



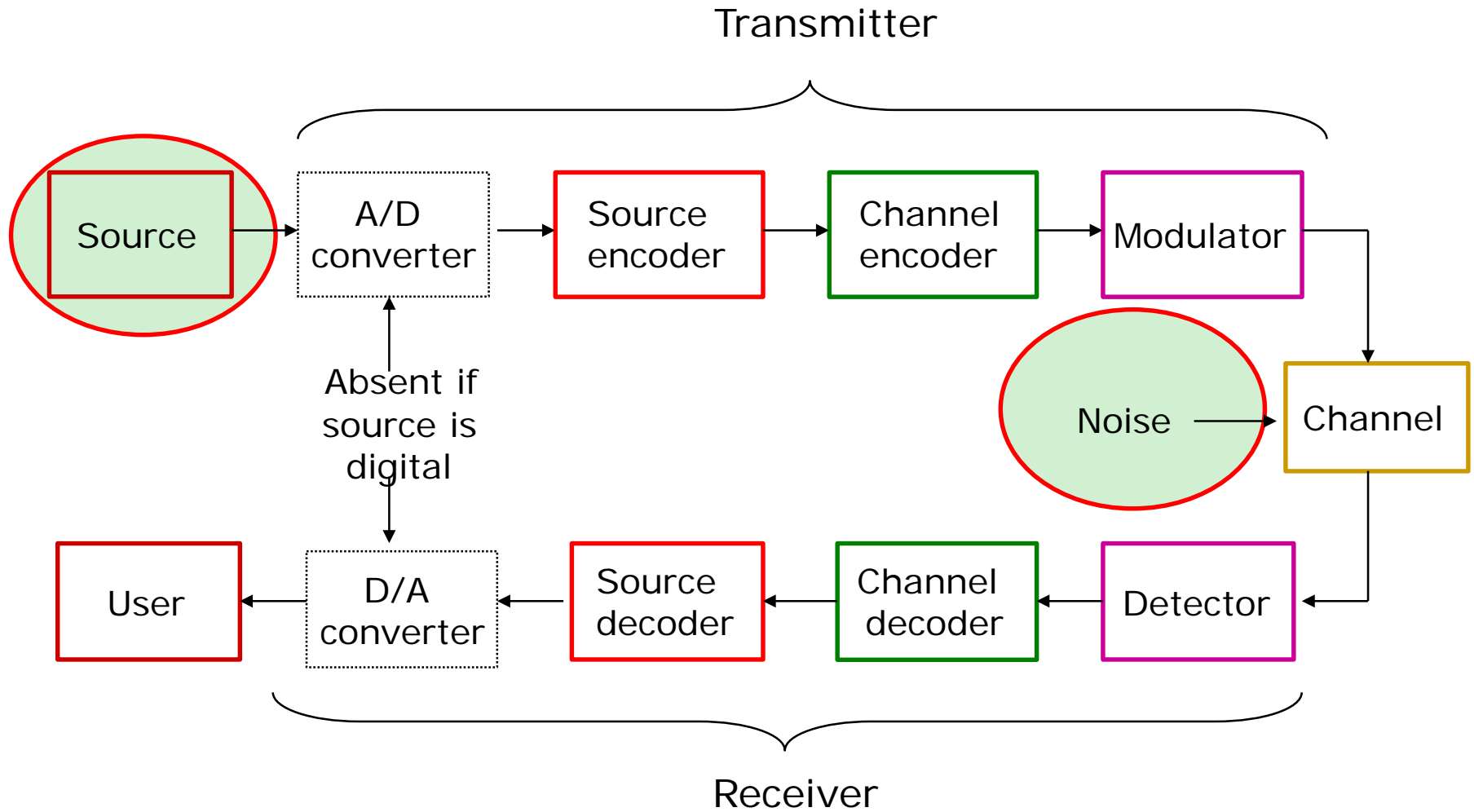
上海科技大学
ShanghaiTech University

EE140 Introduction to Communication Systems

Lecture 3

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ShanghaiTech University, Fall 2022

Architecture of a (Digital) Communication System

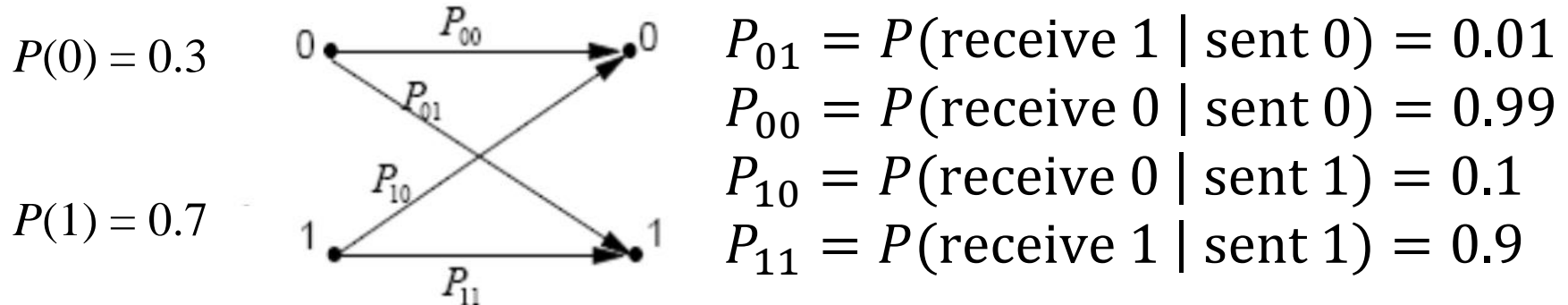


Contents

- Random signals
 - Review of probability and random variables
 - Random processes: basic concepts
 - Gaussian white processes

Probability

- Consider a binary communication system



Maximum posterior probability estimates

$$P(\text{send 0} \mid \text{receive 1}) = ?$$

$$P(\text{send 1} \mid \text{receive 1}) = ?$$

$$P(\text{send 1} \mid \text{receive 0}) = ?$$

$$P(\text{send 0} \mid \text{receive 0}) = ?$$

Conditional Probability

- Consider two events A and B
- Conditional probability $P(A|B)$
- Joint probability

$$\begin{aligned}P(AB) &= P(A \cap B) \\ &= P(B)P(A|B) = P(A)P(B|A)\end{aligned}$$

- A and B are said statistically **independent** iff

$$P(AB) = P(A)P(B) \quad \longrightarrow \quad \begin{aligned}P(A|B) &= P(A) \\ P(B|A) &= P(B)\end{aligned}$$

Law of Total Probability

- Let A_j , $j = 1, 2, \dots, n$ be mutually exclusive events with $A_i \cap A_j = \emptyset, \forall i \neq j$ and $\bigcup_{i=1}^n A_i = S$
- For any event B , we have

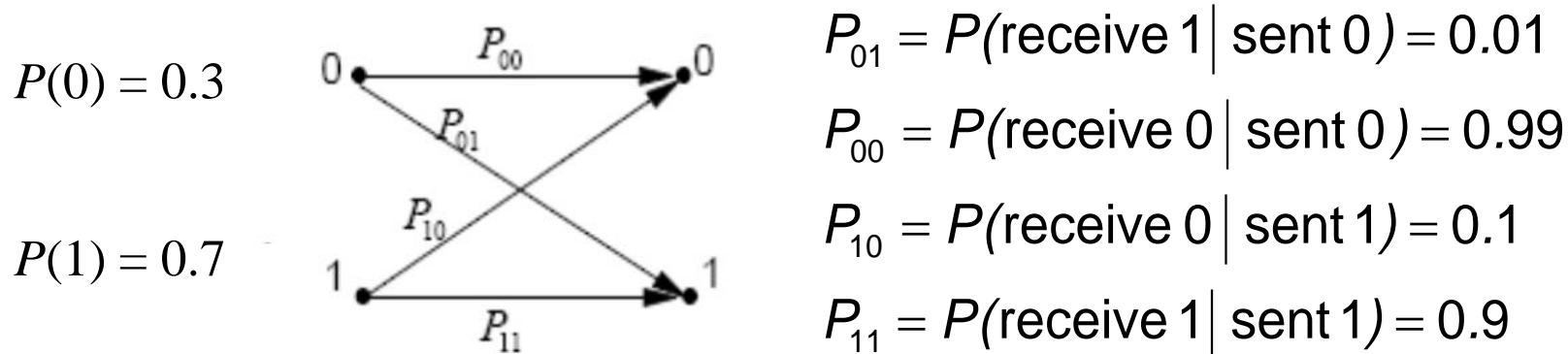
$$P_B = \sum_{j=1}^n P(B \cap A_j) = \sum_{j=1}^n P(B/A_j)P(A_j) \quad \text{Venn diagram}$$

- Bayes' Theorem

$$\begin{aligned} P(A_i|B) &= \frac{P(A_i, B)}{P(B)} \\ &= \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^n P(B|A_j)P(A_j)} \end{aligned}$$

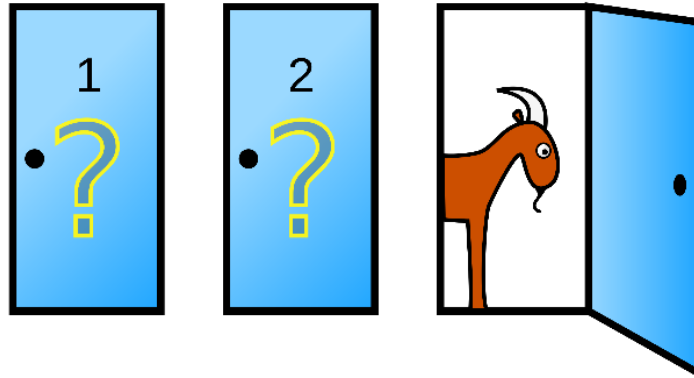
Example

- Consider a binary communication system



- What is the probability that the output of this channel is 1?
- Assuming that we have observed a 1 at the output, what is the probability that the input to the channel was a 1?

Game



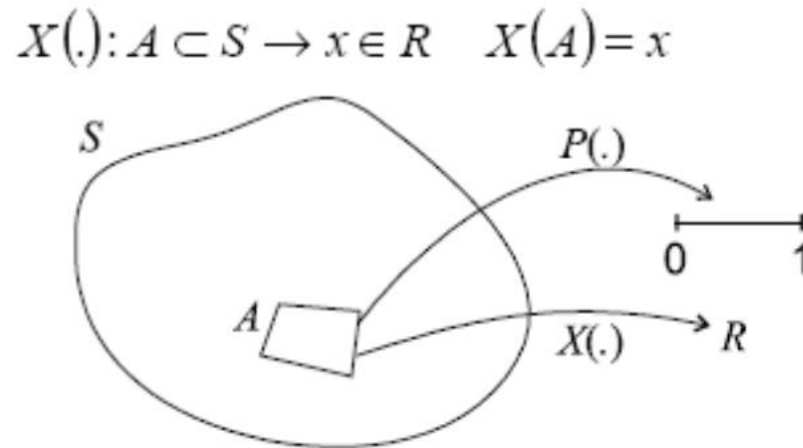
In search of a new car, the player picks a door, say 1. The game host then opens one of the other doors, say 3, to reveal a goat and offers to let the player switch from door 1 to door 2.

Random Variables (r.v.)

- A r.v. is a mapping from the sample space S to the set of real numbers.

$$X(.) : A \subset S \rightarrow x \in R$$

$$X(A) = x$$



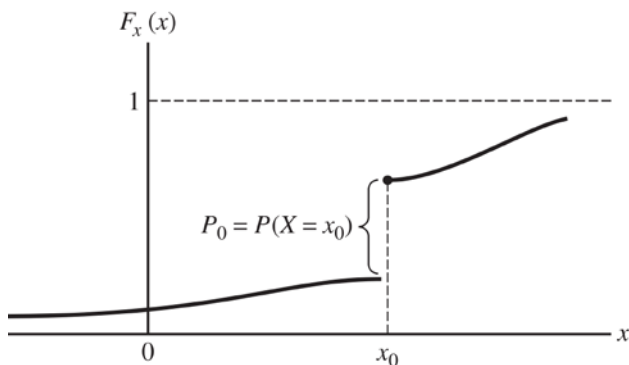
- A r.v. may be
 - Discrete-valued: range is finite (e.g. $\{0,1\}$), or countable infinite (e.g. $\{1,2,3 \dots\}$)
 - Continuous-valued: range is uncountable infinite (e.g. R)

Cumulative Distribution Function (CDF)

- The CDF of a r.v. X , is

$$F_X(x) \triangleq P(X \leq x)$$

- Key properties of CDF
 - $0 \leq F_X(x) \leq 1$ with $F_X(-\infty) = 0$, $F_X(\infty) = 1$
 - $F_X(x)$ is a non-decreasing function of x
 - $F_X(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$



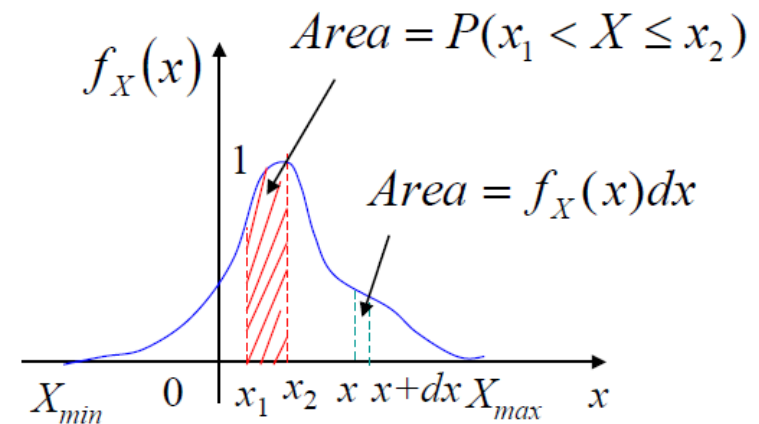
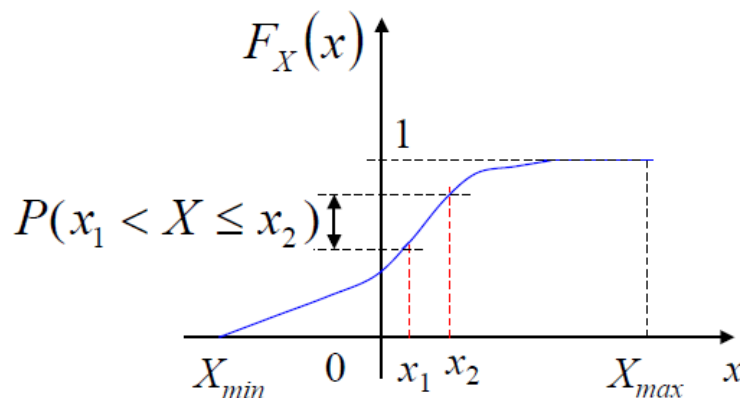
Probability Density Function (PDF)

- The PDF, of a r.v. X , is defined as

$$f_X(x) \triangleq \frac{d}{dx} F_X(x) \quad \text{or} \quad F_X(x) = \int_{-\infty}^x f_X(y) dy$$

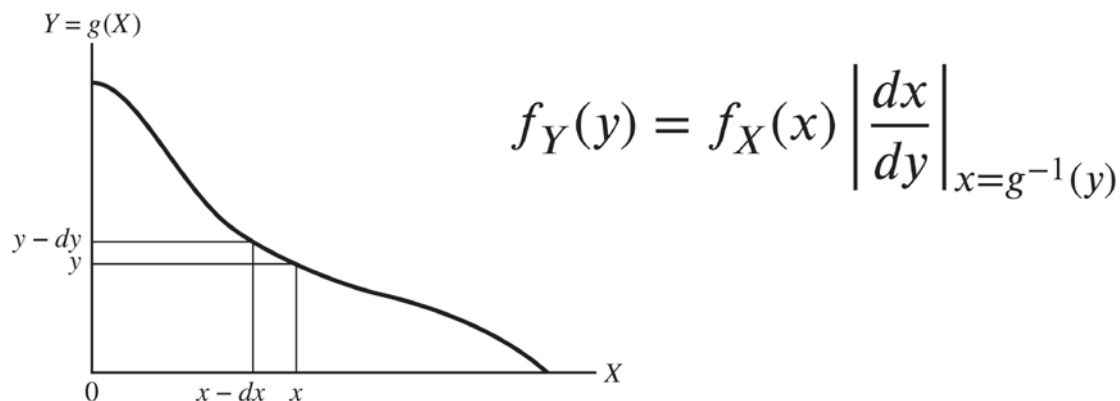
- Key properties of PDF

1. $f_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} f_X(x) dx = 1$
3. $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x) dx$

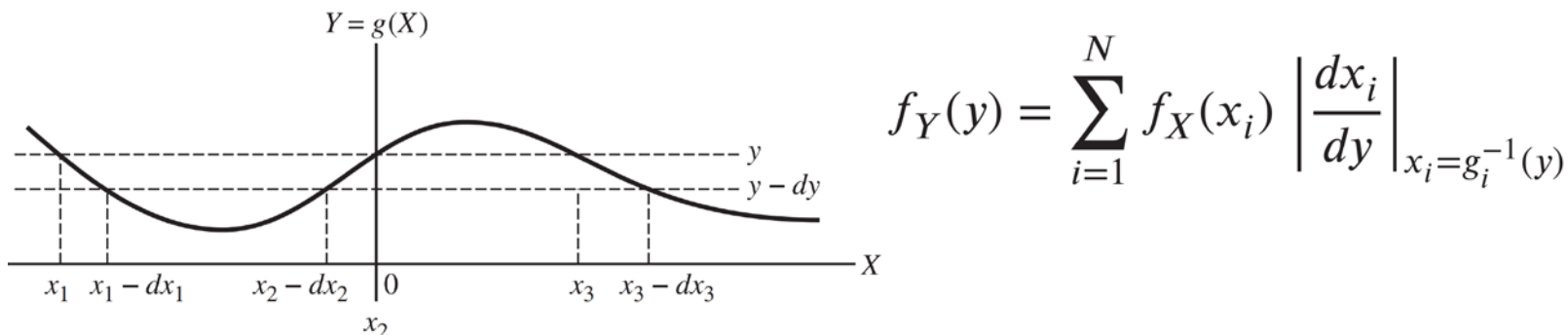


Transformation of R.V.

- The PDF of a function of X , $Y = g(X)$
- Monotonic transformation:



- Nonmonotonic transformation:



Exercise

- Suppose that X is a Gaussian random variable with zero mean and unit variance. Let

$$Y = aX^3 + b, \quad a > 0$$

- Determine the PDF of Y

$$p_Y(y) = \frac{1}{3a\sqrt{2\pi} [(y-b)/a]^{2/3}} e^{-\frac{1}{2}(\frac{y-b}{a})^{2/3}}$$

Statistical Averages

- Consider a discrete r.v. which takes on the possible values x_1, x_2, \dots, x_M with respective probabilities

$$P_1, P_2, \dots, P_M$$

- The mean or expected value of X is

$$m_x = \bar{X} = E[X] = \sum_{i=1}^M x_i P_i$$

- If X is continuous, then

$$m_x = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Average of a function of a r.v. $Y = g(X)$.

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y) dy \quad \overline{g(X)} \triangleq E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x) dx$$

- This is the first moment of X .

Exercise

- The PDF of a Cauchy distributed random variable X is

$$p_x(x) = \frac{a/\pi}{x^2 + a^2}, \quad -\infty < x < \infty$$

- Determine the mean and variance of X

$$E(X) = \int_{-\infty}^{\infty} xp_x(x)dx = 0$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 p_x(x)dx = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{x^2}{x^2 + a^2} dx = \infty$$

Moment

- The n^{th} moment of X

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x) dx$$

- When $n = 2$, we have the mean-square value of X

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

- n^{th} central moment

$$E[(X - E[X])^n] = \int_{-\infty}^{\infty} (x - E[X])^n f_x(x) dx$$

- At $n=2$, we have the variance

$$\sigma_x^2 = E[(X - E[X])^2] = E[X^2 - 2E[X]X + E[X]^2] = E[X^2] - E[X]^2$$

- σ_x is called the standard deviation, the average distance from the mean, a measure of the concentration of X around the mean.

Some Useful Distribution: Bernoulli Distribution

- A discrete r.v. taking two possible values, $X = 1$ or $X = 0$, with probability mass function (pmf)

$$\begin{aligned} p(x) &= P(X = x) \\ &= \begin{cases} 1 - p, & x = 0 \\ p, & x = 1 \end{cases} \end{aligned}$$

- Often used to model binary data

Some Useful Distribution: Binomial Distribution

- A discrete r.v. taking the sum of n-independent Bernoulli r.v. , i.e.

$$Y = \sum_{i=1}^n X_i \quad \text{where} \quad p_X(x) = \begin{cases} 1-p, & x = 0 \\ p, & x = 1 \end{cases}$$

- The PMF is given by

$$p_Y(k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \quad \binom{n}{k} = \frac{n!}{k! (n-k)!}$$

- That is, the probability that $Y = k$ is the probability that k of the X_i are equal to 1 and $n-k$ are equal to 0.
- Expectation and variance:

$$\overline{K} = E[K] = np(p+q)^\ell = np \quad \sigma_K^2 = E[K^2] - E^2[K] = npq$$

Example

- Suppose that we transmit a 31-bit long sequence with error correction capability up to 3 bit errors
- If the probability of a bit error is $p = 0.001$, what is the probability that this sequence is received in errors?

$$P(\text{sequence error}) = 1 - P(\text{correct sequence})$$

$$= 1 - \sum_{i=0}^3 \binom{31}{i} (0.001)^i (0.999)^{31-i} \approx 3.10^{-8}$$

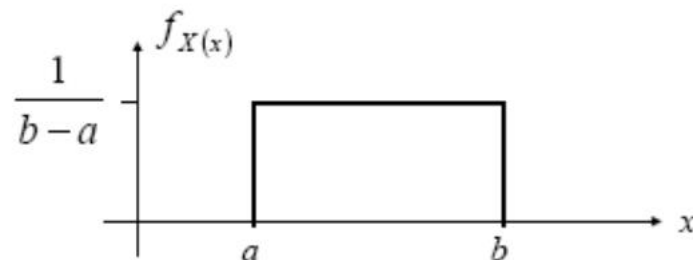
- If no error correction is used, the error probability is

$$P_e = 1 - 0.999^{31} \approx 0.0305$$

Some useful distribution: Uniform Distribution

- A continuous r.v. taking values between a and b with equal probabilities
- The PDF is given by

$$f_X(x) = \begin{cases} \frac{1}{b-a}, & a \leq x \leq b \\ 0, & \text{otherwise} \end{cases}$$

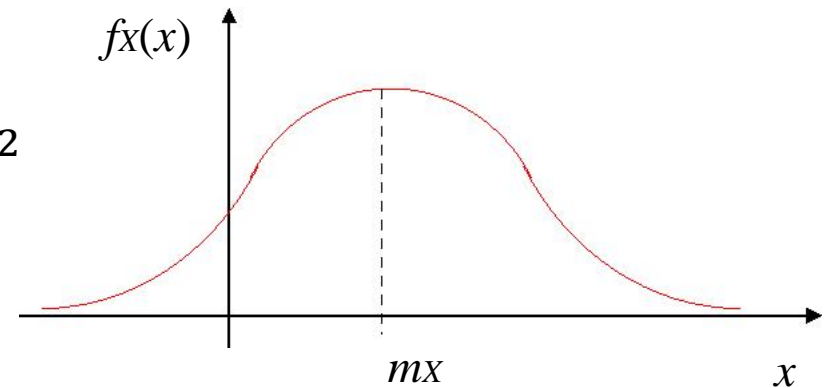


- The random phase of a sinusoid is often modeled as a uniform r.v. between 0 and 2π .

Some Useful Distribution: Gaussian Distribution

- Gaussian or normal distribution is a continuous r.v. with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp\left[-\frac{1}{2\sigma_x^2} (x - m_x)^2\right]$$



- A Gaussian r.v. is completely determined by its mean and variance, and hence usually denoted as

$$x \sim N(m_x, \sigma_x^2)$$

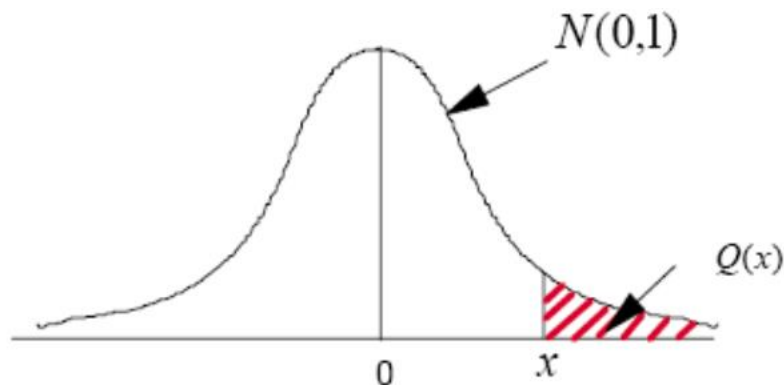
- By far the most important distribution in communications

The Q-Function

- The Q-function is a standard form to express error probabilities without a closed form

$$Q(x) = \int_x^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du$$

- The Q-function is the area under the tail of a Gaussian pdf with mean 0 and variance 1



Extremely important in error probability analysis!!!

More about Q-Function

- Q-function is monotonically decreasing
- Some features

$$Q(-\infty) = 1 \quad Q(0) = \frac{1}{2} \quad Q(\infty) = 0 \quad Q(-x) = 1 - Q(x)$$

- Craig's alternative form of Q-function (IEEE MILCOM'91)

$$Q(x) = \frac{1}{\pi} \int_0^{\pi/2} \exp\left(-\frac{x^2}{2 \sin^2 \theta}\right) d\theta, \text{ } x \geq 0$$

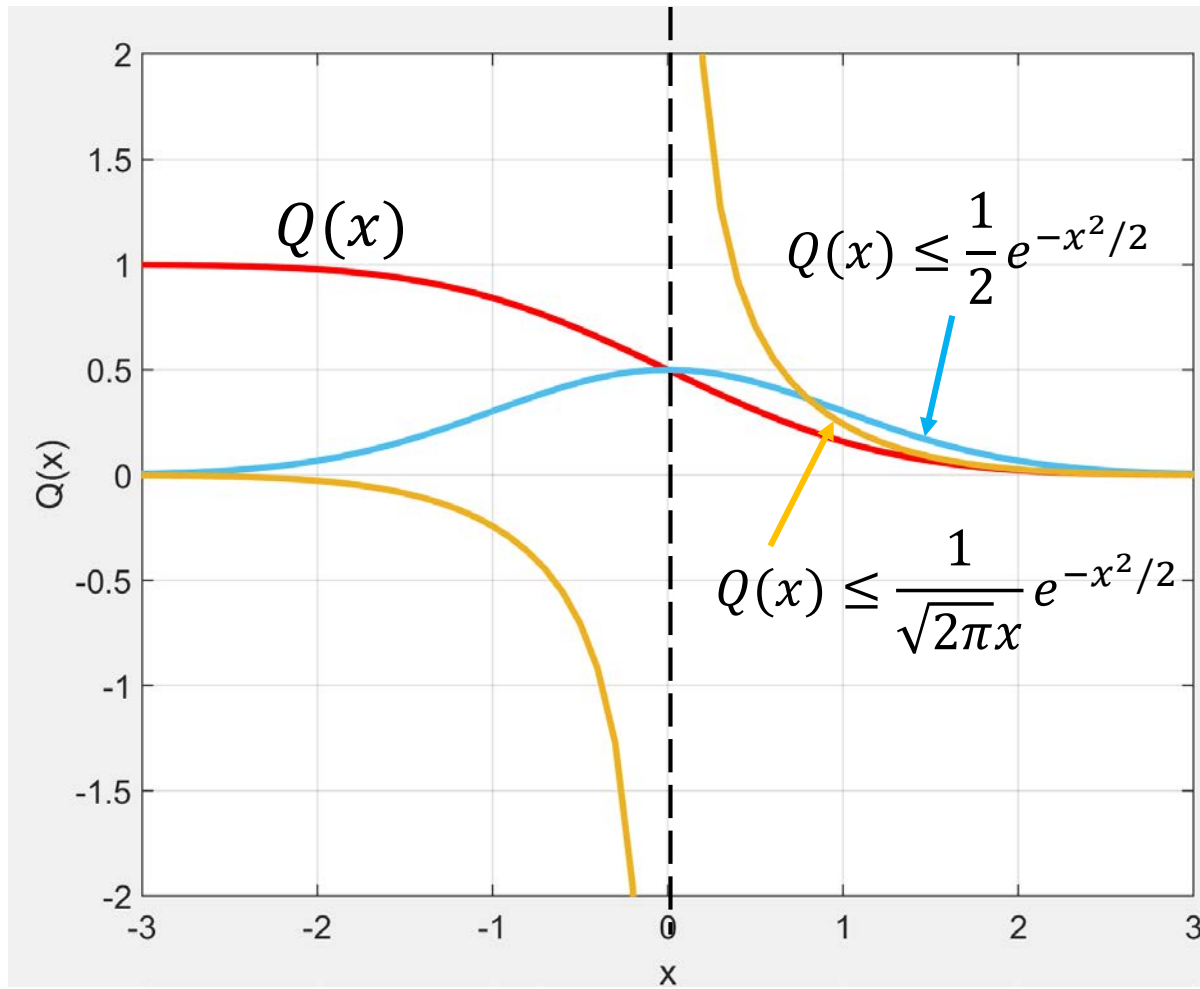
- Upper bound

$$Q(x) \leq \frac{1}{2} e^{-x^2/2}$$

- If we have a Gaussian variable $x \sim N(\mu, \sigma^2)$, then

$$P(X > x) = Q\left(\frac{x - \mu}{\sigma}\right)$$

More about Q-Function



Joint Distribution

- Consider 2 r.v.'s X and Y , joint distribution function is defined as

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

and joint PDF

$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$

- Key properties of joint distribution

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$$

$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = \int_{y_1}^{y_2} \int_{x_1}^{x_2} f_{XY}(x, y) dx dy$$

Joint Distribution (cont'd)

- Marginal distribution

$$F_X(x) = P(X \leq x, -\infty < Y < \infty) = \int_{-\infty}^{\infty} \int_{-\infty}^x f_{XY}(\alpha, \beta) d\alpha d\beta$$

$$F_Y(y) = \int_{-\infty}^y \int_{-\infty}^{\infty} f_{XY}(\alpha, \beta) d\alpha d\beta$$

- Marginal density

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x, \beta) d\beta, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(\alpha, y) d\alpha$$

- X and Y are said to be **independent** iff

$$F_{XY}(x, y) = F_X(x)F_Y(y)$$

$$f_{XY}(x, y) = f_X(x)f_Y(y)$$

$$f_{X|Y}(x|y) = f_X(x), f_{Y|X}(y|x) = f_Y(y)$$

A Linear Combination of N R.V.s

- Average of a linear combination of N r.v.s

$$E \left[\sum_{i=1}^N a_i X_i \right] = \sum_{i=1}^N a_i E[X_i]$$

- Variance of a linear combination of **independent** r.v.s

$$\text{var} \left[\sum_{i=1}^N a_i X_i \right] = \sum_{i=1}^N a_i^2 \text{var} \{ X_i \}$$

- The pdf of the sum of two independent r.v.s

$$Z = X + Y \quad f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_X(z-u) f_Y(u) du$$

Correlation of the R.V.

- Correlation of the two r.v. X and Y is defined as

$$R_{xy} = E[XY] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy f_{XY}(x, y) dx dy$$

- Correlation of the two centered r.v. $X - E[X]$ and $Y - E[Y]$, is called the covariance of X and Y

$$\sigma_{XY} = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$$

- The **correlation coefficient** of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

- If $\rho_{XY} = 0$, i.e., $E[XY] = E[X]E[Y]$, then X and Y are called **uncorrelated**.

Correlation (cont'd)

- If X and Y are independent, then they are uncorrelated.

$$\text{Independent} \rightarrow E[XY] = E[X]E[Y] \rightarrow \rho_{XY} = 0.$$

- The converse is not true (except for the Gaussian case)

Joint Gaussian Random Variables

- $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are jointly Gaussian iff

$$p(x_1, x_2, \dots, x_n) = \frac{1}{(2\pi)^{n/2} (\det(\mathbf{C}))^{1/2}} \exp \left[-\frac{(\mathbf{x} - \mathbf{m})^T \mathbf{C}^{-1} (\mathbf{x} - \mathbf{m})}{2} \right]$$

- \mathbf{x} is a column vector $\mathbf{x} = (x_1, \dots, x_n)^T$
- \mathbf{m} is the vector of the means $\mathbf{m} = (m_1, \dots, m_n)^T$
- \mathbf{C} is the $n \times n$ covariance matrix

$$\mathbf{C} = [C_{i,j}], C_{i,j} = E[(X_i - m_i)(X_j - m_j)]$$

Two-Variate Gaussian PDF

- Given two r.v.s: X_1 and X_2 that are joint Gaussian

$$\begin{aligned} \mathcal{C} &= \begin{bmatrix} E[X_1 - m_1]^2 & E[(X_1 - m_1)(X_2 - m_2)] \\ E[(X_1 - m_1)(X_2 - m_2)] & E(X_2 - m_2)^2 \end{bmatrix} \\ &= \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix} \end{aligned}$$

- We have

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp \left\{ -\frac{1}{2(1-\rho^2)} \left[\frac{(x_1 - m_1)^2}{\sigma_1^2} - \frac{2\rho(x_1 - m_1)(x_2 - m_2)}{\sigma_1\sigma_2} \right] \right\}$$

Two-Variate Gaussian (cont'd)

- For uncorrelated X_1 and X_2 , i.e. $\rho=0$

$$\begin{aligned} p(x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2} \exp \left\{ -\frac{1}{2} \left[\frac{(x_1 - m_1)^2}{\sigma_1^2} + \frac{(x_2 - m_2)^2}{\sigma_2^2} \right] \right\} \\ &= \frac{1}{2\pi\sigma_1\sigma_2} e^{-(x_1 - m_1)^2 / 2\sigma_1^2} \cdot \frac{1}{\sqrt{2\pi}\sigma_2} e^{-(x_2 - m_2)^2 / 2\sigma_2^2} \\ &= p(x_1)p(x_2) \end{aligned}$$



X_1 and X_2 are independent!

- If X_1 and X_2 are Gaussian and uncorrelated, then they are independent.

Some Properties of Jointly Gaussian r.v.s

- If n random variables $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are jointly Gaussian, then any set of them is also jointly Gaussian. In particular, all individual r.v.s are Gaussian.
- Jointly Gaussian r.v.s are completely characterized by the mean vector and the covariance matrix, i.e. the second-order properties.
- Any linear combination of $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ is a Gaussian r.v.

Law of Large Numbers

- Consider a sequence of r.v. $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$
- Let $y = \frac{1}{n} \sum_{i=1}^n x_i$
- If X 's are uncorrelated with the same mean and variance
- Then

$$\lim_{n \rightarrow \infty} P(|Y - m_X| \geq \varepsilon) = 0, \forall \varepsilon > 0$$

the sample average converges to
the expected value!

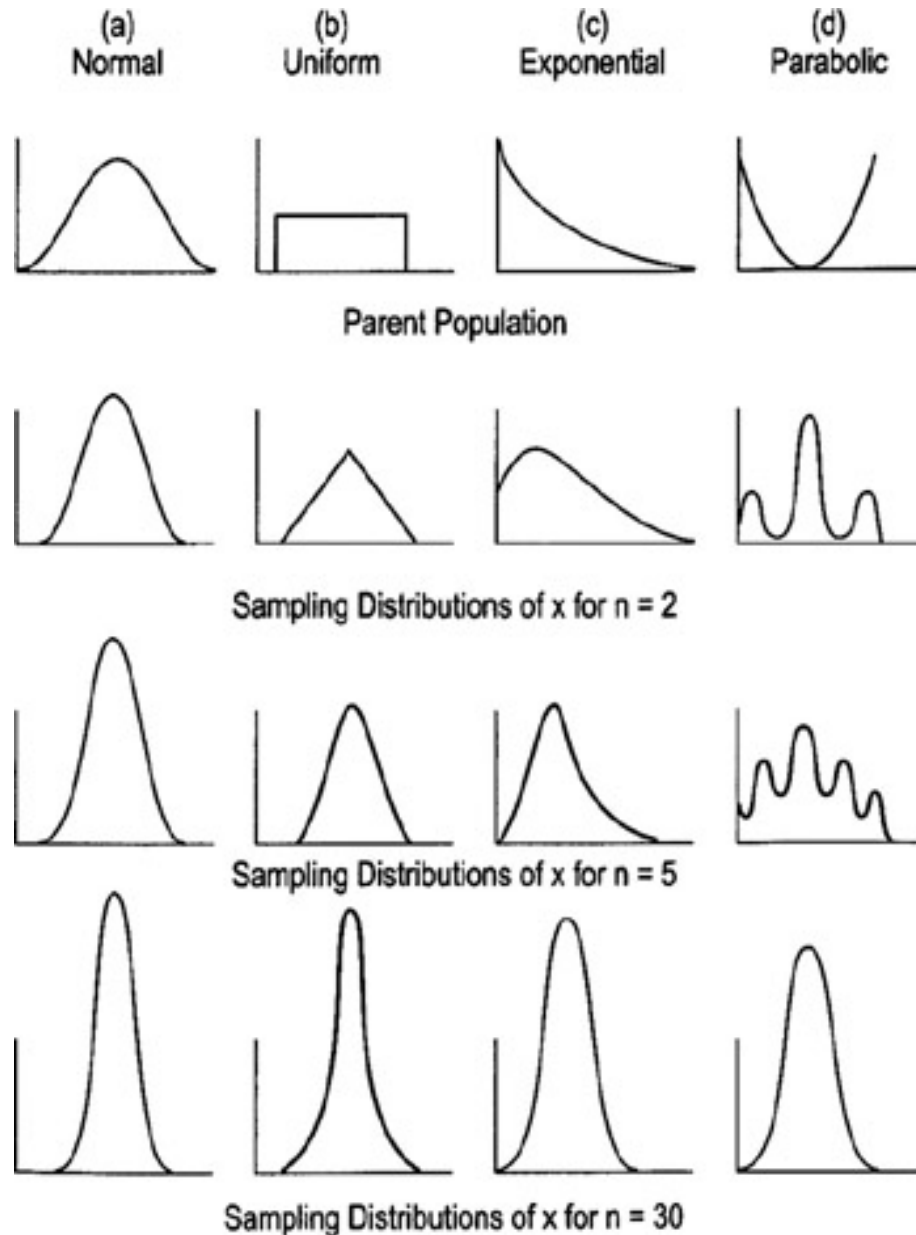
Central Limit Theorem

- If $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_n$ are **i.i.d.** random variables with common mean m_x and common variance σ_x^2 then $y = \frac{1}{n} \sum_{i=1}^n x_i$ converges to $N(m_x, \frac{\sigma_x^2}{n})$

the sum of many i.i.d random variables converges to a Gaussian random variable

- Thermal noise results from the random movement of many electrons – it is well modeled by a Gaussian distribution.

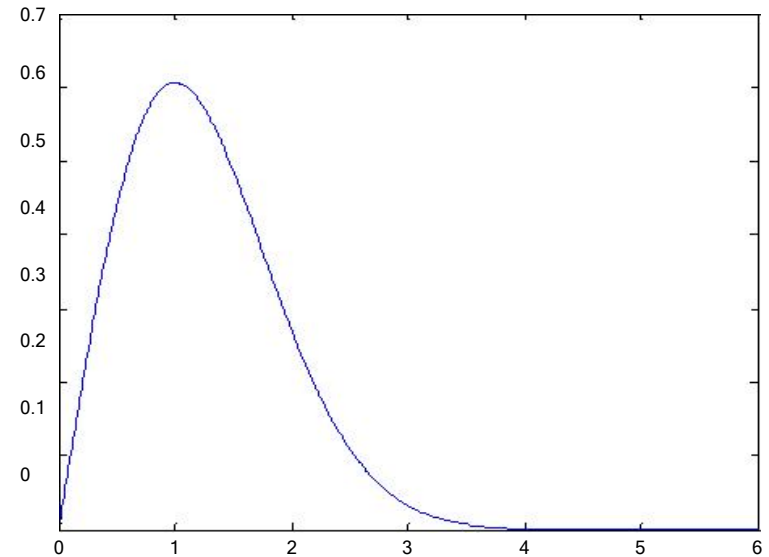
Example



Some useful distribution: Rayleigh Distribution

- PDF

$$f_X(x) = \begin{cases} \frac{x}{\alpha^2} \exp\left(-\frac{x^2}{2\alpha^2}\right), & x \geq 0 \\ 0, & x < 0 \end{cases}$$



- Rayleigh distributions are frequently used to model fading for non-line of sight (NLOS) signal transmission
- Very important for mobile and wireless communications

Exercise

- Let $Z = X + jY$, where x and Y are i.i.d. Gaussian random variables with mean 0 and variance σ^2
- Show that the magnitude of Z follows Rayleigh distribution and its phase follows a uniform distribution.

$$R = \sqrt{X^2 + Y^2} \quad \Theta = \tan^{-1} \left(\frac{Y}{X} \right)$$

$$\begin{aligned} X &= R \cos \Theta = g_1^{-1}(R, \Theta) \\ Y &= R \sin \Theta = g_2^{-1}(R, \Theta) \end{aligned} \quad \frac{\partial (x, y)}{\partial (r, \theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} = r$$

$$f_{R\Theta}(r, \theta) = \frac{re^{-r^2/2\sigma^2}}{2\pi\sigma^2}, \quad \begin{aligned} 0 \leq \theta < 2\pi \\ 0 \leq r < \infty \end{aligned}$$

Rayleigh distribution

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad 0 \leq r < \infty$$



Thanks for your kind attention!

Questions?