IN THE GRAPHICAL SIERPINSKI GASKET, THE REVERSE RIESZ TRANSFORM IS UNBOUNDED ON L^p, $p \in (1, 2)$.

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Abstract. In this article, we proved that the reverse Riesz transform on the graphical Sierpinski gasket is unbounded on L^p for $p \in (1, 2)$. Together with previous results, it shows that the Riesz transform on the graphical Sierpinski gasket is bounded on L^p if and only if $p \in (1, 2]$ and the reverse Riesz transform is bounded on L^p if and only if $p \in [2, \infty)$.

Moreover, our method is quite flexible - but requires explicit computations - and hints to the fact that the reverse Riesz transforms is never bounded on L^p, $p \in (1, 2)$, on graphs with slow diffusions.

Keywords: Riesz transform, Sierpinski gasket, reverse Hölder inequality, Faber-Krahn inequality.

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Contents

1. Introduction

In $\mathbb{R}^n$, the Fourier transform $\mathcal{F}$ is used to define the Riesz transform

$$\mathcal{R}f := \mathcal{F}^{-1}\Big[\frac{i\xi}{|\xi|}\mathcal{F}f\Big]$$

Parseval's identity immediately gives that the Riesz transform $\mathcal{R}$ is an isometry on $L^2(\mathbb{R}^n)$. For smooth functions, we can equivalently write $\mathcal{R}$ as the singular integral

$$\mathcal{R}f(x) := p.v. \int_{\mathbb{R}^n} \frac{x-y}{|x-y|^{n+1}} f(y)\, dy$$

and using the Calderón-Zygmund theory and a bit of duality, one will get that (see [Ste70, Chapter 2, Theorem 1])

$$C_p\|f\|_p \leq \|\mathcal{R}f\|_p \leq C_p\|f\|_p \qquad \text{for } p \in (1, \infty). \tag{E_p}$$

and that method can be easily adapted to prove the same inequality on Lie groups instead of $\mathbb{R}^n$.

If we define $(-\Delta)^{-1/2}$ on a dense subdomain of $L^2(\mathbb{R}^n)$, for instance $C_0^\infty(\mathbb{R}^n)$, via the spectral theory, then one can observe through the definition using Fourier transform that we actually have

$$\mathcal{R}f = \nabla(-\Delta)^{\frac{1}{2}}f \qquad \text{for } f \in C_0^\infty(\mathbb{R}^n).$$

This last identity is more flexible, as it allows us to define the Riesz transform in a very general setting. Indeed, if we have a metric measured space (Γ, d, m) equipped with a non-negative self-adjoint operator on $L^2(M, \mu)$ - called Δ - with dense domain, we can define a Dirichlet form

$$\mathcal{E}(f, g) := \int_\Gamma f \Delta g \, dm$$

and a 'length of a gradient'

$$\nabla f := \mathcal{E}(f, f).$$

The identity $\|\nabla f\|_2 = \|\Delta^{\frac{1}{2}} f\|_2$ is then automatic by construction and we can further study the inequalities

$$(\mathrm{R}_p) \qquad \|\nabla f\|_{L^p} \leq C\|\Delta^{1/2} f\|_{L^p}$$

and

$$(\mathrm{RR}_p) \qquad \|\Delta^{1/2} f\|_{L^p} \leq C\|\nabla f\|_{L^p},$$

for $p \in (1, \infty)$, where f is chosen in a dense domain that makes sense.

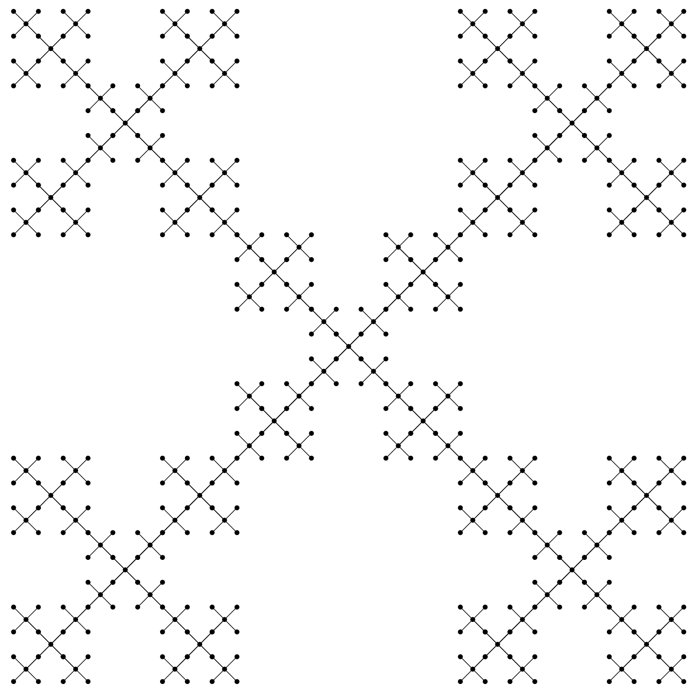
These questions sparked a lot of interest from the mathematic community from the 1980s after Strichartz asked in [Str83] for which Riemannian manifolds and which p the equivalence (E_p) holds. Let us mention a non-exhaustive list of results related to the topic:

(i) (R_p) holds for $1 \leq p \leq 2$ in doubling Riemannian manifolds that have a pointwise Gaussian bound on the heat kernel (see [CD99]) and in graphs that have pointwise Gaussian bound on the Markov process (see [Rus00]).
(ii) Same as in (i) but for sub-Gaussian bounds on the heat kernel / Markov process, see [CCFR17].
(iii) (R_p) holds for $2 < p \leq 2 + \epsilon$ for doubling Riemannian manifold (see [ACDH04, AC05]) and graphs (see [BR09]) with a L^2 Poincaré inequality on balls. See also [RD22] for results that link the reverse Riesz transform (RR_p) and Poincaré inequalities..
(iv) The assumptions of [ACDH04, BR09] are weakened in [BF16].
(v) The Riesz transform is bounded in some Hardy space H^1 (that is, the limit case $p = 1$), see [Rus01, AMR08, HLM+11, BD14, Fen16].

The difficulties of proving (R_p) and (RR_p) - at least for Riemannian manifolds and graphs - lie in the behavior of the space in a large scale. Indeed, (smooth doubling) Riemannian manifolds behave locally like a plane, and graphs and locally just a finite number of vertex, so the difficulty of having (R_p) and (RR_p) would be limited if we have a compact manifold or a finite graph. In the sequel, we choose to restrict the presentation to graphs, since we believe that graphs already encapsulate all the difficulties that we can have (see [BCG01, Section 6] to construct a smooth Riemannian manifold that has the same property as a given graph).

In [CCFR17], we also gave the complete range of validity of the estimates (R_p) and (RR_p) for the Vicsek graphs, an example of which can be found in Figure 1.

FIGURE 1. Vicsek graph (4th step of the construction) with volume $V(x,r) \approx r^{\log_3 5}$ and diffusion parameter $\beta = \log_3 5 + 1$



Theorem 1.1 ([CCFR17, Theorem 5.3])**.** *For Vicsek graphs,* (R_p) *holds if and only if* $p \in (1,2]$ *and* (RR_p) *holds if and only if* $p \in [2,\infty)$.

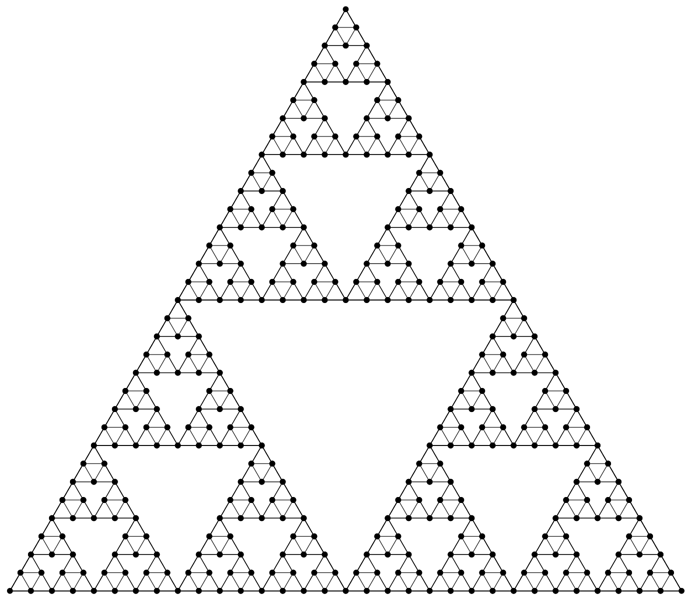
The Vicsek graphs are extreme in the sense that they have the slowest diffusion possible, or equivalently, they have the smallest amount of paths, for a given growth rate. It raises the question whether this picture on the Riesz transform and reverse Riesz transform - namely that the Riesz transform and reverse Riesz transform are mutually exclusive except when $p = 2$ - is specific to this extreme case of Vicsek graph or a property of all graphs that do not have a pointwise Gaussian upper bound (we shall call them "fractal-type" graphs).

The purpose of this article is to hint towards the latest. We shall give a strategy to prove the same result for other fractal-like graphs where we can compute solutions (Green functions or minimizers of energy) explicitly, and to apply it for the graphical Sierpinski gasket (Figure 2).

Theorem 1.2. *For the graphical Sierpinski gasket,* (RR_p) *does not hold if* $p \in (1,2)$.

Corollary 1.3. *For the graphical Sierpinski gasket,* (R_p) *holds if and only if* $p \in (1,2]$ *and* (RR_p) *holds if and only if* $p \in [2,\infty)$.

FIGURE 2. Graphical Sierpinski gasket (6th step of the construction)



Proof of the Corollary from the Theorem. The Theorem gives that (RR_p) is false for $p \in (1,2)$, while Theorem 4.1 in [CCFR17] shows that (R_p) is true for $p \in (1,2]$. The full picture is then deduced by duality, since (R_p) $\implies$ ($RR_{p'}$), see for instance in [CD03, Proposition 2.1]. □

We conclude this introduction by giving or recalling a collection of conjectures to aim going forward. All the definitions (in particular, what is a graph) and the notations will be defined in the next paragraph.

Conjectures. All the conjectures are for a graph equipped with an analytic Markov semi-group.

A. (Coulhon, Duong) (R_p) holds for $p \in (1,2]$.

A'. If the graph is doubling, then (R_p) holds for $p \in (1,2]$.

B. If a doubling graph does not satisfy (UE_2), then (RR_p) is false for $p \in (1,2)$, hence (R_p) is false for $p \in (2,\infty)$.

B'. If a doubling graph that satisfies the sub-Gaussian bound (UE_β) with $\beta > 2$, (RR_p) is false for $p \in (1,2)$ and hence (R_p) is false for $p \in (2,\infty)$.

C. For any $p \in (1,\infty)$, there exists $\gamma^* := \gamma^*(p) \in (0,1)$ such that

$$\|\nabla f\|_{L^p} \leq C\|\Delta^\gamma f\|_{L^p} \quad \text{when } \gamma < \gamma^*$$

and

$$\|\Delta^\gamma f\|_{L^p} \leq C\|\nabla f\|_{L^p} \quad \text{when } \gamma > \gamma^*$$

for any $f \in L^p(\Gamma)$.

C'. For any $p \in (1,2)$, there exists $C > 0$ such that

$$\|\nabla f\|_{L^p} \leq C\|\Delta^{\gamma^*(p)} f\|_{L^p} \quad \text{for } f \in L^p(\Gamma).$$

C". If the Markov kernel satisfies, whenever $d(x,y) \leq n$, the bounds

$$\frac{c}{V(x,k^{1/\beta})} \exp\Big(-C\big[\frac{d(x,y)^\beta}{k+1}\big]^{1/(\beta-1)}\Big) \leq p_k(x,y) \leq \frac{C}{V(x,k^{1/\beta})} \exp\Big(-c\big[\frac{d(x,y)^\beta}{k+1}\big]^{1/(\beta-1)}\Big)$$

for some $0<c<C$ independent of x, y, and n, then there exists $\epsilon>0$ such that, when $p \in (2,2+\epsilon)$, we have

$$\|\nabla f\|_{L^p} \leq C\|\Delta^{\gamma^*(p)} f\|_{L^p} \quad \text{for } f \in L^p(\Gamma).$$

Comments on the conjectures.

- The conjecture A is supported by the fact that there are non-doubling Lie groups and tree graphs where (R_p) holds for all $p \in (1,2)$, see [Sjö99, SV08, LMS+23]. However, all the proofs rely in a critical manner on the group structure or the specific choice of weight on the tree, and so it is not clear that the doubling condition of the space can be unconditionally removed. The conjecture A' is more modest and probably easier to prove ... once we find a way to remove the dependence of the proofs on explicit pointwise estimates on the Markov kernel.
- Conjectures B and B' are natural sequels to this article: we need to remove in our computations the need to have explicit expression of the Green function.
- Conjectures C, C', and C" are tentatives to generalize [CD99, Rus00, ACDH04, AC05, BR09] to the sub-Gaussian setting.

  When the Markov process verifies the Gaussian estimate

  $$p_k(x,y) \leq \frac{C}{V(x,k^{1/\beta})} \exp\Big(-c\frac{d(x,y)^2}{k+1}\Big),$$

  then $\gamma^*(p)=2$, C' is [Rus00], and C" is [BR09].

  When the graph under consideration is a Vicsek graph - which satisfies $V(x,r) \approx r^D$ and (UE_{D+1}) - then the value $\gamma^*(p) = \frac{D-1+p}{p(D+1)}$ was suggested in [DRY23, DR24], and partial results for the conjecture C are given there.

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2. General results on graphs

2.1. **The setting.** A (weighted unoriented) graph (Γ,μ) is the pairing of a countable (infinite) set Γ and a symmetric nonnegative weight μ on $\Gamma \times \Gamma$. The set $E := \{\{(x,y) \in \Gamma^2,\ \mu_{xy}>0\}$ is the set of edges Γ, and we write $x \sim y$ when $(x,y) \in E$ (we say that x and y are neighbors).

The graph is naturally equipped with a measure, a distance, and a random walk. The weight m on Γ is defined as $m(x) = \sum_{x\sim y} \mu_{xy}$, and more generally the measure of a set is $m(E) := \sum_{x\in E} m(x)$.

The distance $d(x,y)$ between two vertices $x,y \in \Gamma$ is given by the length of the shortest path linking x and y, that is the smallest value N for which we can find a chain $x =$

x_0, x_1 dist, $x_N = y$ such that $x_{i-1} \sim x_i$ for all $i \in \{1, \dots, N\}$. We write $B(x,r)$ for the "ball" $\{y \in \Gamma,\, d(x,r) \leq r\}$, and $V(x,r)$ for $m(B(x,r))$.

The weight μ_{xy} provides us with a discrete-time reversible Markov kernel p defined as $p(x,y) = \frac{\mu_{xy}}{m(x)m(y)}$. The random walk is then given by the discrete kernel $p_k(x,y)$, which is defined recursively for all $k \geq 0$ by

$$\begin{cases} p_0(x,y) = \frac{\delta(x,y)}{m(y)} \\ p_{k+1}(x,y) = \sum_{z\in\Gamma} p(x,z)p_k(z,y)m(z). \end{cases} \tag{2.1}$$

Note that for all $k \geq 1$, the kernel p_k is symmetric, that is, $p_k(x,y) = p_k(y,x)$ for all $x, y \in \Gamma$. Moreover, $p_k(x,y)m(y)$ denotes the probability to move from x to y in k steps. The Markov operator P is the operator with kernel p (with respect to the measure m), that is, for any $x \in \Gamma$ and any function f, $Pf(x) = \sum_{y\in\Gamma} p(x,y)f(y)m(y)$. It is easy to check that p_k is the kernel of the operator P^k, and that P^k contracts $L^p(\Gamma)$ for all $k \in \mathbb{N}$ and all $p \in [1, +\infty]$.

The following assumptions on the graph will be added:

- We always assume that (Γ, μ) is locally uniformly finite, that is exists $M \in \mathbb{N}$ such that, for all $x \in \Gamma$,
$$\#B_0(x,1) = \#\left(\{y \in \Gamma,\, y \sim x\} \cup \{x\}\right) \leq M.$$
- We say that the Markov semigroup $(P^k)_{k\in\mathbb{N}}$ is **analytic** if there exists $C > 0$ such that
$$\|(I-P)P^k f\|_{L^2(\Gamma)} \leq \frac{C}{k+1}\|f\|_{L^2(\Gamma)} \qquad \text{for } k \in \mathbb{N} \text{ and } f \in L^2(\Gamma).$$
- We say that the graph (Γ, μ) is doubling if there exists $C > 0$ such that
$$V(x,2r) \leq CV(x,r) \qquad \text{for } x \in \Gamma \text{ and } r \in \mathbb{N}. \tag{D}$$
- We say that (Γ, μ) supports a diffusion with parameter β if, for any $N \in \mathbb{N}^*$, there exists $C_N > 0$ such that the Markov kernel $p_k(x,y)$ satisfies the pointwise bound
$$p_k(x,y) \leq \frac{C_N}{V(x,k^{1/\beta})}\left(1 + \frac{d(x,y)^\beta}{k+1}\right)^{-N}. \tag{UE$_\beta$}$$

Some comments are in order. First, the analyticity of Markov operators - discrete analogue of the analyticity of semigroups - has been studied in, for instance, [CSC90, Dun06, LM14, Fen18]. In the case of reversible Markov processes, the situation is quite simple. Indeed, assuming that (Γ, μ) is locally uniformly finite and doubling, we have the following equivalence from [Fen18, Theorem 1.7]:

Proposition 2.2. *(i) $(P^k)_{k\in\mathbb{N}}$ is analytic (in L^2);*
(ii) $(P^k)_{k\in\mathbb{N}}$ is analytic in L^q for all $q \in (1,\infty)$, i.e. there exists C_q such that
$$\|(I-P)P^k f\|_{L^q(\Gamma)} \leq \frac{C_q}{k+1}\|f\|_{L^q(\Gamma)} \qquad \textit{for } k \in \mathbb{N} \textit{ and } f \in L^q(\Gamma);$$
(iii) -1 is not in the L^2 spectrum of P;

(iv) there exists $\ell \in \mathbb{N}$ and $\epsilon > 0$ such that

$$p_{2\ell+1}(x,x)m(x) \geq \epsilon \qquad \text{for } x \in \Gamma.$$

If we assume moreover that there exists ϵ_0 such that $p(x,y)m(y) \geq \epsilon_0$ whenever $x \sim y$, then analyticity is also equivalent to

(v) There exists $\ell \in \mathbb{N}$ such that, for any vertex $x \in \Gamma$, we can find a closed path $x = x_0, x_1, \dots, x_{2\ell+1} = x$ of length $2\ell + 1$.

The doubling property implies that the volume of growth of the balls is at most polynomial, meaning that there exists $C > 0$ and $D > 0$ such that

$$V(x, \Lambda r) \leq C \Lambda^D V(x,r) \qquad \text{for } x \in \Gamma,\ r \in \mathbb{N}, \text{ and } \Lambda \geq 1. \tag{2.3}$$

The parameter β is the diffusion (UE_β) is related to the escape rate of the diffusion: that is, the average time needed for a particle in x to escape the ball $B(x,r)$ is equivalent to $k = r^\beta$. Barlow showed in [Bar04] that we can find a graph $(\Gamma^{D,\beta}, \mu^{D,\beta})$ satisfying $V(x,r) \approx r^D$ and (UE_β) when (and only when) $D \in (1,\infty)$ and $\beta \in [2, D+1]$. The cases $\beta = D+1$ are the Vicsek graphs mentioned in Theorem 1.1 (Figure 1). The case $D = \log_2(3)$ and $\beta = \log_2(5)$ is known as the graphical Sierpinski gasket (Figure 2). Usually, a diffusion with parameter β will have bounds of the form

$$p_k(x,y) \leq \frac{C}{V(x,k^{1/\beta})} \exp\left(-c \left[\frac{d(x,y)^\beta}{k+1} \right]^{\frac{1}{\beta-1}} \right) \tag{2.4}$$

for some $C, c > 0$. Note that (UE_β) is weaker and has a simpler form than (2.4), and since (UE_β) is enough for our purpose, we prefer to use (UE_β) instead of (2.4).

As one might have figured out at this point, our Laplacian is $\Delta := (I-P)$, and the 'length of gradient' is

$$\nabla f(x) := \left(\frac{1}{2} \int_{y\sim x} p(x,y) |f(y) - f(x)|^2 m(y) \right)^{\frac{1}{2}}.$$

We let the reader check that we have indeed the isometry

$$\|\nabla f\|_{L^2}^2 = \sum_{x\in\Gamma} (I-P)[f](x)\, f(x) = \|(I-P)^{1/2} f\|_{L^2}^2. \tag{2.5}$$

2.2. **The results.** We start by introducing the class of "cut-off" functions.

Definition 2.6. *Given a ball $B \subset \Gamma$, we say that a function f on Γ belongs to $\mathcal{CO}(B)$ if*

(a) $\operatorname{supp} f \in 4B$,
(b) $0 \leq f \leq 1$,
(c) $f \equiv 1$ on B,

Note that with our definition, we have $\|f\|_{L^p} \approx V(B)^{1/p}$ whenever $f \in \mathcal{CO}(B)$ and the underlying graph (Γ, μ) is doubling.

Proposition 2.7. *Let (Γ,μ) be a doubling graph whose Markov semigroup is analytic and satisfies (UE_β). Then for any $p\in(1,\infty)$ and any $\alpha>0$, there exists $C=C(\alpha,p)>0$ such that for any ball $B=B(x,r)\subset\Gamma$ and any function $f\in\mathcal{CO}(B)$, we have*

$$\frac{\|(I-P)^\alpha f\|_{L^p}}{\|f\|_{L^p}}\geq C^{-1}r^{-\alpha\beta}. \tag{2.8}$$

Remark 2.9. We choose to write the result in the form (2.8) to make a parallel with Faber-Krahn inequalities. Note that an immediate consequence of the theorem and (2.5) is the Faber-Krahn inequality

$$\frac{\|\nabla f\|^2_{L^2}}{\|f\|^2_{L^2}}\geq C^{-1}r^{-\beta}\qquad \text{for } f\in\mathcal{CO}(B(x,r)). \tag{2.10}$$

Proof: We start with the case $\alpha\in(0,1)$, which is less technical, and we will give the explanations to treat the case $\alpha\geq 1$.

Let $k=\lambda^\beta r^\beta$, with $\lambda\geq 1$ to be determined later. We can write f as

$$f=P^{k+1}f+\sum_{\ell=0}^{k}(I-P)P^\ell f. \tag{2.11}$$

Step 1. The norm of $P^{k+1}f$ can be estimated using (UE_β), and we shall see that with a good choice of λ, we will get that

$$\|P^{k+1}f\|_p\leq\frac{1}{2}\|f\|_p \tag{2.12}$$

Indeed,

$$\begin{aligned}\|P^{k+1}f\|_{L^p}&:=\left(\sum_{x\in\Gamma}\left|\sum_{y\in\Gamma}p_{k+1}(x,y)f(y)m(y)\right|^p m(x)\right)^{\frac{1}{p}}\\ &\leq\sup_{y\in\Gamma}\left(\sum_{x\in\Gamma}|p_{k+1}(x,y)|^p m(x)\right)^{\frac{1}{p}}\|f\|_{L^1}.\end{aligned} \tag{2.13}$$

From (D) and the conditions on f, we have

$$\|f\|_{L^1}\approx V(B)\approx V(B)^{1-\frac{1}{p}}\|f\|_p. \tag{2.14}$$

Moreover, we can use the estimate (UE_β) with $N=D+1$, where D is the "dimension" given in (2.3). In this case,

$$\begin{aligned}\sum_{x\in\Gamma}|p_{k+1}(x,y)|^p m(x)&=\sum_{x\in 2\Lambda B}|p_{k+1}(x,y)|^p m(x)+\sum_{j\in\mathbb{N}^*}\sum_{x\in 2^{j+1}\lambda B\backslash 2^j\lambda B}|p_{k+1}(x,y)|^p m(x)\\ &\lesssim\sum_{j\in N}V(2^{j+1}\lambda B)\left(\frac{(2^j\lambda)^{-D-1}}{V(\lambda B)}\right)^p\lesssim\sum_{j\in N}2^{(j+1)D}V(\lambda B)\left(\frac{(2^j\lambda)^{-D-1}}{V(\lambda B)}\right)^p\lesssim\frac{\lambda^{-(D+1)p}}{V(\lambda B)^{p-1}}\end{aligned}$$

By choosing λ sufficiently large (independently of x or r), we get that

$$\sup_{y\in\Gamma}\left(\sum_{x\in\Gamma}|p_{k+1}(x,y)|^p m(x)\right)^{\frac{1}{p}} \leq \frac{1}{2}V(B)^{\frac{1}{p}-1}\underbrace{\frac{\|f\|_p V(B)^{1-\frac{1}{p}}}{\|f\|_1}}_{\approx 1 \text{ by } (2.14)} = \frac{1}{2}\frac{\|f\|_p}{\|f\|_1}.$$

We reinject the above estimate in (2.13) to get the claim (2.12).

Step 2. The last term of (2.11) can be rewritten as

$$\sum_{\ell=0}^{k}(I-P)P^\ell f = \sum_{\ell=0}^{k}(I-P)^{1-\alpha}P^\ell[(I-P)^\alpha f].$$

But $(P^\ell)_{\ell\in\mathbb{N}}$ is an analytic Markov semigroup in L^p, so [CSC90, Proposition 2] gives that

$$\|(I-P)^{1-\alpha}P^\ell g\|_{L^p} \leq C(\ell+1)^{\alpha-1}\|g\|_{L^p} \qquad \text{for } g\in L^p,$$

which means that

$$\begin{aligned}\Big\|\sum_{\ell=0}^{k}(I-P)P^\ell f\Big\|_{L^p} &\lesssim \left(\sum_{\ell=0}^{k}(\ell+1)^{\alpha-1}\right)\|(I-P)^\alpha f\|_{L^p}\\ &\lesssim (k+1)^\alpha\|(I-P)^\alpha f\|_{L^p} \lesssim r^{\alpha\beta}\|(I-P)^\alpha f\|_{L^p}\end{aligned}$$

with our choice of k.

Step 3: Conclusion We reinject the estimates from Steps 1 and 2 in (2.11), and we obtain the existence of $C>0$ independent of B and f such that

$$\|f\|_{L^p(\Gamma)} \leq \frac{1}{2}\|f\|_{L^p(\Gamma)} + Cr^{\alpha\beta}\|(I-P)^\alpha f\|_{L^p(\Gamma)}.$$

The desired estimate (2.8) follows (when $0<\alpha<1$).

Step 4: Case $\alpha\geq 1$. The strategy is the same: we just need to find a appropriate identity like (2.11). If $\alpha\in[1,2)$, we have

$$f = P^{k+1}f + \sum_{\ell=0}^{k}(I-P)P^\ell \underbrace{f}_{=P^{k+1}f+\sum_{\ell=0}^{k}(I-P)P^\ell f}, \tag{2.15}$$

so

$$f = \underbrace{P^{k+1}f + \sum_{\ell=0}^{k}(I-P)P^{\ell+k+1}f}_{I_1(f)\text{: treated like in Step 1}} + \underbrace{\sum_{\ell=0}^{k}\sum_{n=0}^{k}(I-P)^2P^{\ell+n}f}_{I_2(f)\text{: treated like in Step 2}}.$$

Note that by using the boundedness of P, we easily have that

$$\|I_1(f)\|_{L^p} \lesssim \|P^{k+1}f\|_{L^p}$$

with a constant independent of k, so for λ large enough, we have

$$\|I_1(f)\|_{L^p} \leq \frac{1}{2}\|f\|_{L^p}.$$

As for $I_2(f)$, by the analyticity of $(P^\ell)_{\ell\in\mathbb{N}}$, we have

$$\|I_2(f)\|_{L^p} \lesssim \sum_{\ell=0}^k \sum_{n=0}^k (\ell+n+1)^{\alpha-2}\|(I-P)^\alpha f\|_{L^p} \lesssim (k+1)^\alpha \|(I-P)^\alpha f\|_{L^p}$$

Altogether,

$$\|f\|_{L^p} \leq \frac{1}{2}\|f\|_{L^p} + Cr^{\alpha\beta}\|(I-P)^\alpha f\|_{L^p},$$

which proves (2.8) when $\alpha \in [1,2)$.

To treat the general case $\alpha > 0$, we need to take N to be the first integer strictly bigger than α, then obtain an appropriate reproductive formula that has some terms in $(I-P)^N P^\ell f$ by iterating the process in (2.15) N times. □

Proposition 2.16. *Let (Γ,μ) be a doubling graph. Assume that there exists $\beta > 0$ such that*

(i) (Γ,μ) satisfies that for all $p \in (1,2)$, there exists $C = C(p) > 0$ such that for any ball $B = B(x,r) \subset \Gamma$ and any function $f \in \mathcal{CO}(B)$, we have

$$\frac{\|(I-P)^{\frac{1}{2}} f\|_{L^p}}{\|f\|_{L^p}} \geq C^{-1} r^{-\beta/2}. \tag{2.17}$$

(ii) There exists $C > 0$, a collection of balls $B_n := B(x_n, r_n)$, and a collections of functions $f_n \in \mathcal{CO}(B_n)$ such that

$$\frac{\|\nabla f_n\|_{L^2}}{\|f_n\|_{L^2}} \leq C r_n^{-\beta/2} \tag{2.18}$$

and

$$\lim_{n\to\infty} \frac{\|\nabla f_n\|_{L^1}}{\|f_n\|_{L^1}} \frac{\|f_n\|_{L^2}}{\|\nabla f_n\|_{L^2}} = 0. \tag{2.19}$$

Then the reverse Riesz transform is unbounded in L^p for all $p \in (1,2)$. More precisely,

$$\lim_{n\to\infty} \frac{\|\nabla f_n\|_{L^p}}{\|(I-P)^{1/2} f_n\|_{L^p}} = 0.$$

Remark 2.20. The above proposition is quite empty: the reader will see that the proof is almost immediate. We just want to emphasize here that we need to show to prove that the reverse Riesz transform is unbounded in L^p for all $p \in (1,2)$:

- first we need to figure out what is the optimal rate of decay of L^2 Faber-Krahn inequality (it is given by the pointwise estimate on the heat kernel for instance, see Proposition 2.7);
- and second, we need to check that the gradients of the functions that minimize the Faber-Krahn inequality does **not** satisfy a reverse Hölder inequality. This second condition reflects a fractal structure of the graph, as it means that the gradient takes low values most of the time, and is big in few concentrated places.

Proof: Let $p \in (1,2)$ and assume by contradiction that

$$\inf_{n\in\mathbb{N}} \frac{\|\nabla f_n\|_{L^p}}{\|f_n\|_{L^p}} \frac{\|f_n\|_{L^2}}{\|\nabla f_n\|_{L^2}} = \epsilon > 0. \tag{2.21}$$

But since $\|f_n\|_q^q \approx V(B_n)$ for all n, with a constant independent of q, we have that

$$\left(\frac{1}{V(B_n)}\sum_{x\in\Gamma}|f_n(x)|^2 m(x)\right)^{\frac{1}{2}} \leq C_\epsilon \left(\frac{1}{V(B_n)}\sum_{x\in\Gamma}|f_n(x)|^p m(x)\right)^{\frac{1}{p}}$$

But the left-hand side can be bounded using interpolation by

$$\left(\frac{1}{V(B_n)}\sum_{x\in\Gamma}|f_n(x)|^p m(x)\right)^{\frac{1}{p}} \leq \left(\frac{1}{V(B_n)}\sum_{x\in\Gamma}|f_n(x)| m(x)\right)^{\frac{2}{p}-1} \left(\frac{1}{V(B_n)}\sum_{x\in\Gamma}|f_n(x)|^2 m(x)\right)^{1-\frac{1}{p}}$$

So we obtain

$$\left(\frac{1}{V(B_n)}\sum_{x\in\Gamma}|f_n(x)|^2 m(x)\right)^{\frac{1}{2}(\frac{2}{p}-1)} \leq C_\epsilon \left(\frac{1}{V(B_n)}\sum_{x\in\Gamma}|f_n(x)| m(x)\right)^{\frac{2}{p}-1},$$

that is

$$\left(\frac{1}{V(B_n)}\sum_{x\in\Gamma}|f_n(x)|^2 m(x)\right)^{\frac{1}{2}} \leq (C_\epsilon)^{p/(2-p)} \frac{1}{V(B_n)}\sum_{x\in\Gamma}|f_n(x)| m(x).$$

Using again the fact that $\|f_n\|_q^q \approx V(B_n)$, it entails that

$$\inf_{n\in\mathbb{N}} \frac{\|\nabla f_n\|_{L^1}}{\|f_n\|_{L^1}} \frac{\|f_n\|_{L^2}}{\|\nabla f_n\|_{L^2}} > 0,$$

which contradicts the assumption (2.19). Consequently, we have

$$\lim_{n\to\infty} \frac{\|\nabla f_n\|_{L^p}}{\|f_n\|_{L^p}} \frac{\|f_n\|_{L^2}}{\|\nabla f_n\|_{L^2}} = 0$$

and then, by invoking successively (2.17) and (2.18),

$$0 \leq \frac{\|\nabla f_n\|_{L^p}}{\|(I-P)^{\frac{1}{2}} f_n\|_{L^p}} \lesssim r^{\beta/2} \frac{\|\nabla f_n\|_{L^p}}{\|f_n\|_{L^p}} \lesssim \frac{\|\nabla f_n\|_{L^p}}{\|f_n\|_{L^p}} \frac{\|f_n\|_{L^2}}{\|\nabla f_n\|_{L^2}} \to 0 \text{ as } n \to \infty.$$

The proposition follows. □

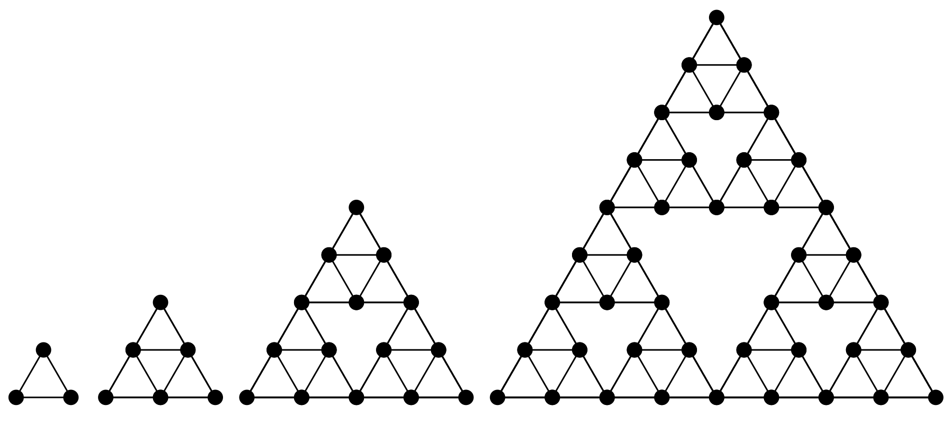
3. The case of the graphical Sierpinski gasket

In the case of the graphical Sierpinski gasket (Figures 3 and 2), we have $p(x,y) = \mu_{xy} = \frac{1}{4}$ whenever $x \neq y$ and an edge is linking x to y[1], and so $m(x) = 1$. The underlying Markov process is analytic, as we can easily see that any vertex of Γ is a part of a triangle of sidelength 1, meaning that (v) of Proposition 2.2 is satisfied. It is fairly easy to see that the volume of the balls in the graphical Sierpinski gasket satisfies

$$V(x,r) \approx r^{\log_2 3}.$$

[1]We could also take $p(x,y) = \mu_{xy} = \frac{1}{5}$ whenever $x = y$ or an edge is joining x to y.

Figure 3. Graphical Sierpinski gasket (4 first steps of construction)



Moreover, as shown in [Jon96], the Markov kernel verify (UE_β) with the diffusion parameter $\beta := \log_2(5)$, or more precisely, when $d(x,y) \leq n$, we have

$$\frac{c}{V(x,k^{1/\beta})}\exp\Big(-C\big[\frac{d(x,y)^\beta}{k+1}\big]^{1/(\beta-1)}\Big) \leq p_k(x,y) \leq \frac{C}{V(x,k^{1/\beta})}\exp\Big(-c\big[\frac{d(x,y)^\beta}{k+1}\big]^{1/(\beta-1)}\Big)$$

where $0 < c < C$. Proposition 2.7 shows that the first assumption of Proposition 2.16 is satisfied with $\beta = \log_2(5)$. As a consequence, Theorem 1.2 will be a consequence of Proposition 2.16 once if we can find a sequence of cut-off functions f_n such that (2.18)–(2.19) are satisfied.

Let Γ_n be the triangle subset of Γ of sidelength 2^n, as shown in Figure 3. For $n \in \mathbb{N}$ and $z \in [0,1]$, let $g_n = g_n^{(a,b,c)}$ be the function that minimizes the quantity

$$\sum_{x\in\Gamma_n}\sum_{\substack{y\in\Gamma_n\\ y\sim x}} \frac{1}{4}|g(x)-g(y)|^2$$

when g is taken in the set of functions whose values at the three corners of Γ_n are $a,b,c \in \mathbb{R}$. To obtain $g_{n+1}^{(a,b,c)}$ from $g_n^{(a,b,c)}$, it is classical: it suffices to add a vertex in the middle of each segment of each Γ_0 sub-triangle of Γ_n, and add a value on the new vertices as given in Figure 4.

We write $H_n^{(p)}(a,b,c)$ to be the quantity

$$\sum_{x\in\Gamma_n}\sum_{\substack{y\in\Gamma_n\\ y\sim x}} \frac{1}{4}|g_n^{(a,b,c)}(x)-g_n^{(a,b,c)}(y)|^p,$$

FIGURE 4. How to find $g_1^{(a,b,c)}$ from $g_0^{(a,b,c)}$

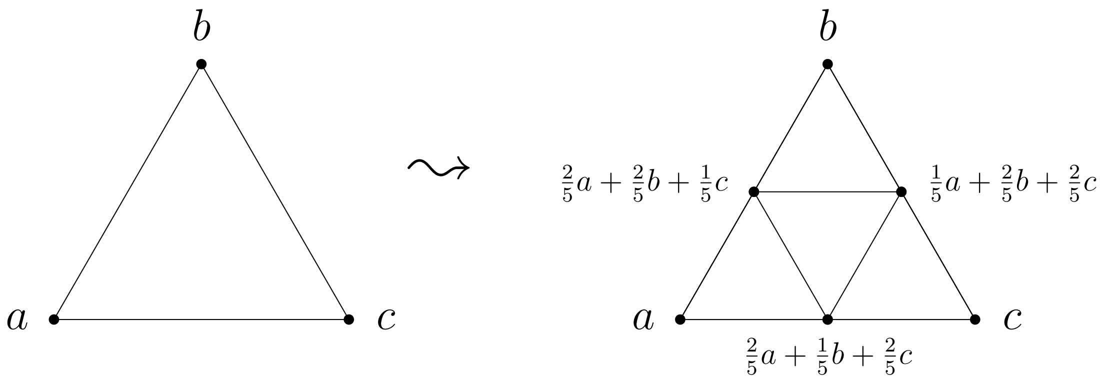
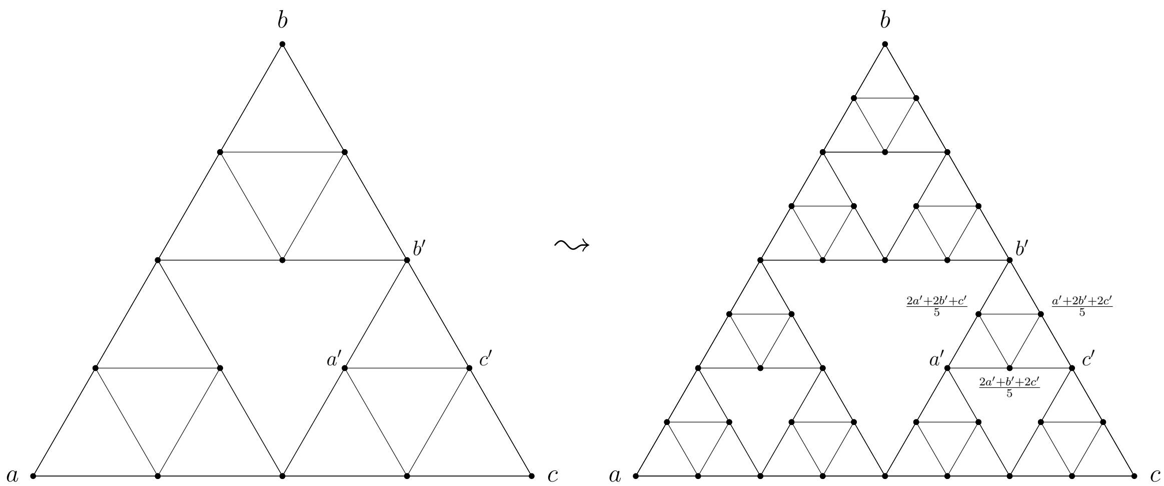


FIGURE 5. Passage from $g_n^{(a,b,c)}$ to $g_{n+1}^{(a,b,c)}$, where we use Figure 4 on each elementary triangle.



in particular $H_n^{(2)}(a,b,c)$ is the minimum value of the functional that $g_n^{(a,b,c)}$ minimizes. We let the reader check that

$$H_1^{(2)}(a,b,c) = \frac{3}{5} \underbrace{\frac{(a-b)^2 + (b-c)^2 + (c-a)^2}{4}}_{H_0^{(2)}(a,b,c)},$$

so we obtain by iteration that

$$H_n^{(2)}(a,b,c) = \left(\frac{3}{5}\right)^n H_0^{(2)}(a,b,c).$$

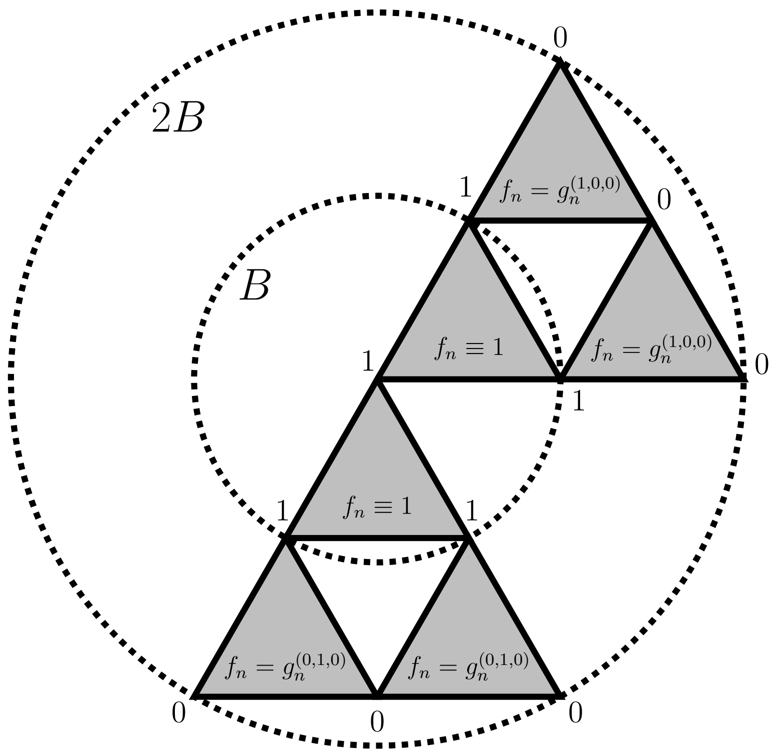
Similarly, we have

$$H_1^{(1)}(a,b,c) = \frac{6}{5} \underbrace{\frac{|a-b|+|b-c|+|c-a|}{4}}_{H_0^{(1)}(a,b,c)},$$

so

$$H_n^{(1)}(a,b,c) = \left(\frac{6}{5}\right)^n H_0^{(1)}(a,b,c).$$

FIGURE 6. Choice of f_n. Each gray triangle represents Γ_n.



Finally, we construct f_n as in Figure 6. We have in one hand that

$$\|\nabla f_n\|_{L^2} = \sqrt{4H_n^{(2)}(1,0,0)} = \sqrt{\frac{1}{2}\left(\frac{3}{5}\right)^n},$$

hence, since $\|f_n\|_2^2 \approx m(\Gamma_n) = 3^n$,

$$\frac{\|\nabla f_n\|_{L^2}}{\|f_n\|_{L^2}} \approx \sqrt{\left(\frac{1}{5}\right)^n} \approx (2^n)^{\frac{\log_2 5}{2}}. \tag{3.1}$$

On the other hand, we also have

$$\|\nabla f_n\|_{L^1} \approx 4H_n^{(1)}(1,0,0) = \frac{1}{2}\left(\frac{6}{5}\right)^n,$$

and since $\|f_n\|_{L^1} \approx 3^n$

$$\frac{\|\nabla f_n\|_{L^1}}{\|f_n\|_{L^1}} \frac{\|f_n\|_{L^2}}{\|\nabla f_n\|_{L^2}} \approx \Big(\frac{2\sqrt{5}}{5}\Big)^n \to 0 \text{ as } n \to \infty, \tag{3.2}$$

The estimates (3.1) and (3.2) show that the remaining assumption of Proposition 2.16 - assumption (ii) - is satisfied, so Proposition 2.16 implies that the reverse Riesz transform is unbounded on $L^p(\Gamma)$ for all $p \in (1, 2)$. Theorem 1.2 follows.

References

[AC05] P. Auscher and Thierry Coulhon, *Riesz transform on manifolds and Poincaré inequalities*, Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5) **4** (2005), no. 3, 531–555. 2, 5

[ACDH04] Pascal Auscher, Thierry Coulhon, Xuan Thinh Duong, and Steve Hofmann, *Riesz transform on manifolds and heat kernel regularity*, Ann. Sci. Éc. Norm. Supér. (4) **37** (2004), no. 6, 911–957. 2, 5

[AMR08] Pascal Auscher, Alan McIntosh, and Emmanuel Russ, *Hardy spaces of differential forms on Riemannian manifolds*, J. Geom. Anal. **18** (2008), no. 1, 192–248. 2

[Bar04] Martin T. Barlow, *Which values of the volume growth and escape time exponent are possible for a graph?*, Rev. Mat. Iberoam. **20** (2004), no. 1, 1–31. 7

[BCG01] Martin Barlow, Thierry Coulhon, and Alexander Grigor'yan, *Manifolds and graphs with slow heat kernel decay*, Invent. Math. **144** (2001), no. 3, 609–649. 3

[BD14] The Anh Bui and Xuan Thinh Duong, *Hardy spaces associated to the discrete Laplacians on graphs and boundedness of singular integrals*, Trans. Am. Math. Soc. **366** (2014), no. 7, 3451–3485. 2

[BF16] Frédéric Bernicot and Dorothee Frey, *Riesz transforms through reverse Hölder and Poincaré inequalities*, Math. Z. **284** (2016), no. 3-4, 791–826. 2

[BR09] Nadine Badr and Emmanuel Russ, *Interpolation of Sobolev spaces, Littlewood-Paley inequalities and Riesz transforms on graphs*, Publ. Mat., Barc. **53** (2009), no. 2, 273–328. 2, 5

[CCFR17] Li Chen, Thierry Coulhon, Joseph Feneuil, and Emmanuel Russ, *Riesz transform for* $1 \le p \le 2$ *without Gaussian heat kernel bound*, J. Geom. Anal. **27** (2017), no. 2, 1489–1514. 2, 3, 4

[CD99] Thierry Coulhon and Xuan Thinh Duong, *Riesz transforms for* $1 \le p \le 2$, Trans. Am. Math. Soc. **351** (1999), no. 3, 1151–1169. 2, 5

[CD03] Thierry Coulhon and Xuan Thinh Duong, *Riesz transform and related inequalities on non-compact Riemannian manifolds*, Commun. Pure Appl. Math. **56** (2003), no. 12, 1728–1751. 4

[CSC90] Thierry Coulhon and Laurent Saloff-Coste, *Puissances d'un opérateur régularisant. (Powers of a regularizing operator)*, Ann. Inst. Henri Poincaré, Probab. Stat. **26** (1990), no. 3, 419–436 (French). 6, 9

[DR24] Baptiste Devyver and Emmanuel Russ, *Reverse inequalities for quasi-Riesz transform on the Vicsek cable system*, Preprint, arXiv:2403.02779 [math.AP] (2024), 2024. 5

[DRY23] Baptiste Devyver, Emmanuel Russ, and Meng Yang, *Gradient estimate for the heat kernel on some fractal-like cable systems and quasi-Riesz transforms*, Int. Math. Res. Not. **2023** (2023), no. 18, 15537–15583. 5

[Dun06] Nick Dungey, *A note on time regularity for discrete time heat kernels*, Semigroup Forum **72** (2006), no. 3, 404–410. 6

[Fen16] Joseph Feneuil, *Hardy and* BMO *spaces on graphs, application to Riesz transform*, Potential Anal. **45** (2016), no. 1, 1–54. 2

[Fen18] ———, *About the* L^2 *analyticity of Markov operators on graphs*, Proc. Am. Math. Soc. **146** (2018), no. 4, 1793–1805. 6

[HLM+11] Steve Hofmann, Guozhen Lu, Dorina Mitrea, Marius Mitrea, and Lixin Yan, *Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates*, Mem. Am. Math. Soc., vol. 1007, Providence, RI: American Mathematical Society (AMS), 2011. 2

[Jon96] Owen Dafydd Jones, *Transition probabilities for the simple random walk on the Sierpinski graph*, Stochastic Processes Appl. **61** (1996), no. 1, 45–69. 12

[LM14] Christian Le Merdy, *H^∞ functional calculus and square function estimates for Ritt operators*, Rev. Mat. Iberoam. **30** (2014), no. 4, 1149–1190. 6

[LMS+23] Matteo Levi, Alessio Martini, Federico Santagati, Anita Tabacco, and Maria Vallarino, *Riesz transform for a flow Laplacian on homogeneous trees*, J. Fourier Anal. Appl. **29** (2023), no. 2, 29, Id/No 15. 5

[RD22] Emmanuel Russ and Baptiste Devyver, *Reverse inequality for the Riesz transforms on Riemannian manifolds*, Preprint, arXiv:2209.05083 [math.AP] (2022), 2022. 2

[Rus00] Emmanuel Russ, *Riesz transforms on graphs for $1 \leq p \leq 2$*, Math. Scand. **87** (2000), no. 1, 133–160. 2, 5

[Rus01] ______, *$H^1 - L^1$ boundedness of Riesz transforms on Riemannian manifolds and on graphs*, Potential Anal. **14** (2001), no. 3, 301–330. 2

[Sjö99] Peter Sjögren, *An estimate for a first-order Riesz operator on the affine group*, Trans. Am. Math. Soc. **351** (1999), no. 8, 3301–3314. 5

[Ste70] Elias M. Stein, *Singular integrals and differentiability properties of functions*, Princeton Math. Ser., vol. 30, Princeton University Press, Princeton, NJ, 1970. 1

[Str83] Robert S. Strichartz, *Analysis of the Laplacian on a complete Riemannian manifold*, J. Funct. Anal. **52** (1983), 48–79. 2

[SV08] Peter Sjögren and Maria Vallarino, *Boundedness from H^1 to L^1 of Riesz transforms on a Lie group of exponential growth*, Ann. Inst. Fourier **58** (2008), no. 4, 1117–1151. 5

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