EE140 Introduction to Communication Systems

Lecture 4

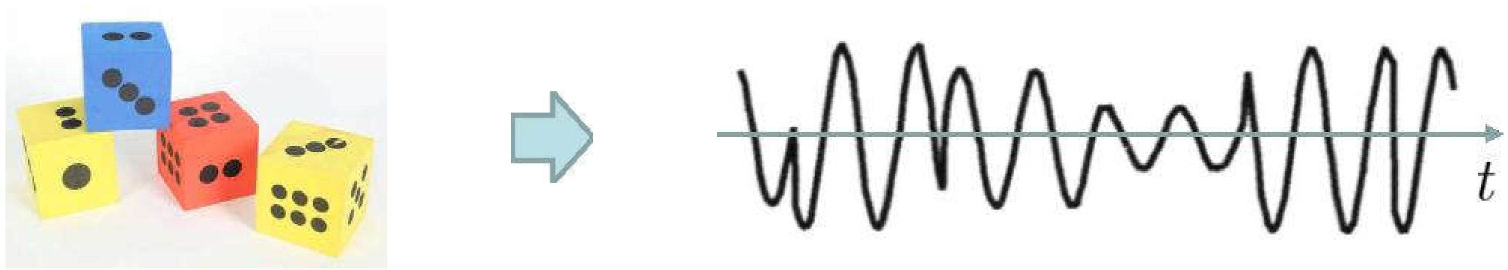
Instructor: Prof. Lixiang Lian
ShanghaiTech University, Fall 2022

Contents

- Random signals
  - Review of probability and random variables
  - Random processes: basic concepts
  - Gaussian white processes

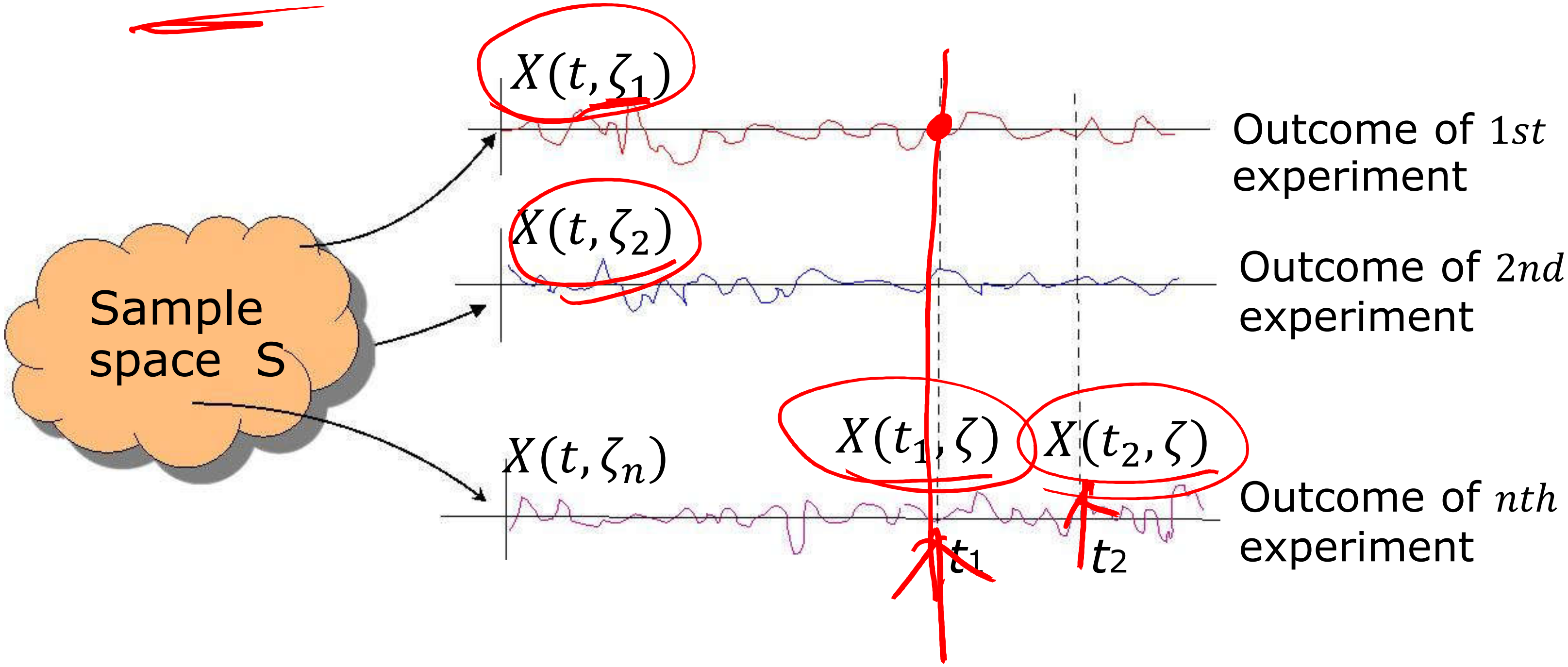
Random Process

- A random process (stochastic process, or random signal) is the evolution of random variables over time.



Description of Random Process

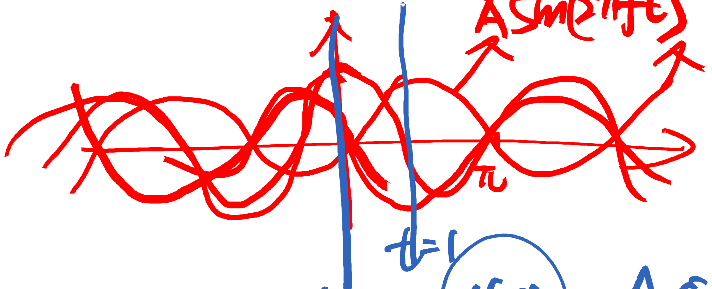
- $X(t,\zeta)$: random process
- $X(t,\zeta_i)$: sample function of the random process, ζ_i is a member of a sample space $\mathcal{S}$.
- $X(t_j,\zeta)$: a random variable
- $X(t_j,\zeta_i)$: a number



Example: $x(t) = A\sin(2\pi f t + \phi)$

① A, f constant $\phi \sim U[0, 2\pi]$

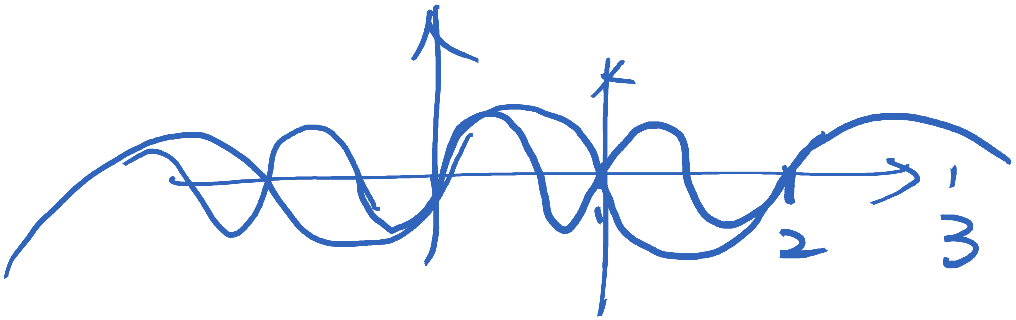
$A\sin(2\pi f t)$ $\phi = \pi$ $-A\sin(2\pi f t)$



$x(0) = A\sin(\phi) \in [-A, A]$

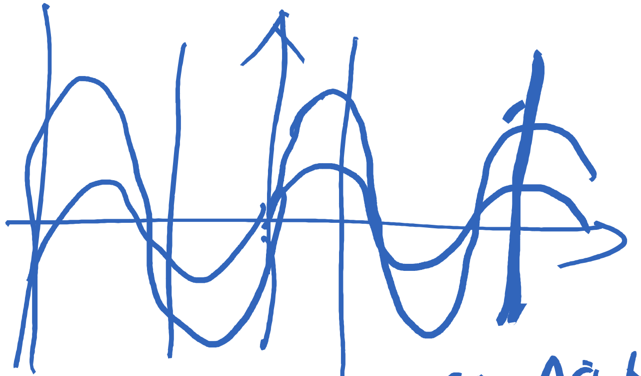
$x(1) = A\sin(2\pi f + \phi) \in [-A, A]$

② A, ϕ constant $f \sim U[0, 1]$



$x(1) = A\sin(2\pi f + \phi)$

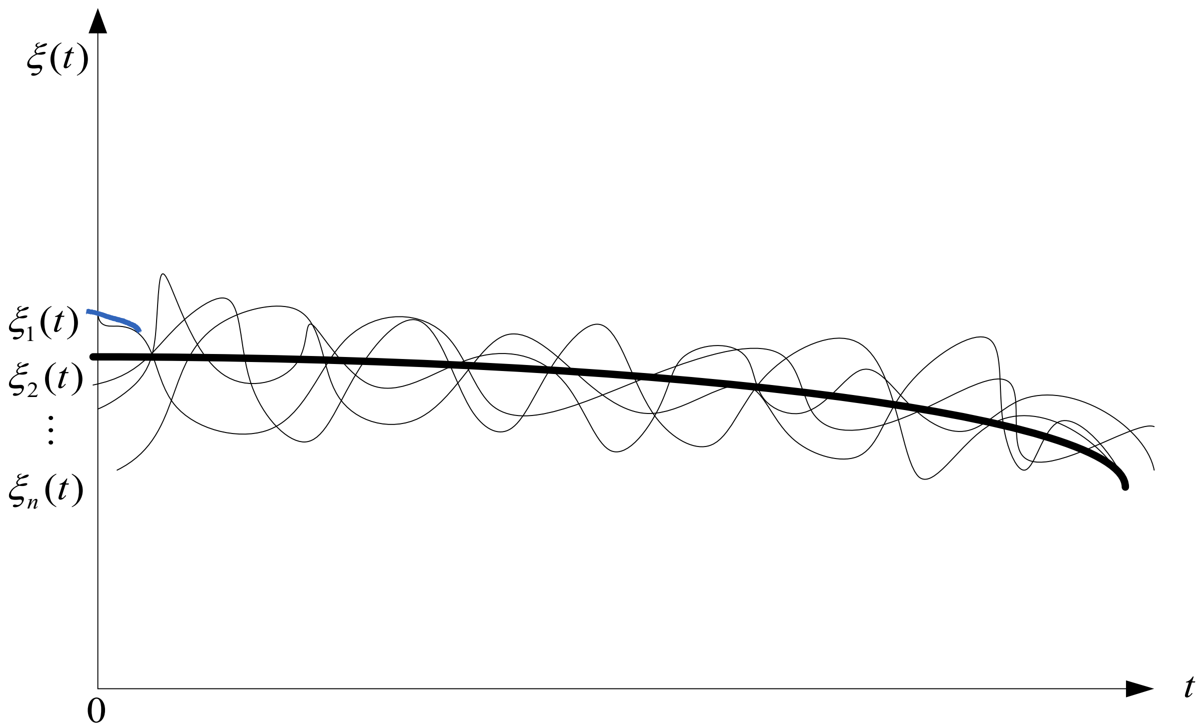
③ f, ϕ constant. $A \sim N(0, 1)$



$t=1$ $x(1) = A\sin(2\pi f)$

Example

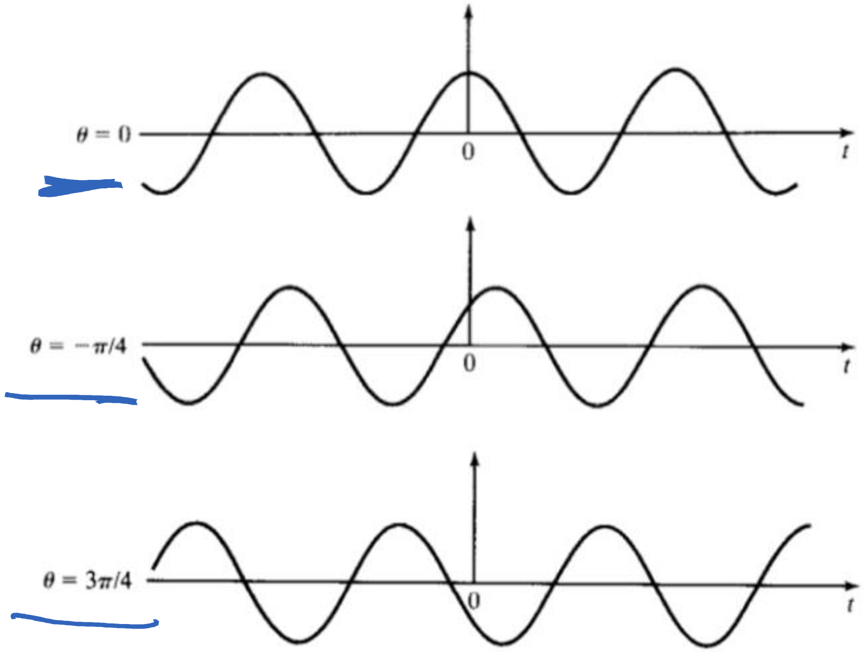
- Observation of noise
  - $\xi_i(t)$, one realization, deterministic
  - $\xi(t) = \{\xi_1(t), \xi_2(t), \ldots, \xi_n(t)\}$, random process, the set of all realizations.

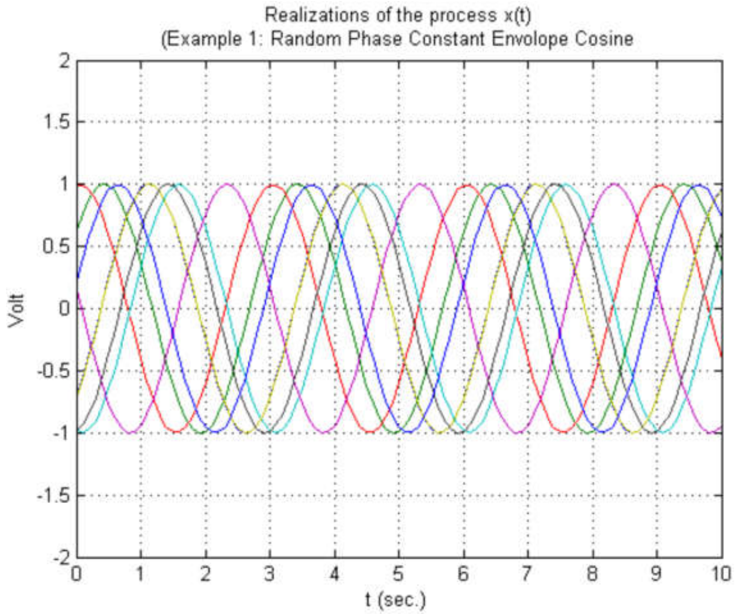


Example

- Uniformly choose a phase θ between $[0, 2\pi]$ and generate a sinusoid with a fixed amplitude and frequency but with a random phase θ.
- In this case, the random process is

$$X(t) = A\cos(2\pi f_0 t + \theta)$$





Statistics of Random Processes

- An infinite collection of random variables specified at time $t_1, t_2, \cdots, t_n, \forall n$

$$\{X(t_1), X(t_2), \ldots, X(t_n)\}$$

- Joint pdf (different notations)

$$f_{X(t_1),X(t_2),\ldots,X(t_n)}(x_1, x_2, \ldots, x_n), \forall n$$

$$f_{X_1,X_2,\cdots,X_n}(x_1, x_2, \cdots, x_n), \forall n$$

lectures

$$f_{X_1,X_2,\cdots,X_n}(x_1, t_1; x_2, t_2; \cdots; x_n, t_n), \forall n$$

textbook

First Order Statistics

- Probability density function of $X(t)$ at time t:

$$\boxed{f_{X(t)}(x)}$$

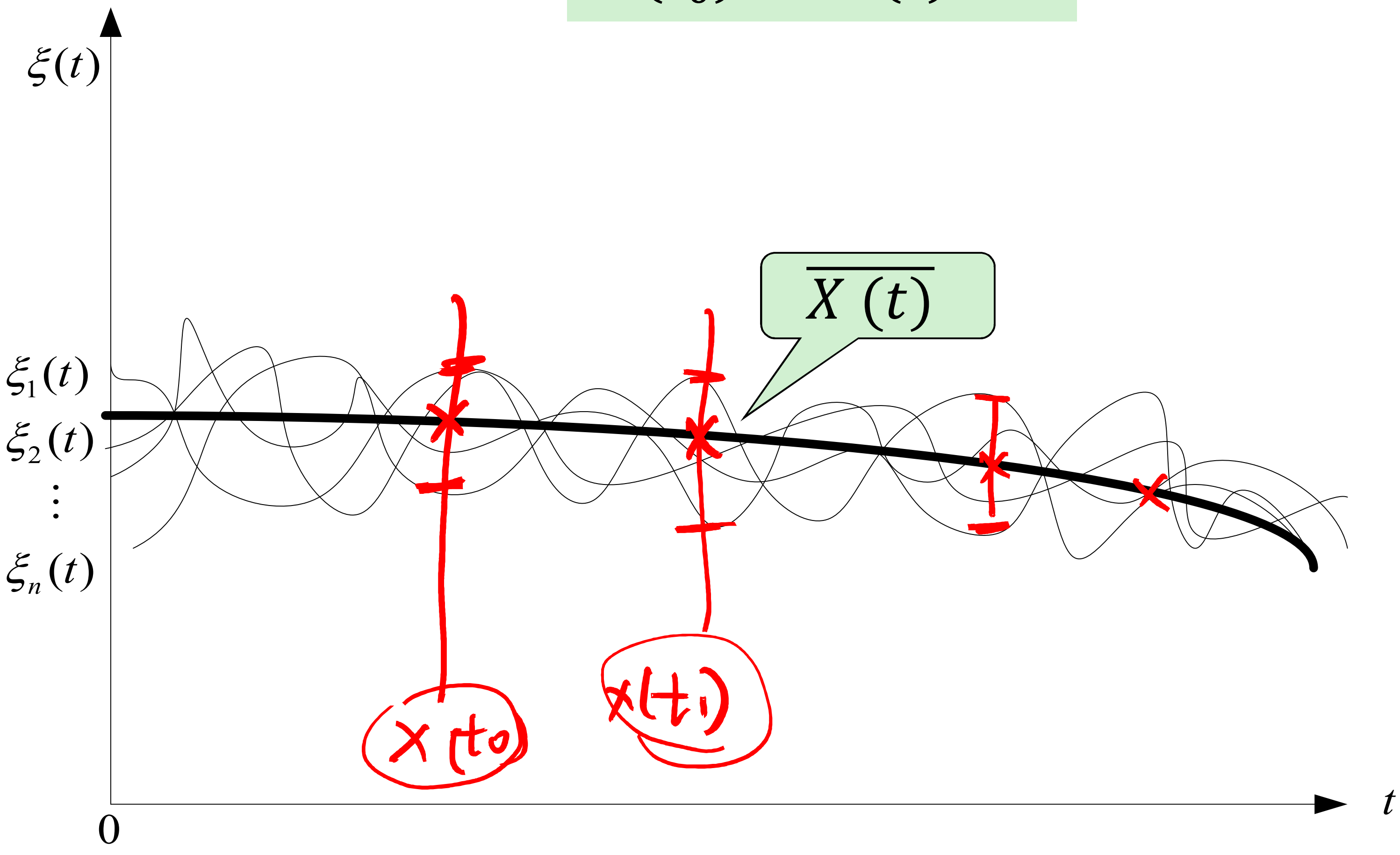
- Mean $E[X(t_0)] = E[X(t = t_0)] = \int_{-\infty}^{\infty} x f_{X(t_0)}(x)dx$

$$= \overline{X\ (t_0)}$$

- Variance $E\left[\left|X(t_0) - \overline{X\ (t_0)}\right|^2\right] = \sigma_X^2(t_0)$

Example

$$\overline{X\ (t_0)} \Rightarrow \overline{X\ (t)}$$



Second-Order Statistics

- Joint pdf of the random variables $X(t_1), X(t_2)$

$$f_{X(t_1),X(t_2)}(x_1, x_2) \triangleq f_{X_1,X_2}(x_1, x_2),$$
$$X_1 = X(t_1), X_2 = X(t_2).$$

- Autocorrelation function of the process $X(t)$ (correlation within a process):

$$R_X(t_1, t_2) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(x_1, x_2) dx_1 dx_2$$

- Autocovariance function

$$\mu_X(t_1, t_2) = E\{[X(t_1) - \overline{X(t_1)}][X(t_2) - \overline{X(t_2)}]\}$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [x_1 - \overline{X(t_1)}][x_2 - \overline{X(t_2)}] f_{X_1,X_2}(x_1, x_2) dx_1 dx_2$$

$$= R_X(t_1, t_2) - \overline{X(t_1)}\,\overline{X(t_2)}$$

$t_1 = t_2$, variance of X(t)

Example 1

- Consider $\boxed{X(t) = A\cos(2\pi f t + \theta),}$ where θ is uniform in $[-\pi, \pi]$
- Mean

$$E[X(t)] = \int_{-\pi}^{\pi} A\cos(2\pi f t + \theta)\frac{1}{2\pi}d\theta = 0$$

- Autocorrelation

$$\int_{-\pi}^{\pi} g(\theta)\cdot f_\theta(\theta)\,d\theta$$

Let $t_1 = t, t_2 = t + \tau$

$$E[X(t_1)X(t_2)] = E[A\cos(2\pi f t + \theta)A\cos(2\pi f(t + \tau) + \theta)]$$

$$= \frac{A^2}{2}E[\cos(4\pi f t + 2\pi f\tau + 2\theta) + \cos(2\pi f\tau)]$$

$$= \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(4\pi f t + 2\pi f\tau + 2\theta)d\theta + \frac{A^2}{2}\int_{-\pi}^{\pi}\frac{1}{2\pi}\cos(2\pi f\tau)d\theta$$

$$= 0 + \frac{A^2}{2}\cos(2\pi f\tau)$$

$$\Rightarrow R_X(t, t+\tau) = \frac{A^2}{2}\cos(2\pi f\tau)$$

Example 2

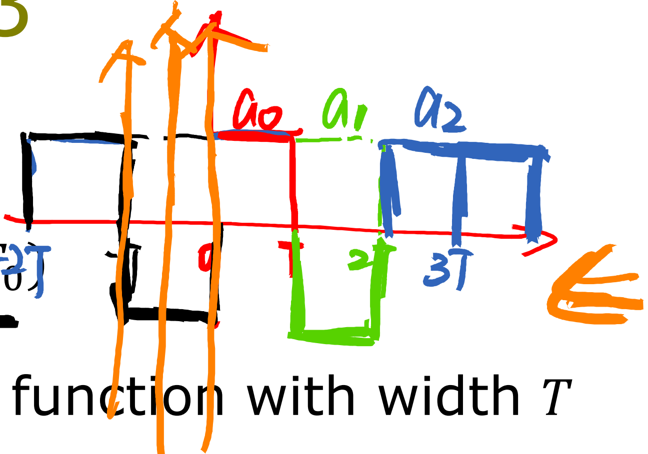
- Consider $Y(t) = B\cos\omega_c t$, where $B \sim N(0, b^2)$
- Find its mean and autocorrelation function

$E[Y(t)] = 0$ $E(B\cos w_c t) = E(B)\cos w_c t = 0$

$E[Y(t)Y(t+\tau)] = E[B^2]\cos\omega_c t\cos\omega_c(t+\tau)$
$= b^2 \cos\omega_c t\cos\omega_c(t+\tau)$

Example 3

- Given a binary random signal



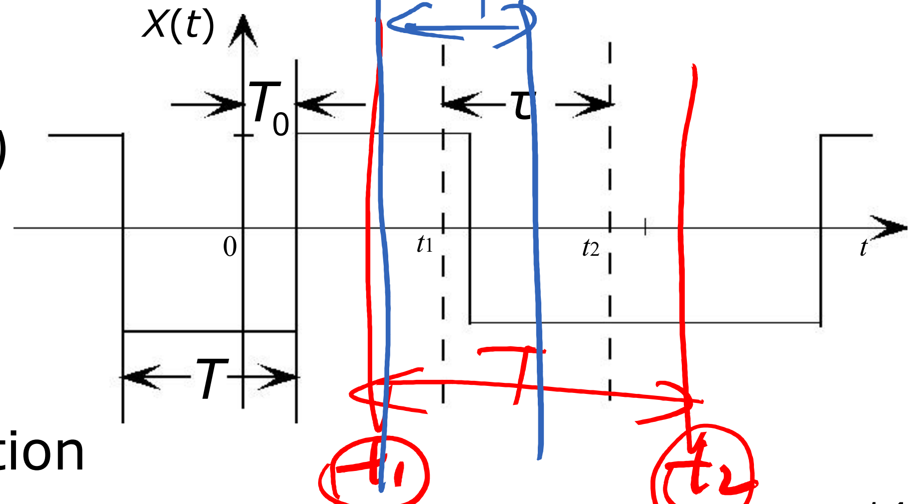
$$X(t) = \sum_n a_n p(t - nT - T_0)$$

  - $p(t)$ is a rectangular pulse shaping function with width T
  - a_n is a random variable that takes +1 or -1 with equal probability, and it is independent for different n
  - T_0 is a random time delay uniformly distributed within $[0, T]$
  - A typical sample function of X(t) is

$R_X(\tau) = 0$

$R_{X(t_1)X(t_2)}(x_1, x_2) = 0$

$E[X_1 X_2] = E(X_1)E(X_2)$



Find its autocorrelation function

Example 3

- Given a binary random signal

$$X(t) = \sum_n a_n p(t - nT - T_0)$$

$n=m \quad E[a_n^2]=1$

$n \neq m \quad E[a_n a_m]=0$

- Solution

$$R_X(t, t+\tau) = E[X(t)X(t+\tau)]$$

$$= E\left[\sum_n a_n p(t - nT - T_0) \sum_m a_m p(t + \tau - mT - T_0)\right]$$

Independence of a_n

$$= \sum_n E[a_n^2] E[p(t - nT - T_0)\, p(t + \tau - nT - T_0)]$$

$$= \sum_n \int_0^T \frac{1}{T} p(t - nT - T_0)\, p(t + \tau - nT - T_0) \mathrm{dT_0}$$

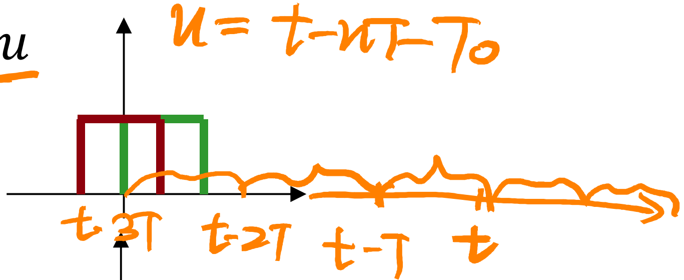
Let $t - nT - T_0 = u$

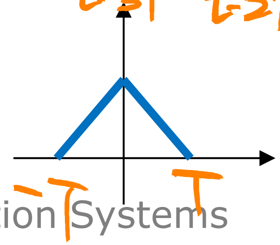
$u = t - nT - T_0$

$$= \sum_n \int_{t-nT-T}^{t-nT} \frac{1}{T} p(u)\, p(u+\tau) du$$

$$= \int_{-\infty}^{\infty} \frac{1}{T} p(u) p(u+\tau) du$$

$$= \begin{cases} \frac{T-|\tau|}{T}, |\tau| < T \\ 0, |\tau| \geq T \end{cases}$$





Stationary Processes

- A stochastic process is said to be stationary if for any *n* and *τ*:

$$f_{X(t_1),X(t_2),\cdots,X(t_n)}(x_1, x_2, ..., x_n) = f_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_n+\tau)}(x_1, x_2, ..., x_n), \ \forall n, \tau$$

- First-order statistics is independent of t

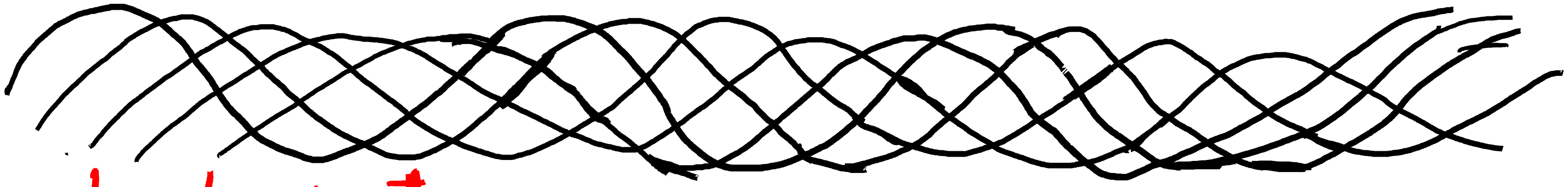
$$E\{X(\mathrm{t})\} = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx = m_X, E\left\{X(\mathrm{t}) - \overline{X(t)}\right\}^2 = \sigma_X^2.$$

- Second-order statistics only depends on the gap

$\tau = t_2 - t_1$

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)\, dx_1 dx_2$$
$$= R_X(t_2 - t_1) = R_X(\tau), \text{where } \tau = t_2 - t_1$$

$X(t)$



$$f_{X(t_1)X(t_2)\cdots X(t_n)} = f_{X(t_1+\tau)X(t_2+\tau)\cdots X(t_n+\tau)} \quad \forall n, \forall \tau$$

$n=1$, $$f_{X(t)} = f_{X(t+\tau)} \qquad E[X(t)] = C_1$$

$$Var[X(t)] = C_2$$

$n=2$ $$f_{X(t_1)X(t_2)} = f_{X(t_1+\tau)X(t_2+\tau)}$$

$$R_{X(t_1)X(t_2)}(t_1, t_2) = R(\tau) \qquad \tau = t_2 - t_1$$

Wide-Sense Stationary (WSS)

- A random process is said to be WSS when

$$E\{X(\mathrm{t})\} = \int_{-\infty}^{\infty} x f_{X(t)}(x)dx = m_X,\ E\{X(\mathrm{t}) - \overline{X(t)}\}^2 = \sigma_X^2.$$

$$R_X(t_1, t_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(t_1),X(t_2)}(x_1, x_2)\, dx_1 dx_2$$
$$= R_X(t_2 - t_1) = R_X(\tau)$$

– Defined with the first order and second order statistics only

- Compare the strictly stationary

$$f_{X(t_1),X(t_2),\cdots,X(t_n)}(x_1, x_2, \ldots, x_n)$$
$$= f_{X(t_1+\tau),X(t_2+\tau),\cdots,X(t_n+\tau)}(x_1, x_2, \ldots, x_n),\quad \forall n, \tau$$

Examples

- Example 1: Determine if $X(t)$ is WSS

$$X(t) = A\cos(2\pi f t + \theta),\ \text{where } \theta \sim U(-\pi, \pi)$$

  - Check the first order and second order statistics

$$E[X(\mathrm{t})] = 0$$

$$R_X(t, t+\tau) = \frac{A^2}{2}\cos(2\pi f \tau)$$

$\Rightarrow$ $X(t)$ is WSS

- Example 2: Determine if $Y(t)$ is WSS

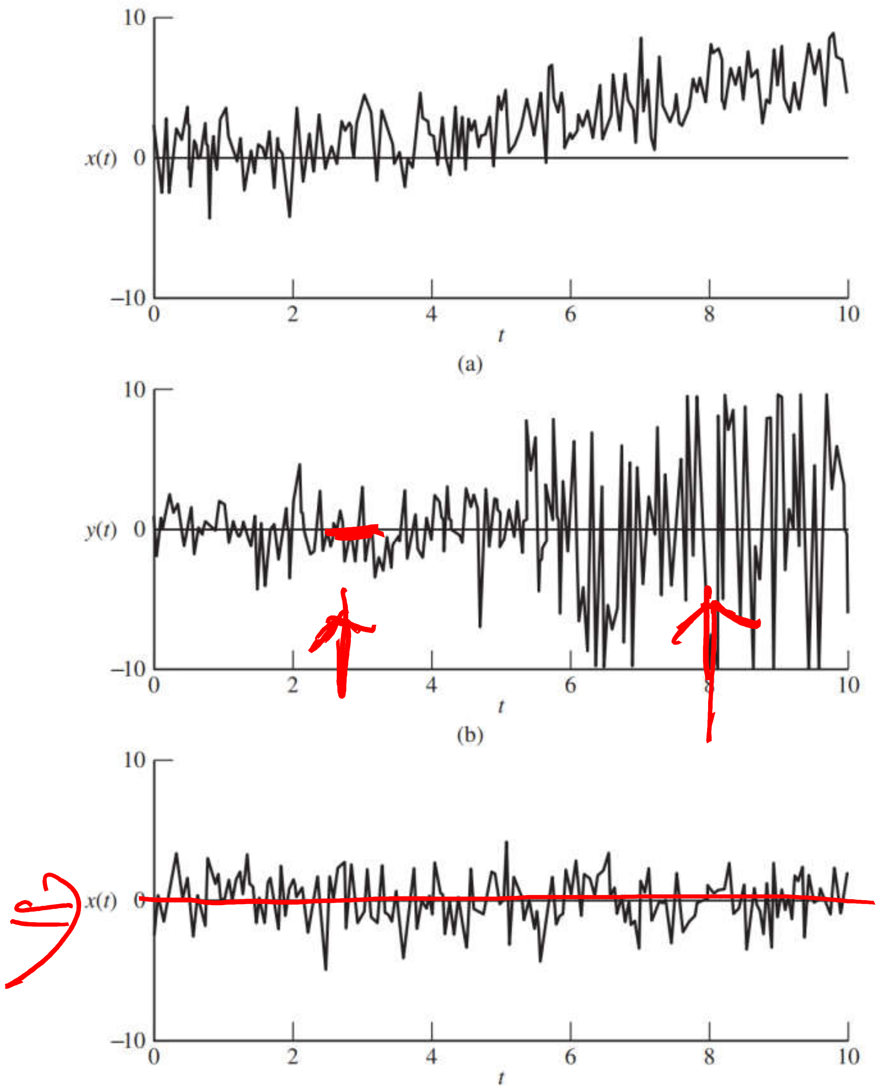
$$Y(t) = B\cos\omega_c t\text{, where } B \sim N(0, b^2)$$

$$E[Y(t)] = 0$$

$$R_Y(t, t+\tau) = E[Y(t)Y(t+\tau)] = E[B^2]\cos\omega_c t\cos\omega_c(t+\tau)$$

$$= b^2\cos\omega_c t\cos\omega_c(t+\tau)$$

Examples



Time-varying mean

Time-varying variance

Stationary

Averages and Ergodic

$X(t, \xi)$

- Ensemble (or statistical) averaging

$$\overline{X(t)} \triangleq E[X(t)] = \int_{-\infty}^{\infty} x f_{X(t)}(x) dx,$$

$(t) = m_x$

$$R_X(t, t+\tau) = E[X(t_1)X(t_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X_1,X_2}(t, t+\tau) dx_1 dx_2$$

$(t, \tau) \Rightarrow (\tau)$

Function of time

- Time averaging

$$< X(t) > \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)dt$$

$(\xi) = m_x$

$$< X(t)X(t+\tau) > = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

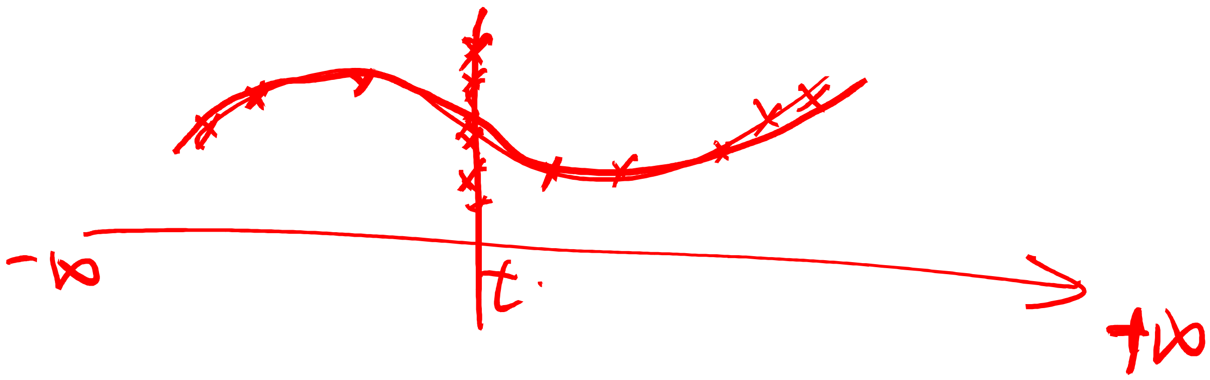
$(\tau, \xi) \Rightarrow (\tau)$

Random variable

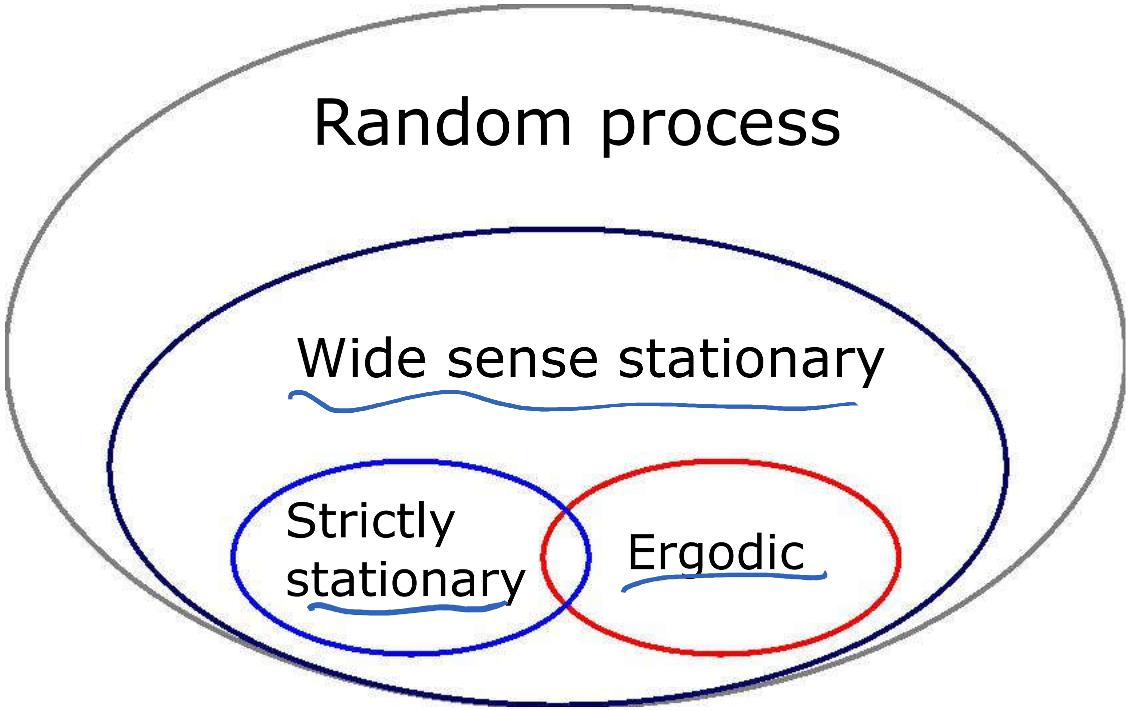
- If ensemble average = time average, $X(t)$ is said to be Ergodic (各态历经)

$\overline{X(t)} = < X(t) >$, $R_X(t, t+\tau) = < X(t) X(t+\tau) >$

Ergodic $\Rightarrow$ stationary


$-\infty$
t
$+\infty$

Applications of Random Process



- Applications
  - Signal: WSS
  - Noise: (strictly) stationary
  - Time-varying channel: ergodic

Examples

- Example 1: Determine if $X(t)$ is Ergodic

$$X(t) = A\cos(2\pi f t + \theta), \text{ where } \theta \sim U(-\pi, \pi)$$

Statistical average

$$E[X(\mathrm{t})] = 0$$

$$R_X(\tau) = \frac{A^2}{2}\cos(2\pi f \tau)$$

Time average

$$< X(\mathrm{t}) >= \frac{1}{T}\int_0^T A\cos(2\pi f t + \theta)\,dt = 0$$

$$< X(t)X(t+\tau) >$$

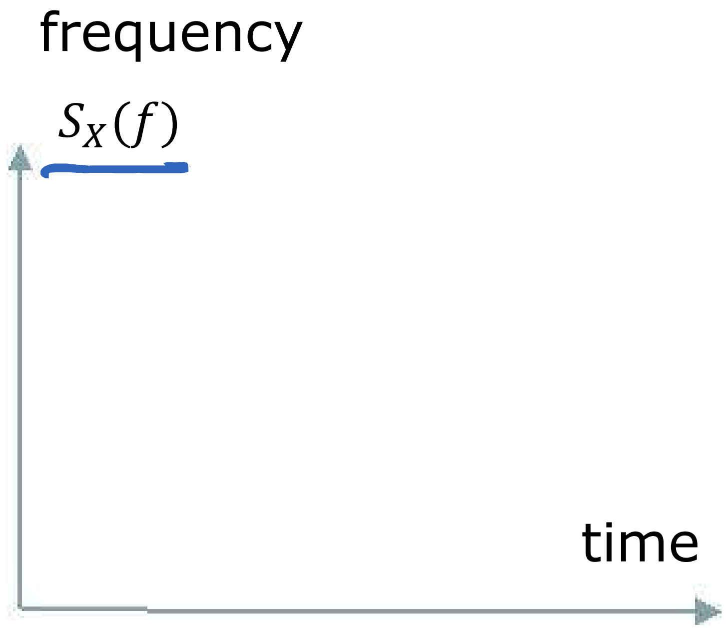
$$= \frac{1}{T}\int_0^T A^2\cos(2\pi f t + \theta)\cos(2\pi f(t+\tau) + \theta)\,dt$$

$$= \frac{A^2}{2}\cos(2\pi f \tau)$$

$\Rightarrow$ $X(t)$ is Ergodic

Frequency Domain Characteristics of Random Process

- Power Spectral Density



$$m(t) = E[X(\mathrm{t})]$$

$$R_X(t,\tau) = E[X(t)X(t+\tau)]$$

PSD of Random Process

- PSD of deterministic signal

$$S_X(f) = \lim_{T\to\infty} \frac{1}{T} |X_T(f)|^2$$

- Consider x(t) as a sample function of a random process X(t). The PSD of X(t) is given by

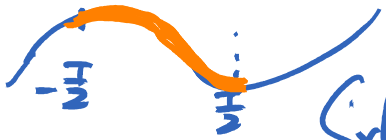
$$S_X(f) = \lim_{T\to\infty} \frac{E\{|X_T(f)|^2\}}{T} \geq 0$$

$$x(t,\xi) \to x(t,\xi_i) \to \underline{x_T(t,\xi_i)} \to \underline{X_T(f,\xi_i)}$$

$$\downarrow$$

$$\underline{X_T(f)}$$

$$\downarrow$$



$$S_x(f) = \lim_{T\to\infty} \frac{E\left[|X_T(f)|^2\right]}{T}$$

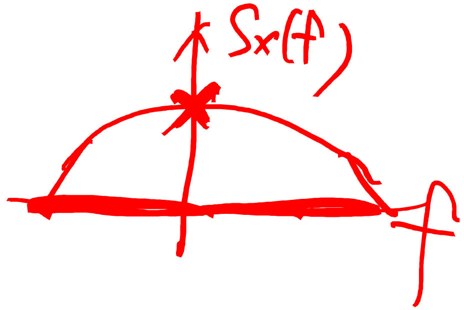
PSD of WSS Process

- Wiener-Khinchine theorem (Page 318)
- For WSS process

$\tau=0$

$$S_X(f) \leftrightarrow R_X(\tau) \begin{cases} R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\exp(j2\pi f\tau)df \\ S_X(f) = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau \end{cases}$$

$f=0$



- Property:

Autocorrelation function $R_X(\tau)$	**PSD** $S_X(f)$
Total power $R_X(0) = E[X(t)^2] = \int_{-\infty}^{\infty} S_X(f)df$	$S_X(0) = \int_{-\infty}^{\infty} R_X(\tau)d\tau$
$\lvert R(\tau)\rvert \le R(0)$	$S_X(f) \ge 0, \forall f$
$R(-\tau) = R(\tau)$	$S_X(f) = S_X(-f)$
$\lim_{\lvert\tau\rvert\to\infty} R(\tau) = \overline{X(t)}^2$ if X(t) does not contain a periodic component	

$R(\tau) = E[X(t)X(t+\tau)]$

$\tau \to \infty$ $= E[X(t)]\,E[X(t+\tau)]$

$= E[$

Prove: $|R_X(\tau)| \le R_X(0)$

$E|X(t) \pm X(t+\tau)|^2 \geq 0$, for stationary $X(t)$

$\Rightarrow \overline{X^2(t)} + \overline{X^2(t+\tau)} \pm 2R_X(\tau) \geq 0$

$\Rightarrow R_X(0) + R_X(0) \pm 2R_X(\tau) \geq 0$

$\Rightarrow R_X(0) \pm R_X(\tau) \geq 0$

$\Rightarrow -R_X(0) \le R_X(\tau) \le R_X(0)$

$\Rightarrow |R_X(\tau)| \le R_X(0) \quad \forall \tau$

Prove $R_X(\tau) = R_X(-\tau)$

$R_X(\tau) = \overline{X(t)X(t+\tau)} = \overline{X(t'-\tau)X(t')} = R_X(-\tau)$

PSD of Ergodic Random Process

- By definition:
  - the time average of the auto-correlation function of the sample function equals the auto-correlation function of the random process, or

$$< X(t)X(t+\tau) > = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$\Updownarrow$$

$$R_X(\tau) = \overline{X(t)X(t+\tau)}$$

- We have

$$< X(t)X(t+\tau) > \Leftrightarrow S_X(f)$$

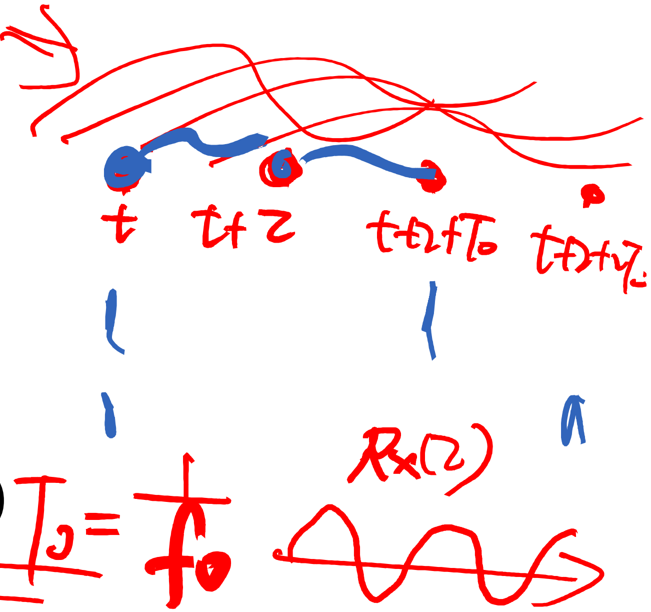
Example 1

- For the random process

$$X(t) = A\cos(2\pi f_0 t + \theta)$$

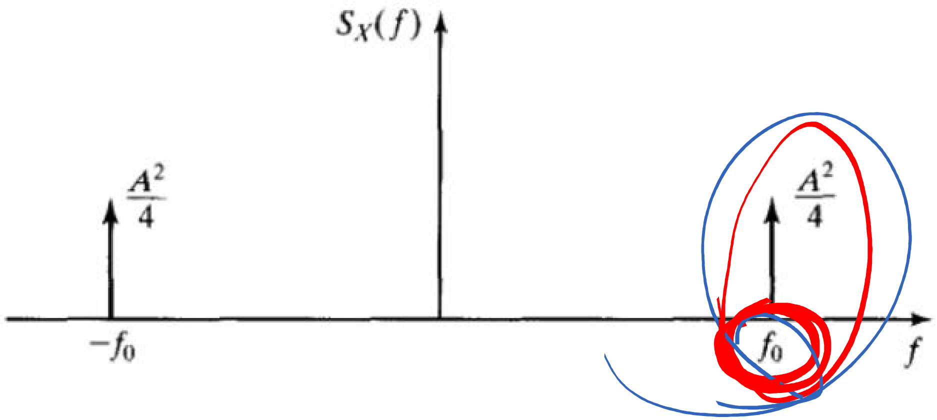
- Autocorrelation

$$\Rightarrow R_X(\tau) = \frac{A^2}{2}\cos(2\pi f_0 \tau)$$



- PSD

$$S_X(f) = \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$$

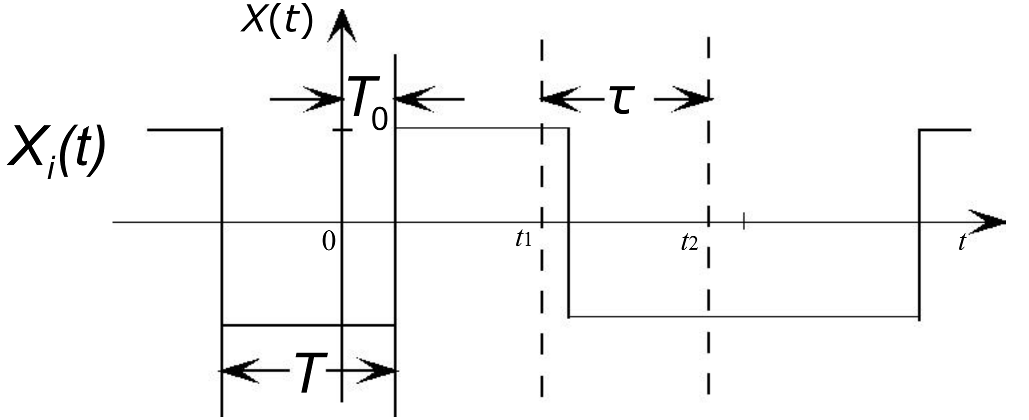


Example 2

- Given a binary random signal

$$X(t) = \sum_n a_n p(t - nT - T_0)$$

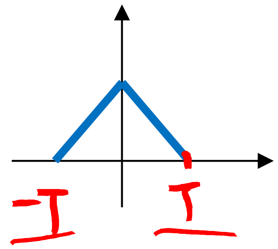
  - $p(t)$ is a rectangular pulse shaping function with width T
  - a_n is a random variable that takes +1 or -1 with equal probability, and it is independent for different n
  - T_0 is a random time delay uniformly distributed within $[0, T]$
  - A typical sample function of X(t) is

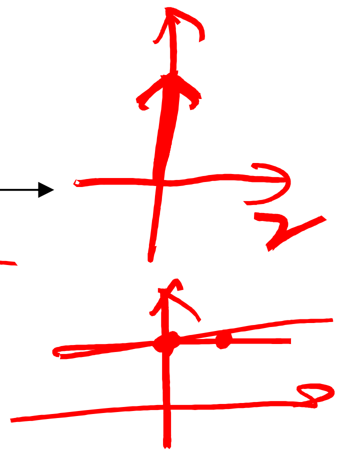


Example (cont'd)

- Autocorrelation function

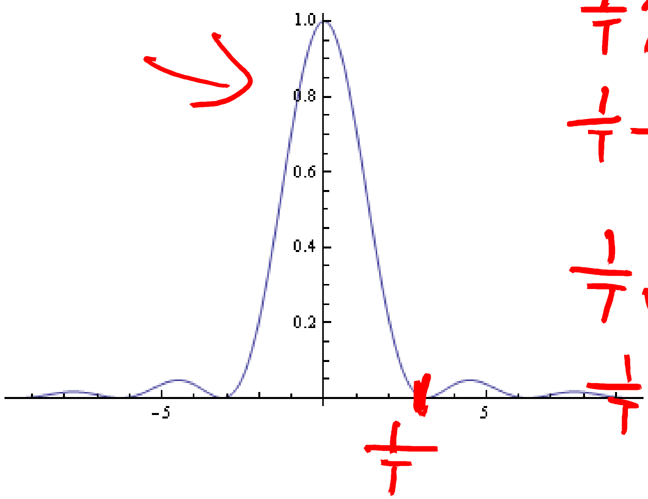
$$R_X(\tau) = \begin{cases} 1 - \frac{|\tau|}{T}, -T < \tau < T \\ 0, \text{ otherwise} \end{cases}$$





- PSD

$$S_X(f) = T sinc^2(fT)$$



$\frac{1}{T}\uparrow$ correlation $\downarrow$

$\frac{1}{T}\to\infty$ $S_X(f)=C$ $R_X(\tau)\to\delta(\tau)$

correlation $=0$

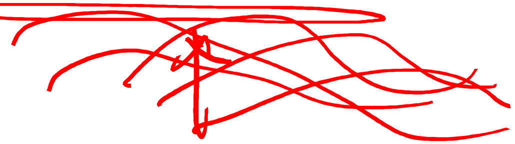
$\frac{1}{T}\downarrow$ correlation $\uparrow$

$\frac{1}{T}\to 0$ $R_X(\tau)\to C$

Cross Correlation

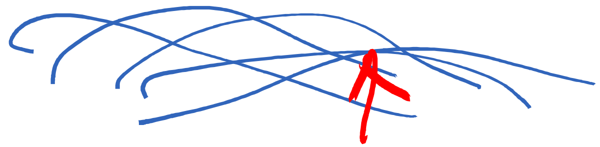
- X(t), Y(t): each WSS, jointly WSS.
- $n(t) = X(t) + Y(t)$, calculate the power of n(t)

$$E[n^2(t)] = E\{[X(t) + Y(t)]^2\} = P_X + 2\underbrace{E[X(t)Y(t)]}_{R_{XY}(0)} + P_Y$$



- Cross-correlation function

$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$

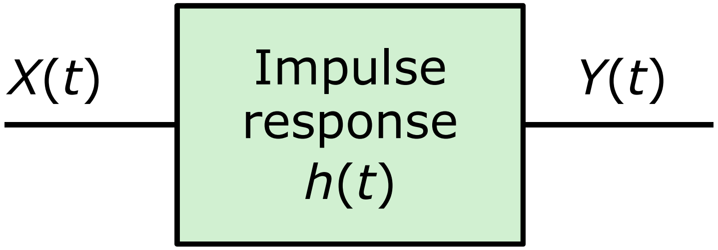


  - $R_{XY}(\tau) = 0, \forall\tau \Rightarrow$ X(t) and Y(t) are orthogonal
  - Property: $R_{XY}(\tau) = R_{YX}(-\tau)$

$$E[X(t)Y(t+2)] = E[X(t'-2)Y(t')]$$
$$= E[Y(t')X(t'-2)]$$
$$= R_{YX}(-2)$$

- Cross PSD: $S_{XY}(f) = \mathfrak{F}[R_{XY}(\tau)]$

- Consider a linear system (channel)



$$Y(t) = X(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)\, X(t-\tau)\, d\tau$$

- Mean

$E[X] = \overline{X}$ $\langle X \rangle$

$$\overline{Y}(t) = E[Y(t)] = \int_{-\infty}^{\infty} h(\tau)\, E[X(t-\tau)]\, d\tau$$

$$= \int_{-\infty}^{\infty} h(\tau)\, \overline{X}(t-\tau)\, d\tau$$

If $X(t)$ is WSS

$$= \overline{X} \int_{-\infty}^{\infty} h(\tau)\, d\tau = \boxed{\overline{X} \cdot H(0)}$$

Random Process Transmission Through Linear Systems

- Autocorrelation of Y(t)

$t-u=\tau$

$$R_Y(t,u) = E[Y(t)Y(u)]$$
$$= E\left[\int_{-\infty}^{\infty} h(\tau_1)\,X(t-\tau_1)d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)\,X(u-\tau_2)d\tau_2\right]$$
$$= \int_{-\infty}^{\infty} h(\tau_1)\,d\tau_1 \int_{-\infty}^{\infty} h(\tau_2)E[X(t-\tau_1)X(u-\tau_2)]d\tau_2$$

If $X(t)$ is WSS $\Downarrow$

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\boxed{R_X(\tau-\tau_1+\tau_2)}d\tau_1 d\tau_2$$

If input is a WSS process, the output is also a WSS process!

Relation Among the Input-Output PSD

- Autocorrelation of Y(t)

$$R_Y(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(\tau_1)h(\tau_2)\, R_X(\tau - \tau_1 + \tau_2)d\tau_1 d\tau_2$$

$$= \int h(\tau_2)[h(\tau) * R_X(\tau + \tau_2)]d\tau_2$$

$$= h(-\tau) * h(\tau) * \boxed{R_X(\tau)}$$

$$x(-\tau) * h(\tau) = \int x(t_1)h(\tau + t_1)dt_1$$

- PSD of Y(t):

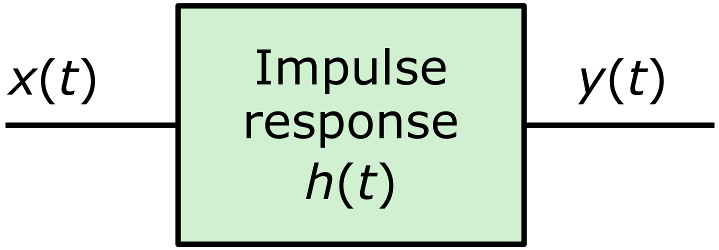
$F[R_Y(\tau)] = H^*(f)\,H(f)\,S_X(f)$

$F[h(-\tau)] = H^*(f)$

$$\boxed{S_Y(f) = |H(f)|^2 S_X(f)}$$

Deterministic vs. Random

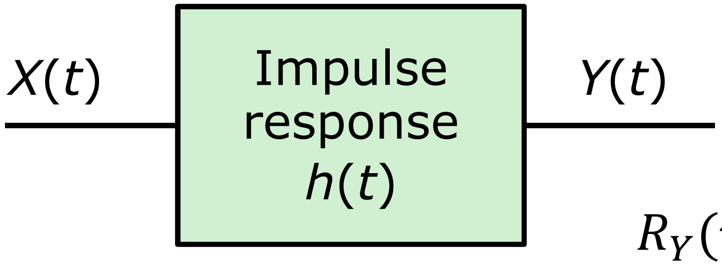
Deterministic signal



$$y(t) = x(t) * h(t)$$

$$Y(f) = X(f)H(f)$$

Radom signal (WSS)



$$R_Y(\tau) = h(-\tau) * h(\tau) * R_X(\tau)$$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

Example

- LTI = a differentiator $H(f) = j2\pi f$
- Input random signal $X(t) = A\cos(2\pi f_0 t + \theta)$
- Output PSD

$$
\begin{aligned}
&S_Y(f) \\
&= 4\pi^2 f^2 \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)] \\
&= A^2\pi^2 {f_0}^2[\delta(f - f_0) + \delta(f + f_0)]
\end{aligned}
$$