EE140 Introduction to Communication Systems

Lecture 5

Instructor: Prof. Lixiang Lian
ShanghaiTech University, Fall 2022

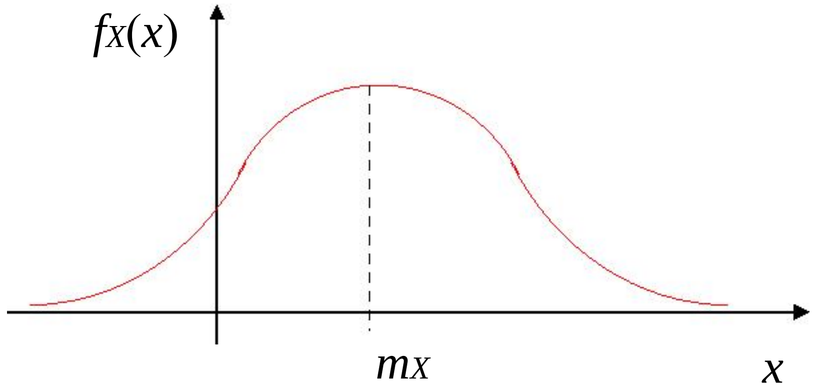
Contents

- Random signals
  - Review of probability and random variables
  - Random processes: basic concepts
  - Gaussian white processes

Recall: Gaussian Distribution

- Gaussian or normal distribution is a continuous r.v. with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp[-\frac{1}{2\sigma_x^2}(x - m_x)^2]$$



- A Gaussian r.v. is completely determined by its mean and variance, and hence usually denoted as

$$x \sim N(m_x, \sigma_x^2)$$

Gaussian Process

- Definition: $X(t)$ is a Gaussian process if for all n and all $t_1, t_2, \cdots, t_n$, the sample values $X(t_1), X(t_2), \ldots,$ $X(t_n)$ have a joint Gaussian density function

$$f_{X(t_1)X(t_2),\cdots,X(t_n)}(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}(\det(\mathbf{C}))^{1/2}} \exp\left[-\frac{(x-\mathbf{m})^T\mathbf{C}^{-1}(x-\mathbf{m})}{2}\right]$$

$C_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$

- Properties:
  - If it is wide-sense stationary, it is also strictly stationary (Gaussian process is completely defined by its first order statistics $\mathbf{m}$ and second order statistics $\mathbf{C}$.)

Gaussian Process

- Properties:
  - If the samples of Gaussian process $X(t_1), X(t_2), \ldots, X(t_n)$ are uncorrelated in time, they are also independent

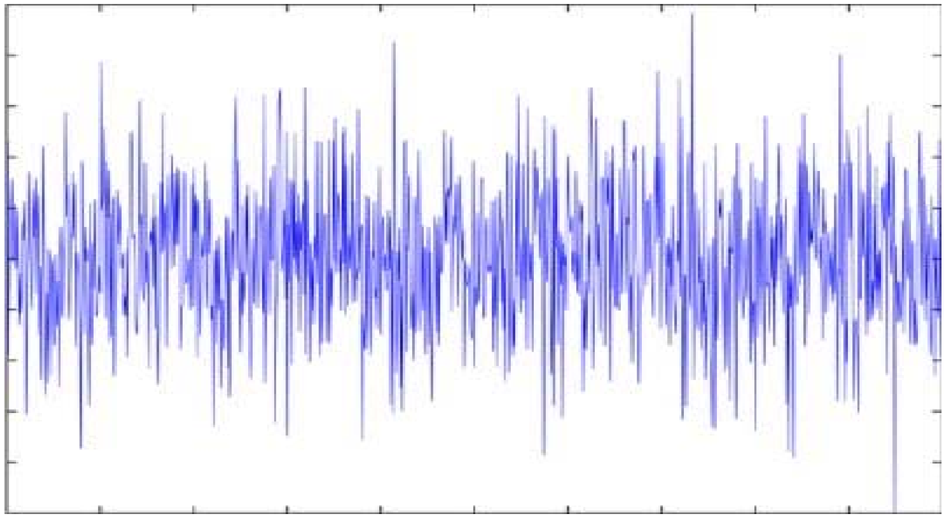
$$f_{X(t_1)X(t_2),\cdots,X(t_n)}(x_1, x_2, \ldots, x_n) = \prod_{\mathrm{k}=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_{\mathrm{k}}} \exp[-\frac{(x_{\mathrm{k}} - a_{\mathrm{k}})^2}{2\sigma_{\mathrm{k}}^2}]$$
$$= f_{X(t_1)}(x_1) \cdot f_{X(t_2)}(x_2) \cdot \cdots \cdot f_{X(t_n)}(x_n)$$

  - If the input to a linear system is a Gaussian process, the output is also a Gaussian process

$$Y_o(t) = \int_{-\infty}^{\infty} h(\tau) X_i(t - \tau) d\tau$$

Noise

- Gaussian Noise:
  - often modeled as Gaussian and stationary with 0 mean



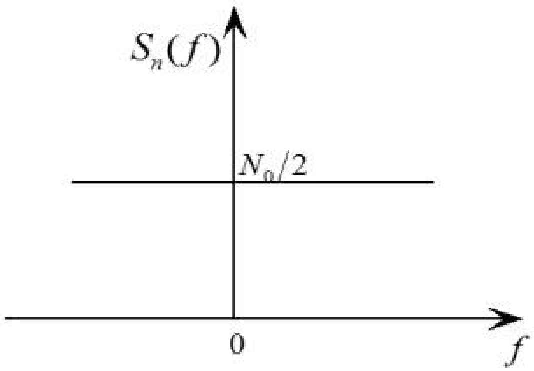
- White noise (stationary and zero mean)

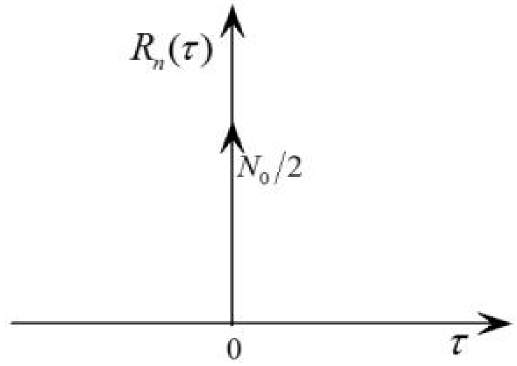
Noise

- White Noise (stationary and zero mean)

$$S_n(f) = \frac{N_0}{2} \quad \Longrightarrow \quad R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

Infinite power





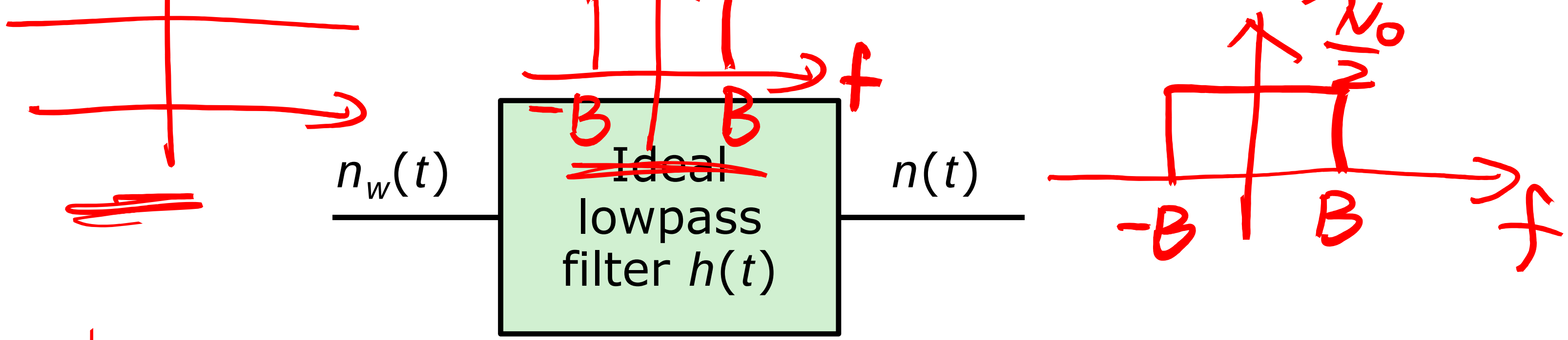
White noise is completely uncorrelated!

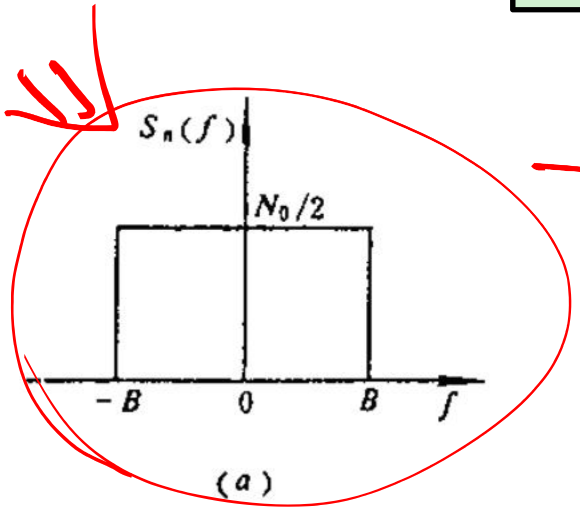
$N_0 = KT = 4.14 \times 10^{-21}$
$= -174$ dBm/Hz

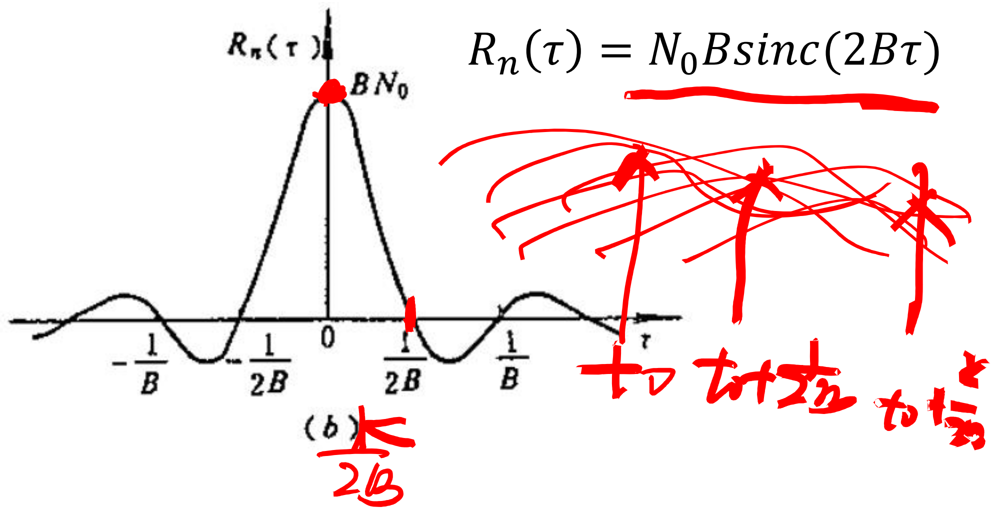
N_0: single-sided power spectral density
$\frac{N_0}{2}$: two-sided power spectral density

- White Gaussian Noise (stationary and zero mean)

Bandlimited Noise







$R_n(\tau) = N_0 B sinc(2B\tau)$

- Q1. At what rate to sample the noise can we get uncorrelated realizations? (2B/second)
- Q2. What is the power of each sample? (BN_0) $\frac{N_0}{2} \times B \times 2 = N_0 B$

Noise Equivalent Bandwidth

- White noise: zero mean, two-sided PSD=$\frac{N_0}{2}$
- Arbitrary filter: $H(f)$
- Average output noise power

$n_w(t) \rightarrow \boxed{H(f)} \rightarrow P_{n_o}$

$n_w(t) \rightarrow \boxed{B_N, H_0} \rightarrow P_{n_o}$

$$P_{n_o} = \int_{-\infty}^{\infty} \frac{N_0}{2} |H(f)|^2 \, df = N_0 \int_0^{\infty} |H(f)|^2 \, df$$

$|H(f)| = |H(-f)|$, if $h(t)$ is real.

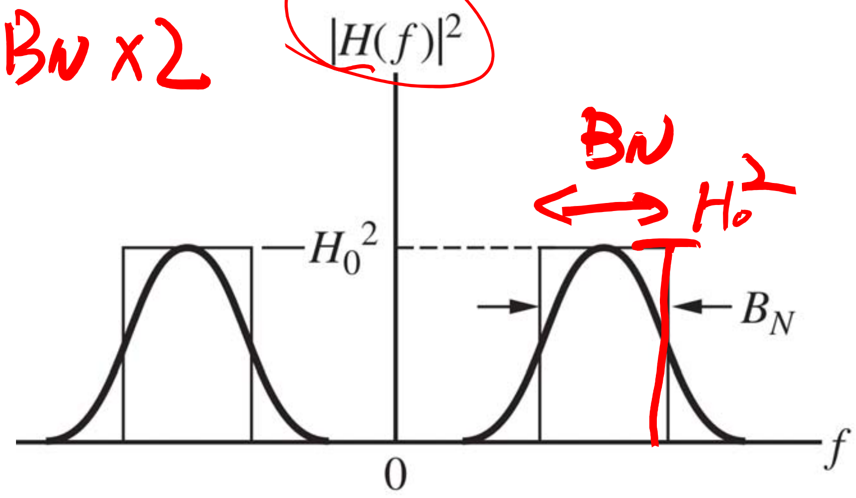
- Ideal filter: B_N, H_0

$$P_{n_o} = N_0 H_0^2 B_N$$

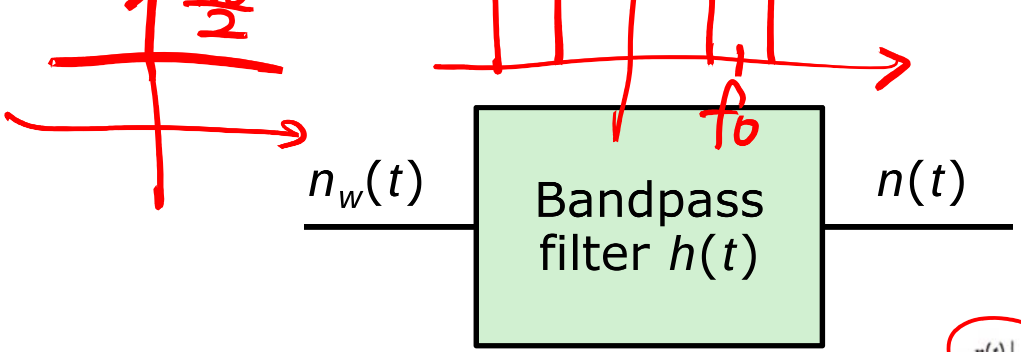
$\frac{N_0}{2} \times H_0^2 \cdot B_N \times 2$

- Noise equivalent bandwidth

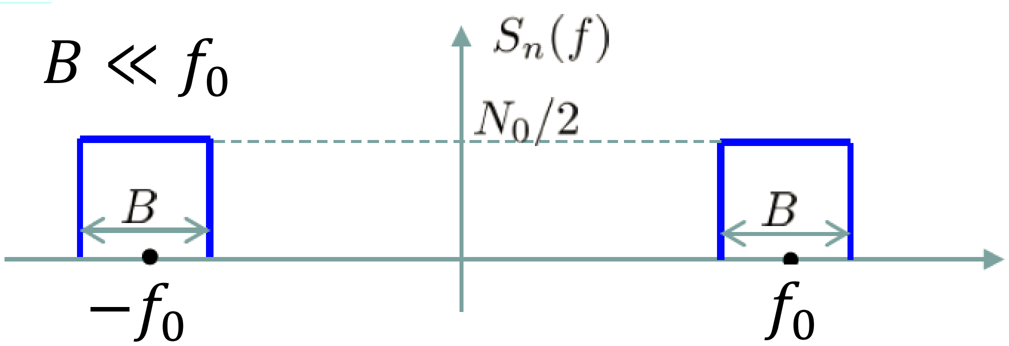
$$B_N = \frac{\int_0^{\infty} |H(f)|^2 df}{H_0^2}$$

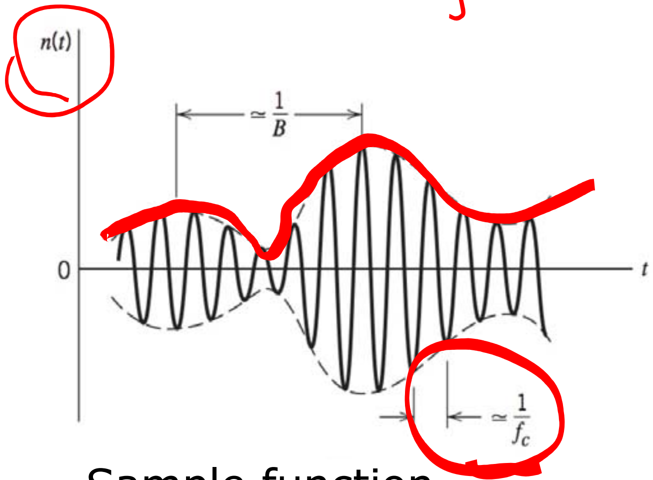


Narrowband Noise



$B << f_0$

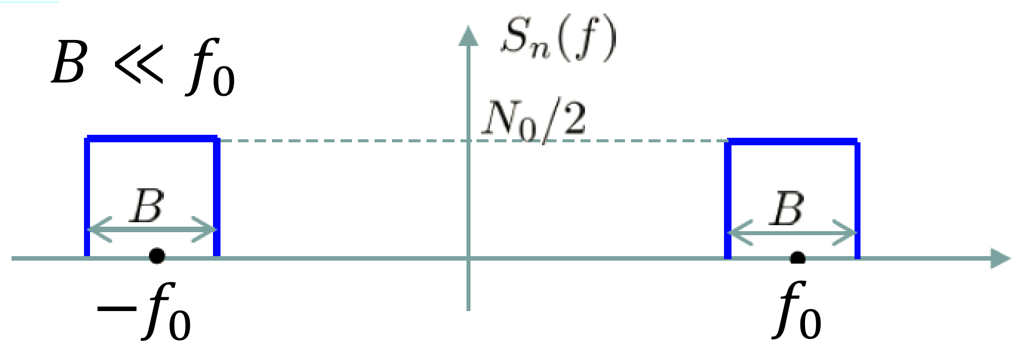




Sample function

- Two specific representation of narrowband noise
  - In-phase and quadrature components
  - Envelope and phase

Narrowband Noise



- Canonical form of a band-pass noise process

$$n(t) = n_c(t)\cos(2\pi f_0 t + \theta) - n_s(t)\sin(2\pi f_0 t + \theta)$$

In-phase component

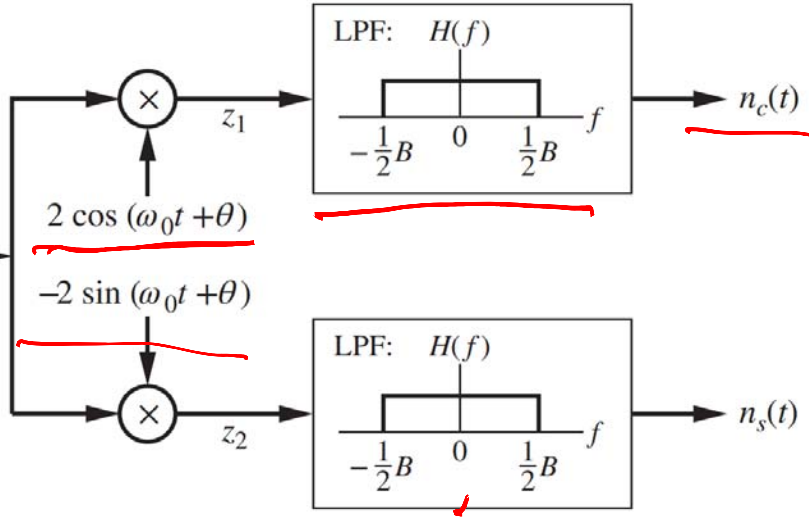
Quadrature component

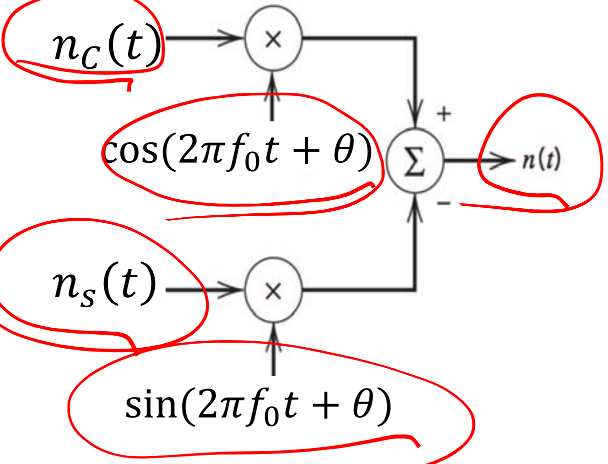
Low-pass noise process

θ is an arbitrary phase angle

Narrowband Noise

- How to produce $n_c(t)$ and $n_s(t)$



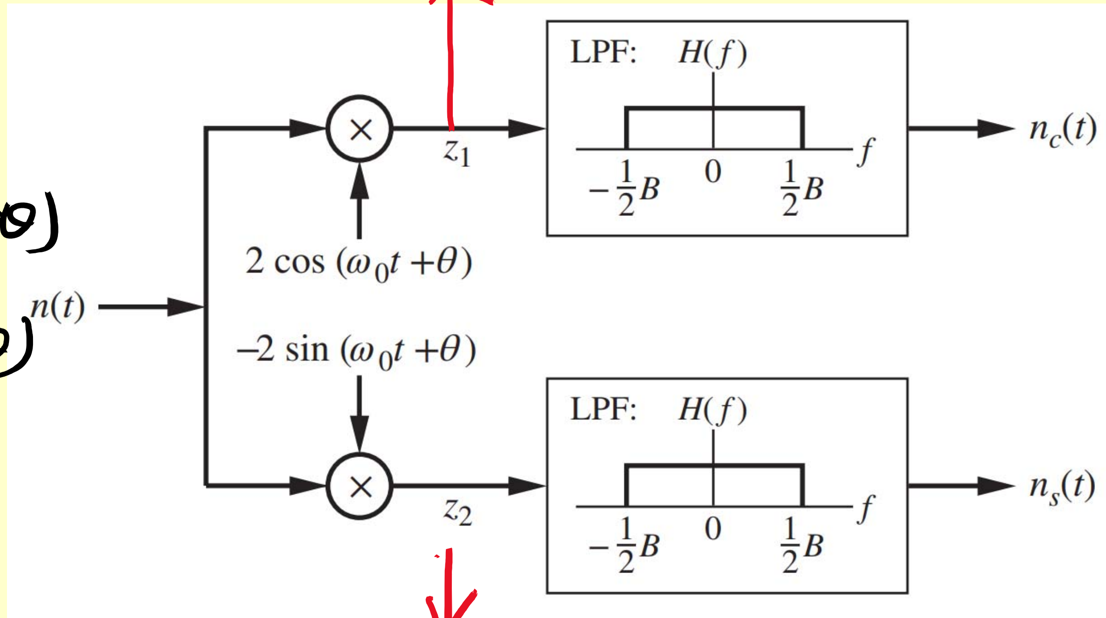


Why equality holds? (Proof: Page 712, Appendix C)

$$E\left\{\left[n(t) - [n_c(t)\cos(2\pi f_0 t + \theta) - n_s(t)\sin(2\pi f_0 t + \theta)]\right]^2\right\} = 0$$

$$z_1(t) = n(t)\cdot 2\cos(w_0 t+\theta)$$
$$= 2n_c(t)\cos^2(2\pi f_0 t+\theta) - 2n_s(t)\sin(2\pi f_0 t+\theta)\cos(2\pi f_0 t+\theta)$$
$$= n_c(t) + n_c(t)\cos(4\pi f_0 t+2\theta) - n_s(t)\sin(4\pi f_0 t+2\theta)$$

$$n(t) = n_c(t)\cos(2\pi f_0 t+\theta) - n_s(t)\sin(2\pi f_0 t+\theta)$$



$$z_2(t) = n(t)\cdot(-2\sin(2\pi f_0 t+\theta))$$
$$= -2n_c(t)\sin(2\pi f_0 t+\theta)\cos(2\pi f_0 t+\theta) + 2n_c(t)\sin^2(2\pi f_0 t+\theta)$$
$$= -n_c(t)\sin(4\pi f_0 t+2\theta) + n_s(t) - n_s(t)\cos(4\pi f_0 t+\theta)$$

Properties

If $n(t)$ is Gaussian RP → $n_s(t)$ and $n_C(t)$ are joint Gaussian RP

If $n(t)$ is stationary → $n_s(t)$ and $n_C(t)$ are jointly stationary

E

R_{n_s}, R_{n_c}

$R_{n_s n_c}$

$n_s(t)$ and $n_C(t)$ →

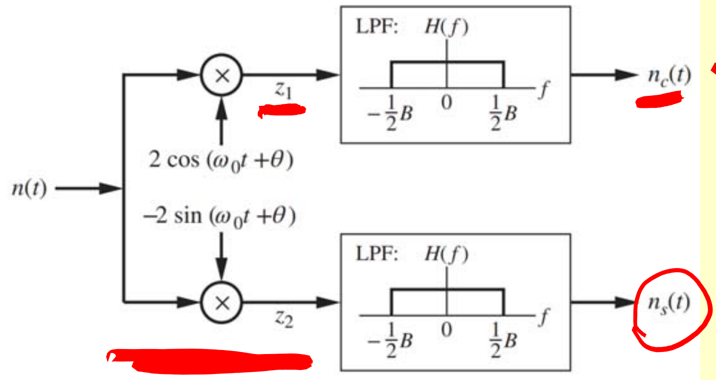
Zero mean

Same PSD (autocorrelation, variance)

$$S_{n_c}(f) = S_{n_s}(f)$$
$$= \mathrm{Lp}[S_n(f - f_0) + S_n(f + f_0)]$$

Cross-PSD (odd Cross-correlation)

$$S_{n_c n_s}(f) = j\mathrm{Lp}[S_n(f - f_0) - S_n(f + f_0)]$$



$z_1(t) = 2n(t)\cos(w_0 t+\theta)$ $\quad E[z_1(t)]=0$

$n_c(t) = LP[z_1(t)]$ $\quad E[n_c(t)] = 0\cdot H(0) = 0$

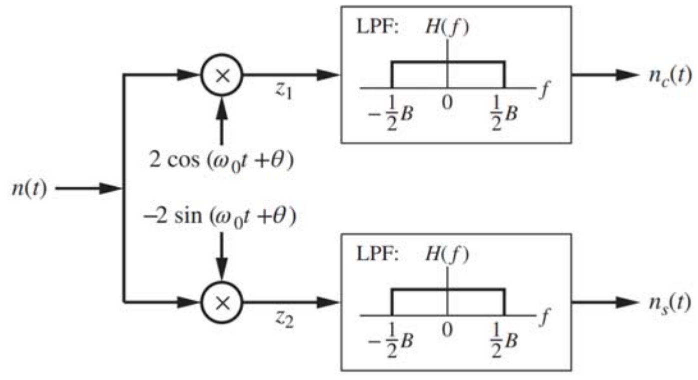
$z_2(t) = -2n(t)\sin(w_0 t+\theta)$

$n_s(t) = LP[z_2(t)]$ $\quad E[n_s(t)]=0$

$$R_{z_1}(t,t+\tau) = E[4n(t)n(t+\tau)\cos(w_0 t+\theta)\cos(w_0(t+\tau)+\theta)]$$
$$= 2R_n(\tau)\cos(2\pi f_0\tau)$$

$$R_{z_2}(t,t+\tau) = E[4n(t)n(t+\tau)\sin(2\pi f_0 t+\theta)\sin(2\pi f_0(t+\tau)+\theta)]$$
$$= 2R_n(\tau)\left(\cos(2\pi f_0\tau) - E[\cos(2\pi f_0(2t+\tau)+\theta)]\right)$$
$$= 2R_n(\tau)\cos(2\pi f_0\tau)$$

$$R_{z_1z_2}(t,t+\tau) = E[z_1(t)z_1(t+\tau)]$$
$$= -2R_n(\tau)\,E[2\cos(w_0 t+\theta)\sin(w_0(t+\tau)+\theta)]$$
$$= -2R_n(\tau)\sin(2\pi f_0\tau)$$



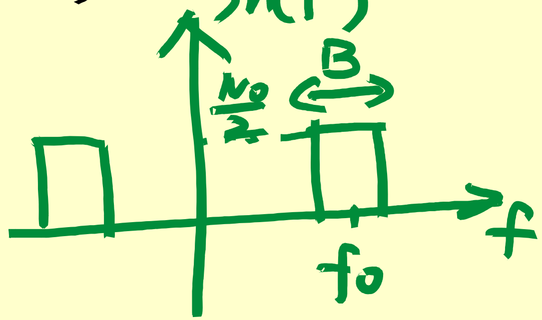
$z_1(t) = 2n(t)\cos(w_0t+\theta)$

$n_c(t) = LP[z_1(t)]$

$z_2(t) = -2n(t)\sin(w_0t+\theta)$

$n_s(t) = LP[z_2(t)]$

$P = N_0B$



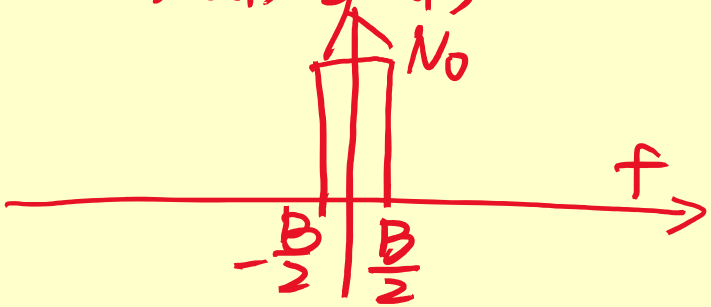
$R_{z_1}(\tau) = 2R_n(\tau)\cos(2\pi f_0\tau)$

$S_{z_1}(f) = S_n(f-f_0)+S_n(f+f_0)$

$S_{n_c}(f) = LP[S_n(f-f_0)+S_n(f+f_0)]$

$\Downarrow$

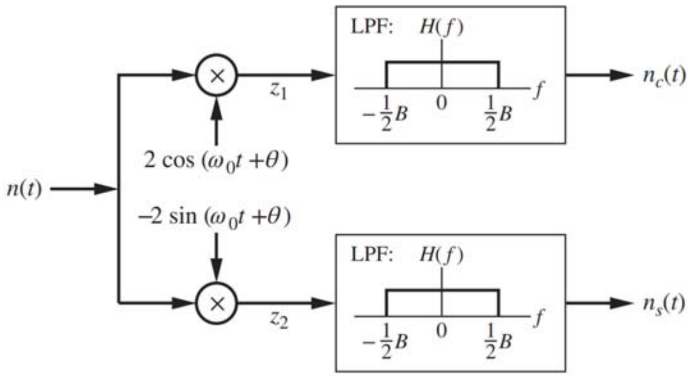
$S_{n_c}(f) = S_{n_s}(f)$ $P=$



$R_{z_2}(\tau) = 2R_n(\tau)\cos(2\pi f_0\tau)$

$S_{z_2}(f) = S_n(f-f_0)+S_n(f+f_0)$

$S_{n_s}(f) = LP[S_n(f-f_0)+S_n(f+f_0)]$

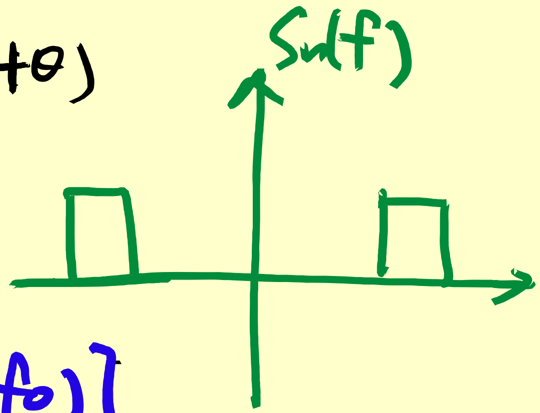


$$z_1(t) = 2n(t)\cos(\omega_0 t+\theta)$$

$$n_c(t) = LP[z_1(t)]$$

$$z_2(t) = -2n(t)\sin(\omega_0 t+\theta)$$

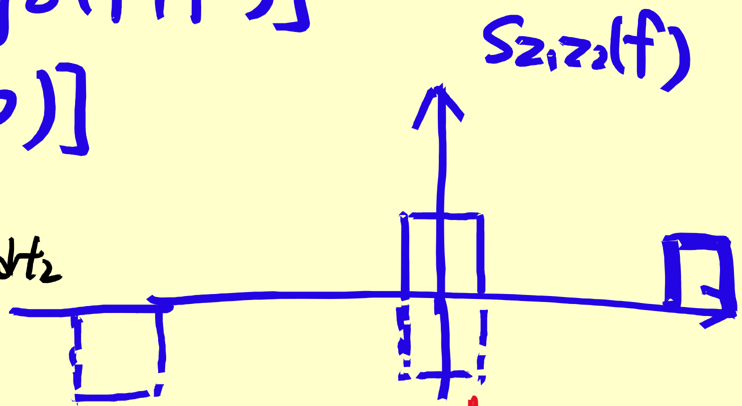
$$n_s(t) = LP[z_2(t)]$$



$$R_{z_1z_2}(t,t+\tau) = -2R_n(\tau)\sin(2\pi f_0\tau)$$

$$S_{z_1z_2}(f) = S_n(f) * [j\delta(f-f_0) - j\delta(f+f_0)]$$

$$= j[S_n(f-f_0) - S_n(f+f_0)]$$



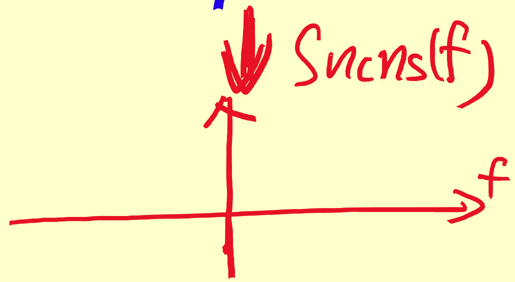
$$R_{n_cn_s}(\tau) = E[h(t)*z_1(t)\cdot h(t)*z_2(t+\tau)]$$

$$= E\int_{t_1}\int_{t_2} h(t_1)\,h(t_2)\,z_1(t-t_1)\,z_2(t+\tau-t_2)\,dt_1dt_2$$

$$= \int\int h(t_1)\,h(t_2)\,R_{z_1z_2}(\tau+t_1-t_2)\,dt_1dt_2$$

$$= \int h(t_1)\cdot[h(\tau)*R_{z_1z_2}(\tau+t_1)]\,dt_1$$

$$= h(-\tau)*h(\tau)*R_{z_1z_2}(\tau)$$



$$S_{n_cn_s}(f) = |H(f)|^2 S_{z_1z_2}(f) = jLP[S_n(f-f_0) - S_n(f+f_0)]$$

Properties

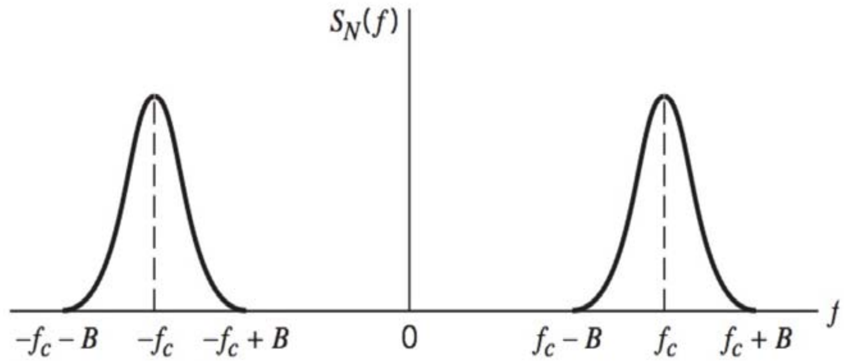
- Let n(t) be a zero-mean, stationary and Gaussian noise, then $n_c(t)$ and $n_s(t)$ satisfy the following properties
  - $n_c(t)$ and $n_s(t)$ are zero-mean, jointly stationary and jointly Gaussian process
  - Means: $E[n(t)] = E[n_c(t)] = E[n_s(t)] = 0$
  - PSD: $S_{n_c}(f) = S_{n_s}(f) = \mathrm{Lp}[S_n(f - f_0) + S_n(f + f_0)]$
  - Variances(power): $E[n^2(t)] = E[n_c^2(t)] = E[n_s^2(t)] = \mathrm{N}_0 B \triangleq \sigma^2$
  - Correlation function:
    - $R_{n_c}(\tau) = R_{n_S}(\tau)$, $R_n(0) = R_{n_c}(0) = R_{n_S}(0)$
    - $R_{n_c n_s}(\tau) = -R_{n_c n_s}(-\tau)$ (odd), $R_{sc}(0) = R_{cs}(0) = 0.$
  - Cross-PSD: $S_{n_c n_s}(f) = j\mathrm{Lp}[S_n(f - f_0) - S_n(f + f_0)]$
    - $R_{n_c n_s}(\tau) \equiv 0, \forall\tau,$ if $\mathrm{Lp}[S_n(f - f_0) - S_n(f + f_0)] = 0.$

Properties

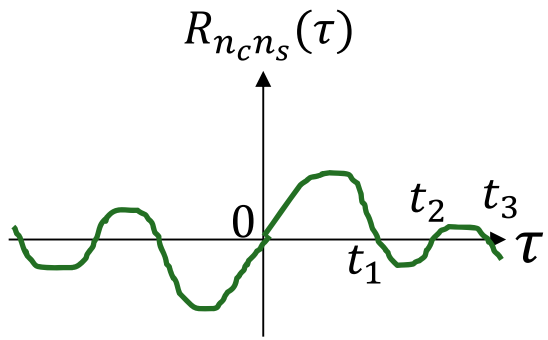
- Let n(t) be a zero-mean, stationary and Gaussian noise, then $n_c(t)$ and $n_s(t)$ satisfy the following properties
  - $n_c(t)$ and $n_s(t)$ are uncorrelated (independent) Gaussian process

1. If PSD of n(t) is symmetric about f_0

$$R_{n_c n_s}(\tau) \equiv 0, \forall\tau$$

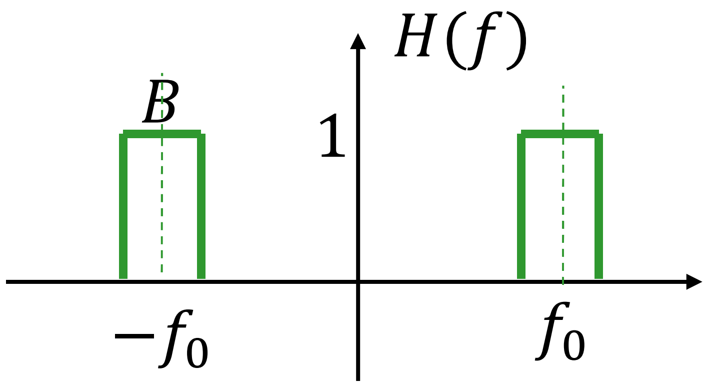


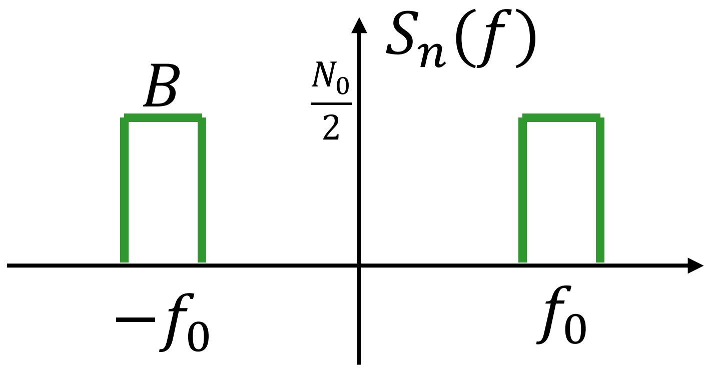
2. $\{\tau : R_{n_c n_s}(\tau) = 0\}$.



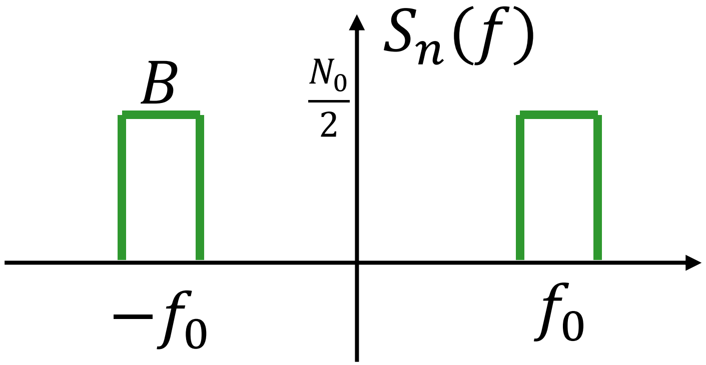
Example: Ideal band-pass filtered white noise

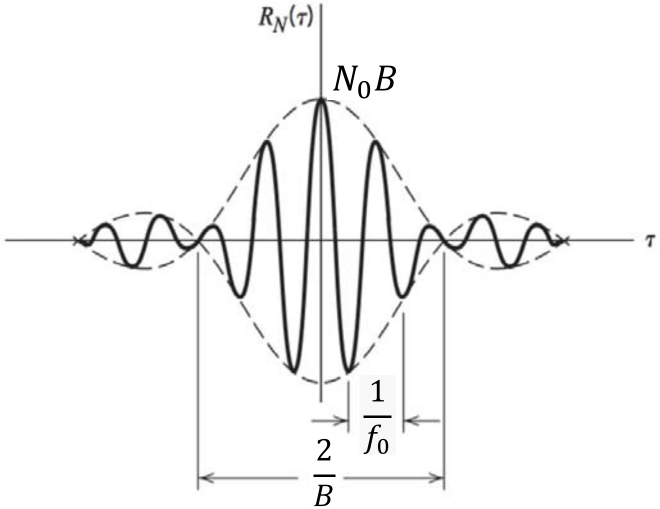
- Consider a white Gaussian noise of zero mean and PSD N0/2, which is passed through an ideal band-pass filter.
- Determine the autocorrelation function of n(t) and its in-phase and quadrature components.





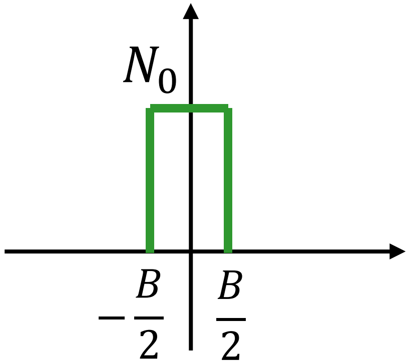
Example: Ideal band-pass filtered white noise

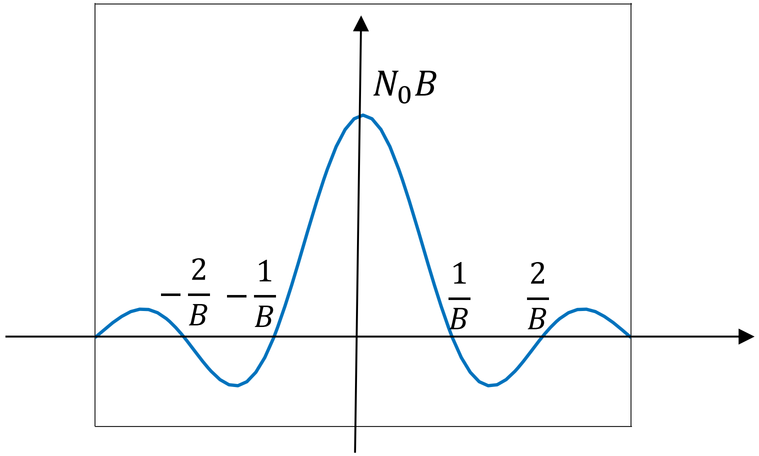




$$R_n(\tau) = N_0 B sinc(B\tau)\cos(2\pi f_0 \tau)$$

$$S_{n_c}(f) = S_{n_s}(f)$$





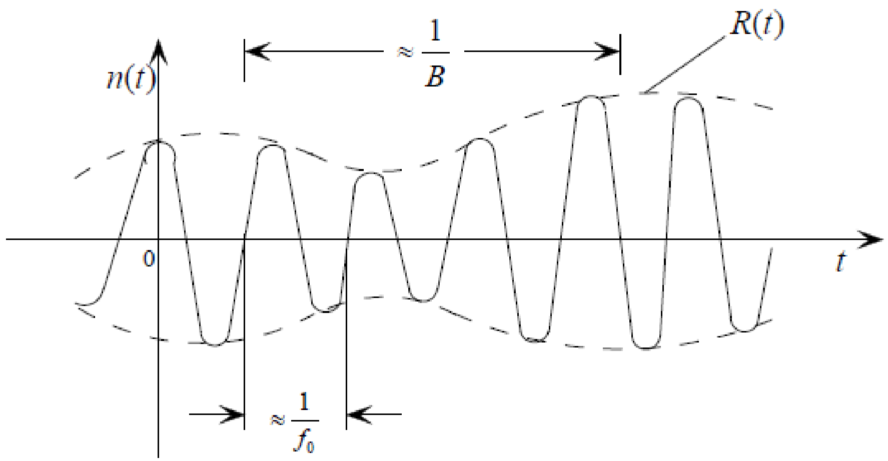
$$R_{n_c}(\tau) = R_{n_s}(\tau) = N_0 B sinc(B\tau)$$

Envelope and Phase

- Angular representation of n(t)

$$n(t) = R(t)\cos(\omega_0 t + \theta + \varphi(t))$$

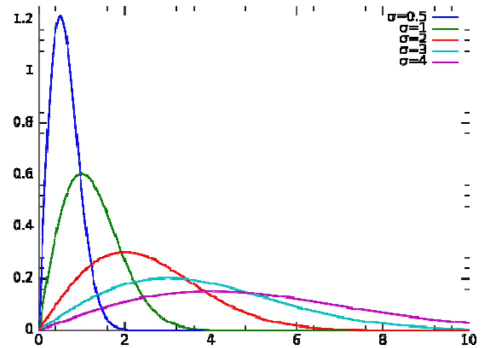
$$\text{where} \begin{cases} R(t) = \sqrt{n_c^2(t) + n_s^2(t)} & \text{envelope} \\ \varphi(t) = \tan^{-1}\dfrac{n_s(t)}{n_c(t)}, [0 \le \varphi(t) \le 2\pi] & \text{phase} \end{cases}$$



Example

- Let n(t) be a zero-mean, stationary Gaussian process, find the statistics of the envelop and phase
- Result:
  - Envelop follows Rayleigh distribution
  - Phase follows uniform distribution

$$\begin{cases} f(R) = \int_0^{2\pi} f(R,\varphi)d\varphi = \dfrac{R}{\sigma^2}\exp\left\{-\dfrac{R^2}{2\sigma^2}\right\}, R \geq 0 \\ f(\varphi) = \int_0^{\infty} f(R,\varphi)dR = \dfrac{1}{2\pi}, 0 \leq \varphi \leq 2\pi \end{cases}$$



- For the same t, the envelop variable R and phase variable φ are independent (but not the two processes)

Rayleigh fading channel: model the fading channel with random scatters.

Derivation

- Derivation

$$f(R,\varphi) = f(n_c,n_s)\left|\frac{\partial(n_c,n_s)}{\partial(R,\varphi)}\right|$$

$$f(n_c,n_s) = f(n_c)\cdot f(n_s) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{n_c^2+n_s^2}{2\sigma^2}\right]$$

$$\left|\frac{\partial(n_c,n_s)}{\partial(R,\varphi)}\right| = \begin{vmatrix} \dfrac{\partial n_c}{\partial R} & \dfrac{\partial n_s}{\partial R} \\ \dfrac{\partial n_c}{\partial \varphi} & \dfrac{\partial n_s}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos\varphi & \sin\varphi \\ -R\sin\varphi & R\cos\varphi \end{vmatrix} = R$$

$$f(R,\varphi) = Rf(n_c,n_s) = \frac{R}{2\pi\sigma^2}\exp\left[-\frac{(R\cos\varphi)^2+(R\sin\varphi)^2}{2\sigma^2}\right]$$

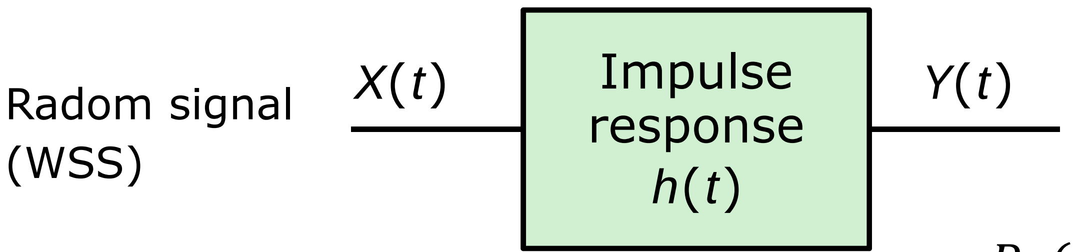
$$= \frac{R}{2\pi\sigma^2}\exp\left\{-\frac{R^2}{2\sigma^2}\right\}$$

Summary

- For WSS:

$$S_X(f) \leftrightarrow R_X(\tau)$$

- WSS transmission through a linear system



$$R_Y(\tau) = h(-\tau) * h(\tau) * R_X(\tau)$$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

- White noise (zero-mean)
  - Bandlimited noise
  - Narrowband noise: Gaussian, stationary and zero-mean
  - Non-Gaussian?

Sine Wave with Bandpass Noise

- Received signal

$$r(t) = A\cos(\omega_c t + \theta) + n(t)$$

where A, ω_c are deterministic, θ is random phase uniformly distributed in $[-\pi, \pi]$, $n(t)$ is narrowband noise (zero-mean, stationary Gaussian process).

$$\begin{aligned} r(t) &= [A\cos\theta + n_c(t)]\cos\omega_c t - [A\sin\theta + n_s(t)]\sin\omega_c t \\ &= z_c(t)\cos\omega_c t - z_S(t)\sin\omega_c t \\ &= z(t)\cos[\omega_c t + \varphi(t)] \end{aligned}$$

$$z_c(t) = A\cos\theta + n_c(t)$$

$$z_s(t) = A\sin\theta + n_s(t)$$

Sine Wave with Bandpass Noise (cont'd)

- Envelop

$$z(t) = \sqrt{z_c^2(t) + z_s^2(t)}, z \geq 0$$

- Phase

$$\varphi(t) = \tan^{-1}\frac{z_s(t)}{z_c(t)}$$

- Given θ

$$E[z_c] = A\cos\theta$$
$$E[z_s] = A\sin\theta$$
$$\sigma_c^2 = \sigma_s^2 = \sigma_n^2$$

- Joint distribution

$$f(z_c, z_s|\theta) = \frac{1}{2\pi\sigma_n^2}\exp\left\{-\frac{1}{2\sigma_n^2}[(z_c - A\cos\theta)^2 + (z_s - A\sin\theta)^2]\right\}$$

PDF of the Amplitude

$$f(z_c, z_s|\theta) = \frac{1}{2\pi\sigma_n^2} \exp\left\{-\frac{1}{2\sigma_n^2}\left[(z_c - A\cos\theta)^2 + (z_s - A\sin\theta)^2\right]\right\}$$

$$f(z,\varphi|\theta) = f(z_c, z_s|\theta)\left|\frac{\partial(z_c, z_s)}{\partial(z,\varphi)}\right|$$

$$= \begin{vmatrix} \cos\varphi & \sin\varphi \\ -z\sin\varphi & z\cos\varphi \end{vmatrix} f(z_c, z_s|\theta) = z \cdot f(z_c, z_s|\theta)$$

- For PDF of the amplitude

$$f(z|\theta) = \int_0^{2\pi} f(z,\varphi|\theta)d\varphi$$

$$= \frac{z}{2\pi\sigma_n^2} \exp\left[-\frac{z^2 + A^2}{2\sigma_n^2}\right] \int_0^{2\pi} \exp\left[\frac{Az}{\sigma_n^2}\cos(\theta - \varphi)\right] d\varphi$$

PDF of the Amplitude (cont'd)

- Amplitude

$$f(z|\theta) = \int_0^{2\pi} f(z,\varphi|\theta)d\phi$$

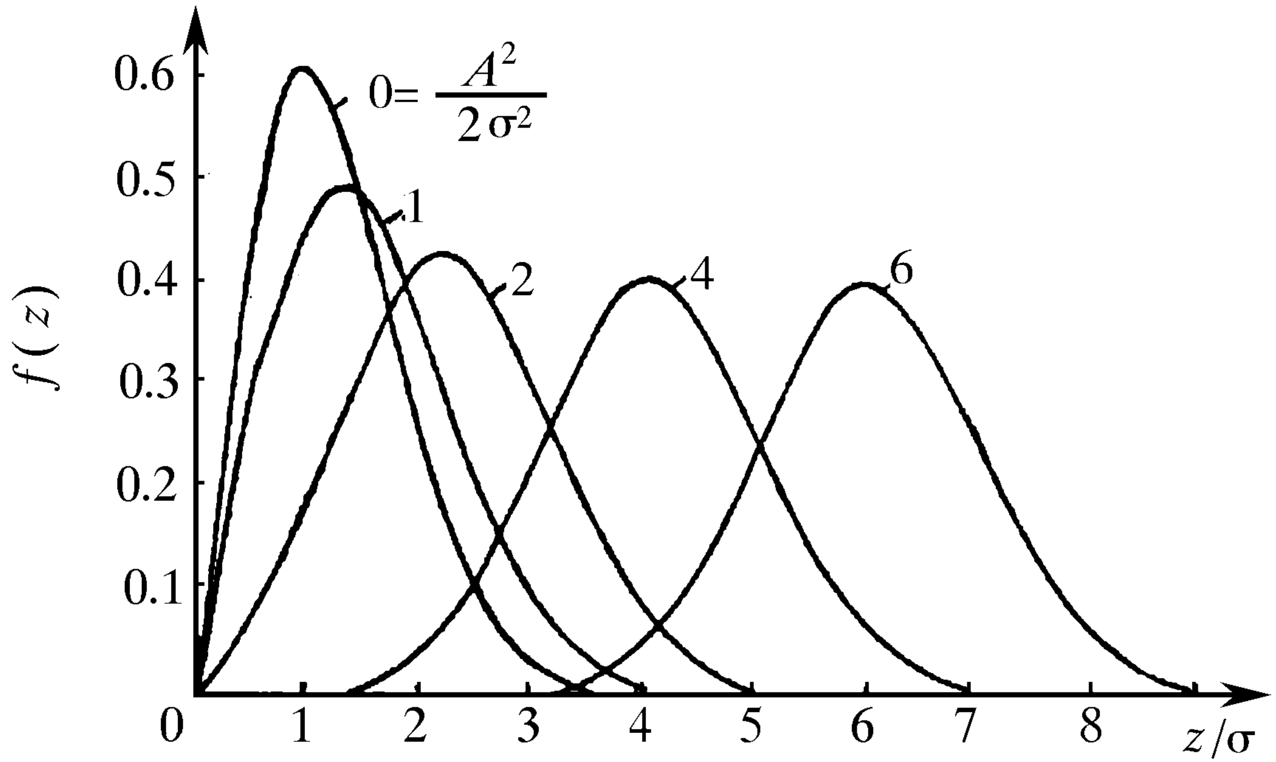
$$= \frac{z}{2\pi\sigma_n^2}\exp\left[-\frac{z^2+A^2}{2\sigma_n^2}\right]\int_0^{2\pi}\exp\left[\frac{Az}{\sigma_n^2}\cos(\theta-\phi)\right]d\phi$$

$$= \frac{z}{2\pi\sigma_n^2}\exp\left[-\frac{z^2+A^2}{2\sigma_n^2}\right]I_0\left(\frac{Az}{\sigma_n^2}\right)$$

Ricean distribution

$I_0\left(\frac{Az}{\sigma_n^2}\right)$ 0th order Bessel functions of the first kind

PDF of the Amplitude (cont'd)



- A small,

$$I_0\left(\frac{Az}{\sigma_n^2}\right) \approx 1$$

Rayleigh distribution

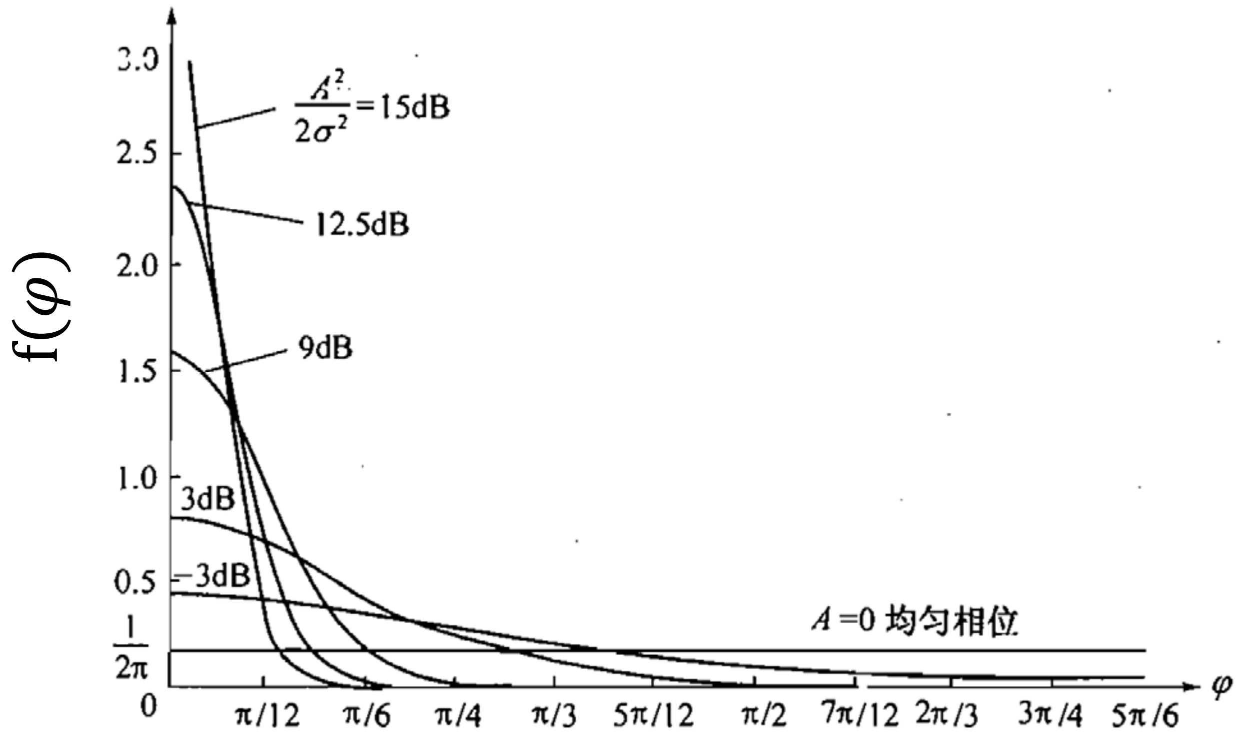
- A large,

$$I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$$

Gaussian distribution

Ricean fading channel: model the fading channel with a direct path and scatters.

PDF of the Amplitude (cont'd)



- A small,

Uniform distribution

- A large,

Concentrate around θ

Ricean fading channel: model the fading channel with a direct path and scatters.

Thanks for your kind attention!

Questions?