Minimal non-comparability graphs and semi-transitivity

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Abstract. The concept of word-representable graphs has been widely explored in the literature. The class of word-representable graphs is characterized by the existence of a semi-transitive orientation. Specifically, a graph is word-representable if and only if it admits such an orientation. Comparability graphs form a subclass of word-representable graphs. Both word-representable and comparability graphs belong to hereditary graph classes. Every hereditary class can be characterized in terms of their forbidden induced subgraphs. The minimal forbidden induced subgraphs of comparability graphs and word-representable graphs are referred to as minimal non-comparability graphs and minimal non-word-representable graphs, respectively.

While the complete set of minimal non-comparability graphs is known, identifying the set of all minimal non-word-representable graphs remains an open problem. In this paper, we precisely determine the set of all minimal non-comparability graphs that are minimal non-word-representable graphs as well. To achieve this, we categorize all minimal non-comparability graphs into those that are semi-transitive and those that are not.

Furthermore, as a byproduct of our classification, we establish a characterization and a complete list of minimal non-word-representable graphs that contain an all-adjacent vertex. This is accomplished by introducing an all-adjacent vertex to each minimal non-comparability graph that is semi-transitive. As a result of our study, we identify several infinite families of minimal non-word-representable graphs, expanding the understanding of their structural properties.

Keywords: comparability graph \cdot minimal non-comparability graph \cdot word-representable graph \cdot minimal non-word-representable graph \cdot semi-transitivity.

1 Introduction

Comparability Graphs Comparability graphs form an important class of graphs characterized by the existence of a transitive orientation. These are graphs whose edges can be oriented such that for any three vertices x, y, and z, if there is a directed edge from x to y and from y to z, then a directed edge from x to z

exists. Comparability graphs belong to hereditary graph classes. A graph class X is said to be hereditary if, for any graph in X, all of its induced subgraphs also belong to X. Every hereditary graph class can be characterized in terms of forbidden induced subgraphs [5]. The list of minimal forbidden induced subgraphs for comparability graphs was identified by Gallai in [2]. These graphs, known as minimal non-comparability graphs, do not admit a transitive orientation, but every proper induced subgraph of them does.

Word-representable graphs and semi-transitivity A graph is said to be wordrepresentable if there exists a word over its vertex set such that two vertices are adjacent if and only if their occurrences alternate within the word. The notion of word-representable graphs was introduced by Sergey Kitaev in [7]. Since its inception, this concept has garnered significant attention ([1], [4], [6]). Word-representable graphs hold significance within graph theory, as they encompass various graph classes such as 3-colorable graphs, sub-cubic graphs, and comparability graphs [5]. Word-representable graphs also belong to hereditary graph classes. However, the identification of the complete set of minimal forbidden induced subgraphs (also called minimal non-word-representable graphs) for word-representable graphs remains an open problem. Word-representable graphs are characterized by the existence of a semi-transitive orientation. Specifically, a graph is word-representable if and only if it admits such an orientation [3]. The notion of semi-transitivity extends the concept of transitive orientations. A graph is said to be semi-transitive if it admits an acyclic orientation satisfying a specific ordering condition on directed paths.

We address the following question in this paper.

Problem 1. Are there minimal non-comparability graphs that also qualify as minimal non-word-representable graphs? If so, which graphs belong to this intersection?

Contributions In this paper, we determine which minimal non-comparability graphs are also minimal non-word-representable graphs. We establish that this intersection consists of two infinite families of graphs and a graph on 7 vertices. To identify this intersection, we classify all minimal non-comparability graphs into those that are semi-transitive and those that are not. Building on this classification, we identify and characterize minimal non-word-representable graphs containing an all-adjacent vertex. As a result, we identify several infinite families of minimal non-word-representable graphs.

Organization The paper is organized as follows. In Section 2, we provide the basic definitions and results that we follow in this paper. In Section 3, we identify the minimal non-comparability graphs, which are semi-transitive. In Section 4, we identify the minimal non-comparability graphs, which are not semi-transitive. Section 5 provides the concluding remarks.

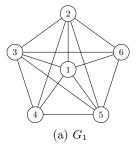
2 Preliminaries

The graphs that we consider in this paper are all simple graphs. For a graph G, V(G) denotes the vertex set of G, and E(G) denotes the edge set of G.

Definition 1. Consider a word w on the alphabet Σ . For any $i, j \in \Sigma$, the word w_{ij} is the word w restricted to the copies of i and j in the order of their appearance in w. The letters i and j are said to alternate in w if there is no substring of the form ii or jj in the word w_{ij} .

Example 1. Let w=3123143. The letters 1 and 2 alternate in w, as the substrings 11 and 22 are absent in $w_{12}=121$. The letters 1 and 3 are alternating in w, as $w_{13}=31313$. Letters 2 and 4 alternate in w, as $w_{24}=24$. Letters 2 and 3 does not alternate in w, as $w_{23}=3233$, and it has a substring 33 in it. Letters 1 and 4 are not alternating in w, as $w_{14}=114$ contains the substring 11 in it. Letters 3 and 4 are not alternating in w, as $w_{34}=3343$ contains the substring 33 in it.

Definition 2. A graph G is said to be word-representable if there exists a word w over V(G) such that for any distinct pair of vertices $i, j \in V(G)$, $\{i, j\} \in E(G)$ if and only if the letters i and j alternate in w. The word w is said to represent G.



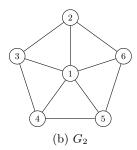


Fig. 1: G_1 is a word-representable graph and G_2 is a non-word-representable graph.

Example 2. The graph G_1 in Figure 1a is an example of a word-representable graph. The word w = 6123564 represents G_1 . The graph G_2 in Figure 1b is the Wheel graph W_5 on six vertices.

Definition 3. A class of graphs X is said to be hereditary if, for any graph $G \in X$, all induced subgraphs of G belong to the class X.

 $Example \ 3.$ Planar graphs, comparability graphs, and word-representable graphs are all examples of hereditary graph classes.

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The class of word-representable graphs is hereditary [5], which implies that any induced subgraph of a word-representable graph is word-representable. One of the important properties of hereditary graph classes is that they admit forbidden induced subgraph characterization.

Definition 4. A graph G is a forbidden induced subgraph for a hereditary class X if G is not an induced subgraph of any graph $H \in X$. This means that no graph in X can have G as an induced subgraph.

Example 4. Wheel graph W_5 shown in Figure 2a is a forbidden induced subgraph for word-representable graphs. W_5 is also a minimal forbidden induced subgraph for the class of word-representable graphs.

Definition 5. A graph G is a minimal forbidden induced subgraph for a hereditary class X if G is a forbidden induced subgraph for X and if any proper induced subgraph of G belongs to X.

There exists a unique set of minimal forbidden induced subgraphs for every hereditary class X [5]. The minimal forbidden induced subgraphs for word-representable graphs are the minimal non-word-representable graphs, and hence, it is of interest to study about them.

Definition 6. Let G be a non-word-representable graph. G is a minimal non-word-representable graph if every proper induced subgraph of G is word-representable.



Fig. 2: G_1 is minimal non-word-representable graph, where as G_2 is non-minimal non-word-representable graph.

Example 5. The graph G_1 in Figure 2a is an example of a minimal non-word-representable graph. G_1 is the Wheel graph W_5 on six vertices. The graph G_2 in Figure 2b contains Wheel graph W_5 as an induced subgraph, and hence G_2 is a non-minimal non-word-representable graph.

The notion of semi-transitivity is very important in the study of word-representable graphs.

Definition 7. A graph G is semi-transitive if it admits an acyclic orientation such that for any directed path $v_1 \rightarrow v_2 \rightarrow \cdots \rightarrow v_k$, with $v_i \in V(G)$, either

- there is no edge $v_1 \rightarrow v_k$, or
- the edge $v_1 \rightarrow v_k$ is present and there are edges $v_i \rightarrow v_j$, for all $1 \le i < j \le k$.

Such an orientation is called semi-transitive orientation. The property of having a semi-transitive orientation is referred to as the semi-transitivity property.

Note 1. A directed path $P = v_1 \to v_2 \to \cdots \to v_k$ is said to violate semi-transitivity if the edge $v_1 \to v_k$ exists, but there is at least one missing edge $v_i \to v_j$ for some $1 \le i < j \le k$. A directed graph G is semi-transitive if it is acyclic, and no path in G violates semi-transitivity.

Example 6. An example of a semi-transitive orientation is shown in Figure 3. Observe that the longest path of the graph in Figure 3 is of length 3, which is the path $3 \to 2 \to 6 \to 1$. Since $6 \to 1$ is not an edge, it does not violate the semi-transitivity. One thing to note is that any directed path of length 2 will not be a violation of semi-transitivity. It does not matter if there exists an edge between the first vertex and the third vertex in the path.

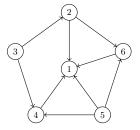


Fig. 3: A semi-transitive orientation.

Theorem 1 is significant as it establishes that a graph is word-representable if and only if it is semi-transitive. This result suggests that to prove a graph G is word-representable, it is sufficient to demonstrate that G is semi-transitive.

Theorem 1. [5] A graph G is word-representable if and only if it admits a semi-transitive orientation.

Definition 8. Let G be a directed graph and $v \in V(G)$. The vertex v is called a source vertex if all edges incident to v are directed away from it. Similarly, v is called a sink vertex if all edges incident to v are directed towards it.

Example 7. Consider the Figure 3. 1 is a sink vertex, and 3 is a source vertex.

Theorem 2 is an interesting result about the existence of semi-transitive orientations with any vertex as the source.

Theorem 2. [8] If a graph G is word-representable, then there is a semi-transitive orientation of G with any vertex $v \in V(G)$ as a source vertex.

Theorem 3 states that any 3-colorable graph is word-representable.

Theorem 3. [3] Any 3-colorable graph G is semi-transitive.

Definition 9. A graph G is a comparability graph if and only if the edges of G admit a transitive orientation. That is, if there is an edge directed from a to b and an edge directed from b to c, then there is an edge directed from a to c. A graph G, which is not a comparability graph, is called a non-comparability graph.

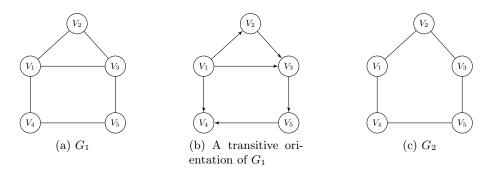


Fig. 4: G_1 is a comparability graph and G_2 is a non-comparability graph.

Example 8. Figure 4a shows an example of a comparability graph, and Figure 4b provides one of its transitive orientations. G_2 in Figure 4c is C_5 , which is known to be a non-comparability graph [2].

Comparability graphs form a subclass of word-representable graphs [5]. The class of comparability graphs is hereditary, and hence, a forbidden induced subgraph characterization exists. In the case of comparability graphs, the set of all minimal forbidden induced subgraphs (minimal non-comparability graphs) is known and was discovered by Gallai in [2].

Definition 10. Let G be a non-comparability graph. G is said to be a minimal non-comparability graph if every proper induced subgraph of G is a comparability graph. If G is not a minimal non-comparability graph, then G is a non-minimal non-comparability graph.

Example 9. Figure 5a shows C_5 , which is known to be a minimal non-comparability graph [2], and since G_2 in Figure 5b contains G_1 as a proper induced subgraph, G_2 is a non-minimal non-comparability graph.



Fig. 5: G_1 is a minimal non-comparability graph and G_2 is non-minimal non-comparability graph.

Gallai provided the forbidden induced subgraph characterization for comparability graphs in [2]. Gallai identified the complete list of minimal forbidden induced subgraphs for comparability graphs, which are precisely what we call minimal non-comparability graphs. The list of all minimal non-comparability graphs is given in Figure 6, 7, 8, and 9.

Several infinite classes of graphs are illustrated in Figures 6, 7, and 8. The individual graphs that are minimal non-comparability graphs are shown in Figure 9. In particular, Figure 7 presents two distinct classes of graphs, where only the missing edges are highlighted. Dashed edges represent absent connections, while all other edges are present. Additionally, Figure 8 depicts three graph classes, each structured into three levels. Given the complexity of these graphs, dashed edges are used in certain levels to indicate the only missing edges within that level. All other edges in the induced subgraph of vertices at that level remain present.

In the following section, we identify all the minimal non-comparability graphs that are semi-transitive.

3 Minimal non-comparability graphs that are semi-transitive

In this section, we prove that several minimal non-comparability graphs are semi-transitive. Theorem 4 shows that three of the infinite classes of minimal non-comparability graphs are 3-colorable and hence word-representable.

Theorem 4. The graph classes G_n^1 , G_n^2 , and G_n^3 shown in Figure 6 constitute a subclass of 3-colorable graphs.

Proof. For any graph $G \in G_n^1$, the vertex set can be partitioned into three independent sets: A, B, and C, where $A = \{1, 3, \dots, 2n-1\}, B = \{2, 4, \dots, 2n\},$ and $C = \{2n+1\}$. Similarly, any graph $G \in G_n^2$ can be vertex partitioned into

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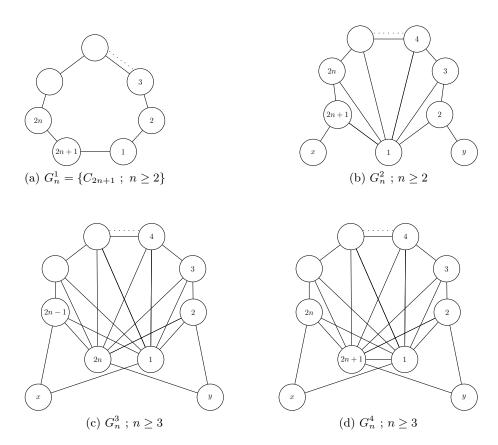


Fig. 6: Minimal non-comparability graphs - 1 (Four infinite classes of graphs.)

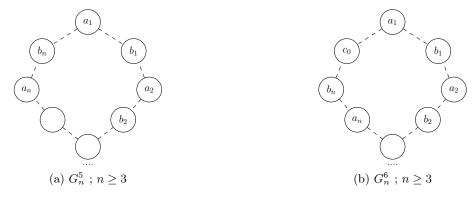


Fig. 7: Minimal non-comparability graphs - 2 (Two infinite classes of graphs, where only the missing edges are highlighted. Dashed edges represent absent connections, while all other edges are present.)

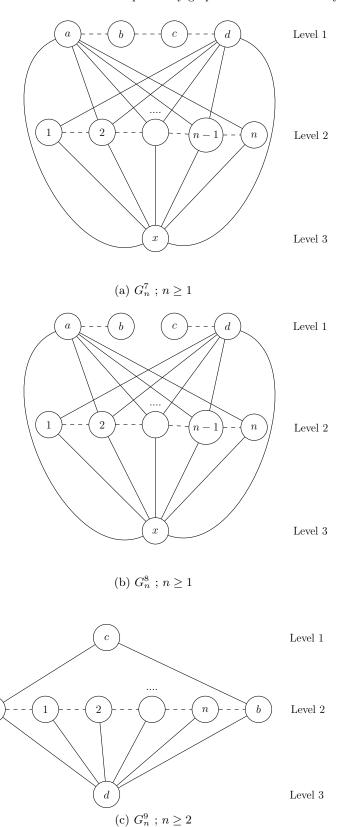


Fig. 8: Minimal non-comparability graphs - 3 (Three infinite graph classes, each structured into three levels. Due to the complexity of these graphs, dashed edges are used in certain levels to indicate the **only** missing edges within that level.)

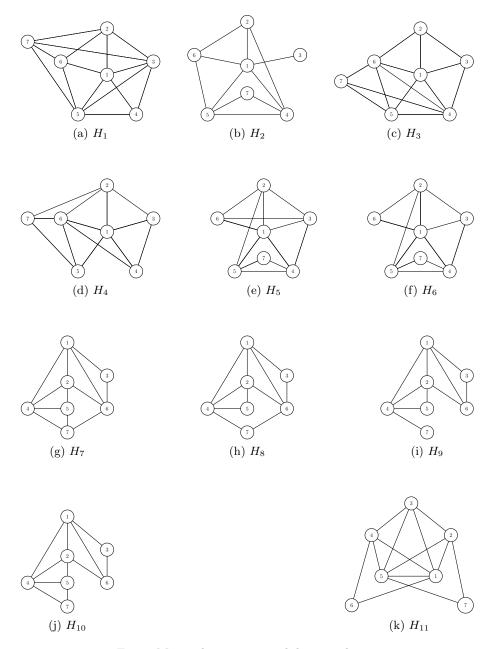


Fig. 9: Minimal non-comparability graphs - $4\,$

three independent sets: A, B, and C, where $A = \{1, x, y\}$, $B = \{2, 4, 6, \ldots, 2n\}$, and $C = \{3, 5, 7, \ldots, 2n+1\}$. Likewise, for any graph $G \in G_n^3$, the vertex set can be partitioned into three independent sets: A, B, and C, with $A = \{1, 2n\}$, $B = \{2, 4, 6, \ldots, 2n-2, x\}$, and $C = \{3, 5, 7, \ldots, 2n-1, y\}$. Figure 10 demonstrates the 3-colorability of these graph classes.

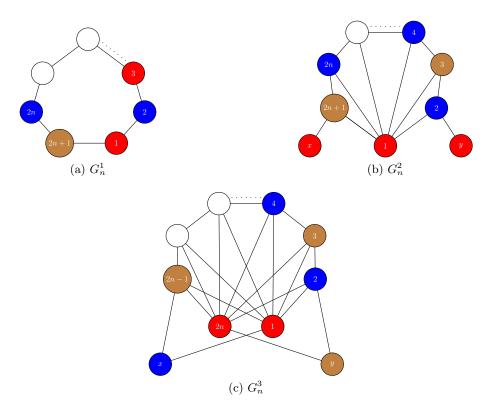


Fig. 10: 3-colorable minimal non-comparability graph classes

From Theorem 4 and Theorem 3, we have the following Corollary 1.

Corollary 1. The class of graphs G_n^1, G_n^2 , and G_n^3 form a subclass of word-representable graphs.

Theorem 5. The class of graphs G_n^5 depicted in Figure 7a form a subclass of word-representable graphs.

Proof. Consider an arbitrary graph $G \in G_n^5$, where |V(G)| = 2n and $|E(G)| = \binom{2n}{2} - 2n$. The 2n edges, which are absent in G, form a cycle as shown in Figure 7. Let A denote the set $\{a_1, a_2, \cdots a_n\}$ and B denote the set $\{b_1, b_2, \cdots b_n\}$.

Note that each of the sets A and B induce a clique in G. We provide a semi-transitive orientation of G. Consider the following orientation for edges in G. An edge between two vertices in G is oriented from the vertex with a smaller index value to the vertex with a larger index value. This is shown in Figure 11. This orientation is acyclic. For the sake of contradiction, assume that the orientation provided in Figure 11 is not semi-transitive. That implies that at least one directed path exists that violates semi-transitivity. Consider a path $P = u_1 \rightarrow u_2 \rightarrow \cdots \rightarrow u_k$ that violates semi-transitivity, where u_r is either a_i or b_j . The edge $u_1 \rightarrow u_k$ is present and some edge $\{u_i, u_j\}$ is missing, where $1 \leq i < j \leq k$.

Note 2. Any directed path in G under the orientation provided in Figure 11 has the index of vertices in strictly increasing order.

The missing edges in G can be categorized into two types.

- 1. Type $1 : \{a_i, b_j\}$, where $|i j| \le 1$.
- 2. Type $2: \{a_1, b_n\}$

If the missing edge $\{u_i, u_j\}$ in the path P is a Type 1 edge, then from Note 2, it can only be the adjacent vertices in P. That's a contradiction, and hence the missing edge $\{u_i, u_j\}$ in the path P cannot be a Type 1 edge. Hence, $\{u_i, u_j\}$ can only be $\{a_1, b_n\}$. Due to the specific orientation of edges in the graph G, if a_1 and b_n are present in the path P, then a_1 must be the starting vertex, and b_n must be the ending vertex. That is, $u_1 = a_1$ and $u_k = b_n$. However, since u_1 is not connected to u_k , this leads to a contradiction. Therefore, our initial assumption is false, confirming that the orientation is semi-transitive.



Fig. 11: A semi-transitive orientation of edges of $G \in G_n^5$

Theorem 6. The class of graphs G_n^6 depicted in Figure 7b form a subclass of word-representable graphs.

Proof. Consider an arbitrary graph $G \in G_n^6$, where |V(G)| = 2n+1 and $|E(G)| = \binom{2n+1}{2} - (2n+1)$. The 2n+1 edges, which are absent in G, form a cycle as shown in Figure 7. Let A denote the set $\{a_1, a_2, \cdots a_n\}$ and B denote the set $\{b_1, b_2, \cdots b_n\}$. Note that each of the sets A and B induce a clique in G. We provide a semi-transitive orientation of G. Consider the following orientation for edges in G. An edge between two vertices in G is oriented from the vertex with a smaller index value to the vertex with a larger index value. This is shown in

Figure 12. This orientation is acyclic. For the sake of contradiction, assume that the orientation provided in Figure 12 is not semi-transitive. That implies that at least one directed path exists that violates semi-transitivity. Consider a path $P = u_1 \to u_2 \to \cdots \to u_k$ that violates semi-transitivity, where u_r is either a_i or b_j . The edge $u_1 \to u_k$ is present and some edge $\{u_i, u_j\}$ is missing, where $1 \le i < j \le k$.

Note 3. Any directed path in G under the orientation provided in Figure 12 has the index of vertices in strictly increasing order.

The missing edges in G can be categorized into two types.

- 1. Type 1: $\{a_i, b_j\}$ in G, where $|i j| \le 1$.
- 2. Type $2: \{c_0, b_n\}$

If the missing edge $\{u_i, u_j\}$ in the path P is a Type 1 edge, then from Note 3, it can only be the adjacent vertices in P. That's a contradiction, and hence the missing edge $\{u_i, u_j\}$ in the path P cannot be a Type 1 edge. Hence, $\{u_i, u_j\}$ can only be $\{c_0, b_n\}$. Due to the specific orientation of edges in the graph G, if c_0 and b_n are present in the path P, then c_0 must be the starting vertex, and b_n must be the ending vertex. That is, $u_1 = c_0$ and $u_k = b_n$. However, since u_1 is not connected to u_k , this leads to a contradiction. Therefore, our initial assumption is false, confirming that the orientation is semi-transitive.

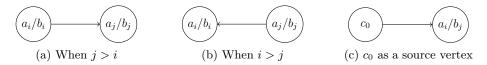


Fig. 12: A semi-transitive orientation of edges of $G \in G_n^6$

Theorem 7. The class of graphs G_n^7 depicted in Figure 8a form a subclass of word-representable graphs.

Proof. Consider an arbitrary graph $G \in G_n^7$. As illustrated in Figure 8a, the graph is structured into three levels. The vertex sets corresponding to levels 1, 2, and 3 are $\{a, b, c, d\}$, $\{1, 2, \ldots, n\}$, and $\{x\}$, respectively. Consider the orientation of the edges of G as shown in Figure 13. This orientation is acyclic. We show that this is a semi-transitive orientation of G by showing that none of the paths in G violate semi-transitivity. In Figure 13c, L1, L2 and L3 denote any vertex from Level 1, Level 2 and Level 3 respectively.

Note 4. No path originates from a level 2 vertex and terminates at a level 1 vertex. Similarly, no path starts at level 3 and ends at any other level.

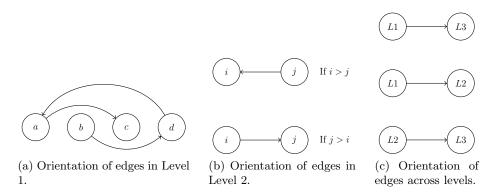


Fig. 13: A semi-transitive orientation of edges of $G \in G_n^7$.

Lemma 1. Consider G with its orientation as shown in Figure 13. Any path that starts and ends at a level 2 vertex does not violate the semi-transitivity property.

Proof. Consider an arbitrary path $P = u_1 \to u_2 \to \cdots \to u_k$, where each vertex u_i , for $1 \le i \le k$, belongs to level 2. For any consecutive vertices u_i and u_j in P, we have $|u_i - u_j| \ge 2$. Suppose P violates semi-transitivity. This implies that the edge $u_1 \to u_k$ is present, but some edge $\{u_i, u_j\}$ is missing for some $1 \le i < j \le k$. Since the only missing edges in level 2 are of the form $\{i, i+1\}$ for $1 \le i \le (n-1)$, this leads to a contradiction. Hence, our assumption is false, confirming that P does not violate semi-transitivity.

Observation 1 Consider G with its orientation as shown in Figure 13. Any path that starts and ends at a level 1 vertex does not violate the semi-transitivity property.

Proof. The longest path within level 1 has a length of 3, given by $P=b\to d\to a\to c$. The path P does not violate semi-transitivity, as the edge $b\to c$ is absent. Additionally, any path with a length of at most 2 does not violate semi-transitivity.

Lemma 2. Consider the graph G with its orientation as depicted in Figure 13. Any path in G that begins at a level 1 vertex and terminates at a level 2 vertex adheres to the semi-transitivity property.

Proof. The only vertices in level 1 that are adjacent to vertices in level 2 are d and a. Therefore, it suffices to demonstrate that the paths starting at a or d do not violate the semi-transitivity property. Consider an arbitrary path P that begins at vertex a and terminates at a vertex in level 2. For P to end at a level 2 vertex, its neighboring vertex must be from level 2. Vertex a is connected to all vertices in level 2, except for vertex 1. Therefore, if a is the starting vertex of path P, vertex 1 cannot appear in P. Since a is connected to every vertex in level

2, and by Lemma 1, we conclude that P does not violate the semi-transitivity property. Consider an arbitrary path P that begins at vertex d and terminates at a vertex in level 2. The path P can be of two types:

- 1. Type 1: $d \to a \to \text{level 2 vertices}$.
- 2. Type 2: $d \rightarrow$ level 2 vertices.

Consider P of Type 2. Vertex d is adjacent to all vertices in level 2, except n. If n appears in P, it must be at the end. By Lemma 1, P does not violate the semi-transitivity property. For P of Type 1, a similar argument applies. Paths of the form $a \to \text{level } 2$ vertices satisfy semi-transitivity. Prepending d does not violate this property, as d is adjacent to all level 2 vertices except n. If n appears in P, it must be at the end. By Lemma 1, P does not violate semi-transitivity.

Observation 2 Consider G with its orientation as shown in Figure 13. Any path in G that terminates at a level 3 vertex preserves the semi-transitivity property.

Proof. The vertex x is adjacent to all vertices except b and c. However, b and c (both in level 1) are not connected to any vertices from other levels. Therefore, by Observation 1, Lemma 1, and Lemma 2, any path in G ending at x maintains semi-transitivity.

Theorem 8. The class of graphs G_n^8 depicted in Figure 8b form a subclass of word-representable graphs.

The proof of Theorem 8 follows the same approach as the proof of Theorem 7. Although the orientation in level 1 differs slightly, it does not affect the validity of the proof. A semi-transitive orientation of the edges in an arbitrary graph $G \in G_n^8$ is shown in Figure 14.

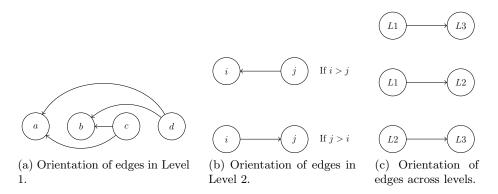


Fig. 14: A semi-transitive orientation of edges of $G \in G_n^8$.

Remark 1. The graphs identified as semi-transitive from the set of minimal comparability graphs depicted in Figure 9 are presented along with their corresponding semi-transitive orientations in Figure 15. Since these graphs are relatively small and can be individually verified as semi-transitive through straightforward inspection, we omit formal proofs.

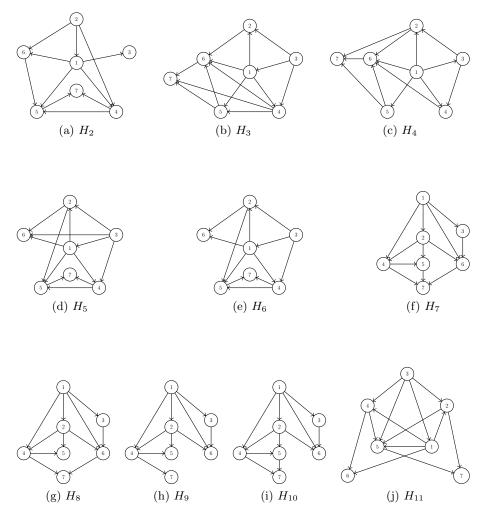


Fig. 15: Semi-transitive orientations of graphs in Figure 9 (H_1 is not semi-transitive)

Theorem 9 provides a characterization of the class of minimal non-word-representable graphs that contain an all-adjacent vertex.

Theorem 9. Let G be a graph with n vertices, where a vertex $x \in V(G)$ has degree (n-1), meaning x is adjacent to all other vertices in G. Let H be the induced subgraph of G with $V(H) = V(G) \setminus \{x\}$. Then, G is minimal non-word-representable if and only if both of the following conditions hold:

- 1. H is minimal non-comparability.
- 2. H is semi-transitive.

Proof. Suppose H is a minimal non-comparability graph and is also semi-transitive. Since H is non-comparability, it follows from Lemma 3 that G is a non-word-representable graph. As H is minimal non-comparability, any proper induced subgraph H_1 of H is a comparability graph. Consider an induced subgraph G' of G where $V(G') = V(H_1) \cup \{x\}$. Since H_1 is a comparability graph, Lemma 3 implies that G' is word-representable. Consequently, every proper induced subgraph of G containing the vertex x is word-representable. Furthermore, since H is semi-transitive, it follows from Theorem 1 that H is word-representable. As word-representable graphs are hereditary, every proper induced subgraph of G is word-representable, confirming that G is a minimal non-word-representable graph.

Lemma 3. [6] Let G be a graph with n vertices, where a vertex $x \in V(G)$ has degree (n-1). Let H be the induced subgraph of G with $V(H) = V(G) \setminus \{x\}$. Then, G is word-representable if and only if H is a comparability graph.

Suppose G is a minimal non-word-representable graph with an all-adjacent vertex $x \in V(G)$. Since G is minimal, every proper induced subgraph of G is word-representable. By Theorem 1, it follows that H is semi-transitive. Now, assume H is a comparability graph. By Lemma 3, G must then be word-representable, contradicting the assumption that G is non-word-representable. Hence, H is a non-comparability graph. Next, consider the case where H fails to be a minimal non-comparability graph. Then, there exists a proper induced subgraph H_2 of H that is also non-comparability. The induced subgraph of G formed by the vertex set $V(H_2) \cup \{x\} \subset V(G)$ would then be non-word-representable by Lemma 3, contradicting the minimality of G. Thus, H must be a minimal non-comparability graph.

We have identified all the minimal non-comparability graphs that are semi-transitive. Therefore, by applying Theorem 9, if we add an all-adjacent vertex to each of these minimal non-comparability graphs that are semi-transitive, we obtain all the minimal non-word-representable graphs that contain an all-adjacent vertex.

In the following section, we identify the minimal non-comparability graphs that are not semi-transitive. They are the graphs which falls in the intersection of minimal non-comparability graphs and minimal non-word-representable graphs.

4 Minimal non-comparability graphs that are not semi-transitive

In this section, we determine which graphs in the list of minimal non-comparability graphs are not semi-transitive.

Theorem 10. The graph class G_n^9 , depicted in Figure 8c, forms an infinite subclass of minimal non-word-representable graphs.

Proof. Let $G \in G_n^9$ be an arbitrary graph. If G is word-representable, then by Theorem 1 and Theorem 2, there exists a semi-transitive orientation of G' where vertex d is the source (i.e., all edges incident to d are oriented away from it). Consider such an orientation where d is the source. By symmetry between vertices a and b, we may assume without loss of generality that the edge between them is oriented as $a \to b$.

Note 5. The orientations $a \to c$ and $c \to b$ are not possible, as the path $d \to a \to c \to b$ violates semi-transitivity.

Note 6. Similarly, $b \to c$ and $c \to a$ are not possible, as the path $d \to b \to c \to a$ violates semi-transitivity.

Thus, the only remaining valid orientations are $a \to c, \ b \to c$ and $c \to a, c \to b.$

Case 1: $a \to c$, $b \to c$ The edge between 1 and b must be oriented as $1 \to b$. If $b \to 1$, then the path $d \to a \to b \to 1$ violates semi-transitivity.

Observation 3 In a semi-transitive orientation of G, where d is the source and the edges among a, c, and b are oriented as $a \to c$, $b \to c$, and $a \to b$, the edges from any vertex i, $2 \le i \le (n-1)$, to a and b must be oriented as $i \to a$ and $i \to b$.

Proof. The following cases can be ruled out:

- 1. $a \to i$ and $i \to b$, since the path $a \to i \to b \to c$ violates semi-transitivity.
- 2. $b \to i$ and $i \to a$, since the path $b \to i \to a \to c$ violates semi-transitivity.

Thus, the remaining possibilities are:

- 1. $a \to i$ and $b \to i$.
- 2. $i \to a$ and $i \to b$.

For $i=2,\ b\to 2$ is not valid, as the path $d\to 1\to b\to 2$ violates semitransitivity. Hence, the valid orientation for i=2 is $i\to a$ and $i\to b$. Now, assume that at least one vertex i has the orientation $a\to i$ and $b\to i$. Let i be the smallest such vertex. Then, the path $d\to (i-1)\to b\to i$ violates semi-transitivity, leading to a contradiction. Thus, the only valid orientation is $i\to a$ and $i\to b$.

Note 7. The edge between a and n cannot be oriented as $a \to n$ since the path $d \to (n-1) \to a \to n$ violates semi-transitivity.

Note 8. The edge between a and n cannot be oriented as $n \to a$ since the path $d \to n \to a \to b$ violates semi-transitivity.

From Notes 7 and 8, we reach a contradiction. Hence, no semi-transitive orientation exists for G. Since G is a minimal non-comparability graph, every proper induced subgraph of G is word-representable, proving its minimality.

Case 2: $c \to a$, $c \to b$ The edge between 1 and b must be oriented as $1 \to b$. If $b \to 1$, the path $d \to a \to b \to 1$ violates semi-transitivity.

Note 9. In a semi-transitive orientation of G, where d is the source and the edges among a, c, and b are oriented as $c \to a$, $c \to b$, and $a \to b$, the edges from any vertex $i, 2 \le i \le (n-1)$, to a and b must be oriented as $i \to a$ and $i \to b$.

Proof. The following cases can be ruled out:

- 1. $a \to i$ and $i \to b$, since the path $c \to a \to i \to b$ violates semi-transitivity.
- 2. $b \to i$ and $i \to a$, since the path $c \to b \to i \to a$ violates semi-transitivity.

Thus, the remaining possibilities are:

- 1. $a \to i$ and $b \to i$.
- 2. $i \to a$ and $i \to b$.

For $i=2,\ b\to 2$ is not valid, as the path $d\to 1\to b\to 2$ violates semi-transitivity. Hence, the valid orientation for i=2 is $i\to a$ and $i\to b$. Suppose at least one vertex i has the orientation $a\to i$ and $b\to i$. Let i be the smallest such vertex. Then, the path $d\to (i-1)\to b\to i$ violates semi-transitivity, leading to a contradiction. Thus, the only valid orientation is $i\to a$ and $i\to b$.

Note 10. The edge between a and n cannot be oriented as $a \to n$ since the path $d \to (n-1) \to a \to n$ violates semi-transitivity.

Note 11. The edge between a and n cannot be oriented as $n \to a$ since the path $d \to n \to a \to b$ violates semi-transitivity.

From Notes 10 and 11, we reach a contradiction. Hence, no semi-transitive orientation exists for G. Since G is a minimal non-comparability graph, every proper induced subgraph of G is word-representable, proving its minimality.

Theorem 11. The graph class G_n^4 , depicted in Figure 6d, forms an infinite subclass of minimal non-word-representable graphs.

Proof. Consider an arbitrary graph $G \in G_n^4$. Assume G is word-representable. By Theorem 1 and Theorem 2, it follows that there exists a semi-transitive orientation of G in which vertex 1 serves as the source, meaning that all edges incident to vertex 1 are oriented away from it. Let G' denote such a directed version of the graph G, where G' is semi-transitive and every edge incident to vertex 1 is directed away from it. Consider the induced subgraph G'', where $V(G'') = V(G') \setminus \{x,y\}$. If G'' is not semi-transitive, then it must follow that G' is also not semi-transitive, as the same violation of semi-transitivity would appear in G'. Therefore, G'' must be semi-transitive. Next, we examine the undirected version of G'', denoted as G''_U . Lemma 4 addresses the number of possible semi-transitive orientations of G''_U with vertex 1 as the source vertex.

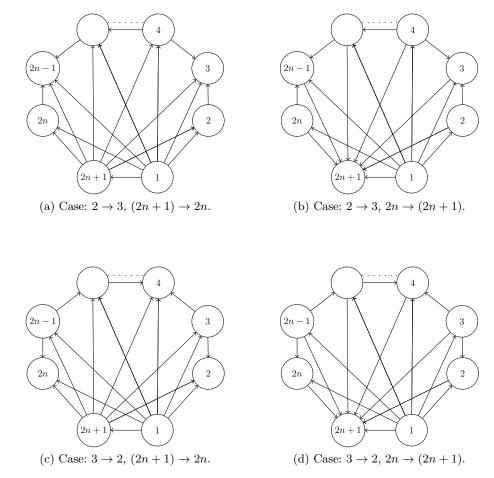


Fig. 16: Four semi-transitive orientations of G_U'' , with 1 as a source.

Observation 4 There are exactly four semi-transitive orientations of the graph G''_{IJ} , with vertex 1 as a source, as illustrated in Figure 16.

Lemma 5 and Lemma 7 gives the proof for Observation 4. The orientation of the edge $\{2,3\} \in E(G_U'')$ can either be $2 \to 3$ or $3 \to 2$. We will examine the possible semi-transitive orientations of G_U'' for both cases.

Case 1: $2 \rightarrow 3$

Lemma 4. If G''_U is oriented with vertex 1 as the source and has the orientation $2 \to 3$, then to achieve a semi-transitive orientation for G''_U , any even-numbered vertex i, where $4 \le i \le 2n-2$, must have the following orientation:

1.
$$i \to (i+1)$$

2. $i \to (i-1)$

Proof. Consider the case when i=4. If the orientation is $3\to 4$, the path $1\to 2\to 3\to 4$ violates semi-transitivity. Thus, the valid orientation is $4\to 3$. Similarly, if the orientation is $5\to 4$, the path $1\to 5\to 4\to 3$ violates semi-transitivity, so the correct orientation is $4\to 5$. Therefore, Lemma 4 holds when i=4. Now, assume that Lemma 4 holds for i=j, where j is even, j>4, and $j\le 2k-4$, i.e., the orientations $j\to (j-1)$ and $j\to (j+1)$ are valid. Consider the case when i=j+2. If the orientation is $(j+1)\to (j+2)$, then the path $1\to j\to (j+1)\to (j+2)$ violates semi-transitivity. Thus, the valid orientation is $(j+2)\to (j+1)$. On the other hand, if the orientation is $(j+3)\to (j+2)$, the path $1\to (j+3)\to (j+2)\to (j+1)$ violates semi-transitivity, so the correct orientation is $(j+2)\to (j+3)$.

Note 12. If $(2n-1) \to 2n$, then the path $1 \to (2n-2) \to (2n-1) \to 2n$ violates semi-transitivity. Thus, the valid orientation is $2n \to (2n-1)$.

Lemma 5. There are only two semi-transitive orientations possible under the case of orienting the edge $\{2,3\} \in E(G''_U)$ as $2 \to 3$, based on the following two conditions. For $i, 2 \le i \le 2n-1$,

```
1. If (2n+1) \to 2n, then (2n+1) \to i.
2. If 2n \to (2n+1), then i \to (2n+1).
```

They are shown in Figure 16a and Figure 16b.

Proof. Suppose we orient $(2n+1) \to 2n$. Consider the case when i=2. If $2 \to (2n+1)$, then the path $1 \to 2 \to (2n+1) \to 2n$ violates semi-transitivity. Therefore, the valid orientation is $(2n+1) \to 2$. Now, assume for i=j, the orientation is $(2n+1) \to j$. When i=j+1, if the orientation is $(j+1) \to (2n+1)$, then the path $1 \to (j+1) \to (2n+1) \to 2$ violates semi-transitivity. Hence, the valid orientation is $(2n+1) \to (j+1)$. Now, consider the orientation $2n \to (2n+1)$. Examining the case when i=2, if the edge is directed as

 $(2n+1) \to 2$, then the path $1 \to 2n \to (2n+1) \to 2$ violates the semi-transitivity condition. Consequently, the correct orientation must be $2 \to (2n+1)$. Suppose for i=j, the orientation is $j \to (2n+1)$. When i=j+1, if the orientation is $(2n+1) \to (j+1)$, the path $1 \to 2 \to (2n+1) \to (j+1)$ violates semi-transitivity. Hence, the correct orientation is $(2n+1) \to (j+1)$.

Case 2: $3 \rightarrow 2$

Lemma 6. If G''_U is oriented with vertex 1 as the source and has the orientation $3 \to 2$, then to achieve a semi-transitive orientation for G''_U , any even-numbered vertex i, where $4 \le i \le 2n - 2$, must have the following orientation:

1.
$$i \leftarrow (i+1)$$

2. $i \leftarrow (i-1)$

Proof. Consider the case when i=4. If the orientation is $4\to 3$, the path $1\to 4\to 3\to 2$ violates semi-transitivity. Hence, the valid orientation is $3\to 4$. Similarly, if the orientation is $4\to 5$, the path $1\to 3\to 4\to 5$ violates semi-transitivity, so the correct orientation is $5\to 4$. Thus, Lemma 6 holds when i=4. Now, assume that Lemma 6 holds for i=j, where j is even, j>4, and $j\le 2n-4$, i.e., the orientations $j\leftarrow (j-1)$ and $j\leftarrow (j+1)$ are valid. Consider the case when i=j+2. If the orientation is $(j+1)\leftarrow (j+2)$, the path $1\to (j+2)\to (j+1)\to j$ violates semi-transitivity. Therefore, the valid orientation is $(j+2)\leftarrow (j+1)$. If the orientation is $(j+3)\leftarrow (j+2)$, the path $1\to (j+1)\to (j+2)\to (j+3)$ violates semi-transitivity, so the correct orientation is $(j+2)\leftarrow (j+3)$.

Note 13. If $(2n-1) \leftarrow 2n$, the path $1 \rightarrow 2n \rightarrow (2n-1) \rightarrow (2n-2)$ violates semi-transitivity. Thus, the valid orientation is $2n \leftarrow (2n-1)$.

Lemma 7. There are only two semi-transitive orientations possible under the case of orienting the edge $\{2,3\} \in E(G''_U)$ as $3 \to 2$, based on the following two conditions. For $i, 2 \le i \le 2n-1$,

```
1. If (2n+1) \to 2n, then (2n+1) \to i.
2. If 2n \to (2n+1), then i \to (2n+1).
```

They are shown in Figure 16c and Figure 16d.

Proof. Suppose we orient $(2n+1) \to 2n$ in G_P'' . Consider the case when i=2. If $2 \to (2n+1)$, the path $1 \to 2 \to (2n+1) \to 2n$ violates semi-transitivity. Thus, the valid orientation is $(2n+1) \to 2$. Suppose for i=j, the orientation is $(2n+1) \to j$. When i=j+1, if the orientation is $(j+1) \to (2n+1)$, then the path $1 \to (j+1) \to (2n+1) \to 2$ violates semi-transitivity, and hence the valid orientation is $(2n+1) \to (j+1)$. Suppose we orient $2n \to (2n+1) \to 2$. Consider the case when i=2. If $(2n+1) \to 2$, the path $1 \to 2n \to (2n+1) \to 2$

violates semi-transitivity. Thus, the valid orientation is $2 \to (2n+1)$. Suppose for i = j, the orientation is $j \to (2n+1)$. When i = j+1, if the orientation is $(2n+1) \to (j+1)$, the path $1 \to 2 \to (2n+1) \to (j+1)$ violates semi-transitivity. Hence, the correct orientation is $(2n+1) \to (j+1)$.

The graph G' is an extension of the four types of graphs depicted in Figure 16. To construct G', two additional vertices, x and y, are introduced, along with the edges $\{1, x\}$, $\{2n, x\}$, $\{2, y\}$, and $\{2n + 1, y\}$. The orientation of the edge $\{1, x\}$ is predetermined as $1 \to x$. We demonstrate that none of these extensions preserve the semi-transitivity of G', thereby leading to a contradiction.

- 1. Extending the graph in Figure 16a. The edge $\{2n,x\}$, cannot be oriented as $2n \to x$, as the path $1 \to (2n+1) \to 2n \to x$ violates semi-transitivity. The edge $\{2n,x\}$, cannot be oriented as $x \to 2n$, as the path $1 \to x \to 2n \to (2n-1)$ violates semi-transitivity. Hence, we do not get a semi-transitive G'. The orientation of the edge $\{1,x\}$ is known, which is $1 \to x$.
- 2. Extending the graph in Figure 16b. The orientations of the edges $\{2, y\}$ and $\{2n+1, y\}$ can be of four types, and none of them maintains semi-transitivity.
 - (a) If the orientations were $2 \to y$, and $y \to (2n+1)$, the path $1 \to 2 \to y \to (2n+1)$ violates semi-transitivity.
 - (b) If the orientations were $2 \to y$, and $(2n+1) \to y$, the path $2 \to 3 \to (2n+1) \to y$ violates semi-transitivity.
 - (c) If the orientations were $y \to 2$, and $y \to (2n+1)$, the path $y \to 2 \to 3 \to (2n+1)$ violates semi-transitivity.
 - (d) If the orientations were $y \to 2$, and $(2n+1) \to y$, the path $1 \to (2n+1) \to y \to 2$ violates semi-transitivity.
- 3. Extending the graph in Figure 16c.

The orientations of the edges $\{2, y\}$ and $\{2n+1, y\}$ can be of four types, and none of them maintains semi-transitivity.

- (a) If the orientations were $2 \to y$, and $y \to (2n+1)$, the path $1 \to 2 \to y \to (2n+1)$ violates semi-transitivity.
- (b) If the orientations were $2 \to y$, and $(2n+1) \to y$, the path $(2n+1) \to 3 \to 2 \to y$ violates semi-transitivity.
- (c) If the orientations were $y \to 2$, and $y \to (2n+1)$, the path $y \to (2n+1) \to 3 \to 2$ violates semi-transitivity.
- (d) If the orientations were $y \to 2$, and $(2n+1) \to y$, the path $1 \to (2n+1) \to y \to 2$ violates semi-transitivity.
- 4. Extending the graph in Figure 16d.

The edge $\{2n, x\}$, cannot be oriented as $2n \to x$, as the path $1 \to (2n-1) \to 2n \to x$ violates semi-transitivity. The edge $\{2n, x\}$, cannot be oriented as $x \to 2n$, as the path $1 \to x \to 2n \to (2n+1)$ violates semi-transitivity. Hence, we do not get a semi-transitive G'.

Remark 2. The graph H_1 , depicted in Figure 9a, is a minimal non-word-representable graph. The proof of this result has already been provided in [8], and therefore, we do not include it here.

5 Concluding remarks

We have determined the set of all graphs that lie in the intersection of minimal non-comparability graphs and minimal non-word-representable graphs. We identify that this intersection consists of two infinite families of graphs, namely G_n^9 and G_n^4 , as well as the graph H_1 , which are illustrated in Figure 8c, Figure 6d, and Figure 9a, respectively. Furthermore, we classify all minimal non-comparability graphs into two categories: those that are semi-transitive and those that are not. Building on this classification, we identify and characterize the set of all minimal non-word-representable graphs containing an all-adjacent vertex. As a byproduct of our study, we discover several minimal non-word-representable graphs.

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