COSC1285/2123: Algorithms & Analysis

Dynamic Programming

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Lecture 10

Overview

Levitin – The design and analysis of algorithms

This week we will be covering the material from Chapter 8.

Learning outcomes:

- Understand and be able to apply dynamic programming to solving problems.
- Examples:
  - Coin-row problem
  - Computing the edit distance
  - Knapsack
  - Transitive closure – Warshall's algorithm

Outline

Overview

Dynamic Programming

Dynamic Programming is a general algorithm approach for solving problems using the solutions of (overlapping) subproblems.

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Main idea:

1. Setup a recurrence relating a solution of larger instances to the solutions of smaller instances.
2. Solve smaller instances once.
3. Record solutions in a table.
4. Extract solutions to the initial instance from the table, i.e., use solutions of smaller instances to construct solutions of larger initial problem instance.

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Difference?

- Dynamic programming can be thought of as divide-and-conquer and storing sub-solutions.
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- Dynamic programming can be thought of as divide-and-conquer and storing sub-solutions.
- Why have both then?
  - Divide-and-conquer algorithms are preferred when the sub-problems/instances are independent, e.g., Mergesort.
  - Dynamic programming approach better when the sub-problems/instances are dependent, i.e., the solution to a sub-problem may be needed multiple times. Hence saving solutions allow them to be reused rather than recomputed. Tradeoff space (more) for time (faster).

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Two basic approaches to dynamic programming.
For both first: Study a recursive divide-and-conquer algorithm and figure out the dependencies between the subproblems.

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   - Start with a divide-and-conquer algorithm, and begin dividing recursively.
   - Only solve/recurse on a subproblem if the solution is not available in the table.
   - Save solutions to subproblems in a table.
2. Bottom-Up
   - Solve all subproblems, and use solutions to subproblems to construct solutions to larger problems.

Dynamic Programming example: Coin-row Problem

Coin-row Problem

Given a row of n coins with positive integer values $c_1, c_2, \ldots, c_n$ (not necessarily distinct), pick up the maximum amount of money with the constraint that no two adjacent coins can be selected.

EXAMPLE: $5, 1, 2, 10, 6, 2$. What is the best selection?

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Formally:

$$F(n) = \max\{c_n + \text{total value of solution to the } n-2 \text{ coin sub-problem}, \\ \text{total value of solution to the } n-1 \text{ coin sub-problem}\} \\ \text{for } n > 1,$$

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index (i)	0	1	2	3	4	5	6
coins	–	5	1	2	10	6	2
$F(i)$							

Coin-row Problem (continued)

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index (i)	0	1	2	3	4	5	6
coins	–	5	1	2	10	6	2
$F(i)$	0						

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index (i)	0	1	2	3	4	5	6
coins	–	5	1	2	10	6	2
$F(i)$	0	5					

$F[0] = 0, F[1] = c_1 = 5$

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index (i)	0	1	2	3	4	5	6
coins	–	5	1	2	10	6	2
$F(i)$	0	5	5				

$F[2] = \max(1 + 0, 5) = 5$

Coin-row Problem (continued)

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index (i)	0	1	2	3	4	5	6
coins	–	5	1	2	10	6	2
$F(i)$	0	5	5	7			

$F[3] = \max(2 + 5, 5) = 7$

Coin-row Problem (continued)

$$F(n) = \max\{c_n + F(n-2), F(n-1)\} \text{ for } n > 1,$$
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index (i)	0	1	2	3	4	5	6
coins	–	5	1	2	10	6	2
$F(i)$	0	5	5	7	15		

$F[4] = \max(10 + 5, 7) = 15$

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index (i)	0	1	2	3	4	5	6
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$F[5] = \max(6 + 7, 15) = 15$

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index (i)	0	1	2	3	4	5	6
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$F(i)$	0	5	5	7	15	15	**17**

$F[6] = \max(2 + 15, 15) = 17$

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- Continue until we reach $F(0)$

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Question

What is the worst case time and space complexity of this solution?

Overview

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Consider the problem of comparing two sequences/strings. You may:

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Solution? Direct position by position comparison will not work.

One possible solution is the edit distance.

Edit Distance

Definition

The Levenshtein distance or edit distance of two strings/sequences S_1 and S_2 is the minimum number of point mutations required to change S_1 into S_2, where a point mutation is one of:

1. change a symbol,
2. insert a symbol, or
3. delete a symbol.

- A classic problem for dynamic programming solutions.

Edit Distance

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$$
\begin{array}{ccccccccccc}
S_1 & = & A & T & A & T & A & T & A & \color{red}{T} \\
 & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\
S_2 & = & \color{red}{T} & A & T & A & T & A & T & A
\end{array}
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- The Hamming distance $d(S_1, S_2) = 8$.
- Because of its operations, the Edit Distance can allow shifts in order to maximise the matching. Example:

$$\begin{array}{cccccccccccc} S_1 & = & - & A & T & A & T & A & T & A & T \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ S_2 & = & T & A & T & A & T & A & T & A & - \end{array}$$

- The Edit distance $Ed(S_1, S_2) = 2$.

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Lets study how to match $X[1 \dots n]$ and $Y[1 \dots m]$ and develop a recurrence relationship that will tell us how.

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Case 1 ($X[n] == Y[m]$): This means the last characters match, so no additional distance to add to total edit distance so far.

- Hence, edit distance $M[n, m]$ = edit distance of matching the substrings $X[1 \dots n-1]$ and $Y[1 \dots m-1]$.

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- Or, $M[n, m] = M[n-1, m-1]$.
- E.g., matching strings

$$\begin{array}{cccc} X & = & a & b & c \\ Y & = & u & v & c \end{array}$$

Edit Distance

Case 2 ($X[n]! = Y[m]$): This means the last characters mismatch, and we need to apply one of the edit operations (delete, insert, change). Which one to choose, and which calculated edit distance to reuse?

- "Delete" $X[n]$ from X, at the cost of 1 unit to total edit distance. Hence $M[n, m] = 1 + M[n-1, m]$.

$$\begin{array}{lclll} X & = & a & b & d \\ Y & = & a & b & \end{array}$$

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- "Change" either $X[n]$ or $Y[m]$ into the other, so the characters now match. Hence $M[n, m] = 1 + M[n-1, m-1]$.

$$\begin{array}{lclll} X & = & a & b & d \\ Y & = & a & b & e \end{array}$$

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Computing the Edit Distance

$$M[n, m] = \begin{cases} M[n-1, m-1] & \text{if } X[n] = Y[m] \\ 1 + \min\left(M[n-1, m-1], M[n-1, m], M[n, m-1]\right) & \text{otherwise} \end{cases}$$

$$M[n, 0] = n, M[0, m] = m$$

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Recursive definition! Similar to row-coin problem, we build a dynamic programming table/matrix for $M[n, m]$.

I.e., To compute $M[n, m]$, a table $M[1 \ldots n, 1 \ldots m]$ is computed and filled.

Edit Distance – Example

		a	n	n	e	a	l	i	n	g
	0	1	2	3	4	5	6	7	8	9
a	1									
n	2									
n	3									
u	4									
a	5									
l	6									

Compute $Ed(\texttt{annual}, \texttt{annealing})$.

Edit Distance – Example

		a	n	n	e	a	l	i	n	g
	0	1	2	3	4	5	6	7	8	9
a	1	0								
n	2									
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a	5									
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Compute Ed(`annual`, `annealing`).

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Compute $Ed(\texttt{annual}, \texttt{annealing})$.

Edit Distance – Example

		a	n	n	e	a	l	i	n	g
	0	1	2	3	4	5	6	7	8	9
a	1	0	1	2						
n	2									
n	3									
u	4									
a	5									
l	6									

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Edit Distance – Example

		a	n	n	e	a	l	i	n	g
	0	1	2	3	4	5	6	7	8	9
a	1	0	1	2	3					
n	2									
n	3									
u	4									
a	5									
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Edit Distance – Example

		a	n	n	e	a	l	i	n	g
	0	1	2	3	4	5	6	7	8	9
a	1	0	1	2	3	4				
n	2									
n	3									
u	4									
a	5									
l	6									

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	0	1	2	3	4	5	6	7	8	9
a	1	0	1	2	3	4	5			
n	2									
n	3									
u	4									
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		a	n	n	e	a	l	i	n	g
	0	1	2	3	4	5	6	7	8	9
a	1	0	1	2	3	4	5	6		
n	2									
n	3									
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n	2									
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	0	1	2	3	4	5	6	7	8	9
a	1	0	1	2	3	4	5	6	7	8
n	2	1								
n	3									
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Compute $Ed(\texttt{annual}, \texttt{annealing})$.

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	0	1	2	3	4	5	6	7	8	9
a	1	0	1	2	3	4	5	6	7	8
n	2	1	0							
n	3									
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a	1	0	1	2	3	4	5	6	7	8
n	2	1	0	1						
n	3									
u	4									
a	5									
l	6									

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		a	n	n	e	a	l	i	n	g
	0	1	2	3	4	5	6	7	8	9
a	1	0	1	2	3	4	5	6	7	8
n	2	1	0	1	2					
n	3									
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Edit Distance – Example

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	0	1	2	3	4	5	6	7	8	9
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n	2	1	0	1	2	3				
n	3									
u	4									
a	5									
l	6									

Compute $Ed(\texttt{annual}, \texttt{annealing})$.

Edit Distance – Example

And so on ...

Edit Distance – Example

		a	n	n	e	a	l	i	n	g
	0	1	2	3	4	5	6	7	8	9
a	1	0	1	2	3	4	5	6	7	8
n	2	1	0	1	2	3	4	5	6	7
n	3	2	1	0	1	2	3	4	5	6
u	4	3	2	1	1	2	3	4	5	6
a	5	4	3	2	2	1	2	3	4	5
l	6	5	4	3	3	2	1	2	3	4

Compute $Ed(\texttt{annual}, \texttt{annealing})$.

Edit Distance – Backtrace

- In addition to minimum edit distance, we would like to know what is the sequence of operations (insertion, deletion, substitution, match) to obtain this edit distance and how the strings align.
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- From (m,n) cell (bottom right corner of table), evaluate which operation and adjacent cell we used to arrive to (m,n). Repeat this process until traverse to (0,0) cell.
- This produces a path from (0,0) to (m,n) that is non-decreasing in edit distance.
- May not be unique (i.e., several alignments/sequence of operations) lead to same edit distance. This means multiple backtraces.

Edit Distance – Backtrace

		a	n	n	e	a	l	i	n	g
	0	1	2	3	4	5	6	7	8	9
a	1	0	1	2	3	4	5	6	7	8
n	2	1	0	1	2	3	4	5	6	7
n	3	2	1	0	1	2	3	4	5	6
u	4	3	2	1	1	2	3	4	5	6
a	5	4	3	2	2	1	2	3	4	5
l	6	5	4	3	3	2	1	2	3	4

Compute *Ed*(`annual`, `annealing`).

Edit Distance

- The worst-case and average-case time complexity for table construction is is $\Theta(nm)$ where $|S_1| = n$ and $|S_2| = m$. Backtrace complexity is $\Theta(m + n)$. The solution also uses $\Theta(nm)$ space.

Edit Distance

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- DEMO: `http://www.let.rug.nl/kleiweg/lev/`
- The canonical reference on Edit Distance is: D. Gusfield. *Algorithms on Strings, Trees, and Sequences.* Cambridge University Press, New York, New York, USA, 1997.

Overview

Knapsack Problem

Knapsack Problem

Given n items of known weights $w_1, \ldots, w_n$ and the values $v_1, \ldots, v_n$ and a knapsack of capacity W, find the most valuable subset of the items that fit into the knapsack.

- Recall that the exact solution for all instances of this problem has been proven to be $\Omega(2^n)$.
- We can solve the problem using dynamic programming in "pseudo-polynomial" time.

DP Knapsack Problem - Sketch

- Consider an instance of the knapsack problem defined by the first i items, $1 \leq i \leq n$, with weights $w_1 \ldots w_i$, values $v_1 \ldots v_i$, and capacity j, $1 \leq j \leq W$.

DP Knapsack Problem - Sketch

- Consider an instance of the knapsack problem defined by the first i items, $1 \leq i \leq n$, with weights $w_1 \ldots w_i$, values $v_1 \ldots v_i$, and capacity j, $1 \leq j \leq W$.
- Let $V[i, j]$ be an optimal value to the subproblem instance of having the first i items and a knapsack capacity of j.

DP Knapsack Problem - Sketch

- Consider we are solving a (sub)instance of the knapsack problem $V[i, j]$
- We can divide all the subsets of the first i items that fit into the knapsack of capacity j into two categories:

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  - Among the subsets that do not include the i-th item, the value of the optimal subset is, by definition, $V[i-1, j]$
  - Among the subsets that do include the i-th item ($j - w_i \geq 0$), an optimal subset is made up of this item and an optimal subset of the first $i-1$ items that fit into the knapsack of capacity $j - w_i$. The value of such an optimal subset is $v_i + V[i-1, j-w_i]$.

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  - Among the subsets that do include the i·th item ($j - w_i \geq 0$), an optimal subset is made up of this item and an optimal subset of the first $i-1$ items that fit into the knapsack of capacity $j - w_i$. The value of such an optimal subset is $v_i + V[i-1, j-w_i]$.
- Whether we choose to include i·th item dependent on whether the i·th item can fit into knapsack and if so, which option leads to larger value ($V[i, j]$).

DP Knapsack Problem - Sketch

This leads to the following recursion:

$$V[i,j] = \begin{cases} \max(V[i-1,j], v_i + V[i-1,j-w_i]) & \text{if } j - w_i \geq 0, \\ V[i-1,j] & \text{if } j - w_i < 0. \end{cases}$$

$$V[0,j] = 0 \text{ for } j \geq 0 \text{ and } V[i,0] = 0 \text{ for } i \geq 0.$$

Bottom-Up DP algorithm

Bottom-up Dynamic Programming: What we have been doing up to this point, computing solutions to all entries in the dynamic programming table.

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For example the dynamic programming solution to the *coin row problem* and calculating the *edit distance*.

Given the following problem, how do we solve it using a Bottom-Up Dynamic Programming algorithm?

Bottom-Up DP algorithm

Bottom-up Dynamic Programming: What we have been doing up to this point, computing solutions to all entries in the dynamic programming table.

For example the dynamic programming solution to the *coin row problem* and calculating the *edit distance*.

Given the following problem, how do we solve it using a Bottom-Up Dynamic Programming algorithm?

Knapsack capacity $W = 6$.

i	1	2	3	4	5
weight(w_i)	3	2	1	4	5
value(v_i)	\$25	\$20	\$15	\$40	\$50

Bottom-Up DP algorithm

We record the solutions to each smaller problems in table.

$\downarrow i$ $\quad W \rightarrow$	0	1	2	3	4	5	6
0							
1							
2							
3							
4							
5							**GOAL**

Bottom-Up DP algorithm

We record the solutions to each smaller problems in table.

$\downarrow i \quad W \rightarrow$	0	1	2	3	4	5	6
0							
1							
2							
3					?		
4							
5							**GOAL**

$V[4, 3] =$? stores the optimal value for a knapsack with capacity 4 of the first 3 items

Bottom-Up DP algorithm

$$V[i,j] = \begin{cases} \max(V[i-1,j], v_i + V[i-1,j-w_i]) & \text{if } j - w_i \geq 0, \\ V[i-1,j] & \text{if } j - w_i < 0. \end{cases}$$

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$\downarrow i$ $\quad W \rightarrow$		0	1	2	3	4	5	6
	0							
$w_1 = 3$ $v_1 = 25$	1							
$w_2 = 2$ $v_2 = 20$	2							
$w_3 = 1$ $v_3 = 15$	3							
$w_4 = 4$ $v_4 = 40$	4							
$w_5 = 5$ $v_5 = 50$	5							

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$w_1 = 3$ $v_1 = 25$	1	0	0					
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$w_2 = 2$ $v_2 = 20$	2	0	0	20				
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$w_1 = 3$ $v_1 = 25$	1	0	0	0	25			
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$w_2 = 2$ $v_2 = 20$	2	0	0	20	25			
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$w_3 = 1$ $v_3 = 15$	3	0	15	20				
$w_4 = 4$ $v_4 = 40$	4	0	15	20				
$w_5 = 5$ $v_5 = 50$	5	0	15	20				

Bottom-Up DP algorithm

$$V[i,j] = \begin{cases} \max(V[i-1,j], v_i + V[i-1, j-w_i]) & \text{if } j - w_i \geq 0, \\ V[i-1,j] & \text{if } j - w_i < 0. \end{cases}$$

$$V[0,j] = 0 \text{ for } j \geq 0 \text{ and } V[i,0] = 0 \text{ for } i \geq 0.$$

$\downarrow i$ $\quad W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25			
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25			
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35			
$w_4 = 4$ $v_4 = 40$	4	0	15	20				
$w_5 = 5$ $v_5 = 50$	5	0	15	20				

Bottom-Up DP algorithm

$$V[i,j] = \begin{cases} \max(V[i-1,j], v_i + V[i-1, j-w_i]) & \text{if } j - w_i \geq 0, \\ V[i-1,j] & \text{if } j - w_i < 0. \end{cases}$$

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$\downarrow i$ $\quad W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25			
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25			
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35			
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35			
$w_5 = 5$ $v_5 = 50$	5	0	15	20	35			

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	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25		
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25		
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35			
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35			
$w_5 = 5$ $v_5 = 50$	5	0	15	20	35			

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$\downarrow i$ $W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25		
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25		
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	?		
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35			
$w_5 = 5$ $v_5 = 50$	5	0	15	20	35			

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$\downarrow i$ $W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25		
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25		
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	?		
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$\downarrow i$ $W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25		
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25		
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35			
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	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25		
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25		
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40		
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35			
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$\downarrow i$ $\quad W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25		
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25		
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40		
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40		
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	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40		
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	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40	55	
$w_5 = 5$ $v_5 = 50$	5	0	15	20	35	40		

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	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40	55	
$w_5 = 5$ $v_5 = 50$	5	0	15	20	35	40		

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$\downarrow i$ $W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40	55	
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$\downarrow i$ $W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	25
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	45
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	60
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$\downarrow i$ $\quad W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	25
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	45
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	60
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40	55	60
$w_5 = 5$ $v_5 = 50$	5	0	15	20	35	40	55	65

Bottom-Up DP algorithm – Backtrace

How to find the set of items to include? Use backtrace, similar to edit distance and coin-row problem.

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1. From $V[n, W]$, trace back how we arrived at this table cell – either from $V[n-1, W]$ or $V[n-1, W-w_n]$.
2. Repeat this step until reach $V[0, 0]$.
3. Items that were included in the backtrace form the final solution for knapsack problem.

Bottom-Up DP algorithm – Backtrace

$\downarrow i$ $W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	25
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	45
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	60
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40	55	60
$w_5 = 5$ $v_5 = 50$	5	0	15	20	35	40	55	65

Lets do the backtrace!

Bottom-Up DP algorithm – Backtrace

$\downarrow i$ $\quad W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	25
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	45
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	60
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40	55	60
$w_5 = 5$ $v_5 = 50$	5	0	15	20	35	40	55	65

Bottom-Up DP algorithm – Backtrace

$\downarrow i$ $\quad$ $W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	25
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	45
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	60
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40	55	60
$w_5 = 5$ $v_5 = 50$	5	0	15	20	35	40	55	65

Bottom-Up DP algorithm – Backtrace

$\downarrow i$ $W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	25
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	45
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	60
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40	55	60
$w_5 = 5$ $v_5 = 50$	5	0	15	20	35	40	55	65

Bottom-Up DP algorithm – Backtrace

$\downarrow i$ $\quad W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	25
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	45
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	60
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40	55	60
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Bottom-Up DP algorithm – Backtrace

$\downarrow i$ $W \rightarrow$		0	1	2	3	4	5	6
	0	0	0	0	0	0	0	0
$w_1 = 3$ $v_1 = 25$	1	0	0	0	25	25	25	25
$w_2 = 2$ $v_2 = 20$	2	0	0	20	25	25	45	45
$w_3 = 1$ $v_3 = 15$	3	0	15	20	35	40	45	60
$w_4 = 4$ $v_4 = 40$	4	0	15	20	35	40	55	60
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Question: In general, using the dynamic programming table, how can we tell if there is multiple optimal solutions to a Knapsack problem?

DP Knapsack Problem

- The complexity of constructing the dynamic table is $\Theta(nW)$ in time and space.
- The complexity of performing the backtrack to find the optimal subset is $\Theta(n + W)$.

NOTE : The running time of this algorithm is not a polynomial function of n; rather it is a polynomial function of n and W, the largest integer involved in defining the problem. Such algorithms are known as *pseudo-polynomial*. They are efficient when the values $\{w_i\}$ are small, but less practical as these values grow large.

DP Knapsack Problem – Top-Down

- Divide and conquer type of (top down) approach of solving knapsack generally recompute many previously computed sub-problems, hence inefficient.

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DP Knapsack Problem – Top-Down

- Divide and conquer type of (top down) approach of solving knapsack generally recompute many previously computed sub-problems, hence inefficient.
- Bottom up dynamic programming approach avoids recomputation, but can compute many unnecessary solutions to sub-problems.
- Combine space saving of divide and conquer and speed up of bottom up approaches?

DP Knapsack Problem – Top-Down

ALGORITHM **MFKnapsack** (i, j)

/* Implement the memory function method (top-down) for the knapsack problem. */

/* INPUT : A non-negative integer i indicating the number of the first items being considered and a non-negative integer j indicating the knapsack capacity. */

/* OUTPUT : The value of an optimal, feasible subset of the first i items. */

/* NOTE: Requires global arrays $w[1 \dots n]$ and $v[1 \dots n]$ of weights and values of n items, and table $F[0 \dots n, 0 \dots W]$ initialized with -1s, except for row 0 and column 0 being all 0s. */

if $F[i, j] < 0$ **then**
    if $j < w[i]$ **then**
        $x =$ **MFKnapsack**$(i-1, j)$
    else
        $x = \max($ **MFKnapsack**$(i-1, j), v[i]+$ **MFKnapsack**$(i-1, j-w[i]))$
    end if
    $F[i, j] = x$
end if
return $F[i, j]$

DP Knapsack Problem – Top-Down

```
if F[i, j] < 0 then
    if j < w[i] then
        x = MFKnapsack(i − 1, j)
    else
        x = max( MFKnapsack(i − 1, j), v[i]+ MFKnapsack(i − 1, j − w[i]))
    end if
    F[i, j] = x
end if
```

$\downarrow i$	$W \rightarrow$	0	1	2	3	4	5	6
		0	0	0	0	0	0	0
$w_1 = 3, v_1 = 25$	1	0	-1	-1	-1	-1	-1	-1
$w_2 = 2, v_2 = 20$	2	0	-1	-1	-1	-1	-1	-1
$w_3 = 1, v_3 = 15$	3	0	-1	-1	-1	-1	-1	-1
$w_4 = 4, v_4 = 40$	4	0	-1	-1	-1	-1	-1	-1
$w_5 = 5, v_5 = 50$	5	0	-1	-1	-1	-1	-1	-1

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    end if
    F[i, j] = x
end if
```

$\downarrow i \quad W \rightarrow$		0	1	2	3	4	5	6
		0	0	0	0	0	0	0
$w_1 = 3, v_1 = 25$	1	0	-1	-1	-1	-1	-1	-1
$w_2 = 2, v_2 = 20$	2	0	-1	-1	-1	-1	-1	-1
$w_3 = 1, v_3 = 15$	3	0	-1	-1	-1	-1	-1	-1
$w_4 = 4, v_4 = 40$	4	0	-1	-1	-1	-1	-1	-1
$w_5 = 5, v_5 = 50$	5	0	-1	-1	-1	-1	-1	$\boxed{}$

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  end if
  $F[i,j] = x$
end if

$\downarrow i$ $W \rightarrow$		0	1	2	3	4	5	6
		0	0	0	0	0	0	0
$w_1 = 3, v_1 = 25$	1	0	-1	-1	-1	-1	-1	-1
$w_2 = 2, v_2 = 20$	2	0	-1	-1	-1	-1	-1	-1
$w_3 = 1, v_3 = 15$	3	0	-1	-1	-1	-1	-1	-1
$w_4 = 4, v_4 = 40$	4	0	$\boxed{}$	-1	-1	-1	-1	$\boxed{}$
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DP Knapsack Problem – Top-Down

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  if $j < w[i]$ **then**
    $x =$ **MFKnapsack**$(i - 1, j)$
  else
    $x = \max($ **MFKnapsack**$(i - 1, j), v[i] +$ **MFKnapsack**$(i - 1, j - w[i]))$
  end if
  $F[i, j] = x$
end if

$\downarrow i \quad W \rightarrow$		0	1	2	3	4	5	6
		0	0	0	0	0	0	0
$w_1 = 3, v_1 = 25$	1	0	-1	-1	-1	-1	-1	-1
$w_2 = 2, v_2 = 20$	2	0	-1	-1	-1	-1	-1	-1
$w_3 = 1, v_3 = 15$	3	0		-1	-1	-1	-1	-1
$w_4 = 4, v_4 = 40$	4	0		-1	-1	-1	-1	
$w_5 = 5, v_5 = 50$	5	0	-1	-1	-1	-1	-1	

DP Knapsack Problem – Top-Down

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  if $j < w[i]$ **then**
    $x =$ **MFKnapsack**$(i - 1, j)$
  else
    $x = \max($ **MFKnapsack**$(i - 1, j), v[i] +$ **MFKnapsack**$(i - 1, j - w[i]))$
  end if
  $F[i, j] = x$
end if

$\downarrow i$ $\quad W \rightarrow$		0	1	2	3	4	5	6
		0	0	0	0	0	0	0
$w_1 = 3, v_1 = 25$	1	0		-1	-1	-1	-1	-1
$w_2 = 2, v_2 = 20$	2	0		-1	-1	-1	-1	-1
$w_3 = 1, v_3 = 15$	3	0		-1	-1	-1	-1	-1
$w_4 = 4, v_4 = 40$	4	0		-1	-1	-1	-1	
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$w_5 = 5, v_5 = 50$	5	0	–	–	–	–	–	**65**

Top-Down vs. Bottom-Up

In general, when to use top-down or bottom-up dynamic programming?

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Top-down incurs additional space and time cost of maintaining recursion stack. Hence:

Top-Down vs. Bottom-Up

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Top-down incurs additional space and time cost of maintaining recursion stack. Hence:

- Bottom-up: When the final problem instance requires most or all of the sub-problem instances to be solved, e.g., edit distance.
- Top-down: When the final problem instance only requires a subset of the sub-problem instances to be solved, e.g., possibly knapsack.

Overview

Transitive closure

Definition

The transitive closure of a directed graph with n vertices is to determine if there is a path between any pair of vertices in the graph. Yes/True/1 if there is a path, No/False/0 if not. We want to determine for all pairs of vertices.

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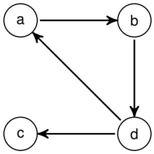


(a) digraph

$$A = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

(b) adjacency matrix

$$T = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

(c) transitive closure

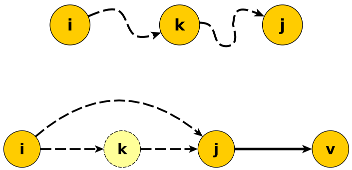
Why compute transitive closure?

- In spreadsheet software, when a cell is changed, what are the other cells whose computation directly or indirectly depend on it (i.e., might need to update also)?
- Critical communication systems: Is it possible for any nodes in this system to communicate with any other node?

Warshall's Algorithm – Sketch

Idea: Progressively use each vertex as an intermediate to join two paths.

If we know there is a path between vertex *i* to intermediate node *k*, and a path from intermediate node *k* to *j*, then we know there is a path from *i* to *j*.



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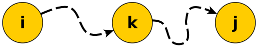

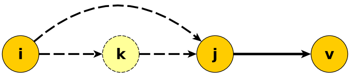

Warshall's algorithm progressively considers using each vertex as the intermediate, and when all the vertices have been considered as intermediates, we have the transitive closure of the graph.

Warshall's Algorithm – Sketch

Idea: Construct the transitive closure of a given digraph with *n* vertices through a series of *n*-by-*n* binary matrices:

$$R^{(0)}, \dots, R^{(k-1)}, R^{(k)}, \dots, R^{(n)}.$$

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Updating from R^{k-1} to R^{k}:

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- If we can use vertex *k* as intermediate vertex to traverse from *i* to *j* vertices and we previously didn't know there was such a path, then update our transitivity information, i.e.,
  - if $R[i,j]^{(k-1)} = 0$, and $R[i,k]^{(k-1)} = 1$ and $R[k,j]^{(k-1)} = 1$, then change $R[i,j]^{(k)}$ to 1.

Warshall's Algorithm – Example

$$R^{(0)} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Applying Warshall's algorithm to calculate the transitive closure of a digraph.

Warshall's Algorithm – Example

$$R^{(1)} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

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Applying Warshall's algorithm to calculate the transitive closure of a digraph.

$$R^{(2)} = \begin{array}{c c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

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Warshall's Algorithm – Example

$$T = R^{(4)} = \begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \end{array}$$

Applying Warshall's algorithm to calculate the transitive closure of a digraph.

Warshall's Algorithm – Pseudocode

ALGORITHM **Warshall** ($A[1 \ldots n, 1 \ldots n]$)

/* Compute the transitive closure of a graph using Warshall's algorithm. */

/* INPUT : The adjacency matrix A of a digraph with n vertices. */

/* OUTPUT : The transitive closure of the digraph. */

$R^{(0)} = A$
for $k = 1$ **to** n **do**
    for $i = 1$ **to** n **do**
        for $j = 1$ **to** n **do**
            $R^{(k)}[i, j] = R^{(k-1)}[i, j]$ **or** $(R^{(k-1)}[i, k]$ **and** $(R^{(k-1)}[k, j])$
        end for
    end for
end for
return $R^{(n)}$

Warshall's Algorithm

- The time efficiency of Warshall's algorithm is $\Theta(n^3)$ and uses $\Theta(n^2)$ space.
- For sparse graphs, the transitive closure can be calculated more efficiently using an adjacency list representation with a depth-first or breadth-first traversal.
- Using a DFS or BFS traversal with n vertices and m edges takes $\Theta(n + m)$ time. Doing it n times takes $\Theta(n^2 + nm)$ time.
- If the graph is sparse, i.e. $m \approx n$, the time efficiency is $\Theta(n^2)$.

Overview

Summary

- Described dynamic programming and how it is useful to solve problems that require solving sub-problems multiple times.
- Examples:
  - Coin-row problem
  - Computing the edit distance
  - Knapsack – Bottom up and top down
  - Transitive closure – Warshall's algorithm