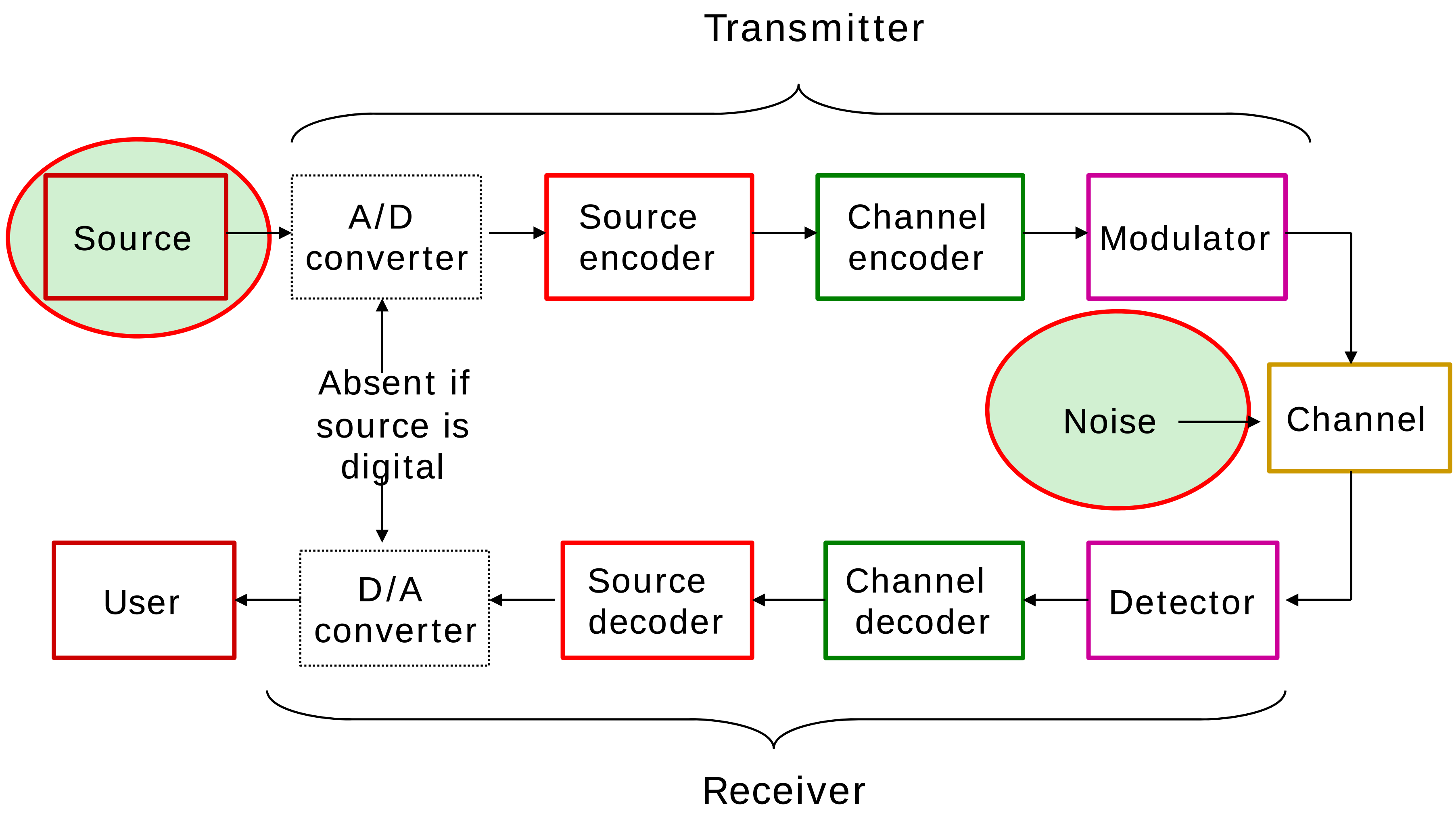
EE140 Introduction to Communication Systems

Lecture 3

Instructor: Prof. Lixiang Lian

ShanghaiTech University, Fall 2022

Architecture of a (Digital) Communication System



Contents

- Random signals
  - Review of probability and random variables
  - Random processes: basic concepts
  - Gaussian white processes

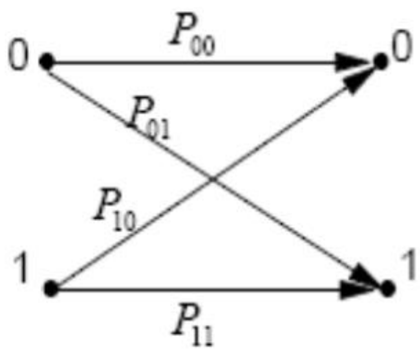
Probability

- Consider a binary communication system

prior

$$P(0) = 0.3$$

$$P(1) = 0.7$$



$$P_{01} = P(\text{receive } 1 \mid \text{sent } 0) = 0.01$$
$$P_{00} = P(\text{receive } 0 \mid \text{sent } 0) = 0.99$$
$$P_{10} = P(\text{receive } 0 \mid \text{sent } 1) = 0.1$$
$$P_{11} = P(\text{receive } 1 \mid \text{sent } 1) = 0.9$$

Maximum posterior probability estimates

$$P(\text{send } 0 | \text{receive } 1) = ?$$
$$P(\text{send } 1 | \text{receive } 1) = ?$$
$$P(\text{send } 1 | \text{receive } 0) = ?$$
$$P(\text{send } 0 | \text{receive } 0) = ?$$

$$\arg\max P(X = x \mid Y = y) = \frac{P(X = x, Y = y)}{P(Y = y)}$$

Conditional Probability

- Consider two events A and B
- Conditional probability P(A|B)
- Joint probability

$$
\begin{aligned}
P(AB) &= P(A \cap B) \\
&= P(B)P(A|B) = P(A)P(B|A)
\end{aligned}
$$

- A and B are said statistically independent iff

$$
P(AB) = P(A)P(B) \Rightarrow \begin{cases} P(A|B) = P(A) \\ P(B|A) = P(B) \end{cases}
$$

Law of Total Probability

- Let A_j, $\mathrm{j} = 1,2,\ldots,n$ be mutually exclusive events with $A_\mathrm{i} \cap A_j = 0, \forall i \neq j$ and $\bigcup_{i=1}^{\mathrm{n}} A_i = S$
- For any event B, we have

$$P_B = \sum_{j=1}^{\mathrm{n}} P(B \cap A_j) = \sum_{j=1}^{n} P(B/A_j)P(A_j)$$

Venn diagram

- Bayes' Theorem

$$P(A_\mathrm{i}|B) = \frac{P(A_i, B)}{P(B)}$$

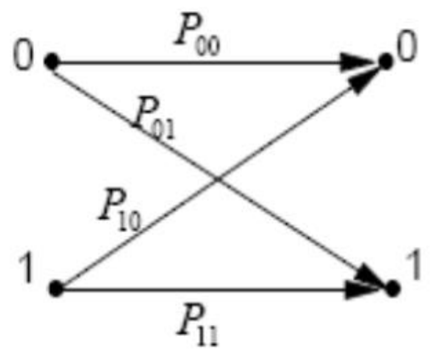
$$= \frac{P(B|A_i)P(A_i)}{\sum_{j=1}^{n} P(B|A_j)P(A_j)}$$

Example

- Consider a binary communication system

$P(0) = 0.3$

$P(1) = 0.7$



$$P_{01} = P(\text{receive } 1 \mid \text{sent } 0) = 0.01$$

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$$P_{11} = P(\text{receive } 1 \mid \text{sent } 1) = 0.9$$

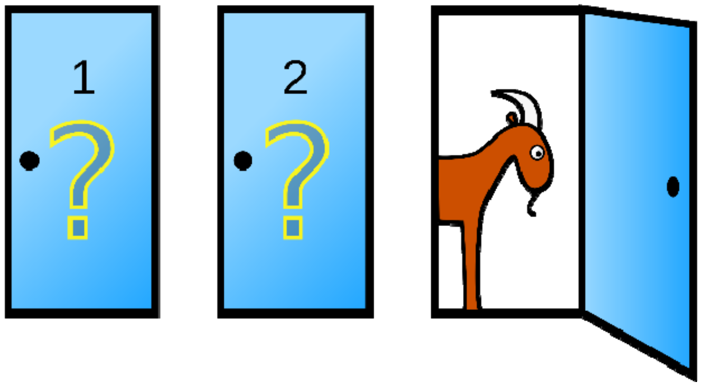
- What is the probability that the output of this channel is 1?

$$P(Y=1) = P(Y=1 \mid X=0)P(X=0) + P(Y=1 \mid X=1)P(X=1)$$

- Assuming that we have observed a 1 at the output, what is the probability that the input to the channel was a 1?

$$P(X=1 \mid Y=1) = \frac{P(Y=1 \mid X=1)P(X=1)}{P(Y=1)}$$

Game



In search of a new car, the player picks a door, say *1*. The game host then opens one of the other doors, say *3*, to reveal a goat and offers to let the player switch from door *1* to door *2*.

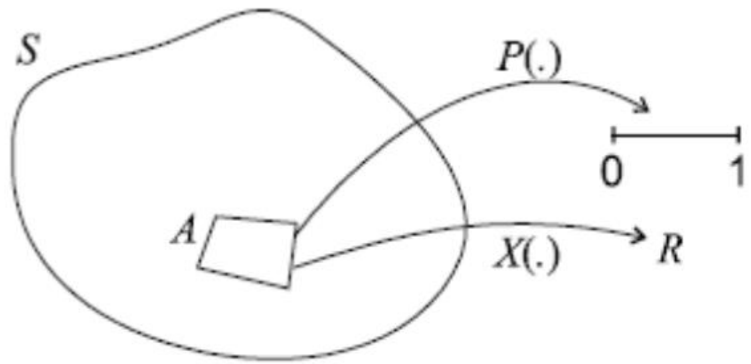
Random Variables (r.v.)

- A r.v. is a mapping from the sample space *S* to the set of real numbers.

$$X(.): A \subset S \rightarrow x \in R \qquad X(A) = x$$

$$X(.) : A \in S \rightarrow x \in R$$

$$X(A) = x$$



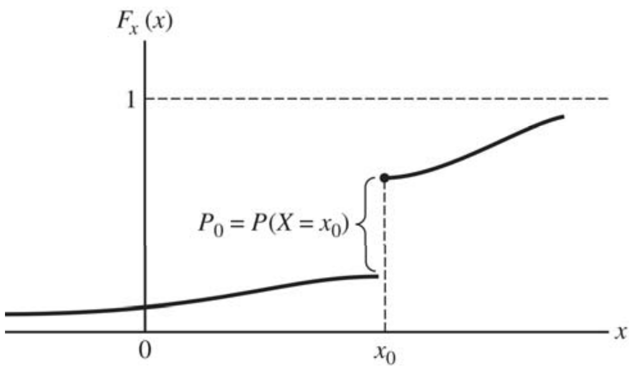
- A r.v. may be
  - Discrete-valued: range is finite (e.g. {0,1}), or countable infinite (e.g. {1,2,3 ...})
  - Continuous-valued: range is uncountable infinite (e.g. *R*)

Cumulative Distribution Function (CDF)

- The CDF of a r.v. X, is

$$F_X(x) \triangleq P(X \leq x)$$

- Key properties of CDF
  - $0 \leq F_X(x) \leq 1$ with $F_X(-\infty) = 0$, $F_X(\infty) = 1$
  - $F_X(x)$ is a non-decreasing function of x
  - $F_X(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1)$



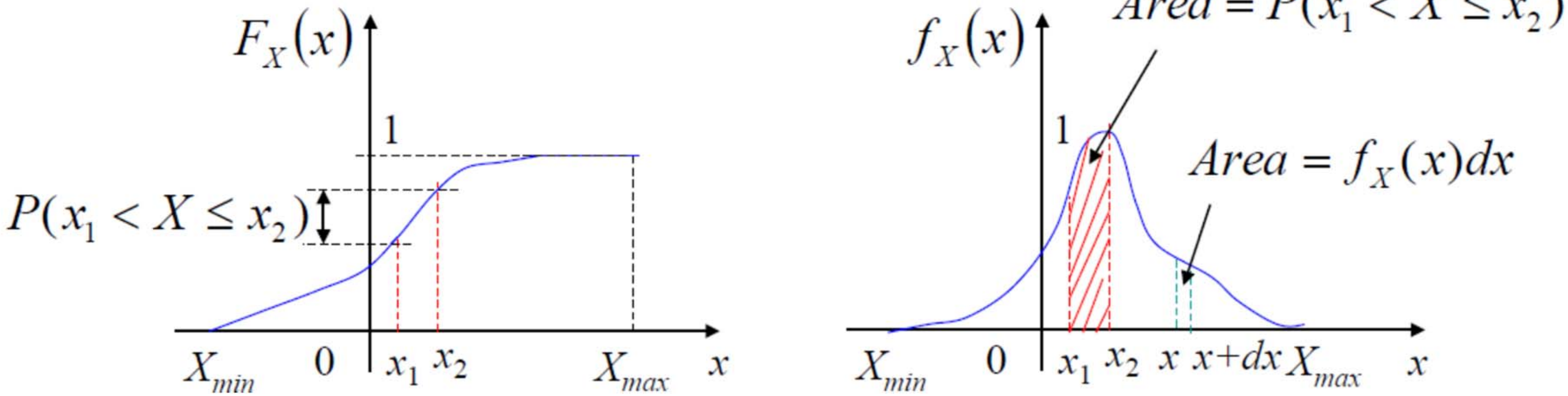
Probability Density Function (PDF)

- The PDF, of a r.v. X, is defined as

$$f_X(x) \triangleq \frac{d}{dx} F_X(x) \quad \text{or} \quad F_X(x) = \int_{-\infty}^{x} f_X(y)dy$$

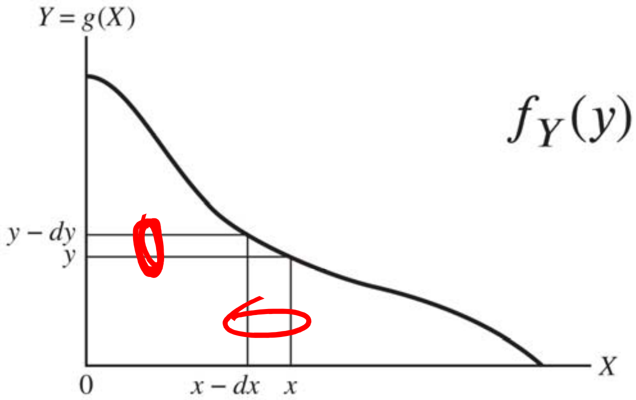
- Key properties of PDF

1. $f_X(x) \geq 0$
2. $\int_{-\infty}^{\infty} f_X(x)dx = 1$
3. $P(x_1 < X \leq x_2) = F_X(x_2) - F_X(x_1) = \int_{x_1}^{x_2} f_X(x)dx$



Transformation of R.V.

- The PDF of a function of X, $Y = g(X)$
- Monotonic transformation:

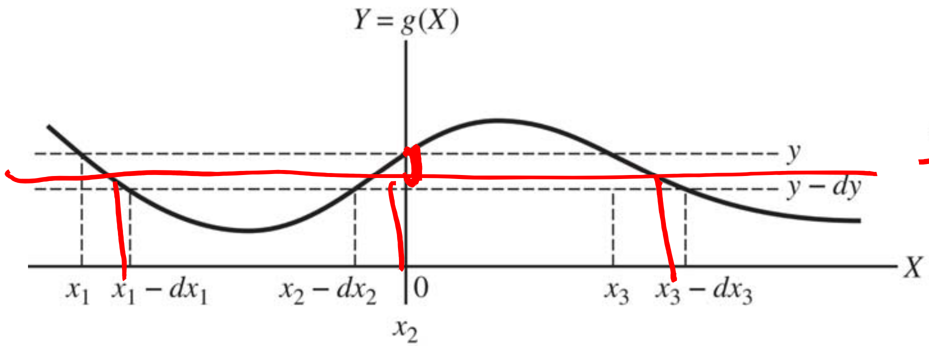


$$f_Y(y) = f_X(x)\left|\frac{dx}{dy}\right|_{x=g^{-1}(y)}$$

$$f_Y(y)\,dy = f_X(x)\,dx$$

$$f_Y(y) = \left[f_X(x) \left|\frac{dx}{dy}\right| \right]_{x=g(y)}$$

- Nonmonotonic transformation:



$$f_Y(y) = \sum_{i=1}^{N} f_X(x_i)\left|\frac{dx_i}{dy}\right|_{x_i=g_i^{-1}(y)}$$

Exercise

- Suppose that X is a Gaussian random variable with zero mean and unit variance. Let

$$\underline{Y = aX^3 + b,} \quad a > 0$$

- Determine the PDF of Y $f_Y(y) = \left[f_X(x) \left| \frac{dx}{dy} \right| \right]$ $\quad x = \left(\frac{y-b}{a}\right)^{\frac{1}{3}}$

$$= \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$p_Y(y) = \frac{1}{3a\sqrt{2\pi}\,[(y-b)/a]^{2/3}} e^{-\frac{1}{2}\left(\frac{y-b}{a}\right)^{2/3}}$$

Statistical Averages

- Consider a discrete r.v. which takes on the possible values $x_1, x_2, \ldots, x_M$ with respective probabilities $P_1, P_2, \ldots, P_M$
- The mean or expected value of X is

$$m_x = \bar{X} = E[X] = \sum_{i=1}^{M} x_i P_i$$

- If X is continuous, then

$$m_x = E[X] = \int_{-\infty}^{\infty} x f_X(x) dx$$

- Average of a function of a r.v. $Y = g(X)$.

$$E[Y] = \int_{-\infty}^{\infty} y f_Y(y)\, dy \qquad \overline{g(X)} \triangleq E[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

- This is the first moment of X.

Moment

- The n[th] moment of X

$$E[X^n] = \int_{-\infty}^{\infty} x^n f_X(x)\, dx$$

  – When n = 2, we have the mean-square value of X

$$E[X^2] = \int_{-\infty}^{\infty} x^2 f_X(x) dx$$

- n[th] central moment

$$E[(X - E[X])^n] = \int_{-\infty}^{\infty} (x - E[X])^n f_x(x) dx$$

  – At n=2, we have the variance

$$\sigma_x^2 = E[(X - E[X])^2] = E[X^2 - 2E[X]X + E[X]^2] = E[X^2] - E[X]^2$$

  – σ_x is called the standard deviation, the average distance from the mean, a measure of the concentration of X around the mean.

Some Useful Distribution: Bernoulli Distribution

- A discrete r.v. taking two possible values, X = 1 or X = 0, with probability mass function (pmf)

$$\mathrm{p}(x) = P(X = x) = \begin{cases} 1 - p, x = 0 \\ p, x = 1 \end{cases}$$

- Often used to model binary data

Some Useful Distribution: Binomial Distribution

- A discrete r.v. taking the sum of n-independent Bernoulli r.v. , i.e.

$$Y = \sum_{i=1}^{n} X_i \quad \text{where} \quad p_X(x) = \begin{cases} 1-p, x = 0 \\ p, x = 1 \end{cases}$$

- The PMF is given by

$$p_Y(k) = \binom{n}{k} \cdot p^k \cdot (1-p)^{n-k} \qquad \binom{n}{k} = \frac{n!}{k!\,(n-k)!}$$

  – That is, the probability that Y = k is the probability that k of the X_i are equal to 1 and n-k are equal to 0.

- Expectation and variance:

$$\overline{K} = E[K] = np(p+q)^{\ell} = \underline{np} \qquad \sigma_K^2 = E[K^2] - E^2[K] = \underline{npq}$$

Example

- Suppose that we transmit a 31-bit long sequence with error correction capability up to 3 bit errors
- If the probability of a bit error is p = 0.001, what is the probability that this sequence is received in errors?

$$P(\text{sequence error}) = 1 - P(\text{correct sequence})$$

$$= 1 - \sum_{i=0}^{3} \binom{31}{i} (0.001)^i (0.999)^{31-i} \approx 3.10^{-8}$$

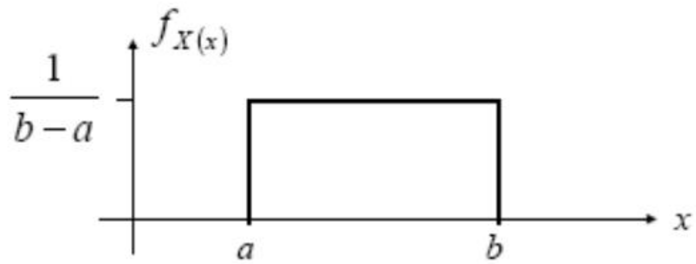
- If no error correction is used, the error probability is

$$P_e = 1 - 0.999^{31} \approx 0.0305$$

Some useful distribution: Uniform Distribution

- A continuous r.v. taking values between a and b with equal probabilities
- The PDF is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a}, & a \le x \le b \\ 0, & otherwise \end{cases}$$



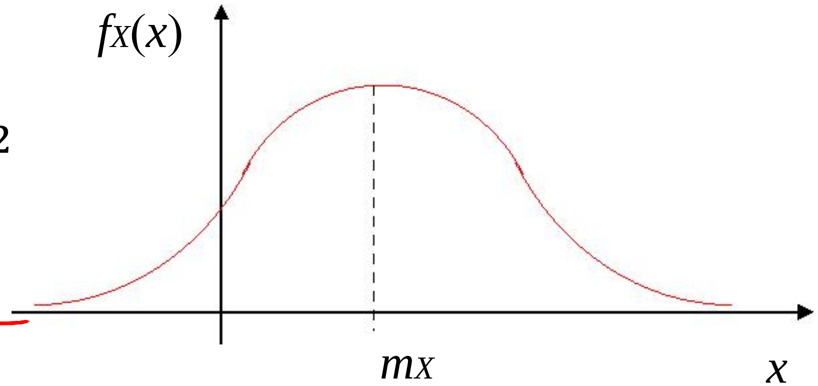
- The random phase of a sinusoid is often modeled as a uniform r.v. between 0 and 2π.

$\sin(2\pi \underline{f_0} t + \phi)$ $\phi \in U[0, 2\pi]$

Some Useful Distribution: Gaussian Distribution

- Gaussian or normal distribution is a continuous r.v. with pdf

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma_x^2}} \exp[-\frac{1}{2\sigma_x^2}(x - m_x)^2$$



- A Gaussian r.v. is completely determined by its mean and variance, and hence usually denoted as

$$x \sim N(m_x, \sigma_x^2)$$

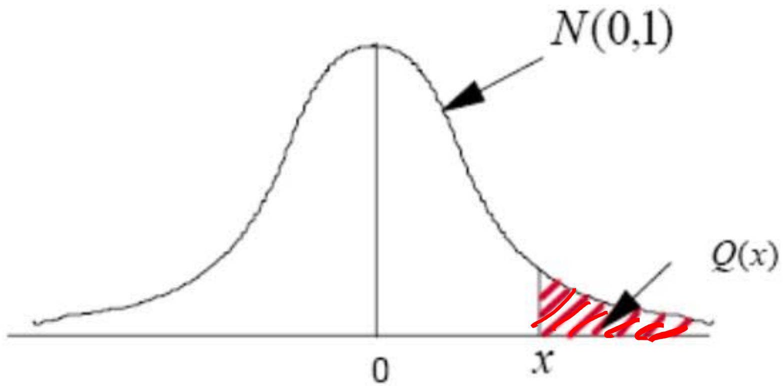
  – By far the most important distribution in communications

The Q-Function

- The Q-function is a standard form to express error probabilities without a closed form

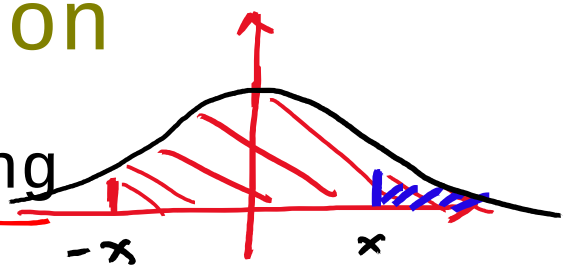
$$Q(x) = \int_x^\infty \frac{1}{\sqrt{2\pi}} \exp(-\frac{u^2}{2}) du$$

- The Q-function is the area under the tail of a Gaussian pdf with mean 0 and variance 1



Extremely important in error probability analysis!!!

More about Q-Function



- Q-function is monotonically decreasing
- Some features

$$Q(-\infty)=1 \qquad Q(0)=\frac{1}{2} \qquad Q(\infty)=0 \qquad Q(-x)=1-Q(x)$$

- Craig's alternative form of Q-function (IEEE MILCOM'91)

$$Q(x)=\frac{1}{\pi}\int_0^{\pi/2}\exp\left(-\frac{x^2}{2\sin^2\theta}\right)d\theta,\ x\geq 0$$

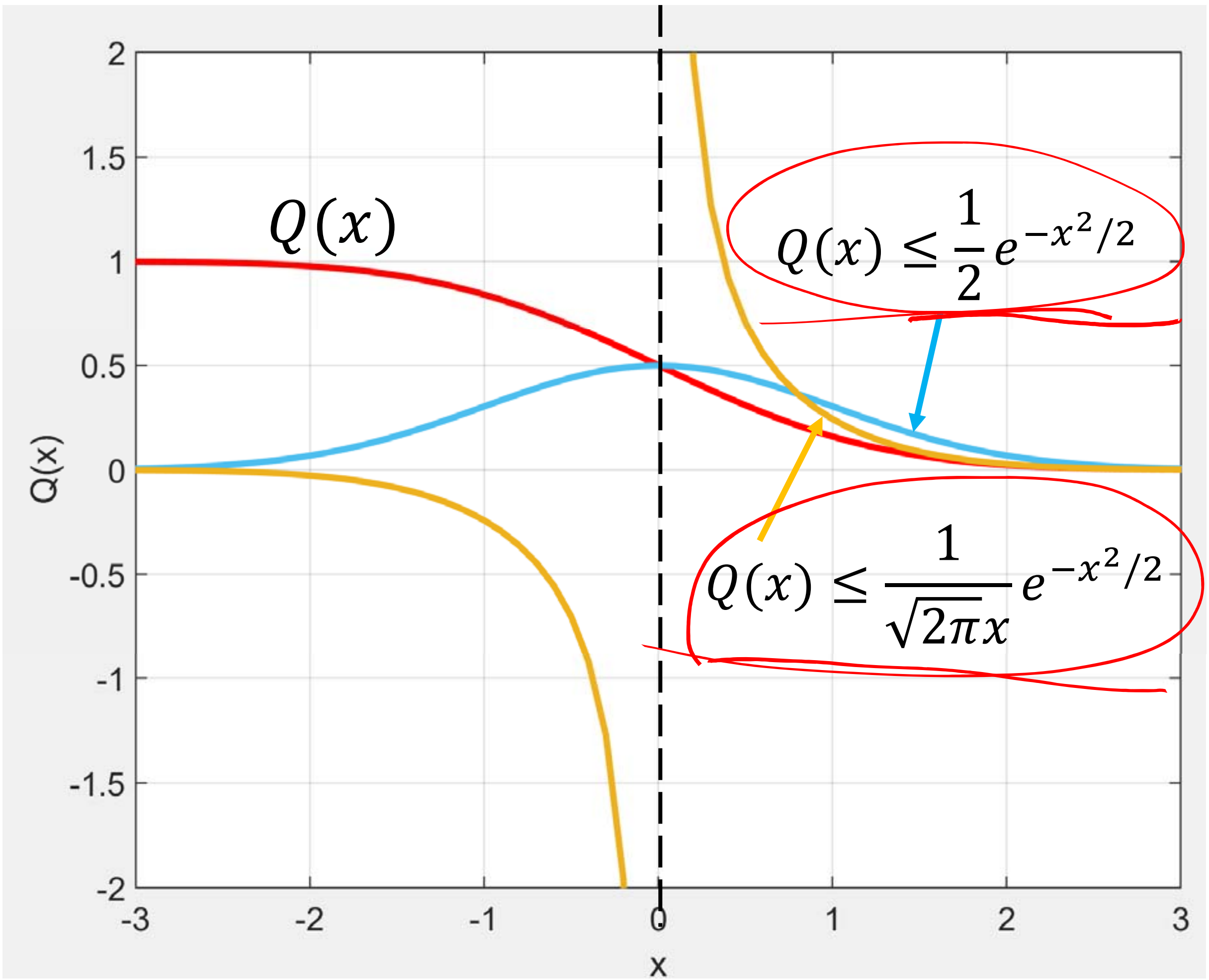
- Upper bound

$$Q(x)\leq\frac{1}{2}e^{-x^2/2}$$

- If we have a Gaussian variable $X\sim N(\mu,\sigma^2)$, then

$$P(X>x)=Q\left(\frac{x-\mu}{\sigma}\right)$$

More about Q-Function



Joint Distribution

- Consider 2 r.v.'s X and Y, joint distribution function is defined as

$$F_{XY}(x, y) = P(X \leq x, Y \leq y)$$

and joint PDF
$$f_{XY}(x, y) = \frac{\partial^2 F_{XY}(x, y)}{\partial x \partial y}$$

- Key properties of joint distribution

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f_{XY}(x, y) dx dy = 1$$

$$P(x_1 < X \leq x_2, y_1 < Y \leq y_2) = \int_{y1}^{y2} \int_{x1}^{x2} f_{XY}(x, y) dx dy$$

Joint Distribution (cont'd)

- Marginal distribution

$$F_X(x) = P(X \le x, -\infty < Y < \infty) = \int_{-\infty}^{\infty}\int_{-\infty}^{x} f_{XY}(\alpha,\beta)\, d\alpha d\beta$$

$$F_Y(y) = \int_{-\infty}^{y}\int_{-\infty}^{\infty} f_{XY}(\alpha,\beta) d\alpha d\beta$$

- Marginal density

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,\beta)\, d\beta, \quad f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(\alpha, y)\, d\alpha$$

- X and Y are said to be independent iff

$$F_{XY}(x,y) = F_X(x)F_Y(y)$$

$$f_{XY}(x,y) = f_X(x)f_Y(y)$$

$$f_{X|Y}(x|y) = f_X(x), f_{Y|X}(y|x) = f_Y(y)$$

A Linear Combination of N R.V.s

- Average of a linear combination of N r.v.s

$$E\left[\sum_{i=1}^{N} a_i X_i\right] = \sum_{i=1}^{N} a_i E[X_i]$$

- Variance of a linear combination of independent r.v.s

$$\mathrm{var}\left[\sum_{i=1}^{N} a_i X_i\right] = \sum_{i=1}^{N} a_i^2 \mathrm{var}\{X_i\}$$

- The pdf of the sum of two independent r.v.s

$$Z = X + Y \quad f_Z(z) = f_X(x) * f_Y(y) = \int_{-\infty}^{\infty} f_X(z-u) f_Y(u)\, du$$

Correlation of the R.V.

- Correlation of the two r.v. X and Y is defined as

$$R_{xy} = E[XY] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy f_{XY}(x,y)\,dx\,dy$$

- Correlation of the two centered r.v. X-E[X] and Y-E[Y], is called the covariance of X and Y

$$\sigma_{XY} = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]E[Y] = 0$$

- The correlation coefficient of X and Y

$$\rho_{XY} = \frac{\sigma_{XY}}{\sigma_X \sigma_Y}$$

$0 \leq |\rho| \leq 1$

$\rho = 0 \Rightarrow E[XY] = E[X]E[Y]$

- If $\rho_{XY} = 0$, i.e., $E[XY] = E[X]E[Y]$, then X and Y are called uncorrelated.

$|\rho| = 1 \Rightarrow$

Correlation (cont'd)

- If X and Y are independent, then they are uncorrelated.

Independent→$E[XY] = E[X]E[Y]$→ $\rho_{XY} = 0$.

- The converse is not true (except for the Gaussian case)

Joint Gaussian Random Variables

- $x_1, x_2, \cdots, x_n$ are jointly Gaussian iff

$$p(x_1, x_2, \ldots, x_n) = \frac{1}{(2\pi)^{n/2}(\det(\mathbf{C}))^{1/2}} \exp\left[-\frac{(\mathbf{x}-\mathbf{m})^T \mathbf{C}^{-1}(\mathbf{x}-\mathbf{m})}{2}\right]$$

  - **x** is a column vector $\mathbf{x} = (x_1, \ldots, x_n)^T$
  - **m** is the vector of the means $\mathbf{m} = (m_1, \ldots, m_n)^T$
  - **C** is the $n \times n$ covariance matrix

$$C = [C_{i,j}], C_{i,j} = E[(X_i - m_i)(X_j - m_j)]$$

Two-Variate Gaussian PDF

- Given two r.v.s: X_1 and X_2 that are joint Gaussian

$$C = \begin{bmatrix} E[X_1 - {m_1}^2] & E[(X_1 - m_1)(X_2 - m_2)] \\ E[(X_1 - m_1)(X_2 - m_2)] & E(X_2 - m_2)^2 \end{bmatrix} = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

- We have

$$p(x_1, x_2) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x_1-m_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-m_1)(x_2-m_2)}{\sigma_1\sigma_2} + \frac{(x_2-m_2)^2}{\sigma_2^2}\right]\right\}$$

Two-Variate Gaussian (cont'd)

- For uncorrelated X_1 and X_2, i.e. $\rho = 0$

$$
\begin{aligned}
p(x_1, x_2) &= \frac{1}{2\pi\sigma_1\sigma_2} \exp\left\{-\frac{1}{2}\left[\frac{(x_1 - m_1)^2}{\sigma_1^2} + \frac{(x_2 - m_2)^2}{\sigma_2^2}\right]\right\} \\
&= \frac{1}{2\pi\sigma_1\sigma_2} e^{-(x_1 - m_1)^2/2\sigma_1^2} \cdot \frac{1}{\sqrt{2\pi\sigma_2}} e^{-(x_2 - m_2)^2/2\sigma_2^2} \\
&= p(x_1)p(x_2)
\end{aligned}
$$

⇒ X_1 and X_2 are independent!

- If X_1 and X_2 are Gaussian and uncorrelated, then they are independent.

Some Properties of Jointly Gaussian r.v.s

- If n random variables $x_1, x_2, \ldots, x_n$ are jointly Gaussian, then any set of them is also jointly Gaussian. In particular, all individual r.v.s are Gaussian.
- Jointly Gaussian r.v.s are completely characterized by the mean vector and the covariance matrix, i.e. the second-order properties.
- Any linear combination of $x_1, x_2, \ldots, x_n$ is a Gaussian r.v.

Law of Large Numbers

- Consider a sequence of r.v. $X_1, X_2, \cdots, X_n$
- Let $Y = \frac{1}{n}\sum_{i=1}^{n} X_i$
- If X's are uncorrelated with the same mean and variance
- Then

$$\lim_{n\to\infty} P\left(|Y - m_X| \geq \varepsilon\right) = 0, \forall \varepsilon > 0$$

the sample average converges to the expected value!

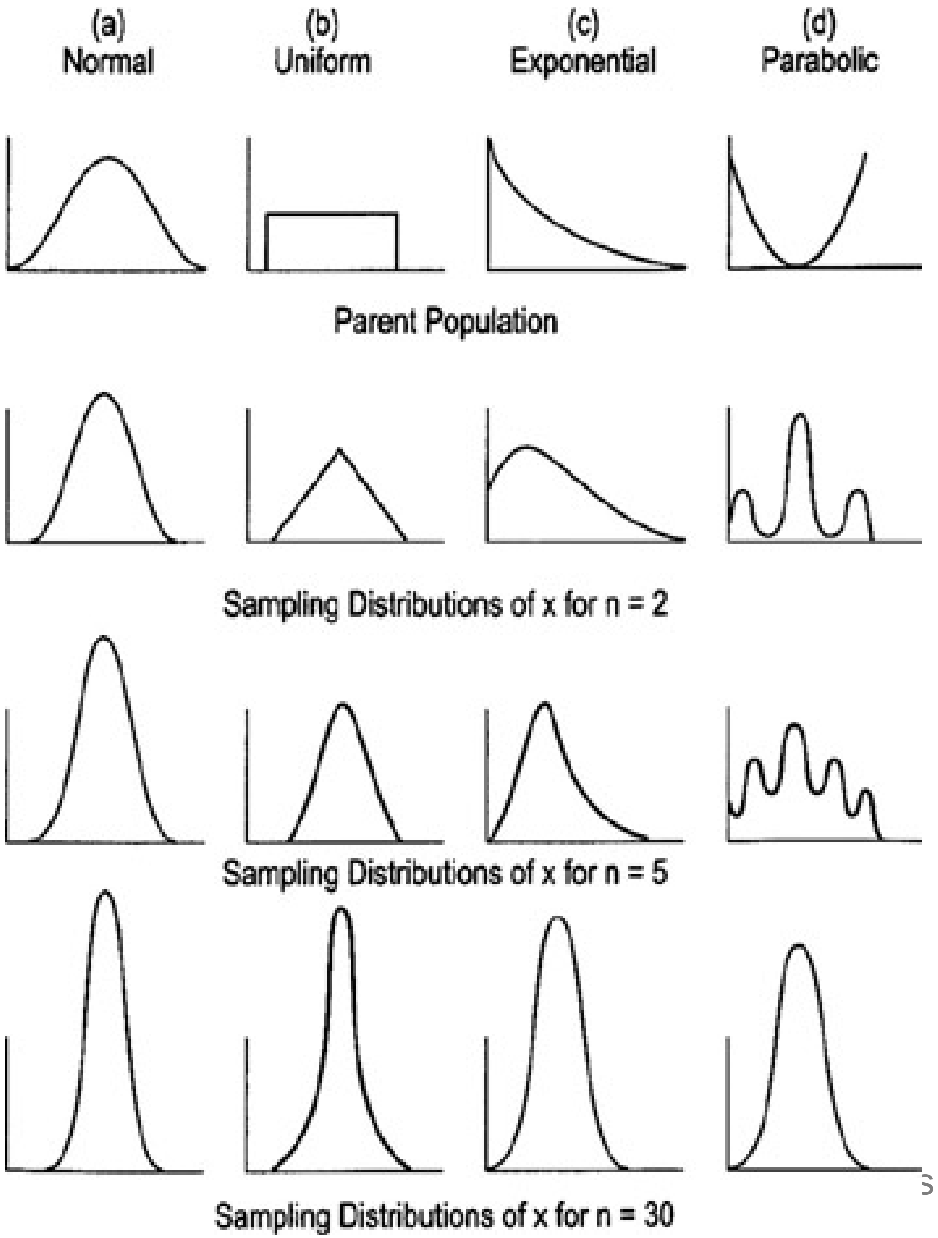
Central Limit Theorem

- If $x_1, x_2, \cdots, x_n$ are i.i.d. random variables with common mean m_X and common variance σ_X^2 then $y = \frac{1}{n}\sum_{i=1}^{n} x_i$ converges to $N(m_X, \frac{\sigma_X^2}{n})$

$$\boxed{\text{the sum of many i.i.d random variables converges to a Gaussian random variable}}$$

- Thermal noise results from the random movement of many electrons – it is well modeled by a Gaussian distribution.

Example

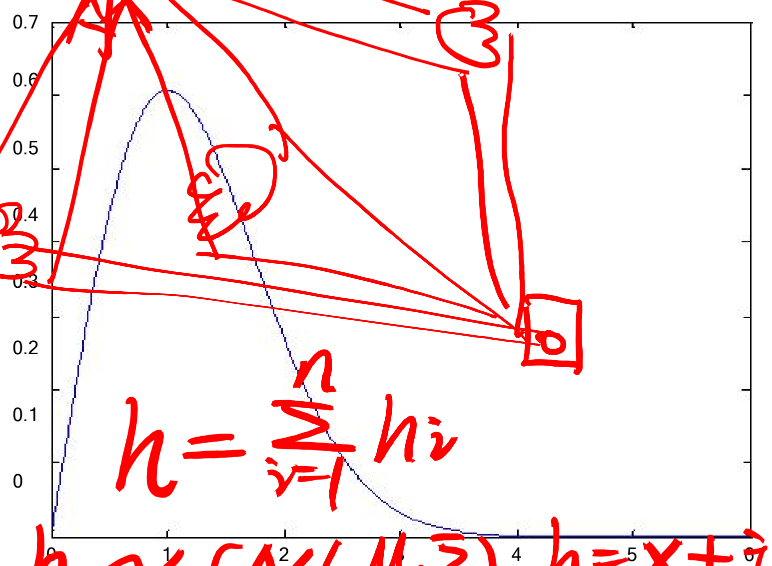


(a)
Normal
(b)
Uniform
(c)
Exponential
(d)
Parabolic
Parent Population
Sampling Distributions of x for n = 2
Sampling Distributions of x for n = 5
Sampling Distributions of x for n = 30

Some useful distribution: Rayleigh Distribution

- PDF

$$f_X(x) = \begin{cases} \frac{x}{\alpha^2}\exp(-\frac{x^2}{2\alpha^2}), x \geq 0 \\ 0, x < 0 \end{cases}$$



$h = \sum_{i=1}^{n} h_i$

$n \to \infty$ $h \sim CN(\mu, \Sigma)$ $h = x + jy$

- Rayleigh distributions are frequently used to model fading for non-line of sight (NLOS) signal transmission

$|h| \sim$ Rayleigh

$\angle h \sim$ uniform $[0, 2\pi]$

- Very important for mobile and wireless communications

Exercise

- Let $Z = X + jY$, where x and Y are i.i.d. Gaussian random variables with mean 0 and variance σ^2
- Show that the magnitude of Z follows Rayleigh distribution and its phase follows a uniform distribution.

$(u,v) = g(X,Y)$ $\quad f(u,v) = \left[f(x,y) \left| \frac{\partial(x,y)}{\partial(u,v)} \right| \right]_{(x,y)=g^{-1}(u,v)}$

$$R = \sqrt{X^2 + Y^2} \qquad \Theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

$f(x,y) = f(x)f(y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2}}$

$$X = R\cos\Theta = g_1^{-1}(R,\Theta)$$

$$Y = R\sin\Theta = g_2^{-1}(R,\Theta)$$

$$\frac{\partial(x,y)}{\partial(r,\theta)} = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

$$\boxed{f_{R\Theta}(r,\theta) = \frac{r e^{-r^2/2\sigma^2}}{2\pi\sigma^2} \qquad \begin{matrix} 0 \le \theta < 2\pi \\ 0 \le r < \infty \end{matrix}}$$

$f(r) = \int_0^{2\pi} \frac{r e^{-r^2/2\sigma^2}}{2\pi\sigma^2} d\theta = \frac{r^2 e^{-r^2/2\sigma^2}}{\sigma^2}$

Rayleigh distribution

$$f_R(r) = \frac{r}{\sigma^2} e^{-r^2/2\sigma^2}, \quad 0 \le r < \infty$$

$f(\theta) = \frac{1}{2\pi}$

Thanks for your kind attention!

Questions?