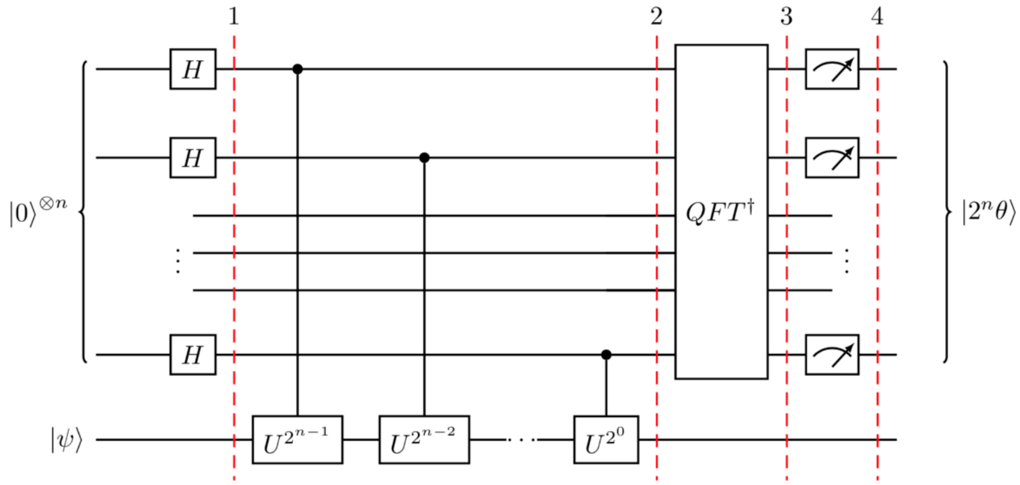
b07901152 張力元

1. Background --- Quantum Phase Estimation (QPE)



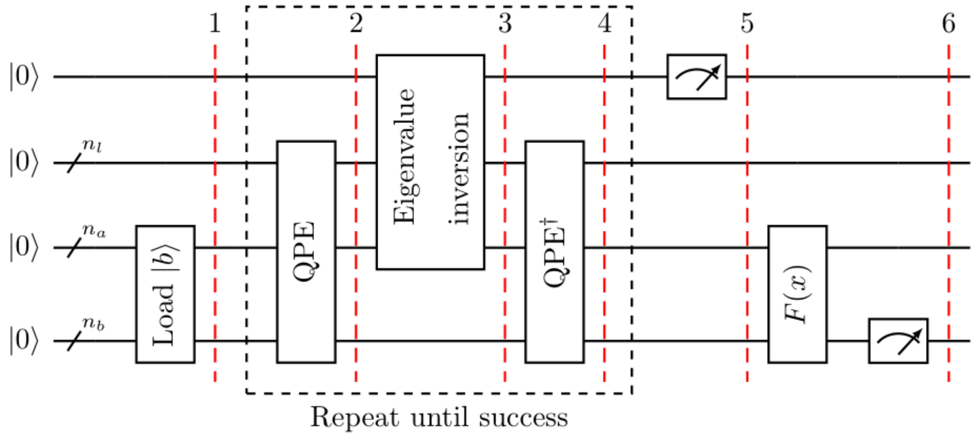
Let U be a unitary operator. The circuit above estimates θ in $U|\psi\rangle = e^{i2\pi\theta}|\psi\rangle$, where $|\psi\rangle$ is an eigenvector of U and $e^{i2\pi\theta}$ is the corresponding eigenvalue.

Proof：

1. **Setup**： $|\psi_0\rangle = |0\rangle^{\otimes n}|\psi\rangle$
2. **Superposition**： $|\psi_1\rangle = \frac{1}{2^{\frac{n}{2}}}(|0\rangle + |1\rangle)^{\otimes n}|\psi\rangle$
3. **Controlled Unitary Operations**：
   The controlled unitary CU applies the unitary operator U on the target register only if its corresponding control bit is $|1\rangle$.
   $\because U|\psi\rangle = e^{i2\pi\theta}|\psi\rangle$
   $\therefore U^{2^j}|\psi\rangle = U^{2^j-1}U|\psi\rangle = U^{2^j-1}e^{i2\pi\theta}|\psi\rangle = \cdots = e^{i2\pi\cdot 2^j\theta}|\psi\rangle$
   After applying all the n controlled operations CU^{2^j} on $|\psi_1\rangle$ with $0 \le \mathrm{j} \le \mathrm{n}-1$, we can get
   $$|\psi_2\rangle = \frac{1}{2^{\frac{n}{2}}}\left(|0\rangle + e^{i2\pi\cdot 2^{n-1}\theta}|1\rangle\right)\otimes\left(|0\rangle + e^{i2\pi\cdot 2^{n-1}\theta}|1\rangle\right)\otimes \ldots \otimes\left(|0\rangle + e^{i2\pi\cdot 2^{n-1}\theta}|1\rangle\right)\otimes|\psi\rangle$$
   $$\Longrightarrow |\psi_2\rangle = \frac{1}{2^{\frac{n}{2}}}\sum_{k=0}^{2^n-1} e^{i2\pi\cdot k\theta}\,|k\rangle\otimes|\psi\rangle$$
   where k denotes the integer representation of n– bit binary numbers.
4. Inverse Fourier Transform and measurement：
   known that $QFT|x\rangle = \frac{1}{2^{\frac{n}{2}}}\left(|0\rangle + e^{\frac{i2\pi}{2}x}|1\rangle\right)\otimes\left(|0\rangle + e^{\frac{i2\pi}{2^2}x}|1\rangle\right)\otimes \ldots \otimes\left(|0\rangle + e^{\frac{i2\pi}{2^n}x}|1\rangle\right)$
   replace x with $2^n\theta$, we can obtain $|\psi_2\rangle = QFT|2^n\theta\rangle\otimes|\psi\rangle$
   $$\therefore QFT^{-1}|\psi_2\rangle = \frac{1}{2^n}\sum_{x=0}^{2^n-1}\sum_{k=0}^{2^n-1} e^{-\frac{i2\pi k}{2^n}(x-2^n\theta)}|x\rangle\otimes|\psi\rangle \quad \Longrightarrow \text{measuring in the computational}$$
   basis gets $|2^n\theta\rangle$ in the ancilla register with high probability.

b07901152 張力元

2. Solving Linear Systems of Equations using HHL Algorithm



Problem：given a **Hermitian** matrix $A \in \mathbb{C}^{N\times N}$ and a vector $\mathbf{b} \in \mathbb{C}^N$, find $\mathbf{x} \in \mathbb{C}^{\mathrm{N}}$ satisfying $A\mathbf{x} = \mathbf{b}$

Reformulate the problem：$A|x\rangle = |b\rangle$ where the ith component of $\mathbf{b}$ ($\mathbf{x}$) corresponds to the amplitude of the ith basis state of the quantum state $|b\rangle$ ($|x\rangle$)

Background：

1. Spectral decomposition of Hermitian matrix A

$$A = \sum_{j=0}^{N-1} \lambda_j \left|u_j\right\rangle\!\left\langle u_j\right| \quad \text{where } \left|u_j\right\rangle \text{ is the jth eigenvector of } A \text{ with corresponding eigenvalue } \lambda_j$$

Moreover, $\{\, |u_j\rangle \mid j \in [0, N-1] \,\}$ forms an **orthonormal basis.**

$$\Rightarrow A^{-1} = \sum_{j=0}^{N-1} \frac{1}{\lambda_j} |u_j\rangle\langle u_j|$$

2. $|b\rangle$ can be represented in the eigenbasis of A iff $A|x\rangle = |b\rangle$ has any solution

$$|b\rangle = \sum_{j=0}^{N-1} b_j \left|u_j\right\rangle$$

3. A is a Hermitian matrix, and thus A is invertible.

$$|x\rangle = A^{-1}|b\rangle = \sum_{j=0}^{N-1} \frac{b_j}{\lambda_j} |u_j\rangle$$

Description of the HHL algorithm:

1. **Load the data $|b\rangle$:** $|0\rangle_{n_b} \mapsto |b\rangle_{n_b}$
2. **Apply QPE on $|0\rangle_{n_l}|b\rangle_{n_b}$ with $U = e^{iAt}$**

$$U = e^{iAt} = \sum_{j=0}^{N-1} e^{i\lambda_j t} \left|u_j\right\rangle\!\left\langle u_j\right|$$

$$\because U|u_j\rangle_{n_b} = e^{i\lambda_j t}|u_j\rangle_{n_b} = e^{i2\pi\theta}|u_j\rangle_{n_b} \quad \Longrightarrow \theta = \frac{\lambda_j t}{2\pi}$$

$$\therefore QPE(U, |0\rangle_{n_l}|b\rangle_{n_b}) = QPE\left(U, \sum_{j=0}^{N-1} b_j\, |0\rangle_{n_l}|u_j\rangle_{n_b}\right) = \sum_{j=0}^{N-1} b_j\, |\widetilde{\lambda_j}\rangle_{n_l}|u_j\rangle_{n_b}$$

where $\widetilde{\lambda_j}$ is an n_l- bit approximation to $2^{n_l}\theta = 2^{n_l}\dfrac{\lambda_j t}{2\pi}$

If each λ_j can be exactly represented with n_l bits, then $\widetilde{\lambda_j} = 2^{n_l}\frac{\lambda_j t}{2\pi}$

3. **Add an ancilla qubit $|0\rangle$ and apply a y– axis rotation conditioned on $|\widetilde{\lambda_j}\rangle_{n_l}$**

$$\text{y– axis rotation operator} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \quad \text{where } \theta = \sin^{-1}\left(\frac{C}{\widetilde{\lambda_j}}\right) = \sin^{-1}\left(\frac{1}{\lambda_j}\right)$$

$$\text{ancilla qubit } |0\rangle \longmapsto \left(\sqrt{1 - \frac{C^2}{\widetilde{\lambda_j}^2}}|0\rangle + \frac{C}{\widetilde{\lambda_j}}|1\rangle\right)$$

$$\therefore \text{the result}: \sum_{j=0}^{N-1} b_j\, |\widetilde{\lambda_j}\rangle_{n_l}|u_j\rangle_{n_b}\left(\sqrt{1 - \frac{C^2}{\widetilde{\lambda_j}^2}}|0\rangle + \frac{C}{\widetilde{\lambda_j}}|1\rangle\right)$$

4. **Apply QPE†**

$$\sum_{j=0}^{N-1} b_j\, |\widetilde{\lambda_j}\rangle_{n_l}|u_j\rangle_{n_b}\left(\sqrt{1 - \frac{C^2}{\widetilde{\lambda_j}^2}}|0\rangle + \frac{C}{\widetilde{\lambda_j}}|1\rangle\right)$$

$$\longmapsto \sum_{j=0}^{N-1} b_j\, |0\rangle_{n_l}|u_j\rangle_{n_b}\left(\sqrt{1 - \frac{C^2}{\widetilde{\lambda_j}^2}}|0\rangle + \frac{C}{\widetilde{\lambda_j}}|1\rangle\right) = \sum_{j=0}^{N-1} b_j\, |0\rangle_{n_l}|u_j\rangle_{n_b}\left(\sqrt{1 - \frac{1}{\lambda_j^2}}|0\rangle + \frac{1}{\lambda_j}|1\rangle\right)$$

5. **Measure the ancilla qubit in the computational basis**

If the measurement outcome is $|1\rangle$, then the post– measurement state becomes

$$\frac{\sum_{j=0}^{N-1}\frac{b_j}{\lambda_j}|0\rangle_{n_l}|u_j\rangle_{n_b}}{\sqrt{\sum_{j=0}^{N-1}\frac{|b_j|^2}{|\lambda_j|^2}}} \Longrightarrow \text{select the lower } n_b \text{ qubits}: \frac{\sum_{j=0}^{N-1}\frac{b_j}{\lambda_j}|u_j\rangle_{n_b}}{\sqrt{\sum_{j=0}^{N-1}\frac{|b_j|^2}{|\lambda_j|^2}}} = \frac{|x\rangle}{\|x\|}$$

In addition, the probability of measuring $|1\rangle$ in the ancilla qubit $= \displaystyle\sum_{j=0}^{N-1}\frac{|b_j|^2}{|\lambda_j|^2} = \|x\|^2$