EE140 Introduction to Communication Systems

Lecture 5

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ShanghaiTech University, Fall 2022

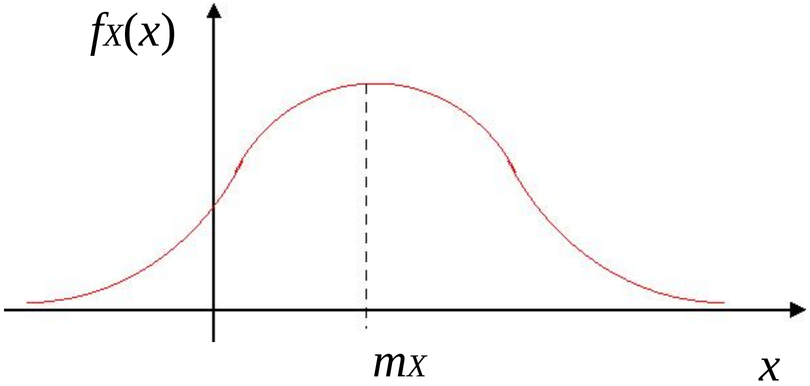
Contents

- Random signals
  - Review of probability and random variables
  - Random processes: basic concepts
  - Gaussian white processes

Recall: Gaussian Distribution

- Gaussian or normal distribution is a continuous r.v. with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp[-\frac{1}{2\sigma_x^2}(x - m_x)^2]$$



- A Gaussian r.v. is completely determined by its mean and variance, and hence usually denoted as

$$x \sim N(m_x, \sigma_x^2)$$

Gaussian Process

- Definition: $X(t)$ is a Gaussian process if for all n and all $t_1, t_2, \ldots, t_n$ the sample values $X(t_1), X(t_2), \ldots, X(t_n)$ have a joint Gaussian density function

$$f_{X(t_1)X(t_2),\cdots,X(t_n)}(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}(\det(\mathbf{C}))^{1/2}} \exp\left[-\frac{(x-\mathbf{m})^T\mathbf{C}^{-1}(x-\mathbf{m})}{2}\right]$$

- Properties:
  - If it is wide-sense stationary, it is also strictly stationary (Gaussian process is completely defined by its first order statistics **m** and second order statistics **C** .)

Gaussian Process

- Properties:
  - If the samples of Gaussian process $X(t_1), X(t_2), \ldots, X(t_n)$ are uncorrelated in time, they are also independent

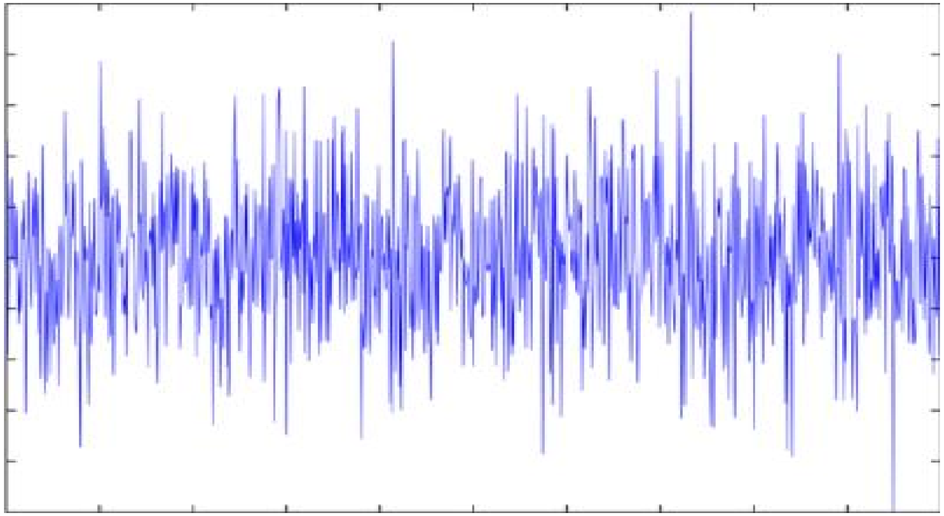
$$f_{X(t_1)X(t_2),\cdots,X(t_n)}(x_1, x_2, \ldots, x_n) = \prod_{\mathrm{k}=1}^{n} \frac{1}{\sqrt{2\pi}\sigma_{\mathrm{k}}} \exp[-\frac{(x_{\mathrm{k}} - a_{\mathrm{k}})^2}{2\sigma_{\mathrm{k}}^2}]$$
$$= f_{X(t_1)}(x_1) \cdot f_{X(t_2)}(x_2) \cdot \cdots \cdot f_{X(t_n)}(x_n)$$

  - If the input to a linear system is a Gaussian process, the output is also a Gaussian process

$$Y_o(t) = \int_{-\infty}^{\infty} h(\tau) X_i(t - \tau) d\tau$$

Noise

- Gaussian Noise:
  - often modeled as Gaussian and stationary with 0 mean



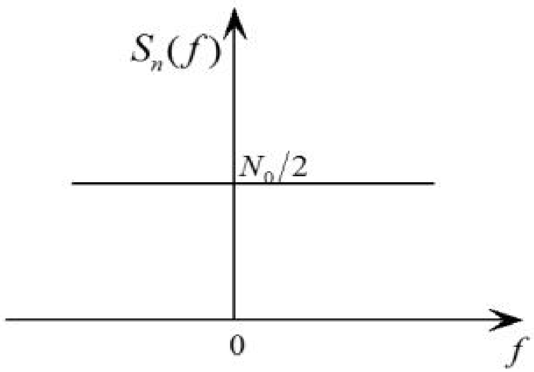
- White noise (stationary and zero mean)

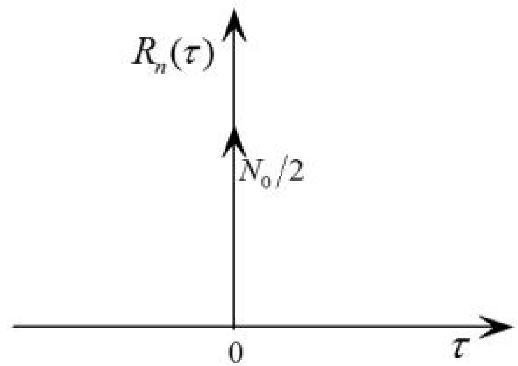
Noise

- White Noise (stationary and zero mean)

$$S_n(f) = \frac{N_0}{2} \qquad \Rightarrow \qquad R_n(\tau) = \frac{N_0}{2}\delta(\tau)$$

Infinite power





White noise is completely uncorrelated!

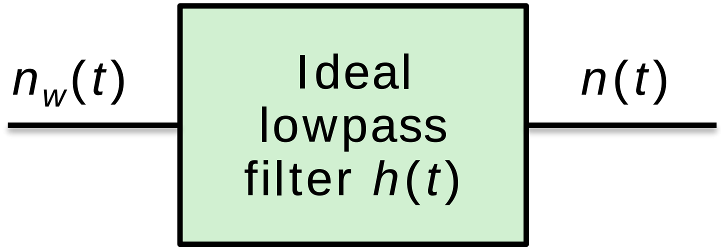
$$N_0 = KT = 4.14 \times 10^{-21}$$
$$= -174\ \text{dBm/Hz}$$

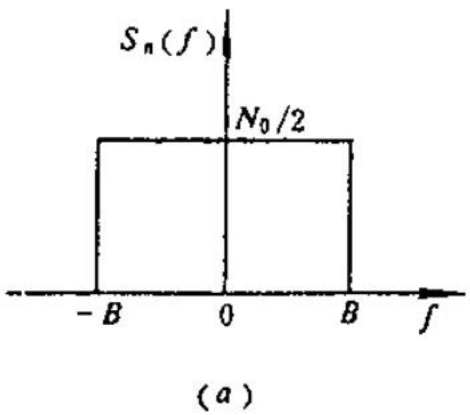
N_0: single-sided power spectral density

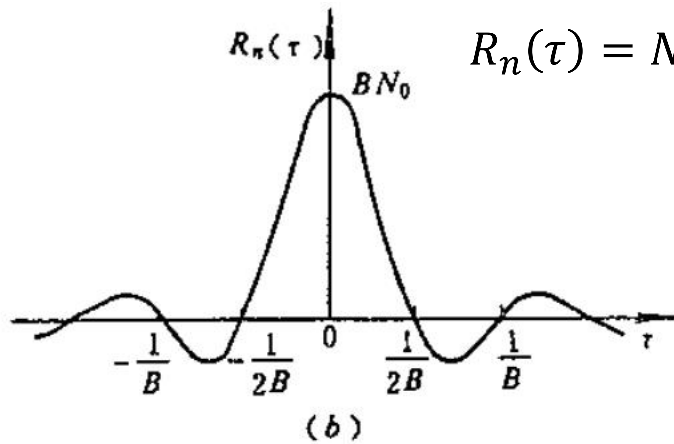
$\frac{N_0}{2}$: two-sided power spectral density

- White Gaussian Noise (stationary and zero mean)

Bandlimited Noise







$R_n(\tau) = N_0 B sinc(2B\tau)$

- Q1. At what rate to sample the noise can we get uncorrelated realizations? (2B/second)
- Q2. What is the power of each sample? (BN_0)

Noise Equivalent Bandwidth

- White noise: zero mean, two-sided PSD$=\frac{N_0}{2}$
- Arbitrary filter: $H(f)$
- Average output noise power

$$P_{n_o} = \int_{-\infty}^{\infty} \frac{N_0}{2} |H(f)|^2\, df = N_0 \int_0^{\infty} |H(f)|^2\, df$$

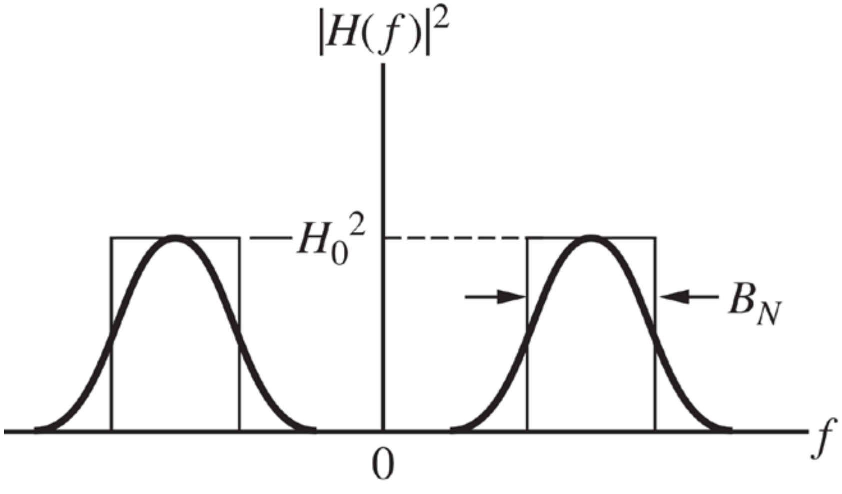
$|H(f)| = |H(-f)|$, if $h(t)$ is real.

- Ideal filter: B_N, H_0

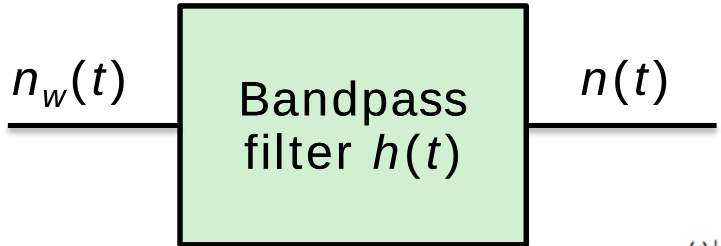
$$P_{n_o} = N_0 H_0^2 B_N$$

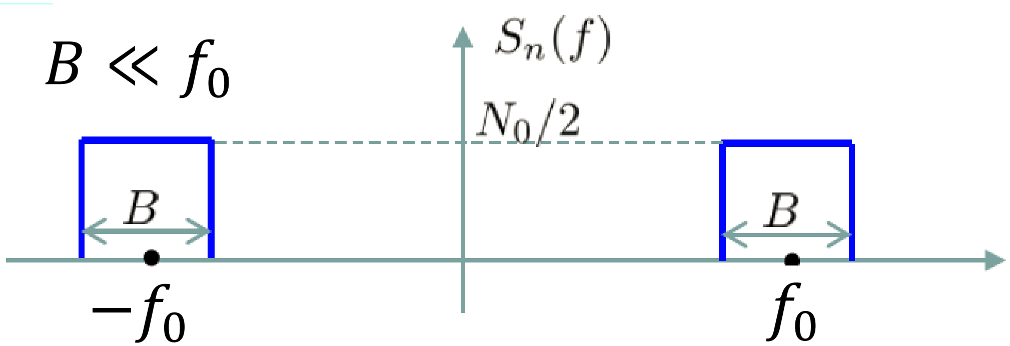
- Noise equivalent bandwidth

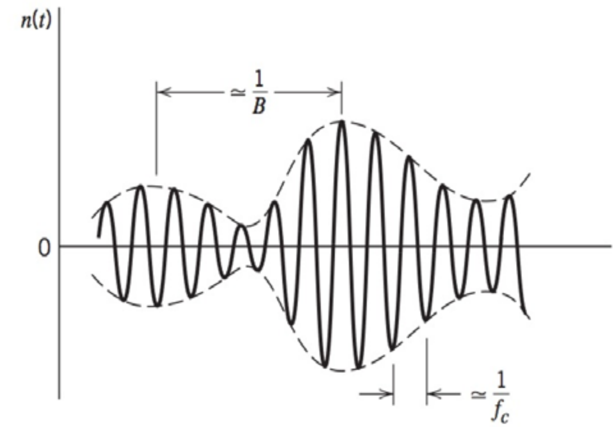
$$B_N = \frac{\int_0^{\infty} |H(f)|^2 df}{H_0^2}$$



Narrowband Noise



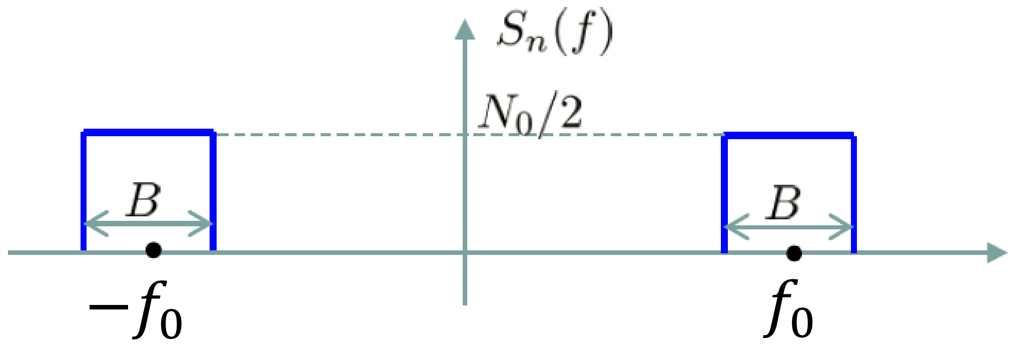




Sample function

- Two specific representation of narrowband noise
  - In-phase and quadrature components
  - Envelope and phase

Narrowband Noise



- Canonical form of a band-pass noise process

$$\mathrm{n}(t) = n_c(t)\cos(2\pi f_0 t + \theta) - n_s(t)\sin(2\pi f_0 t + \theta)$$

In-phase component

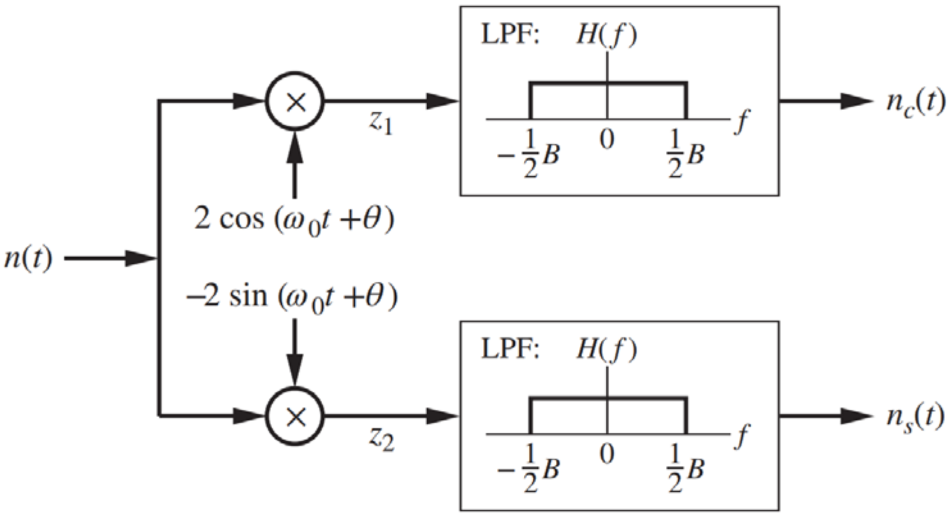
Quadrature component

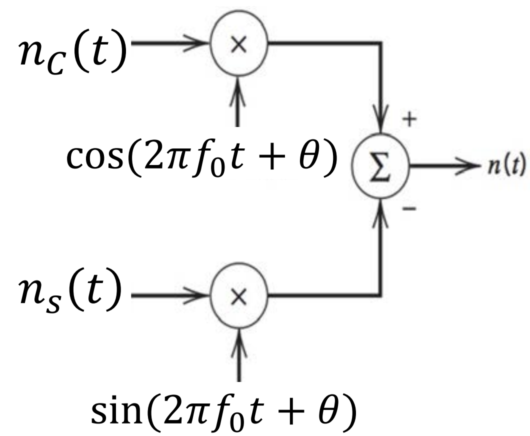
Low-pass noise process

$\boxed{\theta \text{ is an arbitrary phase angle}}$

Narrowband Noise

- How to produce $n_c(t)$ and $n_s(t)$





Why equality holds? (Proof: Page 712, Appendix C)

$$\boxed{E\left\{\left[n(t) - [n_c(t)\cos(2\pi f_0 t + \theta) - n_s(t)\sin(2\pi f_0 t + \theta)]\right]^2\right\} = 0}$$

Properties

If $n(t)$ is Gaussian RP → $n_s(t)$ and $n_C(t)$ are joint Gaussian RP

If $n(t)$ is stationary → $n_s(t)$ and $n_C(t)$ are jointly stationary

$n_s(t)$ and $n_C(t)$ →

- Zero mean
- Same PSD (autocorrelation, variance)

$$S_{n_c}(f) = S_{n_s}(f)$$
$$= \mathrm{Lp}[S_n(f - f_0) + S_n(f + f_0)]$$

- Cross-PSD (odd Cross-correlation)

$$S_{n_c n_s}(f) = j\mathrm{Lp}[S_n(f - f_0) - S_n(f + f_0)]$$

Properties

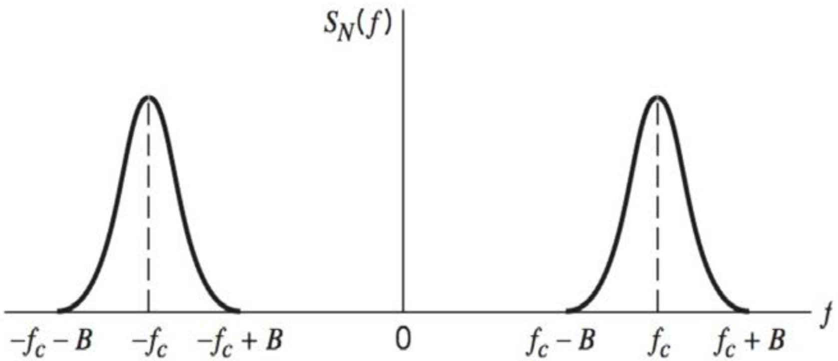
- Let n(t) be a zero-mean, stationary and Gaussian noise, then n_c(t) and n_s(t) satisfy the following properties
  - $n_c(t)$ and $n_s(t)$ are zero-mean, jointly stationary and jointly Gaussian process
  - Means: $E[n(t)] = E[n_c(t)] = E[n_s(t)] = 0$
  - PSD: $S_{n_c}(f) = S_{n_s}(f) = \mathrm{Lp}[S_n(f - f_0) + S_n(f + f_0)]$
  - Variances(power): $E[n^2(t)] = E[n_c^2(t)] = E[n_s^2(t)] = \mathrm{N}_0 B \triangleq \sigma^2$
  - Correlation function:
    - $R_{n_c}(\tau) = R_{n_s}(\tau)$, $R_n(0) = R_{n_c}(0) = R_{n_s}(0)$
    - $R_{n_c n_s}(\tau) = -R_{n_c n_s}(-\tau)$ (odd), $R_{sc}(0) = R_{cs}(0) = 0.$
  - Cross-PSD: $S_{n_c n_s}(f) = j\mathrm{Lp}[S_n(f - f_0) - S_n(f + f_0)]$
    - $R_{n_c n_s}(\tau) \equiv 0, \forall \tau,\ \text{ if } \mathrm{Lp}[S_n(f - f_0) - S_n(f + f_0)] = 0.$

Properties

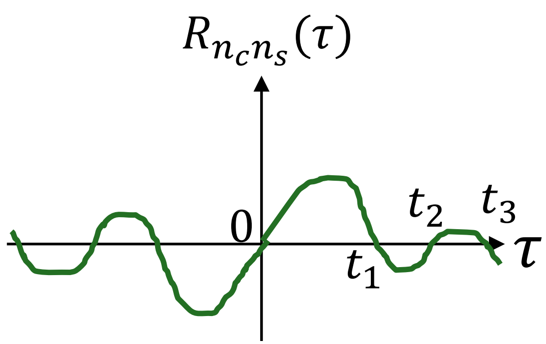
- Let n(t) be a zero-mean, stationary and Gaussian noise, then $n_c(t)$ and $n_s(t)$ satisfy the following properties
  - $n_c(t)$ and $n_s(t)$ are uncorrelated (independent) Gaussian process

1. If PSD of n(t) is symmetric about f_0

$$R_{n_c n_s}(\tau) \equiv 0, \forall\tau$$

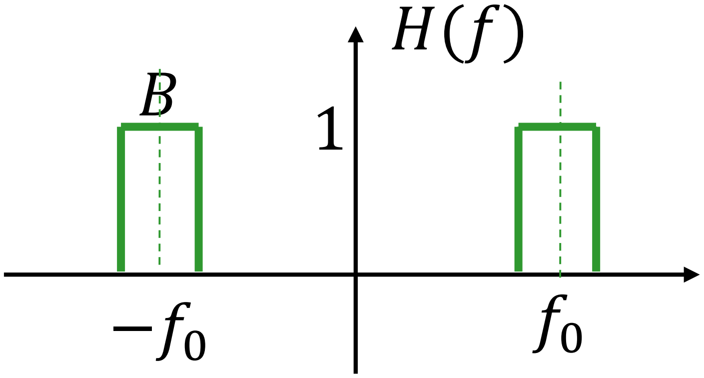


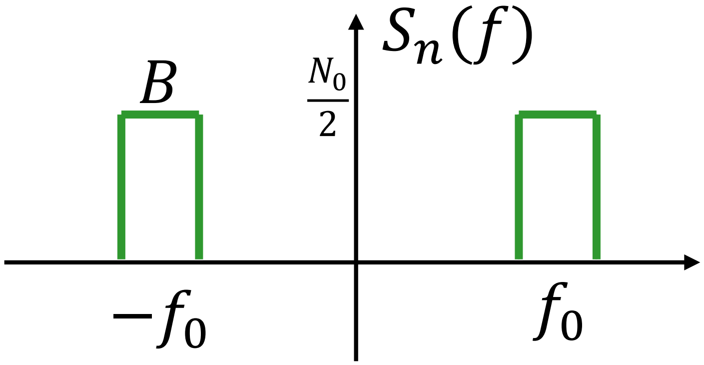
2. $\{\tau: R_{n_c n_s}(\tau)=0\}$.



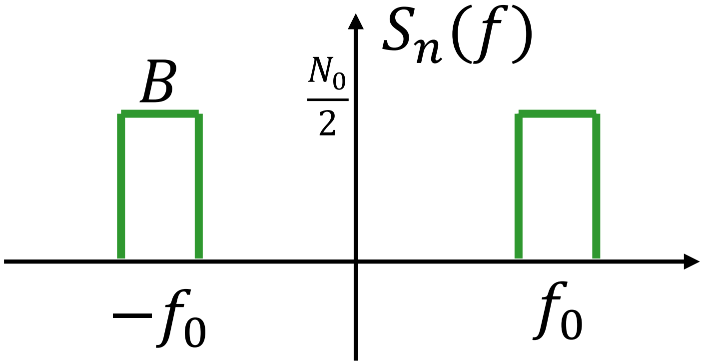
Example: Ideal band-pass filtered white noise

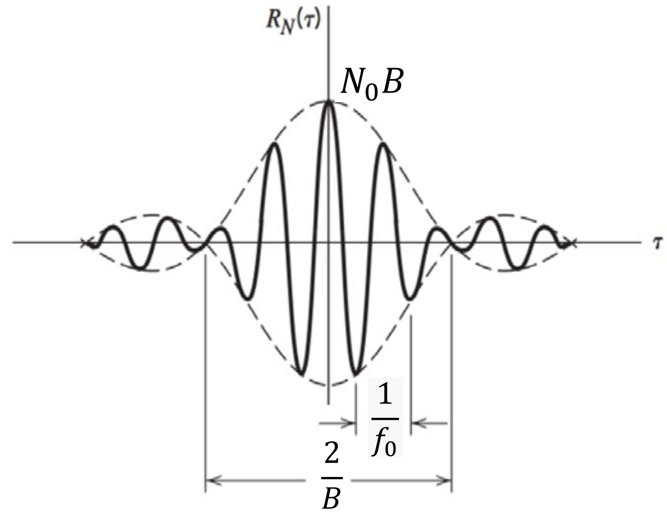
- Consider a white Gaussian noise of zero mean and PSD N0/2, which is passed through an ideal band-pass filter.
- Determine the autocorrelation function of n(t) and its in-phase and quadrature components.





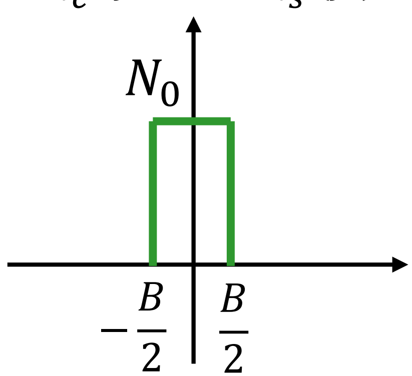
Example: Ideal band-pass filtered white noise

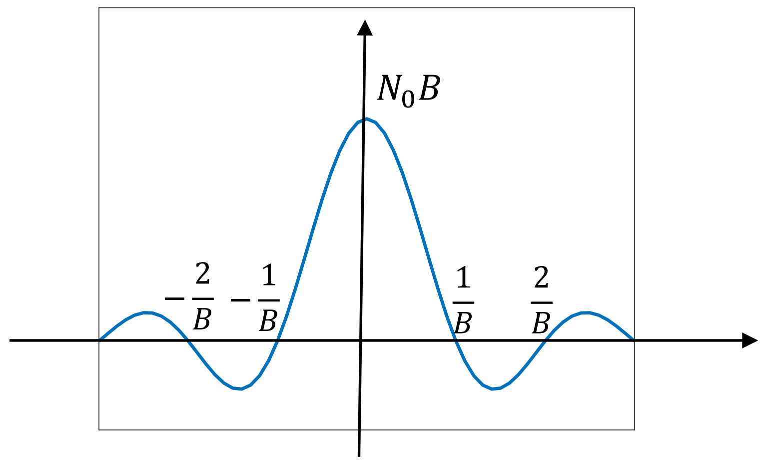




$$R_n(\tau) = N_0 B sinc(B\tau)\cos(2\pi f_0 \tau)$$

$$S_{n_c}(f) = S_{n_s}(f)$$





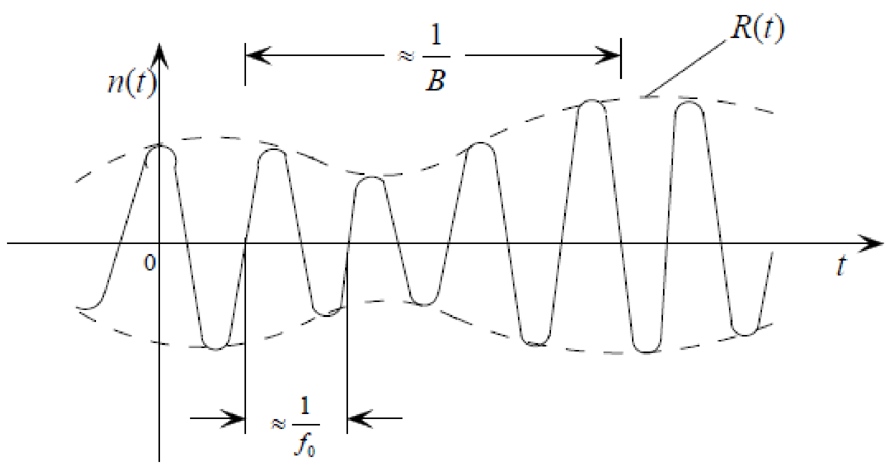
$$R_{n_c}(\tau) = R_{n_s}(\tau) = N_0 B sinc(B\tau)$$

Envelope and Phase

- Angular representation of n(t)

$$n(t) = R(t)\cos(\omega_0 t + \theta + \varphi(t))$$

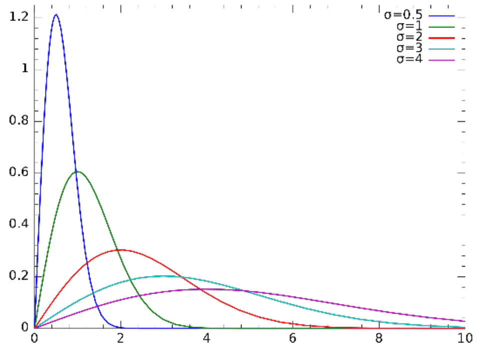
$$\text{where}\begin{cases} R(t) = \sqrt{n_c^2(t) + n_s^2(t)} & \text{envelope} \\ \varphi(t) = \tan^{-1}\dfrac{n_s(t)}{n_c(t)}, [0 \le \varphi(t) \le 2\pi] & \text{phase} \end{cases}$$



Example

- Let n(t) be a zero-mean, stationary Gaussian process, find the statistics of the envelop and phase
- Result:
  - Envelop follows Rayleigh distribution
  - Phase follows uniform distribution

$$\begin{cases} f(R) = \int_0^{2\pi} f(R,\varphi)d\varphi = \dfrac{R}{\sigma^2}\exp\left\{-\dfrac{R^2}{2\sigma^2}\right\}, R \geq 0 \\ f(\varphi) = \int_0^{\infty} f(R,\varphi)dR = \dfrac{1}{2\pi}, 0 \leq \varphi \leq 2\pi \end{cases}$$



- For the same t, the envelop variable R and phase variable φ are independent (but not the two processes)

Rayleigh fading channel: model the fading channel with random scatters.

Derivation

- Derivation

$$f(R,\varphi) = f(n_c, n_s)\left|\frac{\partial(n_c, n_s)}{\partial(R,\varphi)}\right|$$

$$f(n_c, n_s) = f(n_c)\cdot f(n_s) = \frac{1}{2\pi\sigma^2}\exp\left[-\frac{n_c^2+n_s^2}{2\sigma^2}\right]$$

$$\left|\frac{\partial(n_c, n_s)}{\partial(R,\varphi)}\right| = \begin{vmatrix}\frac{\partial n_c}{\partial R} & \frac{\partial n_s}{\partial R}\\ \frac{\partial n_c}{\partial \varphi} & \frac{\partial n_s}{\partial \varphi}\end{vmatrix} = \begin{vmatrix}\cos\varphi & \sin\varphi\\ -R\sin\varphi & R\cos\varphi\end{vmatrix} = R$$

$$f(R,\varphi) = Rf(n_c, n_s) = \frac{R}{2\pi\sigma^2}\exp\left[-\frac{(R\cos\varphi)^2+(R\sin\varphi)^2}{2\sigma^2}\right]$$

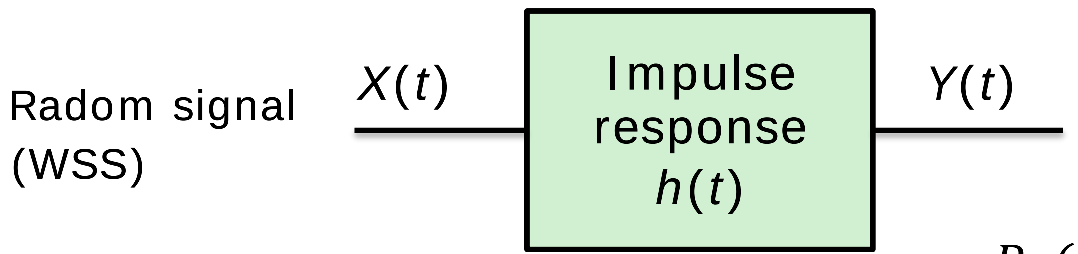
$$= \frac{R}{2\pi\sigma^2}\exp\left\{-\frac{R^2}{2\sigma^2}\right\}$$

Summary

- For WSS:

$$S_X(f) \leftrightarrow R_X(\tau)$$

- WSS transmission through a linear system



$$R_Y(\tau) = h(-\tau) * h(\tau) * R_X(\tau)$$

$$S_Y(f) = |H(f)|^2 S_X(f)$$

- White noise (zero-mean)
  - Bandlimited noise
  - Narrowband noise: Gaussian, stationary and zero-mean
  - Non-Gaussian?

Sine Wave with Bandpass Noise

- Received signal

$$r(t) = A\cos(\omega_c t + \theta) + n(t)$$

where A, ω_c are deterministic, θ is random phase uniformly distributed in $[-\pi, \pi]$, $n(t)$ is narrowband noise (zero-mean, stationary Gaussian process).

$$\begin{aligned} r(t) &= [A\cos\theta + n_c(t)]\cos\omega_c t - [A\sin\theta + n_s(t)]\sin\omega_c t \\ &= z_c(t)\cos\omega_c t - z_S(t)\sin\omega_c t \\ &= z(t)\cos[\omega_c t + \varphi(t)] \end{aligned}$$

$$z_c(t) = A\cos\theta + n_c(t)$$

$$z_s(t) = A\sin\theta + n_s(t)$$

Sine Wave with Bandpass Noise (cont'd)

- Envelop

$$z(t) = \sqrt{z_c^2(t) + z_s^2(t)}, z \geq 0$$

- Phase

$$\varphi(t) = \tan^{-1} \frac{z_s(t)}{z_c(t)}$$

- Given θ

$$E[z_c] = A\cos\theta$$
$$E[z_s] = A\sin\theta$$
$$\sigma_c^2 = \sigma_s^2 = \sigma_n^2$$

- Joint distribution

$$f(z_c, z_s|\theta) = \frac{1}{2\pi\sigma_n^2} \exp\left\{-\frac{1}{2\sigma_n^2}\left[(z_c - A\cos\theta)^2 + (z_s - A\sin\theta)^2\right]\right\}$$

PDF of the Amplitude

$$f(z_c, z_s|\theta) = \frac{1}{2\pi\sigma_n^2} \exp\left\{-\frac{1}{2\sigma_n^2}\left[(z_c - A\cos\theta)^2 + (z_s - A\sin\theta)^2\right]\right\}$$

$$f(z, \varphi|\theta) = f(z_c, z_s|\theta)\left|\frac{\partial(z_c, z_s)}{\partial(z, \varphi)}\right|$$

$$= \begin{vmatrix} \cos\varphi & \sin\varphi \\ -z\sin\varphi & z\cos\varphi \end{vmatrix} f(z_c, z_s|\theta) = z \cdot f(z_c, z_s|\theta)$$

- For PDF of the amplitude

$$f(z|\theta) = \int_0^{2\pi} f(z, \varphi|\theta) d\varphi$$

$$= \frac{z}{2\pi\sigma_n^2} \exp\left[-\frac{z^2 + A^2}{2\sigma_n^2}\right] \int_0^{2\pi} \exp\left[\frac{Az}{\sigma_n^2}\cos(\theta - \varphi)\right] d\varphi$$

PDF of the Amplitude (cont'd)

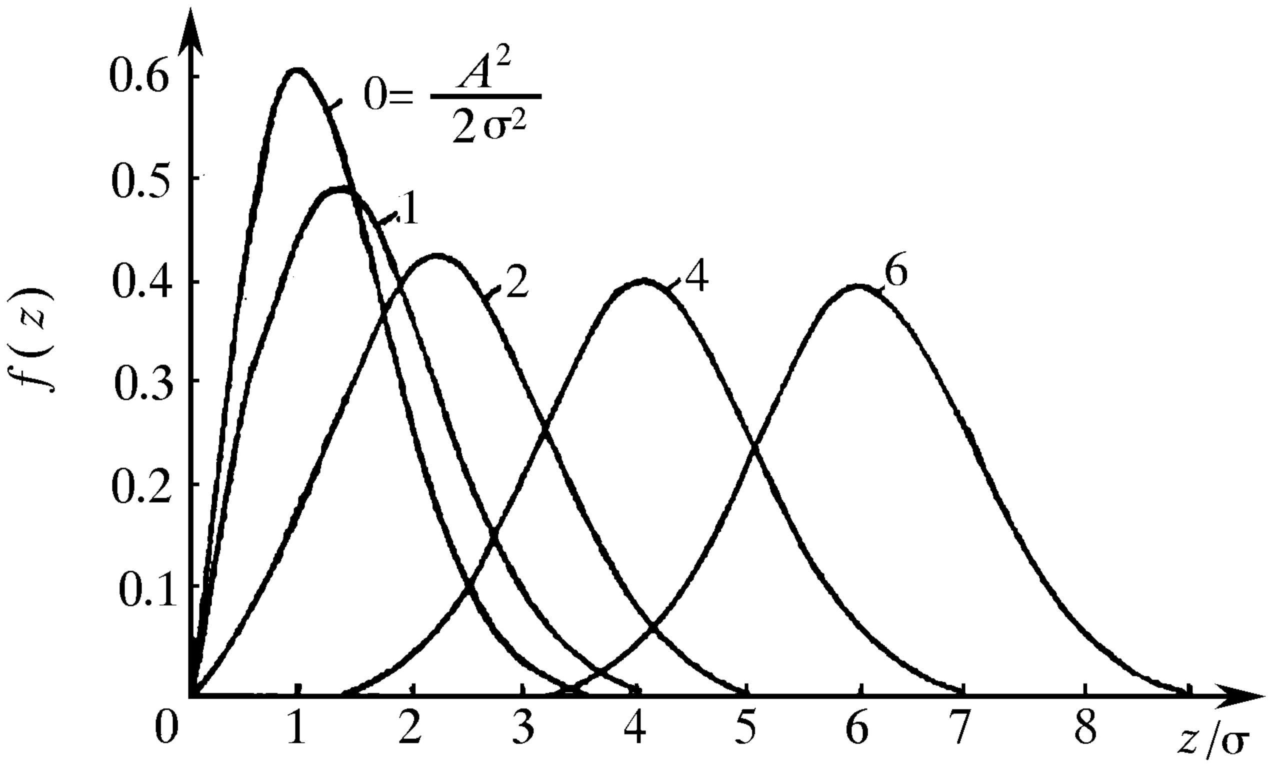
- Amplitude

$$f(z|\theta) = \int_0^{2\pi} f(z,\varphi|\theta)d\phi$$
$$= \frac{z}{2\pi\sigma_n^2}\exp\left[-\frac{z^2+A^2}{2\sigma_n^2}\right]\int_0^{2\pi}\exp\left[\frac{Az}{\sigma_n^2}\cos(\theta-\phi)\right]d\phi$$
$$= \frac{z}{2\pi\sigma_n^2}\exp\left[-\frac{z^2+A^2}{2\sigma_n^2}\right]I_0\left(\frac{Az}{\sigma_n^2}\right)$$

Ricean distribution

$I_0\left(\frac{Az}{\sigma_n^2}\right)$ 0th order Bessel functions of the first kind

PDF of the Amplitude (cont'd)



- A small, $I_0\left(\frac{Az}{\sigma_n^2}\right) \approx 1$

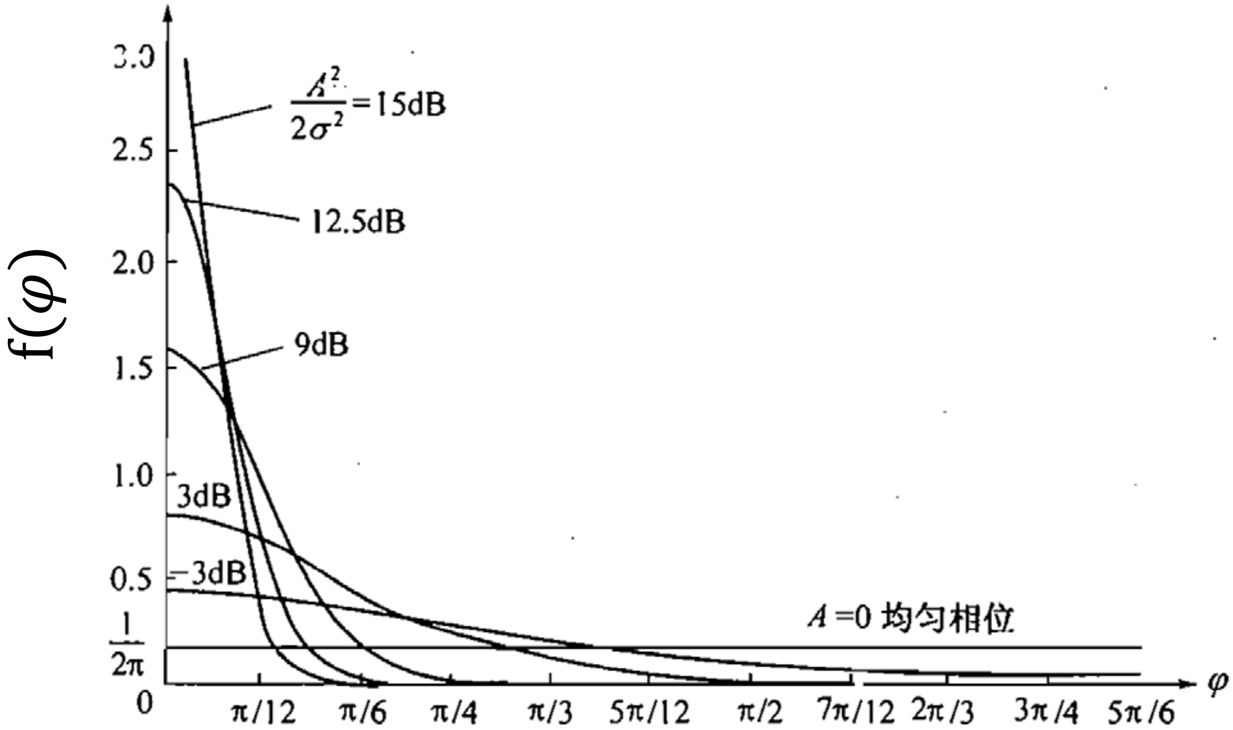
Rayleigh distribution

- A large, $I_0(x) \approx \frac{e^x}{\sqrt{2\pi x}}$

Gaussian distribution

Ricean fading channel: model the fading channel with a direct path and scatters.

PDF of the Amplitude (cont'd)



- A small, Uniform distribution
- A large, Concentrate around θ

Ricean fading channel: model the fading channel with a direct path and scatters.

Thanks for your kind attention!

Questions?