EE140 Introduction to Communication Systems

Lecture 12

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ShanghaiTech University, Fall 2022

Signal Space

- Norm $|x(t)|$. Energy $\mathcal{E}_x$

$$\|x(t)\| = \left(\int_{-\infty}^{\infty} |x(t)|^2 dt\right)^{1/2} = \sqrt{\mathcal{E}_x}$$

- Inner produce:

$$\langle x_1(t), x_2(t)\rangle = \int_{-\infty}^{\infty} x_1(t)x_2^*(t)dt$$

- $x_1(t), x_2(t)$ are orthogonal if their inner product is 0.
- Signals are orthonormal if they are orthogonal and their norm are unity
- Signals are linearly independent if no signal can be represented as a linear combination of remaining signals
- $\|x_1(t) + x_2(t)\| \leq \|x_1(t)\| + \|x_2(t)\|$
- $\|\langle x_1(t), x_2(t)\rangle\| \leq \|x_1(t)\|\|x_2(t)\|$

Gram-Schmidt for Signals

Given an finite energy signal waveforms $\{s_m(t)\}_{m=1}^{M}$, generate an orthonormal waveforms $\{\phi_1(t), \phi_2(t), ..., \phi_M(t)\}$

1. $\phi_1(t) = s_1(t)/\sqrt{E_1}$.
2. $c_{21} = \langle s_2(t), \phi_1(t)\rangle$
3. $\gamma_2(t) = s_2(t) - c_{21}\phi_1(t)$, and let $E_2 = \int_{-\infty}^{\infty} \gamma_2^2(t)dt$
4. $\phi_2(t) = \gamma_2(t)/\sqrt{E_2}$
5. $\phi_k(t) = \gamma_k(t)/\sqrt{E_k}$
   where

$$\gamma_k(t) = s_k(t) - \sum_{i=1}^{k-1} c_{ki}\phi_i(t)$$

$$c_{ki} = \langle s_k(t), \phi_i(t)\rangle$$

$$E_k = \int_{-\infty}^{\infty} \gamma_k^2(t)dt$$

Orthonormal Expansions in $\mathcal{L}_2$

- Given an orthonormal basis $\{\phi_1(t), \phi_2(t), \cdots\}$ of $\mathcal{L}_2$, any $\mathcal{L}_2$ signal x(t) can be represented as $x(t) = \sum_j x_j \phi_j(t)$.
  - $x_j = \langle x(\mathrm{t}), \phi_j(t) \rangle = \int x(t) \phi_j^*(t) dt$.
  - $\boldsymbol{x} = (\mathrm{x}_1, \mathrm{x}_2, \cdots)$
  - Equality holds in the sense of MSE=0.
    - $\int_{-\infty}^{\infty} \left| x(t) - \sum_j x_j \phi_j(t) \right|^2 dt = 0$
  - $E_x = \int_{-\infty}^{\infty} |x(t)|^2 dt = \sum_j \left| x_j \right|^2 = \|\boldsymbol{x}\|^2$
  - $\langle x(t), s(t) \rangle = \langle \boldsymbol{x}, \boldsymbol{s} \rangle$
- Example
  - Fourier Series
  - Sampling Theorem

Orthonormal Expansions in $\mathcal{L}_2$

- Fourier Series

Given a function $v(t)$, with duration $[-T/2, T/2]$

- Define $\theta_k(t) = e^{2\pi ikt/T}\text{rect}(t/T)$
- $\{\theta_k(t)\}_k$ are orthogonal functions, with $||\theta_k(t)||^2 = T$
- Let $\phi_k(t) = \theta_k(t)/\sqrt{T}$:

$$\phi_k(t) = \frac{1}{\sqrt{T}}e^{2\pi ikt/T}\text{rect}(t/T)$$

- By FSE , $v(t)$ can be written as

$$v(t) = \sum_k \alpha_k \phi_k(t)$$

where $\alpha_k =< v(t), \phi_k(t) >$.

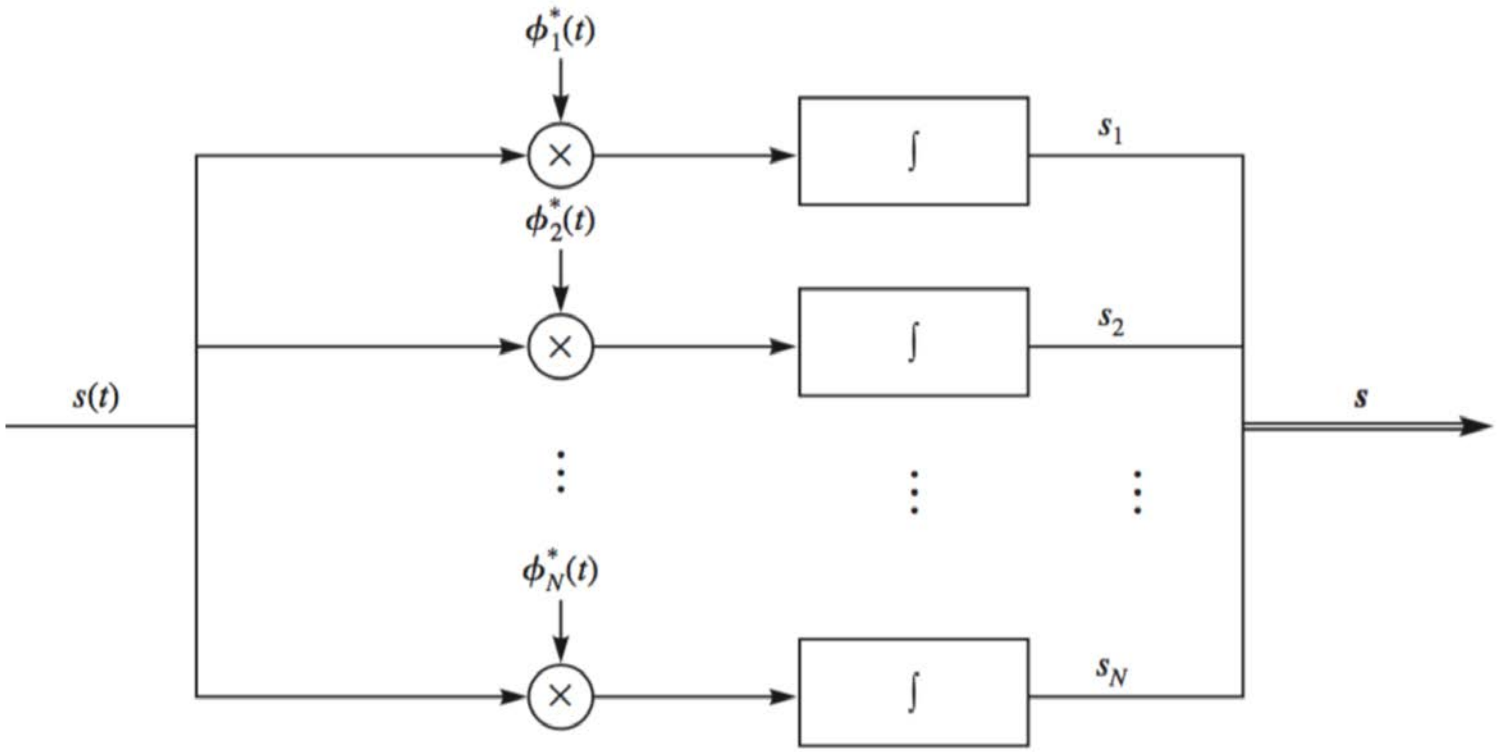
Orthonormal Expansions in $\mathcal{L}_2$

- Sampling Theorem

> Sampling Theorem: If $x(t)$ is a band-limited in W, then
>
> $$x(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}\left[\left(\frac{t}{T} - n\right)\right]$$
>
> where $\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$, $T = 1/2W$.

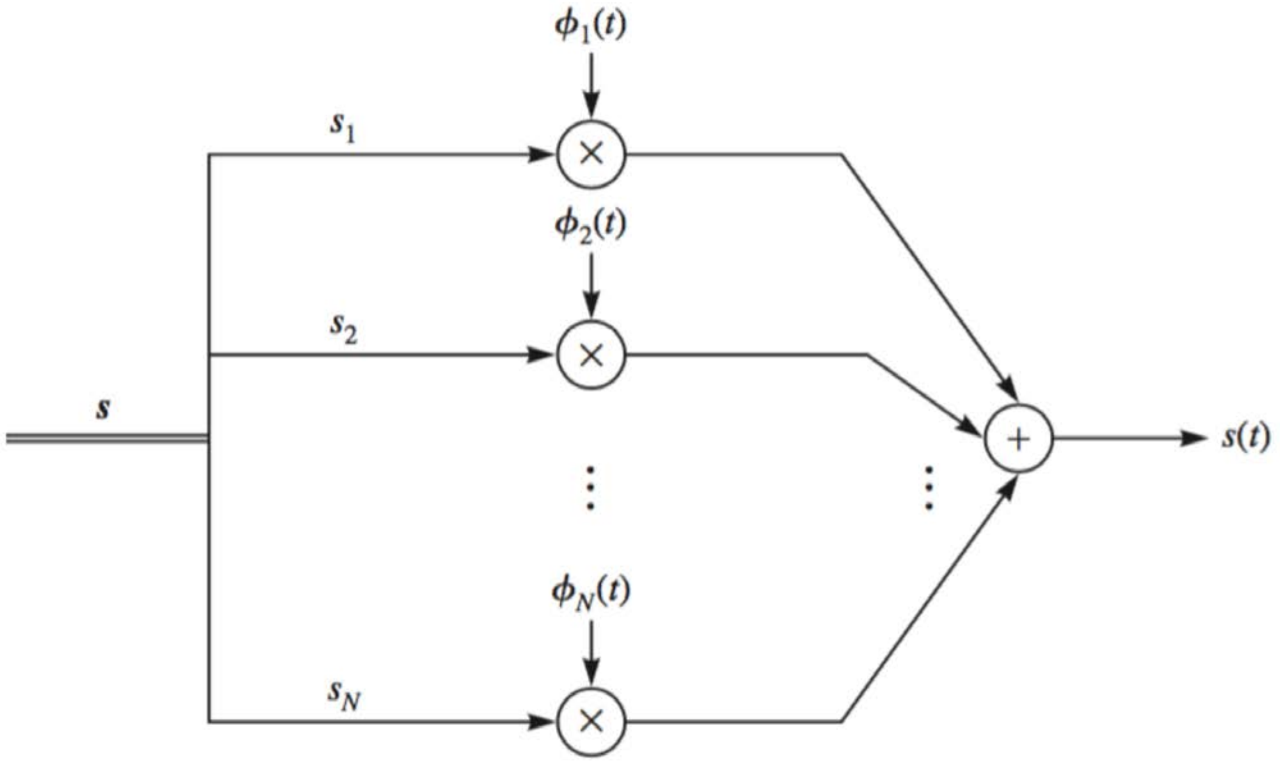
- $x(nT)$ is coefficient (x_k)
- $\{\mathrm{sinc}\left[\left(\frac{t}{T} - n\right)\right]\}_{n=-\infty}^{\infty}$ are orthogonal function
- Equality holds in sense of $E_e = \int_{-\infty}^{\infty} |x(t) - \hat{x}(t)|^2 = 0$
- View signal $x(t)$ as vectors $x(nT)$

Map Signal to Vector



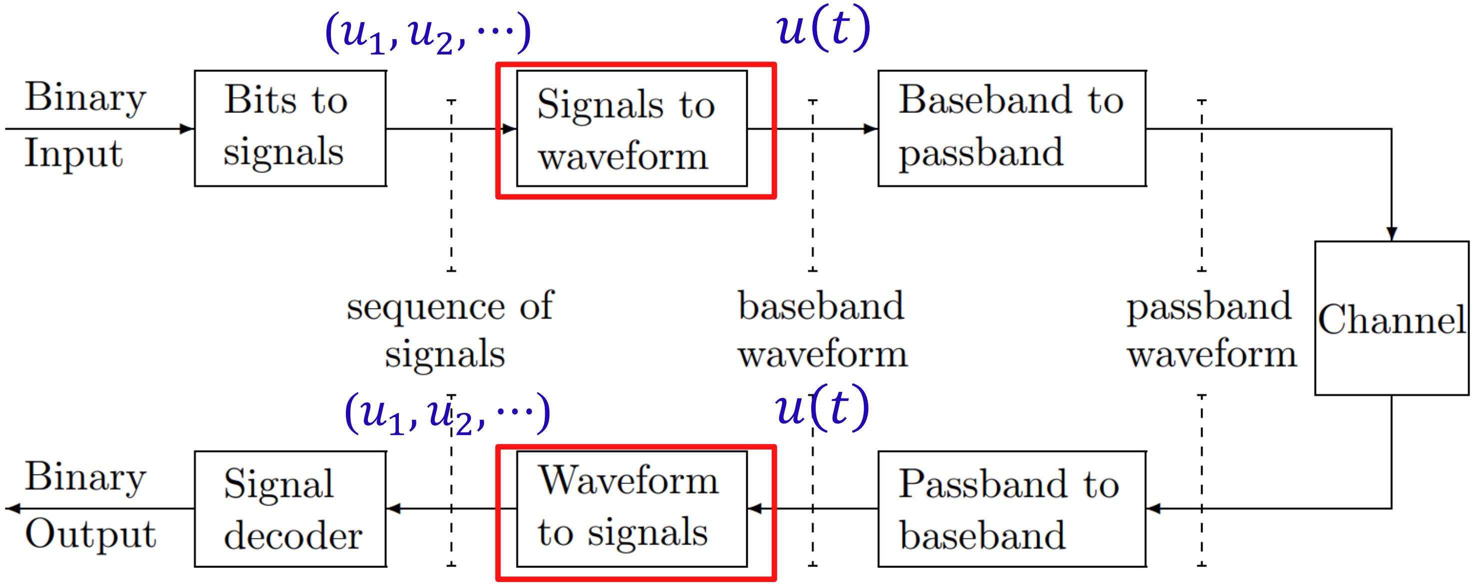
$$s_k = \int_{-\infty}^{\infty} s(t)\phi_k^*(t)dt = \langle s(t), \phi_k(t) \rangle$$

Map Vector to Signal



$$s(t) = \sum_{k=1}^{N} s_k \phi_k(t)$$

Modulation and Demodulation



$$u(t) = \sum_k u_k p(t - kT_s) \longrightarrow (u_1, u_2, \cdots)$$

- Step 1: Binary digits $\rightarrow$ A sequence of signal[1](numbers)
- Step 2: A sequence of numbers $\rightarrow$ Waveforms
- Step 3: baseband $\rightarrow$ pass band

Example: Modulation and Demodulation

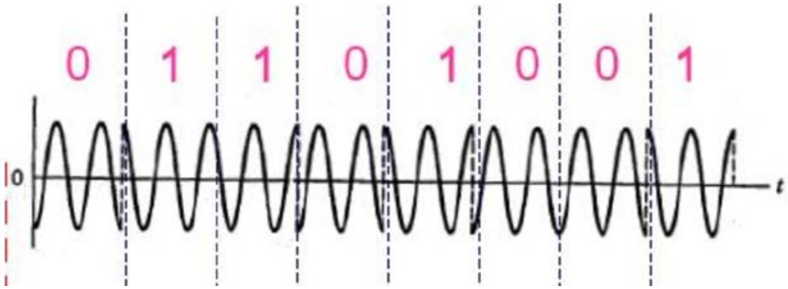
1. Map information bits into $(u_1, u_2, \cdots)$.
   - (0010..) → (+1,+1,-1,+1,...)
2. Map sequence of numbers (or signals) into a waveform
   - $(u_1, u_2, \cdots)$ → $u_1 p(t), u_2 p(t-T_s), u_3 p(t-3T_s), \ldots$
   - $u(t) = \sum_k u_k p(t-kT_s)$, $u_k = \{+1, -1\}$, T_s is the signal interval
   - $\{p(t-kT_s)\}$: baseband pulse waveform (explain later)
3. Map a baseband waveform $u(t)$ into a passband waveform $x(t) = Re\left(u(t)e^{j2\pi f_c t}\right) = u(t)\cos 2\pi f_c t$ (DSB-AM).

- Various Modulations: we can change
  - Amplitude: mapping {00,01,10,11} → -3,-1,+1,+3 (4PAM)
  - Phase: 0 → $p_1(t)$, 1 → $p_2(t) = p_1(t)e^{j\pi}$ (phase-shift-keying)
  - Frequency: 0 → $p_1(t) = a\cos(2\pi\Delta f t)$, 1 → $p_2(t) = a\cos(4\pi\Delta f t)$ (frequency-shift-keying)

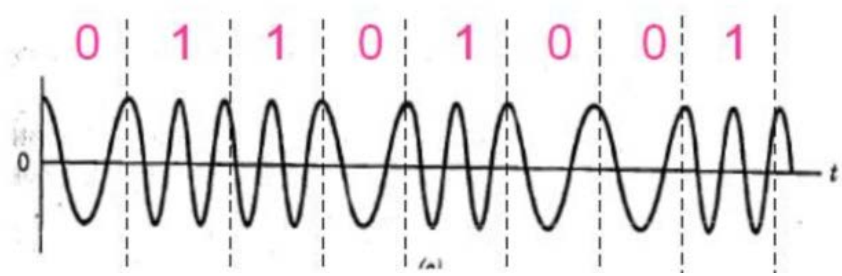
Example: Modulation and Demodulation

1. Map information bits into $(u_1, u_2, \cdots)$.
   – (0010..) → (+1,+1,-1,+1,...)
2. Map sequence of numbers (or signals) into a waveform
   – $(u_1, u_2, \cdots) \rightarrow u_1 p(t), u_2 p(t-T_s), u_3 p(t-3T_s), \ldots$
   – $u(t) = \sum_k u_k p(t-kT_s)$, $u_k = \{+1, -1\}$, T_s is the signal interval
   – $\{p(t-kT_s)\}$: baseband pulse waveform (explain later)
3. Map a baseband waveform $u(t)$ into a passband waveform $x(t) = Re(u(t)e^{j2\pi f_c t})$.

- Various Modulations: we can change



PSK



FSK

Pulse Amplitude Modulation (PAM)

Bit rate: R bps

Bit interval: $T_b = 1/R$ s.

0	1	1	0	1	0	1	1	1	0	1	0

a_4	a_3	a_7	a_3

b bits blocks (b=3)

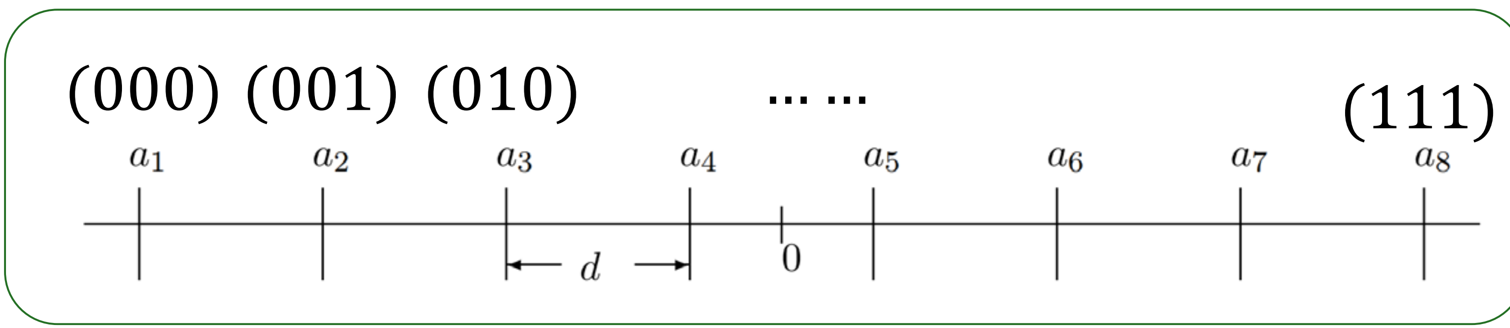
$M = 2^b$

$\downarrow$

signal constellation $\mathcal{A} = \{a_1, a_2, \ldots, a_M\}$ of real numbers

Signal (Symbol) Rate: $R_s = R/b$ signals/s

Signal (Symbol) interval: $T_s = \frac{1}{R_s} = bT_b$ s



Pulse Amplitude Modulation (PAM)

0	1	1	0	1	0	1	1	1	0	1	0

a_4	a_3	a_7	a_3

$\downarrow$

$$u(t) = \sum_k u_k p(t - kT_s)$$

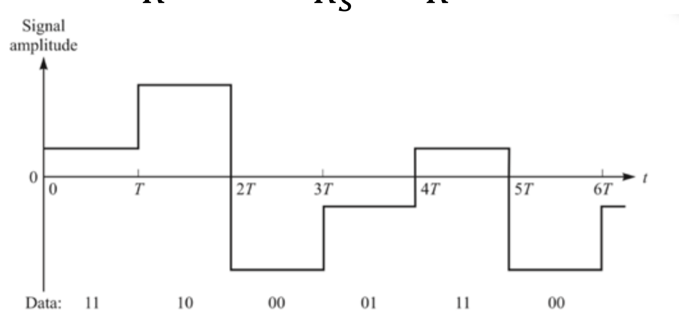
b bits blocks (b=3)

$M = 2^b$

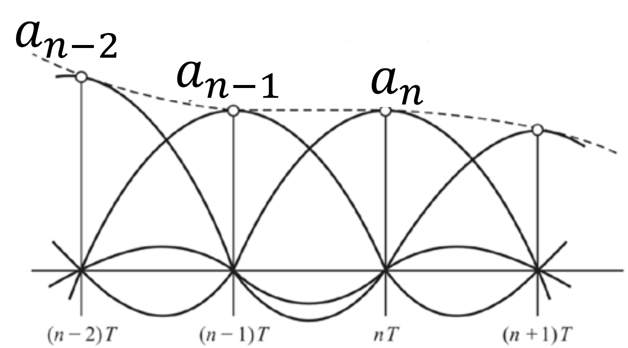
$\downarrow$

signal constellation $\mathcal{A} = \{a_1, a_2, \dots, a_M\}$, $a_j \in R$

$T_b = \frac{1}{R}, T_s = \frac{1}{R_s} = \frac{b}{R} = bT_b$ (interval between signals)



(a) Baseband PAM signal



{00,01,10,11} → -3,-1,+1,+3

Pulse Amplitude Modulation (PAM)

$$\underbrace{0\,1\,1}\;\underbrace{0\,1\,0}\;\underbrace{1\,1\,1}\;\underbrace{0\,1\,0}$$

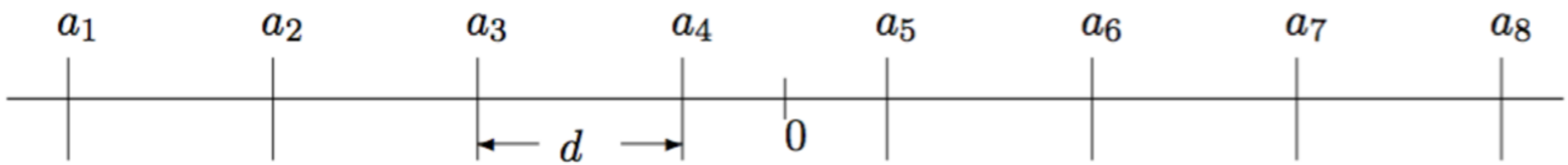
$U_1, U_2, \ldots$ $\quad$ $\boxed{a_4}\boxed{a_3}\boxed{a_7}\boxed{a_3}$

$$\downarrow$$

$$u(t) = \sum_k u_k p(t - kT_s)$$

- Assume: $U_1, U_2, \ldots,$ iid $\sim P(U_k = a_m) = p_m$ (k is time index, $m \in \{1, \ldots, M\}$ is index in $\mathcal{A}$)
- Energy $E_m = \int |a_m p(t)|^2 dt$
- Average signal energy $E_{\text{avg}} = \sum_{m=1}^{M} p_m E_m$
- Average energy per bit $E_{\text{bavg}} = \frac{E_{\text{avg}}}{b}$, power $P_{\text{avg}} = E_{\text{bavg}}/T_b$
- Binary PAM: $b = 1$; M-PAM : $M = 2^b$

PAM Signal Constellation



- Assume incoming bits are equiprobable RVs.
- Each signal $U_k = a_m$ is equiprobable in $\mathcal{A}$

$$\mathcal{A} = \{\frac{-d(M-1)}{2}, \frac{-d(M-3)}{2}, \ldots, \frac{-d}{2}, \frac{d}{2}, \ldots, \frac{d(M-3)}{2}, \frac{d(M-1)}{2}\}$$

- The choose of values in $\mathcal{A}$ is similar to finding representation points in quantization problem
- Average power per signal: E_{avg}?

PAM Signal Constellation

$$\mathcal{A} = \{\frac{-d(M-1)}{2}, \frac{-d(M-3)}{2}, \ldots, \frac{-d}{2}, \frac{d}{2}, \ldots, \frac{d(M-3)}{2}, \frac{d(M-1)}{2}\}$$

For symbol $U_k = a_m \in \mathcal{A} \Rightarrow$ send $u_k(t) = a_m p(t - kT)$,

$$E_m = \int_{-\infty}^{\infty} a_m^2 p^2(t-kT)dt = a_m^2 E_p$$

Thus,

$$\begin{aligned} E_{\text{avg}} &= \sum_{m=1}^{M} \left(\frac{1}{M} \cdot a_m^2 E_p \right) \\ &= 2\frac{E_p}{M} \left(\frac{d}{2} \right)^2 \left(1^2 + 3^2 + \ldots + (M-1)^2 \right) \\ &= \frac{d^2}{2} \frac{E_p}{M} \times \frac{M(M^2-1)}{6} \\ &= \frac{d^2(M^2-1)E_p}{12} \end{aligned}$$

PAM Signal Constellation

The signal energy, i.e., the mean square signal value assuming equiprobable signals is:

$$E_{\text{avg}} = \frac{d^2(M^2 - 1)E_p}{12}$$

- E_{avg} increases as d^2 and M^2
- d is determined by the noise
- Errors is reception are primarily dues to noise exceeding $d/2$
- For many channels, the noise is independent of the signal, which explains the standard equal spacing between signal constellation values.

Quadrature Amplitude Modulation

$$u(t) = \sum_k u_k p(t - kT) = \sum_k u_k(t)$$

- Segment the incoming binary symbols into b-bits blocks.
- Map the b-bits into a signal constellation $\mathcal{A} = \{a_1, \ldots, a_M\}$, $a_j \in \mathbb{C}$
- $u(t)$ is a complex baseband waveform
- Convert $u(t)e^{j2\pi f_c t}$ to a real waveform

$$u(t) = \sum_k u_k p(t - kT_s)$$

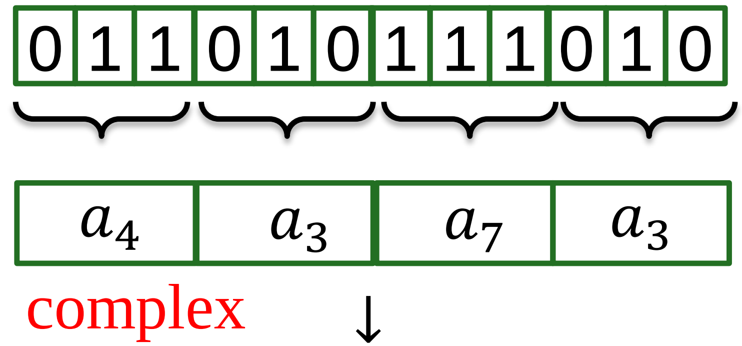
$$\begin{aligned} x(t) &= 2\text{Re}[u(t)e^{j2\pi f_c t}] \\ &= 2\text{Re}[u(t)] \cos(2\pi f_c t) - 2\text{Im}[u(t)] \sin(2\pi f_c t) \\ &= u(t)e^{j2\pi f_c t} + u^*(t)e^{-j2\pi f_c t} \end{aligned}$$

- QAM solves the frequency waste problem of DSB
- When $u(t)$ is real, QAM reduces to DSB PAM
- The factor of 2 is an arbitrary scale factor, and can be left out.

QAM Constellation

- Bit rate: R bps
- Segment binary b bits, map into complex numbers $u_k \in \mathcal{A}$, $|\mathcal{A}| = M = 2^b$.
- Signal (symbol) rate: $R_s = R/b$.
- Standard QAM Constellation
  - Cartesian product: $\mathcal{A} \times \mathcal{B}$, e.g.,

$$\mathcal{A} = \{1, 2\}, \mathcal{B} = \{3, 4\}$$
$$\mathcal{A} \times \mathcal{B} = \{(1,3), (1,4), (2,3), (2,4)\}$$
$$\mathcal{B} \times \mathcal{A} = \{(3,1), (3,2), (4,1), (4,2)\}$$



$$u(t) = \sum_k u_k p(t - kT_s)$$

QAM Constellation

- Standard QAM Constellation
  - A standard $(\sqrt{M}, \sqrt{M})$-QAM signal set is the Cartesian product of two $\sqrt{M}$-PAM set, i.e.,

$$\mathcal{A} = \{(a' + ja'')|a' \in \mathcal{A}', a'' \in \mathcal{A}'\}$$

where $\mathcal{A}' = \{\frac{-d(\sqrt{M}-1)}{2}, \ldots, \frac{-d}{2}, \frac{d}{2}, \ldots, \frac{d(\sqrt{M}-1)}{2}\}$

  - It is a square array signal points located as below for $M = 16$

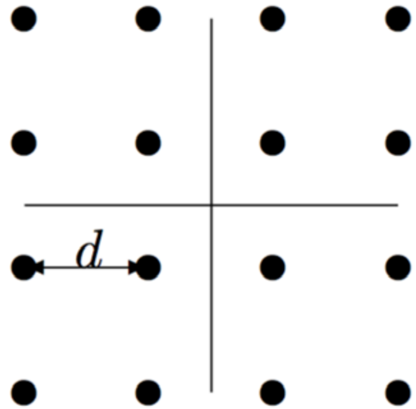
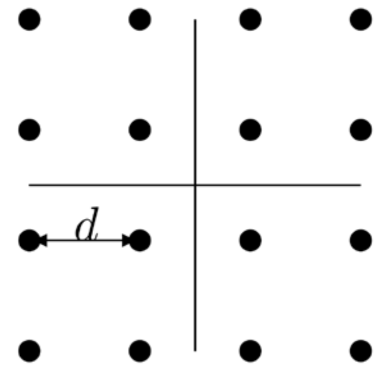


Figure: Standard 16QAM constellation [Gallagar'Book]

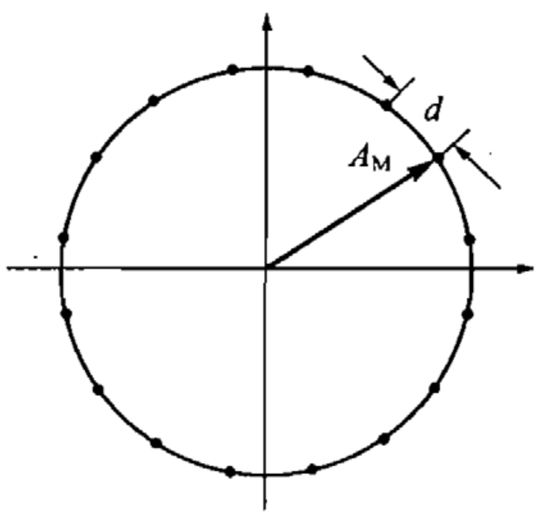
QAM Constellation



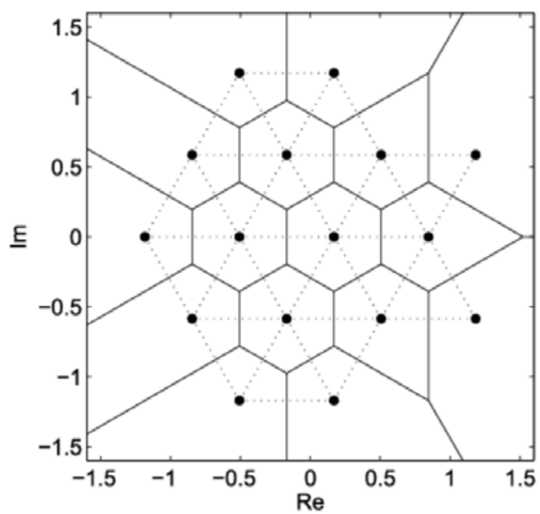
- Standard QAM Constellation
  - Average energy per 2D signal:

$$E_s = \frac{d^2\left(M'^2-1\right)E_p}{6} = \frac{d^2(M-1)E_p}{6}$$

- Q: How to arrange the signal points
  - Choose constellations that minimize Es given d and M.
  - MPSK



(b) 16PSK



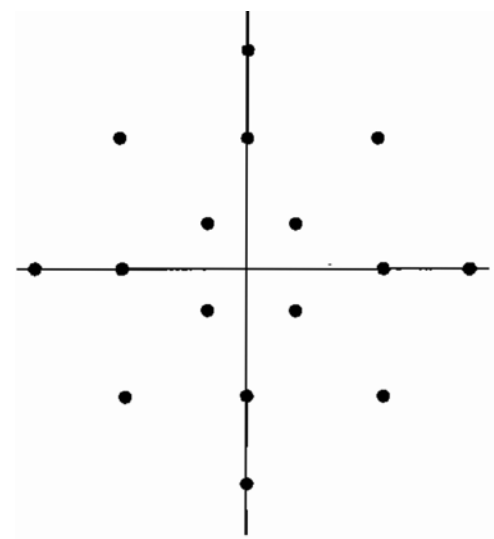
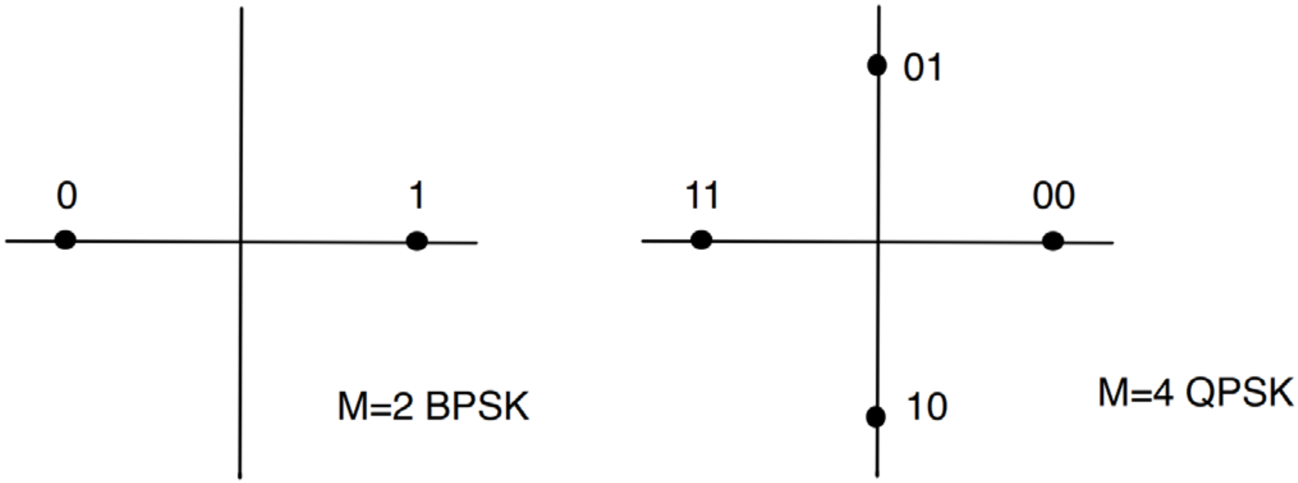


图 8－5　改进的 16QAM 方案

QAM Constellation

- PSK: Special Case of QAM
  - $u_k \in \mathcal{A}$

$$\mathcal{A} = \{e^{j\frac{2\pi(m-1)}{M}}, \text{ for } m = 1, \ldots, M\}$$



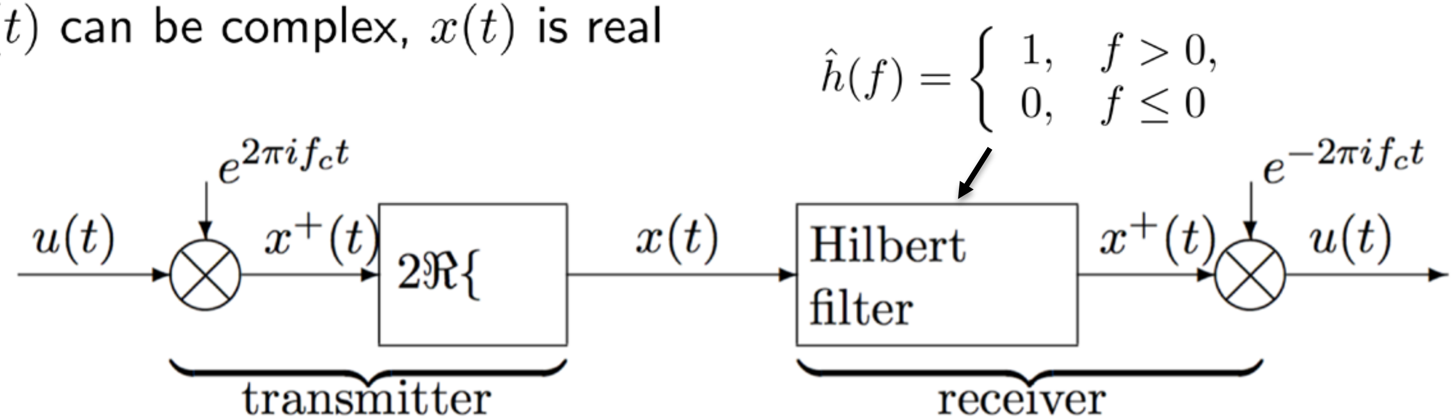
- Signal points have same amplitude
- PSK is rarely used for large M (signal points are very closed)
- Combing PSK and PAM

QAM: baseband to passband

Shift complex $u(t)$ to f_c, then add complex conjugate to form real $x(t)$

$$\begin{aligned} x(t) &= u(t)e^{j2\pi f_c t} + u^*(t)e^{-j2\pi f_c t} \\ &= x^+(t) + (x^+(t))^* \\ &= 2\mathrm{Re}[u(t)e^{j2\pi f_c t}] \end{aligned}$$

- Baseband Bandwidth: $B_b < f_c$
  - $u(t)$ can be complex, $x(t)$ is real



  - This is nice for analysis, but not usually so for implenmentation

QAM: baseband to passband

- Easier way

$$u(t) = \sum_k u_k p(t - kT)$$

$$x(t) = 2\text{Re}[u(t)e^{j2\pi f_c t}]$$

$$= 2\text{Re}[u(t)]\cos(2\pi f_c t) - 2\text{Im}[u(t)]\sin(2\pi f_c t)$$

Assume $p(t)$ is real

$$\text{Re}[u(t)] = \sum_k \text{Re}[u_k]p(t - kT)$$

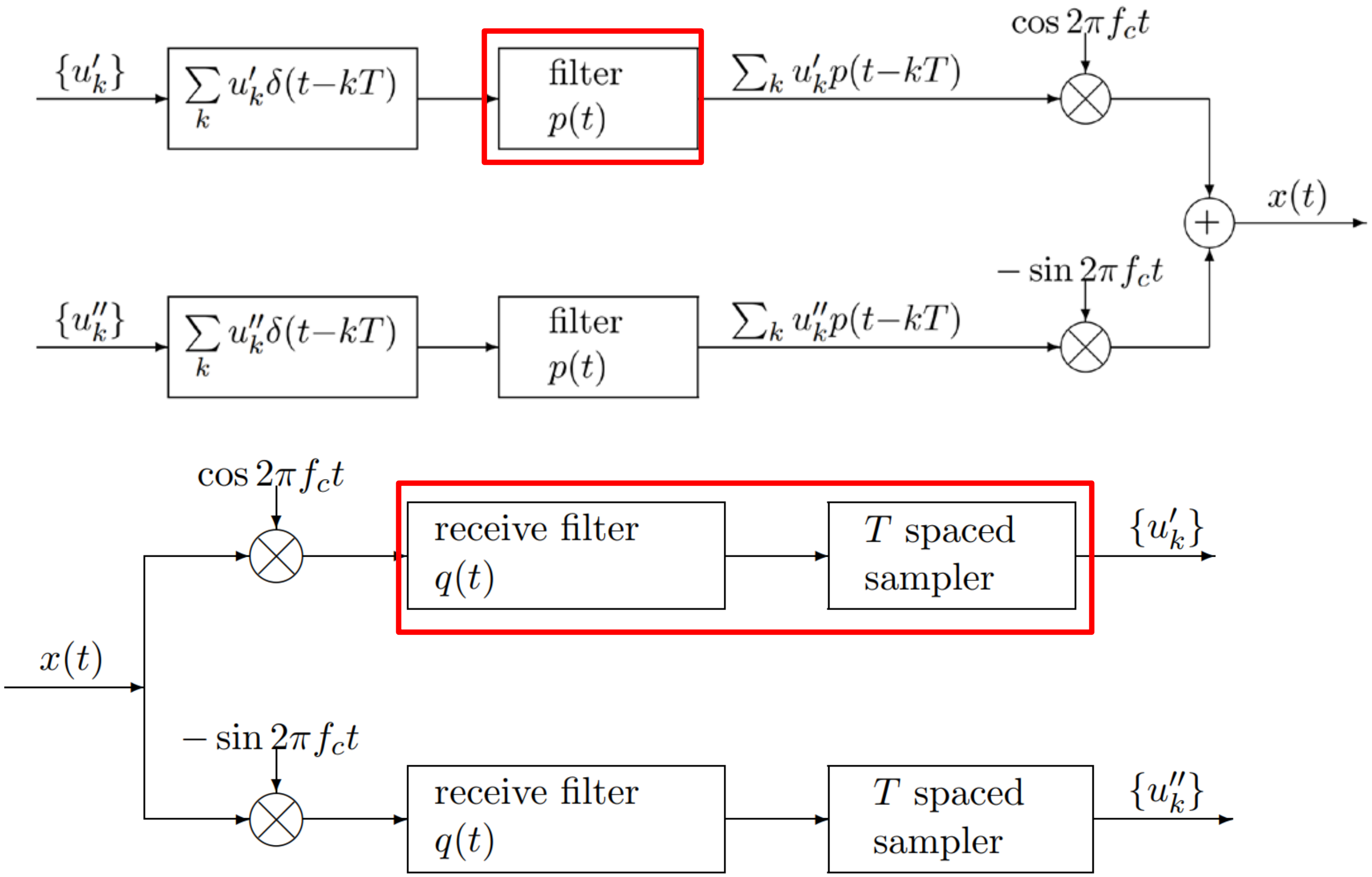
$$\text{Im}[u(t)] = \sum_k \text{Im}[u_k]p(t - kT)$$

with $u'_k = \text{Re}[u_k]$ and $u''_k = \text{Im}[u_k]$

$$x(t) = 2\left(\sum_k u'_k p(t - kT)\right)\cos 2\pi f_c t - 2\left(\sum_k u''_k p(t - kT)\right)\sin 2\pi f_c t$$

QAM: baseband to passband

- Easier way



q(t) should be chosen to such that $\hat{g}(f) = \hat{p}(f)\hat{q}(f)$ satisfy the Nyquist criterion. (Explained Later)

QAM: Implementation

PAM is a special case of QAM

Transmitter:

- Binary digits $\overset{b \text{ bits}}{\Longrightarrow}$ symbols $\overset{\text{Constellation}}{\Longrightarrow}$ $\boxed{\text{complex signals} \overset{\text{multiply } p(t)}{\Longrightarrow} u(t)}$ $\overset{\text{shift to } f_c}{\Longrightarrow}$ $u(t)e^{j2\pi f_c t}$ $\overset{\text{add } u^*(t)e^{-j2\pi f_c t}}{\Longrightarrow}$ $x(t)$

Receiver:

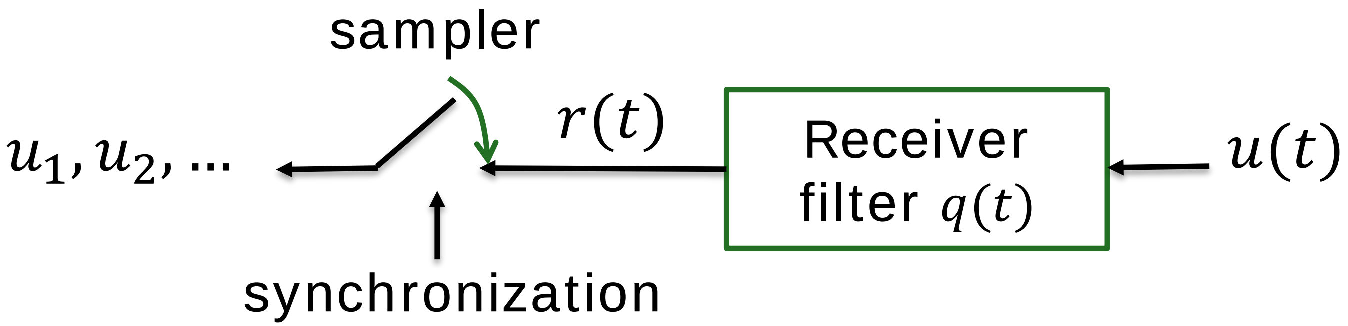
- $x(t)$ $\overset{\text{positive part}}{\Longrightarrow}$ $u(t)e^{j2\pi f_c t}$ $\overset{\text{remove} f_c}{\Longrightarrow}$ $\boxed{u(t) \overset{\text{filter and sampling}}{\Longrightarrow} \text{complex signals}}$ $\overset{\text{Constellation}}{\Longrightarrow}$ symbols $\overset{b \text{ bits}}{\Longrightarrow}$ Binary digits

Q: How to choose pulse $p(t)$ and receive filter $q(t)$, so that there is no inter-symbol interference (ISI)??

The Nyquist Criterion

$$u(t) = \sum_k u_k p(t - kT)$$

- Assume no noise in channel
- Received baseband waveform u(t), retrieve the signals $\{u_1, u_2, \ldots\}$



- $r(t) = \int_{-\infty}^{\infty} u(\tau) q(t - \tau) d\tau$ is sampled as $r(0), r(T), \ldots$
- Objective: choose $p(t)$ and $q(t)$ so that $r(kT) = u_k$ (No ISI)

The Nyquist Criterion

$$r(t) = \int u(\tau)q(\tau - t)d\tau = \int \sum_k u_k p(\tau - kT)q(t-\tau)d\tau$$

$$= \sum_k u_k g(t - kT) \quad \text{where } g(t) = p(t) * q(t)$$

- While ignoring noise, r(t) is determined by g(t)
- Ideal Nyquist: a wave form g(t) is ideal Nyquist with period T if $g(0) = 1 \; and \; g(kT) = 0$ for $k \neq 0$ (same property as sinc function)

$$r(jT) = \sum_k u_k g(jT - kT) \overset{g(t) \text{ is ideal Nyquist}}{=} u_j$$

- If g(t) is ideal Nyquist, then $r(kT) = u_k$ for all k. If g(t) is not ideal Nyquist, then $r(kT) \neq u_k$ for some u_k (intersymbol interference)

Recall: Aliasing Problem

Given a signal $u(t)$, its sampling approximation

$$s(t) = \sum_k u(kT)\,\mathrm{sinc}(\frac{t}{T} - k),$$

and the Fourier transform of $s(t)$ satisfies:

$$\hat{s}(f) = \mathcal{F}(s(t)) = \sum_k \hat{u}(f + \frac{k}{T})\mathrm{rect}(fT)$$

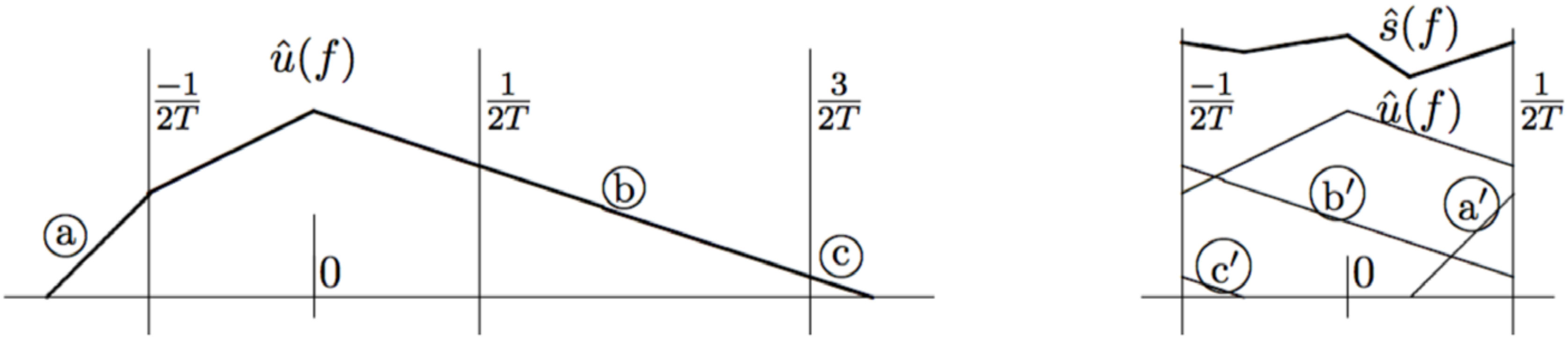


Figure: Aliasing when $1/(2T) < B_b$ [Gallagar'Book]

The Nyquist Criterion

Let $s(t)$ be signal reconstructed by samples of $g(t)$:

$$s(t) = \sum_k g(kT)\,\mathrm{sinc}(\frac{t}{T} - k) \tag{11}$$

g(t) is idea Nyquist: $g(kT) = \delta(k)$, substitute it into (11), we have

$$s(t) = \mathrm{sinc}(\frac{t}{T}) \longrightarrow \hat{s}(f) = T\mathsf{rect}(fT) \tag{12}$$

From (11) and the aliasing theorem

$$\hat{s}(f) = \sum_k \hat{g}(f + \frac{k}{T})\mathsf{rect}(fT) \tag{13}$$

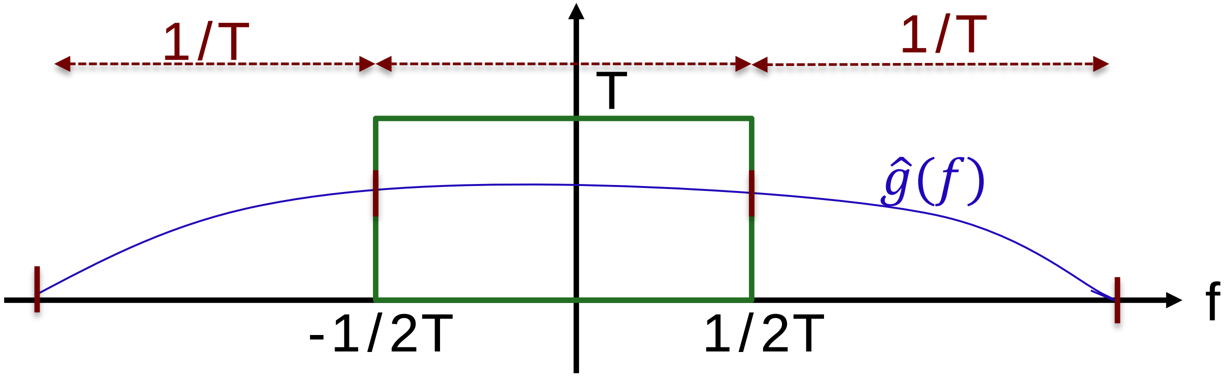
Thus from (12) and (13), we have $g(t)$ is ideal Nyquist iff

$$\sum_k \hat{g}(f + \frac{k}{T})\mathsf{rect}(fT) = T\mathsf{rect}(fT)$$

The Nyquist Criterion

- Nyquist Criterion

$$\boxed{\sum_k \hat{g}(f + \frac{k}{T})\text{rect}(fT) = T\text{rect}(fT)}$$



The Nyquist Criterion

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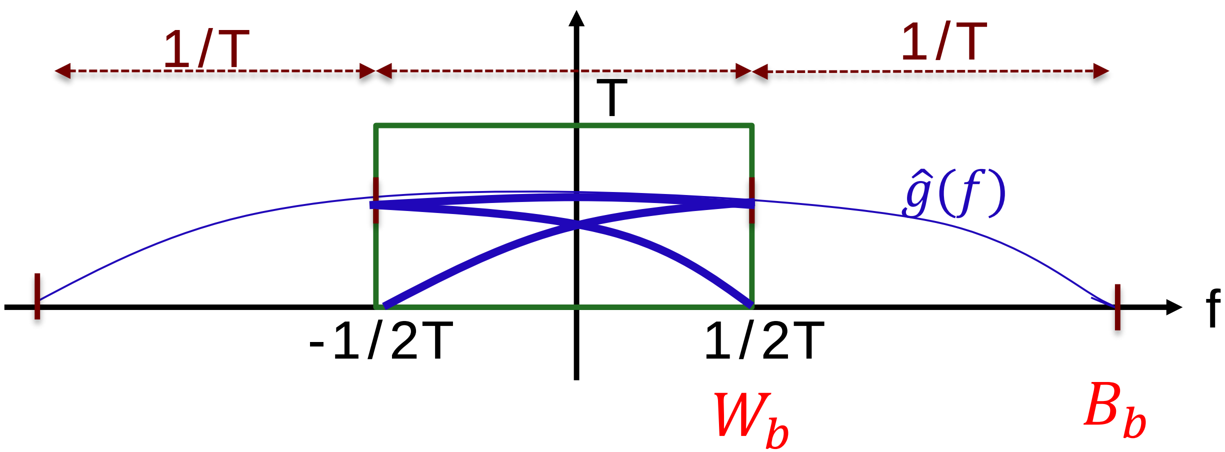
$$\boxed{\sum_k \hat{g}(f+\frac{k}{T})\text{rect}(fT) = T\text{rect}(fT)}$$

  - Signal Interval of g(t): T
  - Nyquist bandwidth: $W_b = \frac{1}{2T}$; Signal rate $R_s = \frac{1}{T} = 2W_b$
  - Actual baseband bandwidth: B_b ($\hat{g}(f) = 0, |f| > B_b$)



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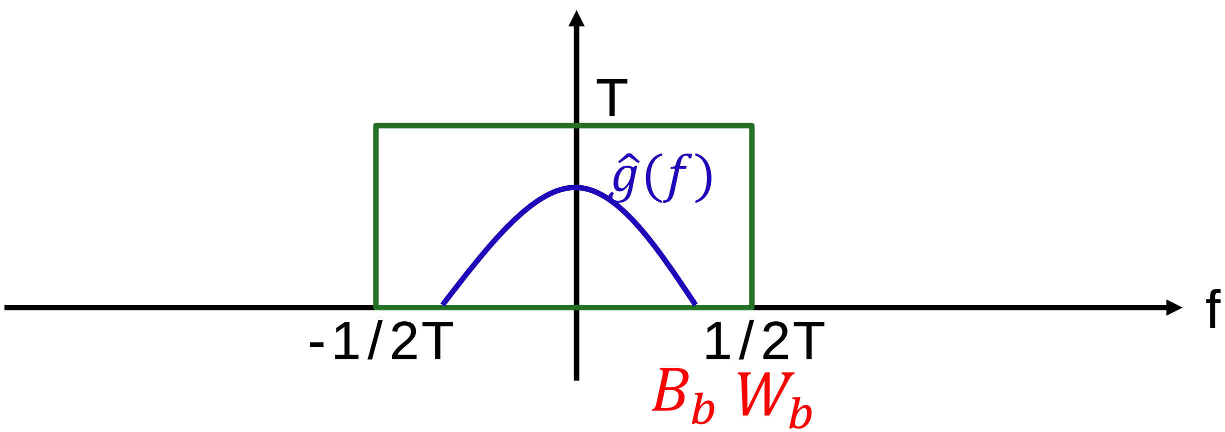
  - Actual baseband bandwidth: B_b ($\hat{g}(f) = 0, |f| > B_b$)
  - Case1: $B_b < W_b$, ISI is not avoidable



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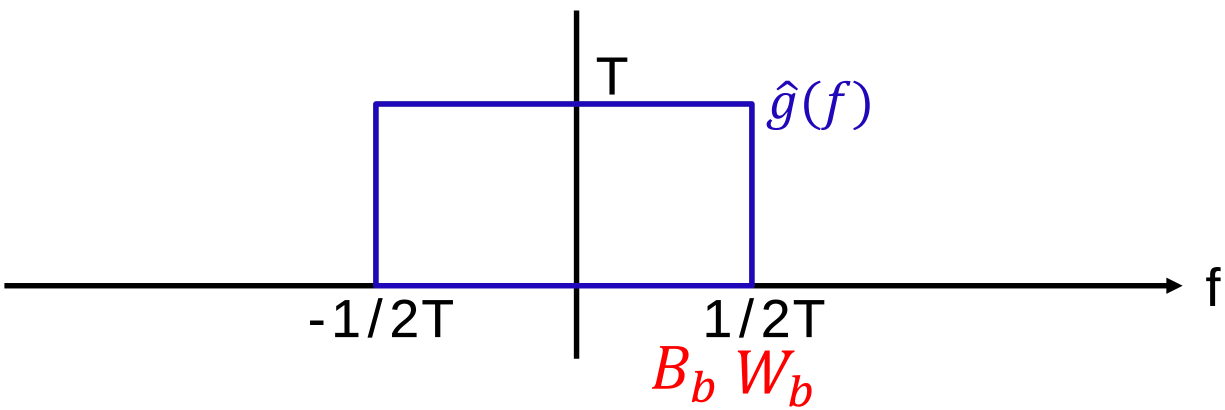
  - Actual baseband bandwidth: B_b ($\hat{g}(f) = 0, |f| > B_b$)
  - Case2: $B_b = W_b$

$$\hat{g}(f) = T\text{rect}(fT) \leftrightarrow g(t) = \text{sinc}(\frac{t}{T})$$



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  - Actual baseband bandwidth: B_b ($\hat{g}(f) = 0, |f| > B_b$)
  - Case3: $B_b > W_b$
    - If B_b is much larger than W_b, than W_b can be increased (T can be decreased), thus increasing the rate Rs at which signal can be transmitted.
    - g(t) should be chosen B_b exceed W_b by a relatively small amount.
    - $W_b \leq B_b < 2W_b$: Keep $\hat{g}(f)$ almost baseband limited to 1/(2T)

The Nyquist Criterion

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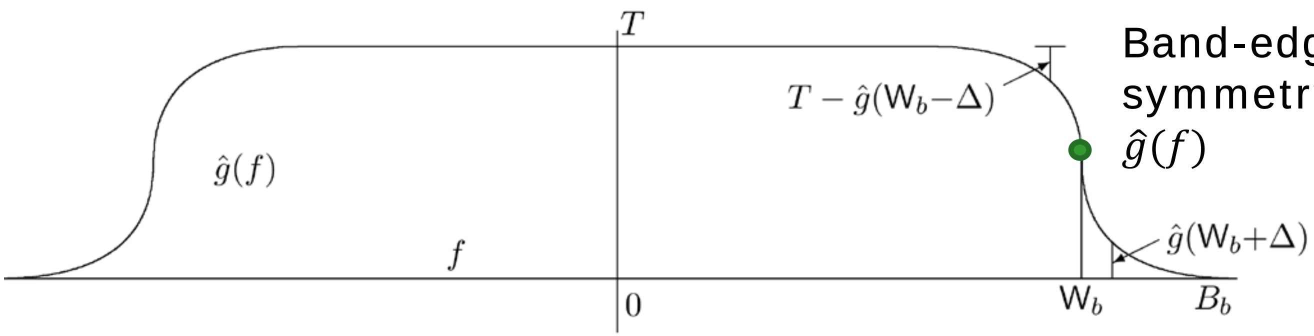
  - Actual baseband bandwidth: B_b ($\hat{g}(f) = 0, |f| > B_b$)
  - Case3: $W_b \leq B_b < 2W_b$
    - Rolloff factor: $\alpha = \frac{B_b}{W_b} - 1$
    - Tradeoff between rolloff and smoothness (slow time decay and bandwidth)



Band-edge symmetry for real $\hat{g}(f)$

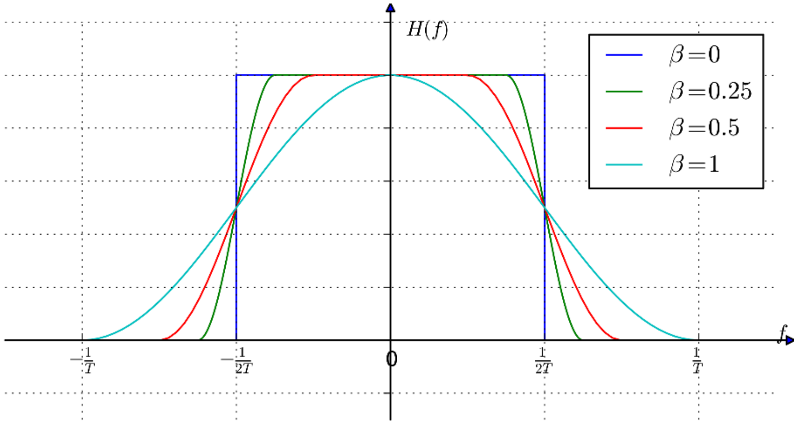
The Nyquist Criterion

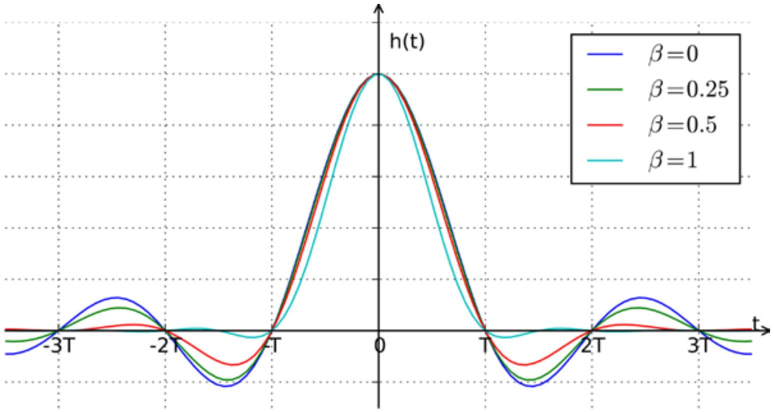
- Nyquist Criterion
  - PAM filters in practice often have raised cosine transform

$$\hat{g}_\alpha(f) = \begin{cases} T, & 0 \le |f| \le \frac{1-\alpha}{2T}; \\ T \cos^2\left[\frac{\pi T}{2\alpha}\left(|f| - \frac{1-\alpha}{2T}\right)\right], & \frac{1-\alpha}{2T} \le |f| \le \frac{1+\alpha}{2T}; \\ 0, & |f| \ge \frac{1+\alpha}{2T}. \end{cases}$$

  - The inverse transform of $\hat{g}_\alpha(f)$

$$\hat{g}_\alpha(t) = \operatorname{sinc}\left(\frac{t}{T}\frac{\cos(\pi\alpha\frac{t}{T})}{1-4\alpha^2t^2/T^2}\right)$$





The Nyquist Criterion

- Q: Restrict $\hat{g}(f)$ real and nonnegative, $\hat{g}(f) > 0$, how to choose $p(t)$ and $q(t)$, s.t. $\hat{p}(f)\hat{q}(f) = \hat{g}(f)$
- Choose $\hat{q}(f) = \hat{p}^*(f)$ → $|\hat{p}(f)| = |\hat{q}(f)| = \sqrt{\hat{g}(f)}$ and $q(t) = p^*(-t)$.
  - $p(t)$: square root of Nyquist;
  - $q(t)$: matched filter to $p(t)$
  - Same bandwidth for $\hat{p}(f)$, $\hat{q}(f)$ and $\hat{g}(f)$ (slightly larger than 1/2T), truncated in time to allow finite delay.
- Orthonormal Shifts:
  - Let $p(t) \in \mathcal{L}_2$ and $\hat{g}(f) = |\hat{p}(f)|^2$ satisfies the Nyquist criterion for T. Then $\{p(t-kT); k \in \mathbb{Z}\}$ is a set of orthonormal functions. Conversely, if $\{p(t-kT); k \in \mathbb{Z}\}$ is a set of orthonormal functions, then $|\hat{p}(f)|^2$ satisfies the Nyquist criterion.
  - Proof:

The Nyquist Criterion

$$g(t) = \int p(\tau)q(t-\tau)d\tau = \int p(\tau)p^*(\tau - t)d\tau$$

$$g(t) = \int p(\tau)p^*(\tau - t)d\tau$$

$$g(kT) = \int p(\tau)p^*(\tau - kT)d\tau$$

- Shift τ by jT for any integer j $(\tau = \tau - jT)$

$$\begin{aligned} g(kT) &= \int p(\tau - jT)p^*(\tau - (k+j)T)d\tau \\ &= \int p(t - jT)p^*(t - (k+j)T)dt \end{aligned}$$

- If $g(t)$ is idea Nyquist, then $g(kT) = 1$ for $k = 0$ and 0 otherwise. Thus,

$$g(kT) = \int p(t - jT)p^*(t - (k+j)T)dt = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

$\Longrightarrow$ $\{p(t - kT)\}$ is a set of orthonormal functions

The Nyquist Criterion

- if $\{p(t-kT); k \in \mathbb{Z}\}$ is a set of orthonormal functions, then let $\hat{q}(f) = \hat{p}^*(f)$, that is $q(t) = p^*(-t)$. We have

$$g(t) = \int p(\tau)q(t-\tau)d\tau = \int p(\tau)p^*(\tau - t)d\tau$$

$$g(kT) = \int p(\tau)p^*(\tau - kT)d\tau$$

- Shift τ by jT for any integer j $(\tau = \tau - jT)$

$$\begin{aligned} g(kT) &= \int p(\tau - jT)p^*(\tau - (k+j)T)d\tau \\ &= \int p(t - jT)p^*(t - (k+j)T)dt \end{aligned}$$

$$g(kT) = \int p(t-jT)p^*(t-(k+j)T)dt = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{for } k \neq 0 \end{cases}$$

$\implies$ $g(t)$ is idea Nyquist

The Nyquist Criterion

- Orthonormal Shifts → recover u_k from vector space perspective

$$\hat{s}(t) = \sum_{k=1}^{N} s_k \phi_k(t)$$

$$s_k = \int_{-\infty}^{\infty} s(t)\phi_k^*(t)dt = \langle s(t), \phi_k(t) \rangle$$

- Since $\{p(t - kT)\}$ is an orthonormal basis and $u(t) = \sum u_k p(t - kT)$ is the orthonormal expansion, thus

$$u_k = \langle u(t), p(t - kT) \rangle$$

- Retrieving u_k corresponding to projecting $u(t)$ onto $p(t - kT)$
- Note this projection is done by $u(t) * q(t)$ and then sampling at time kT

Summarize

- Bit sequence to symbol sequence
  - PAM: b-tuple of bits → one of $M = 2^b$ signal points in $\mathbb{R}^1$
  - QAM: b-tuple of bits → one of $M = 2^b$ signal points in $\mathbb{C}^1$
  - Signal constellation: small average energy with a large distance between points
  - Standard Mapping
- Symbol sequences to baseband waveform
  - $u(t) = \sum_k u_k p(t - kT)$
  - How to choose p(t): $\hat{g}(f) = \hat{p}(f)\hat{q}(f)$ must satisfy Nyquist criterion to avoid ISI

$$\boxed{\text{if } g(kT) = 1 \text{ for } k = 0, \text{ and } 0 \text{ for } k \in \mathbb{Z} \neq 0}$$

$$\boxed{\sum_k \hat{g}(f + \frac{k}{T})\text{rect}(fT) = T\text{rect}(fT)}$$

Summarize

- Symbol sequences to baseband waveform

$$\boxed{\sum_k \hat{g}(f+\frac{k}{T})\mathrm{rect}(fT) = T\mathrm{rect}(fT)}$$

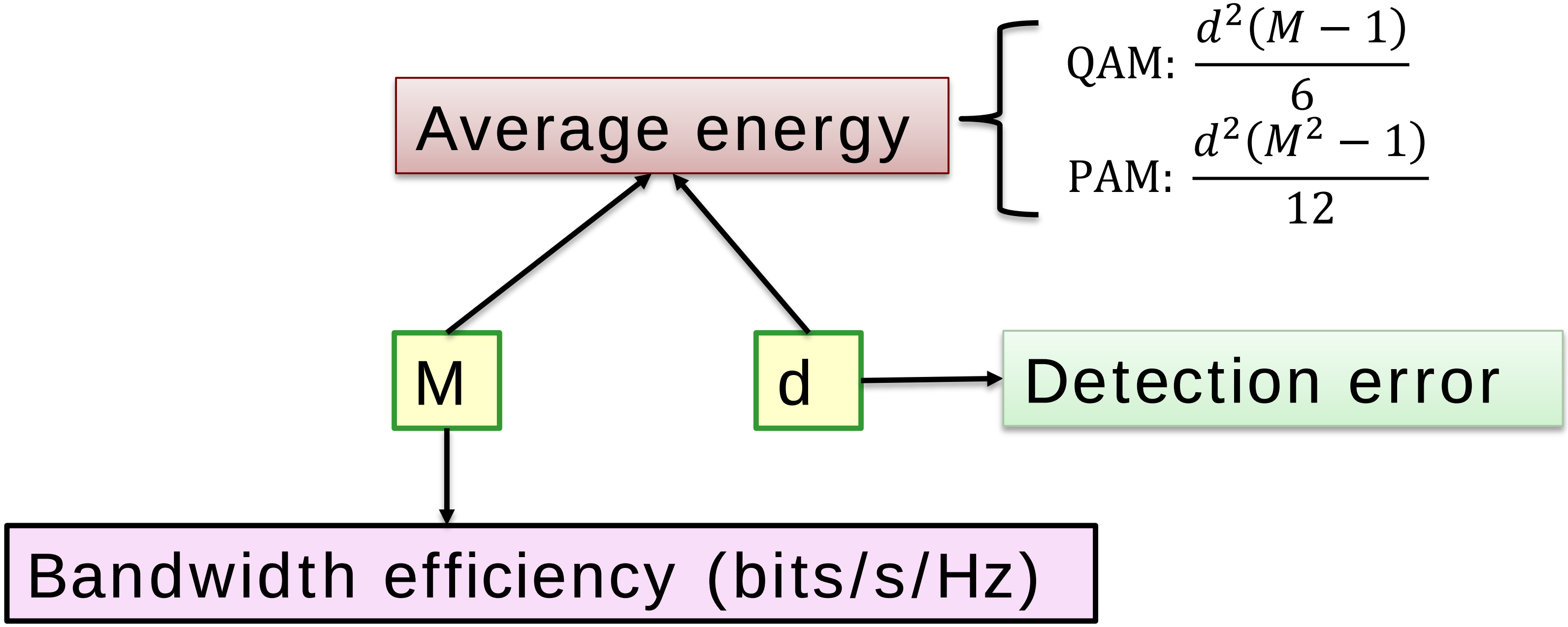
  - $\hat{g}(f)$: almost baseband limited to 1/2T; smooth such that g(t) goes to zero quickly
  - If $\hat{g}(f) > 0$: $|\hat{p}(f)| = |\hat{q}(f)| = \sqrt{\hat{g}(f)}$. In this case, $\{p(t-kT); k \in \mathbb{Z}\}$ is a set of orthonormal functions
- Baseband to passband: $x(t) = 2R\left(u(t)e^{j2\pi f_c t}\right)$

$$x(t) = 2\left(\sum_k u'_k p(t-kT)\right)\cos 2\pi f_c t - 2\left(\sum_k u''_k p(t-kT)\right)\sin 2\pi f_c t$$

- Tradeoff: Bandwidth Efficiency (M), Energy (M,d), Detection Error (d)

Summarize

- Tradeoff: Bandwidth Efficiency (M), Energy (M,d), Detection Error (d)



Bandwidth=1/(2T)=Rs/2=R/(2logM)

M ↗ Bandwidth ↘ (same R), Bandwidth efficiency ↗

Thanks for your kind attention!

Questions?

Summarize

- Bandwidth Efficiency
  - $\mu = \frac{R}{B} = \frac{\log_2 MR_s}{B} = \log_2 M \frac{1}{B_b T}$ (# of bits/s/Hz)
  - $\mu_s = \frac{1}{B_b T} \times a$ (# of real symbols/s/Hz)
  - PAM:
    - BB $\mathrm{B_b} \approx \frac{1}{2T}$, PB $\mathrm{B_b} \approx \frac{1}{T}$ (DSB, half of bandwidth with information)
    - $\mu_s = 2$ at BB.
    - $\mu = 2\log_2 \mathrm{M}$ at BB.
  - QAM:
    - BB $\mathrm{B_b} \approx \frac{1}{2T}$, PB $\mathrm{B_b} \approx \frac{1}{T}$ (DSB)
    - $\mu_s = 2$ at PB.
    - $\mu = \log_2 \mathrm{M}$ at PB.