A sufficient condition under which a monoid is non-finitely related

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Abstract

We transform the method of Glasson into a sufficient condition under which a monoid is non-finitely related, add a new member to the collection of interlocking word-patterns, and use it to show that the monoid $M(ab^2a, a^2b^2)$ is non-finitely related.

1 Introduction

A finite semigroup is *finitely related* if its term functions are determined by a finite set of finitary relations. The study of finitely related semigroups started in 2011 with Davey et al. [1], who showed among many other things that the finite relatedness is a varietal property, in the sense that finite algebras that satisfy the same identities are simultaneously finitely related or not.

The first example of a non-finitely related semigroup was found by Mayr [13]. It was the six-element Brandt monoid which behaves badly with respect to almost any varietal property. The second example [16] was also a monoid: a semigroup with an identity element. In the light of a long history of studying the finite basis property of finite semigroups, it is not surprising that both examples are monoids. Accidentally, but again not surprisingly, Steindl in [16] used implicitly the so-called *chain* word-patterns.

The chain words were initially introduced in 2010 by Wen Ting Zhang for showing that certain 5 element monoid P_2^1 is hereditary finitely based (HFB). (A monoid M is HFB if every monoid in the variety generated by M is finitely based.) P_2^1 is the only existing example of an HFB finite monoid which generates variety with infinitely many infinite (ascending) chains of subvarieties. The monoid P_2^1 is contained in one of the limit varieties discovered by Sergey Gusev and the result of Zhang later became a part of [6].

Since then the chain words have had an amazing variety of applications. In 2014 Edmond Lee and Wen Ting Zhang used the chain words for constructing non-finitely generated monoid varieties [10, 11, 12]. In 2015 the chain words were used by Jackson and Lee to show that the monoid M(abab) generates a variety with uncountably many subvarieties [8], and by Jackson for constructing a finite monoid with infinite irredundant identity basis [9]. Ren, Jackson, Zhao, and Lei utilized

chain words in the identification of semiring limit varieties [15]. Independently, the chain words were introduced in [5]. Recently, Gusev [4] weaved the chain words into a more complicated pattern to show that the finitely based monoid $M(a^2b^2)$ also generates a variety with uncountably many subvarieties.

Glasson [2] used the chain words to prove that the monoid M(abab) is not finitely related. Also in [2], Glasson introduced two new word-patterns: the crown words to show that the monoid $M(ab^2a)$ is non-finitely related and generates a variety with uncountably many subvarieties [3], and the maelstrom words to show that $M(abab, a^2b^2)$ is non-finitely related.

In Sect. 3 we transform the method of Glasson into a sufficient condition under which a monoid is non-finitely related. In Sect. 4 we add a new word-pattern to the collection of interlocking words, and use it to prove that the monoid $M(ab^2a, a^2b^2)$ is also non-finitely related.

2 Preliminaries

2.1 Isoterms and Dilworth-Perkins construction

A word \mathbf{u} said to be an isoterm for a semigroup S if S does not satisfy any nontrivial identity of the form $\mathbf{u} \approx \mathbf{v}$. For any set of words W, let M(W) denote the Rees quotient of the free monoid over the ideal consisting of words which are not subwords of any word in W. This construction was used by Perkins [14] to build one of the first two examples of non-finitely based finite semigroups. If $W = \{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ for some m > 0 then we write $M(W) = M(\mathbf{u}_1, \dots, \mathbf{u}_m)$.

Lemma 2.1. [7, Lemma 3.3]

Let W be a set of words and M be a monoid. Then every word in W is an isoterm for M if and only if the variety generated by M contains M(W).

Denote

$$A_{\alpha,\beta} := \{ x^{\alpha} \approx x^{\alpha+\beta}, t_1 x t_2 x \dots t_{\alpha} x \approx x^{\alpha} t_1 t_2 \dots t_{\alpha} \}.$$

It is easy to see that if every variable occurs in every word in W less than α times then M(W) satisfies $A_{\alpha,1}$.

Given a variable x and a word \mathbf{w} we use $\mathrm{occ}(x,\mathbf{w})$ to denote the number of times x occurs in \mathbf{w} .

2.2 Schemes and Finite Relatedness

We assume that the order of variables in a word $\mathbf{w} = \mathbf{w}(x_1, \dots, x_n)$ is fixed and sometimes say that \mathbf{w} is an *n*-ary term. For example, if $\mathbf{w}(x, y) = xyxy$ then $\mathbf{w}(y, x) = yxyx$. Given $1 \le i < j \le n$ and $\mathbf{w}(x_1, \dots, x_n)$ we use $\mathbf{w}^{(ij)}$ to denote the result of renaming x_i by x_j in \mathbf{w} .

If $n \in \mathbb{N}$ and k = n - 1, then Definition 2.3 in [1] says that an indexed family of n-ary terms $\{\mathbf{t}_{ij}(x_1,\ldots,x_n) \mid 1 \leq i < j \leq n\}$ is a scheme for a semigroup S if for each $1 \le i < j \le n$ and $1 \le k < l \le n$ the following holds.

Dependency Condition:

$$S \models \mathbf{t}_{ij}(x_1, \dots, x_i, \dots, x_j, \dots, x_n) \approx \mathbf{t}_{ij}(x_1, \dots, x_j, \dots, x_j, \dots, x_n).$$

Consistency Condition:

Case 1: If i, j, k, l are pairwise distinct then $S \models \mathbf{t}_{ij}^{(kl)} \approx \mathbf{t}_{kl}^{(ij)}$;

Case 2: If j = k then $S \models \mathbf{t}_{ij}^{(jl)} \approx \mathbf{t}_{jl}^{(il)}$;

Case 3: If i = k and j < l then $S \models \mathbf{t}_{ij}^{(jl)} \approx \mathbf{t}_{il}^{(jl)}$. A scheme $\{\mathbf{t}_{ij}(x_1, \dots, x_n) \mid 1 \leq i < j \leq n\}$ for S comes from a term \mathbf{t}_n if for each $1 \leq i < j \leq n$ we have $S \models \mathbf{t}_n^{(ij)} \approx \mathbf{t}_{ij}$.

Lemma 2.2. [1, Theorem 2.9]

A semigroup S is finitely related if and only if for each n large enough, every scheme $\{\mathbf{t}_{ij}(x_1,\ldots,x_n) \mid 1 \leq i < j \leq n\}$ for S comes from a term.

3 Sufficient condition under which a monoid is not finitely related

Given variables $\{x, y, z\}$ and a word(term) w, we use $\mathbf{w}^{\not z}$ to denote the result of deleting all occurrences of x in w, and $\mathbf{w}[y,z]$ to denote the result of deleting all occurrences of all variables other than y or z in w. For $n \geq 2$ and $\mathcal{X}_n = \mathcal{X}(x_1, x_2, \dots, x_n)$, define

$$\mathcal{Y}(x_2,\ldots,x_n) := \mathcal{X}^{\not x_1}(x_1,x_2,\ldots,x_n).$$

The following theorem explains the method used by Glasson in [2] to prove Theorems 4.4, 4.13 and 4.18 and can be used to reprove them¹.

Theorem 3.1. Let M be a monoid. Suppose that one can find a sequence of n-ary words $\{\mathcal{X}_n = \mathcal{X}(x_1, \dots, x_n) \mid n = 1, 2, \dots\}$ such that for some (possibly partial) operation * on the free monoid we have:

(i) for every $n \in \mathbb{N}$, if $r = 0, 2, 4, \ldots$ then M satisfies

$$\mathcal{X}^{y_1}(x_1, \dots, x_r, y_1, \dots, y_n) \approx \mathcal{X}(x_1, \dots, x_r) * \mathcal{Y}(y_2, \dots, y_n); \tag{1}$$

if $r = 1, 3, 5, \ldots$ then M satisfies

$$\mathcal{X}^{y_1}(x_1, \dots, x_r, y_1, \dots, y_n) \approx \mathcal{X}(x_1, \dots, x_r) * \mathcal{X}(y_2, \dots, y_n); \tag{2}$$

(ii) for every $r, k \in \mathbb{N}$, M satisfies

$$\mathcal{X}(x_1,\ldots,x_r)*\mathcal{X}(y_1,\ldots,y_k)\approx\mathcal{X}(y_1,\ldots,y_k)*\mathcal{X}(x_1,\ldots,x_r);$$
(3)

¹Theorem 4.18 in [2] is missing (2) for $\mathcal{X} = \mathcal{R}$.

(iii) $\operatorname{occ}(x_1, \mathcal{X}_2) = \operatorname{occ}(x_2, \mathcal{X}_2) = d$ and $M \models A_{\alpha,\beta}$ for some $\alpha, \beta \in \mathbb{N}$ with $\alpha \leq 2d$;

(iv) for any even $n \geq 6$ if $\mathbf{t}_n = \mathbf{t}(x_1, \dots, x_n)$ is an n-ary word such that $M \models \mathbf{t}_n[x_i, x_j] \approx \mathcal{X}_n[x_i, x_j]$ for each $\{i, j\} \neq \{1, n\}$, then $M \not\models \mathbf{t}_n[x_1, x_n] \approx \mathcal{Y}(x_n, x_1)$. Then M is not finitely related.

If we delete x_1 from both sides of (1) then we obtain that for every even r, M satisfies:

$$\mathcal{Y}^{y_1}(x_2,\ldots,x_r,y_1,\ldots,y_n) \approx \mathcal{Y}(x_2,\ldots,x_r) * \mathcal{Y}(y_2,\ldots,y_n). \tag{4}$$

If we delete x_1 from both sides of (2) then we obtain that for every odd r, M satisfies:

$$\mathcal{Y}^{y_1}(x_2,\ldots,x_r,y_1,\ldots,y_n) \approx \mathcal{Y}(x_2,\ldots,x_r) * \mathcal{X}(y_2,\ldots,y_n).$$
 (5)

For each even $n \geq 2$ and $1 \leq m \leq n$ define

$$\mathcal{Z}(x_{m+1},\ldots,x_n,x_1,\ldots,x_{m-1}) := \mathcal{X}(x_{m+1},\ldots,x_n,x_1,\ldots,x_{m-1})$$

if m is even and

$$\mathcal{Z}(x_{m+1},\ldots,x_n,x_1,\ldots,x_{m-1}) := \mathcal{Y}(x_{m+1},\ldots,x_n,x_1,\ldots,x_{m-1})$$

if m is odd.

Lemma 3.2. Let M be a monoid that satisfies (i)–(iii) in Theorem 3.1. Fix even $n \geq 2$. Then the family of terms

$$\{\mathbf{t}_{ij} = \mathcal{Z}^{\neq_j}(x_{i+1}, \dots, x_n, x_1, \dots, x_{i-1})x_i^{\alpha} \mid 1 \le i < j \le n\}$$

is a scheme for M.

Proof. First, let us verify that for every even n and $1 \le s \le n, 1 \le m \le n, M$ satisfies

$$\mathcal{Z}^{\not x_m}(x_{s+1},\dots,x_n,x_1,\dots,x_{s-1}) \approx \mathcal{Z}^{\not x_s}(x_{m+1},\dots,x_n,x_1,\dots,x_{m-1}).$$
 (6)

Case 1: If both s and m are odd, we need to verify that

$$M \models \mathcal{Y}^{\not z_m}(x_{s+1}, \dots, x_n, x_1, \dots, x_{s-1}) \approx \mathcal{Y}^{\not z_s}(x_{m+1}, \dots, x_n, x_1, \dots, x_{m-1}).$$

It enough to assume that s = 1 and 1 < m < n.

$$M \models \mathcal{Y}^{\not =_m}(x_2, \dots, x_n) \stackrel{(4)}{\approx}$$

$$\mathcal{Y}(x_2, \dots, x_{m-1}) * \mathcal{Y}(x_{m+1}, \dots, x_n) \stackrel{(3)}{\approx}$$

$$\mathcal{Y}(x_{m+1}, \dots, x_n) * \mathcal{Y}(x_2, \dots, x_{m-1}) \stackrel{(4)}{\approx}$$

$$\mathcal{Y}^{\not x_1}(x_{m+1},\ldots,x_n,x_1,x_2,\ldots,x_{m-1}).$$

Case 2: If both s and m are even, we need to verify that

$$M \models \mathcal{X}^{\not x_m}(x_{s+1},\ldots,x_n,x_1,\ldots,x_{s-1}) \approx \mathcal{X}^{\not x_s}(x_{m+1},\ldots,x_n,x_1,\ldots,x_{m-1}).$$

It enough to assume that s = 2 and $2 < m \le n$.

$$M \models \mathcal{X}^{\not =_m}(x_3, \dots, x_n, x_1) \stackrel{(2)}{\approx}$$

$$\mathcal{X}(x_3, \dots, x_{m-1}) * \mathcal{X}(x_{m+1}, \dots, x_n, x_1) \stackrel{(3)}{\approx}$$

$$\mathcal{X}(x_{m+1}, \dots, x_n, x_1) * \mathcal{X}(x_3, \dots, x_{m-1}) \stackrel{(2)}{\approx}$$

$$\mathcal{X}^{\not =_2}(x_{m+1}, \dots, x_n, x_1, x_2, x_3, \dots, x_{m-1}).$$

Case 3: If s is odd and m is even, we need to verify that

$$\mathcal{Y}^{\not x_m}(x_{s+1},\ldots,x_n,x_1,\ldots,x_{s-1}) \approx \mathcal{X}^{\not x_s}(x_{m+1},\ldots,x_n,x_1,\ldots,x_{m-1}).$$

If we assume that s = 1 and $1 < m \le n$ then:

$$M \models \mathcal{Y}^{\not x_m}(x_2, \dots, x_n) \stackrel{(4)}{\approx}$$

$$\mathcal{Y}(x_2, \dots, x_{m-1}) * \mathcal{X}(x_{m+1}, \dots, x_n) \stackrel{(3)}{\approx}$$

$$\mathcal{X}(x_{m+1}, \dots, x_n) * \mathcal{Y}(x_2, \dots, x_{m-1}) \stackrel{(1)}{\approx}$$

$$\mathcal{X}^{\not x_1}(x_{m+1}, \dots, x_n, x_1, x_2, \dots, x_{m-1}).$$

If we assume that m = 2 and 2 < s < n then:

$$M \models \mathcal{X}^{\not s_s}(x_3, \dots, x_n, x_1) \stackrel{(1)}{\approx}$$

$$\mathcal{X}(x_3, \dots, x_{s-1}) * \mathcal{Y}(x_{s+1}, \dots, x_n, x_1) \stackrel{(3)}{\approx}$$

$$\mathcal{Y}(x_{s+1}, \dots, x_n, x_1) * \mathcal{X}(x_3, \dots, x_{s-1}) \stackrel{(5)}{\approx}$$

$$\mathcal{Y}^{\not s_2}(x_{s+1}, \dots, x_n, x_1, x_2, x_3, \dots, x_{s-1}).$$

Now we use $A_{\alpha,\beta}$ and (6) to verify that for each even $n \geq 2$ the family of terms

$$\{\mathbf{t}_{ij} = \mathcal{Z}^{\not i_j}(x_{i+1}, \dots, x_n, x_1, \dots, x_{i-1})x_i^{\alpha} \mid 1 \le i < j \le n\}$$

is a scheme for M. Indeed, the dependency condition is trivially satisfied. To verify the consistency consider the three cases.

Case 1: If i, j, k, l are pairwise disjoint then

$$M \models \mathbf{t}_{ij}^{(kl)} \approx \mathcal{Z}^{\not z_j, \not z_k, \not z_l}(x_{i+1}, \dots, x_n, x_1, \dots, x_{i-1}) x_j^{\alpha} x_l^{\alpha} \stackrel{(6)}{\approx}$$

$$\mathcal{Z}^{\not x_l,\not x_i,\not x_j}(x_{k+1},\ldots,x_n,x_1,\ldots,x_{k-1})x_l^\alpha x_i^\alpha \approx \mathbf{t}_{kl}^{(ij)}.$$

Case 2: If j = k then $\mathbf{t}_{il} = \mathcal{Z}^{\not x_l}(x_{i+1}, \dots, x_n, x_1, \dots, x_{i-1})x_l^{\alpha}$. Hence

$$M \models \mathbf{t}_{ij}^{(jl)} \approx \mathcal{Z}^{\not x_j, \not x_l}(x_{i+1}, \dots, x_n, x_1, \dots, x_{i-1}) x_l^{\alpha} \stackrel{(6)}{\approx}$$

$$\mathcal{Z}^{\not i_l,\not i_l}(x_{j+1},\ldots,x_n,x_1,\ldots,x_{j-1})x_l^\alpha \approx \mathbf{t}_{il}^{(il)}.$$

Case 3: If i = k and j < l then $\mathbf{t}_{il} = \mathcal{X}^{\not x_l} \{x_{i+1}, \dots, x_n, x_1, \dots, x_{i-1}\} x_l^{\alpha}$. Hence

$$M \models \mathbf{t}_{ij}^{(jl)} \approx \mathcal{Z}^{\not z_j,\not z_l}(x_i,\ldots,x_n,x_1,\ldots,x_{i-1})x_l^{\alpha} \approx \mathbf{t}_{il}^{(jl)}.$$

Observation 3.3. Let M be a monoid, $n \in \mathbb{N}$ and $\mathcal{X}_n = \mathcal{X}(x_1, \dots, x_n)$ be an n-ary word such that Condition (i) of Theorem (3.1) is satisfied. Then:

(i) for each odd $1 \le i < n$, we have

$$M \models \mathcal{X}_n[x_i, x_{i+1}] \approx \mathcal{X}(x_i, x_{i+1}),$$

and for each even $2 \le i < n$, we have

$$M \models \mathcal{X}_n[x_i, x_{i+1}] \approx \mathcal{Y}(x_i, x_{i+1});$$

(ii) If $occ(x_1, \mathcal{X}_2) = occ(x_2, \mathcal{X}_2) = d$ then for each $i \in \mathbb{N}$ we have $occ(x_i, \mathcal{X}_n) = d$ and for each $1 \le i < j \le n$ with $j - i \ge 2$ we have

$$M \models \mathcal{X}_n[x_i, x_i] \approx x_i^d * x_i^d$$
.

Proof of Theorem 3.1. Take even $n \geq 6$. Suppose that the n-ary scheme

$$\{\mathbf{t}_{ij} = \mathcal{Z}^{\neq_j}(x_{i+1}, \dots, x_n, x_1, \dots, x_{i-1})x_j^{\alpha} \mid 1 \le i < j \le n\}$$

comes from a term $\mathbf{t}_n = \mathbf{t}(x_1, \dots, x_n)$ for V(M), that is,

$$M \models \mathbf{t}_n^{(sm)} \approx \mathcal{Z}^{\not t_m}(x_{s+1}, \dots, x_n, x_1, \dots, x_{s-1}) x_m^{\alpha}$$

for each $1 \le s < m \le n$. In particular,

$$M \models \mathbf{t}_n^{(n-1,n)} \approx \mathcal{X}(x_1,\ldots,x_{n-2})x_n^{\alpha}.$$

Using (2) to delete extra variables, we obtain

$$M \models \mathbf{t}_n[x_i, x_{i+1}] \approx \mathcal{X}(x_i, x_{i+1}) \stackrel{Obs. 3.3}{\approx} \mathcal{X}_n[x_i, x_{i+1}]$$

for each i = 1, 3, ..., n - 3.

Using (1) to delete extra variables, we obtain

$$M \models \mathbf{t}_n[x_i, x_{i+1}] \approx \mathcal{Y}(x_i, x_{i+1}) \stackrel{Obs. 3.3}{\approx} \mathcal{X}_n[x_i, x_{i+1}]$$

for each i = 2, 4, ..., n - 4.

Also,

$$M \models \mathbf{t}_n^{(1,2)} \approx \mathcal{X}(x_3, \dots, x_n) x_2^{\alpha}.$$

Using (2) to delete extra variables, we obtain

$$M \models \mathbf{t}_n[x_{n-1}, x_n] \approx \mathcal{X}(x_{n-1}, x_n) \stackrel{Obs.}{\approx} 3.3 \mathcal{X}_n[x_{n-1}, x_n].$$

Using (1) to delete extra variables, we obtain

$$M \models \mathbf{t}_n[x_{n-2}, x_{n-1}] \approx \mathcal{Y}(x_{n-2}, x_{n-1}) \stackrel{Obs.}{\approx} 3.3 \mathcal{X}_n[x_{n-2}, x_{n-1}].$$

Now fix $2 \le i < j \le n$ with $j - i \ge 2$. Then

$$M \models \mathbf{t}^{(i-1,i+1)} \approx \mathcal{Z}^{\not x_{i+1}}(x_i, x_{i+1}, x_{i+2}, \dots, x_n, x_1, \dots, x_{i-2}) x_{i+1}^{\alpha} \stackrel{(2)or(4)}{\approx}$$
$$[\mathcal{Z}(x_i) * \mathcal{Z}(x_{i+2}, \dots, x_n, x_1, \dots, x_{i-2})] x_{i+1}^{\alpha}.$$

If we delete everything except for $\{x_i, x_i\}$ from both sides then we obtain

$$M \models \mathbf{t}[x_i, x_j] \approx x_i^d * x_j^d \overset{Obs.}{\approx} ^{3.3} \mathcal{X}_n[x_i, x_j].$$

Finally,

$$M \models \mathbf{t}_n^{(2,n)} \approx \mathcal{X}^{\not z_n}(x_3, x_4, \dots, x_{n-1}, x_n, x_1) x_n^{\alpha} \stackrel{(1)}{\approx} [\mathcal{X}(x_3, x_4, \dots, x_{n-1}) * \mathcal{X}(x_1)] x_n^{\alpha}.$$

Deleting extra variables we obtain

$$M \models \mathbf{t}[x_1, x_j] \approx x_j^d * x_1^d \stackrel{(3)}{\approx} x_1^d * x_j^d \stackrel{Obs.}{\approx} \stackrel{3.3}{\approx} \mathcal{X}_n[x_1, x_j]$$

for each $3 \le j \le n-1$.

Overall, we conclude that $M \models \mathbf{t}_n[x_i, x_j] \approx \mathcal{X}_n[x_i, x_j]$ for each $\{i, j\} \neq \{1, n\}$. Now consider

$$M \models \mathbf{t}_n^{(n-2,n-1)} \approx \mathcal{Y}(x_n, x_1, x_2, \dots, x_{n-3}) x_{n-1}^{\alpha}.$$

Using (1) to delete extra variables, we obtain

$$M \models \mathbf{t}_n[x_1, x_n] \approx \mathcal{Y}(x_n, x_1).$$

Since this contradicts Condition (iv), such a term \mathbf{t}_n is impossible. Therefore, the scheme does not come from any n-ary term, and M is non-finitely related by Lemma 2.2.

4 Chain, crown, maelstrom and other interlocking words

We use ix to denote the i-th from the left occurrence of a variable x in a word. To show how new interlocking words S_n are similar to the chain C_n , maelstrom \mathcal{M}_n and crown words \mathcal{R}_n , we define all these words recursively as follows.

Definition 4.1.
$$C_0 = \mathcal{M}_0 = \mathcal{R}_0 = \mathcal{S}_0 = 1, C_1 = \mathcal{M}_1 = \mathcal{R}_1 = \mathcal{S}_1 = x_1^2.$$

- If $C_k = \mathbf{c}_k x_k$ then $C_{k+1} = \mathbf{c}_k x_{k+1} x_k x_{k+1}$.
- If k is odd and $\mathcal{M}_k = \mathbf{m}_k x_k$ then $\mathcal{M}_{k+1} = x_{k+1} \mathbf{m}_k x_{k+1} x_k$; If k is even and $\mathcal{M}_k = x_k \mathbf{m}'_k$ then $\mathcal{M}_{k+1} = x_k x_{k+1} \mathbf{m}'_k x_{k+1}$.
- If k is odd and $\mathcal{R}_k = \mathbf{r}_k x_k$ then $\mathcal{R}_{k+1} = \mathbf{r}_k x_{k+1}^2 x_k$; If k is even and $\mathcal{R}_k = \mathbf{r}'_k x_k^2 x_{k-1}$ then $\mathcal{R}_{k+1} = \mathbf{r}'_k x_{k+1} x_k^2 x_{k-1} x_{k+1}$.
- If k is odd and $S_k = \mathbf{s}_k[{}_2x_k]\mathbf{s}'_k$ then $S_{k+1} = x_{k+1}\mathbf{s}_k[{}_2x_k]x_{k+1}\mathbf{s}'_k$; If k is even and $S_k = x_k\mathbf{s}_kx_{k-1}x_k\mathbf{s}'_k$ then $S_{k+1} = x_kx_{k+1}\mathbf{s}_kx_{k+1}x_{k-1}x_k\mathbf{s}'_k$.

For example,

$$S_8 = (x_8 x_6 x_7 x_4 x_5 x_2 x_3 x_1)(x_7 x_8 x_5 x_6 x_3 x_4 x_1 x_2).$$

We say that a word **u** is double-linear if $\mathbf{u} = (x_1 \dots x_n)(x_{\sigma(1)}, \dots, x_{\sigma(n)})$ for some n > 0 and a permutation σ on $\{1, \dots, n\}$. Notice that maelstrom \mathcal{M}_n and the new words \mathcal{S}_n are double-linear. For the rest of this note we fix * to be the following operation on double-linear words:

• If \mathbf{u} and \mathbf{v} are double linear then $\mathbf{u} * \mathbf{v} := \mathbf{u}_1 \mathbf{v}_1 \mathbf{u}_2 \mathbf{v}_2$, where \mathbf{u}_1 , \mathbf{u}_2 , \mathbf{v}_1 and \mathbf{v}_2 are linear words such that $\mathbf{u} = \mathbf{u}_1 \mathbf{u}_2$ and $\mathbf{v} = \mathbf{v}_1 \mathbf{v}_2$.

This operation is associative and $1 * \mathbf{w} = \mathbf{w} * 1 = \mathbf{w}$ for each double-linear \mathbf{w} .

Notice that S_{k+1} is obtained from S_k by inserting ${}_1x_{k+1}$ before the first linear part of S_k and inserting ${}_2x_{k+1}$ immediately to the right of the first letter in the second linear part of S_k when k is odd, and inserting them vice versa when k is even. Having this in mind, we give three more definitions to the words S_n . First, recall that

$$\mathcal{Y}(x_2,\ldots,x_n):=\mathcal{S}^{\not x_1}(x_1,x_2,\ldots,x_n).$$

Definition 4.2. Let $n \geq 3$ and $S_n = S(x_1, ..., x_n)$ be an n-ary word such that $S_n[x_1, x_2, x_3] = (x_2x_3x_1)(x_3x_1x_2)$. Then the following are equivalent:

$$\mathcal{S}^{\not x_m}(x_1,\ldots,x_n) = \mathcal{S}(x_1,\ldots,x_{m-1}) * \mathcal{Y}(x_{m+1},\ldots,x_n)$$

for each odd $1 \le m \le n$, and

$$\mathcal{S}^{\not x_m}(x_1,\ldots,x_n) = \mathcal{S}(x_1,\ldots,x_{m-1}) * \mathcal{S}(x_{m+1},\ldots,x_n)$$

for each even $1 \le m \le n$.

- (ii) for each odd $1 \le i < n$ we have $S_n[x_i, x_{i+1}] = x_{i+1}x_i^2x_{i+1}$, for each even $2 \le i < n$ we have $S_n[x_i, x_{i+1}] = x_ix_{i+1}^2x_i$, and for each $1 \le i < j \le n$ with $j i \ge 2$ we have $S_n[x_i, x_j] = x_j^2 * x_i^2 = x_jx_ix_jx_i$.
- (iii) for each odd $1 \le i < n$ we have $S_n[x_i, x_{i+1}] = x_{i+1}x_i^2x_{i+1}$, for each even $2 \le i < n$ we have $S_n[x_i, x_{i+1}] = x_ix_{i+1}^2x_i$, for each $1 \le i < j \le n$ with j i = 2 or j i = 3 we have $S_n[x_i, x_j] \in \{x_ix_jx_ix_j, x_jx_ix_j\}$, and for each odd $3 \le j \le n$ we have $S_n[x_1, x_j] \in \{x_1x_jx_1x_j, x_jx_1x_jx_1\}$.
 - (iv) S_n is as in Definition 4.1.

Proof. When **u** contains $_{i}x_{j}y$ as a subword, we write $(_{i}x) \ll_{\mathbf{u}} (_{j}y)$. Clearly, the statement holds for n=3. Suppose that it holds for each $3 \leq n \leq k$. Take n=k+1.

Implication (i) \rightarrow (ii) is by Observation 3.3. (Take M to be the free monoid.) Implication (ii) \rightarrow (iii) is evident.

(iii) \rightarrow (iv) By induction hypothesis, S_k is as in Definition 4.1.

If k is odd then S_k begins with $x_{k-1}x_k$ and the second linear part of S_k begins with x_kx_{k-2} . Since $S_{k+1}[x_k, x_{k+1}] = x_{k+1}x_k^2x_{k+1}$, either ${}_1x_{k+1}$ is the first in S_{k+1} or we have

$$(_1x_{k-1}) \ll_{\mathbf{u}} (_1x_{k+1}) \ll_{\mathbf{u}} (_1x_k),$$

where $\mathbf{u} = \mathcal{S}_{k+1}$. In both cases we must have $\mathcal{S}_{k+1}[x_{k-2}, x_{k+1}] = x_{k+1}x_{k-2}x_{k+1}x_{k-2}$. Hence the only possibility for the second occurrence of x_{k+1} is:

$$({}_2x_k) \ll_{\mathbf u} ({}_2x_{k+1}) \ll_{\mathbf u} ({}_2x_{k-2}),$$

where $\mathbf{u} = \mathcal{S}_{k+1}$. Then we must have $\mathcal{S}_{k+1}[x_{k-1}, x_{k+1}] = x_{k+1}x_{k-1}x_{k+1}x_{k-1}$. Therefore, the letter $_1x_{k+1}$ must be the first in \mathcal{S}_{k+1} . The resulting word is \mathcal{S}_{k+1} as in Definition 4.1.

If k is even then S_k begins with $x_k x_{k-2}$ and the second linear part of S_k begins with $x_{k-1}x_k$. Since $S_{k+1}[x_k, x_{k+1}] = x_k x_{k+1}^2 x_k$ and $S_{k+1}[x_{k-2}, x_{k+1}] = x_{k+1} x_{k-2} x_{k+1} x_{k-2}$, the only possibility for the first occurrence of x_{k+1} is:

$$({}_1x_k) \ll_{\mathbf{u}} ({}_1x_{k+1}) \ll_{\mathbf{u}} ({}_1x_{k-2}),$$

where $\mathbf{u} = \mathcal{S}_{k+1}$. Then $\mathcal{S}_{k+1}[x_{k-1}, x_{k+1}] = x_{k+1}x_{k-1}x_{k+1}x_{k-1}$. Therefore, the second occurrence of x_{k+1} must precede the second occurrence of x_{k-1} . Then $\mathcal{S}_{k+1}[x_1, x_{k+1}] = x_{k+1}x_1x_{k+1}x_1$. This leaves only one possibility for the second occurrence of x_{k+1} :

$$(_1x_1) \ll_{\mathbf{u}} (_2x_{k+1}) \ll_{\mathbf{u}} (_2x_{k-1}),$$

where $\mathbf{u} = \mathcal{S}_{k+1}$. The resulting word \mathcal{S}_{k+1} as in Definition 4.1.

Implication (iv)
$$\rightarrow$$
(i) is a routine.

Let $\bar{*}$ denote the operation on the double-linear words which is dual to the operation *. Here are the four definitions of the words \bar{S}_n .

Definition 4.3. Let $n \geq 3$ and $\bar{S}_n = \bar{S}(x_1, \ldots, x_n)$ be an n-ary word such that $\bar{S}_n[x_1, x_2, x_3] = (x_2x_1x_3)(x_1x_3x_2)$. Then the following are equivalent:

$$\bar{\mathcal{S}}^{\not x_m}(x_1,\ldots,x_n) = \bar{\mathcal{S}}(x_1,\ldots,x_{m-1})\bar{\ast}\mathcal{Y}(x_{m+1},\ldots,x_n)$$

for each odd $1 \le m \le n$, and

$$\bar{\mathcal{S}}^{\not x_m}(x_1,\ldots,x_n) = \bar{\mathcal{S}}(x_1,\ldots,x_{m-1})\bar{*}\bar{\mathcal{S}}(x_{m+1},\ldots,x_n)$$

for each even $1 \le m \le n$.

- (ii) for each odd $1 \leq i < n$ we have $\bar{S}_n[x_i, x_{i+1}] = x_{i+1}x_i^2x_{i+1}$, for each even $2 \leq i < n$ we have $\bar{S}_n[x_i, x_{i+1}] = x_ix_{i+1}^2x_i$, and for each $1 \leq i < j \leq n$ with $j i \geq 2$ we have $\bar{S}_n[x_i, x_j] = x_i^2 \bar{*} x_j^2 = x_i x_j x_i x_j$.
- (iii) for each odd $1 \le i < n$ we have $\bar{S}_n[x_i, x_{i+1}] = x_{i+1}x_i^2x_{i+1}$, for each even $2 \le i < n$ we have $\bar{S}_n[x_i, x_{i+1}] = x_ix_{i+1}^2x_i$, for each $1 \le i < j \le n$ with j i = 2 or j i = 3 we have $\bar{S}_n[x_i, x_j] \in \{x_ix_jx_ix_j, x_jx_ix_jx_i\}$ and for each odd $3 \le j \le n$ we have $\bar{S}_n[x_1, x_j] \in \{x_1x_jx_1x_j, x_jx_1x_jx_1\}$.
 - (iv) If k is odd and $\bar{S}_k = \mathbf{s}'_k[_1x_k]\mathbf{s}_k$ then $\bar{S}_{k+1} = \mathbf{s}'_kx_{k+1}[_1x_k]\mathbf{s}_kx_{k+1}$; If k is even and $\bar{S}_k = \mathbf{s}'_kx_kx_{k-1}\mathbf{s}_kx_k$ then $\bar{S}_{k+1} = \mathbf{s}'_kx_kx_{k-1}x_{k+1}\mathbf{s}_kx_{k+1}x_k$.

Notice that Condition (iii) in Definitions 4.2 and 4.3 is the same. The words \bar{S}_n compared to S_n are enumerated in the opposite direction. For example,

$$\bar{\mathcal{S}}_3 = (x_2 x_1 x_4 x_3 x_6 x_5 x_8 x_7)(x_1 x_3 x_2 x_5 x_4 x_7 x_6 x_8).$$

The following theorem is similar to Theorem 4.13 in [2].

Theorem 4.4. Let M be a monoid that satisfies $A_{4,\beta}$ for some $\beta \geq 1$ and for every $r, k \in \mathbb{N}$ we have

$$M \models \mathcal{S}(x_1,\ldots,x_r) * \mathcal{S}(y_1,\ldots,y_k) \approx \mathcal{S}(y_1,\ldots,y_k) * \mathcal{S}(x_1,\ldots,x_r),$$

where $S_n = S(x_1, ..., x_n)$ is as in Definition 4.2.

Suppose that $S(x,y) = yx^2y$ is an isoterm for M and the words $\{xyxy, yxyx\}$ can form an identity of M only with each other. Then M is non-finitely related.

Proof. If $\mathcal{X} = \mathcal{S}$ then Condition (i) of Theorem 3.1 is satisfied by Definition 4.2(i). Clearly M satisfies Condition (iii) of Theorem 3.1 with d = 2.

Now fix even $n \geq 6$ and let $\mathbf{t}(x_1, \ldots, x_n)$ be an n-ary term such that $M \models \mathbf{t}_n[x_i, x_j] \approx \mathcal{S}_n[x_i, x_j]$ for each $\{i, j\} \neq \{1, n\}$. Then for each odd $1 \leq i \leq n - 1$ we have $M \models \mathbf{t}_n[x_i, x_{i+1}] \approx x_{i+1}x_i^2x_{i+1}$, for each even $2 \leq i \leq n - 2$ we have $M \models \mathbf{t}_n[x_i, x_{i+1}] \approx x_ix_{i+1}^2x_i$. Since n > 4, $M \models \mathbf{t}_n[x_i, x_{i+2}] \approx x_jx_ix_jx_i$ for each $1 \leq i < j \leq n$ with j - i = 2 or j - i = 3. Also for each $j = 3, 5, \ldots, n - 1$ we have $M \models \mathbf{t}_n[x_1, x_j] \approx x_jx_1x_jx_1$.

Since xy^2x is an isoterm for M and the words $\{xyxy, yxyx\}$ can form an identity of M only with each other, for each odd $1 \le i \le n-1$ we have $\mathbf{t}_n[x_i, x_{i+1}] = x_{i+1}x_i^2x_{i+1}$, for each even $2 \le i \le n-2$ we have $\mathbf{t}_n[x_i, x_{i+1}] = x_{i+1}x_i^2x_{i+1}$, and for each

 $1 \le i < j \le n$ with j - i = 2 or j - i = 3 we have $\mathbf{t}_n[x_i, x_j] \in \{x_i x_j x_i x_j, x_j x_i x_j x_i \}$. Also for each $j = 3, 5, \dots, n - 1$ we have $\mathbf{t}_n[x_1, x_j] \in \{x_1 x_j x_1 x_j, x_j x_1 x_j x_1 \}$.

If $\mathbf{t}_n[x_1, x_3] = x_3 x_1 x_3 x_1$, then $\mathbf{t}_n[x_1, x_2, x_3] = \mathcal{S}_3 = (x_2 x_3 x_1)(x_3 x_1 x_2)$. Definition 4.2[(iii) \rightarrow (ii)] implies that $\mathbf{t}_n = \mathcal{S}_n$ and $\mathbf{t}_n[x_1, x_n] = \mathcal{S}_n[x_1, x_n] = x_n x_1 x_n x_1$.

If $\mathbf{t}_n[x_1, x_3] = x_1 x_3 x_1 x_3$, then $\mathbf{t}_n[x_1, x_2, x_3] = \bar{\mathcal{S}}_3 = (x_2 x_1 x_3)(x_1 x_3 x_2)$. Definition 4.3[(iii) \rightarrow (ii)] implies that $\mathbf{t}_n = \bar{\mathcal{S}}_n$ and $\mathbf{t}_n[x_1, x_n] = \bar{\mathcal{S}}_n[x_1, x_n] = x_1 x_n x_1 x_n$.

Since xy^2x is an isoterm for M, the monoid M does not satisfy $\mathbf{t}_n[x_1, x_n] \approx \mathcal{Y}[x_n, x_1] = x_n x_1^2 x_n$. Therefore, M is non-finitely related by Theorem 3.1. \square

Theorem 4.4 together with Lemma 2.1 readily gives us the following.

Example 4.5. The monoid $M(a^2b^2, ab^2a)$ is non-finitely related.

5 More definitions of chain and maelstrom words

Notice that Definitions 4.2 and 4.3 play the key role in proving Theorem 4.4. In this section we want to give similar definitions to the chain and maelstrom words. These definitions can be used in a similar way to reprove Theorems 4.4 and 4.18 in [2].

Let \odot denote the 'wrapping' operation in [2, Definition 4.10], but in the reverse order, that is, $\mathbf{w} \odot \mathbf{v}$ means that \mathbf{v} wraps around \mathbf{w} . This operation is also associative and $1 \odot \mathbf{w} = \mathbf{w} \odot 1 = \mathbf{w}$ for each double-linear \mathbf{w} . Here are three more definitions of maelstrom words.

Definition 5.1. Let $n \geq 3$ and $\mathcal{M}_n = \mathcal{M}(x_1, \ldots, x_n)$ be an n-ary word such that $\mathcal{M}_n[x_1, x_2, x_3] = (x_2 x_3 x_1)(x_2 x_1 x_3)$. Then the following are equivalent:

$$\mathcal{M}^{\not z_m}(x_1,\ldots,x_n)=\mathcal{M}(x_1,\ldots,x_{m-1})\odot\mathcal{Y}(x_{m+1},\ldots,x_n)$$

if $1 \le m \le n$ is odd, and

$$\mathcal{M}^{\mathscr{I}_m}(x_1,\ldots,x_n)=\mathcal{M}(x_1,\ldots,x_{m-1})\odot\mathcal{M}(x_{m+1},\ldots,x_n)$$

if $1 \le m \le n$ is even.

(ii) for each odd $1 \le i < n$ we have $\mathcal{M}_n[x_i, x_{i+1}] = x_{i+1}x_ix_{i+1}x_i$, for each even $2 \le i < n$ we have $\mathcal{M}_n[x_i, x_{i+1}] = x_ix_{i+1}x_ix_{i+1}$, and for each $1 \le i < j \le n$ with $j-i \ge 2$ we have $\mathcal{M}_n[x_i, x_j] = x_i^2 \odot x_j^2 = x_jx_i^2x_j$.

(iii) for each odd $1 \le i < n$ we have $\mathcal{M}_n[x_i, x_{i+1}] = x_{i+1}x_ix_{i+1}x_i$, for each even $2 \le i < n$ we have $\mathcal{M}_n[x_i, x_{i+1}] = x_ix_{i+1}x_ix_{i+1}$, and for each $1 \le i < j \le n$ with j - i = 2 or j - i = 3 we have $\mathcal{M}_n[x_i, x_j] \in \{x_jx_i^2x_j, x_ix_j^2x_i\}$.

(iv) \mathcal{M}_n is as in Definition 4.1.

Proof. As in Definition 4.2 only one implication needs verification.

(iii) \rightarrow (iv) By induction hypothesis, \mathcal{M}_k is a maelstrom word as in Definition 4.1. If k is odd then we need to insert x_{k+1} into \mathcal{M}_k so that $\mathcal{M}_{k+1}[x_k, x_{k+1}] = x_{k+1}x_kx_{k+1}x_k$. Then either ${}_1x_{k+1}$ is the first letter in \mathcal{M}_{k+1} or we have

$$(_1x_{k-1}) \ll_{\mathbf{u}} (_1x_{k+1}) \ll_{\mathbf{u}} (_1x_k),$$

where $\mathbf{u} = \mathcal{M}_{k+1}$. In both cases, we must have $\mathcal{M}_{k+1}[x_{k-2}, x_{k+1}] = x_{k+1}x_{k-2}^2x_{k+1}$. This leaves only one possibility for $2x_{k+1}$:

$$(2x_{k-2}) \ll_{\mathbf{u}} (2x_{k+1}) \ll_{\mathbf{u}} (2x_k).$$

Hence we must have $\mathcal{M}_{k+1}[x_{k-1}, x_{k+1}] = x_{k+1}x_{k-1}^2x_{k+1}$. This implies that ${}_1x_{k+1}$ is the first letter in \mathcal{M}_{k+1} . The resulting word \mathcal{M}_{k+1} is maelstrom by Definition 4.1.

If k is even then we need to insert x_{k+1} into \mathcal{M}_k so that $\mathcal{M}_{k+1}[x_k, x_{k+1}] = x_k x_{k+1} x_k x_{k+1}$. Then either $2x_{k+1}$ is the last letter in \mathcal{M}_{k+1} or we have

$$(2x_k) \ll_{\mathbf{u}} (2x_{k+1}) \ll_{\mathbf{u}} (2x_{k-1}),$$

where $\mathbf{u} = \mathcal{M}_{k+1}$. In both cases, we must have $\mathcal{M}_{k+1}[x_{k-2}, x_{k+1}] = x_{k+1}x_{k-2}^2x_{k+1}$. This leaves only one possibility for ${}_1x_{k+1}$:

$$({}_1x_k) \ll_{\mathbf{u}} ({}_1x_{k+1}) \ll_{\mathbf{u}} ({}_1x_{k-2}).$$

Hence we must have $\mathcal{M}_{k+1}[x_{k-1}, x_{k+1}] = x_{k+1}x_{k-1}^2x_{k+1}$. This implies that $2x_{k+1}$ is the last letter in \mathcal{M}_{k+1} . The resulting word \mathcal{M}_{k+1} is maelstrom by Definition 4.1. \square

Let $\bar{\odot}$ denote the operation on the double-linear words which is dual to the operation $\bar{\odot}$. (This will be the operation in [2, Definition 4.10].) Here are the four definitions of the words $\bar{\mathcal{M}}_n$.

Definition 5.2. Let $n \geq 3$ and $\overline{\mathcal{M}}_n = \overline{\mathcal{M}}(x_1, \ldots, x_n)$ be an n-ary word such that $\overline{\mathcal{M}}_n[x_1, x_2, x_3] = (x_2x_1x_3)(x_2x_3x_1)$. Then the following are equivalent:

$$\bar{\mathcal{M}}^{\not x_m}(x_1,\ldots,x_n) = \bar{\mathcal{M}}(x_1,\ldots,x_{m-1})\bar{\odot}\mathcal{Y}(x_{m+1},\ldots,x_n)$$

if $1 \le m \le n$ is odd, and

$$\bar{\mathcal{M}}^{\not x_m}(x_1,\ldots,x_n) = \bar{\mathcal{M}}(x_1,\ldots,x_{m-1})\bar{\odot}\bar{\mathcal{M}}(x_{m+1},\ldots,x_n)$$

if $1 \le m \le n$ is even.

(ii) for each odd $1 \le i < n$ we have $\bar{\mathcal{M}}_n[x_i, x_{i+1}] = x_{i+1}x_ix_{i+1}x_i$, for each even $2 \le i < n$ we have $\bar{\mathcal{M}}_n[x_i, x_{i+1}] = x_ix_{i+1}x_ix_{i+1}$, and for each $1 \le i < j \le n$ with $j-i \ge 2$ we have $\bar{\mathcal{M}}_n[x_i, x_j] = x_i^2 \bar{\odot} x_j^2 = x_i x_j^2 x_i$.

(iii) for each odd $1 \le i < n$ we have $\bar{\mathcal{M}}_n[x_i, x_{i+1}] = x_{i+1}x_ix_{i+1}x_i$, for each even $2 \le i < n$ we have $\bar{\mathcal{M}}_n[x_i, x_{i+1}] = x_ix_{i+1}x_ix_{i+1}$, and for each $1 \le i < j \le n$ with j - i = 2 or j - i = 3 we have $\bar{\mathcal{M}}_n[x_i, x_j] \in \{x_jx_i^2x_j, x_ix_j^2x_i\}$.

(iv) If k is odd and $\overline{\mathcal{M}}_k = \mathbf{m}_k({}_1x_k)\mathbf{m}'_k$ then $\overline{\mathcal{M}}_{k+1} = \mathbf{m}_kx_{k+1}({}_1x_k)x_{k+1}\mathbf{m}'_k$; If k is even and $\overline{\mathcal{M}}_k = \mathbf{m}_k({}_2x_k)\mathbf{m}'_k$ then $\overline{\mathcal{M}}_{k+1} = \mathbf{m}_kx_{k+1}({}_2x_k)x_{k+1}\mathbf{m}'_k$.

Again, as for S_n and \bar{S}_n , Condition (iii) in Definitions 5.1 and 5.2 is the same. Observe that it is more simple for maelstrom words. The words $\bar{\mathcal{M}}_n$ can be obtained from \mathcal{M}_n by switching the order of linear parts. For example,

$$\mathcal{M}_8 = (x_8 x_6 x_7 x_4 x_5 x_2 x_3 x_1)(x_2 x_1 x_4 x_3 x_6 x_5 x_8 x_7)$$

$$\bar{\mathcal{M}}_8 = (x_2 x_1 x_4 x_3 x_6 x_5 x_8 x_7)(x_8 x_6 x_7 x_4 x_5 x_2 x_3 x_1)$$

Finally, three more definitions of the chain words.

Definition 5.3. Let $n \geq 3$ and $C_n = C(x_1, ..., x_n)$ be an n-ary word such that $C_n[x_1, x_2] = x_1x_2x_1x_2$. Then the following are equivalent:

(i)

$$\mathcal{C}^{\not x_m}(x_1,\ldots,x_n) = \mathcal{C}(x_1,\ldots,x_{m-1})\mathcal{C}(x_{m+1},\ldots,x_n)$$

for any $1 \le m \le n$;

- (ii) for each $1 \leq i \leq n-1$ we have $C_n[x_i, x_{i+1}] = x_i x_{i+1} x_i x_{i+1}$ and for each $1 \leq i < j \leq n$ with $j-i \geq 2$ we have $C_n[x_i, x_j] = x_i^2 x_j^2$.
- (iii) for each $1 \leq i \leq n-1$ we have $C_n[x_i, x_{i+1}] = x_i x_{i+1} x_i x_{i+1}$ and for each $1 \leq i < j \leq n$ with j-i=2 we have $C_n[x_i, x_j] \neq x_i x_j x_i x_j$.
 - (iv) C_n is as in Definition 4.1.

Proof. Again, only one implication needs verification.

(iii) \rightarrow (iv) By induction hypothesis, \mathcal{C}_k is a chain word as in Definition 4.1. Then there is only one possibility to insert x_{k+1} in \mathcal{C}_k so that $\mathcal{C}_{k+1}[x_k, x_{k+1}] = x_k x_{k+1} x_k x_{k+1}$ and $\mathcal{C}_{k+1}[x_{k-1}, x_{k+1}] \neq x_{k-1} x_{k+1} x_{k-1} x_{k+1}$. The resulting word \mathcal{C}_{k+1} is chain by Definition 4.1.

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