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INTERFACE CONDITION FOR THE COUPLING OF A FLUID AND POROUS MEDIA

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Abstract: *Modeling the filtration of incompressible fluids through porous media requires dealing with different types of partial differential equations in the fluid and porous domains of the computational domain. Such equations must be coupled through physically continuity conditions at the interface separating the two domains. We will review different interface conditions, including the well-known Beavers-Joseph-Saffman boundary condition and its recent improvement by Le Bars and Worster when using Navier-Stokes and Darcy's equation or his extensions, Brinkman model or the Forchheimer model.*

Key words*: Navier-Stokes equations, Darcy's law, Interface conditions, Brinkman model, Forchheimer model.*

1. INTRODUCTION

Simulations of flow in porous media can even help modeling tsunami interactions on a shore-line [1]. A modeling of filtration processes would require introducing different systems of partial differential equations in the free fluid and in the porous medium regions. The difficulty in finding effective coupling conditions at the interface between the fluid domain and the porous layer lies in the fact that often the orders of the corresponding differential operators are different, e.g. when using Navier-Stokes and Darcy's equation. Alternatively, using the Brinkman model or Forchheimer model for the porous media this difficulty does not occur.

2. MODELS IN THE FREE FLUID REGION

In this section we will review some mathematical models for the flow in each domain. In the following Ω_f denotes the fluid domain and Ω_p is the porous region. The velocity is represented by u_f and p_f pressure variables in Ω_f and p_p for Ω_p.In order to describe the motion of the fluid in Ω_f, we introduce the Navier-Stokes equations:

$$\rho\left(\frac{\partial u_f}{\partial t}+\left(u_f\cdot\nabla\right)u_f\right)-\mu\Delta u_f+\nabla p_f=f \quad \text{in } \Omega_f$$
$$\nabla\cdot u_p=0 \qquad \text{in } \Omega_f \qquad (1)$$

where ρ and μ are the density and dynamic viscosity of the fluid and f represents the external body force.

∇ and $\nabla\cdot$ are, respectively, the gradient and the divergence operator with respect to the space coordinates. Moreover,

$$\nabla\cdot u=\left(\sum_{j=1}^{d}\partial_j u_{ij}\right)_{i=1,\dots,d}$$

Finally, we recall that

$$(v\cdot\nabla)w=\sum_{i=1}^{d}v_i\partial_i w$$

for all vector functions $v=\left(v_1,\dots,v_d\right)$ and $w=\left(w_1,\dots,w_d\right)$

The Navier-Stokes equations are well-suited for which the Reynolds number

$$\mathrm{Re}_f=\frac{\rho UL}{\mu}$$

where ρ is approximately less than 10^3, U and L being a characteristic velocity and a characteristic length scale of the problem, respectively.

The simplest law for describing the flow of a fluid in porous media is the law obtained by Darcy [6]. The laws provides the simplest linear relation between velocity and pressure in porous media under the physically reasonable assumption that fluid flows are usually very slow and all the inertial (non-linear) terms may be neglected:

$$u_p = -\frac{K}{\mu}\nabla p_p \qquad (2)$$

where μ is the dynamic viscosity coefficient and K is the permeability coefficient.

In order to characterize the importance of the inertial effects, similarly to the Navier-Stokes equations, it is possible to define the Reynolds number associated to the pores

$$\mathrm{Re}_p = \frac{\rho U \delta}{\mu}$$

where δ is the characteristic pore size.

The Darcy law is reliable for values of $\mathrm{Re}_p < 1$, otherwise it is necessary to consider a more general model which can account also for the inertial effects, like the non-linear Forchheimer equation [9]:

$$\nabla p_p = -\frac{\mu}{K}u_p - \frac{\rho C_F}{\sqrt{K}}|u_p|u_p \qquad (3)$$

C_F being the inertial resistance coefficient (or tensor in the non-isotropic case). The Forchheimer equation extends the range of validity of the Darcy model (3) to $1 < \mathrm{Re}_p < 10$.

An extension of this model (2), the Brinkman model [3], is usually used in order to account for the high porosity of the porous media or to impose no-slip conditions on solid walls:

$$-\left(\mu_{eff}\nabla u\right) + \frac{\mu}{K}u_p + \nabla p_p = f \ \text{ in } \Omega_f \qquad (4)$$

where $\mu_{eff} = \mu/\phi$ is the effective viscosity of the fluid in Ω_p and ϕ denotes the porosity of the porous media. The Brinkman model is used if the Reynolds numbers Re_p of the corresponding free flow is greater than 10.

The filtration model is determined by the continuity equation:

$$\nabla \cdot u_p = 0 \ \text{ in } \Omega_p$$

3. THE DARCY MODEL

We consider the Navier-Stokes equations (1) in the free fluid region Ω_f, coupled across an

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interface with the Darcy equation (2) in the porous medium Ω_p. It is not clear what conditions we should be imposed at the interface Γ between Ω_p, and Ω_f. The classical coupling conditions for an inviscid fluid are the continuity of the pressure and the continuity of the normal velocities at the interface. For a viscous flow, one would assume additionally the vanishing of the tangential velocity at the interface Γ.

If the interface would be a boundary, then in the fluid part the system needs, e.g., a prescribed velocity and the equation in Ω_p must be supplied with a given pressure or normal flux. For coupling Darcy's model (2) and Stokes equation (1) some interface conditions are needed to obtain a well-posed problem.

In the following, n_p and n_f denote the unit outward normal vectors to the surfaces $\partial\Omega_p$ and $\partial\Omega_f$ and with $n_f = -n_p$ on Γ. Moreover, we shall indicate $n = n_p$ for simplicity of notation. Usually, these interface conditions describe the continuity of the mass flux

$$u_f \cdot n = u_p \cdot n \ \text{ on } \Gamma \qquad (5)$$

Eq. (5) is not sufficient to calculate the flow in $\partial\Omega_p$, since the flux is yet unknown.

3.1 The Interface Conditions of Ene, Levy and Sanchez-Palencia

In 1975 by Ene, Levy and Sanchez-Palencia [8], [14]: they distinguished two principally different cases of flow:

npf (near parallel flow): the velocity in Ω_f is significantly larger than the filtration velocity in Ω_p. The pressure gradients are of similar magnitude in both domains.

npf: $u_f = 0$,

$$p_p = p_f \ \text{ on } \Gamma \qquad (6)$$

The first condition in (6) originates from the continuity of velocity across the interface where the filtration velocity in Ω_p is neglected. This simplification allows computes the flow solely in the domain Ω_f. The pressure field is known in the fluid part Ω_f and via the continuity of pressure condition in (5) also on the interface Ω_p.

nnf (near normal flow): the velocities are of similar magnitude in both domains and the pressure gradient in Ω_f is significantly smaller than in Ω_p and nearly orthogonal.

For the case of near parallel flow Ene, Levy and Sanchez-Palencia [8], [14] suggested to use the conditions

nnf: $p_p = C \ on \ \Gamma$,

$$u_f \cdot \tau_j = 0 \qquad (7)$$

C denotes an a-priority unknown constant and τ_j are the orthogonal unit tangent vectors to the interface Γ. Since the pressure is usually defined only up to a constant, it is often convenient to assume that the pressure p at the porous interface Γ takes a certain (arbitrary) constant value C. Doing so, one neglects the dependence of p_p on Γ fluid flow in Ω_f (compared to the strong dependence in the porous media Ω_p. For a chosen constant C first the flow in Ω_p can be determined and then the problem in the fluid domain is solved using the mass flux condition (5) and the second condition in (7) for the tangential velocity components.

3.2 The Beavers-Joseph Interface Condition

Several experiments performed by Beavers and Joseph [2] in a fluid channel over a porous media and found out that the mass flux through Ω_f is larger than predicted by the Poiseuille flow (i.e. with no-slip boundary conditions). This flow situation can be classified as a case of near parallel flow with interface conditions (5). Beavers and Joseph explained this observation with a slip velocity at the interface and proposed an empirical slip-flow condition that agreed well with their experiments:

$$\tau_j \cdot \partial_n u_f = \frac{\alpha}{\sqrt{K}}(u_f - u_p)\tau_j \qquad (8)$$

where α only depends on porous media properties and ∂_n and τ_j denotes the uniform tangential (horizontal) Darcy velocity in Ω_p and K denotes the permeability.

The eq. (8) allows for a discontinuity in the tangential velocity, i.e., rapid changes in the velocity in a small boundary layer are substituted through a jump. Using the Beavers-Joseph condition (8) the agreement between measurements from their experiments and the predicted values was quite good, with over 90% of the experimental values having errors of less than 2% [2]. The interface condition eq. (8) was mathematically justified by Jager and Mikelic [10].

3.3 Saffman's Modification of the Beavers-Joseph Interface Condition

In [18] Saffman gave a justification of the Beavers-Joseph interface condition at a physical level of rigor. Saffman proposed in 1971 a modification of the Beavers-Joseph law (8): he found out that the tangential velocity on the interface is proportional to the shear stress and proposed a modification of the Beavers-Joseph condition:

$$\tau_j \cdot \partial_n u_f = \frac{\alpha_{BJ}}{\sqrt{K}} u_f \tau_j + O(\sqrt{K}) \ \text{ on } \Gamma \qquad (9)$$

While the Beavers-Joseph interface condition (8) couples the fluid velocity in Ω_f with the filtration velocity in Ω_p, the modified eq. (8) (Beavers-Joseph-Saffmann condition) contains only variables in the free fluid domain Ω_f where the filtration velocity is usually much smaller than the slip velocity u_p. If the slip velocity is smaller than the maximal filtration velocity then setting the tangential velocity to zero is a reasonable approximation.

4. THE BRINKMAN MODEL

Neale and Nader [16] suggested the usage of the Brinkman correction to the Darcy model (3): they proposed to assume continuity of velocity and stress across the fluid-porous interface since the Stokes and the Brinkman equation are of the same order. They obtained in the fluid region the same solution as Beavers and Joseph provided that the slip coefficient is chosen as $\mu_{eff} = \mu / \phi$.

Sahraoui and Kaviany [19] with a numerical study calculated the slip coefficient: they discovered that the Brinkman extension to the Darcy equation does not satisfactorily model the flow field in Ω_p. This can be overcome using a variable effective viscosity μ_{eff} in the porous medium. On the contrary, for the Brinkman model for the flow in Ω_p this ambiguity does not occur. In this case, the equations in the porous media Ω_p and equations in the fluid region Ω_f are of the same type. Two types of coupling conditions can be found in the literature. The more common choice is conditions of continuous velocity and continuity of the normal component of the stress tensor

$$u_f = u_p \text{ on } \Gamma \qquad (10)$$

$$n \cdot \left(\mu_{eff} \nabla u_f - pI \right) = n \cdot \left(\mu_{eff} \nabla u_p - pI \right) \text{ on } \Gamma \quad (11)$$

4.1 The Stress Jump Conditions of Ochoa-Tapia and Whitaker

Ochoa-Tapia and Whitaker [17] obtained at the interface continuity of the velocity and the continuity of the 'modified' normal stress by a volume averaging technique of the momentum equations in the interface region. They showed that the matching of Stokes equation with the Brinkman model conserves the continuity of velocity but induces a jump in the shear stress and suggested additionally to the condition (10), a stress jump condition that takes into account the momentum transfer at the interface

$$\frac{\mu}{\varepsilon}\frac{\left(\partial u_p\right)_t}{\partial n} - \mu\frac{\left(\partial u_f\right)_t}{\partial n} = \beta\frac{\mu}{\sqrt{K}}\left(u_p\right)_t \text{ on } \Gamma \qquad (12)$$

where $\left(\partial u_p\right)_t$, $(\partial u_f)_n$ represents the Darcy velocity component parallel to the interface aligned with the direction *t* and normal to the direction *n,* respectively normal component to the direction *n and* $\left(u_f\right)_t$ is the fluid velocity component parallel to the interface and β is an adjustable parameter which accounts for the stress jump at the interface. β is a dimensionless parameter of order one that is defined as a solution of a closure problem. These boundary conditions proposed in [17] were used by Kuznetsov [11] to compute solutions in channels partially filled with a porous material.

4.2 The Transition Zone Approach

When studying a Poiseuille flow over a permeable region, e.g., by Chandesris and Jamet [5], it turned out that the sharp interface with its jump conditions is only the limiting case of a transition region, where the physical properties of the medium have a strong but still continuous variations. Actually, this idea goes back to Nield [15]. He proposed to use a Brinkman equation in the transition region between the fluid and the porous medium modeled by the Darcy equation.

4.3 The Interface Conditions of Le Bars and Worster

Le Bars and Worster [13] considered special 'analytically tractable' cases for the one-domain approach with the Brinkman model for the porous medium. Using the Darcy equation and its previously proposed interface conditions, especially the Beavers-Joseph condition (7) Le Bars and Worster considered the Brinkman equation in the configuration studied by Beavers and Joseph and found a new condition at the fluid-porous interface

$$u(x,-\delta) = u_D(x,-\delta) \text{ with } \delta = c/\sqrt{K} \quad (13)$$

where *c* is a constant of order 1. They defined a viscous transition zone inside Ω_p, where the Stokes equation still applies up to a depth δ, and imposed continuity of pressure and velocities (9) at the position $y = -\delta$ (cf. Fig. 2). Here, δ denotes the characteristic size of this transition zone (a few pore lengths). Using this new condition (12) the computed values have a (slightly) better coincidence with the experimental values of Beavers and Joseph.

In [7], M. Ehrhardt studied different conditions for the interface conditions.

5. THE FORCHHEIMER MODEL

The Forchheimer model is used to govern the flow in the porous medium region:

无粘流体的经典耦合条件是压力的连续性和界面上的法向速度的连续性

$$\nabla p_p = \frac{\mu}{K} u_p - \frac{\rho C_F}{\sqrt{K}} |u_p| u_p$$

The two models are not different systems of PDEs. The classical coupling conditions for an inviscid fluid are the continuity of the pressure and the continuity of the normal velocities at the interface. For a viscous flow, one would assume additionally the vanishing of the tangential velocity at the interface Γ. For coupling Forchheimer's model (3) and Stokes equation (1) we can used the condition presented above used at Darcy or Birkman models.

F. Cimolin, M. Discacciati [4] and used the continuity of normal component of the velocity $u_f \cdot n = u_p \cdot n$ on Γ , continuity of the normal stress across Γ $p_f - \mu \frac{\partial u_f}{\partial n_f} \cdot n_f = p_p$ and the Beavers- Joseph condition $\tau_j \cdot \partial_n u_f = \frac{\alpha}{\sqrt{K}}(u_f - u_p)\tau_j$ who are used at Darcy models.

P. Yu [20] used continuity of normal component of the velocity $u_f = u_p$ on Γ continuity of the normal stress $\frac{\mu}{\varepsilon}\frac{(\partial u_f)_n}{\partial n_f} - \mu\frac{(\partial u_f)_n}{\partial n_f} = 0$ and the stress jump condition $\frac{\mu}{\varepsilon}\frac{\left(\partial u_p\right)_t}{\partial n} - \mu\frac{\left(\partial u_f\right)_t}{\partial n} = \beta\frac{\mu}{\sqrt{K}}\left(u_p\right)_t$ where $\left(\partial u_p\right)_t$, $(\partial u_f)_n$ represents the Darcy velocity component parallel to the interface aligned with the direction *t* and normal to the direction *n,* respectively normal component to the direction *n and* $\left(u_f\right)_t$ is the fluid velocity component parallel to the interface and β is an adjustable parameter which accounts for the stress jump at the interface. Furthermore, the effect of the jump condition on the diffusive flux is considered, additional to that on the convective part which has been usually considered.

6. CONCLUSION

The question is whether a good interface condition can be found that the coupled problem can be satisfactorily. The most accepted and successful interface condition is Beavers-Joseph interface condition which is concluded from experimental date. But, due to the mathematical difficulty in treating the Beavers-Joseph another conditions have been adopted such the Beavers-Joseph-Saffman, the Interface Conditions of Le Bars and Worster, the Stress Jump Conditions of Ochoa-Tapia and Whitaker

This conditions help up to solve the models. The numerical algorithms for solving the coupled system of free fluid and porous media are separated into three major categories:

- the first group of methods uses different equations in different domains, e.g., the Navier–Stokes equation in the liquid region and the Darcy / Brinkman/ Forchheimer model in the porous zones and couples them through suitable interface conditions. These kind of algorithms use domain decomposition techniques
- the second group consists of those algorithms, that solely uses one system of equations in the whole domain (Navier–Stokes–Brinkman system) obtaining the transition between both fluid and porous regions through continuous spatial variations of properties ('single-domain approach').
- the two method grid by decoupling the mixed model by a coarse grid approximations to the interfaces conditions

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Condițiile de interfață pentru cuplarea unui fluid cu un mediu poros

Rezumat: *Modelarea filtrării fluidelor incompresibile printr-un mediul poros are nevoie de diferite tipuri de ecuații diferențiale în fluid şi în mediul poros. Aceste ecuații trebuie cuplate printr-o interfață care separă cele două domenii. Prezentăm diferite condiții pe interfața dintre un fluid şi un mediu poros, incluzând mult cunoscuta condiție Beavers-Joseph-Saffman cât şi recenta imbunătățire a lui Le Bars şi Worster pentru ecuațiile Navier-Stokes şi Darcy sau extensiile sale, modelul Brinkman sau Forchheimer.*

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