Assignment – 1

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**Problem 1:**

1. For Matrix , using norm, the vector will maximize the expression, as it will extract the last column which has the highest sum of absolute values. The intuitive explanation is that we want to focus all the vector “weights” on the specific row/column that results as the highest sum of the absolute values.

Using norm, the vector will maximize the expression, as it will “spread” the weights across the vector indexes, thus will minimize the denominator (which will be , the norm of ). In addition, the matrix-vector product will be of the following form:

. The second member in the vector, is the sum of absolute values of the second row of , and the coefficient will cancel out when dividing by ’s norm.

1. To find the vector that maximizes the expression, we will look at ’s eigenvectors and choose the one with the largest eigenvalue.

If we choose to be the eigenvector, with the largest eigenvalue, the expression above will be equal to the largest singular value for , which is in fact the definition of the induced norm.

Using the following snippet of python code, we will find : .

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**Problem 2:**

1. Let there be s.t. . Therefore, . This means that .

Thus, .

Let there be s.t. .

Therefore, . This means that

Thus, .

By showing both sides, we concur that .

1. Using the definition given in the exercise, we conclude the following:

.

In addition, is symmetric because:

This means that , as requested.

1. By definition

From section 2(b) we conclude that

And therefore,

By definition, there must be some vector s.t..

**Problem 4:**

1. We will solve the Least Square problem as taught in class for and , using the Cholesky factorization.

The normal equations for are:

The Cholesky factorization for is .

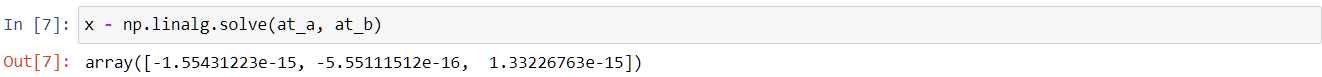
Solving for requires two steps:

1. Forward substitution (for lower triangular matrix ) to solve for ).
2. Backward substitution (for upper triangular matrix ) to solve for x.

Using the following code, we solved for .

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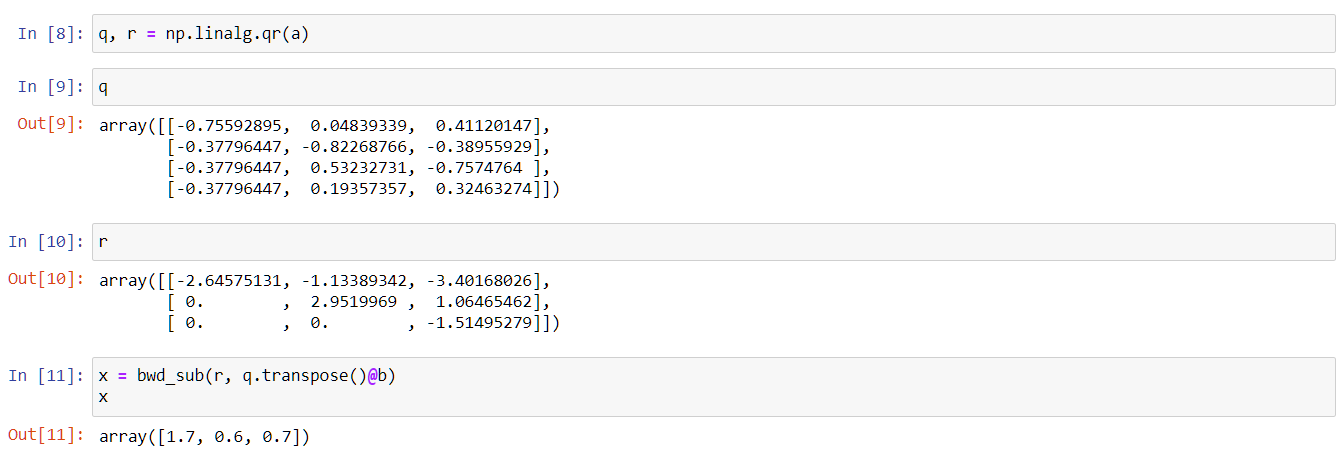
Description automatically generated*Checking our solution:



1. As taught in class, to solve Least Squares problem using the factorization we factorize and do as follows:

And we can solve for this using a backward substitution algorithm for and .

Using the following code, we solved for



1. As taught in class, to solve the Least Squares problem using the SVD factorization we do as follows:

First, we solve . This is implemented using element-wise division by in the following code:

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After finding , and because is orthogonal, we can solve for :

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1. The residual of the Least Squares is: .

We can see from the following code that:

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This is not surprising because we try to solve:

1. Using weighted least squares, we try to find:

We can solve this using any of the techniques above. We have chosen to use the Cholesky factorization using the same steps listed above, only this time we factorize instead of .

This is implemented in the following code:

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Using the weight matrix , we get the solution , with the residuals: .

**Problem 5:**

1. We implemented the Gram Schmidt and Modified Gram Schmidt in the following way:

Text

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1. Running the factorization with receives the exact same factorization in both implementation:

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Running the factorization with receives slightly different results:

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1. Running the factorization with receives very similar results for in both implementations:

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Running the factorization with receives drastically different results between the implementations:

Original:

Modified:

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Given the above results, it is obvious that the modified Graham Schmidt results in a better QR factorization. This can be seen by the fact that the original method, using results in a matrix which is not orthogonal. This can be explained by an accumulation of errors received when multiplying and subtracting in the second loop of the algorithm. The modified algorithm operates on smaller magnitude numbers, thus resulting in a smaller accumulation of errors.

**Problem 6:**

1. Using the equation in section (a): , we get:
2. We start from the previous expression and replace with
3. To solve the regularized LS problem , we will derive the expression (marked as ) and solve .

Using basic derivative rules, the derivative of a sum is the sum of derivates, so we will break the expression into two derivatives:

Now we show that is Positive Definite. Let there be s.t :

1. We need to show that the regularized Least Squares problem is given by:

From section (d) we get that:

And since ,

Since is orthogonal,

Where is a diagonal matrix with

Now we multiply by from left:

Since is orthogonal,

Therefore, after multiplying from the left again:

And since :

We notice that is a diagonal matrix:

We can use the results from sections (a)+(b) by using the diagonal matrix to conclude that:

And following from section (c):

As requested.

1. In the tutorial we were presented with image deblurring example in the following format:

When solving a straightforward least squares problem as follows: , the resulting image was uninterpretable.

This issue can be explained by the fact that is close to being singular, thus some of its eigenvalues are extremely small, and when computing it’s inverse and multiplying by the noise term the noise increased in magnitude.

To fix this issue easily, we added a regularization term that enlarged ’s eigenvalues such that ‘s inverse won’t result in huge eigenvalues: .

In the case presented here, we assume that the noise in the blurred image is represented in the singular value decomposition by a singular vector and a corresponding (close to zero) singular value .

As we can see, trying to solve the regular least squares problem using SVD results in a term that divides by , which becomes very unstable when is close to zero:

When using the regularized version, with an SVD decomposition, this term is numerically stable because of the addition of to the denominator:

This means that even if one of the singular values is close to zero, the resulting coefficient will be stable.

**Problem 7:**

1. For the problem , two unique correspondences would be required to find a solution for .

From the matrix equality we get the following equation system:

We have four variables. To solve them we will need four equations in the equation system.

Each correspondence provides two equations. Given two unique correspondences will be enough to find all four variables of .

1. We will find a good solution for by formalizing our problem as follows:

Now, our least squares problem is:

We can solve this using the normal equations:

The normal equations can be solved by any of the methods we used in problem (4).