COSC1285/2123: Algorithms & Analysis

Iterative Improvment

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Lecture 11

Overview

Levitin – The design and analysis of algorithms

This week we will be covering the material from Chapters 10.

Learning outcomes:

- Understand the paradigm of iterative improvement
- Understand and apply examples of iterative improvement:
  - Maximum-flow problem
  - Stable marriage problem (Gale-Shapeley algorithm)

Outline

Overview

Iterative Improvement

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Iterative Improvement

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In contrast, iternative improvement starts with a feasible solution (one that satisfies all constraints), then proceed to improve it by repeated application of simple steps.

Overview

Maximum-Flow Problem

Imagine you are given this problem:

Problem

Yarra Valley Waters needs to move water from a dam to a local water reserviour. There is a network of pipes and junctions that they can transport water over. Assume there is no loss within the network. How do we maximise the amount of water transported each day?

Maximum-Flow Problem

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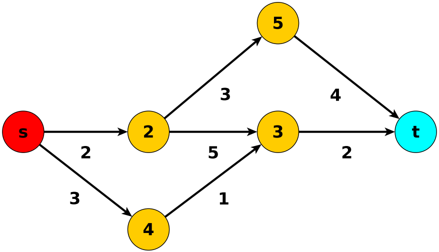
This is an instance of a maximum-flow problem.

Flow Network



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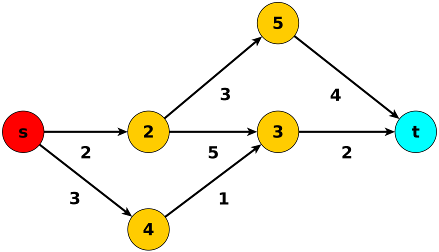
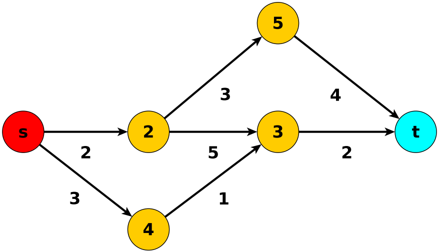


- Source: vertex which has no incoming edges.

Flow Network



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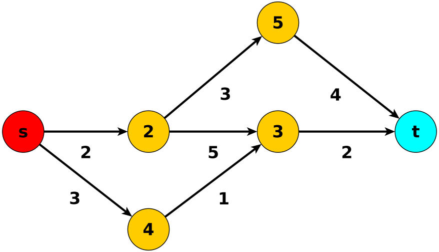

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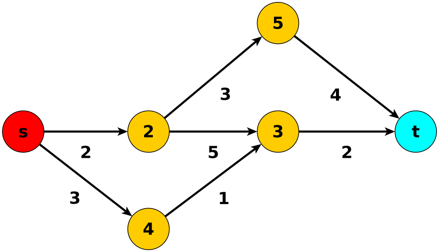
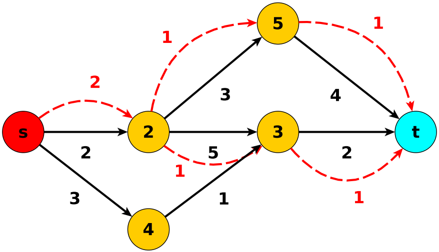
- Sink: vertex with no outgoing edges.
- Edge has a weight representing its capacity.

Flow Network



- Source: vertex which has no incoming edges.
- Sink: vertex with no outgoing edges.
- Edge has a weight representing its capacity.
- Graphs satisfying above properties called flow network.

Maximum-Flow Problem



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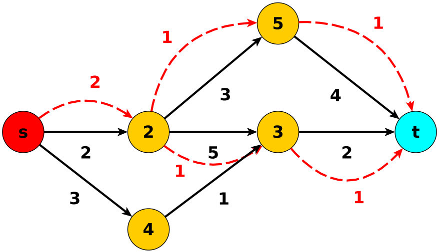
Maximum-Flow Problem



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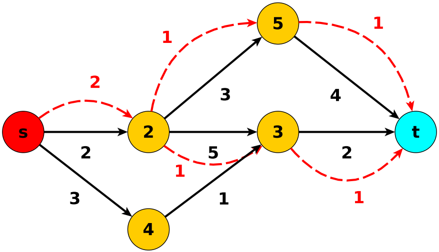

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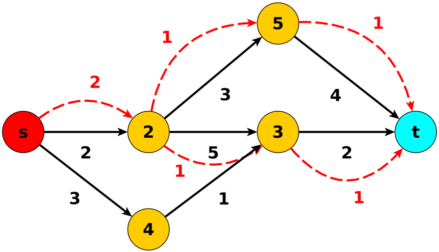
Maximum-Flow Problem



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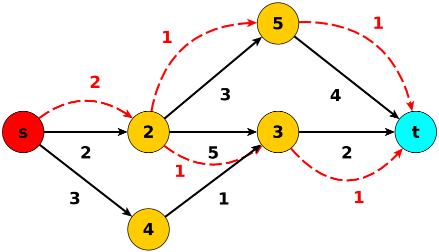
- All other vertices are transit points – flow in = flow out; called flow-conservation.

Maximum-Flow Problem



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- Sink is the final destination of all “material” in the network.
- Maximum amount of flow on an edge cannot exceed its capacity; capacity constraint.
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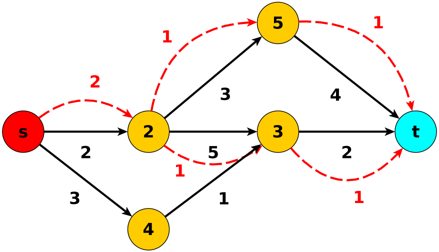
- Total material leaving source = total material flowing into sink; called value of the flow.

Maximum-Flow Problem



Problem

Given a flow network, the maximum-flow problem is find a flow of maximum value, subject to (edge) capacity constraints and flow-conservation.

Ford Fulkerson Method – Sketch

Idea:

- Given an initial flow network, find an initial feasible flow.

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- Find a path from source to sink that can increase the total flow.
- Increase the flow along that path.

Ford Fulkerson Method – Sketch

Idea:

- Given an initial flow network, find an initial feasible flow.
- Find a path from source to sink that can increase the total flow.
- Increase the flow along that path.
- Repeat until no more such paths.

Preliminaries: Residual Network

Purpose: Given a flow, the residual network shows which edges in the flow network can admit more flow.

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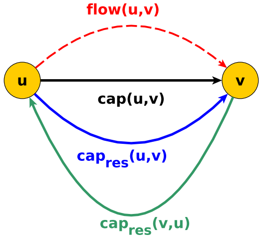
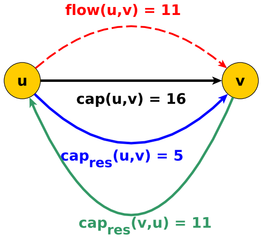
For each edge (u,v) in flow network, we have two edges in the residual network with following residual capacity:

$cap_{res}(u,v) = cap(u,v) - flow(u,v)$, $cap_{res}(u,v) > 0$ (forward edge)
$cap_{res}(v,u) = flow(u,v)$, $cap_{res}(v,u) > 0$ (backward edge)



Example: cap(u,v) = 16, flow(u,v) = 11, then can still increase the flow by cap_{res}(u,v) = 5 in the (u,v) direction, but can also send up to cap_{res}(v,u) = 11 units in the other (v,u) direction to cancel out flow(u,v).



Preliminaries: Augmenting Path

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This path basically tells us which individual flows to increase (when traversing forward edge), and which to decrease (when traversing backward edge) to increase the total flow/value, while satisfying capacity and flow conservation constraints.

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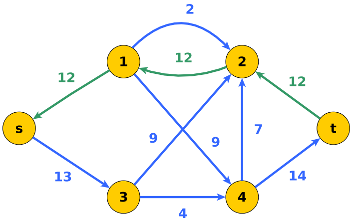

Ford-Fulkerson Method

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    a. For all the edges in path *p* in residual network, find the one with minimum residual capacity (cap_{minres})

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      if ((u,v) is forward) then flow(u,v) += cap_{minres};
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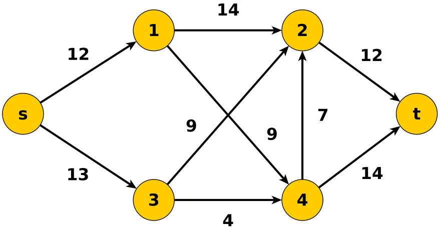
4. repeat step 3 until no more paths from *s* to *t* in residual network

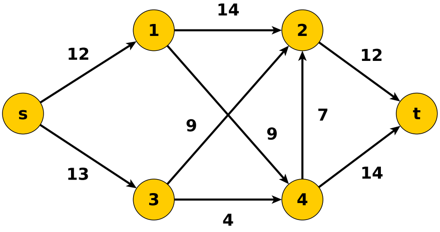
* There are generally many possible augmenting path. We select the shortest one (in terms of number of edges) for step 3.

Ford-Fulkerson Method – Example

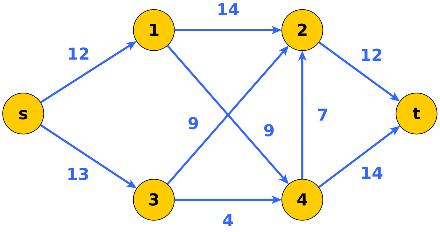


(a) Flow network

Ford-Fulkerson Method – Example

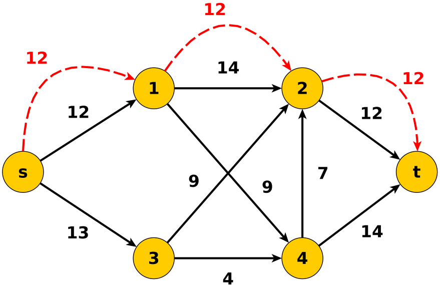


(c) Flow network



(d) Residual network

Ford-Fulkerson Method – Example



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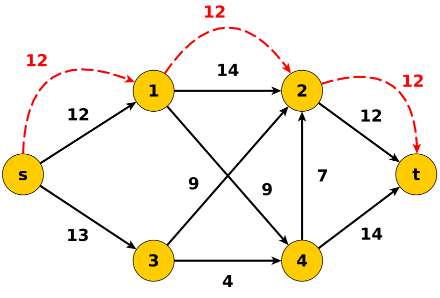
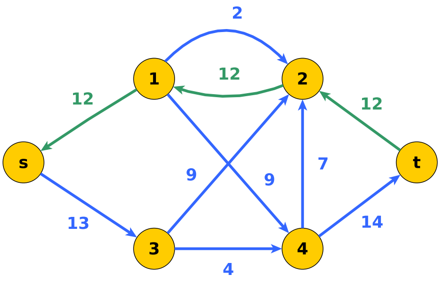
Ford-Fulkerson Method – Example

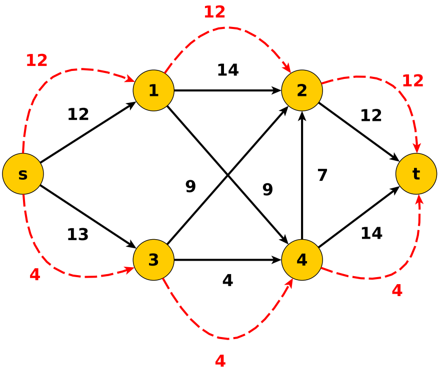


(g) Flow network



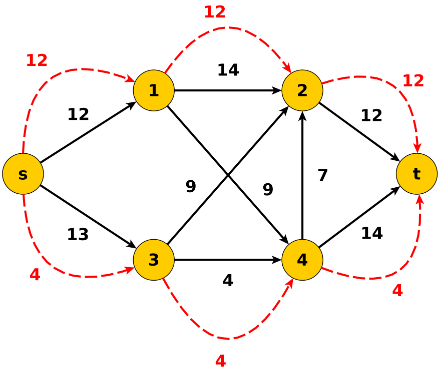
(h) Residual network

Ford-Fulkerson Method – Example

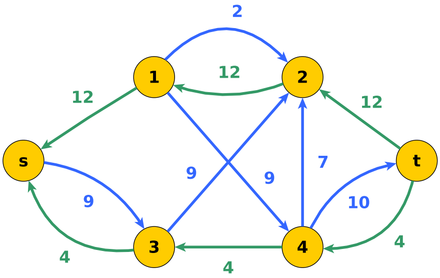


(i) Flow network

Ford-Fulkerson Method – Example

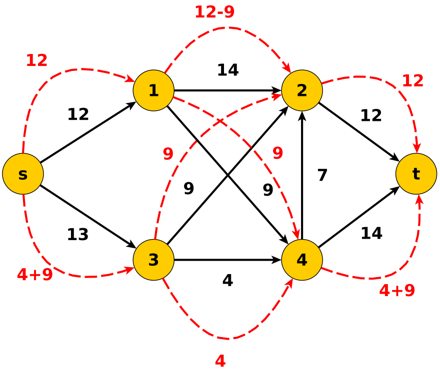


(k) Flow network



(l) Residual network

Ford-Fulkerson Method – Example



(m) Flow network

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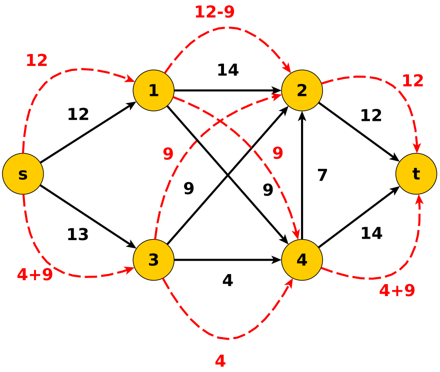
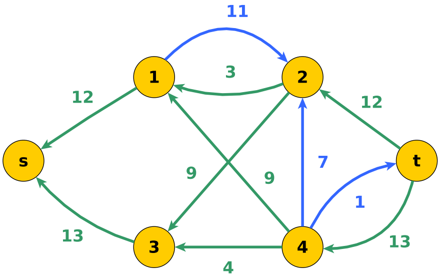


(o) Flow network



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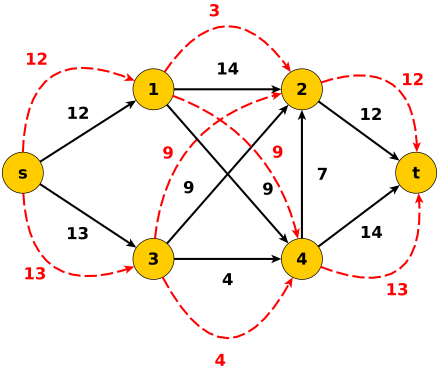


Value of flow?

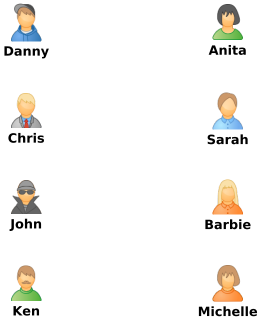
Maximum-Flow Problem

Applications of maximum-flow problem:

`https://www.youtube.com/watch?v=D36MJCXT4Qk`

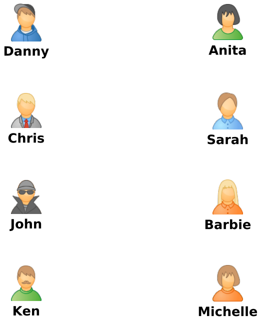
Overview

Stable Marriage Problem

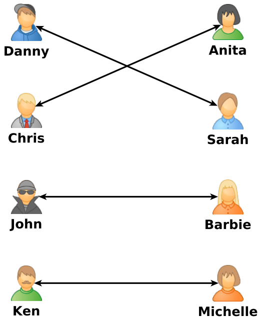


(q) Males and Females.

Stable Marriage Problem



(s) Males and Females.



(t) Matched couples!

Stable Marriage Problem

Stable Marriage Problem

Given a set of n men and n women, who has a list of preferences to the other sex (in terms of a ranking), the problem is how to find a matching between them such that the matching is *stable*?

A matching is stable if:

- No matched pair of man and woman can find other partners and both do better, i.e., both man and woman prefer other partners over their existing match.

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Is a stable marriage (matching) always possible?

- Yes, if equal number of men and women.

Gale-Shapley Algorithm

Idea: Female proposing variant:

1. Over a number of rounds, each unmatched female proposes to their remaining highest male preferences.

Gale-Shapley Algorithm

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2. Each round, males reply "yes" to proposal from their highest female proposer and "no" to all other proposers.

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1. Over a number of rounds, each unmatched female proposes to their remaining highest male preferences.
2. Each round, males reply “yes” to proposal from their highest female proposer and “no” to all other proposers.
3. This continues until all females (and males) are matched.

Gale-Shapley Algorithm

Female Proposing variant:

1. Round 1: Each female proposes to their first male preferences. Each male receives 0 or more proposals. They reply “maybe” to the female they most prefer and “no” to all other proposals. For each “maybe” reply, the corresponding female-male are tentatively matched.

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2. Round 2: Each unmatched female proposes to their next preferred male (2nd ranked), even if the male is tentatively matched already. Each male evaluates their proposals, and again reply “maybe” to the female they most prefer (this can be their existing matched partner) and “no” to all other proposals. For each “maybe” reply, the corresponding female-male are tentatively matched.

Gale-Shapley Algorithm

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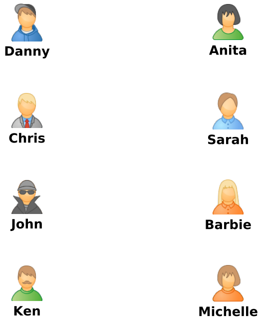
3. Round 3 onwards: We continue with this process until all females and males are matched.

Gale-Shapley Algorithm Example



Gale-Shapley Algorithm Example

Anita	Sarah	Barbie	Michelle
Chris	Chris	Ken	Chris
Ken	Ken	Danny	Ken
Danny	Danny	John	Danny
John	John	Chris	John

Table: Female preferences

Danny	Chris	John	Ken
Michelle	Anita	Barbie	Michelle
Sarah	Sarah	Anita	Barbie
Barbie	Michelle	Michelle	Sarah
Anita	Barbie	Sarah	Anita

Table: Male preferences

Gale-Shapley Algorithm Example

Round 1 (proposing):

Anita	Sarah	Barbie	Michelle
Chris	Chris	Ken	Chris

Table: Female preferences

Danny	Chris	John	Ken
Michelle	Anita	Barbie	Michelle
Sarah	Sarah	Anita	Barbie
Barbie	Michelle	Michelle	Sarah
Anita	Barbie	Sarah	Anita

Table: Male preferences

Gale-Shapley Algorithm Example

Round 1 (end):

Anita	Sarah	Barbie	Michelle
Chris	~~Chris~~	Ken	~~Chris~~

Table: Female preferences

Danny	Chris	John	Ken
Michelle	Anita	Barbie	Michelle
Sarah	Sarah	Anita	Barbie
Barbie	Michelle	Michelle	Sarah
Anita	Barbie	Sarah	Anita

Table: Male preferences

Gale-Shapley Algorithm Example

Round 2 (proposing):

Anita	**Sarah**	**Barbie**	**Michelle**
Chris	~~Chris~~	Ken	~~Chris~~
	Ken		Ken

Table: Female preferences

Danny	**Chris**	**John**	**Ken**
Michelle	Anita	Barbie	Michelle
Sarah	Sarah	Anita	Barbie
Barbie	Michelle	Michelle	Sarah
Anita	Barbie	Sarah	Anita

Table: Male preferences

Gale-Shapley Algorithm Example

Round 2 (end):

Anita	Sarah	Barbie	Michelle
Chris	~~Chris~~	~~Ken~~	~~Chris~~
	~~Ken~~		Ken

Table: Female preferences

Danny	Chris	John	Ken
Michelle	Anita	Barbie	Michelle
Sarah	Sarah	Anita	Barbie
Barbie	Michelle	Michelle	Sarah
Anita	Barbie	Sarah	Anita

Table: Male preferences

Gale-Shapley Algorithm Example

Round 3 (proposing):

Anita	Sarah	Barbie	Michelle
Chris	~~Chris~~	~~Ken~~	~~Chris~~
	~~Ken~~	Danny	Ken
	Danny		

Table: Female preferences

Danny	Chris	John	Ken
Michelle	Anita	Barbie	Michelle
Sarah	Sarah	Anita	Barbie
Barbie	Michelle	Michelle	Sarah
Anita	Barbie	Sarah	Anita

Table: Male preferences

Gale-Shapley Algorithm Example

Round 3 (end):

Anita	Sarah	Barbie	Michelle
Chris	~~Chris~~	~~Ken~~	~~Chris~~
	~~Ken~~	~~Danny~~	Ken
	Danny		

Table: Female preferences

Danny	Chris	John	Ken
Michelle	Anita	Barbie	Michelle
Sarah	Sarah	Anita	Barbie
Barbie	Michelle	Michelle	Sarah
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Table: Male preferences

Gale-Shapley Algorithm Example

Round 4 (proposing):

Anita	Sarah	Barbie	Michelle
Chris	~~Chris~~	~~Ken~~	~~Chris~~
	~~Ken~~	~~Danny~~	Ken
	Danny	John	

Table: Female preferences

Danny	Chris	John	Ken
Michelle	Anita	Barbie	Michelle
Sarah	Sarah	Anita	Barbie
Barbie	Michelle	Michelle	Sarah
Anita	Barbie	Sarah	Anita

Table: Male preferences

Gale-Shapley Algorithm Example

Round 4 (end):

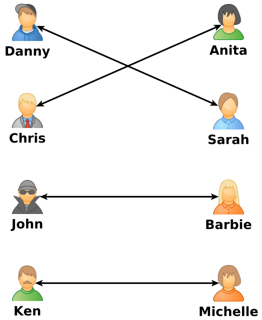
Anita	Sarah	Barbie	Michelle
Chris	~~Chris~~	~~Ken~~	~~Chris~~
	~~Ken~~	~~Danny~~	Ken
	Danny	John	

Table: Female preferences

Danny	Chris	John	Ken
Michelle	Anita	Barbie	Michelle
Sarah	Sarah	Anita	Barbie
Barbie	Michelle	Michelle	Sarah
Anita	Barbie	Sarah	Anita

Table: Male preferences

Gale-Shapley Algorithm Example



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Time complexity?

Stable Marriage Video

https://www.youtube.com/watch?v=fudb8DuzQlM

Overview

Summary

- Maximum flow problem (Ford-Fulkerson method)
- Stable marriage problem (Gale-Shapley algorithm)