Reinforcement Learning and Automated Planning

Class 2: Solving MDPs

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Outline

What is Dynamic Programming?

Dynamic sequential or temporal component to the problem

Programming optimising a "program", i.e. a policy

- c.f. linear programming

- A method for solving complex problems
- By breaking them down into subproblems
  - Solve the subproblems
  - Combine solutions to subproblems

Requirements for Dynamic Programming

Dynamic Programming is a very general solution method for problems which have two properties:

- Optimal substructure
  - *Principle of optimality* applies
  - Optimal solution can be decomposed into subproblems
- Overlapping subproblems
  - Subproblems recur many times
  - Solutions can be cached and reused
- Markov decision processes satisfy both properties
  - Bellman equation gives recursive decomposition
  - Value function stores and reuses solutions

Planning by Dynamic Programming

- Dynamic programming assumes full knowledge of the MDP
- It is used for *planning* in an MDP
- For prediction:
  - Input: MDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$ and policy π
  - or: MRP $\langle \mathcal{S}, \mathcal{P}^\pi, \mathcal{R}^\pi, \gamma \rangle$
  - Output: value function v_π
- Or for control:
  - Input: MDP $\langle \mathcal{S}, \mathcal{A}, \mathcal{P}, \mathcal{R}, \gamma \rangle$
  - Output: optimal value function v_*
  - and: optimal policy π_*

Other Applications of Dynamic Programming

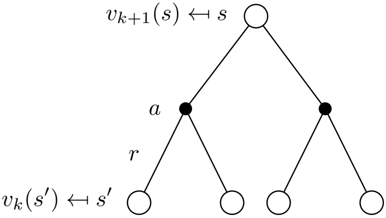
Dynamic programming is used to solve many other problems, e.g.

- Scheduling algorithms
- String algorithms (e.g. sequence alignment)
- Graph algorithms (e.g. shortest path algorithms)
- Graphical models (e.g. Viterbi algorithm)
- Bioinformatics (e.g. lattice models)

Iterative Policy Evaluation

- Problem: evaluate a given policy π
- Solution: iterative application of Bellman expectation backup
- $v_1 \rightarrow v_2 \rightarrow ... \rightarrow v_\pi$
- Using *synchronous* backups,
  - At each iteration $k+1$
  - For all states $s \in \mathcal{S}$
  - Update $v_{k+1}(s)$ from $v_k(s')$
  - where s' is a successor state of s
- We will discuss *asynchronous* backups later
- Convergence to v_π will be proven at the end of the lecture

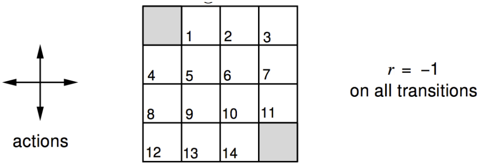
Iterative Policy Evaluation (2)



$$v_{k+1}(s) = \sum_{a \in \mathcal{A}} \pi(a|s) \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

$$\mathbf{v}^{k+1} = \boldsymbol{\mathcal{R}}^{\boldsymbol{\pi}} + \gamma \boldsymbol{\mathcal{P}}^{\boldsymbol{\pi}} \mathbf{v}^k$$

Evaluating a Random Policy in the Small Gridworld



- Undiscounted episodic MDP ($\gamma = 1$)
- Nonterminal states $1, \ldots, 14$
- One terminal state (shown twice as shaded squares)
- Actions leading out of the grid leave state unchanged
- Reward is -1 until the terminal state is reached
- Agent follows uniform random policy

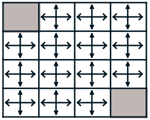
$$\pi(n|\cdot) = \pi(e|\cdot) = \pi(s|\cdot) = \pi(w|\cdot) = 0.25$$

Iterative Policy Evaluation in Small Gridworld

v_k for the Random Policy | Greedy Policy w.r.t. v_k

$k = 0$

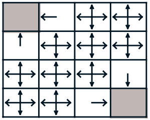
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0
0.0	0.0	0.0	0.0



random policy

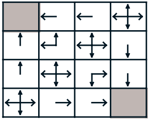
$k = 1$

0.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	-1.0
-1.0	-1.0	-1.0	0.0



$k = 2$

0.0	-1.7	-2.0	-2.0
-1.7	-2.0	-2.0	-2.0
-2.0	-2.0	-2.0	-1.7
-2.0	-2.0	-1.7	0.0



Iterative Policy Evaluation in Small Gridworld (2)

$k = 3$

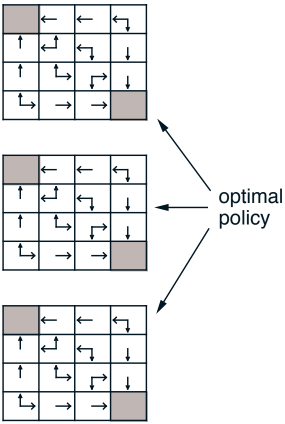
0.0	-2.4	-2.9	-3.0
-2.4	-2.9	-3.0	-2.9
-2.9	-3.0	-2.9	-2.4
-3.0	-2.9	-2.4	0.0

$k = 10$

0.0	-6.1	-8.4	-9.0
-6.1	-7.7	-8.4	-8.4
-8.4	-8.4	-7.7	-6.1
-9.0	-8.4	-6.1	0.0

$k = \infty$

0.0	-14.	-20.	-22.
-14.	-18.	-20.	-20.
-20.	-20.	-18.	-14.
-22.	-20.	-14.	0.0



How to Improve a Policy

- Given a policy π
  - Evaluate the policy π

$$v_\pi(s) = \mathbb{E}\left[R_{t+1} + \gamma R_{t+2} + ... | S_t = s\right]$$

  - Improve the policy by acting greedily with respect to v_π

$$\pi' = \text{greedy}(v_\pi)$$

- In Small Gridworld improved policy was optimal, $\pi' = \pi^*$
- In general, need more iterations of improvement / evaluation
- But this process of policy iteration always converges to $\pi*$

Policy Iteration

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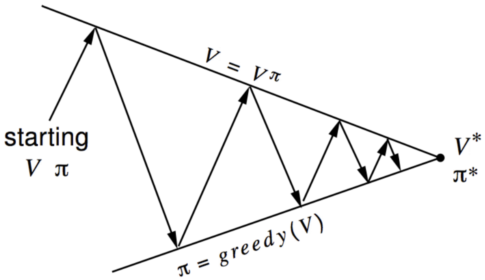
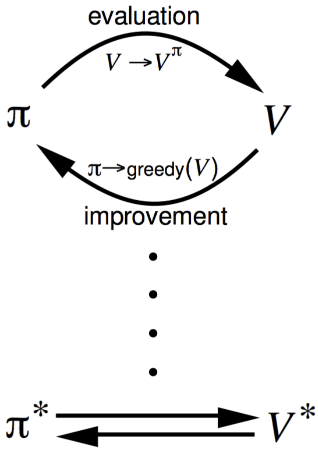




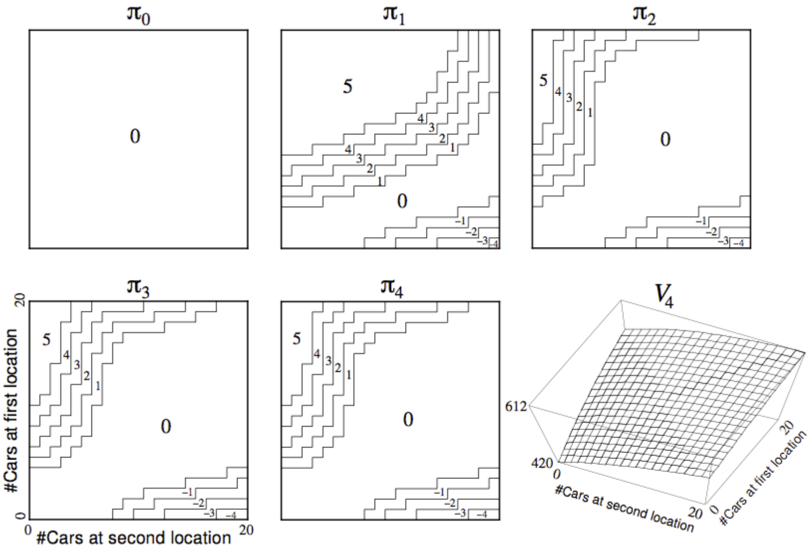
Policy evaluation Estimate v_π
 Iterative policy evaluation

Policy improvement Generate $\pi' \geq \pi$
 Greedy policy improvement

Jack's Car Rental

- States: Two locations, maximum of 20 cars at each
- Actions: Move up to 5 cars between locations overnight
- Reward: \$10 for each car rented (must be available)
- Transitions: Cars returned and requested randomly
  - Poisson distribution, n returns/requests with prob $\frac{\lambda^n}{n!}e^{-\lambda}$
  - 1st location: average requests $= 3$, average returns $= 3$
  - 2nd location: average requests $= 4$, average returns $= 2$

Policy Iteration in Jack's Car Rental



Policy Improvement

- Consider a deterministic policy, $a = \pi(s)$
- We can *improve* the policy by acting greedily

$$\pi'(s) = \underset{a \in \mathcal{A}}{\operatorname{argmax}}\ q_\pi(s, a)$$

- This improves the value from any state s over one step,

$$q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) \geq q_\pi(s, \pi(s)) = v_\pi(s)$$

- It therefore improves the value function, $v_{\pi'}(s) \geq v_\pi(s)$

$$\begin{aligned}
v_\pi(s) \leq q_\pi(s, \pi'(s)) &= \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma v_\pi(S_{t+1}) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma q_\pi(S_{t+1}, \pi'(S_{t+1})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \gamma^2 q_\pi(S_{t+2}, \pi'(S_{t+2})) \mid S_t = s\right] \\
&\leq \mathbb{E}_{\pi'}\left[R_{t+1} + \gamma R_{t+2} + \ldots \mid S_t = s\right] = v_{\pi'}(s)
\end{aligned}$$

Policy Improvement (2)

- If improvements stop,

$$q_\pi(s, \pi'(s)) = \max_{a \in \mathcal{A}} q_\pi(s, a) = q_\pi(s, \pi(s)) = v_\pi(s)$$

- Then the Bellman optimality equation has been satisfied

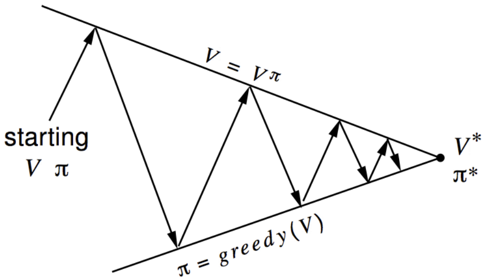
$$v_\pi(s) = \max_{a \in \mathcal{A}} q_\pi(s, a)$$

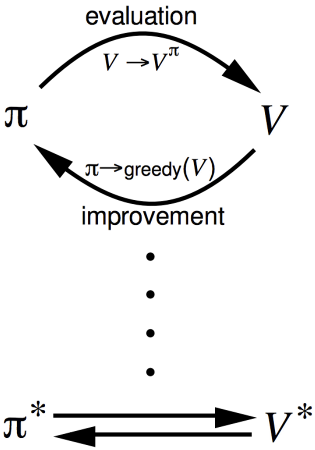
- Therefore $v_\pi(s) = v_*(s)$ for all $s \in \mathcal{S}$
- so π is an optimal policy

Modified Policy Iteration

- Does policy evaluation need to converge to v_π?
- Or should we introduce a stopping condition
  - e.g. ϵ-convergence of value function
- Or simply stop after k iterations of iterative policy evaluation?
- For example, in the small gridworld $k = 3$ was sufficient to achieve optimal policy
- Why not update policy every iteration? i.e. stop after $k = 1$
  - This is equivalent to *value iteration* (next section)

Generalised Policy Iteration





Policy evaluation Estimate v_π

Any policy evaluation algorithm

Policy improvement Generate $\pi' \geq \pi$

Any policy improvement algorithm

Principle of Optimality

Any optimal policy can be subdivided into two components:

- An optimal first action A_*
- Followed by an optimal policy from successor state S'

Theorem (Principle of Optimality)

A policy $\pi(a|s)$ *achieves the optimal value from state* s, $v_\pi(s) = v_*(s)$, *if and only if*

- *For any state* s' *reachable from* s
- π *achieves the optimal value from state* s', $v_\pi(s') = v_*(s')$

Deterministic Value Iteration

- If we know the solution to subproblems $v_*(s')$
- Then solution $v_*(s)$ can be found by one-step lookahead

$$v_*(s) \leftarrow \max_{a \in \mathcal{A}} \mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_*(s')$$

- The idea of value iteration is to apply these updates iteratively
- Intuition: start with final rewards and work backwards
- Still works with loopy, stochastic MDPs

Example: Shortest Path

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Problem

0	0	0	0
0	0	0	0
0	0	0	0
0	0	0	0

V_1

0	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1
-1	-1	-1	-1

V_2

0	-1	-2	-2
-1	-2	-2	-2
-2	-2	-2	-2
-2	-2	-2	-2

V_3

0	-1	-2	-3
-1	-2	-3	-3
-2	-3	-3	-3
-3	-3	-3	-3

V_4

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-4
-3	-4	-4	-4

V_5

0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-5

V_6

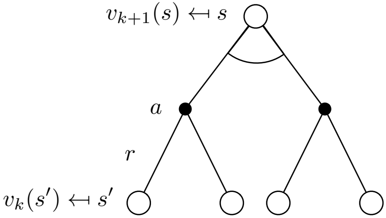
0	-1	-2	-3
-1	-2	-3	-4
-2	-3	-4	-5
-3	-4	-5	-6

V_7

Value Iteration

- Problem: find optimal policy π
- Solution: iterative application of Bellman optimality backup
- $v_1 \rightarrow v_2 \rightarrow ... \rightarrow v_*$
- Using synchronous backups
  - At each iteration $k+1$
  - For all states $s \in \mathcal{S}$
  - Update $v_{k+1}(s)$ from $v_k(s')$
- Convergence to v_* will be proven later
- Unlike policy iteration, there is no explicit policy
- Intermediate value functions may not correspond to any policy

Value Iteration (2)



$$v_{k+1}(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_k(s') \right)$$

$$\mathbf{v}_{k+1} = \max_{a \in \mathcal{A}} \boldsymbol{\mathcal{R}}^{\mathbf{a}} + \gamma \boldsymbol{\mathcal{P}}^{\mathbf{a}} \mathbf{v}_k$$

Example of VI and PI in Practice

https://cs.stanford.edu/people/karpathy/reinforcejs/gridworld_dp.html

Synchronous Dynamic Programming Algorithms

Problem	Bellman Equation	Algorithm
Prediction	Bellman Expectation Equation	Iterative Policy Evaluation
Control	Bellman Expectation Equation + Greedy Policy Improvement	Policy Iteration
Control	Bellman Optimality Equation	Value Iteration

- Algorithms are based on state-value function $v_\pi(s)$ or $v_*(s)$
- Complexity $O(mn^2)$ per iteration, for m actions and n states
- Could also apply to action-value function $q_\pi(s, a)$ or $q_*(s, a)$
- Complexity $O(m^2 n^2)$ per iteration

Asynchronous Dynamic Programming

- DP methods described so far used *synchronous* backups
- i.e. all states are backed up in parallel
- *Asynchronous DP* backs up states individually, in any order
- For each selected state, apply the appropriate backup
- Can significantly reduce computation
- Guaranteed to converge if all states continue to be selected

Asynchronous Dynamic Programming

Three simple ideas for asynchronous dynamic programming:

- *In-place* dynamic programming
- *Prioritised sweeping*
- *Real-time* dynamic programming

In-Place Dynamic Programming

- Synchronous value iteration stores two copies of value function

for all s in $\mathcal{S}$

$$v_{new}(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v_{old}(s') \right)$$

$$v_{old} \leftarrow v_{new}$$

- In-place value iteration only stores one copy of value function

for all s in $\mathcal{S}$

$$v(s) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right)$$

Prioritised Sweeping

- Use magnitude of Bellman error to guide state selection, e.g.

$$\left| \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a v(s') \right) - v(s) \right|$$

- Backup the state with the largest remaining Bellman error
- Update Bellman error of affected states after each backup
- Requires knowledge of reverse dynamics (predecessor states)
- Can be implemented efficiently by maintaining a priority queue

Real-Time Dynamic Programming

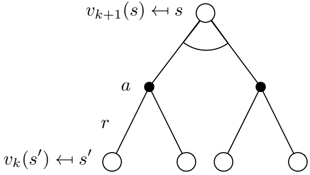
- Idea: only states that are relevant to agent
- Use agent's experience to guide the selection of states
- After each time-step S_t, A_t, R_{t+1}
- Backup the state S_t

$$v(S_t) \leftarrow \max_{a \in \mathcal{A}} \left(\mathcal{R}^a_{S_t} + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}^a_{S_t s'} v(s') \right)$$

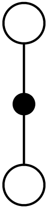
How can we improve this?

Full-Width Backups

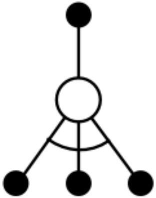
- DP uses *full-width* backups
- For each backup (sync or async)
  - Every successor state and action is considered
  - Using knowledge of the MDP transitions and reward function
- DP is effective for medium-sized problems (millions of states)
- For large problems DP suffers Bellman's *curse of dimensionality*
  - Number of states $n = |\mathcal{S}|$ grows exponentially with number of state variables
- Even one backup can be too expensive



Sample Backups



- In subsequent lectures we will consider *sample backups*
- Using sample rewards and sample transitions $\langle S, A, R, S' \rangle$
- Instead of reward function $\mathcal{R}$ and transition dynamics $\mathcal{P}$
- Advantages:
  - Model-free: no advance knowledge of MDP required
  - Breaks the curse of dimensionality through sampling
  - Cost of backup is constant, independent of $n = |\mathcal{S}|$



Approximate Dynamic Programming

- Approximate the value function
- Using a *function approximator* $\hat{v}(s, \mathbf{w})$
- Apply dynamic programming to $\hat{v}(\cdot, \mathbf{w})$
- e.g. Fitted Value Iteration repeats at each iteration k,
  - Sample states $\tilde{\mathcal{S}} \subseteq \mathcal{S}$
  - For each state $s \in \tilde{\mathcal{S}}$, estimate target value using Bellman optimality equation,

$$\tilde{v}_k(s) = \max_{a \in \mathcal{A}} \left(\mathcal{R}_s^a + \gamma \sum_{s' \in \mathcal{S}} \mathcal{P}_{ss'}^a \hat{v}(s', \mathbf{w_k}) \right)$$

  - Train next value function $\hat{v}(\cdot, \mathbf{w_{k+1}})$ using targets $\{\langle s, \tilde{v}_k(s) \rangle\}$

Some Technical Questions

- How do we know that value iteration converges to v_*?
- Or that iterative policy evaluation converges to v_π?
- And therefore that policy iteration converges to v_*?
- Is the solution unique?
- How fast do these algorithms converge?
- These questions are resolved by *contraction mapping theorem*

Value Function Space

- Consider the vector space $\mathcal{V}$ over value functions
- There are $|\mathcal{S}|$ dimensions
- Each point in this space fully specifies a value function $v(s)$
- What does a Bellman backup do to points in this space?
- We will show that it brings value functions *closer*
- And therefore the backups must converge on a unique solution

Value Function ∞-Norm

- We will measure distance between state-value functions u and v by the ∞-norm
- i.e. the largest difference between state values,

$$||u - v||_\infty = \max_{s \in \mathcal{S}} |u(s) - v(s)|$$

Bellman Expectation Backup is a Contraction

- Define the *Bellman expectation backup operator* T^{π},

$$T^{\pi}(v) = \mathcal{R}^{\pi} + \gamma \mathcal{P}^{\pi} v$$

- This operator is a γ-contraction, i.e. it makes value functions closer by at least γ,

$$\begin{aligned} ||T^{\pi}(u) - T^{\pi}(v)||_{\infty} &= ||\left(\mathcal{R}^{\pi} + \gamma \mathcal{P}^{\pi} u\right) - \left(\mathcal{R}^{\pi} + \gamma \mathcal{P}^{\pi} v\right)||_{\infty} \\ &= ||\gamma \mathcal{P}^{\pi}(u - v)||_{\infty} \\ &\leq ||\gamma \mathcal{P}^{\pi}||u - v||_{\infty}||_{\infty} \\ &\leq \gamma ||u - v||_{\infty} \end{aligned}$$

Contraction Mapping Theorem

Theorem (Contraction Mapping Theorem)

For any metric space $\mathcal{V}$ that is complete (i.e. closed) under an operator $T(v)$, where T is a γ-contraction,

- *T converges to a unique fixed point*
- *At a linear convergence rate of γ*

Convergence of Iter. Policy Evaluation and Policy Iteration

- The Bellman expectation operator T^{π} has a unique fixed point
- v_{π} is a fixed point of T^{π} (by Bellman expectation equation)
- By contraction mapping theorem
- Iterative policy evaluation converges on v_{π}
- Policy iteration converges on v_{*}

Bellman Optimality Backup is a Contraction

- Define the *Bellman optimality backup operator* T^*,

$$T^*(v) = \max_{a \in \mathcal{A}} \mathcal{R}^a + \gamma \mathcal{P}^a v$$

- This operator is a γ-contraction, i.e. it makes value functions closer by at least γ (similar to previous proof)

$$||T^*(u) - T^*(v)||_\infty \leq \gamma ||u - v||_\infty$$

Convergence of Value Iteration

- The Bellman optimality operator T^* has a unique fixed point
- v_* is a fixed point of T^* (by Bellman optimality equation)
- By contraction mapping theorem
- Value iteration converges on v_*

Linear Programming

n - number of states; k - number of actions;

LP: min $\sum_{i=1}^{n} v(s_i)$ subject to:

- $v(s_1) \geq R(s_1, a_1) + \gamma \cdot \sum_{s' \in S} Tr(s_1, a_1), s')v(s')$
- $\ldots$
- $v(s_1) \geq R(s_1, a_k) + \gamma \cdot \sum_{s' \in S} Tr(s_1, a_k), s')v(s')$
- $\ldots$
- $\ldots$
- $v(s_n) \geq R(s_n, a_1) + \gamma \cdot \sum_{s' \in S} Tr(s_n, a_1), s')v(s')$
- $\ldots$
- $v(s_n) \geq R(s_n, a_k) + \gamma \cdot \sum_{s' \in S} Tr(s_n, a_k), s')v(s')$

PI Summary

Policy Iteration

1: Initialize π' to random policy
2: **repeat**
3: $\quad\pi = \pi'$
4: $\quad$Policy Evaluation: Compute V_π
5: $\quad$Policy Improvement: $\pi' = Gready(V_\pi)$, i.e.,
$\quad\pi'(s) = \max_{a\in A} R(s,a) + \gamma \cdot \sum_{s'\in S} Tr(s,a,s')V_\pi(s')$
6: **until** $\pi' = \pi$

Exact Policy Evaluation(π)

Solve $n = |S|$ linear equations with n unknowns

- $V\pi(s_1) = R(s_1, \pi(s_1)) + \gamma \cdot \sum_{s'\in S} Tr(s_1, \pi(s_1), s')V_\pi(s')$
- $\ldots$
- $V\pi(s_n) = R(s_n, \pi(s_n)) + \gamma \cdot \sum_{s'\in S} Tr(s_n, \pi(s_n), s')V_\pi(s')$

Modified PI Summary

Policy Iteration

Initialize π' to random policy
repeat
    $\pi = \pi'$
    Approximate Policy Evaluation: Compute V_π
    Policy Improvement: $\pi' = Gready(V_\pi)$
until $\pi' = \pi$

Approximate Policy Evaluation(π)

Initialize: $i = 0; \forall s \in S : V_\pi^0(s) = 0;$
repeat
    for all $s \in S$ **do**
        $V_\pi^{i+1}(s) = R(s, \pi(s)) + \gamma \cdot \sum_{s' \in S} Tr(s, \pi(s), s') V_\pi^i(s')$
    end for
until $||V_\pi^{i+1} - V_\pi^i(s)|| \leq \epsilon$

VI Summary

For all state s: $V^0(s) = 0$ (or any value)
$i = 0$
repeat
    for all $s \in S$ **do**
        $V^{i+1}(s) = \max_{a \in A} R(s,a) + \gamma \cdot \sum_{s' \in S} Tr(s,a,s') \cdot V^i(s')$
    end for
until $||V^{i+1} - V^i|| \leq \epsilon$
return $V_{i+1} \approx V^*$

VI in *Get the Diamond*: Evolution of Value Function

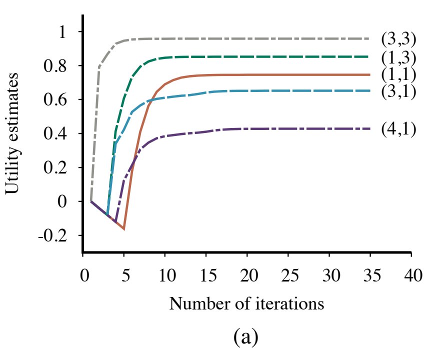


Figure 17.7 (a) Graph showing the evolution of the utilities of selected states using value iteration. (b) The number of value iterations required to guarantee an error of at most $\epsilon = c \cdot R_{\max}$, for different values of c, as a function of the discount factor γ.

VI in *Get the Diamond*: Error vs. Policy Loss

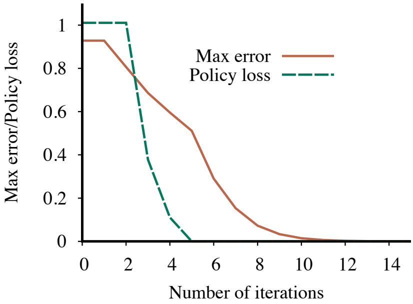


Figure 17.8 The maximum error $\|U_i - U\|$ of the utility estimates and the policy loss $\|U^{\pi_i} - U\|$, as a function of the number of iterations of value iteration on the 4×3 world.