Federated Linear Dueling Bandits

Xuhan Huang ¹ Yan Hu ¹ Zhiyan Li ¹ Zhiyong Wang ² Benyou Wang ¹ Zhongxiang Dai ¹

Abstract

Contextual linear dueling bandits have recently garnered significant attention due to their widespread applications in important domains such as recommender systems and large language models. Classical dueling bandit algorithms are typically only applicable to a single agent. However, many applications of dueling bandits involve multiple agents who wish to collaborate for improved performance yet are unwilling to share their data. This motivates us to draw inspirations from federated learning, which involves multiple agents aiming to collaboratively train their neural networks via gradient descent (GD) without sharing their raw data. Previous works have developed federated linear bandit algorithms which rely on closed-form updates of the bandit parameters (e.g., the linear function parameter) to achieve collaboration. However, in linear dueling bandits, the linear function parameter lacks a closed-form expression and its estimation requires minimizing a loss function. This renders these previous methods inapplicable. In this work, we overcome this challenge through an innovative and principled combination of online gradient descent (for minimizing the loss function to estimate the linear function parameters) and federated learning, hence introducing the first federated linear dueling bandit algorithms. Through rigorous theoretical analysis, we prove that our algorithms enjoy a sub-linear upper bound on its cumulative regret. We also use empirical experiments to demonstrate the effectiveness of our algorithms and the practical benefit of collaboration.

1. Introduction

Contextual dueling bandits (Saha, 2021; Saha & Krishnamurthy, 2022; Bengs et al., 2022; Li et al., 2024b) have

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received significant attention recently, due to its widespread adoption in important real-world applications such as recommender systems (Yue et al., 2012) and large language models (LLMs) (Lin et al., 2024; Ji et al., 2024). In every iteration of a contextual dueling bandit problem, an agent receives a d-dimensional context vector and K arms, selects a pair of arms, and collects a binary observation indicating the relative preference between the selected pair of arms (Bengs et al., 2022). For example, when applying a contextual dueling bandit algorithm to optimize the response from an LLM, we receive a context (i.e., a prompt), use the LLM to generate K responses (i.e., arms), select from them a pair of responses, and then ask the user which response is preferred (Lin et al., 2024). In order to intelligently select the pairs of arms, we often adopt a surrogate function to model the *latent reward function* governing the preference feedback observations (more details in Sec. 2), for which we typically adopt a linear function. That is, we assume that the reward function value at arm x can be expressed as $f(x) = \theta^{\top} \phi(x)$ for some unknown $\theta \in \mathbb{R}^d$ and known feature mapping $\phi(\cdot) \in \mathbb{R}^d$. This is often referred to as the contextual linear dueling bandit problem.

Classical contextual dueling bandit algorithms, as discussed above, are only applicable to single-agent scenarios. However, many real-world applications involve multiple agents, which creates opportunities to enhance performance via collaboration among agents. Of note, in such applications, the agents are often concerned with privacy and are often unwilling to share their data, including the selected arms and the observed preference feedback. For example, users adopting contextual dueling bandits to optimize the response of LLMs may want to collaborate with each other without sharing their personal data involving their selected responses and preference feedback. These requirements naturally align with the paradigm of federated learning (FL), which enables multiple agents to collaboratively train their neural networks (NNs) without requiring them to share their raw data (McMahan et al., 2017). In every round of FL, every agent calculates the local gradient (or parameters) of their NNs and sends them to the central server, who aggregates all received local gradients (typically via simple averaging) and then broadcasts the aggregated information back to the agents for further local updates of their NN parameters (McMahan et al., 2017).

¹The Chinese University of Hong Kong, Shenzhen ²The Chinese University of Hong Kong. Correspondence to: Zhongxiang Dai <dai:zhongxiang@cuhk.edu.cn>.

To extend contextual linear dueling bandits to the federated setting, we can draw inspirations from previous works on federated contextual bandits (Shi & Shen, 2021). Notably, the work of Wang et al. (2019) has proposed a federated contextual linear bandit algorithm, which only requires the agents to share some sufficient statistics, including a d-dimensional vector and a $(d \times d)$ -dimensional matrix. Importantly, the two terms in the linear upper confidence bound (Lin-UCB) policy for arm selection, including (a) the estimated linear function parameter (i.e., the estimate of θ) and (b) an exploration term, can be expressed in closed forms in terms of the summation of these sufficient statistics. As a result, in order to aggregate the information from all agents, the central server only needs to aggregate the sufficient statistics from all agents via a simple summation. This allows every agent to effectively leverage the additional observations from the other agents to accelerate their bandit algorithm without requiring the exchange of raw observations. This paradigm from Wang et al. (2019) has been widely adopted and extended to other bandit problems, such as federated neural bandits (Dai et al., 2022).

However, in contextual linear dueling bandits, the estimated linear function parameter θ lacks a closed-form expression (Bengs et al., 2022). Instead, we often need to estimate θ by minimizing a loss function via gradient descent (GD). This renders the federated bandit paradigm from Wang et al. (2019) inapplicable to our problem of contextual linear dueling bandits. To resolve this challenge, we propose to allow the agents to collaboratively use GD to estimate the linear function parameter. Specifically, the joint loss function (to be minimized for parameter estimation) for all agents can be expressed in terms of a summation among all agents (more details in Sec. 3). Therefore, the gradient of the joint loss function can be decoupled into the contributions from individual agents. As a result, we let every agent calculate the local gradient of its own loss function and send it to the central server. The central server then aggregates (i.e., sums) all local gradients to attain the gradient of the joint loss function, using which GD can be performed. Interestingly, in contrast to previous works on federated bandits (Shi & Shen, 2021; Dai et al., 2022), this novel approach bears a closer resemblance to the original federated learning (FL) paradigm which also involves the exchange of local gradients (McMahan et al., 2017).

We firstly propose an algorithm adopting federated GD (Algo. 2) to estimate the linear function parameter θ , which conducts multiple rounds gradient exchange between the central server and the agents in every iteration. Next, we develop a more practical algorithm which employs federated online GD (Algo. 3) for estimating $\hat{\theta}$, in which only a single round of gradient exchange is needed in every iteration. Our high-level federated linear dueling bandit (FLDB) algorithm can adopt either federated GD or federated online GD

(OGD) for parameter estimation, and we refer to the resulting algorithms as the FLDB-GD algorithm and FLDB-OGD algorithm, respectively.

We perform rigorous theoretical analysis of the regret of our FLDB-GD and FLDB-OGD algorithms and show that they both enjoy sub-linear upper bounds on the cumulative regret (Sec. 4). Our theoretical results show that our FLDB-GD algorithm enjoys a tighter regret upper bound, yet it requires a larger number of communication rounds. On the other hand, our FLDB-OGD algorithm is significantly more communication-efficient (hence more practical) despite having a worse regret upper bound. Through extensive empirical experiments using synthetic functions, we demonstrate that both our algorithms consistently outperform single-agent linear dueling bandits (Sec. 5). In addition, the performance of our algorithm becomes improved as the number of agents is increased, and FLDB-GD, which requires more communication rounds, achieves smaller regrets than FLDB-OGD for the same number of agents.

2. Background and Problem Setting

In a contextual dueling bandit problem, at each iteration t, the environment generates a set of K arms, denoted as $\mathcal{X}_t \subset \mathcal{X} \subset \mathbb{R}^d$, where \mathcal{X} represents the domain of all possible arms. The agent then selects a pair of arms $x_{t,1}, x_{t,2} \in \mathcal{X}_t$ and receives feedback y_t , which is 1 if $x_{t,1}$ is preferred over $x_{t,2}$ and 0 otherwise.

Preference Model. Following the common practice in dueling bandits (Saha, 2021; Bengs et al., 2022; Li et al., 2024b), we assume that the preference feedback follows the Bradley-Terry-Luce (BTL) model (Hunter, 2004; Luce, 2005). Specifically, the utility of the arms are represented by a *latent reward function* $f: \mathbb{R}^d \to \mathbb{R}$, which maps any arm x to its corresponding reward value f(x). Here we assume that f is a linear function $f(x) = \theta^{\top} \phi(x), \forall x$, in which $\theta \in \mathbb{R}^d$ is an unknown parameter and $\phi(\cdot) \in \mathbb{R}^d$ is a known feature mapping. This reduces to standard linear dueling bandits when $\phi(\cdot)$ is the identity mapping. Then, the probability that the first selected arm $(x_{t,1})$ is preferred over the second selected arm $(x_{t,2})$ for the given reward function f is given by

$$\mathbb{P}(x \succ x') = \mathbb{P}\{y_t = 1 | x_{t,1}, x_{t,2}\} = \mu(f(x_{t,1}) - f(x_{t,2})).$$

Here $x_1 \succ x_2$ indicates that $x_{t,1}$ is preferred over $x_{t,2}$, $\mu(x) = 1/(1 + \mathrm{e}^{-x})$ is the *link function* for which we adopt the logistic function.

We list below the assumptions needed for our theoretical analysis, all of which are standard assumptions commonly adopted by previous works on dueling bandits (Bengs et al., 2022; Li et al., 2017a)).

• $\kappa_{\mu} \doteq \inf_{x,x' \in \mathcal{X}} \dot{\mu}(f(x) - f(x')) > 0$ Assumption 1. for all pairs of context-arm.

- The link function $\mu: \mathbb{R} \to [0,1]$ is continuously differentiable and Lipschitz with constant L_{μ} . For logistic function, $L_{\mu} \leq 1/4$.
- The difference between feature maps is bounded $\|\phi(x_{t,1}) - \phi(x_{t,2})\|_2 \le 1$ for all contexts.

We also need the following assumption on the context vec-

Assumption 2. For
$$x_1, x_2 \in \mathcal{X}_{t,i}$$
, $\widetilde{\phi}_{t,i} = \phi(x_1) - \phi(x_2)$
Denote $\Sigma = \widetilde{\phi}_{t,i}\widetilde{\phi}_{t,i}^T$ and $\lambda_f = \inf_{i,t,x_1,x_2 \in \mathcal{X}_{t,i}} \lambda_{\min}(\Sigma)$. We assume λ_f is a positive constant.

Assumption 2 is also commonly used in the analysis of generalized linear bandits and dueling bandits (Li et al., 2017b). For example, this assumption is in a similar vein to Assumption 1.5 from Wu et al. (2020) and Assumption 3 from Ding et al. (2021). Intuitively, Assumption 2 is reasonable since it ensures that the context vectors sampled from the environment are sufficiently diverse to guarantee a non-degenerate difference between feature vectors.

Performance measure. The goal of an agent in contextual dueling bandits is to minimize its regret. After selecting a pair of arms, denoted by $x_{t,1}$ and $x_{t,2}$, in iteration t, the learner incurs an instantaneous regret. In our theoretical analysis, we aim to analyze the following regret:

$$r_t = 2f(x_t^*) - f(x_{t,1}) - f(x_{t,2})$$

in which $x_t^* = \arg\max_{x \in \mathcal{X}_t} f(x)$ denotes the optimal arm in iteration t. After observing preference feedback for T pairs of arms, the cumulative regret (or regret, in short) of a sequential policy is given by

$$R_T = \sum_{t=1}^{T} r_t = \sum_{t=1}^{T} \left(2f(x_t^*) - f(x_{t,1}) - f(x_{t,2}) \right),$$

in which $x_t^* = \arg \max_{x \in \mathcal{X}_t} f(x)$ denotes the optimal arm in iteration t. Any good policy should have sub-linear regret, i.e., $\lim_{T\to\infty} R_T/T = 0$. A policy with a sub-linear regret implies that the policy will eventually find the best arm and recommend only the best arm in the duel for the given contexts.

Federated Contextual Dueling Bandits. In federated contextual dueling bandits involving N agents, we assume that all agents share the same latent reward function $f(x) = \theta^{\top} \phi(x)$. This is consistent with the previous works on federated bandits (Shi & Shen, 2021; Dai et al., 2022). In iteration t, every agent i receives a separate set of arms

 $\mathcal{X}_{t,i} \subset \mathcal{X}$ and chooses from them a pair of arms denoted as $x_{t,1,i}$ and $x_{t,2,i}$. In this case, we analyze the total cumulative regret from all N agents:

$$R_{T,N} = \sum_{t=1}^{T} \sum_{i=1}^{N} \left(2f(x_{t,i}^*) - f(x_{t,1,i}) - f(x_{t,2,i}) \right),$$

in which $x_{t,i}^* = \arg\max_{x \in \mathcal{X}_{t,i}} f(x)$.

3. Federated Linear Dueling Bandits

We firstly introduce our Algo. 1, which is the high-level framework for our federated linear dueling bandit (FLDB) algorithm. When Algo. 1 employs GD (Algo. 2) for estimating the parameter θ_{sync} (see line 9 of Algo. 1), we refer to the resulting algorithm as FLDB-GD; when Algo. 1 adopts *OGD* (Algo. 3) to estimate θ_{sync} , we refer to the algorithm as FLDB-OGD.

Algorithm 1 Federated Linear Dueling Bandits

- 1: **for** t = 1, ..., T **do**
- for Agent $i = 1, \dots, N$ in parallel do
- 3: [Agent i] Receive contexts $\mathcal{X}_{t,i}$
- 4: [Agent i] Choose the first arm $x_{t,1,i}$ $\arg\max_{x\in\mathcal{X}_{t,i}}\theta_{\mathsf{sync}}^{\top}\phi(x)$
- [Agent i] Choose the second 5: $x_{t,2,i} = \arg\max_{x \in \mathcal{X}_{t,i}} \theta_{\text{sync}}^{\top} \left(\phi(x) - \phi(x_{t,1,i}) \right) +$ $\frac{\beta_t}{\kappa_\mu} \|\phi(x) - \phi(x_{t,1,i})\|_{W^{-1}_{\text{sync}}} \\ \text{[Agent i]} \text{ Observe the preference feedback } y_{t,i} =$
- 6: $\mathbb{1}(x_{t,1,i} \succ x_{t,2,i})$
- [Agent i] Calculate $W_{\text{new},i}$ $(\phi(x_{t,1}) \phi(x_{t,2})) (\phi(x_{t,1}) \phi(x_{t,2}))^{\top}$, 7: and send it to central server
- 8:
- 9: [Central Server] Run Algorithms 2 or 3 to obtain
- [Central Server] Update $W_{\text{sync}} \leftarrow W_{\text{svnc}} +$ 10:
- $\frac{\sum_{i=1}^{N}W_{\text{new},i}}{\text{[Central Server] Broadcast }\{\theta_{\text{sync}},W_{\text{sync}}\}\ \ \text{to}\ \ \text{all}}$ 11: agents
- 12: **end for**

In iteration t, each agent $i \in [N]$ receives a set of K contexts, denoted as $\mathcal{X}_{t,i}$ (line 3 of Algo. 1), and selects two arms using the synchronized parameters θ_{sync} and W_{sync} following the classic linear dueling bandit (LDB) algorithm (Bengs et al., 2022). Specifically, the first arm $x_{t,1,i}$ is selected greedily based on $\theta_{\rm sync}$, which is the current estimated linear function parameter based on the data from all agents (line 4 of Algo. 1). Next, the second arm $x_{t,2,i}$ is chosen following an upper confidence bound strategy to balance exploration and exploitation (line 5 of Algo. 1). In this way, we encourage $x_{t,2,i}$ to both achieve high predicted reward

and be different from $x_{t,1,i}$ as well as the previously selected arms. After observing the preference feedback $y_{t,i}$, the agent computes the updated information matrix $W_{\text{new},i}$ and sends it to a central server (line 6-7 of Algo. 1). Next, the central server coordinates all agents to estimate the parameter θ_{sync} (to be used in selecting the pair of arms in the next iteration) through either Algo. 2 or Algo. 3 (line 9 of Algo. 1). Finally, the central server aggregates the updated information matrix $W_{\text{new},i}$ from all agents (lines 10 of Algo. 1) and broadcasts the aggregated information matrix W_{sync} and the estimated θ_{sync} back to all agents (line 11 of Algo. 1) to start the next iteration.

In the next two sections, we introduce how we estimate the parameter $\theta_{\rm sync}$ via either federated GD (Sec. 3.1) or federated OGD (Sec. 3.2).

3.1. Federated GD for Estimating θ_{sync} (Algo. 2)

In a communication round of our algorithm to estimate θ_{sync} , the goal of the central server is to fine $\theta_{\text{sync}} = \arg\min_{\theta'} \mathcal{L}_t^{\text{fed}}(\theta')$, in which the loss function is given by:

$$\mathcal{L}_t^{\text{fed}}(\theta') = \sum_{i=1}^N \mathcal{L}_t^i(\theta') + \frac{1}{2}\lambda \|\theta'\|_2^2, \tag{1}$$

where

$$\mathcal{L}_{t}^{i}(\theta') = -\sum_{s=1}^{t-1} \left(y_{s}^{i} \log \mu \left(\theta'^{\top} \left[\phi(x_{s,1}^{i}) - \phi(x_{s,2}^{i}) \right] \right) + (1 - y_{s}^{i}) \log \mu \left(\theta'^{\top} \left[\phi(x_{s,2}^{i}) - \phi(x_{s,1}^{i}) \right] \right) \right).$$

$$(22)$$

Equivalently, $\theta_{\text{sync}} = \arg\min_{\theta'} \mathcal{L}_t^{\text{fed}}(\theta')$ is the maximum likelihood estimate (MLE) of the unknown parameter θ given the data from all N agents up to iteration t-1 (Bengs et al., 2022).

Since the loss function (1) is convex for any t, it is natural to apply gradient descent (GD) to optimize (1). Interestingly, the loss function (1) is naturally decoupled across different agents, which allows the central server to estimate the gradient of (1) using the gradients of the individual local loss functions (2). As a result, the agents only need to send their local gradients to the central server and are hence not required to share their local data $\{x_{t,1,i}, x_{t,2,i}, y_{t,i}\}$. This is formalized by Algo. 2.

Details of Algo. 2. In every step of Algo. 2, the central server broadcasts its current estimate $\theta^{(s)}$ to all N agents (line 2 of Algo. 2). Next, each agent i calculates its own gradient $\nabla \mathcal{L}_t^i(\theta^{(s)})$, and sends it back to the central server (line 4-5 of Algo. 2). Then, the central server aggregates (i.e., sums) the gradients from all agents (line 7 of Algo. 2) to

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Algorithm 2 Federated GD for Estimating \theta_{\text{sync}}
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- 1: **for** s = 0, ..., M 1 **do**
- 2: [Central Server] Central Server broadcasts $\theta^{(s)}$ to all agents
- 3: **for** Every Agent i = 1, ..., N **do**
- 4: [Agent i] Calculate local gradient $\nabla \mathcal{L}_t^i(\theta^{(s)})$ (2)
- 5: [Agent *i*] Send $\nabla \mathcal{L}_t^i(\theta^{(s)})$ back to Central Server
- 6: end for
- 7: [Central Server] Central Server aggregates local gradients: $\nabla \mathcal{L}_t^{\text{fed}}(\theta^{(s)}) = \sum_{i=1}^N \nabla \mathcal{L}_t^i(\theta^{(s)}) + \lambda \theta^{(s)}$ (1)
- 8: [Central Server] Central Server updates parameter: $\theta^{(s+1)} = \theta^{(s)} \eta \nabla \mathcal{L}_{t}^{\text{fed}}(\theta^{(s)})$
- 9: end for
- 10: [Central Server] $\theta_{\text{sync}} = \theta^{(M)}$

obtain $\nabla \mathcal{L}_t^{\text{fed}}(\theta^{(s)})$, and uses it to perform a step of GD (line 8 of Algo. 2): $\theta^{(s+1)} = \theta^{(s)} - \eta \nabla \mathcal{L}_t^{\text{fed}}(\theta^{(s)})$. After that, the central server broadcasts the updated $\theta^{(s+1)}$ to all agents to initiate the next step s+1. After repeating these procedures for M steps, the central server obtains $\theta_{\text{sync}} = \theta^{(M)}$ (line 10 of Algo. 2) and broadcasts it to all agents (line 11 of Algo. 1), who then use it to select their pairs of arms in the next iteration t+1 (lines 4-5 of Algo. 1).

However, although the objective function (1) is convex, finding the optimal solution of the loss function at time t requires multiple rounds of gradient descent. This necessitates multiple rounds of gradient exchanges between the central server and the agents in each iteration. To address this, we propose Algo. 3 with OGD to estimate $\theta_{\rm sync}$, in which only a single round of GD is needed in every iteration.

3.2. Federated OGD for Estimating θ_{sync} (Algo. 3)

The loss function which Algo. 3 aims to minimize is the same as (1) but reformulated into the following form:

$$\mathcal{L}_t^{\text{fed}}(\theta') = \sum_{s=1}^t f_s(\theta'), \tag{3}$$

where

$$f_s^{\text{fed}}(\theta') = \begin{cases} \sum_{i=1}^{N} l_s^i(\theta'), & \text{if } s \neq 1, \\ \sum_{i=1}^{N} l_1^i(\theta') + \frac{\lambda}{2} \|\theta'\|_2^2, & \text{if } s = 1. \end{cases}$$
(4)

$$l_s^i(\theta') = -\left(y_s^i \log \mu \left(\theta'^{\top} \left[\phi(x_{s,1}^i) - \phi(x_{s,2}^i)\right]\right) + (1 - y_s^i) \log \mu \left(\theta'^{\top} \left[\phi(x_{s,2}^i) - \phi(x_{s,1})\right]\right)\right)$$

$$(5)$$

Algorithm 3 Federated Online GD for Estimating θ_{sync}

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1: if t = 1 then
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[Central Server] Calculate the optimal value $\widehat{\theta}^{(t)}$ of the loss function (1) using algorithm 2, let $\widetilde{\theta}^{(t)} = \widehat{\theta}^{(t)}$

[Central Server] Compute r and maintain convex set $S = \left\{ \theta : \left\| \theta - \widehat{\theta}^{(t)} \right\| \le 2r \right\}.$

[Central Server] Central Server broadcasts $\widehat{\theta}^{(t)}$ to all 5:

for Every Agent i = 1, ..., N do 6:

[Agent i] Calculate local gradient $\nabla l_t^i(\widehat{\theta}^{(t)})$ (5) 7:

[Agent i] Send $\nabla l_t^i(\widehat{\theta}^{(t)})$ back to Central Server 8:

9:

10: [Central Server] Central Server aggregates local gradients: $\nabla f_t^{\text{fed}}(\widehat{\theta}^{(t)}) = \sum_{i=1}^N \nabla l_t^i(\widehat{\theta}^{(t)})$ ((3) and (4)) [Central Server] Central Server updates parameter:

11:

 $\eta_t = \frac{1}{\alpha t}, \widehat{\theta}^{(t+1)} = \pi_S \left(\widehat{\theta}^{(t)} - \eta_t \nabla f_t^{\text{fed}}(\widehat{\theta}^{(t)}) \right)$ [Central Server] Central Server updates parameter: $\widehat{\theta}^{(t+1)} = \frac{1}{t+1} \sum_{j=1}^{t+1} \widehat{\theta}^{(j)}$ 12:

13: end if

14: [Central Server] $\theta_{
m sync} = \widetilde{ heta}^{(t+1)}$

Intuitively, every individual loss function f_s^{fed} (4) corresponds to losses of N agents in iteration s. Inspired by the work of Ding et al. (2021) who have used OGD to estimate the parameters in generalized linear bandits, in iteration t of our Algo. 3, we only use the latest information (i.e., the gradient of f_t^{fed} (4)) to update our estimation of θ_{sync} . That is, instead of GD, we use OGD to estimate θ_{sync} .

Details of Algo. 3. At the beginning of the estimation (t=1), the central server calculates the minimizer of (3) at iteration 1 (line 2 of Algo. 3), which is also the minimizer of $f_1^{\mathrm{fed}}(\theta')$. This ensures the subsequent estimated θ_{sync} always lies in a bounded ball centered at the groundtruth parameter θ , which is needed in our theoretical analysis (Sec. 4). The server also maintains a ball centered at $\widehat{\theta}^{(t)}$ as the projection set during OGD (line 3 of Algo. 3).

Starting from the second round of communication ($t \ge 2$), which corresponds to the second iteration of the overall algorithm, the server computes the gradient at the current $\widehat{\theta}^{(t)}$ and performs online gradient descent. Specifically, the central server firstly broadcasts its current parameter $\widehat{\theta}^{(t)}$ to all N agents (line 5 of Algo. 3). Each agent i then computes its own gradient $\nabla l_t^i(\widehat{\theta}^{(t)})$ of (5) and sends it back to the central server (lines 7-8 of Algo. 3). After that, the central server aggregates the individual gradients $\nabla l_t^i(\widehat{\theta}^{(t)})$ from all agents to obtain $\nabla f_t^{\text{fed}}(\widehat{\theta}^{(t)})$ following (4) (line 10 of Algo. 3). Then, the central server performs one step of projected gradient descent with step size $\eta = \frac{1}{\alpha t}$

(line 11). Finally, to leverage historical information, the server averages (i.e., averages) the past estimates $\widehat{\theta}^{(s)}$ and uses the resulting $\widetilde{\theta}^{(t+1)}$ as the updated estimate θ_{sync} in round t (lines 12-14 of Algo. 3). This estimate θ_{sync} is then broadcast to all agents (line 11 of Algo. 1). Compared to Algo. 2, Algo. 3 requires only one round of communication per iteration for t > 1. Consequently, Algo. 3 is more communication-efficient.

Communication Efficiency. In both FLDB-GD and FLDB-OGD, communication between the central server and the agents occurs in every iteration, which is unlike some previous works on federated bandits (Wang et al., 2019; Dai et al., 2022). This is because of the inherent difficulty in the problem setting of contextual dueling bandits. Specifically, if we start a communication round only after multiple iterations, then when an agent selects arms in an intermediate iteration, it has collected some additional local observations since the last communication round. As a result, the algorithm and analysis from Wang et al. (2019) require a mechanism to combine the newly collected local information with the synchronized information from all agents in the last communication round. This can be easily achieved in (nondueling) linear bandits thanks to the closed-form expression of the estimated θ in terms of a summation across iterations (Wang et al., 2019), which allows us to combine the new local information and synchronized global information via a simple summation. However, this becomes infeasible in our problem of contextual dueling bandits where a closed-form expression of the estimated θ is unavailable. This makes combining the local and global information, and hence designing an algorithm not requiring communication in every iteration, highly challenging, which we leave to future work.

4. Theoretical Analysis

Here we analyze the regret of our FLDB-GD and FLDB-OGD algorithms.

4.1. Analysis of FLDB-GD

We make the following assumption in our analysis of FLDB-GD.

Assumption 3. In our analysis here, we assume that when running Algo. 2, the resulting $\theta_{\rm sync}=\theta^{(M)}$ can exactly minimize (1), i.e., $\nabla \mathcal{L}_t^{\rm fed}(\theta^{(M)})=0$.

This assumption is commonly adopted by previous works on (non-federated) generalized linear bandits and dueling bandits (Li et al., 2017b; Saha, 2021; Bengs et al., 2022). That is, it is a common practice in the literature to assume that the maximum likelihood estimation of the parameter θ is obtained exactly. In our problem of federated dueling bandits, this may incur a large number of communication rounds between the central server and the agents. We will remove the need for this assumption in our analysis of our FLDB-OGD algorithm (Sec. 4.2). The theorem below provides an upper bound on the cumulative regret of our FLDB-GD algorithm.

Denote $\theta_t \triangleq \arg\min_{\theta'} \mathcal{L}_t^{\text{fed}}(\theta')$, i.e., θ_t is the minimizer of the loss function $\mathcal{L}_t^{\text{fed}}$. Note that according to Assumption 3, we have that $\theta_{\text{sync}} = \theta_t$. In our proof of the regret upper bound for FLDB-GD, a crucial step is to derive the following concentration bound.

Lemma 4.1. Let

$$\beta_t \triangleq \sqrt{2\log(1/\delta) + d\log\left(1 + tN\kappa_{\mu}/(d\lambda)\right)}$$

Then with probability of at least $1 - \delta$, we have that for all t = 1, ..., T

$$\|\theta - \theta_t\|_{V_t} \le \frac{\beta_t}{\kappa_\mu}.$$

Lemma 4.1 suggests that after running the federated GD algorithm in Algo. 2, the resulting $\theta_{\rm sync}=\theta_t$ is a good approximation of the groundtruth parameter θ . The regret of FLDB-GD can then be upper-bounded by the following theorem.

Theorem 4.2 (FLDB-GD). With probability at least $1 - \delta$, the overall regrets of all agents in all iterations satisfy:

$$R_{T,N} \le 2Nd\log\left(1 + TN\kappa_{\mu}/(d\lambda)\right) + 3\sqrt{2}\frac{\beta_T}{\kappa_{\mu}}\sqrt{T2d\log\left(1 + TN\kappa_{\mu}/(d\lambda)\right)}$$
 (6)

Ignoring all log factors, we have that

$$R_{T,N} = \widetilde{O}\left(Nd + \frac{d}{\kappa_{\mu}}\sqrt{T}\right) \tag{7}$$

The regret upper bound for FLDB-GD (Theorem 4.2) is sublinear in T. Theorem 4.2 suggests that the average regret of N agents in our FLDB-GD algorithm is upper-bounded by $\widetilde{O}\left(d+\frac{d}{N\kappa_{\mu}}\sqrt{T}\right)$, which becomes smaller with a larger number N of agents. This demonstrates the benefit of collaboration, because it shows that if a larger number N of agents join the federation of our algorithm, they are guaranteed (on average) to achieve a smaller regret compared with running their contextual dueling bandit algorithms in isolation (i.e., N=1). When there is a single agent, the regret upper bound from Theorem 4.2 reduces to $\widetilde{O}(d+d\sqrt{T}/\kappa_{\mu})=\widetilde{O}(d\sqrt{T}/\kappa_{\mu})$, which is of the same order as the classical contextual linear dueling bandit algorithm (Bengs et al., 2022).

4.2. Analysis of FLDB-OGD

In the analysis of FLDB-OGD, we do not require Assumption 3. In other words, the $\theta_{\text{sync}} = \widetilde{\theta}^{(t)}$ returned by Algo. 3

is no longer the minimizer of $\mathcal{L}_t^{\mathrm{fed}}$ in (3). However, here we prove that as long as we have a sufficiently large number of agents, the difference between $\theta_{\mathrm{sync}} = \widetilde{\theta}^{(t)}$ and θ_t is still bounded.

To begin with, the following proposition proves that as long as the number N of agents is large enough, every loss function $f_s^{\rm fed}(\theta')$ in OGD (4) is strongly convex.

Proposition 4.3 (α -strongly convex). *Denote* $\mathbb{B}_{\eta} := \{\theta' : \|\theta' - \theta\| \le \eta\}$. *For a constant* $\alpha > 0$, *let*

$$N \ge \left(\frac{C_1\sqrt{d} + C_2\sqrt{\log\frac{2}{\delta}}}{\lambda_f}\right)^2 + \frac{2\alpha}{\kappa_\mu \lambda_f},$$

where C_1 and C_2 are two universal constants. Then $f_s^{fed}(\theta')$ is an α -strongly convex function in \mathbb{B}_{3r} , with probability at least $1 - \delta$.

Recall that every $f_s^{\rm fed}(\theta') = \sum_{i=1}^N l_s^i(\theta')$ corresponds to an individual loss function in our overall loss function (3) during our OGD algorithm (Algo. 3). Therefore, we can make use of Proposition 4.3 and the convergence result of OGD for strongly convex functions (Hazan et al., 2016) to derive the following lemma.

Lemma 4.4. Under the same condition on N as Proposition 4.3, with probability at least $1 - \delta$, the following holds for all $t \ge 1$,

$$\left\|\widetilde{\theta}^{(t)} - \theta_t\right\|_{V_t} \le \frac{N\sqrt{N + \frac{\lambda}{\kappa_{\mu}}}}{\alpha} \sqrt{1 + \log t}.$$

Lemma 4.4 shows that the difference between $\theta_{\rm sync} = \widetilde{\theta}^{(t)}$ (returned by Algo. 3) and $\theta_t \triangleq \arg\min_{\theta'} \mathcal{L}_t^{\rm fed}(\theta')$ is bounded. As a result, Lemma 4.4, combined with Lemma 4.1 which has provided an upper bound on the difference between θ_t and θ , allows us to bound the difference between $\widetilde{\theta}^{(t)}$ and θ . That is, the parameter $\widetilde{\theta}^{(t)}$ returned by Algo. 3 is an accurate estimation of the groundtruth linear function parameter θ .

With these important supporting lemmas, the following theorem can be proved, which gives an upper bound on the cumulative regret of FLDB-OGD.

Theorem 4.5 (FLDB-OGD). *Ignoring all log factors, with probability of at least* $1 - \delta$, we have the overall regret can be bounded by

$$R_{T,N} = \widetilde{O}\left(Nd + \frac{d}{\kappa_{\mu}}\sqrt{T} + \frac{N^{\frac{3}{2}}\sqrt{d}}{\alpha}\sqrt{T}\right)$$
 (8)

The regret upper bound of our FLDB-OGD algorithm is also sub-linear in T. Compared with FLDB-GD (Theorem 4.2),

the regret upper bound for FLDB-OGD has an additional term of $\widetilde{O}(\frac{N^{3/2}\sqrt{d}}{\alpha}\sqrt{T})$. This is the loss we suffer for not ensuring that the $\theta_{\rm sync}=\widetilde{\theta}^{(t)}$ returned by Algo. 3 could achieve the minimum of $\mathcal{L}_t^{\rm fed}$ (3). On the other hand, by paying this cost in terms of regret, FLDB-OGD gains a considerably smaller communication cost than FLDB-GD, because FLDB-OGD only needs a single round of communication whereas FLDB-GD requires a potentially large number of communication rounds to find the minimum of $\mathcal{L}_t^{\rm fed}(\theta')$ (1).

Unlike the theoretical result for FLDB-GD (Theorem 4.2), the regret upper bound in (Theorem 4.5) is not improved as the number N of agents is increased. This is because in the proof of Lemma 4.4 (App. C), we have made use of the convergence of OGD for strongly convex functions (Hazan et al., 2016), which requires an upper bound on the expected norm of the gradient: $G^2 \geq E \|\nabla f_s\|^2$. We have shown that the worst-case choice of G is G = N (see (32) in App. C), which contributed a dependency of N to the last term in the regret upper bound in Theorem 4.5. In other words, if we additionally assume that there exists an upper bound G on expected gradient norm which is independent of N, then the last term in Theorem 4.5 can be replaced by $\frac{G\sqrt{N}\sqrt{d}}{\sqrt{T}}$. This would then make our average regret upper bound (averaged over N agents) become tighter with a larger number N of agents.

5. Experiments

To corroborate our theoretical results and demonstrate the empirical effectiveness of our algorithms, we evaluate the performance of our algorithms using synthetic experiments.

5.1. Experimental Settings.

We generate the groundtruth linear function parameter θ in each experiment by random sampling from the standard Gaussian distribution. In each round, every agent receives K arms (i.e., contexts) randomly generated from the standard Gaussian distribution with dimension d. In order to ensure fair comparisons, for all methods, we use the same set of hyperparameter of $\lambda = \frac{1}{T}$, where T represents the horizon. In this paper, we chose the horizon T=500. In our simulation of FLDB-GD, the server is required to find the minimizer of the loss function (1). As we have assumed in Assumption. 3, the θ_{sync} exactly solve the minimization problem. And since (1) is convex, gradient descent procedure is guaranteed to converge to its optimum. Thus, when we run our FLDB-GD algorithm (Algo. 2) and compute the cumulative regret, we simply use the LBFGS method from PyTorch (Paszke et al., 2019) to efficiently solve for the optimum. For the simulation of FLDB-OGD, we choose $\alpha = 1000$. All figures in this section plot the average cumulative regret of all N clients up to a given iteration, which allows us to inspect the benefit brought by the federated

setting to an agent on average. Each curve depicts the mean and standard error computed over three independent runs. We verify the benefit of the federated setting, we compare our algorithms with the single-agent baseline of linear dueling bandits (LDB) (Bengs et al., 2022), which we denote as LDB. Specifically, the baseline of LDB simply performs a local dueling bandit algorithm in isolation.

5.2. Experimental Results.

We evaluate our FLDB-GD and FLDB-OGD algorithms, as well as the baseline of LDB, under the same settings of the number of agents N, the number of arms K and the input dimension d. To begin with, we fix K=10 and d=5, and evaluate the impact of different number of agents N. The results in Figures 1a, 1b, and 1c demonstrate that in all settings, our FLDB-GD and FLDB-OGD algorithms significantly outperform the baseline of LDB. In addition, our FLDB-GD consistently achieves smaller regrets than FLDB-OGD, which aligns with our theoretical results (Sec. 4.2) because FLDB-GD enjoy a tighter regret upper bound that FLDB-OGD. However, as we have pointed out in Sec. 3.2 and Sec. 4.2, this performance advantage in terms of regret comes at the expense of increased communication costs.

5.3. Impact of the Number of Agents N.

Next, we use Figure 2 to provide a better illustration of the impact of the number of agents N. In particular, we fix K=10 and d=5, and vary the value of N for both <code>FLDB-GD</code> and <code>FLDB-OGD</code>. The results in Figures 2a and 2b show that as the number of agents N is increased, both <code>FLDB-GD</code> and <code>FLDB-OGD</code> achieve smaller regrets. As a further verification of robustness of the performance of our <code>FLDB-OGD</code>, which is a more practical algorithm than <code>FLDB-GD</code>, we adopt a more challenging setting by increasing the number of arms K to 50 in Figure 2c. The results show that the performance improvement of <code>FLDB-OGD</code> due to a larger number of agents N is still consistent in this more challenging setting.

These results serve as validation for the capability of our algorithms to benefit from more collaboration, and provide incentives to encourage the agents to join the federated of our algorithms. The benefit of a larger N for <code>FLDB-GD</code> is consistent with our theoretical results in Sec. 4.1, because according to Theorem 4.2, a larger number of agents N leads to a tighter upper bound on the average regret across all agents. Of note, the empirical dependency of <code>FLDB-OGD</code> on N appears inconsistent with the theoretical guarantee offered by Theorem 4.5. This may imply that the regret upper bound in Theorem 4.2 is likely overly conservative and hence provide justifications for our extra assumption of a G that is independent of N (the last paragraph of Sec. 4.2).

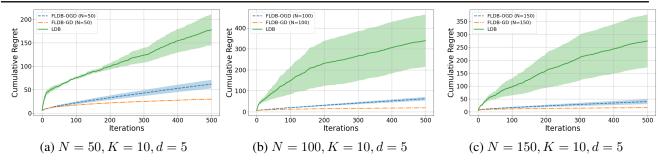


Figure 1. Cumulative regret for different methods with varying numbers of agents (a) N = 50, (b) N = 100 and (c) N = 150 under the number of arms K = 10 and dimension d = 5.

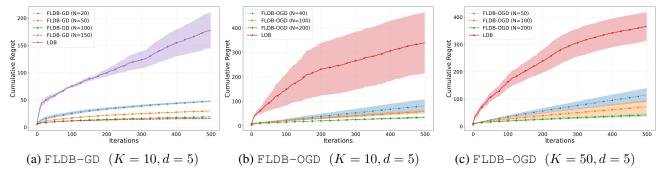


Figure 2. Cumulative regret with varying number of agents for the (a) FLDB-GD, (b) FLDB-OGD method. (c) Illustration of the performance of FLDB-OGD under large number of arms K=50.

Because this extra assumption leads to an average regret upper bound for FLDB-OGD which is improved with a larger N and hence results in a better alignment between our theoretical and empirical results.

6. Related work

Federated Bandits. Recent studies have extended the classical K-armed bandit problem to the federated setting. Li & Song (2022); Li et al. (2020) introduced privacy-preserving federated K-armed bandits in centralized and decentralized settings, respectively. Shi & Shen (2021) formulated a global bandit model where arm rewards are averaged across agents, which was later extended to incorporate personalization (Shi et al., 2021). For federated linear contextual bandits, Wang et al. (2019) proposed a distributed algorithm using sufficient statistics to compute the Linear UCB policy, which was later extended to incorporate differential privacy (Dubey & Pentland, 2020), agent-specific contexts (Huang et al., 2021), and asynchronous communication (Li & Wang, 2022a). Federated kernelized and neural bandits have been developed for hyperparameter tuning (Dai et al., 2020; 2021; 2022). In addition, many recent works have extended federated bandits to various settings and applied them to solve real-world problems (Li et al., 2022; Li & Wang, 2022b; Zhu et al., 2021; Ciucanu et al., 2022; Blaser et al., 2024; Fan et al., 2024; Yang et al., 2024; Solanki et al., 2024; Fourati et al., 2024; Wang et al., 2024; Li et al., 2024a; Wei et al., 2024; Li et al., 2024c).

Dueling Bandits. Thanks to its ability to learn from pairwise preference feedback, dueling bandits have received considerable attention in recent years (Yue & Joachims, 2009; 2011; Yue et al., 2012; Zoghi et al., 2014b; Ailon et al., 2014; Zoghi et al., 2014a; Komiyama et al., 2015; Gajane et al., 2015; Saha & Gopalan, 2018; 2019a;b; Saha & Ghoshal, 2022; Zhu et al., 2023). To account for complicated real-world scenarios, a number of contextual dueling bandits have been developed which model the reward function using either a linear function (Saha, 2021; Bengs et al., 2022; Li et al., 2024b; Saha & Krishnamurthy, 2022; Di et al., 2023) or a neural network (Verma et al., 2024).

7. Conclusion

In this work, we introduced the first federated linear dueling bandit algorithms, enabling multi-agent collaboration while preserving data privacy. By integrating online gradient descent with federated learning, our approach effectively estimates linear function parameters in dueling bandit settings. Theoretical analysis guarantees a sub-linear cumulative regret bound, and empirical results demonstrate the benefits of collaboration. A key challenge, as discussed in Sec. 3.2, is designing algorithms with less frequent communication. Future work will explore novel strategies to address this. Besides, extending our methods to non-linear reward functions using kernelized and neural bandits is a promising direction.

Impact Statement

This paper presents work whose goal is to advance the field of Machine Learning. There are many potential societal consequences of our work, none which we feel must be specifically highlighted here.

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A. Proof of Lemma 4.1

We first consider the classic linear dueling bandit setting, namely the single agent(N=1) scenario. In single agent scenario, we ignore the subscript i (we use $\{x_{t,1}, x_{t,2}, y_t, r_t\}$ instead of $\{x_{t,1,i}, x_{t,2,i}, y_{t,i}, r_{t,i}\}$).

Lemma A.1 (Single Agent). Let
$$\beta_t \triangleq \sqrt{2\log(1/\delta) + d\log(1 + t\kappa_{\mu}/(d\lambda))}$$

With probability of at least $1 - \delta$, we have that for all t = 1, ..., T

$$\|\theta - \theta_t\|_{V_t} \le \frac{\beta_t}{\kappa_u}.$$

Proof. Let $\phi: \mathbb{R}^{d'} \to \mathbb{R}^d$ is a feature map such that $f(x) = \theta^\top \phi(x)$ and $d \geq d'$. In the case of linear bandits, $\phi(x) = x$ and d = d'. In iteration s, define $\widetilde{\phi}_s = \phi(x_{s,1}) - \phi(x_{s,2})$. Define $\widetilde{f}_s = f(x_{s,1}) - f(x_{s,2}) = \theta_f^\top \widetilde{\phi}_s$.

For any $\theta_f \in \mathbb{R}^d$, define

$$G_t(\theta_{f'}) = \sum_{s=1}^t \left(\mu(\theta_{f'}^\top \widetilde{\phi}_s) - \mu(\theta^\top \widetilde{\phi}_s) \right) \widetilde{\phi}_s + \lambda \theta_{f'}.$$

For $\lambda' \in (0,1)$, setting $\theta_{\bar{f}} = \lambda' \theta_{f'_1} + (1-\lambda') \theta_{f'_2}$ and using mean-value theorem, we get:

$$G_{t}(\theta_{f_{1}'}) - G_{t}(\theta_{f_{2}'}) = \left[\sum_{s=1}^{t} \dot{\mu}(\theta_{f}^{\top} \widetilde{\phi}_{s}) \widetilde{\phi}_{s} \widetilde{\phi}_{s}^{\top} + \lambda \mathbf{I}\right] (\theta_{f_{1}'} - \theta_{f_{2}'}) \qquad (\theta_{f} \text{ is constant})$$
(9)

Define $M_t = \left[\sum_{s=1}^t \dot{\mu}(\theta_{\bar{f}}^\top \widetilde{\phi}_s) \widetilde{\phi}_s \widetilde{\phi}_s^\top + \lambda \mathbf{I}\right]$, and define $V_t = \sum_{s=1}^t \widetilde{\phi}_s \widetilde{\phi}_s^\top + \frac{\lambda}{\kappa_{\mu}} \mathbf{I}$. Then we have that $M_t \ge \kappa_{\mu} V_t$ and that $V_t^{-1} \ge \kappa_{\mu} M_t^{-1}$.

$$\begin{aligned} \left\| G_t(\theta_t) \right\|_{V_t^{-1}}^2 &= \left\| G_t(\theta) - G_t(\theta_t) \right\|_{V_t^{-1}}^2 = \left\| M_t(\theta - \theta_t) \right\|_{V_t^{-1}}^2 \\ &= (\theta - \theta_t)^\top M_t V_t^{-1} M_t (\theta - \theta_t) \\ &\geq (\theta - \theta_t)^\top M_t \kappa_\mu M_t^{-1} M_t (\theta - \theta_t) \\ &= \kappa_\mu (\theta - \theta_t)^\top M_t (\theta - \theta_t) \\ &\geq \kappa_\mu (\theta - \theta_t)^\top \kappa_\mu V_t (\theta - \theta_t) \\ &= \kappa_\mu^2 (\theta - \theta_t)^\top V_t (\theta - \theta_t) \\ &= \kappa_\mu^2 \|\theta - \theta_t\|_{V_t}^2 \end{aligned} \qquad \text{(as } ||x||_A^2 = x^\top Ax \text{)}$$

The first inequality is because $V_t^{-1} \ge \kappa_\mu M_t^{-1}$, and the second inequality follows from $M_t \ge \kappa_\mu V_t$.

Let $f_{t,s} = \theta_t^\top \widetilde{\phi}_s$, in which θ_t is our empirical estimate of the parameter θ . Now using 9, we have that

$$\begin{split} \left\|G_t(\theta_t)\right\|_{V_t^{-1}}^2 &= \left\|G_t(\theta) - G_t(\theta_t)\right\|_{V_t^{-1}}^2 & \left(G_t(\theta) = 0 \text{ by definition}\right) \\ &\geq \left(\kappa_\mu V_t(\theta - \theta_t)\right)^\top V_t^{-1} \kappa_\mu V_t(\theta - \theta_t) & \left(\text{as } ||x||_A^2 = x^\top Ax\right) \\ &= \kappa_\mu^2 (\theta - \theta_t)^\top V_t V_t^{-1} V_t(\theta - \theta_t) & \left(\text{as } V_t^\top = V_t \text{ and } \kappa_\mu \text{ is constant}\right) \\ &= \kappa_\mu^2 \|\theta - \theta_t\|_{V_t}^2 & \left(\text{as } ||x||_A^2 = x^\top Ax\right) \end{split}$$

$$\Rightarrow \|\theta - \theta_t\|_{V_t}^2 \leq \frac{1}{\kappa_{\mu}^2} \|G_t(\theta_t)\|_{V_t^{-1}}^2$$

$$= \frac{1}{\kappa_{\mu}^2} \left\| \sum_{s=1}^t (\mu(\theta_t^\top \widetilde{\phi}_s) - \mu(\theta^\top \widetilde{\phi}_s)) \widetilde{\phi}_s + \lambda \theta_t \right\|_{V_t^{-1}}^2$$

$$= \frac{1}{\kappa_{\mu}^2} \left\| \sum_{s=1}^t (\mu(f_{t,s}) - \mu(\widetilde{f}_s)) \widetilde{\phi}_s + \lambda \theta_t \right\|_{V_t^{-1}}^2$$

$$= \frac{1}{\kappa_{\mu}^2} \left\| \sum_{s=1}^t (\mu(f_{t,s}) - (y_s - \epsilon_s)) \widetilde{\phi}_s + \lambda \theta_t \right\|_{V_t^{-1}}^2$$

$$= \frac{1}{\kappa_{\mu}^2} \left\| \sum_{s=1}^t (\mu(f_{t,s}) - (y_s - \epsilon_s)) \widetilde{\phi}_s + \lambda \theta_t \right\|_{V_t^{-1}}^2$$

$$= \frac{1}{\kappa_{\mu}^2} \left\| \sum_{s=1}^t (\mu(f_{t,s}) - y_s) \widetilde{\phi}_s + \sum_{s=1}^t \epsilon_s \widetilde{\phi}_s + \lambda \theta_t \right\|_{V_t^{-1}}^2$$

$$\leq \frac{1}{\kappa_{\mu}^2} \left\| \sum_{s=1}^t \epsilon_s \widetilde{\phi}_s \right\|_{V_t^{-1}}^2.$$
(by definition of $G_t(\theta_t)$)
$$= \frac{1}{\kappa_{\mu}^2} \left\| \sum_{s=1}^t (\mu(f_{t,s}) - y_s) \widetilde{\phi}_s + \lambda \theta_t \right\|_{V_t^{-1}}^2$$

$$\leq \frac{1}{\kappa_{\mu}^2} \left\| \sum_{s=1}^t \epsilon_s \widetilde{\phi}_s \right\|_{V_t^{-1}}^2.$$

The last step follows from the fact that θ_t is computed using MLE by solving the following equation:

$$\sum_{s=1}^{t} \left(\mu \left(\theta_t^{\top} \widetilde{\phi}_s \right) - y_s \right) \widetilde{\phi}_s + \lambda \theta_t = 0, \tag{10}$$

which is ensured by Assumption 3.

With this, we have

$$\|\theta - \theta_t\|_{V_t}^2 \le \frac{1}{\kappa_\mu^2} \left\| \sum_{s=1}^t \epsilon_s \widetilde{\phi}_s \right\|_{V_t^{-1}}^2.$$
 (11)

Denote $V_t \triangleq \sum_{s=1}^t \widetilde{\phi}_s \widetilde{\phi}_s^\top + \frac{\lambda}{\kappa_\mu} \mathbf{I}$ and $V \triangleq \frac{\lambda}{\kappa_\mu} \mathbf{I}$. Denote the observation noise $\epsilon_s = y_s - \mu(f(x_{s,1}) - f(x_{s,2}))$. Note that the sequence of observation noises $\{\epsilon_s\}$ is 1-sub-Gaussian (justification omitted).

Next, we can apply Theorem 1 from (Abbasi-Yadkori et al., 2011), to obtain

$$\left\| \sum_{s=1}^{t} \epsilon_s \widetilde{\phi}_s \right\|_{V_t^{-1}}^2 \le 2 \log \left(\frac{\det(V_t)^{1/2}}{\delta \det(V)^{1/2}} \right), \tag{12}$$

which holds with probability of at least $1 - \delta$.

Next, based on our assumption that $\|\widetilde{\phi}_s\|_2 \le 1$ (Assumption 1), according to Lemma 10 from (Abbasi-Yadkori et al., 2011), we have that

$$\det(V_t) \le \left(\lambda/\kappa_\mu + t/d\right)^d. \tag{13}$$

Therefore,

$$\sqrt{\frac{\det V_t}{\det(V)}} \le \sqrt{\frac{\left(\lambda/\kappa_\mu + t/d\right)^d}{\left(\lambda/\kappa_\mu\right)^d}} = \left(1 + t\kappa_\mu/(d\lambda)\right)^{\frac{d}{2}} \tag{14}$$

This gives us

$$\left\| \sum_{s=1}^{t} \epsilon_s \widetilde{\phi}_s \right\|_{V_s^{-1}}^2 \le 2 \log \left(\frac{\det(V_t)^{1/2}}{\delta \det(V)^{1/2}} \right) \le 2 \log(1/\delta) + d \log \left(1 + t\kappa_{\mu} / (d\lambda) \right)$$
(15)

Combining (11) and (15), we have that

$$\|\theta - \theta_t\|_{V_t}^2 \le \frac{1}{\kappa_u^2} \left(2\log(1/\delta) + d\log\left(1 + t\kappa_\mu/(d\lambda)\right) \right) = \frac{\beta_t^2}{\kappa_u^2},\tag{16}$$

which completes the proof.

To establish the multi-agent scenario, it suffices to replace (13) with

$$\det(V_t) \le \left(\lambda/\kappa_\mu + tN/d\right)^d,\tag{17}$$

while keeping the rest unchanged.

B. Proof of Theorem 4.2

To prove Theorem 4.2, we first need the following lemma:

Lemma B.1.

$$\sum_{t=1}^{T} \left\| \phi(x_{t,1}) - \phi(x_{t,2}) \right\|_{V_{t-1}^{-1}}^{2} \le 2d \log \left(1 + TN \kappa_{\mu} / (d\lambda) \right).$$

Similar to the proof of Lemma 4.1 in Appendix A, we first prove the single agent case:

Lemma B.2 (Single Agent).

$$\sum_{t=1}^{T} \|\phi(x_{t,1}) - \phi(x_{t,2})\|_{V_{t-1}^{-1}}^{2} \le 2d \log (1 + T\kappa_{\mu}/(d\lambda)).$$

Proof. We denote $\widetilde{\phi}_t = \phi(x_{t,1}) - \phi(x_{t,2})$. Recall that we have assumed that $\left\|\phi(x_{t,1}) - \phi(x_{t,2})\right\|_2 \le 1$. It is easy to verify that $V_{t-1} \succeq \frac{\lambda}{\kappa_\mu} I$ and hence $V_{t-1}^{-1} \preceq \frac{\kappa_\mu}{\lambda} I$. Therefore, we have that $\left\|\widetilde{\phi}_t\right\|_{V_{t-1}^{-1}}^2 \le \frac{\kappa_\mu}{\lambda} \left\|\widetilde{\phi}_t\right\|_2^2 \le \frac{\kappa_\mu}{\lambda}$. We choose λ such that $\left\|\widetilde{\phi}_t\right\|_{V_{t-1}^{-1}}^2 \le 1$, which ensures that $\left\|\widetilde{\phi}_t\right\|_{V_{t-1}^{-1}}^2 \le 1$. Our proof here mostly follows from Lemma 11 of (Abbasi-Yadkori et al., 2011). To begin with, note that $x \le 2\log(1+x)$ for $x \in [0,1]$. Then we have that

$$\sum_{t=1}^{T} \left\| \widetilde{\phi}_{t} \right\|_{V_{t-1}^{-1}}^{2} \leq \sum_{t=1}^{T} 2 \log \left(1 + \left\| \widetilde{\phi}_{t} \right\|_{V_{t-1}^{-1}}^{2} \right)$$

$$= 2 \left(\log \det V_{T} - \log \det V \right)$$

$$= 2 \log \frac{\det V_{T}}{\det V}$$

$$\leq 2 \log \left(\left(1 + T \kappa_{\mu} / (d\lambda) \right)^{d} \right)$$

$$= 2d \log \left(1 + T \kappa_{\mu} / (d\lambda) \right).$$
(18)

The second inequality follows from (14). This completes the proof.

Lemma B.2 can be applied to (17) to extend the conclusion to the N-agent case, resulting in Lemma B.1.

Then we can derive the regret bound of single agent case:

Lemma B.3. In any iteration t = 1, ..., T, for all $x, x' \in \mathcal{X}_t$, with probability of at least $1 - \delta$, we have that

$$|(f(x) - f(x')) - \theta_t^{\top} (\phi(x) - \phi(x'))| \le \frac{\beta_t}{\kappa_{\mu}} ||\phi(x) - \phi(x')||_{V_{t-1}^{-1}}.$$

Proof.

$$| (f(x) - f(x')) - \theta_t^{\top} (\phi(x) - \phi(x')) | = |\theta^{\top} [(\phi(x) - \phi(x')] - \theta_t^{\top} [\phi(x) - \phi(x')] |$$

$$= | (\theta - \theta_t)^{\top} [\phi(x) - \phi(x')] |$$

$$\leq ||\theta - \theta_t||_{V_{t-1}} ||\phi(x) - \phi(x')||_{V_{t-1}^{-1}}$$

$$\leq \frac{\beta_t}{\kappa_u} ||\phi(x) - \phi(x')||_{V_{t-1}^{-1}},$$
(19)

in which the last inequality follows from Lemma A.1.

Theorem B.4 (Single Agent). Let $\beta_t \triangleq \sqrt{2\log(1/\delta) + d\log(1 + t\kappa_{\mu}/(d\lambda))}$. With probability of at least $1 - \delta$, we have that

$$R_T \leq \frac{3}{2} \frac{1}{\kappa_{\mu}} \sqrt{2 \log(1/\delta) + d \log \left(1 + T \kappa_{\mu} / (d\lambda)\right)} \sqrt{T 2 d \log \left(1 + T \kappa_{\mu} / (d\lambda)\right)}.$$

Proof.

$$\begin{aligned} & 2r_{t} = f(x_{t}^{*}) - f(x_{t,1}) + f(x_{t}^{*}) - f(x_{t,2}) \\ & \stackrel{(a)}{\leq} \theta_{t}^{\top} \left(\phi(x_{t}^{*}) - \phi(x_{t,1}) \right) + \frac{\beta_{t}}{\kappa_{\mu}} \left\| \phi(x_{t}^{*}) - \phi(x_{t,1}) \right\|_{V_{t-1}^{-1}} + \theta_{t}^{\top} \left(\phi(x_{t}^{*}) - \phi(x_{t,2}) \right) + \frac{\beta_{t}}{\kappa_{\mu}} \left\| \phi(x_{t}^{*}) - \phi(x_{t,2}) \right\|_{V_{t-1}^{-1}} \\ & = \theta_{t}^{\top} \left(\phi(x_{t}^{*}) - \phi(x_{t,1}) \right) + \frac{\beta_{t}}{\kappa_{\mu}} \left\| \phi(x_{t}^{*}) - \phi(x_{t,1}) \right\|_{V_{t-1}^{-1}} + \\ & \theta_{t}^{\top} \left(\phi(x_{t}^{*}) - \phi(x_{t,1}) \right) + \theta_{t}^{\top} \left(\phi(x_{t,1}) - \phi(x_{t,2}) \right) + \frac{\beta_{t}}{\kappa_{\mu}} \left\| \phi(x_{t}^{*}) - \phi(x_{t,1}) + \phi(x_{t,1}) - \phi(x_{t,2}) \right\|_{V_{t-1}^{-1}} \\ & \stackrel{(b)}{\leq} 2\theta_{t}^{\top} \left(\phi(x^{*}) - \phi(x_{t,1}) \right) + 2 \frac{\beta_{t}}{\kappa_{\mu}} \left\| \phi(x_{t}) - \phi(x_{t,1}) \right\|_{V_{t-1}^{-1}} + \\ & \theta_{t}^{\top} \left(\phi(x_{t,1}) - \phi(x_{t,2}) \right) + \frac{\beta_{t}}{\kappa_{\mu}} \left\| \phi(x_{t,1}) - \phi(x_{t,2}) \right\|_{V_{t-1}^{-1}} + \\ & \stackrel{(c)}{\leq} 2\theta_{t}^{\top} \left(\phi(x_{t,2}) - \phi(x_{t,1}) \right) + 2 \frac{\beta_{t}}{\kappa_{\mu}} \left\| \phi(x_{t,2}) - \phi(x_{t,1}) \right\|_{V_{t-1}^{-1}} + \\ & \theta_{t}^{\top} \left(\phi(x_{t,2}) - \phi(x_{t,1}) \right) + 3 \frac{\beta_{t}}{\kappa_{\mu}} \left\| \phi(x_{t,2}) - \phi(x_{t,1}) \right\|_{V_{t-1}^{-1}} \\ & \stackrel{(d)}{\leq} 3 \frac{\beta_{t}}{\kappa_{\mu}} \left\| \phi(x_{t,1}) - \phi(x_{t,2}) \right\|_{V_{t-1}^{-1}} \end{aligned}$$

Step (a) follows from Lemma B.3. Step (b) makes use of the triangle inequality. Step (c) follows from the way in which we choose the second arm $x_{t,2}$: $x_{t,2} = \arg\max_{x \in \mathcal{X}_t} \theta_t^\top \left(\phi(x) - \phi(x_{t,1}) \right) + \frac{\beta_t}{\kappa_\mu} \left\| \phi(x) - \phi(x_{t,1}) \right\|_{V_{t-1}^{-1}}$. Step (d) results from the way in which we select the first arm: $x_{t,1} = \arg\max_{x \in \mathcal{X}_t} \theta_t^\top \phi(x)$.

Recall that
$$\beta_T = \sqrt{2\log(1/\delta) + d\log(1 + T\kappa_{\mu}/(d\lambda))}$$

$$R_{T} = \sum_{t=1}^{T} r_{t} \leq \sum_{t=1}^{T} \frac{3}{2} \frac{\beta_{t}}{\kappa_{\mu}} \|\phi(x_{t,1}) - \phi(x_{t,2})\|_{V_{t-1}^{-1}}$$

$$\leq \frac{3}{2} \frac{\beta_{T}}{\kappa_{\mu}} \sqrt{T \sum_{t=1}^{T} \|\phi(x_{t,1}) - \phi(x_{t,2})\|_{V_{t-1}^{-1}}^{2}}$$

$$\leq \frac{3}{2} \frac{\beta_{T}}{\kappa_{\mu}} \sqrt{T 2d \log (1 + T \kappa_{\mu} / (d\lambda))}.$$
(21)

Now we begin to prove Theorem 4.2.

We use V_t to denote the covariance matrix after iteration t: $V_t \triangleq \sum_{\tau=1}^t \sum_{i=1}^N \widetilde{\phi}_{\tau,i} \widetilde{\phi}_{\tau,i}^\top + \frac{\lambda}{\mu} \mathbf{I}$, in which $\widetilde{\phi}_{t,i} = \phi(x_{t,1,i}) - \phi(x_{t,2,i})$. Consider a hypothetical agent which chooses all $T \times N$ pairs of arms in a round-robin fashion, i.e., it chooses $\{(x_{1,1,1},x_{1,2,1}),(x_{1,1,2},x_{t,2,2}),\ldots,(x_{1,1,N},x_{t,2,N}),\ldots,(x_{T,1,N},x_{T,2,N})\}$. Define the covariance matrix for this hypothetical agent as $\widetilde{V}_{t,i} \triangleq V_t + \sum_{j=1}^i \widetilde{\phi}_{t,j} \widetilde{\phi}_{t,j}^\top$. That is, the covariance matrix $\widetilde{V}_{t,i}$ consists of the information from the previous t iterations, as well as the information from first t agents in iteration t+1.

Define the set of "bad iterations" as $\mathcal{C} = \{t = 1, \dots, T | \frac{\det V_t}{\det V_{t-1}} > 2\}$, and we use $|\mathcal{C}|$ to represent the size of the set \mathcal{C} . Denoting $V_0 = V = \frac{\lambda}{\kappa_u} \mathbf{I}$, we can show that

$$\frac{\det V_T}{\det V} = \prod_{t=1,\dots,T} \frac{\det V_t}{\det V_{t-1}} \ge \prod_{t\in\mathcal{C}} \frac{\det V_t}{\det V_{t-1}} > 2^{|\mathcal{C}|}.$$
(22)

This implies that $|\mathcal{C}| \leq \log \frac{\det V_T}{\det V}$.

Next, we analyze the instantaneous regret in a good iteration, i.e., an iteration t for which $\frac{\det V_t}{\det V_{t-1}} \leq 2$. Let $\beta_t \triangleq \sqrt{2\log(1/\delta) + d\log\left(1 + tN\kappa_\mu/(d\lambda)\right)}$. Note that we still have $\|\theta - \theta_t\|_{V_{t-1}} \leq \frac{\beta_t}{\kappa_\mu}$, where θ_t refers to θ_{sync} at time t. Following the analysis of 20, we can show that with probability of at least $1 - \delta$,

$$r_{t,i} = f(x_{t,i}^*) - f(x_{t,1,i}) + f(x_{t,i}^*) - f(x_{t,2,i})$$

$$\leq 3 \frac{\beta_t}{\kappa_\mu} \|\phi(x_{t,1,i}) - \phi(x_{t,2,i})\|_{V_{t-1}^{-1}}.$$
(23)

Next, making use of Lemma 12 of (Abbasi-Yadkori et al., 2011), we can show that

$$\|\phi(x_{t,1,i}) - \phi(x_{t,2,i})\|_{V_{t-1}^{-1}} \leq \|\phi(x_{t,1,i}) - \phi(x_{t,2,i})\|_{\widetilde{V}_{t-1}^{-1}} \sqrt{\frac{\det \widetilde{V}_{t-1,i}}{\det V_{t-1}}}$$

$$\leq \|\phi(x_{t,1,i}) - \phi(x_{t,2,i})\|_{\widetilde{V}_{t-1}^{-1}} \sqrt{\frac{\det V_{t}}{\det V_{t-1}}}$$

$$\leq \sqrt{2} \|\phi(x_{t,1,i}) - \phi(x_{t,2,i})\|_{\widetilde{V}_{t-1}^{-1}}.$$
(24)

This allows us to show that

$$r_{t,i} \le 3\sqrt{2} \frac{\beta_t}{\kappa_\mu} \|\phi(x_{t,1,i}) - \phi(x_{t,2,i})\|_{\widetilde{V}_{t-1}^{-1}}$$
(25)

Now the overall regrets of all agents in all iterations can be analyzed as:

$$R_{T,N} = \sum_{t=1}^{T} \sum_{i=1}^{N} r_{t,i} = |\mathcal{C}|N2 + \sum_{t \notin \mathcal{C}} \sum_{i=1}^{N} r_{t,i}$$

$$\leq 2N \log \frac{\det V_T}{\det V} + \sum_{t \notin \mathcal{C}} \sum_{i=1}^{N} r_{t,i}$$

$$\leq 2N \log \frac{\det V_T}{\det V} + \sum_{t=1}^{T} \sum_{i=1}^{N} r_{t,i}$$

$$\leq 2N \log \frac{\det V_T}{\det V} + 3\sqrt{2} \frac{\beta_T}{\kappa_{\mu}} \sum_{t=1}^{T} \sum_{i=1}^{N} \|\phi(x_{t,1,i}) - \phi(x_{t,2,i})\|_{\widetilde{V}_{t-1}^{-1}}$$

$$\leq 2N d \log (1 + TN\kappa_{\mu}/(d\lambda)) + 3\sqrt{2} \frac{\beta_T}{\kappa_{\mu}} \sqrt{T2d \log (1 + TN\kappa_{\mu}/(d\lambda))}$$
(26)

Ignoring all log factors, we have that

$$R_{T,N} = \widetilde{O}\left(Nd + \frac{\sqrt{d}}{\kappa_{\mu}}\sqrt{Td}\right) = \widetilde{O}\left(Nd + \frac{d}{\kappa_{\mu}}\sqrt{T}\right)$$
(27)

C. Proof of Lemma 4.4

Lemma C.1 ensures all the $\theta_t = \arg \max_{\theta'}$ (optimal value of the 1 at iteration t) lie in a ball centered with the ground truth θ

Lemma C.1. For horizon T, let $r \triangleq \sqrt{\frac{2\log(1/\delta) + d\log(1 + TN\kappa_{\mu}/(d\lambda))}{\lambda\kappa_{\mu}}}$. With probability at least $1 - \delta$, we have $\|\theta_t - \theta\| \leq r$ for $t = 1 \dots T$

Proof. Since $\|\theta - \theta_t\|_{V_{t-1}} \leq \frac{\beta_t}{\kappa_\mu}$ with probability at least $1 - \delta$, we have

$$\frac{\beta_T}{\kappa_{\mu}} \ge \|\theta - \theta_t\|_{V_{t-1}} \ge \sqrt{\lambda_{min}(V_{t-1})} \|\theta - \theta_t\| \ge \sqrt{\frac{\lambda}{\kappa_{\mu}}} \|\theta - \theta_t\|$$
(28)

Thus,

$$\|\theta - \theta_t\| \le \frac{\beta_T}{\sqrt{\kappa_\mu \lambda}} = \sqrt{\frac{2\log(1/\delta) + d\log\left(1 + TN\kappa_\mu/(d\lambda)\right)}{\lambda \kappa_\mu}}$$
 (29)

We will also use the Proposition 1 in (Li et al., 2017b):

Proposition C.2 (Proposition 1 in (Li et al., 2017b)). Define $V_{n+1} = \sum_{t=1}^{n} X_t X_t^T$, where X_t is drawn IID from some distribution in unit ball \mathbb{B}^d . Furthermore, let $\Sigma := E[X_t X_t^T]$ be the second moment matrix, let $B, \delta > 0$ be two positive constants. Then there exists positive, universal constants C_1 and C_2 such that $\lambda_{\min}(V_{n+1}) \geq B$ with probability at least $1 - \delta$, as long as

$$n \ge \left(\frac{C_1\sqrt{d} + C_2\sqrt{\log(1/\delta)}}{\lambda_{\min}(\Sigma)}\right)^2 + \frac{2B}{\lambda_{\min}(\Sigma)}.$$

To prove Lemma 4.4, we also need to show Propsition 4.3.

Proof of Proposition 4.3. Recall

$$f_s^{\text{fed}}(\theta') = \begin{cases} \sum_{i=1}^{N} l_s^i(\theta'), & \text{if } s \neq 1, \\ \sum_{i=1}^{N} l_1^i(\theta') + \frac{\lambda}{2} \|\theta'\|_2^2, & \text{if } s = 1. \end{cases}$$

we have

$$\nabla^{2} l_{s}^{i}(\theta') = \mu'({\theta'}^{T} \widetilde{\phi}_{s,i}) \widetilde{\phi}_{s,i} \widetilde{\phi}_{s,i}^{T},$$

$$\nabla^{2} f_{s}^{\text{fed}}(\theta') = \sum_{i=1}^{N} \mu'({\theta'}^{T} \widetilde{\phi}_{s,i}) \widetilde{\phi}_{s,i} \widetilde{\phi}_{s,i}^{T} + \lambda \mathbb{1}(s=1) \succeq \kappa_{\mu} \sum_{i=1}^{N} \widetilde{\phi}_{s,i} \widetilde{\phi}_{s,i}^{T} \text{ for } \forall \theta' \in \mathbb{B}_{3r}.$$
(30)

The last inequality comes from Assumption 1 and $\lambda>0$. Since we update $\widetilde{\theta}_t$ every N rounds (for clients), for the next N rounds (for client), the pulled arms are only dependent on $\widetilde{\theta}_t$. Therefore, the feature vectors of pulled arms among the next N rounds are IID. Thus, $\widetilde{\phi}_i$ is drawn IID in unit ball \mathbb{B}^d . By applying Proposition C.2 letting $B=\frac{\alpha}{\kappa_\mu}$ and with Assumption 2, we have with probability at least $1-\delta$,

$$\nabla^2 f_s\left(\theta'\right) \succeq \alpha I_d \text{ if } N \ge \left(\frac{C_1 \sqrt{d} + C_2 \sqrt{\log\left(1/\delta\right)}}{\lambda_f}\right)^2 + \frac{2\alpha}{\kappa_\mu \lambda_f}$$

Namely, $f_s^{\mathrm{fed}}\left(\theta'\right)$ is α -stronly convex for $\forall~1\leqslant s\leqslant t.$

Then we will prove Lemma 4.4.

Proof. Now we have:

$$\theta_t = \arg\min_{\theta'} \mathcal{L}_t^{\text{fed}}(\theta')$$

$$\widehat{\theta}^{(t+1)} = \pi_S \left(\widehat{\theta}^{(t)} - \eta_t \nabla f_t^{\text{fed}}(\widehat{\theta}^{(t)}) \right) \text{ for } t \ge 1$$

$$\widetilde{\theta}^{(t)} = \frac{1}{t} \sum_{j=1}^t \widehat{\theta}^{(j)}$$

From Lemma C.1 and Algorithm 3 we have $\theta_t, \widehat{\theta}^{(t+1)}, \widetilde{\theta}^{(t)} \in \mathbb{B}_{3r}$ with probability at least $1 - \frac{\delta}{2}$. Then by Jensen's inequality, we have

$$\sum_{s=1}^{t} f_s^{\text{fed}}(\widehat{\theta}^{(s)}) \ge \sum_{s=1}^{t} f_s^{\text{fed}}(\widetilde{\theta}^{(t)}). \tag{31}$$

Since S is convex, we apply Theorem 3.3 of Section 3.3.1 in (Hazan, 2023) and get

$$\sum_{s=1}^t \left(f_s^{\text{fed}}(\widehat{\theta}^{(s)}) - f_s^{\text{fed}}(\theta_t) \right) \leq \frac{G^2}{2\alpha} (1 + \log t) \ \, \forall \, t \geq 1$$

where G satisfies $G^2 \geq E \|\nabla f_s^{\text{fed}}\|^2$. Additionally, we have

$$\|\nabla f_s^{\text{fed}}(\theta')\| = \|\sum_{i=1}^N \left(\mu\left(\theta'^{\top}\widetilde{\phi}_{s,i}\right) - y_{s,i}\right)\widetilde{\phi}_{s,i}\| \le N, \text{ for } s > 1.$$
(32)

Thus, we have

$$\frac{N^2}{2\alpha}(1 + \log t) \ge \sum_{s=1}^{t} \left(f_s^{\text{fed}}(\widehat{\theta}^{(s)}) - f_s^{\text{fed}}(\theta_t) \right) \ge \sum_{s=1}^{t} \left(f_s^{\text{fed}}(\widetilde{\theta}^{(t)}) - f_s^{\text{fed}}(\theta_t) \right)$$

By Proposition 4.3, we know $f_s^{\rm fed}(\theta')$ is α -strongly convex for $\forall \ 1 \leq s \leq t$ with probability at least $1 - \frac{\delta}{2}$. Meanwhile, $\theta_t, \widehat{\theta}^{(t+1)}, \widetilde{\theta}^{(t)} \in \mathbb{B}_{3r}$ with probability at least $1 - \frac{\delta}{2}$. Then using the property of α -strongly convex we have

$$f_s^{\text{fed}}(\widetilde{\theta}^{(t)}) \ge f_s^{\text{fed}}(\theta_t) + \nabla f_s^{\text{fed}}(\theta_t)^\top \left(\widetilde{\theta}^{(t)} - \theta_t\right) + \frac{\alpha}{2} \|\widetilde{\theta}^{(t)} - \theta_t\|^2$$
(33)

Sum over s we have

$$\sum_{s=1}^{t} f_s^{\text{fed}}(\widetilde{\theta}_t) \ge \sum_{s=1}^{t} f_s^{\text{fed}}(\theta_t) + \left(\sum_{s=1}^{t} \nabla f_s^{\text{fed}}(\theta_t)^{\top}\right) \left(\widetilde{\theta}_t - \theta_t\right) + \frac{\alpha t}{2} \|\widetilde{\theta}_t - \theta_t\|^2$$
(34)

Since $\sum_{s=1}^{t} \nabla f_s(\theta_t) = 0$ we get

$$\frac{\alpha t}{2} \|\widetilde{\theta}^{(t)} - \theta_t\|^2 \le \sum_{s=1}^t f_s^{\text{fed}}(\widetilde{\theta}^{(t)}) - \sum_{s=1}^t f_s^{\text{fed}}(\theta_t) = \sum_{s=1}^t \left(f_s^{\text{fed}}(\widetilde{\theta}^{(t)}) - f_s^{\text{fed}}(\theta_t) \right) \le \frac{(N)^2}{2\alpha} (1 + \log t)$$

That is

$$\|\widetilde{\theta}^{(t)} - \theta_t\| \le \frac{N}{\alpha} \sqrt{\frac{1 + \log t}{t}} \tag{35}$$

From $\|\widetilde{\phi}_{s,i}\| \leq 1$ we get $\lambda_{max}(V_t) \leq tN + \frac{\lambda}{\kappa_{\mu}}$. Therefore,

$$\left\|\widetilde{\theta}^{(t)} - \theta_t\right\|_{V_*} \le \sqrt{\lambda_{max}(V_t)} \|\widetilde{\theta}_t - \theta_t\|$$
(36)

$$\leq \frac{\sqrt{tN + \frac{\lambda}{\kappa_{\mu}}}N}{\alpha} \sqrt{\frac{1 + \log t}{t}} \tag{37}$$

$$\leq \frac{N\sqrt{N + \frac{\lambda}{\kappa_{\mu}}}}{\alpha} \sqrt{1 + \log t} \tag{38}$$

The last inequality holds since $t \geq 1$.

D. Proof of Theorem 4.5

Proof. Following the definition in Section 2, we will analyze the instantaneous regret in a good iteration. First we have

$$\left\|\theta - \widetilde{\theta}^{(t)}\right\|_{V_t} \le \left\|\theta - \theta_t\right\|_{V_t} + \left\|\theta_t - \widetilde{\theta}^{(t)}\right\|_{V_t} \le \frac{\beta_t}{\kappa_\mu} + \left\|\theta_t - \widetilde{\theta}^{(t)}\right\|_{V_t} \le \frac{\beta_t}{\kappa_\mu} + \frac{N\sqrt{N + \frac{\lambda}{\kappa_\mu}}}{\alpha}\sqrt{1 + \log t}$$

And

$$\begin{split} |\left(f(x) - f(x')\right) - \left(\widetilde{\theta}^{(t)}\right)^{\top} \left(\phi(x) - \phi(x')\right)| &= |\theta^{\top} \left[\left(\phi(x) - \phi(x')\right] - \left(\widetilde{\theta}^{(t)}\right)^{\top} \left[\phi(x) - \phi(x')\right]| \\ &= |\left(\theta - \widetilde{\theta}^{(t)}\right)^{\top} \left[\phi(x) - \phi(x')\right]| \\ &\leq \left\|\theta - \widetilde{\theta}^{(t)}\right\|_{V_{t-1}} \left\|\phi(x) - \phi(x')\right\|_{V_{t-1}^{-1}} \\ &\leq \left(\frac{\beta_t}{\kappa_{\mu}} + \frac{N\sqrt{N + \frac{\lambda}{\kappa_{\mu}}}}{\alpha} \sqrt{1 + \log t}\right) \left\|\phi(x) - \phi(x')\right\|_{V_{t-1}^{-1}} \end{split}$$

Thus, for the instantaneous regret in good iteration, with probability at least $1 - \delta$, we have

$$r_{t,i} = f(x_{t,i}^*) - f(x_{t,1,i}) + f(x_{t,i}^*) - f(x_{t,2,i})$$

$$\leq 3 \left(\frac{\beta_t}{\kappa_\mu} + \frac{N\sqrt{N + \frac{\lambda}{\kappa_\mu}}}{\alpha} \sqrt{1 + \log t} \right) \left\| \phi(x_{t,1,i}) - \phi(x_{t,2,i}) \right\|_{V_{t-1}^{-1}}.$$
(39)

By following the proof in Appendix B, the overall regrets of all agents in all iterations can be analyzed as:

$$R_{T,N} \le 2Nd\log\left(1 + TN\kappa_{\mu}/(d\lambda)\right) + 3\sqrt{2}\left(\frac{\beta_T}{\kappa_{\mu}} + \frac{N\sqrt{N + \frac{\lambda}{\kappa_{\mu}}}}{\alpha}\sqrt{1 + \log T}\right)\sqrt{T2d\log\left(1 + TN\kappa_{\mu}/(d\lambda)\right)}$$
(40)

Ignoring all log factors, we have that

$$R_{T,N} = \widetilde{O}\left(Nd + \left(\frac{\sqrt{d}}{\kappa_{\mu}} + \frac{N^{\frac{3}{2}}}{\alpha}\right)\sqrt{Td}\right) = \widetilde{O}\left(Nd + \left(\frac{d}{\kappa_{\mu}} + \frac{N^{\frac{3}{2}}\sqrt{d}}{\alpha}\right)\sqrt{T}\right)$$
(41)