COSC1285/2123: Algorithms & Analysis

Algorithmic Analysis

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Lecture 2

Outline

Overview

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Levitin – The design and analysis of algorithms

This week we will be covering the material from Chapter 2.

Learning outcomes:

- Understand why it is important to be able to compare the complexiy of algorithms.
- Be able to:
  - measure complexity of algorithms and compare complexity classes.
  - analysis of non-recursive algorithms.
  - analysis of recursive algorithms.
- Be able to perform empirical analysis of algorithms.

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- Solution A: Sequential search.

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- Solution A: Sequential search.
- Solution B: Sort then search.

https://goo.gl/forms/34JJe2n9bAlX9eBP2

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Space is also important but we focus on time in this course.

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These operations are called **basic operations** and the number of times is based on the **input size** of the problem.

Running example: a^n

This algorithm computes a^n:

```
// INPUT : a, n
// OUTPUT : s = a^n
set s = 1
for i = 1 to n do
    s = s * a
end for
return s
```

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More examples of basic operation and input size

Problem Type	Basic Operation	Input Size
Iterating through an array, of size n, to print its contents	?	n
Comparing between all pairs of points to find closest pair (n number of points).	Comparison	?

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$$t(n) \approx c_{op} \times C(n)$$

- $t(n)$ is the running time.
- n is the input size.
- c_{op} is the execution time for a basic operation.
- $C(n)$ is the number of times the basic operation is executed.

Estimating algorithm for a^n

What is the theoretical running time for our algorithm for computing a^n?

Recall: $t(n) \approx c_{op} \times C(n)$

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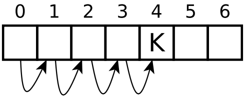
Example: Searching for a key in n items using Sequential Search



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What is $C(n)$, the number of times the basic operation is executed?

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NOTE : **Average Case** is *not* the average of the worst and best case. Rather, it is the average performance across all possible inputs.

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Best-case : The best case input is when the item being searched for is the first item in the list, so $C_b(n) = 1$.

Worst-case : The worst case input is when the item being searched for is not present in the list, so $C_w(n) = n$

Sequential Search

Average-case : What does average-case mean?

- Recall: average across all possible inputs – how to analyse this?
- Typically not straight forward.

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Sequential Search Example:

- How often do we search for items in the array?
    - Where in the array do we find the item? 1st, 2nd, . . ., last position?
- How often do we search for items **not** in the array?

Sequential Search

Average-case Analysis : Skipping a few steps (notes of full derivation available on Canvas) and p is the probability of a successful search.

$$C_{avg}(n) = \frac{p(n+1)}{2} + n(1-p)$$

If $p = 1$, then $C_{avg}(n) = (n+1)/2$.

If $p = 0$, then $C_{avg}(n) = n$.

Summary for this part

- Input size, basic operation
- Time complexity estimate using input size and basic operation
- Best, worst, and average cases

Overview

Asymptotic Complexity

Problem:

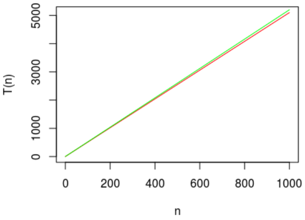
- We now have a way to analyse the running time (aka time complexity) of an algorithm, but every algorithms have their own time complexities. How to compare in a meaningful way?

Asymptotic Complexity

Consider the running times estimates of two algorithms:

$$\text{Algorithm 1: } T_1(n) = 5.1n$$
$$\text{Algorithm 2: } T_2(n) = 5.2n$$

Similar timing profiles as n grows?



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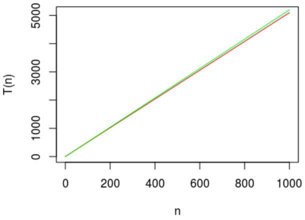

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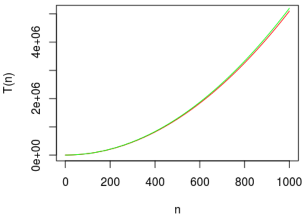

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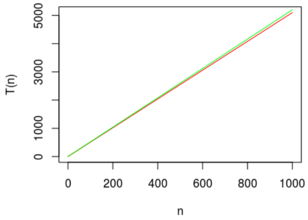

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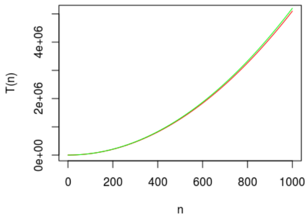

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Solution:

- Group them into equivalence classes (for easier comparison and understanding), with respect to the input size.
- Focus of this part, asymptotic complexity and equivalence classes.

Asymptotic Complexity - bounds

Warning: Some (mathematical) definitions coming up!

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Idea: Use bounds and asymptotic complexity (as n becomes large, what is the dominant term contributing to the running time ($t(n)$)?)

Asymptotic Complexity, Upper bounds

Definition: Given a function $t(n)$ (the running time of an algorithm):

- Let $c \times g(n)$ be a function that is an **upper** bound on $t(n)$ for some $c > 0$ and for *"large"* n.

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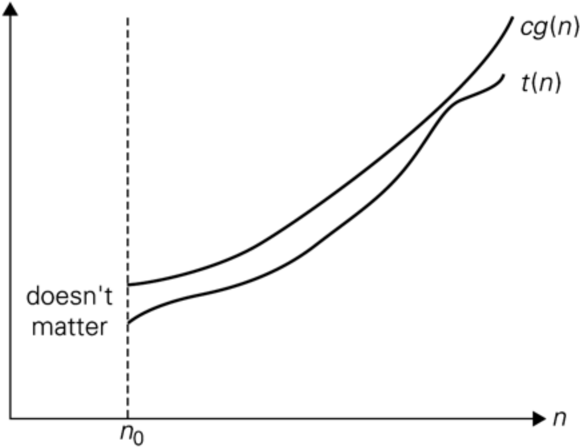
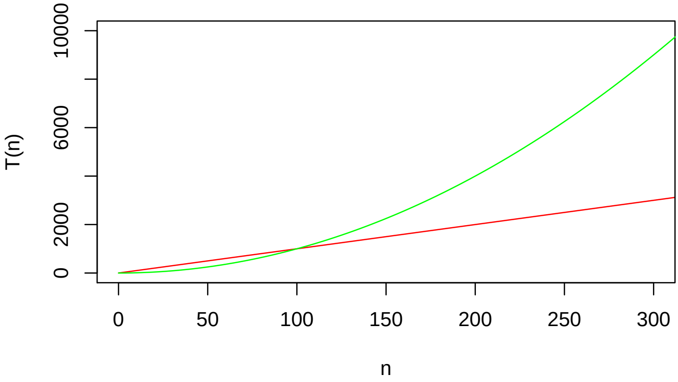


Upper bounds Examples



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What $g(n)$ to use if $t(n) = 5.2n$? Any of the above $g(n)$ functions are possible!

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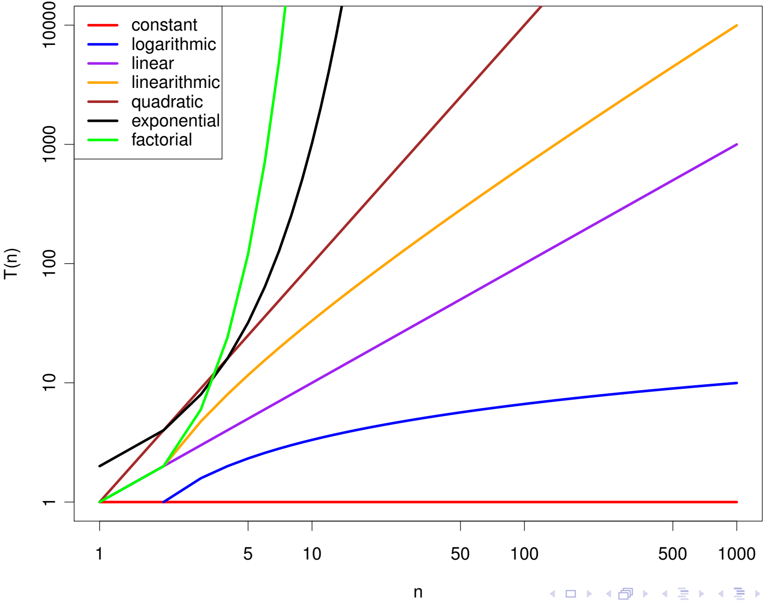
First we look at common equivalence classes:

- Constant - $\mathcal{O}(1)$: Access array element
- Logarithmic - $\mathcal{O}(\log n)$: Binary search
- Linear - $\mathcal{O}(n)$: Link list search
- Linearithmic (Supralinear) - $\mathcal{O}(n \log n)$: Merge Sorting
- Quadratic - $\mathcal{O}(n^2)$: Selection Sorting
- Exponential - $\mathcal{O}(2^n)$: Generating all subsets
- Factorial - $\mathcal{O}(n!)$: Generating all permutations

Common Complexity Bounds



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Returning to previous example, we can write the two upper bounds $\mathcal{O}(0.01n - 6)$ and $\mathcal{O}(n)$ as $\mathcal{O}(n)$.

Example

Let

$$g(n) = 2n^4 + 43n^3 - n + 50$$

be an **upper** bound for the running time $t(n)$ of an algorithm.

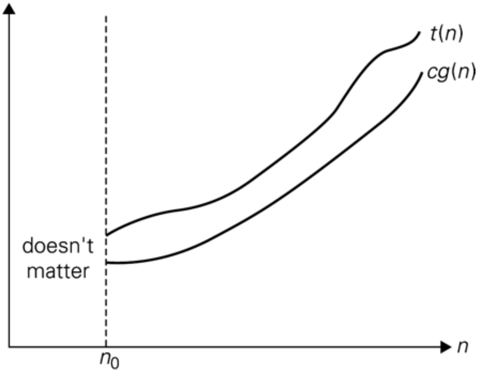
Using the previous guidelines, what equivalence **class** should you use to describe the **order of growth** of the algorithm?

`https://goo.gl/forms/nybF74i02wBFUXWP2`

Asymptotic Complexity, Lower Bound ($\Omega(n)$)

Lower Bound Definition: Given a function $t(n)$,

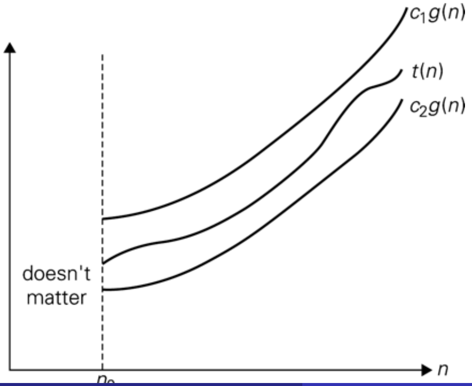
- Informally: $t(n) \in \Omega(g(n))$ means $g(n)$ is a function that, as n increases, is a lower bound of $t(n)$
- Formally: $t(n) \in \Omega(g(n))$, if $g(n)$ is a function and $c \times g(n)$ is a **lower** bound on $t(n)$ for some $c > 0$ and for *"large"* n.



Asymptotic Complexity, Exact Bounds ($\Theta(n)$)

Exact Bound Definition: Given a function $t(n)$,

- Informally: $t(n) \in \Theta(g(n))$ means $g(n)$ is a function that, as n increases, is both a upper and lower bound of $t(n)$
- Formally: $t(n) \in \Theta(g(n))$, if $g(n)$ is a function and $c_1 \times g(n)$ is an **upper** bound on $t(n)$ and $c_2 \times g(n)$ is an **lower** bound on $t(n)$, for some $c_1 > 0$ and $c_2 > 0$ and for *"large" n*



Examples

$t(n)$	$\mathcal{O}(n)$	$\mathcal{O}(n^2)$	$\mathcal{O}(n^3)$	$\Omega(n)$	$\Omega(n^2)$	$\Omega(n^3)$	Θ
$\log_2 n$	T	T	T	F	F	F	$\Theta(\log_2 n)$
$10n + 5$	T	T	T	T	F	F	$\Theta(n)$
$n(n-1)/2$	F	T	T	T	T	F	$\Theta(n^2)$
$(n+1)^3$	F	F	T	T	T	T	$\Theta(n^3)$
2^n	F	F	F	T	T	T	$\Theta(2^n)$

For example, $10n + 5$ is in $\mathcal{O}(n)$.

Terminology Clarification

- $\mathcal{O}(n)$ is not the same thing as "Worst Case Efficiency".
- $\Omega(n)$ is not the same thing as "Best Case Efficiency".
- $\Theta(n)$ is not the same thing as "Average Case Efficiency".

It is perfectly reasonable to want the $\Omega(n)$ *and* $\mathcal{O}(n)$ worst case efficiency bounds for a class of algorithms.

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It is perfectly reasonable to want the $\Omega(n)$ *and* $\mathcal{O}(n)$ worst case efficiency bounds for a class of algorithms.

Why all the bounds?

- Generally $\mathcal{O}(n)$ is most commonly used , but exact bounds tell us the bounds are tight and the algorithm doesn't have anything outside what we expect.
- Lower bounds are useful to describe the (theoretical) limits of whole classes of algorithms, and also sometimes useful to state how fast can the best case reach.

Overview

Time Efficiency of Algorithms

Typically, we are given the pseudo code of an algorithm, not a nice $t(n)$ function.

Question

So how do we determine **bounds** on the **order of growth** of an algorithm?

- Next two parts, we focus on answering this question.

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Question

So how do we determine **bounds** on the **order of growth** of an algorithm?

- Next two parts, we focus on answering this question.
- We first study how to do this for non-recursive algorithms, then recursive ones.
- Keep in mind: Generally, we first need to estimate the running time $t(n)$, then determine a bound and order of growth.

Time Efficiency of Non-Recursive Algorithms

Question

So how do we determine **bounds** on the **order of growth** of an algorithm?

Recall $t(n) = c_{op}C(n)$.

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3. Determine the number of basic operation executions in terms of the input size? (i.e., How to determine $C(n)$?)
   - Setup a summation for $C(n)$ reflecting the algorithm's loop structure.
   - Simplify the summation by using standard formulas in Appendix A of textbook.

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4. Determine a equivalence class $g(n)$ that bounds $t(n)$. Recall we generally want the tightest bound possible.

Non Recusive Example: a^n

// INPUT : a, n
// OUTPUT : $s = a^n$

set $s = 1$
for $i = 1$ to n **do**
    $s = s * a$
end for
return

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4: **end for**
5: **return**

Input size: n
Basic operation: multiplication
$C(n)$**:** ? $\underbrace{1 + 1 + 1 + \ldots + 1}_{n}$

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Input size: n
Basic operation: multiplication
$C(n)$**:** ? $\underbrace{1 + 1 + 1 + \ldots + 1}_{n} = \sum_{i=1}^{n} 1$

Given two (square) matrices A and B, both of dimensions n by n, the following algorithm computes $C = A + B$.

```
for (int i = 0; i <= n-1; i++) {
  for (int j = 0; j <= n-1; j++) {
    C[i,j] = A[i,j] + B[i,j];
  }
}
```

Non-Recursive Examples (continued): Adding two matrices

Given two (square) matrices A and B, both of dimensions n by n, the following algorithm computes $C = A + B$.

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for (int i = 0; i <= n-1; i++) {
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   }
}
```

Input size: ? n
Basic operation: ?

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}
```

Input size: ? n
Basic operation: ? addition
$C(n)$**:** ?

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for (int i = 0; i <= n-1; i++) {
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   }
}
```

Input size: ? n

Basic operation: ? addition

$C(n)$**:** ?

$$\sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1$$

Time Efficiency of Non-Recursive Algorithms

Question

So how do we determine **bounds** on the **order of growth** of an algorithm?

Recall $t(n) = c_{op}C(n)$.

1. Determine what is used to measure the input size.
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Useful series (from Appendix A)

- R1 (Sum): $\sum_{i=l}^{n} 1 = n - l + 1$.
- R2 (Geometric): $\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$.
- R3 (Distributive): $\sum_{i=1}^{n} c * a_i = c \sum_{i=1}^{n} a_i$.
- R4 (Associative): $\sum_{i=1}^{n} (a_i \pm b_i) = \sum_{i=1}^{n} a_i \pm \sum_{i=1}^{n} b_i$.

Non Recusive Example a^n: Simplification

// INPUT : a, n
// OUTPUT : $s = a^n$

```
set s = 1
for i = 1 : n do
    s = s * a
end for
return
```

$C(n) = \sum_{i=1}^{n} 1$

Non Recusive Example a^n: Simplification

$$C_(n) = \sum_{i=1}^{n} 1 \tag{1}$$

(2)

(3)

Non Recusive Example a^n: Simplification

$$C_(n) = \sum_{i=1}^{n} 1 \tag{1}$$

$$= n - 1 + 1 \text{ (Use R1)} \tag{2}$$

$$\tag{3}$$

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$$= n \tag{3}$$

Non-Recursive Example Adding two matrices: Simplication

Given two (square) matrices A and B, both of dimensions n by n, the following algorithm computes $C = A + B$. **Example:**

```
for (int i = 0; i <= n-1; i++) {
  for (int j = 0; j <= n-1; j++) {
    C[i,j] = A[i,j] + B[i,j];
  }
}
```

$C(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1$

Non-Recursive Example Adding two matrices: Simplication

$$C_(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1 \tag{1}$$

(2)

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(4)

(5)

(6)

Non-Recursive Example Adding two matrices: Simplication

$$C_(n) = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1 \tag{1}$$

$$= \sum_{i=0}^{n-1} (n - 1 + 1) \text{ (Use R1 on inner summation)} \tag{2}$$

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Non-Recursive Example Adding two matrices: Simplication

$$
\begin{align}
C_(n) &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1 \tag{1}\\
&= \sum_{i=0}^{n-1} (n - 1 + 1) \text{ (Use R1 on inner summation)} \tag{2}\\
&= \sum_{i=0}^{n-1} (n) \text{ (Simplify)} \tag{3}\\
& \tag{4}\\
& \tag{5}\\
& \tag{6}
\end{align}
$$

Non-Recursive Example Adding two matrices: Simplication

$$
\begin{aligned}
C_(n) &= \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} 1 && (1) \\
&= \sum_{i=0}^{n-1} (n - 1 + 1) \text{ (Use R1 on inner summation)} && (2) \\
&= \sum_{i=0}^{n-1} (n) \text{ (Simplify)} && (3) \\
&= n \sum_{i=0}^{n-1} 1 \text{ (Use R3 to take } n \text{ out of summation)} && (4) \\
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\end{aligned}
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&= n * (n - 1 + 1) \text{ (Use R1 on remaining summation)} \tag{5}\\
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Non-Recursive Example Adding two matrices: Simplication

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$$= n * n \text{ (Simplify)} \tag{6}$$

Overview

Recursion

- Recursion is fundamental tool in computer science.
- A recursive program (or function) is one that calls itself.
- It must have a termination condition defined.

Recursion

- Recursion is fundamental tool in computer science.
- A recursive program (or function) is one that calls itself.
- It must have a termination condition defined.
- Many interesting algorithms are simply expressed with a recursive approach.

Recursive Algorithm Example: Factorial

Factorial: $\mathcal{F}(n) = n * (n-1) * (n-2) * (n-3) * \ldots * 3 * 2 * 1$

ALGORITHM $\mathcal{F}(n)$

if $n = 1$ **then**
    return 1
else
    return $\mathcal{F}(n-1) * n$
end if

Time Efficiency of Recursive Algorithms

1. Determine what is used to measure the input size.
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1. Determine what is used to measure the input size.
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3. Setup a recurrence relation for $C(n)$, including termination condition(s).
4. Simplify the recurrence relation using methods in Appendix B of textbook. (**Backward substitution**)
5. Determine a equivalence class $g(n)$ that bounds $t(n) = c_{op}C(n)$. Recall we generally want the tightest bound possible.

Recurrence Relations

A **recurrence relation** represents a sequence of terms, that results from a recursion on one or more of the previous terms.

The recurrence relation include the termination condition.

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$$F(n) = F(n-1) * n, \; F(1) = 1$$

Time Efficiency of Recursive Algorithms

1. Determine what is used to measure the input size.
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Recursive Algorithm Example: Factorial

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ALGORITHM F(n)
if n = 1 then
    return 1
else
    return F(n − 1) ∗ n
end if
```

Recursive Algorithm Example: Factorial

ALGORITHM $\mathcal{F}(n)$

```
if n = 1 then
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Basic operation: ?

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Recursive Algorithm Example: Factorial

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if $n = 1$ **then**
  return 1
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  return $\mathcal{F}(n-1) * n$
end if

Basic operation: ? multiplication **Input size:** ? n,

Recurrence relation and conditions: The number of multiplications is $C(n) = C(n-1) + 1$ for $n > 1$, and $C(1) = 0$.

Recursive Algorithm Example: Factorial

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if n = 1 then
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- "$+1$" is the number of multiplication operations at each recursive step.

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- "$+1$" is the number of multiplication operations at each recursive step.
- When $n = 1$, we have our termination/base case, where we stop the recursion. When we reach this base case, the number of multiplcations is 0, hence $C(1) = 0$.

Time Efficiency of Recursive Algorithms

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Backward Substitution: Factorial Example

Recurrence: $C(n) = C(n-1) + 1$ for $n > 1$, and $C(1) = 0$

Aim of simplification and backward substitution: Convert $C(n) = C(n-1) + 1$ to $C(n) =$ function(n), e.g., $C(n) = n + 1$.

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4. Spot the pattern of $C(n)$ and introduce a variable to express this pattern.

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5. Determine when the value of this variable that relates $C(n) = C(1) + \ldots$

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5. Determine when the value of this variable that relates $C(n) = C(1) + \ldots$
6. Substitue the value of $C(1)$ and get $C(n)$ in terms of some expression of n.

Backward Substitution: Factorial Example

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Recurrence: $C(n) = C(n-1) + 1$ for $n > 1$, and $C(1) = 0$

1. $C(n) = C(n-1) + 1$
2. Substitute $C(n-1) = C(n-2) + 1$ into original equation
3. $C(n) = [C(n-2) + 1] + 1 = C(n-2) + 2$

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Hence $t(n) = c_{op} \cdot (n-1) \in \mathcal{O}(n)$.

Summary of these parts

- (non-recursive algorithm) Discussed now to write down expression for C(n) (number of basic operations executed in algorithm).
- Discussed how to simply this for non-recursive algorithm.
- (recursive algorithm) Discussed the idea of recurrence and how to write C(n).
- Discussed how to simply via backward substitution.

Next week: Empirical evaluation & Rule of thumb estimation of complexity!

Overview

Theory vs Practice

Formal Analysis (theoretical):

- **Pro:** No interference from implementation / hardware details.
- **Con:** Hides constant factors. Requires a different mindset.

Empirical Analysis (measure):

- **Pro:** Discovers the true impact of constant factors.
- **Con:** May be running on the “wrong” inputs.

“*In theory, theory and practice are the same. In practice, they are not.*”
– Albert Einstein

Empirical Analysis

- The complexity of an algorithm gives an estimate of the running time (we discussed this in detail).
- Measuring the *actual* time an implementation takes is also important, especially when comparing two algorithms with the same time complexity.
- When in doubt, **measure!**

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7. Analyse the data obtained.

Benchmarking Algorithms: Significance of input

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Empirical: Running Times (in seconds) for different sorting algorithms on a randomised list:

List size	500	2,500	10,000
Merge-Sort	0.8	8.1	39.8
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For ordered lists for $N = 2,500$:

	Random	In Order	In Reverse Order
Merge-Sort	8.1	7.8	7.9
Quick-Sort	1.3	35.1	37.1
Selection-Sort	35.0	34.4	35.3

Comparing Search Algorithms: Significance of input size

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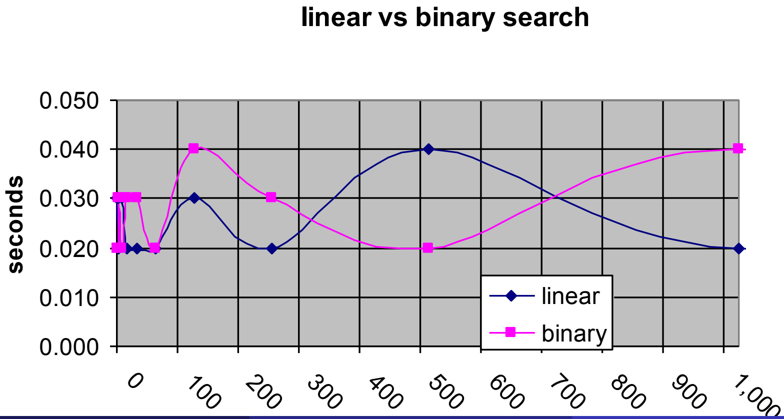
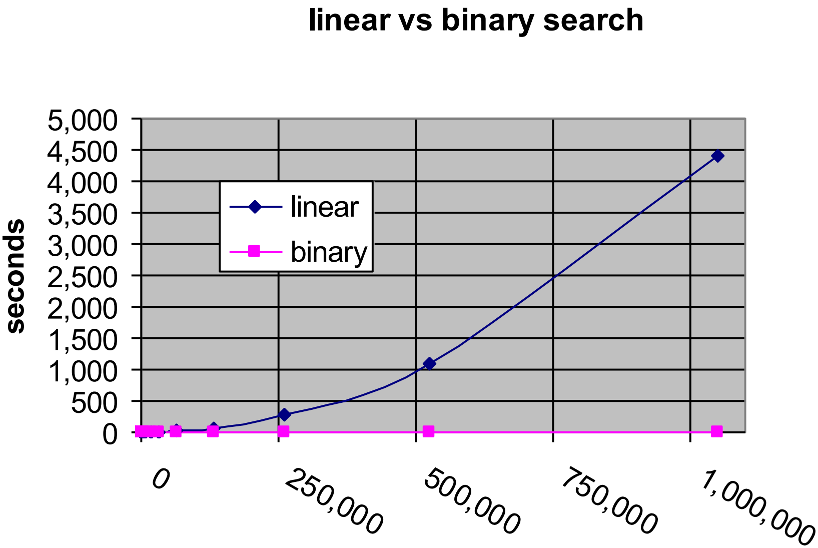
Thereotical: Linear search have $\mathcal{O}(n)$ complexity while binary search have $\mathcal{O}(\log(n))$ complexity on a sorted list.




linear vs binary search
seconds
5,000
4,500
4,000
3,500
3,000
2,500
2,000
1,500
1,000
500
0
linear
binary
0
250,000
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750,000
1,000,000

Overview

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- Requires experience.

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Overview

Summary

- We have covered the core theoretical framework for algorithmic analysis which will be used for the rest of the semester.
  - Problem input size, basic operation, time complexity, asymptotic complexity, worst/best/average cases.
  - Analysis of Non-recursive, resurive algorithms.
  - Empirical analysis.
  - Approximate analysis.

Next week, we look at first of the algorithmic paradigms in this course, brute force algorithms.