COSC1285/2123: Algorithms & Analysis

Time and Space Tradeoffs

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Lecture 7

Overview

Levitin – The design and analysis of algorithms

This week we will be covering the material from Chapter 7.

Learning outcomes:

- Understand space-time tradeoffs in algorithm design.
- Two of the paradigms:
  - input enhancement (sort by counting)
  - prestructuring (hashing)

Outline

Overview

1. Overview
2. Sort by Counting
3. Hash Tables
4. Separate Chaining Hashing
5. Open Address Hashing
6. Summary

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1. **Input Enhancement** - preprocess the input to store extra information that will accelerate the solving of the problem.
   - counting sorts
2. **Prestructuring** - use extra space to make accessing its elements easier or faster.
   - hashing

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- Distribution sort does exactly this, in a smart way.

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12	4	12	9	12	15	9	4	9

4	9	12	15
0	0	0	0

1	2	3	4	5	6	7	8	9

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Distribution Sorting

ALGORITHM **DistCountSort** ($A[0 \ldots n-1], u, l$)
// Sort an array by distribution counting
// INPUT : An array $A[0 \ldots n-1]$ of orderable integers between l and u ($l \leq u$), and $n_{\text{max}} = u - l$
// OUTPUT : An array $S[0 \ldots n-1]$ of integers sorted in nondecreasing order.

for $j = 0$ **to** n_{max} **do** ▷ Initialize frequencies
  $\sigma[j] = 0$
end for
for $i = 0$ **to** $n - 1$ **do** ▷ Compute frequencies
  $\sigma[A[i] - l] = \sigma[A[i] - l] + 1$
end for
for $j = 1$ **to** n_{max} **do** ▷ Compute cumulative frequencies
  $\sigma[j] = \sigma[j-1] + \sigma[j]$
end for
for $i = n - 1$ **down to** 0 **do**
  $j = A[i]$
  $S[\sigma[j] - 1] = A[i]$
  $\sigma[j] = \sigma[j] - 1$
end for
return S

Distribution Sorting

The worst-case analysis for distribution sorting is:

$$
\begin{aligned}
C(n) &= \underbrace{\sum_{j=0}^{n_{\max}} 1}_{\text{initialise freqs}} + \underbrace{\sum_{i=0}^{n-1} 1}_{\text{compute freqs}} + \underbrace{\sum_{j=0}^{n_{\max}} 1}_{\text{compute cumulative freq}} + \underbrace{\sum_{i=0}^{n-1} 1}_{\text{copy values}} \\
&= 2\sum_{i=0}^{n-1} 1 + 2\sum_{j=0}^{n_{\max}} 1 \\
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$$= 2\mathcal{O}(n) + 2\mathcal{O}(n_{\max})$$

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The algorithm also uses an additional $\mathcal{O}(n) + \mathcal{O}(n_{\max})$ space.

Overview

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Is it possible to achieve better efficiency?

Hash Tables

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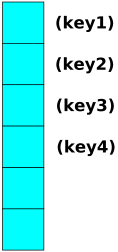

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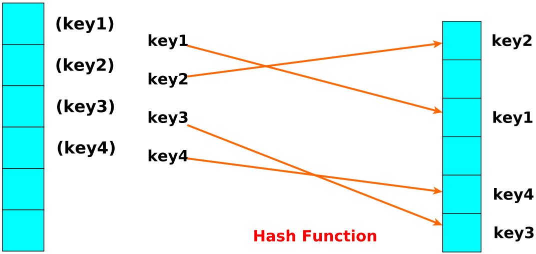

(g) Array (h) Hash table

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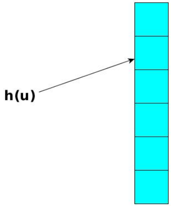
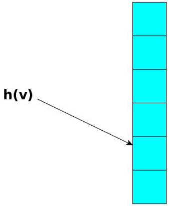
Collisons:

- If two distinct keys u and v map to the same array position/index, i.e., $h(u) = h(v)$, we say that a collision has occurred.

Hash Tables



(i) h(u)



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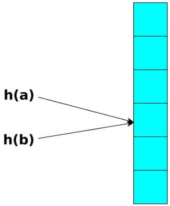

(k) Collision

Hash Table Choices

- Hash function.
- Size of hash table.
- Collison resolution.

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- Example: $h(u) = u \mod n$, produces a position index between 0 and $n - 1$.

Perfect Hash Functions (static set)

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- If we have a static set, we can achieve perfect hashing and $\mathcal{O}(1)$ average (and worst) case timing (given array is big enough).
- One approach to achieve this bound is to generate a perfect hash function for all of the elements *a priori*.
- Example 1 : Given $S_1 = \{10, 21, 32, 43, 54, 65, 76, 87\}$, then the function $h_1(x) = x \mod 10$ is perfect.
- Example 2 : Given $S_2 = \{110, 210, 310, \dots, 810\}$, then the function $h_2(x) = (x - 10)/100$ is perfect.

Size of Hash Table

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Solutions?

- Choose an initial $n \approx p$ and a prime number (reduce collison chance, but not waste space).
- If dynamic set and p becomes bigger than n, increase size of table (n) and rehash all existing keys.

Collision Resolution

There are two major approaches to handle collisons:

1. Separate Chaining Hashing.
2. Open Address Hashing.

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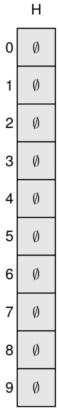
- For completeness, if no key hashed to a position, set linked list pointer to null.

Separate Chaining Hashing – Example



A chained hash table for the sequence $\mathcal{T}_c =$ “APJQDHBWM”.

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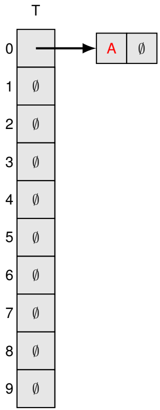

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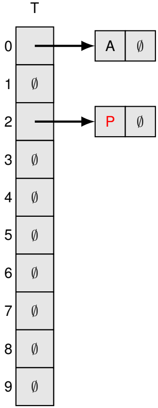

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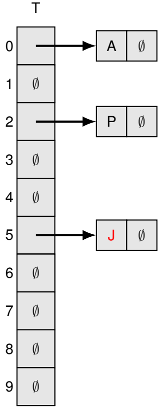

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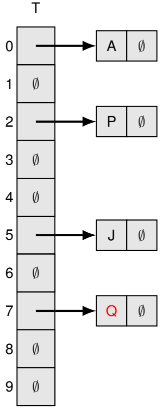

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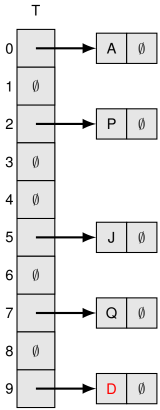

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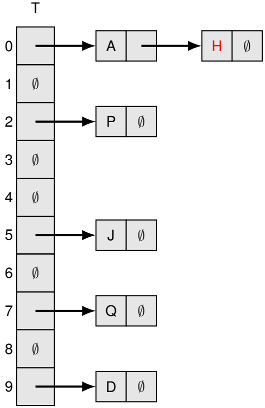

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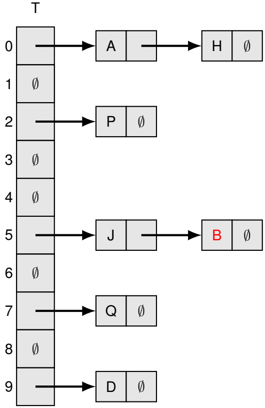

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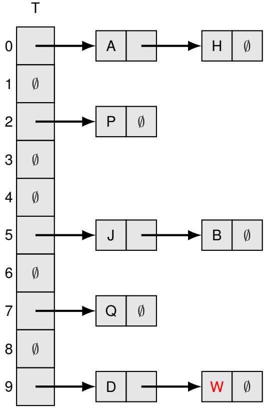

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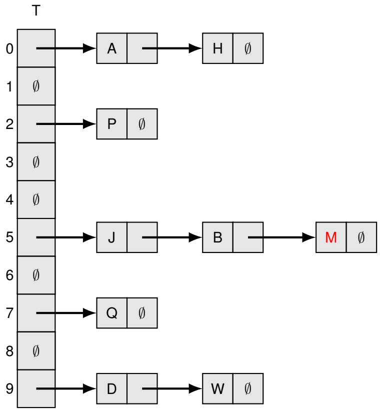

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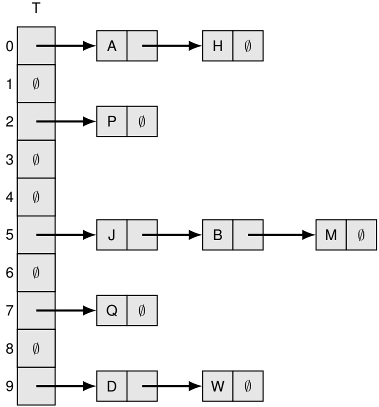

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- Average case time is $\mathcal{O}(1)$ for all operations, assuming simple uniform hashing (distribute keys uniformally).

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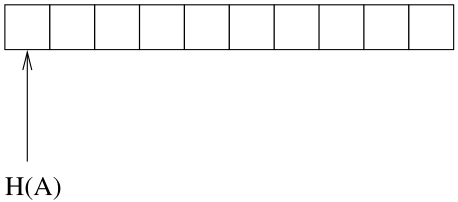
- The load α is typically kept small (and ideally it is 1).

Overview

Open Address Hashing – Overview

- Open address hashing is an alternative method to handle collisions.
- Each cell in the base array can store exactly one item.
- Linear probing - store the item in the next free cell.
- Double hashing - use a second hash function to compute the increment.

Open Address Hashing: Linear probing



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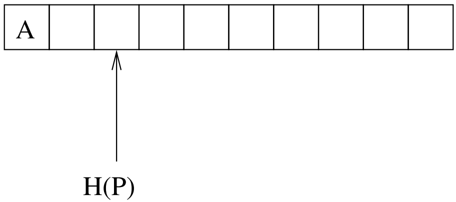

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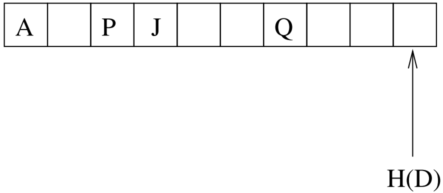

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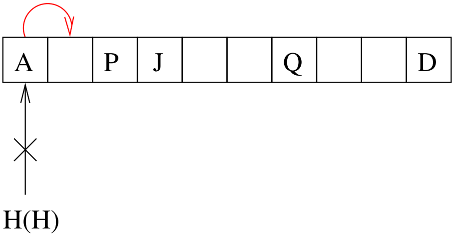

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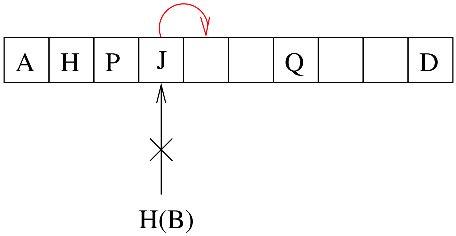

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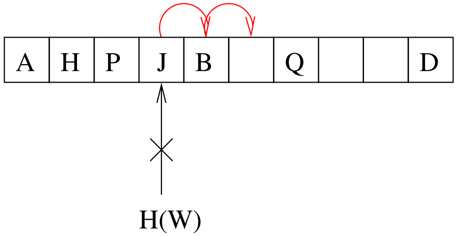

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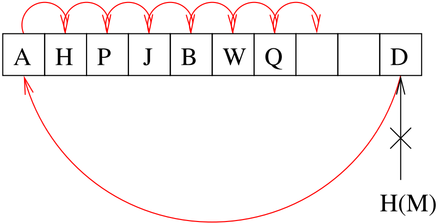

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A	H	P	J	B	W	Q	M		D

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- Number of probes for the three key set operations depend on the load factor $\alpha = p/n$.
- For linear probing, a successful search is $\approx \frac{1}{2}(1 + 1/(1-\alpha))$ probes and an unsuccessful search is $\approx \frac{1}{2}(1 + 1/(1-\alpha)^2)$ probes.
- It is much more difficult to derive these bounds than in the separate chaining hash table.

Linear probing – Analysis

As the table fills ($\alpha \to 1$), the number of probes necessary increases dramatically.

α	$\frac{1}{2}(1 + \frac{1}{1-\alpha})$	$\frac{1}{2}(1 + \frac{1}{(1-\alpha)^2})$
50%	1.5	2.5
75%	2.5	8.5
90%	5.5	50.5

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If h_2 is chosen well, we can avoid clustering effects, which can lead to faster collison resolution.

Double hashing – Comments

- Difficult to analyse the complexity of successful and unsuccessful searches.
- Empirically shown double hashing performs better than linear probing, especially when table is more full.

Other Applications of Hashing

Checksums and signatures:

`https://www.youtube.com/watch?v=b4b8ktEV4Bg`

Locality sensitive hashing:

`https://www.youtube.com/watch?v=Ha7_Vf2eZvQ`

Overview

Summary

The two types of space-time tradeoffs discussed:

- **input enhancement**
  - preprocess input and store relevant information that speeds up solving the problem.
  - e.g., distribution sorting
- **prestructuring**
  - construct data structures (space) that have faster or more flexible access to data.
  - e.g., hashing