IDEAL MHD. PART II: RIGIDITY FROM INFINITY FOR IDEAL ALFVÉN WAVES IN 3D THIN DOMAINS

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ABSTRACT. This paper concerns the rigidity from infinity for Alfvén waves governed by ideal incompressible magnetohydrodynamic equations subjected to strong background magnetic fields along the x_1 -axis in 3D thin domains $\Omega_{\delta} = \mathbb{R}^2 \times (-\delta, \delta)$ with $\delta \in (0, 1]$ and slip boundary conditions. We show that in any thin domain Ω_{δ} , Alfvén waves must vanish identically if their scattering fields vanish at infinities. As an application, the rigidity of Alfvén waves in Ω_{δ} , propagating along the horizontal direction, can be approximated by the rigidity of Alfvén waves in \mathbb{R}^2 when δ is sufficiently small. Our proof relies on the uniform (with respect to δ) weighted energy estimates with a position parameter in weights to track the center of Alfvén waves. The key issues in the analysis include dealing with the nonlinear nature of Alfvén waves and the geometry of thin domains.

Running title: Rigidity of MHD in thin domain

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1. Introduction

Magnetohydrodynamics (MHD for short) is the study of interactions between magnetic fields and electrically conducting fluids. It is known that, in a perfectly conducting fluid, magnetic field lines tend to be frozen into the fluid in the sense that they move with the fluid. On this basis, one may assume that the fluid flows along a strong constant background magnetic field B_0 and it is perturbed by a small velocity field v perpendicular to B_0 . There subsequently exist two kinds of restoring forces in MHD leading to different waves, namely, the magnetic pressure force parallel to magnetic field lines produces magnetosonic waves propagating orthogonal to B_0 , whereas the magnetic tension force perpendicular to magnetic field lines generates a completely new type of waves called Alfvén waves propagating along B_0 . The phenomenon of Alfvén waves is significant in wide applications such as solar wind, dynamo theory, and magnetic reconnection, and it particularly underpins the existing explanations for the origin of the Earth's magnetic field and of the solar field. Readers are referred to the textbook [7] for more details and related topics.

Historically, the MHD theory was pioneered in 1942 [2] by the Swedish plasma physicist Hannes Alfvén, who was awarded the Nobel prize in 1970 for his outstanding contributions to this field, and the Alfvén waves were named after him. An interesting case, together with its linearized analysis considered therein, can be revisited as follows: If the electric conductivity is set to be infinite, the fluid density and permeability to be 1, and B_0 to be homogenous and parallel to the x_1 -axis, then by elementary calculation the 3D MHD

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equations become a 1D wave equation for the magnetic field b as $\frac{\partial^2 b}{\partial x_1^2} = \frac{4\pi}{B_0^2} \frac{\partial^2 b}{\partial t^2}$, which implies that the Alfvén waves move along B_0 in both directions with the velocity $V_A = \frac{B_0}{\sqrt{4\pi}}$. Since then, the MHD equations have been considered extensively by mathematicians and physicists working in this field, especially when studying the dynamics of Alfvén waves.

It behooves us to first review some significant progress made in studying the global existence of Alfvén waves governed by incompressible MHD systems in strong magnetic field backgrounds with small perturbations. In [3], Bardos, Sulem, and Sulem investigated the ideal case via convolutions with fundamental solutions, which gave rise to global existence results in Hölder spaces. In [18,27], Lin, Xu, and Zhang obtained global existence results in energy spaces for the viscous case with strong fluid viscosity but no Ohmic dissipation. However, the latter works required the smallness of data to rely on viscosity so that the Fourier methods effective in studying Navier–Stokes equations can be adapted. Later on, He, Xu, and Yu [13] contributed to the ideal case as well as the viscous case with small diffusion. In their work, with inspiration from the stability of Minkowski spacetime [6], the global nonlinear stability of Alfvén waves was proved by more natural energy methods, and moreover the smallness of data was independent of viscosity. Alternative proofs can also be found in the work of Cai and Lei [4] as well as the works of Wei and Zhang [24,25].

In recent years, the energy methods in [13] have been extended in the following two aspects. One is a crosswise extension [26], where an important and interesting type of 3D thin domains $\Omega_{\delta} = \mathbb{R}^2 \times (-\delta, \delta)$ with a thickness parameter δ (the thickness of domain as 2δ) that is sufficiently small has been considered. In [26], Xu derived global existence and uniform (with respect to δ) energy estimates of Alfvén waves in Ω_{δ} . It was then proved that 3D Alfvén waves in Ω_{δ} propagating along the horizontal direction can be approximated by 2D Alfvén waves in \mathbb{R}^2 as δ goes to zero. The other aspect in which the aforementioned energy methods have been extended is a lengthwise extension [17], where the scattering behavior of 3D Alfvén waves at infinities has been further studied. Specifically, by choosing suitable position parameters in weighted energy estimates, Li and Yu constructed the rigidity from infinity theorems for Alfvén waves in \mathbb{R}^3 (which indeed hold in \mathbb{R}^2 as well, see the appendix in this paper) that Alfvén waves must vanish identically if their scattering fields vanish at infinities. This conclusion is consistent with the physical phenomenon that there are no Alfvén waves emanating from the plasma if no waves are detected by faraway observers. Based on these results, it appears promising that the rigidity from infinity for 3D Alfvén waves in thin domains Ω_{δ} can also be constructed. In fact, to demonstrate this rigidity will be the main purpose of the present paper.

It is meaningful to point out that thin-domain problems are encountered in mathematical models from many applications. For example, in solid mechanics and especially in elasticity, one considers thin rods, beams, plates, and shells; in fluid dynamics, one is concerned with fluid lubrication, blood circulation, ocean dynamics, and meteorology problems; and in MHD, one deals with solar tachocline, wave heating in the solar corona, thin airfoils in MHD, and shallow water MHD. Other applied areas include physiology, nanotechnology, material engineering, etc. Readers can also consult [10, 22] about more details regarding physical backgrounds. Most of the above problems are described by various partial differential equations on thin domains, their solutions are always compared with corresponding ones on domains of lower dimension with the thin directions reduced, and a fairly satisfactory understanding has been achieved for what impact the thickness of thin domains has on solutions. From pioneering works to recent ones, we list some representative contributions to the global existence of solutions in thin domains as follows: See the works of Hale and Raugel [11, 12] for reaction-diffusion equations and damped hyperbolic equations; the work of Raugel and Sell [23] for Navier-Stokes equations; the work of Marsden, Ratiu, and Raugel [20] for Euler equations; as well as the aforementioned work [26] for MHD equations. The works [5,8,14,21] and their related references, in which more equations are considered or further results obtained, are also relevant.

We now turn to review the wave scattering theory and especially its rigidity from infinity aspect. While we do not give a complete history of this classical topic here, special mention should be made of the simplest instance therein, i.e., the smooth solution ϕ to the 3D linear wave equation $\Box \phi = 0$. In the situation where $\phi = O(\frac{1}{t})$, if the initial data are in a sufficiently mathematically elegant space [all we require is that for any radius r and any center y, the integral $\int_{B_r(y)} (|\phi(0,x)| + \partial \phi(0,x)|) d\sigma(x)$, with σ the induced surface measure, is uniformly bounded], then the scattering field is exactly the Radon transform of the initial data, and hence the uniqueness of the Radon transform yields the rigidity from infinity that the solution must vanish identically if its scattering field vanishes at infinities. This result can be found in [16] and other

relevant works, such as [9,19]. We point out that this kind of rigidity is also called unique continuation. Readers are referred to the survey paper of Ionescu and Klainerman [15] for more details on an essential role of unique continuation in studying the uniqueness of black holes. Such an idea was then applied to the above example of linear waves in more general forms by Alexakis and Shao, see [1] for instance. Regarding the method, these unique continuation results were mainly obtained by making use of the Carleman estimates.

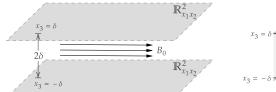
Our construction of scattering fields for 3D Alfvén waves in Ω_{δ} (see Theorem 5.1) is similar to the corresponding one for the above 3D linear waves, and our main rigidity from infinity result for 3D Alfvén waves in Ω_{δ} (see Theorem 6.1) is indeed an analogue of these unique continuation results in spirit. These results are consistent with the physical intuition that the 3D Alfvén waves produced from the plasma in thin domains are characterized by their scattering fields detected by faraway observers, and accordingly there are no Alfvén waves in thin domains if there are no waves detected by faraway observers. However, both the nonlinear nature of Alfvén waves and the geometry of thin domains distinguish our problem from the aforementioned situations. Instead of applying Carleman-type estimates, we will investigate the uniform (with respect to δ) weighted energy estimates with a position parameter in weights to track the center of Alfvén waves. Although difficulties in the nonlinear setting still exist, the energy method will be more natural and effective for Alfvén waves. We will also see that the rigidity of 3D Alfvén waves in Ω_{δ} converges to the rigidity of 2D Alfvén waves in \mathbb{R}^2 in the limit that the thickness parameter δ goes to zero (see Corollary 7.1). This is consistent with the geometric fact that Ω_{δ} can be viewed as \mathbb{R}^2 when $\delta \to 0$ as well as the approximation result in [26] that 3D Alfvén waves in Ω_{δ} converge to 2D Alfvén waves in \mathbb{R}^2 when $\delta \to 0$. Moreover, the rigidity of 2D Alfvén waves in \mathbb{R}^2 deduced from the approximation in Corollary 7.1 coincides with the 2D version (see Theorem A.3) of the rigidity from infinity for Alfvén waves established in [17]. For clarity of presentation, we will put together these relations in Figure 4.

Structure of the paper. In Section 2, we introduce some preliminary notations for MHD equations in 3D thin domains Ω_{δ} , adapt the uniform weighted energy estimates for the study of rigidity and revisit the necessary proof with main steps. This new version of uniform weighted energy estimates with a position parameter in weights also allows us to construct the global solution (Alfvén waves) in Ω_{δ} . We devote Section 3 to the estimates on the nonlinear term and Section 4 to the estimates on the pressure term. All these estimates are the main ingredients in studying the dynamical behavior of Alfvén waves in Ω_{δ} . Based on these estimates, Section 5 then involves constructing scattering fields of Alfvén waves in Ω_{δ} together with their weighted Sobolev spaces at infinities, and describing the large time behavior of Alfvén waves in Ω_{δ} with vanishing scattering fields by smallness conditions of energies; see our first main Theorem 5.1 for details. In Section 6, we prove our second main Theorem 6.1 which concerns the rigidity from infinity for Alfvén waves in Ω_{δ} . In Section 7, the convergence between the rigidity of Alfvén waves in Ω_{δ} propagating along the horizontal direction and the rigidity of Alfvén waves in \mathbb{R}^2 is provided in Corollary 7.1 as an immediate consequence because of the rigidity condition that the scattering fields are vanishing at infinities. Finally, in the appendix, we provide Theorem A.3 concerning the rigidity for Alfvén waves in \mathbb{R}^2 as an extension of [17] concerning the rigidity for Alfvén waves in \mathbb{R}^3 . We remark that though derivations are different, both Corollary 7.1 (as δ goes to zero) and Theorem A.3 demonstrate the the rigidity for Alfvén waves in \mathbb{R}^2 , and in particular these two perspectives are complementary with each other and coexist in a harmonious way.

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2. Notations and energy estimates

Throughout this paper, we consider 3D thin domains $\Omega_{\delta} = \mathbb{R}^2 \times (-\delta, \delta)$ with $\delta \in (0, 1]$. In Ω_{δ} , it is clear that the horizontal variables are x_1 and x_2 while the vertical variable is x_3 and the thin direction is the vertical direction. We are interested in the most interesting situation where a strong background magnetic field B_0 presents along the horizontal direction and therefore a small initial perturbation will generate Alfvén waves which propagate along B_0 . Without loss of generality, we assume B_0 to be a constant magnetic field parallel to the x_1 -axis: $B_0 = (1,0,0)$. The geometry of this part is illustrated in the following Figure 1 with a stereo view on the left hand side and a sectional view on the right hand side.



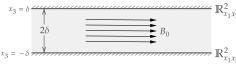


FIGURE 1. $\Omega_{\delta} = \mathbb{R}^2 \times (-\delta, \delta)$

Let us consider the ideal incompressible MHD equations in Ω_{δ} with slip boundary conditions:

$$\begin{cases}
\partial_t v + v \cdot \nabla v = -\nabla p + (\nabla \times b) \times b, \\
\partial_t b + v \cdot \nabla b = b \cdot \nabla v, \\
\operatorname{div} v = 0, & \operatorname{div} b = 0, \\
v^3|_{x_3 = \pm \delta} = 0, & b^3|_{x_3 = \pm \delta} = 0,
\end{cases} \tag{2.1}$$

where $b = (b^1, b^2, b^3)$ is the magnetic field, $v = (v^1, v^2, v^3)$ the velocity of fluid, and p the scalar pressure of fluid. By writing $(\nabla \times b) \times b = -\nabla(\frac{1}{2}|b|^2) + b \cdot \nabla b$ and $p' = p + \frac{1}{2}|b|^2$, the first equation in (2.1) can be rephrased as

$$\partial_t v + v \cdot \nabla v = -\nabla p' + b \cdot \nabla b.$$

For the sake of simplicity, we will still use p to denote p' in what follows.

We now employ the Elsässer variables $Z_{+} = v + b$ and $Z_{-} = v - b$ to diagonalize the system (2.1) as

$$\begin{cases}
\partial_t Z_+ + Z_- \cdot \nabla Z_+ = -\nabla p, \\
\partial_t Z_- + Z_+ \cdot \nabla Z_- = -\nabla p, \\
\operatorname{div} Z_+ = 0, & \operatorname{div} Z_- = 0, \\
Z_+^3|_{x_3 = \pm \delta} = 0, & Z_-^3|_{x_3 = \pm \delta} = 0.
\end{cases} (2.2)$$

The fluctuations $z_+ = Z_+ - B_0$ and $z_- = Z_- - (-B_0) = Z_- + B_0$ then verify

$$\begin{cases}
\partial_t z_+ + (z_- - B_0) \cdot \nabla z_+ = -\nabla p, \\
\partial_t z_- + (z_+ + B_0) \cdot \nabla z_- = -\nabla p, \\
\operatorname{div} z_+ = 0, & \operatorname{div} z_- = 0, \\
z_+^3 |_{x_3 = \pm \delta} = 0, & z_-^3 |_{x_3 = \pm \delta} = 0.
\end{cases} \tag{2.3}$$

When t=0, the initial data $(z_{+,0}(x), z_{-,0}(x))$ also satisfy the divergence conditions div $z_{+,0}=0$, div $z_{-,0}=0$ and the boundary conditions $z_{+,0}^3|_{x_3=\pm\delta}=0$, $z_{-,0}^3|_{x_3=\pm\delta}=0$.

To investigate the influence of the thickness of Ω_{δ} , we first perform the following rescalings for any $x \in \Omega_1$:

$$\begin{split} z_{(\delta)}^h(t,x_h,x_3) &:= z^h(t,x_h,\delta x_3), \\ z_{(\delta)}^3(t,x_h,x_3) &:= \delta^{-1}z^3(t,x_h,\delta x_3), \\ p_{(\delta)}(t,x_h,x_3) &:= p(t,x_h,\delta x_3). \end{split}$$

Hereinafter, we always denote $z=(z^h,z^3)$, $z^h=(z^1,z^2)$ and $x_h=(x_1,x_2)$. Moreover, all above notations for z adapt not only to z_\pm and $z_{\pm(\delta)}$ here but also to $z_{\pm,0}$ and $z_{\pm(\delta),0}$ in the sequel. Then the MHD system (2.3) in Ω_δ for z_\pm can be written as the following rescaled system in Ω_1 for $z_{\pm(\delta)}$:

$$\begin{cases}
\partial_{t} z_{+(\delta)} + (z_{-(\delta)} - B_{0}) \cdot \nabla z_{+(\delta)} = -\nabla_{\delta} p_{(\delta)}, \\
\partial_{t} z_{-(\delta)} + (z_{+(\delta)} + B_{0}) \cdot \nabla z_{-(\delta)} = -\nabla_{\delta} p_{(\delta)}, \\
\operatorname{div} z_{+(\delta)} = 0, \quad \operatorname{div} z_{-(\delta)} = 0, \\
z_{+(\delta)}^{3}|_{x_{3} = \pm 1} = 0, \quad z_{-(\delta)}^{3}|_{x_{3} = \pm 1} = 0,
\end{cases} (2.4)$$

where $\nabla_{\delta} = (\partial_1, \partial_2, \delta^{-2}\partial_3)$, and the initial data $(z_{+(\delta),0}(x), z_{-(\delta),0}(x))$ also satisfy the divergence conditions $\operatorname{div} z_{+(\delta),0} = 0$, $\operatorname{div} z_{-(\delta),0} = 0$ and the boundary conditions $z_{+(\delta),0}^3|_{x_3=\pm 1} = 0$, $z_{-(\delta),0}^3|_{x_3=\pm 1} = 0$. In

particular, the anisotropic property of the rescaled system (2.4) will help us deal with the thickness of Ω_{δ} in the rest of this paper.

We turn to review the following Sobolev lemma for functions defined on thin domains Ω_{δ} . The interested readers can consult Lemma 2.6 (ii) in [26] about its proof.

Lemma 2.1 (Sobolev lemma). For any $f(x) \in H^2(\Omega_{\delta})$, we have

$$||f||_{L_x^{\infty}} \leqslant C \sum_{k+l \leqslant 2} \delta^{l-\frac{1}{2}} ||\nabla_h^k \partial_3^l f||_{L_x^2}.$$

Let us introduce characteristic functions as

$$u_{\pm} = u_{\pm}(t, x_1) = x_1 \mp t, \tag{2.5}$$

and weight functions as

$$\langle u_{\pm} \rangle = (1 + |u_{\pm} \mp a|^2)^{\frac{1}{2}} = (1 + |x_1 \mp (t+a)|^2)^{\frac{1}{2}},$$
 (2.6)

where a represents the position parameter tracking the centers of Alfvén waves. Some elementary properties of $\langle u_{\pm} \rangle$ can be collected as the following lemma. The proof is by direct calculation and so is omitted here.

Lemma 2.2 (Properties of weights). For any $\sigma \in (0, \frac{1}{3})$, there hold:

(i) For $|x_h - y_h| \leq 2$, we have

$$\left(\langle u_{\mp}\rangle^{1+\sigma}\langle u_{\pm}\rangle^{\frac{1}{2}(1+\sigma)}\right)(t,x) \lesssim \left(\langle u_{\mp}\rangle^{1+\sigma}\langle u_{\pm}\rangle^{\frac{1}{2}(1+\sigma)}\right)(t,y). \tag{2.7}$$

(ii) For $|x_h - y_h| \ge 1$, we have

$$\left(\langle u_{\mp}\rangle^{1+\sigma}\langle u_{\pm}\rangle^{\frac{1}{2}(1+\sigma)}\right)(t,x) \lesssim |x_h - y_h|^{\frac{3}{2}(1+\sigma)}\left(\langle u_{\mp}\rangle^{1+\sigma}\langle u_{\pm}\rangle^{\frac{1}{2}(1+\sigma)}\right)(t,y). \tag{2.8}$$

(iii) For all $k \in \mathbb{Z}_{\geq 0}$ with $k \leq 3$, we have

$$\left| \nabla^k \left(\langle u_{\mp} \rangle^{1+\sigma} \langle u_{\pm} \rangle^{\frac{1}{2}(1+\sigma)} \right) \right| \lesssim \langle u_{\mp} \rangle^{1+\sigma} \langle u_{\pm} \rangle^{\frac{1}{2}(1+\sigma)}. \tag{2.9}$$

(iv) For all $k, l \in \mathbb{Z}_{\geq 0}$ with $k + l \leq 2$, we have

$$\left| \nabla_h^k \partial_3^l \left(\frac{\langle u_{\pm} \rangle^{1+\sigma}}{\langle u_{\mp} \rangle^{\frac{1}{2}(1+\sigma)}} \right) \right| \lesssim \frac{\langle u_{\pm} \rangle^{1+\sigma}}{\langle u_{\mp} \rangle^{\frac{1}{2}(1+\sigma)}}. \tag{2.10}$$

(v) For the product of $\langle u_+ \rangle$ and $\langle u_- \rangle$, we have

$$\langle u_+ \rangle \langle u_- \rangle \gtrsim 1 + |t + a|.$$
 (2.11)

Here, the notation $A \lesssim B$ means that there is a universal constant C (independent of a) such that $A \leqslant CB$; the notation $A \gtrsim B$ means that there is a universal constant C (independent of a) such that $A \geqslant CB$.

Now we state the following weighted div-curl lemma. This lemma will help us control the gradient of vectors by their divergence and curl. In particular, the readers are referred to Lemma 2.4 in [26] for a divergence-free version of this lemma.

Lemma 2.3 (Weighted div-curl lemma). Let $\lambda(x) \ge 1$ be a smooth positive function on Ω_{δ} with the additional property $|\nabla \lambda| \lesssim \lambda$. For any smooth vector field $v(x) \in H^1(\Omega_{\delta})$, we have

$$\left\|\sqrt{\lambda}\nabla v\right\|_{L^{2}(\Omega_{\delta})}^{2} \lesssim \left\|\sqrt{\lambda}\operatorname{div}\nabla v\right\|_{L^{2}(\Omega_{\delta})}^{2} + \left\|\sqrt{\lambda}\operatorname{curl}\nabla v\right\|_{L^{2}(\Omega_{\delta})}^{2} + \left\|\sqrt{\lambda}v\right\|_{L^{2}(\Omega_{\delta})}^{2} + \left|\int_{\partial\Omega_{\delta}}\lambda(v^{h}\cdot\nabla_{h}v^{3} - v^{3}\nabla_{h}v^{h})dx_{h}\right|,\tag{2.12}$$

provided $\sqrt{\lambda}v \in L^2(\Omega_\delta)$ and $\sqrt{\lambda}\nabla v \in L^2(\Omega_\delta)$.

Proof. We first recall the vector calculus identity

$$-\Delta v = -\nabla(\operatorname{div} v) + \operatorname{curl} \operatorname{curl} v.$$

Multiplying this identity by λv and then integrating over Ω_{δ} lead us to

$$-\int_{\Omega_{\delta}} \lambda v \cdot \Delta v dx = -\int_{\Omega_{\delta}} \lambda v \cdot \nabla(\operatorname{div} v) dx + \int_{\Omega_{\delta}} \lambda v \cdot \operatorname{curl} \operatorname{curl} v dx.$$

After integration by parts, we infer that

$$\begin{split} &-\int_{\Omega_{\delta}}\lambda v\cdot\Delta v dx = -\int_{\partial\Omega_{\delta}}\lambda v\cdot\nabla v\cdot n dS + \int_{\Omega_{\delta}}\lambda|\nabla v|^2 dx + \int_{\Omega_{\delta}}\nabla\lambda\cdot v\cdot\nabla v dx, \\ &-\int_{\Omega_{\delta}}\lambda v\cdot\nabla(\operatorname{div} v) dx = -\int_{\partial\Omega_{\delta}}\lambda v\cdot(\operatorname{div} v) n dS + \int_{\Omega_{\delta}}\lambda|\operatorname{div} v|^2 dx + \int_{\Omega_{\delta}}\nabla\lambda\cdot v\operatorname{div} v dx, \\ &\int_{\Omega_{\delta}}\lambda v\cdot\operatorname{curl}\operatorname{curl} v dx = -\int_{\partial\Omega_{\delta}}\lambda v\cdot(\operatorname{curl} v\times n) dS + \int_{\Omega_{\delta}}\lambda|\operatorname{curl} v|^2 dx + \int_{\Omega_{\delta}}(\nabla\lambda\wedge v)\cdot\operatorname{curl} v dx, \end{split}$$

where n is the unit outward normal of $\partial\Omega_{\delta}$ and dS is the surface measure of $\partial\Omega_{\delta}$. Therefore we derive

$$\int_{\Omega_{\delta}} \lambda |\nabla v|^{2} dx = \int_{\partial\Omega_{\delta}} \lambda v \cdot \left[\nabla v \cdot n - (\operatorname{div} v)n - (\operatorname{curl} v \times n)\right] dS + \int_{\Omega_{\delta}} \lambda |\operatorname{div} v|^{2} dx + \int_{\Omega_{\delta}} \lambda |\operatorname{curl} v|^{2} dx
- \int_{\Omega_{\delta}} \nabla \lambda \cdot v \cdot \nabla v dx + \int_{\Omega_{\delta}} \nabla \lambda \cdot v \operatorname{div} v dx + \int_{\Omega_{\delta}} (\nabla \lambda \wedge v) \cdot \operatorname{curl} v dx.$$
(2.13)

By direct calculation, we acquire

$$\int_{\partial\Omega_{\delta}} \lambda v \cdot \left[\nabla v \cdot n - (\operatorname{div} v) n - (\operatorname{curl} v \times n) \right] dS = \int_{\partial\Omega_{\delta}} \lambda v^{j} (\partial_{j} v^{i} n_{i} - \partial_{i} v^{i} n_{j}) dS. \tag{2.14}$$

In view of (2.13) and (2.14), we deduce that

$$\int_{\Omega_{\delta}} \lambda |\nabla v|^{2} dx \leq \left| \int_{\partial\Omega_{\delta}} \lambda v^{j} (\partial_{j} v^{i} n_{i} - \partial_{i} v^{i} n_{j}) dS \right| + \int_{\Omega_{\delta}} \lambda |\operatorname{div} v|^{2} dx + \int_{\Omega_{\delta}} \lambda |\operatorname{curl} v|^{2} dx + \int_{\Omega_{\delta}} |\nabla \lambda| |v| |\nabla v| dx + \int_{\Omega_{\delta}} |\nabla \lambda| |v| |\operatorname{div} v| dx + \int_{\Omega_{\delta}} |\nabla \lambda| |v| |\operatorname{curl} v| dx.$$

Noticing that $n\big|_{\partial\Omega_{\delta}} = n\big|_{x_3=+\delta} = (0,0,\pm 1)^T$ and $dS = dx_h$, we obtain

$$\Big| \int_{\partial \Omega_{\delta}} \lambda v^{j} (\partial_{j} v^{i} n_{i} - \partial_{i} v^{i} n_{j}) dS \Big| = \Big| \int_{\partial \Omega_{\delta}} \lambda (v \cdot \nabla v^{3} - v^{3} \operatorname{div} v) dx_{h} \Big| = \Big| \int_{\partial \Omega_{\delta}} \lambda (v^{h} \cdot \nabla_{h} v^{3} - v^{3} \nabla_{h} v^{h}) dx_{h} \Big|.$$

Then using Cauchy-Schwarz inequality gives rise to

$$\int_{\mathbb{R}^3} \lambda |\nabla v|^2 dx \leqslant \left| \int_{\partial \Omega_\delta} \lambda (v^h \cdot \nabla_h v^3 - v^3 \nabla_h v^h) dx_h \right| + \frac{3}{2} \int_{\mathbb{R}^3} \lambda |\operatorname{div} v|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} \lambda |\operatorname{curl} v|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} \lambda |\nabla v|^2 dx + \frac{3}{2} \int_{\mathbb{R}^3} \lambda |\nabla v|^2 dx.$$

Since $|\nabla \lambda| \lesssim \lambda$ gives $\frac{|\nabla \lambda|}{\sqrt{\lambda}} \lesssim \sqrt{\lambda}$, we finally summarize that

$$\left\|\sqrt{\lambda}\nabla v\right\|_{L^{2}(\mathbb{R}^{3})}^{2} \lesssim \left\|\sqrt{\lambda}\operatorname{div}v\right\|_{L^{2}(\mathbb{R}^{3})}^{2} + \left\|\sqrt{\lambda}\operatorname{curl}v\right\|_{L^{2}(\mathbb{R}^{3})}^{2} + \left\|\sqrt{\lambda}v\right\|_{L^{2}(\mathbb{R}^{3})}^{2} + \left\|\int_{\partial\Omega_{\varepsilon}}\lambda(v^{h}\cdot\nabla_{h}v^{3} - v^{3}\nabla_{h}v^{h})dx_{h}\right|.$$

This ends the proof of the lemma.

As a consequence of Lemma 2.3, there also holds:

Lemma 2.4 (Weighted div-curl lemma with higher order derivatives). Let $\lambda(x) \geq 1$ be a smooth positive function on Ω_{δ} with the additional property $|\nabla \lambda| \leq \lambda$. For any smooth vector field $v(x) \in H^m(\Omega_{\delta})$ $(m \in \mathbb{Z}_{\geq 1})$, we have

$$\|\sqrt{\lambda}\nabla^{k}v\|_{L^{2}(\Omega_{\delta})}^{2} \lesssim \sum_{l=0}^{k-1} \|\sqrt{\lambda}\operatorname{div}\nabla^{l}v\|_{L^{2}(\Omega_{\delta})}^{2} + \sum_{l=0}^{k-1} \|\sqrt{\lambda}\operatorname{curl}\nabla^{l}v\|_{L^{2}(\Omega_{\delta})}^{2} + \|\sqrt{\lambda}v\|_{L^{2}(\Omega_{\delta})}^{2} + \sum_{l=0}^{k-1} \left| \int_{\partial\Omega_{\delta}} \lambda(\nabla^{l}v^{h} \cdot \nabla^{l}\nabla_{h}v^{3} - \nabla^{l}v^{3} \cdot \nabla^{l}\nabla_{h}v^{h}) dx_{h} \right|, \tag{2.15}$$

provided $\sqrt{\lambda}v \in L^2(\Omega_\delta)$ and $\sqrt{\lambda}\nabla^k v \in L^2(\Omega_\delta)$, where $1 \leqslant k \leqslant m$.

Proof. For any $1 \leq k \leq m$, we have $\nabla^{k-1}v \in H^{m-k+1}(\Omega_{\delta}) \subset H^1(\Omega_{\delta})$. Thus, applying (2.12) to the vector field $\nabla^{k-1}v$ yields

$$\begin{split} \left\| \sqrt{\lambda} \nabla^k v \right\|_{L^2(\Omega_\delta)}^2 &\lesssim \left\| \sqrt{\lambda} \operatorname{div} \nabla^{k-1} v \right\|_{L^2(\Omega_\delta)}^2 + \left\| \sqrt{\lambda} \operatorname{curl} \nabla^{k-1} v \right\|_{L^2(\Omega_\delta)}^2 + \left\| \sqrt{\lambda} \nabla^{k-1} v \right\|_{L^2(\Omega_\delta)}^2 \\ &+ \left| \int_{\partial \Omega_\delta} \lambda (\nabla^{k-1} v^h \cdot \nabla^{k-1} \nabla_h v^3 - \nabla^{k-1} v^3 \cdot \nabla^{k-1} \nabla_h v^h) dx_h \right|. \end{split}$$

By induction on k, we can infer (2.15) immediately. Hence the lemma is proved.

Let t^* be the lifespan of solutions to the system (2.3), which indeed also adapts to the rescaled system (2.4). Here and subsequently, we follow the notations of energy and flux used in [26]. For all $k, l \in \mathbb{Z}_{\geq 0}$, we denote the weighted energy norms on $[0, t^*] \times \Omega_{\delta}$ as

$$E_{\pm}^{(k,l)}(z_{\pm}(t)) = \sum_{|\alpha_{h}|=k} \|\langle u_{\mp}\rangle^{1+\sigma} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} z_{\pm}(t,x)\|_{L^{2}(\Omega_{\delta})}^{2}, \quad E_{\pm}^{(k,l)}(z_{\pm}) = \sup_{0 \leqslant t \leqslant t^{*}} E_{\pm}^{(k,l)}(z_{\pm}(t)),$$

$$E_{\pm}^{(k,l)}(z_{\pm}^{h}(t)) = \sum_{|\alpha_{h}|=k} \|\langle u_{\mp}\rangle^{1+\sigma} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} z_{\pm}^{h}(t,x)\|_{L^{2}(\Omega_{\delta})}^{2}, \quad E_{\pm}^{(k,l)}(z_{\pm}^{h}) = \sup_{0 \leqslant t \leqslant t^{*}} E_{\pm}^{(k,l)}(z_{\pm}^{h}(t)),$$

$$E_{\pm}^{(k,l)}(z_{\pm}^{3}(t)) = \sum_{|\alpha_{h}|=k} \|\langle u_{\mp}\rangle^{1+\sigma} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} z_{\pm}^{3}(t,x)\|_{L^{2}(\Omega_{\delta})}^{2}, \quad E_{\pm}^{(k,l)}(z_{\pm}^{3}) = \sup_{0 \leqslant t \leqslant t^{*}} E_{\pm}^{(k,l)}(z_{\pm}^{3}(t)),$$

and the weighted flux norms on $[0, t^*] \times \Omega_{\delta}$ as

$$F_{\pm}^{(k,l)}(z_{\pm}(t)) = \sum_{|\alpha_{h}|=k} \int_{[0,t]\times\Omega_{\delta}} \frac{\langle u_{\mp}\rangle^{2(1+\sigma)}}{\langle u_{\pm}\rangle^{1+\sigma}} |\partial_{h}^{\alpha_{h}}\partial_{3}^{l}z_{\pm}(\tau,x)|^{2} dx d\tau, \quad F_{\pm}^{(k,l)}(z_{\pm}) = F_{\pm}^{(k,l)}(z_{\pm}(t^{*})),$$

$$F_{\pm}^{(k,l)}(z_{\pm}^{h}(t)) = \sum_{|\alpha_{h}|=k} \int_{[0,t]\times\Omega_{\delta}} \frac{\langle u_{\mp}\rangle^{2(1+\sigma)}}{\langle u_{\pm}\rangle^{1+\sigma}} |\partial_{h}^{\alpha_{h}}\partial_{3}^{l}z_{\pm}^{h}(\tau,x)|^{2} dx d\tau, \quad F_{\pm}^{(k,l)}(z_{\pm}^{h}) = F_{\pm}^{(k,l)}(z_{\pm}^{h}(t^{*})),$$

$$F_{\pm}^{(k,l)}(z_{\pm}^{3}(t)) = \sum_{|\alpha_{h}|=k} \int_{[0,t]\times\Omega_{\delta}} \frac{\langle u_{\mp}\rangle^{2(1+\sigma)}}{\langle u_{\pm}\rangle^{1+\sigma}} |\partial_{h}^{\alpha_{h}}\partial_{3}^{l}z_{\pm}^{3}(\tau,x)|^{2} dx d\tau, \quad F_{\pm}^{(k,l)}(z_{\pm}^{3}) = F_{\pm}^{(k,l)}(z_{\pm}^{3}(t^{*})).$$

For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ and $l \geq 0$, we see that

$$\begin{split} E_{\pm}^{(\alpha_h,l)}(z_{\pm(\delta)}^h) &= \delta^{2(l-\frac{1}{2})} E_{\pm}^{(\alpha_h,l)}(z_{\pm}^h), \quad E_{\pm}^{(\alpha_h,0)}(z_{\pm(\delta)}^3) = \delta^{-3} E_{\pm}^{(\alpha_h,0)}(z_{\pm}^3), \\ E_{\pm}^{(\alpha_h,l)}(z_{\pm(\delta)}^3) &= \delta^{2(l-\frac{3}{2})} E_{\pm}^{(\alpha_h,l)}(z_{\pm}^3) = \delta^{2(l-\frac{3}{2})} E_{\pm}^{(\alpha_h,l-1)}(\nabla_h \cdot z_{\pm}^h) \quad \text{for } l \geqslant 1, \\ F_{\pm}^{(\alpha_h,l)}(z_{\pm(\delta)}^h) &= \delta^{2(l-\frac{1}{2})} F_{\pm}^{(\alpha_h,l)}(z_{\pm}^h), \quad F_{\pm}^{(\alpha_h,0)}(z_{\pm(\delta)}^3) = \delta^{-3} F_{\pm}^{(\alpha_h,0)}(z_{\pm}^3), \\ F_{\pm}^{(\alpha_h,l)}(z_{\pm(\delta)}^3) &= \delta^{2(l-\frac{3}{2})} F_{\pm}^{(\alpha_h,l)}(z_{\pm}^3) = \delta^{2(l-\frac{3}{2})} F_{\pm}^{(\alpha_h,l-1)}(\nabla_h \cdot z_{\pm}^h) \quad \text{for } l \geqslant 1. \end{split}$$

We denote the energy and flux on $[0, t^*] \times \Omega_1$ as

$$E_{\pm}^{(k)}(z_{\pm(\delta)}) = \sum_{k'+l'=k} E_{\pm}^{(k',l')}(z_{\pm(\delta)}), \quad E_{\pm}^{(k)}(z_{\pm(\delta)}^h) = \sum_{k'+l'=k} E_{\pm}^{(k',l')}(z_{\pm(\delta)}^h), \quad E_{\pm}^{(k)}(z_{\pm(\delta)}^3) = \sum_{k'+l'=k} E_{\pm}^{(k',l')}(z_{\pm(\delta)}^3),$$

$$F_{\pm}^{(k)}(z_{\pm(\delta)}) = \sum_{k'+l'=k} F_{\pm}^{(k',l')}(z_{\pm(\delta)}), \quad F_{\pm}^{(k)}(z_{\pm(\delta)}^h) = \sum_{k'+l'=k} F_{\pm}^{(k',l')}(z_{\pm(\delta)}^h), \quad F_{\pm}^{(k)}(z_{\pm(\delta)}^3) = \sum_{k'+l'=k} F_{\pm}^{(k',l')}(z_{\pm(\delta)}^3).$$

We are now ready to review the global existence and uniform (with respect to δ) weighted energy estimates of the system (2.3):

Theorem 2.5 (Uniform weighted energy estimates in Ω_{δ} ; adapted from Theorem 1.1 in [26] for the study of rigidity). Let $N \in \mathbb{Z}_{\geq 5}$, $\delta \in (0,1]$ and $\sigma \in (0,\frac{1}{3})$. There exists a universal constant $\varepsilon_0 \in (0,1)$ such that if the initial data $(z_{+,0}(x), z_{-,0}(x))$ of the system (2.3) satisfy

$$\mathcal{E}(0) := \sum_{+,-} \left(\sum_{k+l \leqslant 2N} \delta^{2(l-\frac{1}{2})} E_{\pm}^{(k,l)}(z_{\pm,0}) + \sum_{k \leqslant 2N-1} \delta^{-3} E_{\pm}^{(k,0)}(z_{\pm,0}^3) + \sum_{k+l \leqslant N+2} \delta^{2(l-\frac{1}{2})} E_{\pm}^{(k,l)}(\partial_3 z_{\pm,0}) \right) \leqslant \varepsilon_0^2,$$

then the system (2.3) admits a unique global solution $(z_+(t,x), z_-(t,x))$. Moreover, there is a universal constant C such that the following uniform (with respect to δ) weighted energy estimates hold:

$$\mathcal{E} := \sum_{+,-} \left(\sum_{k+l \leqslant 2N} \delta^{2(l-\frac{1}{2})} \left(E_{\pm}^{(k,l)}(z_{\pm}) + F_{\pm}^{(k,l)}(z_{\pm}) \right) + \sum_{k \leqslant 2N-1} \delta^{-3} \left(E_{\pm}^{(k,0)}(z_{\pm}^{3}) + F_{\pm}^{(k,0)}(z_{\pm}^{3}) \right) \right)$$

$$+ \sum_{k+l \leqslant N+2} \delta^{2(l-\frac{1}{2})} \left(E_{\pm}^{(k,l)}(\partial_{3}z_{\pm}) + F_{\pm}^{(k,l)}(\partial_{3}z_{\pm}) \right)$$

$$\leqslant C\mathcal{E}(0).$$

$$(2.16)$$

In particular, both the constants ε_0 and C are independent of the thickness parameter δ and the position parameter a. These facts are indeed important keys to ensuring the further study on rigidity.

The proof of Theorem 2.5 is based on the standard continuity method, and we refer the readers to Theorem 1.1 in [26] for the details. For the sake of use, we recall the main bootstrap argument therein as follows.

(Bootstrap Assumption) We assume that

$$||z_{\pm}^{1}||_{L_{t}^{\infty}L_{x}^{\infty}} \leqslant 1, \tag{2.17}$$

and

$$\delta^{2(l-\frac{1}{2})} E_{\pm}^{(k,l)}(z_{\pm}) \leqslant 2C_{1}\varepsilon^{2}, \qquad \delta^{2(l-\frac{1}{2})} F_{\pm}^{(k,l)}(z_{\pm}) \leqslant 2C_{1}\varepsilon^{2}, \quad \text{for } k+l \leqslant 2N,$$

$$\delta^{-3} E_{\pm}^{(k,0)}(z_{\pm}^{3}) \leqslant 2C_{1}\varepsilon^{2}, \qquad \delta^{-3} F_{\pm}^{(k,0)}(z_{\pm}^{3}) \leqslant 2C_{1}\varepsilon^{2}, \quad \text{for } k \leqslant 2N-1,$$

$$\delta^{2(l-\frac{1}{2})} E_{\pm}^{(k,l)}(\partial_{3}z_{\pm}) \leqslant 2C_{1}\varepsilon^{2}, \quad \delta^{2(l-\frac{1}{2})} F_{\pm}^{(k,l)}(\partial_{3}z_{\pm}) \leqslant 2C_{1}\varepsilon^{2}, \quad \text{for } k+l \leqslant N+2,$$

$$(2.18)$$

where C_1 can be determined by the energy estimates. We remark here that the assumption (2.18) is legitimate since it holds for the initial data and then remains correct for at least a short time interval $[0, t^*]$.

(Bootstrap Expectation) We show that there exists a universal constant ε_0 such that for all $\varepsilon \leqslant \varepsilon_0$, the constants in (2.17)-(2.18) can be improved to their corresponding halves, i.e.

$$||z_{\pm}^{1}||_{L_{t}^{\infty}L_{x}^{\infty}} \leqslant \frac{1}{2},$$
 (2.19)

and

$$\delta^{2(l-\frac{1}{2})} E_{\pm}^{(k,l)}(z_{\pm}) \leqslant C_{1} \varepsilon^{2}, \qquad \delta^{2(l-\frac{1}{2})} F_{\pm}^{(k,l)}(z_{\pm}) \leqslant C_{1} \varepsilon^{2}, \quad \text{for } k+l \leqslant 2N,
\delta^{-3} E_{\pm}^{(k,0)}(z_{\pm}^{3}) \leqslant C_{1} \varepsilon^{2}, \qquad \delta^{-3} F_{\pm}^{(k,0)}(z_{\pm}^{3}) \leqslant C_{1} \varepsilon^{2}, \quad \text{for } k \leqslant 2N-1,
\delta^{2(l-\frac{1}{2})} E_{\pm}^{(k,l)}(\partial_{3} z_{\pm}) \leqslant C_{1} \varepsilon^{2}, \quad \delta^{2(l-\frac{1}{2})} F_{\pm}^{(k,l)}(\partial_{3} z_{\pm}) \leqslant C_{1} \varepsilon^{2}, \quad \text{for } k+l \leqslant N+2.$$
(2.20)

We point out that both the constants ε_0 and C_1 are independent of the thickness parameter δ , the position parameter a as well as the lifespan $[0, t^*]$. Based on the last fact, there exists an endless continuation of the lifespan from $[0, t^*]$ to $[0, +\infty]$, which thus leads to the global existence result. Therefore the proof of this theorem reduces to showing the uniform energy estimates (2.16), that is, we only need to show (2.19)-(2.20) under (2.17)-(2.18). The rest of the proof is similar to Page 22-48 in [26] with the only difference being the appearance of the position parameter a. In fact, as discussed in Lemma 2.2, the position parameter only influences the properties of weights, which also accounts for the independence of universal constants on the position parameter.

Remark 2.6. In the rest of this paper, we will directly use (2.19)-(2.20) to derive other estimates.

Based on the notations before, we have

 $\sim \mathcal{E}$.

$$\mathcal{E}_{\delta} := \sum_{+,-} \left(\sum_{k \leqslant 2N} \left(E_{\pm}^{(k)}(z_{\pm(\delta)}^{h}) + F_{\pm}^{(k)}(z_{\pm(\delta)}^{h}) \right) + \sum_{k \leqslant 2N-1} \left(E_{\pm}^{(k)}(z_{\pm(\delta)}^{3}) + F_{\pm}^{(k)}(z_{\pm(\delta)}^{3}) \right) \right) \\ + \delta^{2} \left(E_{\pm}^{(2N)}(z_{\pm(\delta)}^{3}) + F_{\pm}^{(2N)}(z_{\pm(\delta)}^{3}) \right) + \delta^{-2} \sum_{k \leqslant N+2} \left(E_{\pm}^{(k)}(\partial_{3}z_{\pm(\delta)}^{h}) + F_{\pm}^{(k)}(\partial_{3}z_{\pm(\delta)}^{h}) \right) \right)$$

We remark that similar equivalences will not be repeated in the sequel. Then the following corollary holds as a direct renormalization consequence of Theorem 2.5:

Corollary 2.7 (Uniform weighted energy estimates in Ω_1). Let $N \in \mathbb{Z}_{\geq 5}$, $\delta \in (0,1]$ and $\sigma \in (0,\frac{1}{3})$. There exists a universal constant $\varepsilon_1 \in (0,\varepsilon_0]$ such that if the initial data $(z_{+(\delta),0}(x),z_{-(\delta),0}(x))$ of the rescaled system (2.4) satisfy

$$\mathcal{E}_{\delta}(0) := \sum_{\pm,-} \left(\sum_{k \leq 2N} E_{\pm}^{(k)}(z_{\pm(\delta),0}^h) + \sum_{k \leq 2N-1} E_{\pm}^{(k)}(z_{\pm(\delta),0}^3) + \delta^2 E_{\pm}^{(2N)}(z_{\pm(\delta),0}^3) + \delta^{-2} \sum_{k \leq N+2} E_{\pm}^{(k)}(\partial_3 z_{\pm(\delta),0}^h) \right) \leqslant \varepsilon_1^2,$$

then the rescaled system (2.4) admits a unique global solution $(z_+(t,x), z_-(t,x))$. Moreover, there is a universal constant C such that the following uniform (with respect to δ) weighted energy estimates hold:

$$\mathcal{E}_{\delta} \leqslant C\mathcal{E}_{\delta}(0).$$

In particular, both the constants ε_1 and C are independent of the thickness parameter δ and the position parameter a. These facts are indeed important keys to ensuring the further study on rigidity.

To end this section, we recall the approximation theory of the global solution (Alfvén waves) to the system (2.3) in Ω_{δ} as $\delta \to 0$:

Theorem 2.8 (Asymptotics of the global solution from Ω_{δ} to \mathbb{R}^2 as δ goes to zero; extracted from Theorem 1.3 in [26]). Let $N \in \mathbb{Z}_{\geq 5}$, $\delta \in (0,1]$ and $\sigma \in (0,\frac{1}{3})$. Assume that the initial data $(z_{+(\delta),0},z_{-(\delta),0})$ converge to $(z_{+(0),0},z_{-(0),0})$ in $H^{N+1}(\Omega_1)$ with respect to δ :

$$\lim_{\delta \to 0} \left(z_{\pm(\delta),0}^h(x_h, x_3), z_{\pm(\delta),0}^3(x_h, x_3) \right) = \left(z_{\pm(0),0}^h(x_h), 0 \right) \quad in \ H^{N+1}(\Omega_1),$$

i.e.

$$\lim_{\delta \to 0} \sum_{k \le N+1} \| \langle u_{\mp} \rangle^{1+\sigma} \nabla^k (z_{\pm(\delta),0} - z_{\pm(0),0}) \|_{L^2(\Omega_1)} = 0,$$

where $z_{\pm(0),0}^h$ satisfies $\nabla_h \cdot z_{\pm(0),0}^h = 0$. If $\left(z_{+(\delta)}(t,x), z_{-(\delta)}(t,x)\right)$ is a solution to the rescaled system (2.4), then there exist functions $z_{\pm(0)}^h(t,x_h)$ such that for any $x_3 \in (-1,1)$, there hold

$$\lim_{\delta \to 0} z_{\pm(\delta)}^h(t, x_h, x_3) = z_{\pm(0)}^h(t, x_h) \quad in \ H^N(\mathbb{R}^2),$$

$$\lim_{\delta \to 0} z_{\pm(\delta)}^3(t, x_h, x_3) = 0 \quad in \ H^{N-1}(\mathbb{R}^2).$$

Remark 2.9. In particular, $(z_{+(0)}^h(t,x_h), z_{-(0)}^h(t,x_h))$ solves the 2D version of the rescaled system (2.4) with the initial data $(z_{+(0),0}^h(t,x_h), z_{-(0),0}^h(t,x_h))$. Theorem 2.8 indeed shows that 3D Alfvén waves in Ω_{δ} propagating along the horizontal direction can be approximated by 2D Alfvén waves in \mathbb{R}^2 as δ goes to zero.

3. Estimates on the nonlinear term

In this section, let us provide some important estimates for the nonlinear term.

Lemma 3.1 (Estimate for $\mathbf{I}_{\pm}^{(\alpha_h,l)}$). For any $\alpha_h \in (\mathbb{Z}_{\geqslant 0})^2$ and $l \in \mathbb{Z}_{\geqslant 0}$ with $|\alpha_h| + l \leqslant 2N - 1$, there holds $\delta^{l+\frac{1}{2}} \|\langle u_{\mp} \rangle^{1+\sigma} \langle u_{\pm} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{\pm}^{(\alpha_h,l)} \|_{L^2_{+}L^2_{-}} \lesssim C_1 \varepsilon^2,$

where

$$\mathbf{I}_{\pm}^{(\alpha_h,l)} := -\partial_h^{\alpha_h} \partial_3^l (\nabla z_{\mp} \cdot \nabla z_{\pm}).$$

We remark that given $N \in \mathbb{Z}_{\geqslant 5}$, this result also holds for any $\alpha_h \in (\mathbb{Z}_{\geqslant 0})^2$ and any $l \in \mathbb{Z}_{\geqslant 0}$ with $0 \leqslant |\alpha_h| + l \leqslant N + 2$.

Proof. We only derive the estimate for $\mathbf{I}_{+}^{(\alpha_h,l)}$. The estimate on $\mathbf{I}_{-}^{(\alpha_h,l)}$ can be given in the same way. By the divergence free condition div $z_{-}=0$, we see that

$$\nabla z_{-} \cdot \nabla z_{+} = \nabla z_{-}^{k} \cdot \partial_{k} z_{+} = \nabla z_{-}^{h} \cdot \nabla_{h} z_{+} + \nabla z_{-}^{3} \cdot \partial_{3} z_{+}$$

$$= (\nabla_{h} z_{-}^{h}, \partial_{3} z_{-}^{h}) \cdot \nabla_{h} z_{+} + (\nabla_{h} z_{-}^{3}, \partial_{3} z_{-}^{3}) \cdot \partial_{3} z_{+}$$

$$= (\nabla_{h} z_{-}^{h}, \partial_{3} z_{-}^{h}) \cdot \nabla_{h} z_{+} + (\nabla_{h} z_{-}^{3}, -\nabla_{h} z_{-}^{h}) \cdot \partial_{3} z_{+},$$

and therefore

$$\begin{split} \left|\mathbf{I}_{+}^{(\alpha_{h},l)}\right| \lesssim \sum_{\beta_{h}\leqslant\alpha_{h} \atop l_{1}\leqslant l} \underbrace{\left(\underbrace{\left|\partial_{h}^{\alpha_{h}-\beta_{h}}\partial_{3}^{l-l_{1}}\nabla_{h}z_{-}\right|\cdot\left|\partial_{h}^{\beta_{h}}\partial_{3}^{l_{1}}\nabla_{h}z_{+}\right|}_{\mathbf{I}_{+,1}^{(\beta_{h},l_{1})}} + \underbrace{\left|\partial_{h}^{\alpha_{h}-\beta_{h}}\partial_{3}^{l-l_{1}}\nabla_{h}z_{-}\right|\cdot\left|\partial_{h}^{\beta_{h}}\partial_{3}^{l_{1}}\partial_{3}z_{+}\right|}_{\mathbf{I}_{+,2}^{(\beta_{h},l_{1})}} \\ + \underbrace{\left|\partial_{h}^{\alpha_{h}-\beta_{h}}\partial_{3}^{l-l_{1}}\partial_{3}z_{-}^{h}\right|\cdot\left|\partial_{h}^{\beta_{h}}\partial_{3}^{l_{1}}\nabla_{h}z_{+}\right|}_{\mathbf{I}_{+,3}^{(\beta_{h},l_{1})}}\right). \end{split}$$

In this way we obtain that

$$\begin{split} \delta^{l+\frac{1}{2}} \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{+}^{(\alpha_{h},l)} \|_{L_{t}^{2}L_{x}^{2}} &\lesssim \delta^{l+\frac{1}{2}} \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{+,1}^{(\beta_{h},l_{1})} \|_{L_{t}^{2}L_{x}^{2}} \\ &+ \delta^{l+\frac{1}{2}} \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{+,2}^{(\beta_{h},l_{1})} \|_{L_{t}^{2}L_{x}^{2}} \\ &+ \delta^{l+\frac{1}{2}} \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{+,3}^{(\beta_{h},l_{1})} \|_{L_{t}^{2}L_{x}^{2}}. \end{split} \tag{3.1}$$

According to the size of $|\beta_h| + l_1$, we now have two cases:

$$|\beta_h| + l_1 \leqslant N - 1$$
 and $N \leqslant |\beta_h| + l_1 \leqslant |\alpha_h| + l \leqslant 2N - 1$.

Case 1: $|\beta_h| + l_1 \leq N - 1$. Thanks to $(|\beta_h| + l_1 + 1) + 2 \leq N + 2$, we can use the Sobolev lemma to bound L_x^{∞} norms of $\nabla_h^{|\beta_h|+1} \partial_3^{l_1} z_+$ in $\mathbf{I}_{+,1}^{(\beta_h,l_1)}$ as well as in $\mathbf{I}_{+,3}^{(\beta_h,l_1)}$ and $\nabla_h^{|\beta_h|} \partial_3^{l_1+1} z_+$ in $\mathbf{I}_{+,2}^{(\beta_h,l_1)}$ respectively. From this, we have

$$\begin{split} &\delta^{l+\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{+,1}^{(\beta_{h},l_{1})} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \delta^{l+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l-l_{1}} z_{h}^{h} \right\|_{L_{t}^{\infty}L_{x}^{2}} \cdot \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|+1} \partial_{3}^{l-l_{1}} z_{+}^{h} \right\|_{L_{t}^{\infty}L_{x}^{2}} \\ &\lesssim \delta^{l+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l-l_{1}} z_{-}^{h} \right\|_{L_{t}^{\infty}L_{x}^{2}} \cdot \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \nabla_{h}^{k_{2}} \partial_{3}^{l_{2}} \left(\frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|+1} \partial_{3}^{l_{1}} z_{+} \right) \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{1} \leqslant |\alpha_{h}|+1} \delta^{l-l_{1}-\frac{1}{2}} \left(E_{-}^{(k_{1},l-l_{1})}(z_{-}) \right)^{\frac{1}{2}} \cdot \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|+1} \partial_{3}^{l_{1}} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{1} \leqslant |\alpha_{h}|+1} \delta^{l-l_{1}-\frac{1}{2}} \left(E_{-}^{(k_{1},l-l_{1})}(z_{-}) \right)^{\frac{1}{2}} \cdot \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|+1} \partial_{3}^{l_{1}+l_{2}} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{1} \leqslant |\alpha_{h}|+1} \delta^{l-l_{1}-\frac{1}{2}} \left(E_{-}^{(k_{1},l-l_{1})}(z_{-}) \right)^{\frac{1}{2}} \cdot \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \nabla_{h}^{k_{2}} \partial_{3}^{l_{1}+1} z_{+} \right\|_{L_{t}^{2}L_{x}^{\infty}} \\ &\delta^{l+\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{2}-l_{1}} z_{-} \right\|_{L_{t}^{\infty}L_{x}^{2}} \cdot \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \nabla_{h}^{k_{2}} \partial_{3}^{l_{2}} \left(\frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|} \partial_{3}^{l_{3}+1} z_{+} \right) \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{1} \leqslant |\alpha_{h}|+1} \delta^{l-l_{1}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{2}-l_{1}} z_{-} \right\|_{L_{t}^{\infty}L_{x}^{2}} \cdot \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \nabla_{h}^{k_{2}} \partial_{3}^{l_{2}} \left(\frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|+1} \partial_{3}^{l_{1}+1+l_{2}} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{1} \leqslant |\alpha_{h}|+1} \delta^{l-l_{1}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{1}-l_{1}$$

$$\stackrel{(2.10)}{\lesssim} \delta^{l+\frac{1}{2}} \|\langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|} \partial_{3}^{l-l_{1}+1} z_{-}^{h} \|_{L_{t}^{\infty} L_{x}^{2}} \cdot \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|+1+k_{2}} \partial_{3}^{l_{1}+l_{2}} z_{+} \|_{L_{t}^{2} L_{x}^{2}} \\
\stackrel{\lesssim}{\sum_{k_{1} \leqslant |\alpha_{h}|}} \delta^{l-l_{1}+\frac{1}{2}} \left(E_{-}^{(k_{1},l-l_{1}+1)}(z_{-}) \right)^{\frac{1}{2}} \cdot \sum_{k_{2}+l_{2} \leqslant N+2} \delta^{l_{2}-\frac{1}{2}} \left(F_{+}^{(k_{2},l_{2})}(z_{+}) \right)^{\frac{1}{2}} \stackrel{(2.20)}{\lesssim} C_{1} \varepsilon^{2}. \tag{3.4}$$

In this case, substituting (3.2)-(3.4) into (3.1) immediately gives the desired result.

Case 2: $N \leq |\beta_h| + l_1 \leq |\alpha_h| + l \leq 2N - 1$. However, in this case, L_x^{∞} estimates are not directly applicable to $\nabla_h^{|\beta_h|+1} \partial_3^{l_1} z_+$ in $\mathbf{I}_{+,1}^{(\beta_h,l_1)}$ as well as in $\mathbf{I}_{+,3}^{(\beta_h,l_1)}$ and $\nabla_h^{|\beta_h|} \partial_3^{l_1+1} z_+$ in $\mathbf{I}_{+,2}^{(\beta_h,l_1)}$ anymore. This is because $(|\beta_h| + l_1 + 1) + 2 > N + 2$ and one cannot afford more than N + 2 derivatives to close the energy estimates in flux terms. Instead, we shall adapt L_x^{∞} estimates to $\nabla_h^{|\alpha_h-\beta_h|+1} \partial_3^{l-l_1} z_-^h$ in $\mathbf{I}_{+,1}^{(\beta_h,l_1)}, \nabla_h^{|\alpha_h-\beta_h|+1} \partial_3^{l-l_1} z_-$ in $\mathbf{I}_{+,2}^{(\beta_h,l_1)}$ and $\nabla_h^{|\alpha_h-\beta_h|} \partial_3^{l-l_1+1} z_-^h$ in $\mathbf{I}_{+,3}^{(\beta_h,l_1)}$ via the Sobolev lemma as substitutes:

$$\begin{array}{l} \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{+,1}^{(\beta_{h},l_{1})} \right\|_{L_{t}^{2}L_{x}^{2}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{+,1}^{(\beta_{h},l_{1})} \right\|_{L_{t}^{2}L_{x}^{2}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{h}^{(\beta_{h},l_{1})} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{h}^{(\beta_{h},l_{1})} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1+k_{2}} \partial_{3}^{l_{1}+l_{2}} z_{-}^{h} \right\|_{L_{t}^{\infty}L_{x}^{2}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1+k_{2}} \partial_{3}^{l_{1}+1+l_{2}} z_{-}^{h} \right\|_{L_{t}^{\infty}L_{x}^{2}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{+,2}^{|\beta_{h},l_{1}|} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{+,2}^{|\beta_{h},l_{1}|} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{I}_{+,2}^{|\beta_{h},l_{1}|} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{1}-1} z_{-} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{1}-1} z_{-} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{1}-1} z_{-} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{1}-1} z_{-} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{1}-1} z_{-} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{1}-1} z_{-} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{1}-1} z_{-} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1+\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \nabla_{h}^{|\alpha_{h}-\beta_{h}|+1} \partial_{3}^{l_{1}-1} z_{-} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \\ \lesssim \int_{0}^{1$$

In this case, using (3.5)-(3.7) together with (3.1) also yields the desired result.

Finally, collecting the above two cases finishes the proof of this lemma.

Lemma 3.2 (Estimate for $\mathbf{J}_{\pm}^{(\alpha_h,l)}$). For any $\alpha_h \in (\mathbb{Z}_{\geqslant 0})^2$ and any $l \in \mathbb{Z}_{\geqslant 0}$ with $0 \leqslant |\alpha_h| + l \leqslant 2N - 1$, there holds

$$\delta^{l-\frac{1}{2}} \| \langle u_{\mp} \rangle^{1+\sigma} \langle u_{\pm} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{J}_{\pm}^{(\alpha_h,l)} \|_{L_t^2 L_x^2} \lesssim C_1 \varepsilon^2,$$

where

$$\mathbf{J}_{+}^{(\alpha_{h},l)} := \partial_{h}^{\alpha_{h}} \partial_{3}^{l} (z_{\pm} \cdot \nabla z_{+}).$$

We remark that given $N \in \mathbb{Z}_{\geqslant 5}$, this result also holds for any $\alpha_h \in (\mathbb{Z}_{\geqslant 0})^2$ and any $l \in \mathbb{Z}_{\geqslant 0}$ with $0 \leqslant |\alpha_h| + l \leqslant N + 2$.

Proof. Based on symmetry, we only need to derive bound related to $\mathbf{J}_{+}^{(\alpha_{h},l)}$.

Our proof starts with the observation that

$$\begin{split} \left| \mathbf{J}_{+}^{(\alpha_{h},l)} \right| &= \big| \sum_{\beta_{h} \leqslant \alpha_{h} \atop l_{1} \leqslant l} \partial_{h}^{\alpha_{h} - \beta_{h}} \partial_{3}^{l-l_{1}} z_{-} \cdot \nabla \partial_{h}^{\beta_{h}} \partial_{3}^{l_{1}} z_{+} \big| \lesssim \sum_{\beta_{h} \leqslant \alpha_{h} \atop l_{1} \leqslant l} \left| \partial_{h}^{\alpha_{h} - \beta_{h}} \partial_{3}^{l-l_{1}} z_{-} \cdot \nabla \partial_{h}^{\beta_{h}} \partial_{3}^{l_{1}} z_{+} \right| \\ &= \sum_{\beta_{h} \leqslant \alpha_{h} \atop l_{1} \leqslant l} \left| \partial_{h}^{\alpha_{h} - \beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{h} \cdot \partial_{h} \partial_{h}^{\beta_{h}} \partial_{3}^{l_{1}} z_{+} + \partial_{h}^{\alpha_{h} - \beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{3} \cdot \partial_{3} \partial_{h}^{\beta_{h}} \partial_{3}^{l_{1}} z_{+} \right| \\ &\lesssim \sum_{\beta_{h} \leqslant \alpha_{h} \atop l_{1} \leqslant l} \left(\underbrace{\left| \partial_{h}^{\alpha_{h} - \beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{h} \cdot \left| \nabla_{h}^{|\beta_{h}| + 1} \partial_{3}^{l_{1}} z_{+} \right|}_{\mathbf{J}_{+,1}^{(\beta_{h}, l_{1})}} + \underbrace{\left| \partial_{h}^{\alpha_{h} - \beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{3} \right| \cdot \left| \nabla_{h}^{|\beta_{h}|} \partial_{3}^{l_{1} + 1} z_{+} \right|}_{\mathbf{J}_{+,2}^{(\beta_{h}, l_{1})}} \right). \end{split}$$

As a result, there holds

$$\delta^{l-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{J}_{+}^{(\alpha_{h},l_{1})} \|_{L_{t}^{2}L_{x}^{2}} \lesssim \delta^{l-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{J}_{+,1}^{(\beta_{h},l_{1})} \|_{L_{t}^{2}L_{x}^{2}}
+ \delta^{l-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{J}_{+,2}^{(\beta_{h},l_{1})} \|_{L_{t}^{2}L_{x}^{2}}.$$
(3.8)

Due to the number of derivatives (of z_+ related), we distinguish the following two cases:

$$|\beta_h| + l_1 \leq N - 1$$
 and $N \leq |\beta_h| + l_1 \leq |\alpha_h| + l \leq 2N - 1$.

Case 1: $|\beta_h| + l_1 \leq N - 1$. Due to $(|\beta_h| + l_1 + 1) + 2 \leq N + 2$, we can always derive L_x^{∞} estimates on $\nabla_h^{|\beta_h|+1} \partial_3^{l_1} z_+$ in $\mathbf{J}_{+,1}^{(\beta_h,l_1)}$ and $\nabla_h^{|\beta_h|} \partial_3^{l_1+1} z_+$ in $\mathbf{J}_{+,2}^{(\beta_h,l_1)}$ via the Sobolev lemma. Consequently, we can carry out the estimates as follows:

$$\begin{split} &\delta^{l-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{J}_{+,1}^{(\beta_{h},l_{1})} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\stackrel{\text{Hölder}}{\lesssim} \delta^{l-\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{h} \right\|_{L_{t}^{\infty}L_{x}^{2}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|+1} \partial_{3}^{l_{1}} z_{+} \right\|_{L_{t}^{2}L_{x}^{\infty}} \\ &\stackrel{\text{Lemma 2.1}}{\lesssim} \delta^{l-\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{h} \right\|_{L_{t}^{\infty}L_{x}^{2}} \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \nabla_{h}^{k_{2}} \partial_{3}^{l_{2}} \left(\frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|+1} \partial_{3}^{l_{1}} z_{+} \right) \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\stackrel{(2.10)}{\lesssim} \delta^{l-\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{h} \right\|_{L_{t}^{\infty}L_{x}^{2}} \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|+1+k_{2}} \partial_{3}^{l_{1}+l_{2}} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\stackrel{(2.10)}{\lesssim} \delta^{l-\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{h} \right\|_{L_{t}^{2}L_{x}^{2}} \int_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|} \partial_{3}^{l_{1}+l_{2}} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\stackrel{\text{Hölder}}{\lesssim} \delta^{l-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{3} \right\|_{L_{t}^{\infty}L_{x}^{2}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|} \partial_{3}^{l_{1}+1} z_{+} \right\|_{L_{t}^{2}L_{x}^{\infty}} \\ &\stackrel{\text{Hölder}}{\lesssim} \delta^{l-\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{3} \right\|_{L_{t}^{\infty}L_{x}^{2}} \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \nabla_{h}^{k_{2}} \partial_{3}^{l_{2}} \left(\frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|} \partial_{3}^{l_{1}+1} z_{+} \right) \right\|_{L_{t}^{2}L_{x}^{2}} \end{split}$$

$$\stackrel{(2.10)}{\lesssim} \delta^{l-\frac{1}{2}} \|\langle u_{+} \rangle^{1+\sigma} \partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{3} \|_{L_{t}^{\infty} L_{x}^{2}} \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \|\frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{|\beta_{h}|+k_{2}} \partial_{3}^{l_{1}+1+l_{2}} z_{+} \|_{L_{t}^{2} L_{x}^{2}} \\
\lesssim \sum_{k_{1} \leqslant |\alpha_{h}|} \delta^{l-l_{1}-\frac{1}{2}} \left(E_{-}^{(k_{1},l-l_{1})} (z_{-}^{3}) \right)^{\frac{1}{2}} \sum_{k_{2}+l_{2} \leqslant N+1} \delta^{l_{2}-\frac{1}{2}} \left(F_{+}^{(k_{2},l_{2})} (\partial_{3} z_{+}) \right)^{\frac{1}{2}} \stackrel{(2.20)}{\lesssim} C_{1} \varepsilon^{2}. \tag{3.10}$$

Together with (3.8), these two estimates (3.9)-(3.10) lead us to the desired result for this case.

Case 2: $N \leq |\beta_h| + l_1 \leq |\alpha_h| + l \leq 2N - 1$. By virtue of $(|\beta_h| + l_1 + 1) + 2 > N + 2$, the terms related to z_+ now cannot be controlled by the L_x^{∞} estimates via the Sobolev lemma and energy flux estimates as the previous case. Likewise, we also turn our attention to L_x^{∞} estimates on the terms related to z_- at this time:

$$\begin{split} &\delta^{l-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{J}_{+,1}^{(\beta,l_1)} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\overset{\text{Hölder}}{\lesssim} \delta^{l-\frac{1}{2}} \left\| \langle u_{+} \rangle^{1+\sigma} \partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{h} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{\beta_{h}|+1} \partial_{3}^{l_{1}} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\overset{\text{Lemma 2.1}}{\lesssim} \delta^{l-\frac{1}{2}} \sum_{k_{2}+l_{2} \leqslant 2} \delta^{l_{2}-\frac{1}{2}} \left\| \nabla_{k}^{k_{2}} \partial_{3}^{l_{2}} \left(\langle u_{+} \rangle^{1+\sigma} \partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l-l_{1}} z_{-}^{h} \right) \right\|_{L_{t}^{\infty}L_{x}^{2}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{\beta_{h}|+1} \partial_{3}^{l_{1}} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{2}+l_{2} \leqslant N+1} \delta^{l_{2}-\frac{1}{2}} \left(E_{-}^{(k_{2},l_{2})} (z_{-}^{h}) \right)^{\frac{1}{2}} \sum_{k_{1} \leqslant |\alpha_{h}|+1} \delta^{l_{1}+l_{2}} z_{-}^{h} \right\|_{L_{t}^{\infty}L_{x}^{2}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{\beta_{h}|+1} \partial_{3}^{l_{1}} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{2}+l_{2} \leqslant N+1} \delta^{l_{2}-\frac{1}{2}} \left(E_{-}^{(k_{2},l_{2})} (z_{-}^{h}) \right)^{\frac{1}{2}} \sum_{k_{1} \leqslant |\alpha_{h}|+1} \delta^{l_{1}+l_{2}} z_{-}^{h} \right\|_{L_{t}^{\infty}L_{x}^{2}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{\beta_{h}|+1} \partial_{3}^{l_{1}} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{2}+l_{2} \leqslant N+1} \delta^{l_{2}-\frac{1}{2}} \left\| \nabla_{h}^{\alpha_{h}} \partial_{3}^{l_{2}-l_{1}} z_{-}^{3} \right\|_{L_{t}^{\infty}L_{x}^{\infty}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{\beta_{h}|+1} \partial_{3}^{l_{1}+1} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{2}+l_{2} \leqslant N+1} \delta^{l_{2}-\frac{1}{2}} \left\| \nabla_{h}^{\alpha_{h}} \partial_{3}^{\beta_{2}} \left(\langle u_{+} \rangle^{1+\sigma}} \partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l_{2}-l_{1}} z_{-}^{3} \right\|_{L_{t}^{\infty}L_{x}^{2}} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \nabla_{h}^{\beta_{h}|} \partial_{3}^{l_{1}+1} z_{+} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{2}+l_{2} \leqslant N+1} \delta^{l_{2}-\frac{3}{2}} \left(E_{-}^{(k_{2},l_{2})} (z_{-}^{3}) \right)^{\frac{1}{2}} \cdot \delta^{l_{1}+\frac{1}{2}} \left(F_{+}^{(k_{1},l_{1}+1)} (z_{+}) \right)^{\frac{1}{2}} \\ &\lesssim \sum_{k_{2}+l_{2} \leqslant N+1} \delta^{l_{2}-\frac{3}{2}} \left(E_{-}^{(k_{2},l_{2})} (z_{-}^{3}) \right)^{\frac{1}{2}} \cdot \delta^{l_{1}+\frac{1}{2}} \left(F_{+}^{(k_{1},l_{1}+1)} (z_{+} \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \\ &\lesssim$$

In such a case, putting the estimates (3.8) and (3.11)-(3.12) together also gives the desired result. This ends the proof in the same manner.

Remark 3.3. By virtue of div $z_{\pm} = 0$, we notice that $\partial_3 z_{\pm}^3 = -\partial_h z_{\pm}^h$. In fact, we have adopted this observation in the proof of (3.12). This can help us transform some direct estimates on $\partial_3 z_{\pm}^3$, which should have been bad, to good estimates on $\partial_h z_{\pm}^h$ and thus make up for the possible loss of δ in coefficients. We remark that this observation will be frequently used in this paper without further comment.

Remark 3.4 (Estimate for $\mathbf{J}_{\pm(\delta)}^{(\alpha_h,l)}$). Based on the notations in Section 2, Lemma 3.2 also gives the following renormalization result immediately: For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ and any $l \in \mathbb{Z}_{\geq 0}$ with $0 \leq |\alpha_h| + l \leq N + 2$, there holds

$$\|\langle u_{\mp}\rangle^{1+\sigma}\langle u_{\pm}\rangle^{\frac{1}{2}(1+\sigma)}\mathbf{J}_{\pm(\delta)}^{(\alpha_h,l)}\|_{L_t^2L_x^2} \lesssim C_1\varepsilon^2,$$

where

$$\mathbf{J}_{\pm(\delta)}^{(\alpha_h,l)} := \partial_h^{\alpha_h} \partial_3^l (z_{\mp(\delta)} \cdot \nabla z_{\pm(\delta)}).$$

This result will be used in the approximation part of Section 7.

Lemma 3.5 (Estimate for $\mathbf{K}_{\pm}^{(\alpha_h,l)}$). For any $\alpha_h \in (\mathbb{Z}_{\geqslant 0})^2$ and any $l \in \mathbb{Z}_{\geqslant 0}$ with $0 \leqslant |\alpha_h| + l \leqslant N + 2$, there holds

$$\delta^{l-\frac{1}{2}} \| \langle u_{\mp} \rangle^{1+\sigma} \langle u_{\pm} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{K}_{\pm}^{(\alpha_h,l)} \|_{L_t^2 L_x^2} \lesssim C_1 \varepsilon^2,$$

where

$$\mathbf{K}_{\pm}^{(\alpha_h,l)} := \partial_h^{\alpha_h} \partial_3^l \partial_3(z_{\mp} \cdot \nabla z_{\pm}).$$

Proof. Based on symmetry, it suffices to consider $\mathbf{K}_{+}^{(\alpha_h,l)}$. We also note that

$$\left|\mathbf{K}_{+}^{(\alpha_{h},l)}\right| \lesssim \sum_{\beta_{h} \leqslant \alpha_{h} \atop l_{1} \leqslant l+1} \left(\underbrace{\left|\partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l+1-l_{1}} z_{-}^{h}\right| \cdot \left|\nabla_{h}^{|\beta_{h}|+1} \partial_{3}^{l_{1}} z_{+}\right|}_{\mathbf{K}_{+,1}^{(\beta_{h},l_{1})}} + \underbrace{\left|\partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{3}^{l+1-l_{1}} z_{-}^{3}\right| \cdot \left|\nabla_{h}^{|\beta_{h}|} \partial_{3}^{l_{1}+1} z_{+}\right|}_{\mathbf{K}_{+,2}^{(\beta_{h},l_{1})}} \right).$$

Similar to (3.11)-(3.12), we can infer that

$$\begin{split} &\delta^{l-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{K}_{+,1}^{(\beta_{h},l_{1})} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \sum_{k_{2}+l_{2} \leqslant N+5} \delta^{l_{2}-\frac{1}{2}} \left(E_{-}^{(k_{2},l_{2})}(z_{-}^{h}) \right)^{\frac{1}{2}} \sum_{k_{1} \leqslant |\alpha_{h}|+1} \delta^{l_{1}-\frac{1}{2}} \left(F_{+}^{(k_{1},l_{1})}(z_{+}) \right)^{\frac{1}{2}} \lesssim C_{1} \varepsilon^{2}, \\ &\delta^{l-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{K}_{+,2}^{(\beta_{h},l_{1})} \right\|_{L_{t}^{2}L_{x}^{2}} \\ &\lesssim \left(\sum_{k_{2} \leqslant N+5} \delta^{-\frac{3}{2}} \left(E_{-}^{(k_{2},0)}(z_{-}^{3}) \right)^{\frac{1}{2}} + \sum_{k_{2}+l_{2} \leqslant N+5} \delta^{l_{2}-\frac{1}{2}} \left(E_{-}^{(k_{2},l_{2})}(z_{-}^{h}) \right)^{\frac{1}{2}} \right) \sum_{k_{1} \leqslant |\alpha_{h}|} \delta^{l_{1}-\frac{1}{2}} \left(F_{+}^{(k_{1},l_{1})}(\partial_{3}z_{+}) \right)^{\frac{1}{2}} \lesssim C_{1} \varepsilon^{2}. \end{split}$$

Therefore we can conclude from these estimates to obtain the lemma.

4. Estimates on the pressure term

Due to the facts div $z_{\pm}=0$ and the boundary conditions $z_{+}^{3}|_{x_{3}=\pm\delta}=0$, $z_{-}^{3}|_{x_{3}=\pm\delta}=0$, we turn to take divergence of the first equation in (2.3) to derive the following system in Ω_{δ} :

$$\begin{cases}
-\Delta p = \partial_i z_+^j \partial_j z_-^i, \\
\partial_3 p \big|_{x_3 = \pm \delta} = 0.
\end{cases} (4.1)$$

Using the Green's function $G_{\delta}(x,y)$ on $\Omega_{\delta} \times \Omega_{\delta}$, we solve this system (4.1) and write the pressure as

$$p(t,x) = \int_{\Omega_{\delta}} G_{\delta}(x,y) (\partial_i z_+^j \partial_j z_-^i)(t,y) dy.$$
 (4.2)

However, it is rather difficult to give an explicit formula of $G_{\delta}(x,y)$ herein (up to pointwise convergence), which hinders further investigation on the pressure term p. Nevertheless, as we will see, what really counts in the rest of this paper is its gradient, and therefore we only need to give the explicit formula of ∇p .

In fact, taking gradient of (4.2) immediately yields

$$\nabla p(t,x) = \int_{\Omega_{\delta}} \nabla_x G_{\delta}(x,y) (\partial_i z_+^j \partial_j z_-^i)(t,y) dy, \tag{4.3}$$

where $\nabla_x G_{\delta}(x,y)$ is well defined on $\Omega_{\delta} \times \Omega_{\delta}$ by

$$\nabla_x G_{\delta}(x,y) := \frac{1}{4\pi} \sum_{k=-\infty}^{\infty} \nabla_x \frac{1}{\left(|x_h - y_h|^2 + |(-1)^k (x_3 - 2k\delta) - y_3|^2\right)^{\frac{1}{2}}}.$$
(4.4)

Here we refer the interested readers to Lemma 2.1 in [26] for more details of (4.3) and (4.4). Before proceeding further, we first review Corollary 2.3 in [26] with proof, which provides an important bound on derivatives of the Green's function.

Lemma 4.1. For any $l \in \mathbb{Z}_{\geqslant 1}$, there holds

$$\nabla_x^l G_\delta(x, y) \lesssim \frac{1}{\delta} \frac{1}{|x_h - y_h|^l}.$$
(4.5)

Proof. First of all, (4.4) can be rephrased as

$$\nabla_x G_{\delta}(x,y) = \frac{1}{4\pi} \left(\sum_{k=0}^{\infty} + \sum_{k=1}^{\infty} + \sum_{k=-\infty}^{-1} \right) \nabla_x \frac{1}{\left(|x_h - y_h|^2 + |(-1)^k (x_3 - 2k\delta) - y_3|^2 \right)^{\frac{1}{2}}}$$

$$= \frac{1}{4\pi} \left(\nabla_x \frac{1}{\left(|x_h - y_h|^2 + |x_3 - y_3|^2 \right)^{\frac{1}{2}}} + \sum_{k=1}^{\infty} \left(\nabla_x \frac{1}{\left(|x_h - y_h|^2 + |(x_{+,k})_3 - y_3|^2 \right)^{\frac{1}{2}}} + \nabla_x \frac{1}{\left(|x_h - y_h|^2 + |(x_{-,k})_3 - y_3|^2 \right)^{\frac{1}{2}}} \right) \right)$$

$$= \frac{1}{4\pi} \left(\nabla_x \frac{1}{|x - y|} + \sum_{k=1}^{\infty} \left(\nabla_x \frac{1}{|x_{+,k} - y|} + \nabla_x \frac{1}{|x_{-,k} - y|} \right) \right),$$

where we denote

$$x_{+,k} = (x_h, (-1)^k (x_3 - 2k\delta)), \quad x_{-,k} = (x_h, (-1)^k (x_3 + 2k\delta)), \quad k \in \mathbb{Z}_{\geqslant 1}.$$

For any $l \in \mathbb{Z}_{\geqslant 1}$, it subsequently follows that

$$\nabla_x^l G_{\delta}(x,y) = \frac{1}{4\pi} \Big(\nabla_x^l \frac{1}{|x-y|} + \sum_{k=1}^{\infty} \Big(\nabla_x^l \frac{1}{|x_{+,k}-y|} + \nabla_x^l \frac{1}{|x_{-,k}-y|} \Big) \Big).$$

By virtue of $(x,y) \in \Omega_{\delta} \times \Omega_{\delta}$, we have $x_3, y_3 \in (-\delta, \delta)$ and then $\nabla_x^l G_{\delta}(x,y)$ can be bounded as follows:

$$\begin{split} &|\nabla^l_x G_\delta(x,y)|\\ &\lesssim \frac{1}{4\pi} \bigg(\frac{1}{|x-y|^{l+1}} + \sum_{k=1}^\infty \Big(\frac{1}{|x_{+,k}-y|^{l+1}} + \frac{1}{|x_{-,k}-y|^{l+1}}\Big) \bigg)\\ &= \frac{1}{4\pi} \left(\frac{1}{(|x_h-y_h|^2 + |x_3-y_3|^2)^{\frac{l+1}{2}}} \right.\\ &\qquad \qquad + \sum_{k=1}^\infty \Big(\frac{1}{(|x_h-y_h|^2 + |x_3-(-1)^k y_3 - 2k\delta|^2)^{\frac{l+1}{2}}} + \frac{1}{(|x_h-y_h|^2 + |x_3-(-1)^k y_3 + 2k\delta|^2)^{\frac{l+1}{2}}} \bigg) \bigg)\\ &\leqslant \frac{1}{4\pi} \bigg(\frac{1}{|x_h-y_h|^{l+1}} + \sum_{k=1}^\infty \frac{2}{(|x_h-y_h|^2 + |(2k-1)\delta|^2)^{\frac{l+1}{2}}} \bigg) \leqslant \frac{3}{4\pi} \sum_{k=0}^\infty \frac{1}{(|x_h-y_h|^2 + |2k\delta|^2)^{\frac{l+1}{2}}} \\ &\leqslant \frac{3}{4\pi} \int_0^\infty \frac{1}{(|x_h-y_h|^2 + |2\tau\delta|^2)^{\frac{l+1}{2}}} d\tau = \frac{3}{4\pi} \frac{1}{2\delta} \int_0^\infty \frac{1}{(|x_h-y_h|^2 + |2\delta\tau|^2)^{\frac{l+1}{2}}} d(2\delta\tau) \\ &= \frac{3}{4\pi} \frac{1}{2\delta} \int_0^\infty \frac{1}{(|x_h-y_h|^2 + (|x_h-y_h|u)^2)^{\frac{l+1}{2}}} d(|x_h-y_h|u) \\ &= \frac{3}{4\pi} \frac{1}{2\delta} \frac{1}{|x_h-y_h|^l} \int_0^\infty \frac{1}{(1+|y|^2)^{\frac{l+1}{2}}} du \lesssim \frac{1}{\delta} \frac{1}{|x_h-y_h|^l}. \end{split}$$

This proves (4.5) for all $l \in \mathbb{Z}_{\geq 1}$. Moreover, when l = 1, this bound implies that the right hand side of (4.4) is summable, which guarantees the well-definedness of the definition (4.4). The proof of the lemma is now complete.

Furthermore, to avoid the non-integrable singularity $\frac{1}{|x_h-y_h|}$, the following facts on $\frac{1}{|x_h|^{\gamma}}$ are necessary throughout the pressure estimates:

$$\frac{1}{|x_h|^{\gamma}} \chi_{|x_h| \leqslant 2} \in L^1(\mathbb{R}^2) \iff \gamma < 2; \qquad \frac{1}{|x_h|^{\gamma}} \chi_{|x_h| \geqslant 1} \in L^1(\mathbb{R}^2) \iff \gamma > 2;
\frac{1}{|x_h|^{\gamma}} \chi_{|x_h| \leqslant 2} \in L^2(\mathbb{R}^2) \iff \gamma < 1; \qquad \frac{1}{|x_h|^{\gamma}} \chi_{|x_h| \geqslant 1} \in L^2(\mathbb{R}^2) \iff \gamma > 1.$$
(4.6)

Now we can proceed to derive bounds on derivatives of the pressure term; see the following lemmas.

Lemma 4.2. For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ with $0 \leq |\alpha_h| \leq N+2$, there holds

$$\delta^{-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \nabla p \|_{L_{t}^{2} L_{\infty}^{2}} \lesssim C_{1} \varepsilon^{2}.$$

Proof. We start from (4.3) together with the facts div $z_{\pm} = 0$ and the boundary conditions $z_{+}^{3}|_{x_{3}=\pm\delta} = 0$, $z_{-}^{3}|_{x_{3}=\pm\delta} = 0$. Applying derivatives $\partial_{h}^{\alpha_{h}}$ to ∇p and integrating by parts, we obtain

$$\partial_{h}^{\alpha_{h}} \nabla p(\tau, x) = \int_{\Omega_{\delta}} \partial_{x_{h}}^{\alpha_{h}} \nabla_{x} G_{\delta}(x, y) (\partial_{i} z_{+}^{j} \partial_{j} z_{-}^{i})(\tau, y) dy = \int_{\Omega_{\delta}} \partial_{i} \partial_{j} \partial_{x_{h}}^{\alpha_{h}} \nabla_{x} G_{\delta}(x, y) (z_{+}^{j} z_{-}^{i})(\tau, y) dy$$

$$= (-1)^{|\alpha_{h}|} \int_{\Omega_{\delta}} \partial_{i} \partial_{j} \partial_{y_{h}}^{\alpha_{h}} \nabla_{y} G_{\delta}(x, y) (z_{+}^{j} z_{-}^{i})(\tau, y) dy = -\int_{\Omega_{\delta}} \partial_{i} \partial_{j} G_{\delta}(x, y) \partial_{y_{h}}^{\alpha_{h}} \nabla_{y} (z_{+}^{j} z_{-}^{i})(\tau, y) dy$$

$$= \underbrace{-\int_{\Omega_{\delta}} \partial_{i} \partial_{j} G_{\delta}(x, y) \theta(|x_{h} - y_{h}|) \partial_{h}^{\alpha_{h}} \nabla(z_{+}^{j} z_{-}^{i})(\tau, y) dy}_{\mathbf{L}_{1}^{(\alpha_{h}, 0)}(\tau, x)}$$

$$\underbrace{-\int_{\Omega_{\delta}} \partial_{i} \partial_{j} G_{\delta}(x, y) (1 - \theta(|x_{h} - y_{h}|)) \partial_{h}^{\alpha_{h}} \nabla(z_{+}^{j} z_{-}^{i})(\tau, y) dy}_{\mathbf{L}_{2}^{(\alpha_{h}, 0)}(\tau, x)}$$

$$\underbrace{-\int_{\Omega_{\delta}} \partial_{i} \partial_{j} G_{\delta}(x, y) (1 - \theta(|x_{h} - y_{h}|)) \partial_{h}^{\alpha_{h}} \nabla(z_{+}^{j} z_{-}^{i})(\tau, y) dy}_{\mathbf{L}_{2}^{(\alpha_{h}, 0)}(\tau, x)}$$

$$(4.7)$$

where the smooth cut-off function $\theta(r)$ is chosen so that

$$\theta(r) = \begin{cases} 1, & \text{for } |r| \leq 1, \\ 0, & \text{for } |r| \geq 2. \end{cases}$$

$$(4.8)$$

In view of the property of the cut-off function $\theta(r)$, we derive

$$\delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \nabla p \|_{L_{t}^{2} L_{x}^{2}} \lesssim \delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{1}^{(\alpha_{h},0)} \|_{L_{t}^{2} L_{x}^{2}}$$
$$+ \delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{2}^{(\alpha_{h},0)} \|_{L_{t}^{2} L_{x}^{2}}. \tag{4.9}$$

Estimate of $L_1^{(\alpha_h,0)}$. There holds the following decomposition:

$$\mathbf{L}_{\mathbf{1}}^{(\alpha_{h},0)}(\tau,x) = \sum_{\beta_{h} \leqslant \alpha_{h}} C_{\alpha_{h},\beta_{h}} \underbrace{\int_{\Omega_{\delta}} \partial_{i} \partial_{j} G_{\delta}(x,y) \theta(|x_{h} - y_{h}|) \nabla \left(\partial_{h}^{\alpha_{h} - \beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right)(\tau,y) dy}_{\mathbf{L}_{\mathbf{1}}^{(\beta_{h},0)}(\tau,x)}. \tag{4.10}$$

According to the number of derivatives, we will distinguish two cases:

$$0 \le |\beta_h| \le N - 1$$
 and $N \le |\beta_h| \le |\alpha_h|$.

Case 1: $0 \le |\beta_h| \le N - 1$. Thanks to the fact div $z_+ = 0$ and the boundary condition $z_+^3|_{x_3 = \pm \delta} = 0$, integration by parts similar to (4.7) then yields

$$\mathbf{L}_{\mathbf{1}}^{(\beta_h,0)}(\tau,x) = \underbrace{-\int_{\Omega_{\delta}} \partial_i G_{\delta}(x,y) \theta(|x_h - y_h|) \nabla \left(\partial_h^{\alpha_h - \beta_h} z_+^j \partial_h^{\beta_h} \partial_j z_-^i\right)(\tau,y) dy}_{\mathbf{L}_{\mathbf{1}\mathbf{1}}^{(\beta_h,0)}(\tau,x)}$$

$$\underbrace{-\int_{\Omega_{\delta}} \partial_{i} G_{\delta}(x, y) \partial_{j} \theta(|x_{h} - y_{h}|) \nabla \left(\partial_{h}^{\alpha_{h} - \beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right)(\tau, y) dy}_{\mathbf{L}_{12}^{(\beta_{h}, 0)}(\tau, x)}.$$

$$(4.11)$$

Using definition, we deduce that

$$\begin{split} &\delta^{-\frac{1}{2}} \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{11}^{(\beta,0)}(\tau,x) \|_{L_{t}^{2}L_{x}^{2}} \\ &\leq \\ &\leq \\ &\leq \\ &\leq \\ &\delta^{-\frac{1}{2}} \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \int_{-\delta}^{\delta} \int_{|x_{h}-y_{h}| \leqslant 2} \frac{1}{\delta} \frac{1}{|x_{h}-y_{h}|} |\nabla \left(\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} \partial_{j} z_{-}^{i} \right)(\tau,y) |dy_{h} dy_{3} \|_{L_{t}^{2}L_{x_{h}}^{2}L_{x_{3}}^{2}} \\ &\leq \\ &\leq \\ &\delta^{-\frac{1}{2}} \left(\int_{0}^{\delta} \int_{-\delta}^{\delta} \int_{|x_{h}-y_{h}| \leqslant 2} \frac{1}{\delta} \frac{1}{|x_{h}-y_{h}|} \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} \partial_{j} z_{-}^{i} \right) |\right)(\tau,y) dy_{h} dy_{3} \|_{L_{t}^{2}L_{x_{h}}^{2}L_{x_{3}}^{2}} \\ &= \\ &\delta^{-\frac{3}{2}} \left(\int_{0}^{\delta} \int_{-\delta}^{\delta} \int_{|x_{h}-y_{h}| \leqslant 2} \frac{1}{|x_{h}-y_{h}|} \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} \partial_{j} z_{-}^{i} \right) |\right)(\tau,y) dy_{h} dy_{3} \|_{L_{t}^{2}L_{x_{h}}^{2}L_{x_{3}}^{2}} \\ &= \\ &\delta^{-\frac{3}{2}} \left(\int_{0}^{\delta} \int_{-\delta}^{\delta} \int_{|x_{h}-y_{h}| \leqslant 2} \frac{1}{|x_{h}-y_{h}|} \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} \partial_{j} z_{-}^{i} \right) |\right)(\tau,y) dy_{h} dy_{3} \|_{L_{t}^{2}L_{x_{h}}^{2}} dx_{3} d\tau \right)^{\frac{1}{2}} \\ &= \\ &\delta^{-\frac{3}{2}} \left(\int_{0}^{\delta} \int_{-\delta}^{\delta} \int_{|x_{h}-y_{h}| \leqslant 2} \frac{1}{|x_{h}-y_{h}|} \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} \partial_{j} z_{-}^{i} |\right)(\tau,y) dy_{h} dy_{3} \|_{L_{t}^{2}L_{x_{h}}^{2}}^{2} dx_{3} d\tau \right)^{\frac{1}{2}} \\ &= \\ &\delta^{-\frac{3}{2}} \left(\int_{0}^{\delta} \int_{0}^{\delta} \int_{0}^{\delta} \int_{|x_{h}-y_{h}| \leqslant 2} \frac{1}{|x_{h}-y_{h}|} \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} \partial_{j} z_{-}^{i} |\right) \right) \right) (\tau,y) dy_{h} dy_{3} \|_{L_{t}^{2}L_{x_{h}}^{2}}^{2} dx_{3} d\tau \right)^{\frac{1}{2}} \\ &\leq \\ &\delta^{-1} \left(\int_{0}^{\delta} \int_{0}^{\delta} \int_{0}^{\delta} \int_{|x_{h}-y_{h}| \leqslant 2} \frac{1}{|x_{h}-y_{h}|} \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} \partial_{j} z_{-}^{i} |\right) (\tau,x) dy_{h} dy_{3} \|_{L_{t}^{2}L_{x_{h}}^{2}}^{2} dx_{h}^{2} dy_{5}^{2} dy_{5}^{2} dy_{5}$$

We can continue in this fashion to obtain

$$\delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{12}^{(\beta_{h},0)}(\tau,x) \|_{L_{t}^{2}L_{x}^{2}}
\lesssim \left(\sum_{k \leqslant N+2} \delta^{-\frac{1}{2}} \left(E_{-}^{(k,0)}(z_{-}) \right)^{\frac{1}{2}} + \sum_{k \leqslant N+1} \delta^{-\frac{1}{2}} \left(E_{-}^{(k,0)}(\partial_{3}z_{-}) \right)^{\frac{1}{2}} \right) \cdot \left(\sum_{k \leqslant |\alpha_{h}|+1} \delta^{-\frac{1}{2}} \left(F_{+}^{(k,0)}(z_{+}) \right)^{\frac{1}{2}} + \sum_{k \leqslant |\alpha_{h}|} \delta^{-\frac{1}{2}} \left(F_{+}^{(k,0)}(\partial_{3}z_{+}) \right)^{\frac{1}{2}} \right) \\
\lesssim C_{1} \varepsilon^{2}. \tag{4.13}$$

In this case, combining (4.11), (4.12) and (4.13), we are able to derive

$$\delta^{-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{\mathbf{1}}^{(\beta_{h},0)}(\tau,x) \|_{L_{t}^{2}L_{x}^{2}} \lesssim C_{1} \varepsilon^{2}.$$
(4.14)

Case 2: $N \leq |\beta_h| \leq |\alpha_h|$. By virtue of div $z_- = 0$ and $z_-^3|_{x_3 = \pm \delta} = 0$, integration by parts gives

$$\mathbf{L}_{\mathbf{1}}^{(\beta_{h},0)}(\tau,x) = \underbrace{-\int_{\Omega_{\delta}} \partial_{j} G_{\delta}(x,y) \theta(|x_{h}-y_{h}|) \nabla \left(\partial_{h}^{\alpha_{h}-\beta_{h}} \partial_{i} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right)(\tau,y) dy}_{\mathbf{L}_{\mathbf{13}}^{(\beta_{h},0)}(\tau,x)} \\ - \underbrace{-\int_{\Omega_{\delta}} \partial_{j} G_{\delta}(x,y) \partial_{i} \theta(|x_{h}-y_{h}|) \nabla \left(\partial_{h}^{\alpha_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right)(\tau,y) dy}_{\mathbf{L}_{\mathbf{14}}^{(\beta_{h},0)}(\tau,x)}.$$

In the same manner, we can see that

$$\delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{13}^{(\beta_{h},0)}(\tau,x) \|_{L_{t}^{2}L_{x}^{2}} \\
\lesssim \left(\sum_{k \leqslant |\alpha_{h}|+1} \delta^{-\frac{1}{2}} \left(E_{-}^{(k,0)}(z_{-}) \right)^{\frac{1}{2}} + \sum_{k \leqslant |\alpha_{h}|} \delta^{-\frac{1}{2}} \left(E_{-}^{(k,0)}(\partial_{3}z_{-}) \right)^{\frac{1}{2}} \right) \cdot \left(\sum_{k \leqslant 5} \delta^{-\frac{1}{2}} \left(F_{+}^{(k,0)}(z_{+}) \right)^{\frac{1}{2}} + \sum_{k \leqslant 4} \delta^{-\frac{1}{2}} \left(F_{+}^{(k,0)}(\partial_{3}z_{+}) \right)^{\frac{1}{2}} \right) \\
\lesssim C_{1} \varepsilon^{2}, \tag{4.15}$$

and

$$\delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{\mathbf{14}}^{(\beta_{h},0)}(\tau,x) \|_{L_{t}^{2}L_{x}^{2}} \\
\lesssim \left(\sum_{k \leqslant |\alpha_{h}|+1} \delta^{-\frac{1}{2}} \left(E_{-}^{(k,0)}(z_{-}) \right)^{\frac{1}{2}} + \sum_{k \leqslant |\alpha_{h}|} \delta^{-\frac{1}{2}} \left(E_{-}^{(k,0)}(\partial_{3}z_{-}) \right)^{\frac{1}{2}} \right) \cdot \left(\sum_{k \leqslant 4} \delta^{-\frac{1}{2}} \left(F_{+}^{(k,0)}(z_{+}) \right)^{\frac{1}{2}} + \sum_{k \leqslant 3} \delta^{-\frac{1}{2}} \left(F_{+}^{(k,0)}(\partial_{3}z_{+}) \right)^{\frac{1}{2}} \right) \\
\lesssim C_{1} \varepsilon^{2}. \tag{4.16}$$

These estimates (4.11), (4.15) and (4.16) subsequently lead us to

$$\delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{\mathbf{1}}^{(\beta_{h},0)}(\tau,x) \|_{L_{t}^{2}L_{x}^{2}} \lesssim C_{1} \varepsilon^{2}. \tag{4.17}$$

Thus, for all $\beta_h \leqslant \alpha_h$, by noticing $N \geqslant 5$, we conclude from Case 1 (4.14) and Case 2 (4.17) that

$$\delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{\mathbf{1}}^{(\beta_{h},0)}(\tau,x) \|_{L^{2}L^{2}} \lesssim C_{1} \varepsilon^{2}. \tag{4.18}$$

Summing up (4.18) for all $\beta_h \leq \alpha_h$, we can summarize that

$$\delta^{-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{\mathbf{1}}^{(\alpha_{h},0)}(\tau,x) \|_{L_{\tau}^{2} L_{x}^{2}} \lesssim C_{1} \varepsilon^{2}.$$

$$(4.19)$$

Estimate of $L_2^{(\alpha_h,0)}$. Based on the number of derivatives, we will distinguish two cases:

$$1 \leqslant |\alpha_h| \leqslant N + 2$$
 and $|\alpha_h| = 0$.

Case 1: $1 \le |\alpha_h| \le N + 2$. We can use integration by parts to split $\mathbf{L}_{\mathbf{2}}^{(\alpha_h,0)}$ as follows: for all $\gamma_h \le \alpha_h$ with $|\gamma_h| = 1$, there holds

$$\mathbf{L}_{\mathbf{2}}^{(\alpha_{h},0)}(\tau,x) = -\int_{\Omega_{\delta}} \partial_{i}\partial_{j}G_{\delta}(x,y) \left(1 - \theta(|x_{h} - y_{h}|)\right) \partial_{h}^{\gamma_{h}} \partial_{h}^{\alpha_{h} - \gamma_{h}} \nabla(z_{+}^{j}z_{-}^{i})(\tau,y) dy$$

$$= -\int_{\Omega_{\delta}} \partial_{h}^{\gamma_{h}} \partial_{i}\partial_{j}G_{\delta}(x,y) \left(1 - \theta(|x_{h} - y_{h}|)\right) \partial_{h}^{\alpha_{h} - \gamma_{h}} \nabla(z_{+}^{j}z_{-}^{i})(\tau,y) dy$$

$$-\int_{\Omega_{\delta}} \partial_{i}\partial_{j}G_{\delta}(x,y) \partial_{h}^{\gamma_{h}} \theta(|x_{h} - y_{h}|) \partial_{h}^{\alpha_{h} - \gamma_{h}} \nabla(z_{+}^{j}z_{-}^{i})(\tau,y) dy.$$

$$\mathbf{L}_{\mathbf{22}}^{(\alpha_{h} - \gamma_{h},0)}(\tau,x)$$

$$(4.20)$$

On one hand, it is evident that the integral for y in $\mathbf{L}_{22}^{(\alpha_h - \gamma_h, 0)}$ exists when $y \in \{|x_h - y_h| \leq 2\} \times (-\delta, \delta)$. We can apply the argument for $\mathbf{L}_{1}^{(\alpha_h, 0)}$ as (4.10)-(4.19) again, with $\theta(|x_h - y_h|)$ replaced by $\partial_{y_h}^{\gamma_h} \theta(|x_h - y_h|)$ and α_h by $\alpha_h - \gamma_h$, to derive

$$\delta^{-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{22}^{(\alpha_{h}-\gamma_{h},0)}(\tau,x) \|_{L_{t}^{2}L_{x}^{2}} \lesssim C_{1} \varepsilon^{2}.$$
(4.21)

On the other hand, to evaluate $\mathbf{L}_{21}^{(\alpha_h-\gamma_h,0)}$, we can further decompose it as

$$\mathbf{L}_{\mathbf{21}}^{(\alpha_h - \gamma_h, 0)}(\tau, x) = \sum_{\beta_h \leqslant \alpha_h - \gamma_h} C_{\alpha_h - \gamma_h, \beta_h} \underbrace{\int_{\Omega_{\delta}} \partial_h^{\gamma_h} \partial_i \partial_j G_{\delta}(x, y) (1 - \theta(|x_h - y_h|)) \nabla \left(\partial_h^{\alpha_h - \gamma_h - \beta_h} z_+^j \partial_h^{\beta_h} z_-^i\right) (\tau, y) dy}_{\mathbf{L}_{\mathbf{21}}^{(\beta_h, 0)}(\tau, x)}.$$

Every term $\mathbf{L}_{21}^{(\beta_h,0)}$ therein can be handled in much the same way as $\mathbf{L}_{11}^{(\beta_h,0)}$ and $\mathbf{L}_{12}^{(\beta_h,0)}$:

$$\begin{split} &\delta^{-\frac{1}{2}} \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{2\mathbf{1}}^{(\beta_{h},0)}(\tau,x) \|_{L_{t}^{2}L_{x}^{2}} \\ & \underset{\lesssim}{\text{Lemma 4.1}} \delta^{-\frac{1}{2}} \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \int_{-\delta}^{\delta} \int_{|x_{h}-y_{h}| \geqslant 1} \frac{1}{\delta \frac{1}{|x_{h}-y_{h}|^{3}}} |\nabla \left(\partial_{h}^{\alpha_{h}-\gamma_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right)(\tau,y) |dy_{h} dy_{3} \|_{L_{t}^{2}L_{x_{h}}^{2}L_{x_{3}}^{2}} \\ & \underset{\lesssim}{\text{(2.8)}} \delta^{-\frac{1}{2}} \| \int_{-\delta}^{\delta} \int_{|x_{h}-y_{h}| \geqslant 1} \frac{1}{\delta \frac{1}{|x_{h}-y_{h}|^{3-\frac{3}{2}(1+\sigma)}} \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\gamma_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right) |\right)(\tau,y) dy_{h} dy_{3} \|_{L_{t}^{2}L_{x_{h}}^{2}L_{x_{3}}^{2}} \\ & \underset{\lesssim}{\text{Minkowski}} \delta^{-1} \left(\int_{0}^{t} \int_{-\delta}^{\delta} \| \int_{|x_{h}-y_{h}| \geqslant 1} \frac{1}{|x_{h}-y_{h}|^{3-\frac{3}{2}(1+\sigma)}} \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\gamma_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right) |\right)(\tau,y) dy_{h} \|_{L_{t}^{2}L_{x_{h}}^{2}L_{x_{3}}^{2}}^{2} \\ & \underset{\lesssim}{\text{Soing}} \delta^{-1} \left(\int_{0}^{t} \int_{-\delta}^{\delta} \| \frac{1}{|x_{h}|^{3-\frac{3}{2}(1+\sigma)}} \chi_{|x_{h}| \geqslant 1} \|_{L_{t}^{2}L_{x_{h}}^{2}}^{2} \| \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\gamma_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right) |\right)(\tau,x_{h},y_{3}) \|_{L_{t}^{2}L_{x_{h}}^{2}L_{x_{3}}^{2}}^{2} \\ & \lesssim \delta^{-1} \| \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\gamma_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right) |\right)(\tau,x_{h},y_{3}) \|_{L_{t}^{2}L_{x_{h}}^{2}L_{y_{3}}^{2}}^{2} \\ & \lesssim \delta^{-1} \| \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\gamma_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right) |\right)(\tau,x_{h},y_{3}) \|_{L_{t}^{2}L_{x_{h}}^{2}L_{y_{3}}^{2}}^{2} \\ & \lesssim \delta^{-1} \| \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} |\nabla \left(\partial_{h}^{\alpha_{h}-\gamma_{h}-\beta_{h}} z_{+}^{j} \partial_{h}^{\beta_{h}} z_{-}^{i}\right) |\right)(\tau,x_{h},y_{3}) \|_{L_{t}^{2}L_{x_{h}}^{2}L_{y_{3}}^{2}}^{2} \\ & \lesssim \delta^{-1} \| \left(\langle u_{-} \rangle^{1+\sigma} \partial_{h}^{\beta_{h}} \nabla z_{-}^{i} \|_{L_{t}^{\infty}L_{x_{h}}^{2}L_{x_{3}}^{2}} \cdot \| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)}} \partial_{h}^{\alpha_{h}-\gamma_{h}-\beta_{h}} z_{+}^{j} \|_{L_{t}^{2}L_{x_{h}}^{2}L_{x_{3$$

Consequently, we obtain

$$\delta^{-\frac{1}{2}} \| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{21}^{(\alpha_{h}-\gamma_{h},0)}(\tau,x) \|_{L_{\tau}^{2}L_{z}^{2}} \lesssim C_{1} \varepsilon^{2}.$$
(4.22)

Therefore, combining (4.20), (4.21) and (4.22) gives rise to

$$\delta^{-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{\mathbf{2}}^{(\alpha_{h},0)}(\tau,x) \|_{L^{2}L^{2}} \lesssim C_{1} \varepsilon^{2}. \tag{4.23}$$

Case 2: $|\alpha_h| = 0$. Using integration by parts, we see that

$$\mathbf{L_{2}}^{(\alpha_{h},0)}(\tau,x) = \underbrace{-\int_{\Omega_{\delta}} \nabla \partial_{i} \partial_{j} G_{\delta}(x,y) \left(1 - \theta(|x_{h} - y_{h}|)\right) (z_{+}^{j} z_{-}^{i})(\tau,y) dy}_{\mathbf{L_{23}}(\tau,x)} - \underbrace{\int_{\Omega_{\delta}} \partial_{i} \partial_{j} G_{\delta}(x,y) \nabla \theta(|x_{h} - y_{h}|) (z_{+}^{j} z_{-}^{i})(\tau,y) dy}_{\mathbf{L_{24}}(\tau,x)}.$$

On one hand, by the same method used to derive (4.22), we can obtain the following estimate of L_{23} :

$$\delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{23}(\tau, x) \|_{L_{*}^{2} L_{\infty}^{2}} \lesssim C_{1} \varepsilon^{2}.$$

On the other hand, similar arguments of $\mathbf{L}_{\mathbf{1}}^{(\alpha_h,0)}$ can be applied here to $\mathbf{L}_{\mathbf{24}}$ and hence

$$\delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{\mathbf{24}}(\tau,x) \|_{L_{\tau}^{2}L_{x}^{2}} \lesssim C_{1} \varepsilon^{2}.$$

In this case, gathering the above estimates together shows

$$\delta^{-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{\mathbf{2}}^{(\alpha_{h},0)}(\tau,x) \|_{L_{t}^{2} L_{x}^{2}} \lesssim C_{1} \varepsilon^{2}.$$

$$(4.24)$$

In consequence, for all α_h with $0 \le |\alpha_h| \le N+2$, we derive from Case 1 (4.23) and Case 2 (4.24) that

$$\delta^{-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{L}_{\mathbf{2}}^{(\alpha_{h},0)}(\tau,x) \|_{L_{t}^{2}L_{x}^{2}} \lesssim C_{1} \varepsilon^{2}.$$
(4.25)

Up to now, according to (4.9), (4.19) and (4.25), we obtain for all $0 \le |\alpha_h| \le N+2$ that

$$\delta^{-\frac{1}{2}} \|\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \nabla p(\tau, x) \|_{L_{t}^{2} L_{x}^{2}} \lesssim C_{1} \varepsilon^{2}.$$

This completes the proof of the lemma.

Lemma 4.3. For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ and any $l \in \mathbb{Z}_{\geq 1}$ with $1 \leq |\alpha_h| + l \leq N + 2$, there holds

$$\delta^{l-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} \nabla p \|_{L_{t}^{2}L^{2}} \lesssim C_{1} \varepsilon^{2}.$$

Proof. Applying the weighted div-curl lemma to the vector field $\partial_h^{\alpha_h} \nabla p$ with the weight $\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)}$, we have

$$\left\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}\nabla p\right\|_{L_{x}^{2}}\lesssim \left\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\nabla^{l}\partial_{h}^{\alpha_{h}}\nabla p\right\|_{L_{x}^{2}}$$

$$\lesssim \sum_{l_1=0}^{l-1} \left\| \langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \operatorname{div} \partial_h^{\alpha_h} \nabla^{l_1} \nabla p \right\|_{L_x^2} + \sum_{l_1=0}^{l-1} \left\| \langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \operatorname{curl} \partial_h^{\alpha_h} \nabla^{l_1} \nabla p \right\|_{L_x^2}$$

+
$$\|\langle u_-\rangle^{1+\sigma}\langle u_+\rangle^{\frac{1}{2}(1+\sigma)}\partial_h^{\alpha_h}\nabla p\|_{L_x^2}$$

$$+\sum_{l_1=0}^{l-1} \Big| \int_{\partial\Omega_{\delta}} \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \Big(\nabla^{l_1} \partial_h^{\alpha_h} (\nabla p)^h \cdot \nabla^{l_1} \nabla_h \partial_h^{\alpha_h} (\nabla p)^3 - \nabla^{l_1} \partial_h^{\alpha_h} (\nabla p)^3 \cdot \nabla^{l_1} \nabla_h \partial_h^{\alpha_h} (\nabla p)^h \Big) dx_h \Big|.$$

For the terms on the right hand side, we notice that

$$\operatorname{div} \partial_h^{\alpha_h} \nabla^{l_1} \nabla p = \partial_h^{\alpha_h} \nabla^{l_1} \underbrace{\operatorname{div} \nabla p}_{=\Delta p} \stackrel{(4.1)_1}{=} -\partial_h^{\alpha_h} \nabla^{l_1} (\nabla z_+ \cdot \nabla z_-),$$

$$\operatorname{curl} \partial_h^{\alpha_h} \nabla^{l_1} \nabla p = \partial_h^{\alpha_h} \nabla^{l_1} \underbrace{\operatorname{curl} \nabla p}_{=0} = 0,$$

$$(\nabla p)^{3}\big|_{\partial\Omega_{\delta}} = (\nabla p)^{3}\big|_{x_{3}=\pm\delta} \stackrel{(2.3)_{1}}{=} - \left(\partial_{t}z_{+} + (z_{-} - B_{0}) \cdot \nabla z_{+}\right)^{3}\big|_{x_{3}=\pm\delta} \stackrel{(2.3)_{4}}{=} 0.$$

Hence we obtain

$$\begin{split} & \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} \nabla p \right\|_{L_{x}^{2}} \\ & \lesssim \sum_{l_{1}=0}^{l-1} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \nabla^{l_{1}} (\nabla z_{+} \cdot \nabla z_{-}) \right\|_{L_{x}^{2}} + \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \nabla p \right\|_{L_{x}^{2}}. \end{split}$$

It then follows that

$$\begin{split} &\delta^{l-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} \nabla p \right\|_{L_{x}^{2}} \\ &\lesssim \sum_{|\alpha_{h}^{\prime}|+l_{1} \leqslant l-1} \delta^{l_{1}+\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \partial_{h}^{\alpha_{h}^{\prime}} \partial_{3}^{l_{1}} (\nabla z_{+} \cdot \nabla z_{-}) \right\|_{L_{x}^{2}} + \delta^{-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \nabla p \right\|_{L_{x}^{2}} \\ &\lesssim C_{1} \varepsilon^{2}. \end{split}$$

We have thus proved this lemma.

It is trivial to see that combining the preceding two lemmas yields the pressure estimates for all $\partial_h^{\alpha_h} \partial_3^1 \nabla p$ with the coefficient $\delta^{l-\frac{1}{2}}$.

Lemma 4.4. For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ and any $l \in \mathbb{Z}_{\geq 0}$ with $0 \leq |\alpha_h| + l \leq N+2$, there holds $\delta^{l-\frac{1}{2}} \|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \partial_h^{\alpha_h} \partial_3^l \nabla p \|_{L^2L^2} \lesssim C_1 \varepsilon^2.$

In particular, using the notations in Section 2, we have a direct renormalization corollary of Lemma 4.4. We point out that this corollary will be used in the approximation part of Section 7.

Corollary 4.5. For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ and any $l \in \mathbb{Z}_{\geq 0}$ with $0 \leq |\alpha_h| + l \leq N + 2$, there holds

$$\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}\nabla_{\delta}p_{(\delta)}\|_{L_{t}^{2}L_{x}^{2}} \lesssim C_{1}\varepsilon^{2}.$$

To end this section, we give the pressure estimates for $\partial_h^{\alpha_h} \partial_3^l \partial_3 \nabla p$ with the lower order coefficient $\delta^{l-\frac{1}{2}}$.

Lemma 4.6. For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ and any $l \in \mathbb{Z}_{\geq 0}$ with $0 \leq |\alpha_h| + l \leq N + 2$, there holds

$$\delta^{l-\frac{1}{2}} \|\langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \partial_h^{\alpha_h} \partial_3^l \partial_3 \nabla p \|_{L^2_* L^2_-} \lesssim C_1 \varepsilon^2.$$

Proof. The almost same reasoning used in Lemma 4.3 applies to this lower order coefficient case. Hence we have

$$\begin{aligned} & \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} \partial_{3} \nabla p \right\|_{L_{x}^{2}} \\ & \lesssim \sum_{l_{1}=0}^{l} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \nabla^{l_{1}} (\nabla z_{+} \cdot \nabla z_{-}) \right\|_{L_{x}^{2}} + \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \nabla p \right\|_{L_{x}^{2}}, \end{aligned}$$

which together with Lemma 3.1 and Lemma 4.2 implies that

$$\delta^{l-\frac{1}{2}} \big\| \langle u_- \rangle^{1+\sigma} \langle u_+ \rangle^{\frac{1}{2}(1+\sigma)} \partial_h^{\alpha_h} \partial_3^l \partial_3 \nabla p \big\|_{L^2}$$

$$\lesssim \sum_{|\alpha_h'|+l_1\leqslant l} \delta^{l_1+\frac{1}{2}} \|\langle u_-\rangle^{1+\sigma} \langle u_+\rangle^{\frac{1}{2}(1+\sigma)} \partial_h^{\alpha_h} \partial_h^{l_1} \langle u_+\rangle^{\frac{1}{2}(1+\sigma)} \partial_h^{\alpha_h} \partial_h^{l_1} \langle v_-\rangle^{1+\sigma} \langle v_+\rangle^{\frac{1}{2}(1+\sigma)} \partial_h^{\alpha_h} \nabla p \|_{L_x^2} \lesssim C_1 \varepsilon^2.$$

This proves the lemma.

5. Construction of scattering fields in Ω_{δ}

In this section, we will study the scattering fields associated to the solution $(z_+(t,x),z_-(t,x))$ constructed in Theorem 2.5 for the system (2.3).

Towards this goal, we first define the infinities. Given a point $(0, x_1, x_2, x_3) \in \{t = 0\} \times \Omega_{\delta}$, it determines uniquely a left-traveling straight line ℓ_{-} and a right-traveling straight line ℓ_{+} :

$$\ell_{-}: \mathbb{R} \to \mathbb{R} \times \Omega_{\delta}, \quad t \mapsto (t, x_{1} + t, x_{2}, x_{3}),$$

 $\ell_{+}: \mathbb{R} \to \mathbb{R} \times \Omega_{\delta}, \quad t \mapsto (t, x_{1} - t, x_{2}, x_{3}).$

 $\ell_+: \mathbb{R} \to \mathbb{R} \times \Omega_{\delta}, \ \ t \mapsto (t, x_1 - t, x_2, x_3).$ It is clear that $u_-\big|_{\ell_-} \equiv x_1$ and $u_+\big|_{\ell_+} \equiv x_1$. We also denote the line ℓ_- by $\ell_-(u_-, x_2, x_3)$ where u_-, x_2 and x_3 are constants, and denote the line ℓ_+ by $\ell_+(u_+, x_2, x_3)$ where u_+, x_2 and x_3 are constants. We use \mathcal{C}_+ to denote the collection of all the left-traveling lines and \mathcal{C}_{-} to denote the collection of all the right-traveling lines:

$$C_{+} = \{ \ell_{-}(u_{-}, x_{2}, x_{3}) | t = \infty, (u_{-}, x_{2}, x_{3}) \in \Omega_{\delta} \},$$

$$C_{-} = \{ \ell_{+}(u_{+}, x_{2}, x_{3}) | t = \infty, (u_{+}, x_{2}, x_{3}) \in \Omega_{\delta} \}.$$

We call C_+ the left infinity and C_- the right infinity respectively, and in fact they are the spaces where the scattering fields live. These descriptions can be easily seen from the following Figure 2.

We notice that the first two equations in (2.3) can be written as

$$\begin{cases} \frac{d}{dt} (z_{+}(t, u_{-} - t, x_{2}, x_{3})) = -\nabla p(t, u_{-} - t, x_{2}, x_{3}) - (z_{-} \cdot \nabla z_{+})(t, u_{-} - t, x_{2}, x_{3}), \\ \frac{d}{dt} (z_{-}(t, u_{+} + t, x_{2}, x_{3})) = -\nabla p(t, u_{+} + t, x_{2}, x_{3}) - (z_{+} \cdot \nabla z_{-})(t, u_{+} + t, x_{2}, x_{3}). \end{cases}$$

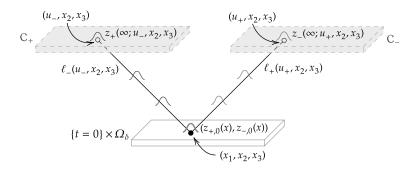


FIGURE 2. Infinities and scattering fields

Integrating these two equations along $\ell_{-}(u_{+}, x_{2}, x_{3})$ and $\ell_{+}(u_{-}, x_{2}, x_{3})$ respectively from time 0 to time t leads us to

$$\begin{cases} z_{+}(t, u_{-} - t, x_{2}, x_{3}) = z_{+}(0, u_{-}, x_{2}, x_{3}) - \int_{0}^{t} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau, \\ z_{-}(t, u_{+} + t, x_{2}, x_{3}) = z_{-}(0, u_{+}, x_{2}, x_{3}) - \int_{0}^{t} (\nabla p + z_{+} \cdot \nabla z_{-})(\tau, u_{+} + \tau, x_{2}, x_{3}) d\tau. \end{cases}$$

When $t \to \infty$, it is natural to expect that the following two formulas can define the scattering fields $\delta^{-\frac{1}{2}}z_{+}(\infty; u_{-}, x_{2}, x_{3})$ (on \mathcal{C}_{+}) and $\delta^{-\frac{1}{2}}z_{-}(\infty; u_{+}, x_{2}, x_{3})$ (on \mathcal{C}_{-}) respectively:

$$\begin{cases}
\delta^{-\frac{1}{2}}z_{+}(\infty; u_{-}, x_{2}, x_{3}) := \delta^{-\frac{1}{2}}z_{+}(0, u_{-}, x_{2}, x_{3}) - \delta^{-\frac{1}{2}} \int_{0}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau, \\
\delta^{-\frac{1}{2}}z_{-}(\infty; u_{+}, x_{2}, x_{3}) := \delta^{-\frac{1}{2}}z_{-}(0, u_{+}, x_{2}, x_{3}) - \delta^{-\frac{1}{2}} \int_{0}^{\infty} (\nabla p + z_{+} \cdot \nabla z_{-})(\tau, u_{+} + \tau, x_{2}, x_{3}) d\tau.
\end{cases} (5.1)$$

In what follows, we turn to prove this expectation valid and study the properties of scattering fields. The first main theorem of this paper is as follows:

Theorem 5.1 (Scattering fields in Ω_{δ}). All the integrals in (5.1) converge. Therefore the vector fields $\delta^{-\frac{1}{2}}z_{+}(\infty; u_{-}, x_{2}, x_{3})$ and $\delta^{-\frac{1}{2}}z_{-}(\infty; u_{+}, x_{2}, x_{3})$ are well-defined by (5.1), and we call them the left scattering field and the right scattering field respectively. Moreover, for any $\alpha_{h} \in (\mathbb{Z}_{\geq 0})^{2}$ and $l \in \mathbb{Z}_{\geq 0}$ with $0 \leq |\alpha_{h}| + l \leq N + 2$, there hold the following two properties of scattering fields:

(i) these scattering fields live in the following functional spaces in the weighted energy sense:

$$\begin{split} \delta^{l-\frac{1}{2}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}z_{\pm}(\infty;u_{\mp},x_{2},x_{3}) &\in L^{2}(\mathcal{C}_{\pm},\langle u_{\mp}\rangle^{2(1+\sigma)}du_{\mp}dx_{2}dx_{3}), \\ \delta^{-\frac{3}{2}}\partial_{h}^{\alpha_{h}}z_{\pm}^{3}(\infty;u_{\mp},x_{2},x_{3}) &\in L^{2}(\mathcal{C}_{\pm},\langle u_{\mp}\rangle^{2(1+\sigma)}du_{\mp}dx_{2}dx_{3}), \\ \delta^{l-\frac{1}{2}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\partial_{3}z_{\pm})(\infty;u_{\mp},x_{2},x_{3}) &\in L^{2}(\mathcal{C}_{\pm},\langle u_{\mp}\rangle^{2(1+\sigma)}du_{\mp}dx_{2}dx_{3}). \end{split}$$

(ii) these scattering fields can be approximated by the large time solution in the weighted energy sense:

$$\lim_{T \to \infty} \left\| \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm}(\infty; u_{\mp}, x_2, x_3) - \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm}(T, u_{\mp} \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3)} = 0,$$

$$\lim_{T \to \infty} \left\| \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm}^3(\infty; u_{\mp}, x_2, x_3) - \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm}^3(T, u_{\mp} \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3)} = 0,$$

$$\lim_{T \to \infty} \left\| \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_{\pm})(\infty; u_{\mp}, x_2, x_3) - \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_{\pm})(T, u_{\mp} \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3)} = 0,$$

We remark here that based on Corollary 2.7 and the notations before, Theorem 5.1 also leads to an immediate consequence via renormalization:

Corollary 5.2 (Scattering fields in Ω_1). Let $(z_{+(\delta)}(t,x), z_{-(\delta)}(t,x))$ be the solution constructed in Corollary 2.7 for the rescaled system (2.4). The vector fields $z_{+(\delta)}(\infty; u_-, x_2, x_3)$ (on the corresponding infinity $\mathcal{C}_+ = \{\ell_-(u_-, x_2, x_3) | t = \infty, (u_-, x_2, x_3) \in \Omega_1\}$) and $z_{-(\delta)}(\infty; u_+, x_2, x_3)$ (on the corresponding infinity $\mathcal{C}_- = \{\ell_-(u_-, x_2, x_3) | t = \infty, (u_-, x_2, x_3) \in \Omega_1\}$)

 $\{\ell_+(u_+,x_2,x_3) | t = \infty, (u_+,x_2,x_3) \in \Omega_1\}$) are well-defined by the following (5.2):

$$\begin{cases}
z_{+(\delta)}(\infty; u_{-}, x_{2}, x_{3}) = z_{+(\delta)}(0, u_{-}, x_{2}, x_{3}) - \int_{0}^{\infty} (\nabla_{\delta} p_{(\delta)} + z_{-(\delta)} \cdot \nabla z_{+(\delta)})(\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau, \\
z_{-(\delta)}(\infty; u_{+}, x_{2}, x_{3}) = z_{-(\delta)}(0, u_{+}, x_{2}, x_{3}) - \int_{0}^{\infty} (\nabla_{\delta} p_{(\delta)} + z_{+(\delta)} \cdot \nabla z_{-(\delta)})(\tau, u_{+} + \tau, x_{2}, x_{3}) d\tau.
\end{cases} (5.2)$$

and we also call them the left scattering field and the right scattering field respectively. Moreover, for any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ and $l \in \mathbb{Z}_{\geq 0}$ with $0 \leq |\alpha_h| + l \leq N + 2$, there hold the following two properties of scattering fields:

(i) these scattering fields live in the following functional spaces in the weighted energy sense:

$$\partial_h^{\alpha_h} \partial_3^l z_{\pm(\delta)}(\infty; u_{\mp}, x_2, x_3) \in L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3),$$

$$\partial_h^{\alpha_h} z_{\pm(\delta)}^3(\infty; u_{\mp}, x_2, x_3) \in L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3),$$

$$\partial_h^{\alpha_h} \partial_3^l (\partial_3 z_{\pm(\delta)})(\infty; u_{\mp}, x_2, x_3) \in L^2(\mathcal{C}_{+}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3).$$

(ii) these scattering fields can be approximated by the large time solution in the weighted energy sense:

$$\lim_{T \to \infty} \left\| \partial_h^{\alpha_h} \partial_3^l z_{\pm(\delta)}(\infty; u_{\mp}, x_2, x_3) - \partial_h^{\alpha_h} \partial_3^l z_{\pm(\delta)}(T, u_{\mp} \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3)} = 0,$$

$$\lim_{T \to \infty} \left\| \partial_h^{\alpha_h} z_{\pm(\delta)}^3(\infty; u_{\mp}, x_2, x_3) - \partial_h^{\alpha_h} z_{\pm(\delta)}^3(T, u_{\mp} \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3)} = 0,$$

$$\lim_{T \to \infty} \left\| \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_{\pm(\delta)})(\infty; u_{\mp}, x_2, x_3) - \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_{\pm(\delta)})(T, u_{\mp} \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3)} = 0.$$

Remark 5.3. When $\delta = 0$, the domain Ω_{δ} naturally becomes \mathbb{R}^2 and we can rewrite (5.2) as

$$\begin{cases}
z_{+(0)}(\infty; u_{-}, x_{2}) = z_{+(0)}(0, u_{-}, x_{2}) - \int_{0}^{\infty} (\nabla p_{(0)} + z_{-(0)} \cdot \nabla z_{+(0)})(\tau, u_{-} - \tau, x_{2}) d\tau, \\
z_{-(0)}(\infty; u_{+}, x_{2}) = z_{-(0)}(0, u_{+}, x_{2}) - \int_{0}^{\infty} (\nabla p_{(0)} + z_{+(0)} \cdot \nabla z_{-(0)})(\tau, u_{+} + \tau, x_{2}) d\tau.
\end{cases} (5.3)$$

It is worth noting that in \mathbb{R}^2 , $z_{+(0)}^h(\infty; u_-, x_2)$ and $z_{-(0)}^h(\infty; u_+, x_2)$, i.e. the 2D version of the scattering fields $z_{+(0)}(\infty; u_-, x_2)$ and $z_{-(0)}(\infty; u_+, x_2)$ constructed in (5.3), coincide with the scattering fields $z_{+}(\infty; u_-, x_2)$ and $z_{-}(\infty; u_+, x_2)$ in (A.2) respectively.

Our task of this section reduces to showing Theorem 5.1. The proof of Theorem 5.1 naturally consists of the following six lemmas.

Lemma 5.4. The scattering fields $\delta^{-\frac{1}{2}}z_{\pm}(\infty; u_{\pm}, x_2, x_3)$ in (5.1) are well-defined.

Proof. By symmetry, it suffices to show that $\delta^{-\frac{1}{2}}z_{+}(\infty; u_{-}, x_{2}, x_{3})$ is well-defined. Using the fact $\langle u_{-} \rangle \geqslant 1$ and the standard Sobolev inequality in \mathbb{R}^{4} , we have

$$\begin{split} &\delta^{-\frac{1}{2}}\langle u_{-}\rangle^{\frac{1}{2}(1+\sigma)}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}|\nabla p|\\ &\lesssim \delta^{-\frac{1}{2}}\big\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\nabla p\big\|_{L_{t,x}^{\infty}}\\ &\lesssim \delta^{-\frac{1}{2}}\big\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\nabla p\big\|_{L_{t,x}^{2}} + \delta^{-\frac{1}{2}}\big\|\nabla^{3}(\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\nabla p)\big\|_{L_{t,x}^{2}}\\ &\lesssim \delta^{-\frac{1}{2}}\big\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\nabla p\big\|_{L_{t,x}^{2}} + \delta^{-\frac{1}{2}}\big\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\nabla^{3}\nabla p\big\|_{L_{t,x}^{2}}\\ &\lesssim \sum_{\substack{0\leqslant |\alpha_{h}|\leqslant 3\\0\leqslant t\leqslant 3}} \delta^{l-\frac{1}{2}}\big\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}\nabla p\big\|_{L_{t}^{2}L_{x}^{2}} &\lesssim C_{1}\varepsilon^{2},\\ &\delta^{-\frac{1}{2}}\langle u_{-}\rangle^{\frac{1}{2}(1+\sigma)}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}|z_{-}\cdot\nabla z_{+}|\\ &\lesssim \delta^{-\frac{1}{2}}\big\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}(z_{-}\cdot\nabla z_{+})\big\|_{L_{t,x}^{\infty}}\\ &\lesssim \delta^{-\frac{1}{2}}\big\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}(z_{-}\cdot\nabla z_{+})\big\|_{L_{t,x}^{2}} + \delta^{-\frac{1}{2}}\big\|\nabla^{3}(\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}(z_{-}\cdot\nabla z_{+}))\big\|_{L_{t,x}^{2}} \end{split}$$

$$\lesssim \delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} (z_{-} \cdot \nabla z_{+}) \|_{L_{t,x}^{2}} + \delta^{-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \nabla^{3} (z_{-} \cdot \nabla z_{+}) \|_{L_{t,x}^{2}}
\lesssim \sum_{0 \leqslant |\alpha_{h}| \leqslant 3 \atop 0 \leqslant l \leqslant 3} \delta^{l-\frac{1}{2}} \|\langle u_{-}\rangle^{1+\sigma} \langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} (z_{-} \cdot \nabla z_{+}) \|_{L_{t}^{2}L_{x}^{2}} \lesssim C_{1} \varepsilon^{2}.$$

Combined with (2.11), these two estimates lead us to

$$\delta^{-\frac{1}{2}}|\nabla p+z_-\cdot\nabla z_+|\lesssim \frac{C_1\varepsilon^2}{\langle u_-\rangle^{\frac{1}{2}(1+\sigma)}\langle u_+\rangle^{\frac{1}{2}(1+\sigma)}}\lesssim \frac{C_1\varepsilon^2}{(1+|t+a|)^{1+\sigma}}\in L^1_t(\mathbb{R}).$$

Therefore the integral in (5.1) converges and hence $\delta^{-\frac{1}{2}}z_{+}(\infty; u_{-}, x_{2}, x_{3})$ is well-defined.

Lemma 5.5. There hold

$$\delta^{-\frac{1}{2}} z_{\pm}(\infty; u_{\mp}, x_2, x_3) \in L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3), \tag{5.4}$$

and

$$\lim_{T \to \infty} \left\| \delta^{-\frac{1}{2}} z_{\pm}(\infty; u_{\mp}, x_2, x_3) - \delta^{-\frac{1}{2}} z_{\pm}(T, u_{\mp} \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_+, \langle u_{\pm} \rangle^{2(1+\sigma)} du_{\pm} dx_2 dx_3)} = 0.$$
 (5.5)

Proof. Based on the symmetry, it suffices to derive the estimates on $z_{+}(\infty; u_{-}, x_{2}, x_{3})$. Firstly, by (2.20), we note that

$$\begin{split} & \left\| \delta^{-\frac{1}{2}} z_{+}(0, u_{-}, x_{2}, x_{3}) \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ &= \delta^{-\frac{1}{2}} \left\| z_{+}(0, u_{-}, x_{2}, x_{3}) \right\|_{L^{2}(\Omega_{\delta}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} = \delta^{-\frac{1}{2}} E_{+}^{(0,0)}(z_{+,0}) \lesssim \varepsilon, \end{split}$$

which shows

$$\delta^{-\frac{1}{2}}z_{+}(0, u_{-}, x_{2}, x_{3}) \in L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3}). \tag{5.6}$$

We also note that for any large T > 0, (2.20) gives

$$\delta^{-\frac{1}{2}}z_{+}(\infty; u_{-}, x_{2}, x_{3}) - \delta^{-\frac{1}{2}}z_{+}(T, u_{-} - T, x_{2}, x_{3}) = -\delta^{-\frac{1}{2}}\int_{T}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3})d\tau. \quad (5.7)$$

Secondly, using coordinate transformations (which allow us to perform the analysis on spacetimes) and Hölder inequality, we obtain

$$\begin{split} & \left\| \delta^{-\frac{1}{2}} \int_{0}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ & \lesssim \delta^{-\frac{1}{2}} \left\| \left(\int_{\mathbb{R}} \frac{1}{\langle u_{+} \rangle^{1+\sigma}} du_{+} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \langle u_{+} \rangle^{1+\sigma} |(\nabla p + z_{-} \cdot \nabla z_{+}) (u_{+}, u_{-}, x_{2}, x_{3})|^{2} du_{+} \right)^{\frac{1}{2}} \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ & \lesssim \delta^{-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} (\nabla p + z_{-} \cdot \nabla z_{+}) (u_{+}, u_{-}, x_{2}, x_{3}) \right\|_{L^{2}(\mathbb{R} \times \Omega_{\delta}, du_{+} du_{-} dx_{2} dx_{3})} \\ & \lesssim \delta^{-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \nabla p \right\|_{L^{2}_{t}L^{2}_{x}} + \delta^{-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} (z_{-} \cdot \nabla z_{+}) \right\|_{L^{2}_{t}L^{2}_{x}} \overset{\text{Lemmas 4.2 \& 3.2}}{\lesssim} C_{1} \varepsilon^{2}. \end{split}$$

This implies

$$\delta^{-\frac{1}{2}} \int_{0}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \in L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3}), \tag{5.8}$$

and

$$\lim_{T \to \infty} \left\| \delta^{-\frac{1}{2}} \int_{T}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} = 0.$$
 (5.9)

Finally, putting (5.1), (5.6) and (5.8) together yields (5.4), while (5.7) and (5.9) give rise to (5.5).

We are now in a position to derive the uniform estimates concerning $\partial_h^{\alpha_h} z_{\pm}(\infty; u_{\mp}, x_2, x_3)$ with the coefficient $\delta^{-\frac{1}{2}}$.

Lemma 5.6. For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ with $1 \leq |\alpha_h| \leq N+2$, there hold

$$\delta^{-\frac{1}{2}}\partial_h^{\alpha_h} z_{\pm}(\infty; u_{\mp}, x_2, x_3) \in L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3),$$

and

$$\lim_{T \to \infty} \left\| \delta^{-\frac{1}{2}} \partial_h^{\alpha_h} z_{\pm}(\infty; u_{\mp}, x_2, x_3) - \delta^{-\frac{1}{2}} \partial_h^{\alpha_h} z_{\pm}(T, u_{\mp} \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3)} = 0.$$

Proof. By the symmetry considerations, we only give details for the estimates on $z_{+}(\infty; u_{-}, x_{2}, x_{3})$. Applying the derivative $\partial_{h}^{\alpha_{h}}$ to (5.1)-(5.7), we have

$$\delta^{-\frac{1}{2}}\partial_h^{\alpha_h}z_+(\infty;u_-,x_2,x_3) = \delta^{-\frac{1}{2}}\partial_h^{\alpha_h}z_+(0,u_-,x_2,x_3) - \delta^{-\frac{1}{2}}\partial_h^{\alpha_h}\int_0^{\infty} (\nabla p + z_- \cdot \nabla z_+)(\tau,u_- - \tau,x_2,x_3)d\tau,$$

and

$$\delta^{-\frac{1}{2}}\partial_{h}^{\alpha_{h}}z_{+}(\infty; u_{-}, x_{2}, x_{3}) - \delta^{-\frac{1}{2}}\partial_{h}^{\alpha_{h}}z_{+}(T, u_{-}, T, x_{2}, x_{3}) = -\delta^{-\frac{1}{2}}\partial_{h}^{\alpha_{h}}\int_{T}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-}, \tau, x_{2}, x_{3})d\tau.$$

$$(5.10)$$

It is clear from (2.20) that

$$\begin{split} & \left\| \delta^{-\frac{1}{2}} \partial_h^{\alpha_h} z_+(0, u_-, x_2, x_3) \right\|_{L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)} \\ &= \delta^{-\frac{1}{2}} \left\| \partial_h^{\alpha_h} z_+(0, u_-, x_2, x_3) \right\|_{L^2(\Omega_\delta, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)} = \delta^{-\frac{1}{2}} E_+^{(|\alpha_h|, 0)}(z_{+, 0}) \lesssim \varepsilon, \end{split}$$

which means

$$\delta^{-\frac{1}{2}}\partial_h^{\alpha_h}z_+(0,u_-,x_2,x_3) \in L^2(\mathcal{C}_+,\langle u_-\rangle^{2(1+\sigma)}du_-dx_2dx_3).$$

Therefore it suffices to show that

$$\delta^{-\frac{1}{2}} \partial_h^{\alpha_h} \int_0^\infty (\nabla p + z_- \cdot \nabla z_+)(\tau, u_- - \tau, x_2, x_3) d\tau \in L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3), \tag{5.11}$$

and

$$\lim_{T \to \infty} \left\| \delta^{-\frac{1}{2}} \partial_h^{\alpha_h} \int_T^{\infty} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \right\|_{L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)} = 0. \tag{5.12}$$

The rest of this proof is divided into four steps.

Step 1: We first prove that

$$\delta^{-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \in L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3}). \tag{5.13}$$

In fact, this can also be proved via coordinate transformations and Hölder inequality:

$$\left\| \delta^{-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})}$$

$$\lesssim \delta^{-\frac{1}{2}} \left\| \left(\int_{\mathbb{R}} \frac{1}{\langle u_{+} \rangle^{1+\sigma}} du_{+} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \langle u_{+} \rangle^{1+\sigma} |\partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (u_{+}, u_{-}, x_{2}, x_{3})|^{2} du_{+} \right)^{\frac{1}{2}} \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})}$$

$$\lesssim \delta^{-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (u_{+}, u_{-}, x_{2}, x_{3}) \right\|_{L^{2}(\mathbb{R} \times \Omega_{\delta}, du_{+} du_{-} dx_{2} dx_{3})}$$

$$\lesssim \delta^{-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \nabla p \right\|_{L^{2}_{t}L^{2}_{x}} + \delta^{-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{J}_{+}^{(\alpha_{h}, 0)} \right\|_{L^{2}_{t}L^{2}_{x}}^{Lemmas} \stackrel{4.2 \& 3.2}{\lesssim C_{1} \varepsilon^{2}}. \tag{5.14}$$

Step 2: We next prove that for any $\gamma_h \leqslant \alpha_h$ with $|\gamma_h| = 1$, there holds

$$\delta^{-\frac{1}{2}}\partial_h^{\gamma_h} \int_0^\infty \partial_h^{\alpha_h - \gamma_h} (\nabla p + z_- \cdot \nabla z_+)(\tau, u_- - \tau, x_2, x_3) d\tau \in L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3). \tag{5.15}$$

In this step, we only consider the case where the outermost derivative of the above term is taken as $\partial_h^{\gamma_h} = \partial_1$, and the ∂_2 case can be treated in the same way. Now it suffices to show that

$$\delta^{-\frac{1}{2}} \partial_1 \int_0^\infty \partial_h^{\alpha_h - \gamma_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \in L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3). \tag{5.16}$$

In fact, by definition, we deduce that

$$\begin{split} & \left\| \delta^{-\frac{1}{2}} \partial_{1} \int_{0}^{\infty} \partial_{h}^{\alpha_{h} - \gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ &= \delta^{-\frac{1}{2}} \left\| \lim_{h \to 0} \int_{0}^{\infty} \frac{1}{h} (\partial_{h}^{\alpha_{h} - \gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} + h - \tau, x_{2}, x_{3}) \right. \\ & \left. - \partial_{h}^{\alpha_{h} - \gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) \right) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ & \stackrel{\text{Fatou}}{\leqslant} \delta^{-\frac{1}{2}} \liminf_{h \to 0} \left\| \int_{0}^{\infty} \frac{1}{h} (\partial_{h}^{\alpha_{h} - \gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} + h - \tau, x_{2}, x_{3}) \right. \\ & \left. - \partial_{h}^{\alpha_{h} - \gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) \right) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ & \stackrel{\text{Newton-Leibniz}}{\leqslant} \delta^{-\frac{1}{2}} \liminf_{h \to 0} \left\| \int_{0}^{\infty} \int_{0}^{1} \partial_{1} \partial_{h}^{\alpha_{h} - \gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} + \theta h - \tau, x_{2}, x_{3}) d\theta d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ & \leqslant \delta^{-\frac{1}{2}} \liminf_{h \to 0} \int_{0}^{1} \int_{0}^{\infty} \partial_{1} \partial_{h}^{\alpha_{h} - \gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} + \theta h - \tau, x_{2}, x_{3}) d\tau d\theta \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ & \leqslant \delta^{-\frac{1}{2}} \liminf_{h \to 0} \int_{0}^{1} \left\| \int_{0}^{\infty} \partial_{1} \partial_{h}^{\alpha_{h} - \gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} + \theta h - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} d\theta \\ & \leqslant \left\| \delta^{-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} d\theta \\ & \leqslant \left\| \delta^{-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} d\theta \\ & \leqslant \left\| \delta^{-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} d\theta \\ & \leqslant \left\| \delta^{-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{$$

Hence (5.16) holds and therefore (5.15) follows.

Step 3: We turn to prove that for any $\gamma_h \leqslant \alpha_h$ with $|\gamma_h| = 1$, as vector fields in $L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)$, there holds

$$\delta^{-\frac{1}{2}} \partial_h^{\gamma_h} \int_0^\infty \partial_h^{\alpha_h - \gamma_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau$$

$$\frac{L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)}{\int_0^\infty \partial_h^{\alpha_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau. \tag{5.17}$$

In view of (5.13) and (5.15), it suffices to show the following equation in the sense of distributions:

$$\delta^{-\frac{1}{2}}\partial_{h}^{\gamma_{h}}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}-\gamma_{h}}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau\xrightarrow{\frac{\mathcal{D}'(\mathcal{C}_{+})}{m}}\delta^{-\frac{1}{2}}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau.$$

$$(5.18)$$

Based on (5.8) and (5.13), both the two time integrals in (5.18) are locally integrable functions. Let us take an arbitrary vector field $\varphi \in \mathcal{D}(\mathcal{C}_+)$. It is clear that $\varphi \in \mathcal{D}(\mathcal{C}_+) \subset L^2(\mathcal{C}_+)$, which also gives $\partial_h^{\gamma_h} \varphi \in \mathcal{D}(\mathcal{C}_+) \subset L^2(\mathcal{C}_+)$. These facts enable us to infer that the following two spacetime integrals are finite:

$$\begin{split} &\delta^{-\frac{1}{2}} \int_{[0,\infty)\times\mathcal{C}_{+}} \left| \partial_{h}^{\alpha_{h}-\gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3}) \cdot \partial_{h}^{\gamma_{h}} \varphi(u_{-}, x_{2}, x_{3}) \right| d\tau du_{-} dx_{2} dx_{3} \\ &\lesssim \delta^{-\frac{1}{2}} \int_{\mathbb{R}\times\mathcal{C}_{+}} \left| \partial_{h}^{\alpha_{h}-\gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+})(u_{+}, u_{-}, x_{2}, x_{3}) \right| \cdot \left| \partial_{h}^{\gamma_{h}} \varphi(u_{-}, x_{2}, x_{3}) \right| du_{+} du_{-} dx_{2} dx_{3} \\ &\lesssim \delta^{-\frac{1}{2}} \left(\int_{\mathbb{R}\times\mathcal{C}_{+}} \langle u_{+} \rangle^{1+\sigma} \left| \partial_{h}^{\alpha_{h}-\gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+})(u_{+}, u_{-}, x_{2}, x_{3}) \right|^{2} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}\times\mathcal{C}_{+}} \frac{\left| \partial_{h}^{\gamma_{h}} \varphi(u_{-}, x_{2}, x_{3}) \right|^{2}}{\langle u_{+} \rangle^{1+\sigma}} \right)^{\frac{1}{2}} \\ &\lesssim \delta^{-\frac{1}{2}} \left(\int_{\mathbb{R}\times\mathcal{C}_{+}} \langle u_{-} \rangle^{2(1+\sigma)} \langle u_{+} \rangle^{1+\sigma} \left| \partial_{h}^{\alpha_{h}-\gamma_{h}} (\nabla p + z_{-} \cdot \nabla z_{+})(u_{+}, u_{-}, x_{2}, x_{3}) \right|^{2} \right)^{\frac{1}{2}} \\ &\times \left(\int_{\mathbb{R}} \frac{1}{\langle u_{+} \rangle^{1+\sigma}} \left(\int_{\mathcal{C}_{+}} \left| \partial_{h}^{\gamma_{h}} \varphi(u_{-}, x_{2}, x_{3}) \right|^{2} du_{-} dx_{2} dx_{3} \right) du_{+} \right)^{\frac{1}{2}} \end{split}$$

$$\lesssim \varepsilon^{2} \|\partial_{h}^{\gamma_{h}} \varphi\|_{L^{2}} < \infty, \tag{5.19}$$

$$\delta^{-\frac{1}{2}} \int_{[0,\infty) \times \mathcal{C}_{+}} |\partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) \cdot \varphi(u_{-}, x_{2}, x_{3}) | d\tau du_{-} dx_{2} dx_{3}$$

$$\lesssim \delta^{-\frac{1}{2}} \int_{\mathbb{R} \times \mathcal{C}_{+}} |\partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (u_{+}, u_{-}, x_{2}, x_{3}) | \cdot |\varphi(u_{-}, x_{2}, x_{3})| du_{+} du_{-} dx_{2} dx_{3}$$

$$\lesssim \delta^{-\frac{1}{2}} \left(\int_{\mathbb{R} \times \mathcal{C}_{+}} \langle u_{+} \rangle^{1+\sigma} |\partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (u_{+}, u_{-}, x_{2}, x_{3})|^{2} \right)^{\frac{1}{2}} \left(\int_{\mathbb{R} \times \mathcal{C}_{+}} \frac{|\varphi(u_{-}, x_{2}, x_{3})|^{2}}{\langle u_{+} \rangle^{1+\sigma}} \right)^{\frac{1}{2}}$$

$$\lesssim \delta^{-\frac{1}{2}} \left(\int_{\mathbb{R} \times \mathcal{C}_{+}} \langle u_{-} \rangle^{2(1+\sigma)} \langle u_{+} \rangle^{1+\sigma} |\partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+}) (u_{+}, u_{-}, x_{2}, x_{3})|^{2} \right)^{\frac{1}{2}}$$

$$\times \left(\int_{\mathbb{R}} \frac{1}{\langle u_{+} \rangle^{1+\sigma}} \left(\int_{\mathcal{C}_{+}} \varphi(u_{-}, x_{2}, x_{3}) |^{2} du_{-} dx_{2} dx_{3} \right) du_{+} \right)^{\frac{1}{2}}$$

$$\lesssim \varepsilon^{2} \|\varphi\|_{L^{2}} < \infty.$$
(5.20)

We remark here that the estimates (5.19)-(5.20) will ensure the subsequent applications of Fubini's theorem to commute the order of integrals.

Using the above estimates, integration by parts and Fubini's theorem repeatedly, we then derive in the sense of distributions that

$$\begin{split} &\left\langle \delta^{-\frac{1}{2}} \partial_h^{\gamma_h} \int_0^\infty \partial_h^{\alpha_h - \gamma_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau, \ \varphi(u_-, x_2, x_3) \right\rangle \\ &= -\delta^{-\frac{1}{2}} \left\langle \int_0^\infty \partial_h^{\alpha_h - \gamma_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau, \ \partial_h^{\gamma_h} \varphi(u_-, x_2, x_3) \right\rangle \\ &= -\delta^{-\frac{1}{2}} \int_{\mathcal{C}_+} \left(\int_0^\infty \partial_h^{\alpha_h - \gamma_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \right) \cdot \partial_h^{\gamma_h} \varphi(u_-, x_2, x_3) du_- dx_2 dx_3 \\ &\stackrel{(5.19)}{=} -\delta^{-\frac{1}{2}} \int_{[0,\infty) \times \mathcal{C}_+} \partial_h^{\alpha_h - \gamma_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) \cdot \partial_h^{\gamma_h} \varphi(u_-, x_2, x_3) d\tau du_- dx_2 dx_3 \\ &= -\delta^{-\frac{1}{2}} \int_{[0,\infty)} \left(\int_{\mathcal{C}_+} \partial_h^{\alpha_h - \gamma_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) \cdot \partial_h^{\gamma_h} \varphi(u_-, x_2, x_3) du_- dx_2 dx_3 \right) d\tau \\ &= \delta^{-\frac{1}{2}} \int_{[0,\infty)} \left(\int_{\mathcal{C}_+} \partial_h^{\alpha_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) \cdot \varphi(u_-, x_2, x_3) du_- dx_2 dx_3 \right) d\tau \\ &= \delta^{-\frac{1}{2}} \int_{[0,\infty) \times \mathcal{C}_+} \partial_h^{\alpha_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) \cdot \varphi(u_-, x_2, x_3) du_- dx_2 dx_3 d\tau \\ &= \delta^{-\frac{1}{2}} \int_{(0,\infty) \times \mathcal{C}_+} \partial_h^{\alpha_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \right) \cdot \varphi(u_-, x_2, x_3) du_- dx_2 dx_3 \\ &= \left\langle \delta^{-\frac{1}{2}} \int_{\mathcal{C}_+} \left(\int_0^\infty \partial_h^{\alpha_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \right) \cdot \varphi(u_-, x_2, x_3) du_- dx_2 dx_3 \right. \\ &= \left\langle \delta^{-\frac{1}{2}} \int_0^\infty \partial_h^{\alpha_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau, \ \varphi(u_-, x_2, x_3) du_- dx_2 dx_3 \right. \\ &= \left\langle \delta^{-\frac{1}{2}} \int_0^\infty \partial_h^{\alpha_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau, \ \varphi(u_-, x_2, x_3) du_- dx_2 dx_3 \right) \right. \\ \end{aligned}$$

This implies (5.18) immediately. Thus we have proved (5.17).

Step 4: We are now ready to show (5.11) and (5.12).

By induction on α_h , we obtain that the following equation holds in the sense of weighted L^2 space $L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)$ as an immediate consequence of (5.17):

$$\delta^{-\frac{1}{2}} \partial_{h}^{\alpha_{h}} \int_{0}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau$$

$$\frac{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})}{\delta^{-\frac{1}{2}}} \delta^{-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau.$$
(5.21)

This proves (5.11).

Moreover, by (5.21) and (5.14), there holds

$$\begin{split} & \left\| \delta^{-\frac{1}{2}} \partial_{h}^{\alpha_{h}} \int_{0}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ & = \left\| \delta^{-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \lesssim C_{1} \varepsilon^{2}. \end{split}$$
 (5.22)

As a direct consequence of (5.21) and (5.22), we obtain

$$\lim_{T \to \infty} \left\| \delta^{-\frac{1}{2}} \partial_h^{\alpha_h} \int_T^{\infty} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \right\|_{L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)}$$

$$= \lim_{T \to \infty} \left\| \delta^{-\frac{1}{2}} \int_T^{\infty} \partial_h^{\alpha_h} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \right\|_{L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)} = 0.$$

Together with (5.10), this gives rise to (5.12).

The proof of this lemma is now complete.

Clearly, the above two lemmas also establish the following lemma as a direct consequence. We remark here that we do not need to consider other terms (such as $\partial_h^{\alpha_h} \partial_3^l z_{\pm}^3$) in this lemma since they can be covered by the subsequent lemmas together with the fact that div $z_{\pm} = 0$ gives $\partial_3 z_{\pm}^h = -\partial_h z_{\pm}^h$.

Lemma 5.7. For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ with $0 \leq |\alpha_h| \leq N+2$, there hold

$$\delta^{-\frac{3}{2}}\partial_h^{\alpha_h} z_+^3(\infty; u_\pm, x_2, x_3) \in L^2(\mathcal{C}_\pm, \langle u_\pm \rangle^{2(1+\sigma)} du_\pm dx_2 dx_3),$$

and

$$\lim_{T \to \infty} \left\| \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm}^3(\infty; u_{\mp}, x_2, x_3) - \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm}^3(T, u_{\mp} \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_{+}, \langle u_{\pm} \rangle^{2(1+\sigma)} du_{\pm} dx_2 dx_3)} = 0.$$

We turn to derive the uniform estimates concerning $\partial_h^{\alpha_h} \partial_3^l z_{\pm}(\infty; u_{\mp}, x_2, x_3)$ with the coefficient $\delta^{l-\frac{1}{2}}$.

Lemma 5.8. For any $\alpha_h \in (\mathbb{Z}_{\geq 0})^2$ and $l \in \mathbb{Z}_{\geq 1}$ with $1 \leq |\alpha_h| + l \leq N + 2$, there hold

$$\delta^{l-\frac{1}{2}}\partial_h^{\alpha_h}\partial_3^l z_\pm(\infty;u_\mp,x_2,x_3)\in L^2(\mathcal{C}_\pm,\langle u_\mp\rangle^{2(1+\sigma)}du_\mp dx_2 dx_3),$$

and

$$\lim_{T \to \infty} \left\| \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm}(\infty; u_{\mp}, x_2, x_3) - \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm}(T, u_{\mp} \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_+, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2 dx_3)} = 0$$

Proof. By the symmetry considerations, it suffices to give details for the estimate on $z_+(\infty; u_-, x_2, x_3)$. Applying the derivative $\partial_h^{\alpha_h} \partial_3^l$ to (5.1) and (5.7) gives rise to

$$\delta^{l-\frac{1}{2}}\partial_h^{\alpha_h}\partial_3^l z_+(\infty;u_-,x_2,x_3)$$

$$= \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_+(0, u_-, x_2, x_3) - \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l \int_0^\infty (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau, \tag{5.23}$$

and

$$\delta^{l-\frac{1}{2}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}z_{+}(\infty; u_{-}, x_{2}, x_{3}) - \delta^{l-\frac{1}{2}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}z_{+}(T, u_{-} - T, x_{2}, x_{3})$$

$$= -\delta^{l-\frac{1}{2}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}\int_{T}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+})(\tau, u_{-} - \tau, x_{2}, x_{3})d\tau.$$

$$(5.24)$$

According to (2.20), we have

$$\begin{split} & \left\| \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_+(0,u_-,x_2,x_3) \right\|_{L^2(\mathcal{C}_+,\langle u_-\rangle^{2(1+\sigma)} du_- dx_2 dx_3)} \\ &= \delta^{l-\frac{1}{2}} \left\| \partial_h^{\alpha_h} \partial_3^l z_+(0,u_-,x_2,x_3) \right\|_{L^2(\Omega_\delta,\langle u_-\rangle^{2(1+\sigma)} du_- dx_2 dx_3)} = \delta^{l-\frac{1}{2}} E_+^{(|\alpha_h|,l)}(z_{+,0}) \lesssim \varepsilon, \end{split}$$

and hence

$$\delta^{l-\frac{1}{2}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}z_{+}(0,u_{-},x_{2},x_{3}) \in L^{2}(\mathcal{C}_{+},\langle u_{-}\rangle^{2(1+\sigma)}du_{-}dx_{2}dx_{3}). \tag{5.25}$$

By virtue of (5.23), (5.24) and (5.25), our task is now reduced to showing that

$$\delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l \int_0^\infty (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \in L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3), \tag{5.26}$$

and

$$\lim_{T \to \infty} \left\| \delta^{-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l \int_T^{\infty} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \right\|_{L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)} = 0.$$
 (5.27)

The rest of this proof is divided into four steps.

Step 1: We first prove that

$$\delta^{l-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \in L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3}). \tag{5.28}$$

In fact, this can be proved via coordinate transformations and Hölder inequality:

$$\left\|\delta^{l-\frac{1}{2}}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau\right\|_{L^{2}(\mathcal{C}_{+},\langle u_{-}\rangle^{2(1+\sigma)}du_{-}dx_{2}dx_{3})}$$

$$\lesssim \delta^{l-\frac{1}{2}}\left\|\left(\int_{\mathbb{R}}\frac{1}{\langle u_{+}\rangle^{1+\sigma}}du_{+}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}}\langle u_{+}\rangle^{1+\sigma}|\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(u_{+},u_{-},x_{2},x_{3})|^{2}du_{+}\right)^{\frac{1}{2}}\right\|_{L^{2}(\mathcal{C}_{+},\langle u_{-}\rangle^{2(1+\sigma)}du_{-}dx_{2}dx_{3})}$$

$$\lesssim \delta^{l-\frac{1}{2}}\left\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(u_{+},u_{-},x_{2},x_{3})\right\|_{L^{2}(\mathbb{R}\times\Omega_{\delta},du_{+}du_{-}dx_{2}dx_{3})}$$

$$\lesssim \delta^{l-\frac{1}{2}}\left\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}\nabla p\right\|_{L^{2}_{*}L^{2}_{x}} + \delta^{l-\frac{1}{2}}\left\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1}{2}(1+\sigma)}\mathbf{J}_{+}^{(\alpha_{h},l)}\right\|_{L^{2}_{*}L^{2}_{x}}^{Lemmas} \stackrel{4.3}{\sim} \stackrel{8}{\sim} \stackrel{3.2}{C_{1}}\varepsilon^{2}. \tag{5.29}$$

Step 2: We show that

$$\delta^{l-\frac{1}{2}}\partial_h^{\gamma_h} \int_0^\infty \partial_h^{\alpha_h - \gamma_h} \partial_3^l (\nabla p + z_- \cdot \nabla z_+)(\tau, u_- - \tau, x_2, x_3) d\tau \in L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)$$
 (5.30)

for any $\gamma_h \leqslant \alpha_h$ with $|\gamma_h| = 1$, and

$$\delta^{l-\frac{1}{2}} \partial_3 \int_0^\infty \partial_h^{\alpha_h} \partial_3^{l-1} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \in L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3). \tag{5.31}$$

We note that (5.30) can be proved by similar methods used for (5.15) and (5.31). Thus it is sufficient to show (5.31) now. In fact, there holds

$$\begin{split} & \left\| \delta^{l-\frac{1}{2}} \partial_{3} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} \partial_{3}^{l-1} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ &= \delta^{l-\frac{1}{2}} \left\| \lim_{h \to 0} \int_{0}^{\infty} \frac{1}{h} (\partial_{h}^{\alpha_{h}} \partial_{3}^{l-1} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3} + h) \\ & - \partial_{h}^{\alpha_{h}} \partial_{3}^{l-1} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) \right) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \end{split}$$
 Fatou
$$- \partial_{h}^{\alpha_{h}} \partial_{3}^{l-1} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3} + h) \\ & - \partial_{h}^{\alpha_{h}} \partial_{3}^{l-1} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) \right) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \end{split}$$
 Newton-Leibniz
$$- \partial_{h}^{\alpha_{h}} \partial_{3}^{l-1} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3} + \theta h) d\theta d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ \leqslant \delta^{l-\frac{1}{2}} \liminf_{h \to 0} \left\| \int_{0}^{1} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3} + \theta h) d\tau d\theta \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ \leqslant \delta^{l-\frac{1}{2}} \liminf_{h \to 0} \int_{0}^{1} \left\| \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3} + \theta h) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} d\theta \\ \leqslant \left\| \delta^{l-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} d\theta \\ \leqslant \left\| \delta^{l-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} d\theta \\ \leqslant \left\| \delta^{l-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} d\theta \\ \leqslant \left\| \delta^{l-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} d\theta \right$$

The proof of this step is complete.

Step 3: We prove that as vector fields in $L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)$, there hold

$$\delta^{l-\frac{1}{2}}\partial_{h}^{\gamma_{h}}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}-\gamma_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau$$

$$\frac{L^{2}(\mathcal{C}_{+},\langle u_{-}\rangle^{2(1+\sigma)}du_{-}dx_{2}dx_{3})}{\delta^{l-\frac{1}{2}}}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau$$
(5.32)

for any $\gamma_h \leqslant \alpha_h$ with $|\gamma_h| = 1$, and

$$\delta^{l-\frac{1}{2}}\partial_{3}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l-1}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau$$

$$=\frac{L^{2}(\mathcal{C}_{+},\langle u_{-}\rangle^{2(1+\sigma)}du_{-}dx_{2}dx_{3})}{2}}{2}\delta^{l-\frac{1}{2}}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau. \tag{5.33}$$

Since (5.32) can be proved by similar methods used for (5.17) and (5.33), it is now suffices to show (5.33). By virtue of (5.28) and (5.31), we only need to prove in the sense of distributions that

$$\delta^{l-\frac{1}{2}}\partial_{3}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l-1}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau\xrightarrow{\mathcal{D}'(\mathcal{C}_{+})}\delta^{l-\frac{1}{2}}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau.$$

$$(5.34)$$

Based on (5.8) and (5.28), both the two time integrals in (5.34) are locally integrable functions. For any vector field $\varphi \in \mathcal{D}(\mathcal{C}_+) \subset L^2(\mathcal{C}_+)$, we have $\partial_3 \varphi \in \mathcal{D}(\mathcal{C}_+) \subset L^2(\mathcal{C}_+)$. Hence, by the estimates in proofs of (5.8) and (5.28), we can show the following two spacetime integrals finite to make the use of Fubini's theorem legitimate:

$$\begin{split} &\delta^{l-\frac{1}{2}}\int_{[0,\infty)\times C_{+}} \left|\partial_{h}^{\alpha_{h}}\partial_{3}^{l-1}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})\cdot\partial_{h}^{\gamma_{h}}\varphi(u_{-},x_{2},x_{3})\right| d\tau du_{-}dx_{2}dx_{3}\\ &\lesssim \delta^{l-\frac{1}{2}}\int_{\mathbb{R}\times C_{+}} \left|\partial_{h}^{\alpha_{h}}\partial_{3}^{l-1}(\nabla p+z_{-}\cdot\nabla z_{+})(u_{+},u_{-},x_{2},x_{3})\right|\cdot\left|\partial_{h}^{\gamma_{h}}\varphi(u_{-},x_{2},x_{3})\right| du_{+}du_{-}dx_{2}dx_{3}\\ &\lesssim \delta^{l-\frac{1}{2}}\left(\int_{\mathbb{R}\times C_{+}} \left|\partial_{h}^{\alpha_{h}}\partial_{3}^{l-1}(\nabla p+z_{-}\cdot\nabla z_{+})(u_{+},u_{-},x_{2},x_{3})\right|^{2}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}\times C_{+}} \frac{\left|\partial_{3}\varphi(u_{-},x_{2},x_{3})\right|^{2}}{\langle u_{+}\rangle^{1+\sigma}}\right)^{\frac{1}{2}}\\ &\lesssim \delta^{l-\frac{1}{2}}\left(\int_{\mathbb{R}\times C_{+}} \left|\partial_{h}^{\alpha_{h}}\partial_{3}^{l-1}(\nabla p+z_{-}\cdot\nabla z_{+})(u_{+},u_{-},x_{2},x_{3})\right|^{2}\right)^{\frac{1}{2}}\\ &\times\left(\int_{\mathbb{R}} \frac{1}{\langle u_{+}\rangle^{1+\sigma}}\left(\int_{C_{+}} \left|\partial_{3}\varphi(u_{-},x_{2},x_{3})\right|^{2}du_{-}dx_{2}dx_{3}\right)du_{+}\right)^{\frac{1}{2}}\\ &\lesssim \varepsilon^{2}\|\partial_{3}\varphi\|_{L^{2}} <\infty, \qquad (5.35)\\ &\delta^{l-\frac{1}{2}}\int_{[0,\infty)\times C_{+}} \left|\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})\cdot\varphi(u_{-},x_{2},x_{3})\right|d\tau du_{-}dx_{2}dx_{3}\\ &\lesssim \delta^{l-\frac{1}{2}}\int_{\mathbb{R}\times C_{+}} \left|\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(u_{+},u_{-},x_{2},x_{3})\right|^{2}\int_{\mathbb{R}\times C_{+}} \frac{|\varphi(u_{-},x_{2},x_{3})|^{2}}{\langle u_{+}\rangle^{1+\sigma}}\right)^{\frac{1}{2}}\\ &\lesssim \delta^{l-\frac{1}{2}}\left(\int_{\mathbb{R}\times C_{+}} \left|u_{+}\rangle^{1+\sigma}\left|\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(u_{+},u_{-},x_{2},x_{3})\right|^{2}\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}\times C_{+}} \frac{|\varphi(u_{-},x_{2},x_{3})|^{2}}{\langle u_{+}\rangle^{1+\sigma}}\right)^{\frac{1}{2}}\\ &\lesssim \delta^{l-\frac{1}{2}}\left(\int_{\mathbb{R}\times C_{+}} \left|u_{+}\rangle^{1+\sigma}\left|\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(u_{+},u_{-},x_{2},x_{3})\right|^{2}\right)^{\frac{1}{2}}\\ &\times\left(\int_{\mathbb{R}} \frac{1}{\langle u_{+}\rangle^{1+\sigma}}\left(\int_{C_{+}} \varphi(u_{-},x_{2},x_{3})\right|^{2}du_{-}dx_{2}dx_{3}\right)du_{+}\right)^{\frac{1}{2}}\\ &\lesssim \varepsilon^{l}|\varphi|_{L^{2}}<\infty. \end{cases}$$

Applying (5.35)-(5.36), integration by parts and Fubini's theorem repeatedly then gives in the sense of distributions that

$$\left\langle \delta^{l-\frac{1}{2}} \partial_3 \int_0^\infty \partial_h^{\alpha_h} \partial_3^{l-1} (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau, \ \varphi(u_-, x_2, x_3) \right\rangle$$

$$\begin{split} &=-\delta^{l-\frac{1}{2}}\bigg\langle\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l-1}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau,\;\partial_{3}\varphi(u_{-},x_{2},x_{3})\bigg\rangle\\ &=-\delta^{l-\frac{1}{2}}\int_{\mathcal{C}_{+}}\bigg(\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l-1}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau\bigg)\cdot\partial_{3}\varphi(u_{-},x_{2},x_{3})du_{-}dx_{2}dx_{3}\\ &\stackrel{(5.35)}{=}-\delta^{l-\frac{1}{2}}\int_{[0,\infty)\times\mathcal{C}_{+}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l-1}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})\cdot\partial_{3}\varphi(u_{-},x_{2},x_{3})d\tau du_{-}dx_{2}dx_{3}\\ &=-\delta^{l-\frac{1}{2}}\int_{[0,\infty)}\bigg(\int_{\mathcal{C}_{+}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l-1}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})\cdot\partial_{3}\varphi(u_{-},x_{2},x_{3})du_{-}dx_{2}dx_{3}\bigg)d\tau\\ &=\delta^{l-\frac{1}{2}}\int_{[0,\infty)}\bigg(\int_{\mathcal{C}_{+}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})\cdot\varphi(u_{-},x_{2},x_{3})du_{-}dx_{2}dx_{3}\bigg)d\tau\\ &=\delta^{l-\frac{1}{2}}\int_{[0,\infty)\times\mathcal{C}_{+}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})\cdot\varphi(u_{-},x_{2},x_{3})du_{-}dx_{2}dx_{3}d\tau\\ &=\delta^{l-\frac{1}{2}}\int_{\mathcal{C}_{+}}\bigg(\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau\bigg)\cdot\varphi(u_{-},x_{2},x_{3})du_{-}dx_{2}dx_{3}\\ &=\bigg\langle\delta^{l-\frac{1}{2}}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau,\;\varphi(u_{-},x_{2},x_{3})du_{-}dx_{2}dx_{3}\bigg\rangle. \end{split}$$

which yields (5.34). Thus we have finished this step.

Step 4: Finally, we prove (5.26) and (5.27).

By induction on α_h and l, we can apply (5.32) and (5.33) repeatedly to get the following equation in the sense of weighted L^2 space $L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)$:

$$\delta^{l-\frac{1}{2}}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}\int_{0}^{\infty}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau$$

$$\frac{L^{2}(\mathcal{C}_{+},\langle u_{-}\rangle^{2(1+\sigma)}du_{-}dx_{2}dx_{3})}{\delta^{l-\frac{1}{2}}}\int_{0}^{\infty}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}(\nabla p+z_{-}\cdot\nabla z_{+})(\tau,u_{-}-\tau,x_{2},x_{3})d\tau,$$
(5.37)

which together with (5.28) leads us to (5.26). Moreover, we can derive from (5.37) and (5.29) that

$$\begin{split} & \left\| \delta^{l-\frac{1}{2}} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} \int_{0}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \\ &= \left\| \delta^{l-\frac{1}{2}} \int_{0}^{\infty} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}, x_{3}) d\tau \right\|_{L^{2}(\mathcal{C}_{+}, \langle u_{-} \rangle^{2(1+\sigma)} du_{-} dx_{2} dx_{3})} \lesssim C_{1} \varepsilon^{2}, \end{split}$$

which leads us to (5.27) immediately.

Therefore we have ended the proof of this lemma.

Let us also derive the uniform estimates concerning $\partial_h^{\alpha_h} \partial_3^l(\partial_3 z_{\pm})(\infty; u_{\mp}, x_2, x_3)$ with the lower order coefficient $\delta^{l-\frac{1}{2}}$.

Lemma 5.9. For any
$$\alpha_h \in (\mathbb{Z}_{\geqslant 0})^2$$
 and $l \in \mathbb{Z}_{\geqslant 0}$ with $0 \leqslant |\alpha_h| + l \leqslant N + 2$, there holds
$$\delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_\pm)(\infty; u_\mp, x_2, x_3) \in L^2(\mathcal{C}_\pm, \langle u_\mp \rangle^{2(1+\sigma)} du_\mp dx_2 dx_3),$$

and

$$\lim_{T \to \infty} \left\| \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_\pm)(\infty; u_\mp, x_2, x_3) - \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_\pm)(T, u_\mp \mp T, x_2, x_3) \right\|_{L^2(\mathcal{C}_+, \langle u_\pm \rangle^{2(1+\sigma)} du_\pm dx_2 dx_3)} = 0$$

Proof. By the symmetry considerations, we only need to consider the scattering field $z_{+}(\infty; u_{-}, x_{2}, x_{3})$. Similar arguments in Lemma 5.8 remain valid for this lemma. The only difference lies in the **Step 1**, which can be modified as follows:

$$\begin{split} & \left\| \delta^{l-\frac{1}{2}} \int_0^\infty \partial_h^{\alpha_h} \partial_3^l \partial_3 (\nabla p + z_- \cdot \nabla z_+) (\tau, u_- - \tau, x_2, x_3) d\tau \right\|_{L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)} \\ \lesssim & \delta^{l-\frac{1}{2}} \left\| \left(\int_{\mathbb{R}} \frac{1}{\langle u_+ \rangle^{1+\sigma}} du_+ \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}} \langle u_+ \rangle^{1+\sigma} |\partial_h^{\alpha_h} \partial_3^l \partial_3 (\nabla p + z_- \cdot \nabla z_+) (u_+, u_-, x_2, x_3) |^2 du_+ \right)^{\frac{1}{2}} \right\|_{L^2(\mathcal{C}_+, \langle u_- \rangle^{2(1+\sigma)} du_- dx_2 dx_3)} \end{split}$$

$$\lesssim \delta^{l-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} \partial_{3} (\nabla p + z_{-} \cdot \nabla z_{+}) (u_{+}, u_{-}, x_{2}, x_{3}) \right\|_{L^{2}(\mathbb{R} \times \Omega_{\delta}, du_{+} du_{-} dx_{2} dx_{3})}$$

$$\lesssim \delta^{l-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \partial_{h}^{\alpha_{h}} \partial_{3}^{l} \partial_{3} \nabla p \right\|_{L^{2}_{t}L^{2}_{x}} + \delta^{l-\frac{1}{2}} \left\| \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1}{2}(1+\sigma)} \mathbf{K}_{+}^{(\alpha_{h}, l)} \right\|_{L^{2}_{t}L^{2}_{x}} \overset{\text{Lemmas 4.6 \& 3.5}}{\lesssim} C_{1} \varepsilon^{2}.$$
We have thus proved this lemma. \square

Gathering the above six lemmas (from Lemma 5.4 to Lemma 5.9) gives Theorem 5.1 immediately.

6. Rigidity from infinity in Ω_{δ}

Based on the properties of scattering fields shown in the last section, we are ready to establish the rigidity from infinity for 3D Alfvén waves in thin domains Ω_{δ} . The second main theorem of this paper is as follows:

Theorem 6.1 (Rigidity theorem in Ω_{δ}). If the scattering fields constructed in Theorem 5.1 vanish on the infinities, i.e.

$$\begin{cases} \delta^{-\frac{1}{2}} z_{+}(\infty; u_{-}, x_{2}, x_{3}) \equiv 0 & on \ \mathcal{C}_{+}, \\ \delta^{-\frac{1}{2}} z_{-}(\infty; u_{+}, x_{2}, x_{3}) \equiv 0 & on \ \mathcal{C}_{-}, \end{cases}$$

then the initial Alfvén waves governed by the system (2.3) vanish identically, i.e.

$$(z_{+,0}(x), z_{-,0}(x)) \equiv (0,0)$$
 for all $x \in \Omega_{\delta}$,

and hence the Alfvén waves governed by the system (2.3) vanish identically, i.e.

$$(z_{+}(t,x),z_{-}(t,x)) \equiv (0,0)$$
 for all $(t,x) \in \mathbb{R} \times \Omega_{\delta}$.

To construct this rigidity is mainly motivated by the previous work [17]. In a similar manner, the rigidity in Theorem 6.1 has the physical interpretation that the 3D Alfvén waves produced from the plasma in thin domains Ω_{δ} are characterized by their scattering fields detected by faraway observers, and hence there are no Alfvén waves in thin domains Ω_{δ} if no waves are detected by faraway observers.

We remark here that based on Corollary 5.2 and the notations before, Theorem 6.1 can be immediately rephrased as the following renormalization result:

Corollary 6.2 (Rigidity theorem in Ω_1). If the scattering fields constructed in Corollary 5.2 vanish on the infinities, i.e.

$$\begin{cases} z_{+(\delta)}(\infty; u_{-}, x_{2}, x_{3}) \equiv 0 & on \ \mathcal{C}_{+}, \\ z_{-(\delta)}(\infty; u_{+}, x_{2}, x_{3}) \equiv 0 & on \ \mathcal{C}_{-}, \end{cases}$$

then the initial Alfvén waves governed by the rescaled system (2.4) vanish identically, i.e.

$$(z_{+(\delta),0}(x), z_{-(\delta),0}(x)) \equiv (0,0)$$
 for all $x \in \Omega_1$,

and hence the Alfvén waves governed by the rescaled system (2.4) vanish identically, i.e.,

$$(z_{+(\delta)}(t,x),z_{-(\delta)}(t,x)) \equiv (0,0)$$
 for all $(t,x) \in \mathbb{R} \times \Omega_1$.

The rest of this section is devoted to proving Theorem 6.1. Now we suppose that the scattering fields vanish identically at infinities, that is,

$$\delta^{-\frac{1}{2}} z_{\pm}(\infty; x_1, x_2, u_{\mp}) \equiv 0 \text{ on } \mathcal{C}_{\pm}.$$

Let $\epsilon < \varepsilon_0$ be an arbitrarily given small positive constant. By virtue of Theorem 5.1 (ii), we can rephrase the vanishing property of scattering fields to the large time behavior of the solution, i.e. there exists a large time $T_{\epsilon} > 0$ such that we have the following smallness condition in the weighted energy sense:

$$\begin{split} &\sum_{+,-} \bigg(\sum_{0 \leqslant |\alpha_h| + l \leqslant N+2} \bigg\| \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_\pm(T_\epsilon, u_\mp \mp T_\epsilon, x_2, x_3) \bigg\|_{L^2(\Omega_\delta, \langle u_\mp \rangle^{2(1+\sigma)} du_\mp dx_2 dx_3)}^2 \\ &\quad + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_\pm^3(T_\epsilon, u_\mp \mp T_\epsilon, x_2, x_3) \bigg\|_{L^2(\Omega_\delta, \langle u_\mp \rangle^{2(1+\sigma)} du_\mp dx_2 dx_3)}^2 \\ &\quad + \sum_{0 \leqslant |\alpha_h| + l \leqslant N+2} \bigg\| \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l(\partial_3 z_\pm)(T_\epsilon, u_\mp \mp T_\epsilon, x_2, x_3) \bigg\|_{L^2(\Omega_\delta, \langle u_\mp \rangle^{2(1+\sigma)} du_\mp dx_2 dx_3)}^2 \bigg) < \epsilon^2. \end{split}$$

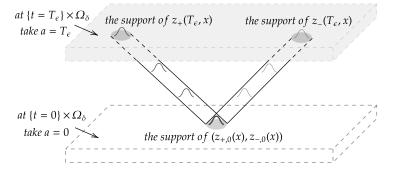


Figure 3. The position parameter in rigidity

We are in a position to study the position parameter a. As depicted in the following Figure 3, at the initial time slice $\{t=0\} \times \Omega_{\delta}$, the position parameter $a=a_0=0$ is given so that we can construct the solution $z_{\pm}(t,x)$. At the time slice $\{t=T_{\epsilon}\} \times \Omega_{\delta}$, within the Cartesian coordinates, we can translate the above smallness condition as

$$\begin{split} & \sum_{+,-} \bigg(\sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} \bigg\| \big(1 + |x_1 \pm T_{\epsilon}|^2 \big)^{\frac{1 + \sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm}(T_{\epsilon}, x) \bigg\|_{L^2(\Omega_{\delta}, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| \leqslant N + 2} \bigg\| \big(1 + |x_1 \pm T_{\epsilon}|^2 \big)^{\frac{1 + \sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm}^3(T_{\epsilon}, x) \bigg\|_{L^2(\Omega_{\delta}, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} \bigg\| \big(1 + |x_1 \pm T_{\epsilon}|^2 \big)^{\frac{1 + \sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l(\partial_3 z_{\pm})(T_{\epsilon}, x) \bigg\|_{L^2(\Omega_{\delta}, dx)}^2 \bigg) < \epsilon^2. \end{split}$$

Let us take $a = T_{\epsilon}$ as the new position parameter and regard $(z_{+}(T_{\epsilon}, x), z_{-}(T_{\epsilon}, x))$ as the new initial data for the MHD system (2.3). In this way, the smallness condition can be rewritten as

$$\sum_{+,-} \left(\sum_{0 \leqslant |\alpha_h| + l \leqslant N+2} \left\| \left(1 + |x_1 \pm a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm}(T_{\epsilon}, x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \right. \\ + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \left\| \left(1 + |x_1 \pm a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm}^3(T_{\epsilon}, x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \\ + \sum_{0 \leqslant |\alpha_h| + l \leqslant N+2} \left\| \left(1 + |x_1 \pm a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l(\partial_3 z_{\pm})(T_{\epsilon}, x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \right) < \epsilon^2.$$

We turn to solve the system backwards in time. Denote $\tilde{t} := T_{\epsilon} - t$. The initial setting now turns into $\{t = T_{\epsilon}\} \times \Omega_{\delta} = \{\tilde{t} = 0\} \times \Omega_{\delta}$. Therefore we have

$$\begin{split} & \sum_{+,-} \bigg(\sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} \bigg\| \big(1 + |x_1 \pm a|^2 \big)^{\frac{1+\sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm}(\widetilde{t} = 0, x) \bigg\|_{L^2(\Omega_{\delta}, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| \leqslant N + 2} \bigg\| \big(1 + |x_1 \pm a|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm}^3(\widetilde{t} = 0, x) \bigg\|_{L^2(\Omega_{\delta}, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} \bigg\| \big(1 + |x_1 \pm a|^2 \big)^{\frac{1+\sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_{\pm}) (\widetilde{t} = 0, x) \bigg\|_{L^2(\Omega_{\delta}, dx)}^2 \bigg) < \epsilon^2. \end{split}$$

We are now ready to apply the uniform (with respect to δ) weighted energy estimates which are independent of the position parameter. It should be noted that the behaviors of z_{\pm} in the process of solving the system backwards in time are the same as those of z_{\mp} in the standard version of uniform (with respect to δ) weighted energy estimates due to the symmetry of time and the symmetry of space. Consequently,

according to Theorem 2.5, we infer that

$$\begin{split} \sum_{+,-} \bigg(\sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} & \left\| \left(1 + |u_{\pm} \pm a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\mp}(\widetilde{t}, x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| \leqslant N + 2} & \left\| \left(1 + |u_{\pm} \pm a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\mp}^3(\widetilde{t}, x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} & \left\| \left(1 + |u_{\pm} \pm a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_{\mp})(\widetilde{t}, x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \bigg) < C \epsilon^2, \end{split}$$

where C is a universal constant. This gives

$$\begin{split} \sum_{+,-} \bigg(\sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} & \left\| \left(1 + |u_\mp \mp a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_\pm(\widetilde{t}, x) \right\|_{L^2(\Omega_\delta, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| \leqslant N + 2} & \left\| \left(1 + |u_\mp \mp a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_\pm^3(\widetilde{t}, x) \right\|_{L^2(\Omega_\delta, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} & \left\| \left(1 + |u_\mp \mp a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_\pm)(\widetilde{t}, x) \right\|_{L^2(\Omega_\delta, dx)}^2 \bigg) < C \epsilon^2. \end{split}$$

By definition of u_{\pm} in (2.5), it then follows that

$$\sum_{+,-} \left(\sum_{0 \leqslant |\alpha_h| + l \leqslant N+2} \left\| \left(1 + |x_1 \pm \widetilde{t} \mp a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm}(\widetilde{t}, x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \right. \\ + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \left\| \left(1 + |x_1 \pm \widetilde{t} \mp a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm}^3(\widetilde{t}, x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \\ + \sum_{0 \leqslant |\alpha_h| + l \leqslant N+2} \left\| \left(1 + |x_1 \pm \widetilde{t} \mp a|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l(\partial_3 z_{\pm})(\widetilde{t}, x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \right) < C\epsilon^2.$$

This estimate indeed holds for all \widetilde{t} . In particular, we take $\widetilde{t} = T_{\epsilon}$ to get

$$\begin{split} & \sum_{+,-} \bigg(\sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1 + \sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm} (\widetilde{t} = T_{\epsilon}, x) \bigg\|_{L^2(\Omega_{\delta}, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| \leqslant N + 2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1 + \sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm}^3 (\widetilde{t} = T_{\epsilon}, x) \bigg\|_{L^2(\Omega_{\delta}, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1 + \sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_{\pm}) (\widetilde{t} = T_{\epsilon}, x) \bigg\|_{L^2(\Omega_{\delta}, dx)}^2 \bigg) < C \epsilon^2. \end{split}$$

We now return to replace \tilde{t} by t. Hence, at the time slice $\{t=0\} \times \Omega_{\delta}$, it holds that

$$\begin{split} \sum_{+,-} \bigg(\sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} & \left\| \left(1 + |x_1|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm}(0,x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| \leqslant N + 2} & \left\| \left(1 + |x_1|^2 \right)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm}^3(0,x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} & \left\| \left(1 + |x_1|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l(\partial_3 z_{\pm})(0,x) \right\|_{L^2(\Omega_{\delta}, dx)}^2 \bigg) < C \epsilon^2, \end{split}$$

which means

$$\sum_{+,-} \bigg(\sum_{0 \leqslant |\alpha_h| + l \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm,0} \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\| \big(1 + |x_1|^2 \big)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \bigg\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N+2} \bigg\|_{L^2(\Omega_\delta, dx)}^2 +$$

$$+\sum_{0\leqslant |\alpha_h|+l\leqslant N+2} \left\| \left(1+|x_1|^2\right)^{\frac{1+\sigma}{2}} \delta^{l-\frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_{\pm,0}) \right\|_{L^2(\Omega_\delta, dx)}^2 < C\epsilon^2.$$

Here, at the time slice $\{t=0\} \times \Omega_{\delta}$, we notice that the new weights associated to a indeed coincide with the original weights, i.e. $(1+|x_1|^2)^{\omega}$.

Since ϵ is arbitrary, we conclude that

$$\begin{split} \sum_{+,-} \bigg(\sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} & \left\| \left(1 + |x_1|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l z_{\pm,0} \right\|_{L^2(\Omega_\delta, dx)}^2 + \sum_{0 \leqslant |\alpha_h| \leqslant N + 2} & \left\| \left(1 + |x_1|^2 \right)^{\frac{1+\sigma}{2}} \delta^{-\frac{3}{2}} \partial_h^{\alpha_h} z_{\pm,0}^3 \right\|_{L^2(\Omega_\delta, dx)}^2 \\ & + \sum_{0 \leqslant |\alpha_h| + l \leqslant N + 2} & \left\| \left(1 + |x_1|^2 \right)^{\frac{1+\sigma}{2}} \delta^{l - \frac{1}{2}} \partial_h^{\alpha_h} \partial_3^l (\partial_3 z_{\pm,0}) \right\|_{L^2(\Omega_\delta, dx)}^2 \bigg) = 0, \end{split}$$

which implies that the initial Alfvén waves $z_{\pm,0}$ indeed vanish. Therefore the Alfvén waves $z_{\pm}(t,x)$ vanish identically. This establishes Theorem 6.1 as desired.

7. Asymptotics of rigidity theorem from Ω_{δ} to \mathbb{R}^2

In view of the above theorems, we are now in a position to investigate the approximation of the rigidity from infinity theorem from thin domains Ω_{δ} to \mathbb{R}^2 . Indeed, an immediate consequence is as follows:

Corollary 7.1 (Asymptotics of rigidity theorem from Ω_{δ} to \mathbb{R}^2 as δ goes to zero). Under the assumptions of Theorem 2.8, if $z_{+(\delta)}(\infty; u_-, x_2, x_3)$ and $z_{-(\delta)}(\infty; u_+, x_2, x_3)$ are the scattering fields constructed in Corollary 5.2 for the rescaled system (2.4), then there exist scattering fields $z_{+(0)}^h(\infty; u_-, x_2)$ and $z_{-(0)}^h(\infty; u_+, x_2)$ such that for any $x_3 \in (-1, 1)$, there hold

$$\lim_{\delta \to 0} z_{\pm(\delta)}^{h}(\infty; u_{-}, x_{2}, x_{3}) = z_{\pm(0)}^{h}(\infty; u_{+}, x_{2}) \quad \text{in } H^{N}(\mathbb{R}^{2}),$$

$$\lim_{\delta \to 0} z_{\pm(\delta)}^{3}(\infty; u_{-}, x_{2}, x_{3}) = 0 \quad \text{in } H^{N-1}(\mathbb{R}^{2}).$$
(7.1)

Moreover, if the scattering fields $z_{+(0)}^h(\infty; u_-, x_2)$ and $z_{-(0)}^h(\infty; u_+, x_2)$ vanish on the infinities, i.e.

$$\begin{cases} z_{+(0)}^h(\infty; u_-, x_2) \equiv 0 & on \ \mathcal{C}_+ \ (the \ 2D \ version \ of \ the \ \mathcal{C}_+ \ above), \\ z_{-(0)}^h(\infty; u_+, x_2) \equiv 0 & on \ \mathcal{C}_- \ (the \ 2D \ version \ of \ the \ \mathcal{C}_- \ above), \end{cases}$$

then the initial Alfvén waves governed by the 2D version of the rescaled system (2.4) vanish identically, i.e.

$$(z_{+(0),0}^h(x), z_{-(0),0}^h(x)) \equiv (0,0)$$
 for all $x \in \mathbb{R}^2$,

and hence the Alfvén waves governed by the 2D version of the rescaled system (2.4) vanish identically, i.e.,

$$(z_{+(0)}^h(t,x), z_{-(0)}^h(t,x)) \equiv (0,0) \text{ for all } (t,x) \in \mathbb{R} \times \mathbb{R}^2.$$

Proof. Similar arguments used in the proof of Lemma 5.4 enable us to derive that

$$\langle u_{\mp}\rangle^{\frac{1}{2}(1+\sigma)}\langle u_{\pm}\rangle^{\frac{1}{2}(1+\sigma)}|\nabla_{\delta}p_{(\delta)}| \lesssim \sum_{0 \leq |\alpha_{h}| \leq 3 \atop 0 \leq l \leq 3} \|\langle u_{\mp}\rangle^{1+\sigma}\langle u_{\pm}\rangle^{\frac{1}{2}(1+\sigma)}\partial_{h}^{\alpha_{h}}\partial_{3}^{l}\nabla_{\delta}p_{(\delta)}\|_{L_{t}^{2}L_{x}^{2}} \overset{\text{Corollary 4.5}}{\lesssim} C_{1}\varepsilon^{2},$$

$$\langle u_{\mp}\rangle^{\frac{1}{2}(1+\sigma)}\langle u_{\pm}\rangle^{\frac{1}{2}(1+\sigma)}|z_{\mp(\delta)}\cdot\nabla z_{\pm(\delta)}|\lesssim \sum_{0\leqslant |\alpha_h|\leqslant 3\atop 0\leqslant l\leqslant 3}\|\langle u_{\mp}\rangle^{1+\sigma}\langle u_{\pm}\rangle^{\frac{1}{2}(1+\sigma)}\partial_h^{\alpha_h}\partial_3^l(z_{\mp(\delta)}\cdot\nabla z_{\pm(\delta)})\|_{L_t^2L_x^2}\overset{\mathrm{Remark }3.4}{\lesssim}C_1\varepsilon^2.$$

In view of (2.11), these two estimates then yield

$$|\nabla_{\delta} p_{(\delta)} + z_{\mp(\delta)} \cdot \nabla z_{\pm(\delta)}| \lesssim \frac{C_1 \varepsilon^2}{\langle u_{\mp} \rangle^{\frac{1}{2}(1+\sigma)} \langle u_{\pm} \rangle^{\frac{1}{2}(1+\sigma)}} \lesssim \frac{C_1 \varepsilon^2}{(1+|t+a|)^{1+\sigma}} \in L_t^1(\mathbb{R}).$$

This means that the two integrands in (5.2) are uniformly integrable.

Therefore, by the Lebesgue's dominated convergence theorem and Theorem 2.8, we infer that

$$\lim_{\delta \to 0} \int_0^\infty (\nabla_\delta p_{(\delta)} + z_{\mp(\delta)} \cdot \nabla z_{\pm(\delta)}) (\tau, u_\mp \mp \tau, x_2, x_3) d\tau = \int_0^\infty \lim_{\delta \to 0} (\nabla_\delta p_{(\delta)} + z_{\mp(\delta)} \cdot \nabla z_{\pm(\delta)}) (\tau, u_\mp \mp \tau, x_2, x_3) d\tau$$

$$= \int_0^\infty (\nabla p_{(0)} + z_{\mp(0)} \cdot \nabla z_{\pm(0)})(\tau, u_{\mp} \mp \tau, x_2) d\tau,$$

where the limit holds in $H^N(\mathbb{R}^2)$ for the horizontal component and in $H^{N-1}(\mathbb{R}^2)$ for the vertical component. Together with (5.2), Theorem 2.8 and Remark 5.3, this gives rise to

$$\lim_{\delta \to 0} z_{\pm(\delta)}(\infty; u_{\mp}, x_2, x_3) = \lim_{\delta \to 0} z_{\pm(\delta)}(0, u_{\mp}, x_2, x_3) - \lim_{\delta \to 0} \int_0^{\infty} (\nabla_{\delta} p_{(\delta)} + z_{\mp(\delta)} \cdot \nabla z_{\pm(\delta)})(\tau, u_{\mp} \mp \tau, x_2, x_3) d\tau$$

$$= z_{\pm(0)}(0, u_{\mp}, x_2) - \int_0^{\infty} (\nabla p_{(0)} + z_{\mp(0)} \cdot \nabla z_{\pm(0)})(\tau, u_{\mp} \mp \tau, x_2) d\tau = z_{\pm(0)}(\infty; u_{\mp}, x_2),$$

where the limit holds in $H^N(\mathbb{R}^2)$ for the horizontal component and in $H^{N-1}(\mathbb{R}^2)$ for the vertical component. Precisely, we obtain (7.1) as asserted.

By virtue of Corollary 6.2, it now follows that if the scattering fields constructed in Remark 5.3 vanish on the infinities, i.e.

$$\begin{cases} z_{+(0)}(\infty; u_{-}, x_{2}) \equiv 0 \text{ on } C_{+}, \\ z_{-(0)}(\infty; u_{+}, x_{2}) \equiv 0 \text{ on } C_{-}, \end{cases}$$

then the initial Alfvén waves governed by the rescaled system (2.4) vanish identically, i.e.

$$(z_{+(0),0}(x), z_{-(0),0}(x)) \equiv (0,0)$$
 for all $x \in \mathbb{R}^2$,

and hence the Alfvén waves governed by the rescaled system (2.4) vanish identically, i.e.,

$$(z_{+(0)}(t,x), z_{-(0)}(t,x)) \equiv (0,0)$$
 for all $(t,x) \in \mathbb{R} \times \mathbb{R}^2$.

We note that the scattering fields $z_{+(0)}^h(\infty; u_-, x_2)$ and $z_{-(0)}^h(\infty; u_+, x_2)$ are the 2D version of the scattering fields $z_{+(0)}(\infty; u_-, x_2)$ and $z_{-(0)}(\infty; u_+, x_2)$; the initial data $\left(z_{+(0),0}^h(x), z_{-(0),0}^h(x)\right)$ are the 2D version of the initial data $\left(z_{+(0),0}(x), z_{-(0),0}(x)\right)$; and the Alfvén waves $\left(z_{+(0)}^h(t,x), z_{-(0)}^h(t,x)\right)$ are the 2D version of the Alfvén waves $\left(z_{+(0)}(t,x), z_{-(0)}(t,x)\right)$. Consequently, the rigidity part of Corollary 7.1 follows immediately. Up to now, we have finished the proof of Corollary 7.1.

Remark 7.2. In particular, the scattering fields $z_{+(0)}^h(\infty; u_-, x_2)$ and $z_{-(0)}^h(\infty; u_+, x_2)$ are the 2D version of the scattering fields $z_{+(0)}(\infty; u_-, x_2)$ and $z_{-(0)}(\infty; u_+, x_2)$ constructed in Remark 5.3 for the rescaled system (2.4). We also remark that the scattering fields $z_{+(0)}^h(\infty; u_-, x_2)$ and $z_{-(0)}^h(\infty; u_+, x_2)$ coincide with the scattering fields $z_{+}(\infty; u_-, x_2)$ and $z_{-(\infty; u_+, x_2)}^h(\infty; u_-, x_2)$ in Theorem A.2 as (A.2) respectively. Furthermore, the rigidity part of Corollary 7.1 indeed coincides with the 2D version of the rigidity from infinity theorem constructed in [17]. For the readers' convenience, this 2D version of the rigidity theorem in [17] is provided as Theorem A.3 in the appendix. For clarity of comparison, we illustrate the relations among these results in the following Figure 4.

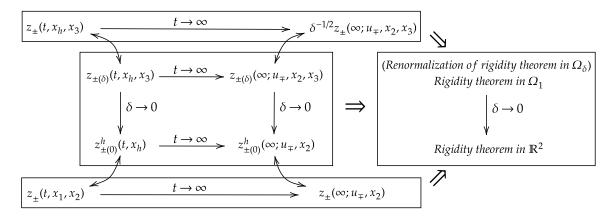


Figure 4. Relations among rigidity theorems

Remark 7.3. Though the approximation of global existence result from Ω_{δ} to \mathbb{R}^2 in [26] is nontrivial, we see that the approximation of rigidity result from Ω_{δ} to \mathbb{R}^2 in this paper can be regarded a direct consequence. This is trivial based on the geometric fact that Ω_{δ} can be viewed as \mathbb{R}^2 when $\delta \to 0$, the approximation result in Theorem 1.3 from [26] that 3D Alfvén waves in Ω_{δ} converge to 2D Alfvén waves in \mathbb{R}^2 when $\delta \to 0$, and in particular the rigidity from infinity conditions that the scattering fields for Alfvén waves are vanishing at their corresponding infinities. However, things will become completely different if we study the approximation of inverse scattering result from Ω_{δ} to \mathbb{R}^2 when given arbitrarily small scattering fields (instead of vanishing scattering fields). This will be treated in a forthcoming paper since the problem is more complicated and requires involved techniques.

Appendix A. Rigidity from infinity in \mathbb{R}^2

In this appendix, we extend the rigidity results in [17] for 3D Alfvén waves in \mathbb{R}^3 to the case of 2D Alfvén waves in \mathbb{R}^2 for the readers' convenience.

Precisely, we study the most interesting physical situation in 2D MHD where a strong background magnetic field generates Alfvén waves. For consistency in this paper, we follow the notations of initial data $(z_{+,0}, z_{-,0})$, weight functions as in (2.6), and so forth (only differences lie in replacing Ω_{δ} by \mathbb{R}^2 and all the 3D coordinates by the corresponding 2D coordinates); and we assume that the strong background magnetic field is taken as the unit vector field along the x_1 -axis in \mathbb{R}^2 : $B_0 = (1,0)$.

Since there never exist boundary conditions in \mathbb{R}^2 , the system for (z_+, z_-) now can be written as the first four equations in (2.3), i.e.

$$\begin{cases} \partial_t z_+ + (z_- - B_0) \cdot \nabla z_+ = -\nabla p, \\ \partial_t z_- + (z_+ + B_0) \cdot \nabla z_- = -\nabla p, \\ \operatorname{div} z_+ = 0, & \operatorname{div} z_- = 0. \end{cases}$$
(A.1)

We now list the three main theorems for the 2D case to give a more precise comparison with the 3D case in [17] and the thin-domain case in the previous sections.

Theorem A.1 (Weighted energy estimates in \mathbb{R}^2). Let $N_* \in \mathbb{Z}_{\geq 5}$ and $\sigma \in (0, \frac{1}{3})$. There exists a universal constant $\varepsilon_0 \in (0, 1)$ such that if the initial data $(z_{+,0}(x), z_{-,0}(x))$ of (A.1) satisfy

$$\mathcal{E}^{N_*}(0) := \sum_{\pm,-} \sum_{k=0}^{N_*+1} \left\| \left(1 + |x_1 \pm a|^2 \right)^{\frac{1+\sigma}{2}} \nabla^k z_{\pm}(0,x) \right\|_{L^2(\mathbb{R}^2)}^2 \leqslant \varepsilon_0^2,$$

then the system (A.1) admits a unique global solution $(z_+(t,x), z_-(t,x))$. Moreover, there exists a universal constant C such that the following weighted energy estimates hold:

$$\sum_{t=-\infty}^{N_*+1} \sup_{t\geqslant 0} \left\| \left(1 + |u_{\mp} \pm a|^2 \right)^{\frac{1+\sigma}{2}} \nabla^k z_{\pm}(t,x) \right\|_{L^2(\mathbb{R}^2)}^2 \leqslant C \mathcal{E}^{N_*}(0).$$

Theorem A.2 (Scattering fields in \mathbb{R}^2). For the solution $(z_+(t,x), z_-(t,x))$ constructed in Theorem A.1, the following two vector fields

$$\begin{cases}
z_{+}(\infty; u_{-}, x_{2}) := z_{+}(0, u_{-}, x_{2}) - \int_{0}^{\infty} (\nabla p + z_{-} \cdot \nabla z_{+}) (\tau, u_{-} - \tau, x_{2}) d\tau \\
z_{-}(\infty; u_{+}, x_{2}) := z_{-}(0, u_{+}, x_{2}) - \int_{0}^{\infty} (\nabla p + z_{+} \cdot \nabla z_{-}) (\tau, u_{+} + \tau, x_{2}) d\tau
\end{cases} (A.2)$$

are well-defined on the infinities C_+ (the 2D version of the C_+ above) and C_- (the 2D version of the C_- above) respectively. We call $z_+(\infty; u_-, x_2)$ as the left scattering field and $z_-(\infty; u_+, x_2)$ as the right scattering field. Moreover, for any $\beta \in (\mathbb{Z}_{\geq 0})^2$ with $0 \leq |\beta| \leq N_*$ there hold the following two properties of scattering fields:

(i) these scattering fields live in the following functional spaces in the weighted energy sense:

$$\nabla^{\beta} z_{\pm}(\infty; u_{\mp}, x_2) \in L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2).$$

(ii) these scattering fields can be approximated by the large time solution in the weighted energy sense:

$$\lim_{T \to \infty} \left\| \nabla^{\beta} z_{\pm}(\infty; u_{\mp}, x_2) - \nabla^{\beta} z_{\pm}(T, u_{\mp} \mp T, x_2) \right\|_{L^2(\mathcal{C}_{\pm}, \langle u_{\mp} \rangle^{2(1+\sigma)} du_{\mp} dx_2)} = 0$$

Theorem A.3 (Rigidity theorem in \mathbb{R}^2). If the scattering fields constructed in Theorem A.2 vanish on the infinities, i.e.

$$\begin{cases} z_{+}(\infty; u_{-}, x_{2}) \equiv 0 & on \ \mathcal{C}_{+} \ (the \ 2D \ version \ of \ the \ \mathcal{C}_{+} \ above), \\ z_{-}(\infty; u_{+}, x_{2}) \equiv 0 & on \ \mathcal{C}_{-} \ (the \ 2D \ version \ of \ the \ \mathcal{C}_{-} \ above), \end{cases}$$

then the initial Alfvén waves vanish identically, i.e.

$$(z_{+,0}(x), z_{-,0}(x)) \equiv (0,0)$$
 for all $x \in \mathbb{R}^2$,

and hence the Alfvén waves vanish identically, i.e.,

$$(z_+(t,x), z_-(t,x)) \equiv (0,0)$$
 for all $(t,x) \in \mathbb{R} \times \mathbb{R}^2$.

Remark A.4. In fact, the rigidity from infinity for Alfvén waves in \mathbb{R}^2 constructed in Theorem A.3 coincides with the rigidity part of Corollary 7.1, and hence coincides with the approximation of the rigidity from infinity for Alfvén waves in Ω_{δ} propagating along the horizontal direction as δ goes to zero. Combined with Remark 7.2, this means that the two perspectives on the rigidity for Alfvén waves in \mathbb{R}^2 from Corollary 7.1 and Theorem A.3 coincide with each other and are perfectly unified. We also remark here that these relations have been nicely depicted in Figure 4.

The proofs of these results are indeed almost the same as that used in [17], and hence we only sketch the differences between them in the rest of this paper.

In fact, it suffices to make suitable modifications on [17]: On one hand, we need to replace $\delta \in (0, \frac{2}{3})$ therein by $\sigma \in (0, \frac{1}{3})$, and replace \mathbb{R}^3 by \mathbb{R}^2 and replace all the 3D coordinates (such as (u_+, x_2, x_3)) by the corresponding 2D coordinates (such as (u_+, x_2)), which will be carried out repeatedly in what follows without further comment. On the other hand, since the pressure term therein involves the Newtonian potential, we need to modify the related estimates on the pressure (we point out that the results are the same while the details are different), especially including the bound on \mathbf{I}' (i.e. from (2.23) to (2.30)) and the proof of Corollary 2.12 therein. In the rest of this paper, we will use two subsections to modify these two pressure estimates respectively.

A.1. Modification on the estimate of I'. By taking divergence on both sides of the first equation of (A.1) and using div $z_{\pm} = 0$, we can obtain the first equation in (4.1), i.e. $-\Delta p = \partial_i z_{-}^j \partial_j z_{+}^i$. According to the Newtonian potential in the 2D case, we can infer on each time slice $\Sigma_{\tau} = \{(t, x) \in \mathbb{R} \times \mathbb{R}^2 \mid t = \tau\}$ that

$$p(\tau, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \log|x - y| \cdot \left(\partial_i z_-^j \partial_j z_+^i\right)(\tau, y) dy.$$

Therefore, the following decomposition of ∇p holds on any time slice Σ_{τ} :

$$\nabla p(\tau, x) = -\frac{1}{2\pi} \nabla \int_{\mathbb{R}^2} \log|x - y| \cdot \left(\partial_i z_-^j \partial_j z_+^i\right)(\tau, y) dy = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \nabla \log|x - y| \cdot \left(\partial_i z_-^j \partial_j z_+^i\right)(\tau, y) dy$$

$$= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \nabla \log|x - y| \cdot \theta(|x - y|) \cdot \left(\partial_i z_-^j \partial_j z_+^i\right)(\tau, y) dy$$

$$-\frac{1}{2\pi} \int_{\mathbb{R}^2} \nabla \log|x - y| \cdot \left(1 - \theta(|x - y|)\right) \cdot \left(\partial_i z_-^j \partial_j z_+^i\right)(\tau, y) dy,$$

where the smooth cutoff function $\theta(r)$ is still taken as (4.8). Since div $z_{\pm} = 0$, we can integrate by parts to derive

$$\nabla p(\tau, x) = -\frac{1}{2\pi} \int_{\mathbb{R}^2} \nabla \log|x - y| \cdot \theta(|x - y|) \cdot \left(\partial_i z_-^j \partial_j z_+^i\right)(\tau, y) dy$$

$$+ \frac{1}{2\pi} \int_{\mathbb{R}^2} \partial_i \left(\nabla \log|x - y| \cdot (1 - \theta(|x - y|))\right) \cdot \left(z_-^j \partial_j z_+^i\right)(\tau, y) dy$$

$$= -\frac{1}{2\pi} \int_{\mathbb{R}^2} \nabla \log|x - y| \cdot \theta(|x - y|) \cdot \left(\partial_i z_-^j \partial_j z_+^i\right)(\tau, y) dy$$

$$-\frac{1}{2\pi} \int_{\mathbb{R}^2} \partial_j \partial_i \left(\nabla \log|x - y| \cdot (1 - \theta(|x - y|))\right) \cdot \left(z_-^j z_+^i\right)(\tau, y) dy. \tag{A.3}$$

Using the property of the cutoff function $\theta(r)$, we can bound ∇p as follows:

$$|\nabla p(\tau, x)| \lesssim \int_{|x-y| \leqslant 2} |\nabla \log|x - y| |\cdot |\theta(|x - y|)| \cdot \left| \left(\partial_{i} z_{-}^{j} \partial_{j} z_{+}^{i} \right)(\tau, y) \right| dy$$

$$+ \int_{|x-y| \geqslant 1} |\partial_{j} \partial_{i} \left(\nabla \log|x - y| \right)| \cdot |1 - \theta(|x - y|)| \cdot \left| \left(z_{-}^{j} z_{+}^{i} \right)(\tau, y) \right| dy$$

$$+ \int_{1 \leqslant |x-y| \leqslant 2} \left(|\partial_{i} \left(\nabla \log|x - y| \right)| \cdot |\theta'(|x - y|)| + |\nabla \log|x - y|| \cdot |\theta''(|x - y|)| \right) \cdot \left| \left(z_{-}^{j} z_{+}^{i} \right)(\tau, y) \right| dy$$

$$\lesssim \underbrace{\int_{|x-y| \leqslant 2} \frac{\left| \left(\nabla z_{-} \cdot \nabla z_{+} \right)(\tau, y) \right|}{|x - y|} dy}_{\mathbf{A}_{1}} + \underbrace{\int_{|x-y| \geqslant 1} \frac{\left| \left(z_{-} \cdot z_{+} \right)(\tau, y) \right|}{|x - y|^{3}} dy}_{\mathbf{A}_{2}} + \underbrace{\int_{1 \leqslant |x-y| \leqslant 2} |\left(z_{-} \cdot z_{+} \right)(\tau, y)| dy}_{\mathbf{A}_{3}}.$$

$$(A.4)$$

Hence we obtain

$$\mathbf{I}' \lesssim \underbrace{\int_{[0,t]\times\mathbb{R}^2} \langle u_{-} \rangle^{2(1+\sigma)} \langle u_{+} \rangle^{1+\sigma} |\mathbf{A_1}|^2 dx d\tau}_{\mathbf{I_1}} + \underbrace{\int_{[0,t]\times\mathbb{R}^2} \langle u_{-} \rangle^{2(1+\sigma)} \langle u_{+} \rangle^{1+\sigma} |\mathbf{A_2}|^2 dx d\tau}_{\mathbf{I_2}} + \underbrace{\int_{[0,t]\times\mathbb{R}^2} \langle u_{-} \rangle^{2(1+\sigma)} \langle u_{+} \rangle^{1+\sigma} |\mathbf{A_3}|^2 dx d\tau}_{\mathbf{I_2}}.$$
(A.5)

Before proceeding further, we collect some preliminary results in [17]. For example, the properties (2.7)-(2.8) now can be improved as follows:

$$(\langle u_{\mp}\rangle^{1+\sigma}\langle u_{\pm}\rangle^{\frac{1+\sigma}{2}})(\tau,x) \lesssim \begin{cases} (\langle u_{\mp}\rangle^{1+\sigma}\langle u_{\pm}\rangle^{\frac{1+\sigma}{2}})(\tau,y) + |x-y|^{\frac{3(1+\sigma)}{2}} & \text{if } |x-y| \geqslant 1, \\ (\langle u_{\mp}\rangle^{1+\sigma}\langle u_{\pm}\rangle^{\frac{1+\sigma}{2}})(\tau,y) & \text{if } |x-y| \leqslant 2. \end{cases}$$
(A.6)

Moreover, there hold the following pointwise estimate and spacetime estimate for z_{\pm} :

(i) (Weighted Sobolev inequality) For all $k \leq N_* - 2$ and multi-indices α with $|\alpha| = k$, we have

$$\langle u_{\mp} \rangle^{\sigma} |z_{\pm}| + \langle u_{\mp} \rangle^{\sigma} |\nabla z_{\pm}^{(\alpha)}| \lesssim C_1 \varepsilon.$$
 (A.7)

(ii) (Weighted spacetime estimate) For all $0 \le k \le N_*$ and multi-indices α with $|\alpha| = k$, we have

$$\int_{[0,t]\times\mathbb{R}^2} \frac{\langle u_{\mp}\rangle^{2(1+\sigma)}}{\langle u_{\pm}\rangle^{\sigma}} |z_{\pm}|^2 dx d\tau + \int_{[0,t]\times\mathbb{R}^2} \frac{\langle u_{\mp}\rangle^{2(1+\sigma)}}{\langle u_{\pm}\rangle^{\sigma}} |\nabla z_{\pm}^{(\alpha)}|^2 dx d\tau \lesssim (C_1)^2 \varepsilon^2. \tag{A.8}$$

The proofs of these results are omitted since the 2D case can also be treated in the same manner. We refer the readers to Lemma 2.4, Lemma 2.5 and Lemma 2.7 in [17] for more details.

We now turn to bound I_1 , I_2 and I_3 one by one:

For I_1 , using the definition of A_1 , we derive

$$\begin{split} \langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1+\sigma}{2}} \, | \mathbf{A_1} | &= \int_{|x-y| \leqslant 2} \frac{\left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1+\sigma}{2}} \right) (\tau,x) \, | (\nabla z_{-} \cdot \nabla z_{+}) \, (\tau,y) |}{|x-y|} dy \\ &\stackrel{(\mathbf{A}.6)}{\lesssim} \int_{|x-y| \leqslant 2} \frac{\left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1+\sigma}{2}} \right) (\tau,y) \, | (\nabla z_{-} \cdot \nabla z_{+}) \, (\tau,y) |}{|x-y|} dy \\ &\leqslant \left\| \langle u_{+} \rangle^{1+\sigma} \, \nabla z_{-} \right\|_{L_{x}^{\infty}} \int_{|x-y| \leqslant 2} \frac{\langle u_{-} \rangle^{1+\sigma} (\tau,y) \, | \nabla z_{+} (\tau,y) |}{\langle u_{+} \rangle^{\frac{1+\sigma}{2}} (\tau,y) | x-y |} dy \\ &\stackrel{(\mathbf{A}.7)}{\lesssim} C_{1} \varepsilon \int_{|x-y| \leqslant 2} \frac{1}{|x-y|} \frac{\langle u_{-} \rangle^{1+\sigma} (\tau,y)}{\langle u_{+} \rangle^{\frac{1+\sigma}{2}} (\tau,y)} \, |\nabla z_{+} (\tau,y)| \, dy. \end{split}$$

By (4.6), we notice that $\frac{1}{|x|}\chi_{|x|\leqslant 2}\in L^1(\mathbb{R}^2)$. Using Young's inequality, we obtain

$$\left\|\langle u_{-}\rangle^{1+\sigma}\langle u_{+}\rangle^{\frac{1+\sigma}{2}}\mathbf{A}_{\mathbf{1}}\right\|_{L^{2}(\mathbb{R}^{2})}^{2} \lesssim \left(C_{1}\right)^{2} \varepsilon^{2} \left\|\frac{1}{|x|}\chi_{|x|\leqslant2}\right\|_{L^{1}(\mathbb{R}^{2})}^{2} \left\|\frac{\langle u_{-}\rangle^{1+\sigma}}{\langle u_{+}\rangle^{\frac{1+\sigma}{2}}}\nabla z_{+}\right\|_{L^{2}(\mathbb{R}^{2})}^{2} \lesssim \left(C_{1}\right)^{2} \varepsilon^{2} \left\|\frac{\langle u_{-}\rangle^{1+\sigma}}{\langle u_{+}\rangle^{\frac{1+\sigma}{2}}}\nabla z_{+}\right\|_{L^{2}(\mathbb{R}^{2})}^{2}.$$

We then apply (A.8) to infer that

$$\mathbf{I}_{1} \lesssim \left(C_{1}\right)^{4} \varepsilon^{4}.\tag{A.9}$$

Here we notice that I_3 can be treated in exactly the same manner. Consequently, we get

$$\mathbf{I_3} \lesssim \left(C_1\right)^4 \varepsilon^4. \tag{A.10}$$

For I_2 , we obtain

$$\mathbf{I_{2}} = \int_{0}^{t} \left\| \int_{|x-y| \geqslant 1} \langle u_{-} \rangle^{1+\sigma}(\tau, x) \langle u_{+} \rangle^{\frac{1+\sigma}{2}}(\tau, x) \frac{|(z_{-} \cdot z_{+})(\tau, y)|}{|x-y|^{3}} dy \right\|_{L^{2}(\mathbb{R}^{2})}^{2} d\tau$$

$$\stackrel{(\mathbf{A}.6)}{\lesssim} \int_{0}^{t} \left\| \int_{|x-y| \geqslant 1} \left(\left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1+\sigma}{2}} \right)(\tau, y) + |x-y|^{\frac{31+\sigma}{2}} \right) \frac{|(z_{-} \cdot z_{+})(\tau, y)|}{|x-y|^{3}} dy \right\|_{L^{2}(\mathbb{R}^{2})}^{2} d\tau$$

$$\lesssim \underbrace{\int_{0}^{t} \left\| \int_{|x-y| \geqslant 1} \left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{\frac{1+\sigma}{2}} \right)(\tau, y) \frac{|(z_{-} \cdot z_{+})(\tau, y)|}{|x-y|^{3}} dy \right\|_{L^{2}(\mathbb{R}^{2})}^{2}} + \underbrace{\int_{0}^{t} \left\| \int_{|x-y| \geqslant 1} \frac{|(z_{-} \cdot z_{+})(\tau, y)|}{|x-y|^{3-\frac{3(1+\sigma)}{2}}} dy \right\|_{L^{2}(\mathbb{R}^{2})}^{2}}_{\mathbf{I_{22}}}.$$

For I_{21} , since $\frac{1}{|x|^3}\chi_{|x|\geqslant 1}\in L^1(\mathbb{R}^2)$, we can derive

$$\mathbf{I_{21}} \overset{\text{Young}}{\lesssim} \int_{0}^{t} \left\| \frac{1}{|x|^{3}} \chi_{|x| \geqslant 1} \right\|_{L^{1}(\mathbb{R}^{2})}^{2} \left\| \langle u_{+} \rangle^{1+\sigma} z_{-} \right\|_{L^{\infty}(\mathbb{R}^{2})}^{2} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1+\sigma}{2}}} z_{+} \right\|_{L^{2}(\mathbb{R}^{2})}^{2} d\tau$$

$$\overset{\text{(A.7)}}{\lesssim} (C_{1})^{2} \varepsilon^{2} \int_{0}^{t} \left\| \frac{\langle u_{-} \rangle^{1+\sigma}}{\langle u_{+} \rangle^{\frac{1+\sigma}{2}}} z_{+} \right\|_{L^{2}(\mathbb{R}^{2})}^{2} d\tau \overset{\text{(A.8)}}{\lesssim} (C_{1})^{4} \varepsilon^{4}.$$

For I_{22} , since $\frac{1}{|x|^{3-\frac{3(1+\sigma)}{2}}}\chi_{|x|\geqslant 1}\in L^2(\mathbb{R}^2)$ holds when $1+\sigma\in(1,\frac{4}{3})$ (this is the place we have constraint on σ), then we have

$$\mathbf{I_{22}} \overset{\text{Young}}{\lesssim} \int_{0}^{t} \left\| \frac{1}{|x|^{3 - \frac{3(1+\sigma)}{2}}} \chi_{|x| \geqslant 1} \right\|_{L^{2}(\mathbb{R}^{2})}^{2} \|z_{-}z_{+}\|_{L^{1}(\mathbb{R}^{2})}^{2} d\tau$$

$$\lesssim \int_{0}^{t} \|z_{-}z_{+}\|_{L^{1}(\mathbb{R}^{2})}^{2} d\tau = \int_{0}^{t} \left\| \frac{1}{\langle u_{+} \rangle^{1+\sigma} \langle u_{-} \rangle^{1+\sigma}} \langle u_{+} \rangle^{1+\sigma} z_{-} \langle u_{-} \rangle^{1+\sigma} z_{+} \right\|_{L^{1}(\mathbb{R}^{2})}^{2} d\tau$$

$$\overset{(2.11)}{\lesssim} \int_{0}^{t} \frac{1}{(1+|\tau+a|)^{2(1+\sigma)}} \left\| \langle u_{+} \rangle^{1+\sigma} z_{-} \right\|_{L^{2}(\mathbb{R}^{2})}^{2} \left\| \langle u_{-} \rangle^{1+\sigma} z_{+} \right\|_{L^{2}(\mathbb{R}^{2})}^{2} d\tau$$

$$\lesssim (C_{1})^{4} \varepsilon^{4} \int_{0}^{t} \frac{1}{(1+|\tau+a|)^{2(1+\sigma)}} d\tau \lesssim (C_{1})^{4} \varepsilon^{4}.$$

Therefore we can similarly obtain

$$\mathbf{I_2} \lesssim (C_1)^4 \,\varepsilon^4. \tag{A.11}$$

Adding (A.9), (A.10) and (A.11) together, and then using (A.5), we can similarly obtain

$$\mathbf{I}' \lesssim (C_1)^4 \, \varepsilon^4.$$

A.2. Modification on the proof of Corollary 2.12. In fact, Corollary 2.12 [17] in the 2D case is the following Corollary A.5:

Corollary A.5. For the solution (z_+, z_-) constructed in Theorem A.1, for l = 1, 2, for all $(\tau, x) \in \mathbb{R} \times \mathbb{R}^2$, we have the following estimates on the pressure term:

$$\left|\nabla^{l} p\left(\tau, x\right)\right| \lesssim \frac{\varepsilon^{2}}{\left(1 + \left|\tau + a\right|\right)^{1 + \sigma}}.$$

Proof. For l = 1, we will estimate ∇p by using (A.4). Next, we will bound $\mathbf{A_1}$, $\mathbf{A_2}$ and $\mathbf{A_3}$ one by one. For $\mathbf{A_1}$, we can also infer

$$\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{1+\sigma} | \mathbf{A}_{1} | = \int_{|x-y| \leqslant 2} \frac{\left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{1+\sigma} \right) (\tau, x) | (\nabla z_{-} \cdot \nabla z_{+}) (\tau, y) |}{|x-y|} dy$$

$$\stackrel{\text{(A.6)}}{\lesssim} \int_{|x-y| \leqslant 2} \frac{\left(\langle u_{-} \rangle^{1+\sigma} \langle u_{+} \rangle^{1+\sigma} \right) (\tau, y) | (\nabla z_{-} \cdot \nabla z_{+}) (\tau, y) |}{|x-y|} dy \stackrel{\text{(A.7)}}{\lesssim} \int_{|x-y| \leqslant 2} \frac{\varepsilon^{2}}{|x-y|} dy \lesssim \varepsilon^{2}.$$

We repeat the estimate on A_1 to the estimate on A_3 . It then follows that

$$\langle u_+ \rangle^{1+\sigma} \langle u_- \rangle^{1+\sigma} |\mathbf{A_3}| \lesssim \varepsilon^2$$

For A_2 , we can similarly use (2.11) to obtain $1 + |\tau + a| \lesssim \langle u_+ \rangle \langle u_- \rangle$, and hence there holds

$$(1+|\tau+a|)^{1+\sigma} |\mathbf{A_2}| \lesssim \int_{|x-y|\geqslant 1} \frac{\langle u_+ \rangle^{1+\sigma}(\tau,y) \langle u_- \rangle^{1+\sigma}(\tau,y) |z_-(\tau,y)| |z_+(\tau,y)|}{|x-y|^3} dy \lesssim \int_{|x-y|\geqslant 1} \frac{\varepsilon^2}{|x-y|^3} dy \lesssim \varepsilon^2.$$

Putting all the estimates on A_i together, we conclude that

$$|\nabla p(\tau, x)| \lesssim \frac{\varepsilon^2}{(1 + |\tau + a|)^{1+\sigma}}.$$

In order to bound $\nabla^2 p$, we take ∇ on both sides of (A.3). Similar to the derivation of (A.4), we can derive

$$\begin{split} |\nabla^{2} p(\tau, x)| \lesssim \underbrace{\sum_{l_{1}, l_{2} = 1}^{2} \int_{|x - y| \leqslant 2} \frac{1}{|x - y|} \left| \left(\nabla^{l_{1}} z_{-} \cdot \nabla^{l_{2}} z_{+} \right) (\tau, y) \right| dy}_{\mathbf{B}_{1}} \\ + \underbrace{\int_{|x - y| \geqslant 1} \frac{1}{|x - y|^{3}} \left| \left(z_{-} \cdot \nabla z_{+} \right) (\tau, y) \right| dy}_{\mathbf{B}_{2}} + \underbrace{\int_{1 \leqslant |x - y| \leqslant 2} \frac{1}{|x - y|^{2}} \left| \left(z_{-} \cdot \nabla z_{+} \right) (\tau, y) \right| dy}_{\mathbf{B}_{2}}, \end{split}$$

where $(l_1, l_2) = (1, 1)$, (1, 2) or (2, 1). We can repeat the above estimate on $\mathbf{A_i}$ to give the estimate on $\mathbf{B_i}$, and thus imply the estimate on $\nabla^2 p$.

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