isomorphisms, meaning they do not change if the graph is transformed in certain ways.

Topological indices [5] describes the molecular structure quantitatively, offering insights into the compounds property such as reactivity and stability. Examples of such indices include the Wiener index, Zagreb indices, and Randic index. Each of these indices highlights different structural features which are useful to study the physical and chemical behaviours of the molecule.

The vertex based Sombor index [5] introduced by I.Gutman is

$$SO(G) = \sum_{e=(v_i,v_j)\in E(G)} \sqrt{d_i^2 + d_j^2}$$

Now recently Gutman(2024) obtained the Euler-Sombor index [4] closely related to the Sombor index.

$$ES = ES(G) = \sum_{e=(v_i,v_j)\in E(G)} \sqrt{d_i^2 + d_j^2 + d_i.d_j}$$

In 2015 B. Furtula et al. [7] gives Forgotten index of a graph G as,

$$FI = FI(G) = \sum_{e=(v_i,v_j)\in E(G)} (d_i^2 + d_j^2)$$

The Zagreb indices [8, 9] of a graph G are defined as,

$$Second\ Zagreb\ Index = SZ = SZ(G) = \sum_{e=(v_i,v_j)\in E(G)} d_i.d_j$$

Let the ES matrix of the graph G, which a square matrix of order n whose entries are given by

$$ES(G)_{ij} = \begin{cases} \sqrt{d_i^2 + d_j^2 + d_i.d_j} & \text{if } (v_i, v_j) \in E(G), \\ 0 & \text{otherwise.} \end{cases}$$

Suppose $\nu_1(G)$, $\nu_2(G)$, ..., $\nu_n(G)$ are the eigenvalues of ES matrix of G. Then ES energy of graph [25, 19] is given by

$$E_{ES}(G) = \sum_{i=1}^{n} |\nu_i(G)|.$$

In this paper, we will explore properties of the ES matrix, ES eigenvalues and ES energy of a graph. The second Section contains preliminaries. In Section 3, we establish some properties of ES eigenvalues of a graph. While in section 4 we obtained some results on ES energy of graph G.

2. Preliminaries

In this section, we begin with some well known results from [1]. The following Lemma 2.1, Lemma 2.2, Lemma 2.3, and Lemma 3.16 gives the eigenvalues of $A(G)$ related with eigenvalues of adjacency matrix of graphs K_n, $K_{m,n}$, C_n, and S_n. The $\lambda_1, \lambda_2, \ldots, \lambda_n$ are adjacency eigenvalues of the graph G. It is known that, if all entries of A are strictly positive, then we say A is positive matrix we write $(A)_{ij} > 0$. If A is a real and symmetric matrix, then all eigenvalues of A are real. We say that as eigenvalue is simple if its algebraic multiplicity is 1.

The ES matrix of graph G is an irreducible non-negative symmetric real matrix having all ES eigenvalues are real with trace 0. For the following results see Bapat [1] and Brouwer [3].

Lemma 2.1. [1] For any positive integer n, the eigenvalues of complete graph K_n are $\lambda_1 = n - 1$ and $\lambda_2 = \cdots = \lambda_n = -1$.

Lemma 2.2. [1] For any positive integer m, n, the eigenvalues of complete bipartite graph $K_{m,n}$ are $\lambda_1 = \sqrt{mn}$, $\lambda_2 = \lambda_3 = \cdots = \lambda_{m+n-2} = 0$ and $\lambda_{m+n} = -\sqrt{mn}$.

Lemma 2.3. [1] For $n \geq 2$, the eigenvalues of $G = C_n$ are $\lambda_i = 2cos\frac{2\pi i}{n}$, where $i = 1, \ldots, n$.

Lemma 2.4. [3] For any positive integer n, the eigenvalues of star graph S_n are $\lambda_1 = \sqrt{n-1}$, $\lambda_2 = \cdots = \lambda_{n-1} = 0$ and $\lambda_n = -\sqrt{n-1}$.

3. ES-Eigenvalues of Some Classes of Graph

In this section, we obtain ES-eigenvalues of some classes of graphs like r-regular, $K_{m,n}$, and S_n graphs, we also obtain some properties of the ES matrix of a graph. In the following result, we give the ES eigenvalues of r-regular graph.

Theorem 3.1. Let G be a simple connected graph of order $n \geq 3$ with eigen values, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ be its ES eigenvalues. If G is r-regular graph, then the ES eigenvalues of G are $\nu_i = r\sqrt{3}\,\lambda_i$ for $i = 1, 2, \ldots, n$.

Proof. Let G be a connected r-regular graph of order n.
We know that $ES(G) = r\sqrt{3}\,A(G)$.
Thus $\nu_i = r\sqrt{3}\,\lambda_i \quad$ for $i = 1, 2, \ldots, n$ $\qquad\square$

Corollary 3.2. If $G = K_n$ complete graph, then the ES eigenvalues of G are $\nu_1 = (n-1)^2\sqrt{3}$ and $\nu_2 = \cdots = \nu_n = -(n-1)\sqrt{3}$.

Proof. From Lemma 2.1, if $G = K_n$, $\lambda_1 = n - 1$ and $\lambda_2 = \cdots = \lambda_n = -1$.
Since $ES(G) = (n-1)\sqrt{3}\,A(G)$.
Therefore $\nu_i = (n-1)\sqrt{3}\,\lambda_i \quad$ for $i = 1, 2, \ldots, n$ $\qquad\square$

Corollary 3.3. If $G = C_n$, then the ES eigenvalues of G are $\nu_i = (4\sqrt{3})cos(\frac{2\pi i}{n})$, where $i = 0, \ldots, n - 1$.

Proof. From Lemma 2.2, if $G = C_n$, $\lambda_i = 2cos\frac{2\pi i}{n}$, for $i = 1, \ldots, n$, then $\nu_i = 2\sqrt{3}\lambda_i$, for $i = 1, \ldots, n$, would yield required result. $\qquad\square$

The following result gives the ES eigenvalues of complete bipartite graph.

Theorem 3.4. Let G be a complete bipartite graph of order $m + n$, with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_{m+n}$ eigenvalues and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_{m+n}$ be its ES eigenvalues. Then ES eigenvalues of $K_{m,n}$ are $\sqrt{m^2 + n^2 + mn}\,\lambda_i$, for $i = 1, \ldots, m + n$.

Proof. From the $ES(K_{m,n}) = (\sqrt{m^2 + n^2 + mn}) \, A(G(K_{m,n}))$.

By Lemma 2.3, the eigenvalues of $A(K_{m,n})$ are, $\lambda_1 = \sqrt{mn}$, $\lambda_2 = \lambda_3 = \cdots = \lambda_{m+n-1} = 0$ and $\lambda_{m+n} = -\sqrt{mn}$.

Therefore, $\nu_1 = (\sqrt{m^2 + n^2 + mn})\lambda_1$, $\nu_2 = \nu_3 = \cdots = \nu_{m+n-1} = (\sqrt{m^2 + n^2 + mn})\lambda_{m+n-1}$ and $\nu_{m+n} = (\sqrt{m^2 + n^2 + mn})\lambda_{m+n}$.

Hence the desired result obtained. $\qquad\square$

The following result gives the ES eigenvalues of star graph S_n.

Theorem 3.5. For any positive integer n, the ES eigenvalues of star graph S_n are $\nu_i = (\sqrt{n^2 - n + 1})\lambda_i$

Proof. The $ES(S_n) = (\sqrt{n^2 - n + 1})A(G(S_n))$.

From Lemma 3.16, the $A(S_n)$ eigenvalues are $\lambda_1 = \sqrt{n-1}$, $\lambda_2 = \cdots = \lambda_{n-1} = 0$ and $\lambda_n = -\sqrt{n-1}$

then the ES eigenvalues of S_n are , $\nu_1 = (\sqrt{n^2 - n + 1})\lambda_1$, $\nu_2 = \cdots = \nu_{n-1} = 0$ with multiplicity $n - 2$ and $\nu_n = (\sqrt{n^2 - n + 1})\lambda_n$. $\qquad\square$

The following result gives the relation between sum of squares of all eigenvalues of ES matrix of r-regular graph in Forgotten and Second Zagreb indices of the graph.

Theorem 3.6. Let G be a connected regular graph of order $n \geq 3$ and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ be the eigenvalues of $ES(G)$. Then

$$\sum_{i=1}^{n} \nu_i^2 \leq (n-1)(FI + SZ),$$

Proof. By the definition of the $ES(G)$ matrix, the trace of $ES(G)$ is 0.

Therefore, we have $\sum_{i=1}^{n} \nu_i = trace(ES(G)) = 0$.

$$\sum_{i=1}^{n} \nu_i^2 = trace(ES(G))^2 = (n-1) \sum_{v_i v_j \in E(G)} (d_i^2 + d_j^2 + d_i d_j)$$

$$\leq (n-1) \left(\sum_{v_i v_j \in E(G)} (d_i^2 + d_j^2) + \sum_{v_i v_j \in E(G)} (d_i d_j) \right).$$

$$\sum_{i=1}^{n} \nu_i^2 \leq (n-1) \left(\sum_{v_i v_j \in E(G)} (d_i^2 + d_j^2) + \sum_{v_i v_j \in E(G)} (d_i d_j) \right).$$

$$\sum_{i=1}^{n} \nu_i^2 \leq (n-1)(FI + SZ)$$

For r-regular graph G degree of every vertex is r, which gives

$$\sum_{i=1}^{n} \nu_i^2 = trace(ES(G)^2) \leq (n-1) \left(\sum_{v_i v_j \in E(G)} (d_i^2 + d_j^2) + \sum_{v_i v_j \in E(G)} (d_i d_j) \right) = r(FI + SZ).$$

$\square$

Theorem 3.7. Let G be a connected graph of order $n \geq 3$ with m edges and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ be the eigenvalues of $ES(G)$. Then

$$\sum_{i=1}^{n} \nu_i^2 = 2(FI + SZ)$$

and

$$\sum_{1 \leq i < j \leq n} \nu_i \nu_j = -(FI + SZ).$$

Proof. By the definition of the $ES(G)$ matrix, $\sum_{i=1}^{n} \nu_i = trace(ES(G)) = 0$.
As G be a connected,

$$\sum_{i=1}^{n} \nu_i^2 = trace(ES(G)^2) = 2 \sum_{v_i v_j \in E(G)} (d_i^2 + d_j^2 + d_i d_j)$$

$$\sum_{i=1}^{n} \nu_i^2 = 2 \left(\sum_{v_i v_j \in E(G)} (d_i^2 + d_j^2) + (\sum_{v_i v_j \in E(G)} d_i d_j) \right)$$

$$\sum_{i=1}^{n} \nu_i^2 = 2(FI + SZ)$$

Hence,

$$\sum_{i=1}^{n} \nu_i^2 = 2(FI + SZ).$$

Moreover,

$$\sum_{1 \leq i < j \leq n} \nu_i \nu_j = \frac{1}{2} \left(\left(\sum_{i=1}^{n} \nu_i \right)^2 - \sum_{i=1}^{n} \nu_i^2 \right) = -(FI + SZ).$$

$\square$

Perron-Frobenius Theorem: If all entries of $n * n$ matrix A are positive, then it has unique maximal eigenvalue and corresponding eigenvector has positive entries.

Lemma 3.8. [26] Let A be an $n \times n$ irreducible non-negative symmetric real matrix, $n \geq 2$. Let α_1 be the maximum eigenvalue of A and $\mathbf{x}$ be the unit Perron-Frobenius eigenvector of A. A has s $(2 \leq s \leq n)$ distinct eigenvalues if and only if there exits $s - 1$ real numbers $\alpha_2, \ldots, \alpha_s$ with $\alpha_1 > \alpha_2 > \cdots > \alpha_s$ such that

$$\prod_{i=2}^{s} (A - \alpha_i I) = \prod_{i=2}^{s} (\alpha_1 - \alpha_i) \mathbf{x} \mathbf{x}^t.$$

Moreover, $\alpha_1 > \alpha_2 > \cdots > \alpha_s$ are exactly the s distinct eigenvalues of A.

Now as ES matrix, $ES(G)$ of a connected graph G, which is an irreducible non-negative symmetric real matrix. Then by lemma 3.8 ,we have

Lemma 3.9. Let G be a connected graph of order $n \geq 2$ and $ES(G)$ be its ES matrix. Let ν_1 be the largest eigenvalue of $ES(G)$ and $\mathbf{x}$ be its corresponding unit column eigenvector. Then $ES(G)$ has k ($2 \leq m \leq n$) distinct eigenvalues if and only if there exist $m-1$ real numbers $\nu_2, \nu_3, \ldots, \nu_m$ with $\nu_1 > \nu_2 > \nu_3 > \cdots > \nu_m$ such that

$$(1) \qquad \prod_{i=2}^{m}(ES(G) - \nu_i I) = \prod_{i=2}^{m}(\nu_1 - \nu_i)\mathbf{x}\mathbf{x}^t.$$

Moreover, $\nu_1, \nu_2, \nu_3, \ldots, \nu_m$ are exactly the m distinct eigenvalues of $ES(G)$.

Theorem 3.10. Let G be a connected graph of order $n \geq 3$. Then G has two distinct the ES eigenvalues if and only if $G = K_n$.

Proof. Suppose G has two distinct ES eigenvalues, $\nu_1 > \nu_2$. Then by lemma 3.9,

$$(2) \qquad\qquad ES(G) = (\nu_1 - \nu_2)\mathbf{x}\mathbf{x}^t + \nu_2 I,$$

From above equation the off diagonal elements of $ES(G)$ are all positive.

Gives G is complete graph ,K_n i.e. $G = K_n$

Conversely,if G is complete graph,then

$$(3) \qquad\qquad ES(G) = (n-1)\sqrt{3}A(G)$$

thus

$$(4) \qquad\qquad \nu_i = (n-1)\sqrt{3}\lambda_i, i = 1, 2, \ldots, n.$$

Hence G has two distinct ES eigenvalues ,$(n-1)^2\sqrt{3}$ and $-(n-1)\sqrt{3}$ $\qquad\qquad\square$

Lemma 3.11. [1] Let$G$ be connected bipartite graph. If λ is eigenvalue of G with multiplicity k then $-\lambda$ is also eigenvalue of G with same multiplicity k. i.e. eigenvalues are symmetric about origin.

Theorem 3.12. A connected bipartite graph G of order $n \geq 3$ has three distinct ES eigenvalues if and only if G is a complete bipartite graph.

Proof. By Perron-Frobenius theorem,$\nu_1 > 0$,ν_n be simple and $\nu_i \neq \nu_1, \nu_n$ be three distinct ES eigenvalues of G. such that,$\nu_i \neq 0$

AS G is connected bipartite graph.Then by lemma 3.11

Then G will have 4-distinct eigenvalues.

Contradiction to that $\nu_i \neq 0$ ES eigenvalue of G.

Therefore , $\nu_i = 0$ is $n-2$ -times ES eigenvalue of G

Gives G has 3-distinct ES eigenvalues viz. $\nu_1 > 0, \nu_n$ and $\nu_i = 0, i \neq 1, n$

Thus by theorem 3.4,G is complete bipartite graph.

Conversly, Let G is complete bipartite graph of order $n \geq 3$ then obviously G is bipartite graph with three distinct ES eigenvalues. □

The following result gives the relation between diameter of graph G and ES eigenvalues of the graph G for this we have following lemma.

Lemma 3.13. Let G be a connected graph with adjacency matrix $A(G)$. Then the number of different paths of length k from vertices i and j is $(i,j)^{th}$ entry of $(A(G))^k$

Theorem 3.14. Let G be a connected graph of order n with m distinct ES eigenvalues. Then diam $(G) \leq m-1$.

Proof. Let $ES(G)$ be the ES matrix of G and $\nu_1 \geq \nu_2 \geq \cdots \geq \nu_n$ be its m distinct eigenvalues.

Let $\mathbf{x}$ be the unit eigenvector of $ES(G)$ corresponding to the largest eigenvalue ν_1. Where $\mathbf{x}$ is positive vector. Now, from Theorem **??** it follows that,

$$\prod_{i=2}^{m}(ES(G)-\nu_i I) = (ES(G))^{m-1}+b_1(ES(G))^{m-2}+\cdots+b_{m-2}(ES(G))+b_{m-1}I = \prod_{i=2}^{m}(\nu_1-\nu_i)\mathbf{x}\mathbf{x}^t = N.$$

Observe that $(N)_{ij} > 0$ for each $i,j = 1,2,\ldots,n$. Therefore, for $i \neq j$, there is a positive integer k with $1 \leq k \leq m-1$ such that $((ES(G))^k)_{ij} > 0$, which implies that is a path of length k between vertices v_i and v_j, that is, diam$(G) \leq m-1$. This complete the proof. □

Now we obtain the lower bound of largest ES eigenvalue of the graph as follows.

Theorem 3.15. Let G be a connected graph of order $n \geq 3$ and m edges. Then

$$(5) \qquad \nu_1 > \sqrt{\frac{2(FI+SZ)}{n}}.$$

Proof. By Theorem (3.7), We have $\sum_{i=1}^{n} \nu_i^2 = 2(FI+SZ)$.
Also $\sum_{i=1}^{n} \nu_i^2 < n\nu_1^2$, this implies that

$$n\nu_1^2 > \sum_{i=1}^{n} \nu_i^2 = 2(FI+SZ)$$

$$n\nu_1^2 > 2(FI+SZ)$$

$$\nu_1 > \sqrt{\frac{2(FI+SZ)}{n}}.$$

□

Now, we will obtain upper bound of largest ES eigenvalue of a graph.

Theorem 3.16. Let G be a connected graph of order $n \geq 3$. Then

$$(6) \qquad \nu_1 \leq \sqrt{\frac{2(n-1)(FI+SZ)}{n}}$$

equality holds (3.12) if $G = K_n$.

Proof. Let G be a connected graph of order $n \geq 3$. Then by Theorem(**??**), we have,

$$\sum_{i=1}^{n} \nu_i^2 = 2(FI + SZ)$$

Thus,

$$\nu_1^2 + s\sum_{i=2}^{n} \nu_i^2 = 2(FI + SZ)$$

$$2(FI + SZ) - \nu_1^2 = \sum_{i=2}^{n} \nu_i^2$$

Now by using Cauchy-Schwarz inequality

$$2(FI + SZ) - \nu_1^2 = \sum_{i=2}^{n} \nu_i^2 \geq \frac{1}{(n-1)}\left(\sum_{i=2}^{n} |\nu_i|\right)^2 \geq \frac{\nu_1^2}{(n-1)}$$

Then,

$$2(FI + SZ) - \nu_1^2 \geq \frac{\nu_1^2}{(n-1)}$$

gives,

$$2(FI + SZ) \geq \nu_1^2 \left(1 + \frac{1}{n-1}\right)$$

$$2(FI + SZ) \geq \nu_1^2 \left(\frac{n}{n-1}\right)$$

$$\nu_1^2 \leq \left(\frac{2(n-1)(FI + SZ)}{n}\right)$$

$$\nu_1 \leq \sqrt{\frac{2(n-1)(FI + SZ)}{n}}$$

The equality holds if $G = K_n$. □

4. ES-Energy of Graphs

In this section, we obtain ES energy of some classes of graph and bounds on the ES energy of a graph.

Theorem 4.1. Let G be a r-regular connected graph of order $n \geq 3$. Then $E_{ES}(G) = r\sqrt{3}E(G)$.

Corollary 4.2. If $G = K_n$ of order $n \geq 3$, then $E_{ES}(K_n) = \sqrt{3}(n-1)E(K_n)$.

Corollary 4.3. If $G = C_n$ of order $n \geq 3$, then $E_{ES}(C_n) = (2\sqrt{3})E(C_n)$.

Corollary 4.4. If $G = K_{m,n}$ of order $(m+n)$, then $E_{ES}(K_{m,n}) = (\sqrt{m^2 + n^2 + mn})E(K_{m,n})$.

The following result gives the lower bound of ES energy of a graph in the form of Forgotten and Second Zagreb index of graph.

Theorem 4.5. Let G be a connected graph with order $n \geq 3$ and m edges. Then $E_{ES}(G) \geq 2\sqrt{(FI + SZ)}$.

Proof. By the definition $E_{ES}(G)$ and Theorem (**??**), we have

$$(E_{ES}(G))^2 = \sum_{i=1}^{n} |\nu_i|^2 + 2 \sum_{1 \leq i < j \leq n}^{n} |\nu_i \nu_j|$$

$$\geq \sum_{i=1}^{n} \nu_i^2 + 2| \sum_{1 \leq i < j \leq n}^{n} \nu_i \nu_j|$$

$$= 2(FI + SZ) + 2| - (FI + SZ)| = 4(FI + SZ)$$

Implies,

$$E_{ES}(G) \geq 2\sqrt{(FI + SZ)}$$

$\square$

Now the following result gives the upper bound of ES energy of a graph in the form of Forgotten and Second Zagreb index of graph.

Theorem 4.6. Let G be a connected graph with order $n \geq 3$ with m edges then $E_{Es}(G) \leq \sqrt{2n(FI + SZ)}$.

Proof. By the definition $E_{Es}(G)$ and Theorem (**??**), we have

$$(E_{Es}(G))^2 = \sum_{i=1}^{n} |\nu_i|^2 + 2 \sum_{1 \leq i < j \leq n}^{n} |\nu_i \nu_j|$$

$$(E_{ES}(G))^2 \leq |4(FI + SZ)| \leq 2n(FI + SZ)$$

then,

$$E_{ES}(G) \leq \sqrt{2n(FI + SZ)}.$$

$\square$

Now to obtain the upper bound for Euler Sombor energy of the path P_n we use the following lemma.

Lemma 4.7. [23] Let M_1 and M_2 be the square matrices of order n. Then

$$\sum_{i=1}^{n} \nu_i(M_1 + M_2) \leq \sum_{i=1}^{n} \nu_i(M_1) + \sum_{i=1}^{n} \nu_i(M_2)$$

equality holds if and only if for an orthogonal matrix M, such that $M.M_1$ and $M.M_2$ are positive semi-definite.

Theorem 4.8. Let P_n be a path with order $n \geq 4$, then it's Euler Sombor energy is given by

$$E_{ES}(P_n) \leq \sqrt{12}E(P_n) + 2(\sqrt{7} - \sqrt{12}).$$

Proof. The Euler Sombor matrix of the path P_n can be written as,

$$ES(P_n) = (\sqrt{12}) \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 1 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & \dots & 1 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & \dots & 0 & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

where, $\alpha = (\sqrt{7} - \sqrt{12})$

Therefore $ES(P_n) = \sqrt{12} A(P_n) + B$

The Euler Sombor eigen values of B are, $\pm\{\sqrt{7} - \sqrt{12}\}$ repeated twice

Then by lemma 4.7, Euler Sombor Energy of path P_n is,

$$E_{ES}(P_n) \leq \sqrt{12}E(P_n) + 2(\sqrt{7} - \sqrt{12}).$$

$\square$

Here we have Euler Sombor ,Forgotten and Second Zagreb energies of Octane isomers

TABLE 1. Euler Sombor ,Forgotten and Second Zagreb energies of octane isomers

Sr.No.	Molucule	E_{ES}	E_{FI}	E_{SZ}
1	octane	30.2172	66.2439	31.6212
2	2-methyl-heptane	30.298	75.489	32.0568
3	3-methyl-heptane	31.61	77.6835	33.8232
4	4-methyl-heptane	30.194	75.0974	32.9665
5	3-ethyl-hexane	31.415	76.9651	34.5159
6	2,2-dimethyl-hexane	32.394	99.8456	33.9879
7	2,3-dimethyl-hexane	31.654	86.9688	35.5943
8	2,4-dimethyl-hexane	31.452	86.2309	34.114
9	2,5-dimethyl-hexane	31.642	87.171	33.3169
10	3,3-dimethyl-hexane	32.464	100.2934	36.4399
11	3,4-dimethyl-hexane	32.98	89.1456	37.1639
12	2-methyl-3-ethyl-pentane	31.502	86.293	36.0791
13	3-methyl-3-ethyl-pentane	33.928	102.0731	38.1993
14	2,2,3-trimethyl-pentane	33.804	111.5061	38.7971
15	2,2,4-trimethyl-pentane	31.964	107.718	33.9633
16	2,3,3-trimethyl-pentane	33.976	111.8089	39.7411
17	2,3,4-trimethyl-pentane	32.98	98.451	37.7739
18	2,2,3,3-tetramethyl-butane	34.632	134.0448	42.332

5. CORRELATION

In this section we have correlation coefficients [21] of ES energy to FI and SZ energy of octane isomers. Also correlation coefficients of ES topological index and some physicochemical properties of octane isomers.

TABLE 2. Correlation Coefficient of ES energy to FI and SZ energy of octane isomers

Sr.No.	Energy	Correlation Coefficient
1	FI	0.8900
2	SZ	0.9310

Now we are giving correlation coefficients of ES energy to some following physicochemical properties of octain isomers.

viz.Boiling point(BP),Entropy, Acentric factor (AF),Enthalpy of vaporization(HVAP) and standard enthalpy of vaporization(DHVAP)

TABLE 3. Some Physicochemical properties of octane isomers

Sr.No.	Molecule	BP	Entropy	AF	HVAP	DHVAP
1	octane	125.7	111.7	0.3979	73.19	9.915
2	2-methyl-heptane	117.6	109.8	0.3792	70.3	9.484
3	3-methyl-heptane	118.9	111.3	0.371	71.3	9.521
4	4-methyl-heptane	117.7	109.3	0.3715	70.91	9.483
5	3-ethyl-hexane	118.5	109.4	0.3625	71.7	9.476
6	2,2-dimethyl-hexane	106.8	103.4	0.3394	67.7	8.915
7	2,3-dimethyl-hexane	115.6	108	0.3483	70.2	9.272
8	2,4-dimethyl-hexane	109.4	107	0.3442	68.5	9.029
9	2,5-dimethyl-hexane	109.1	105.7	0.3568	68.6	9.051
10	3,3-dimethyl-hexane	112	104.7	0.3226	68.5	8.973
11	3,4-dimethyl-hexane	117.7	106.6	0.3404	70.2	9.316
12	2-methyl-3-ethyl-pentane	115.6	106.1	0.3324	69.7	9.209
13	3-methyl-3-ethyl-pentane	118.3	101.5	0.3069	69.3	9.081
14	2,2,3-trimethyl-pentane	109.8	101.3	0.3001	67.3	8.826
15	2,2,4-trimethyl-pentane	99.24	104.1	0.3054	64.87	8.402
16	2,3,3-trimethyl-pentane	114.8	102.1	0.2932	68.1	8.897
17	2,3,4-trimethyl-pentane	113.5	102.4	0.3174	68.37	9.014
18	2,2,3,3-tetramethyl-butane	106.5	93.06	0.2553	66.2	8.41

TABLE 4. Correlation coefficients of BP,Entropy,AF ,HVAP and DHVAP of octane isomers with ES index

Sr.No.	Index	BP	Entropy	AF	HVAP	DHVAP
1	ES	-0.374	-0.8740	-0.9003	-0.6129	-0.6966
2	FI	-0.672	-0.9611	-0.9743	-0.532	-0.9105
3	SZ	-0.248	-0.8557	-0.8967	-0.5053	-0.6139

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