Simple Linear Regression: Model and Estimation

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MSAN 601 - Linear Regression Analysis

Outline

- Measures of association: covariance and correlation
- The Simple Linear Regression Model
- Least Squares Estimation
- The Gauss-Markov Theorem

Setting: Simple Linear Regression

Given:

- Response $y = (y_1, \ldots, y_n)^T$ – *continuous-valued*
- A single explanatory variable X

Aim: Fit / estimate the simple linear regression model of y on X:

$$y = \beta_0 + \beta_1 X + \epsilon$$

Important Considerations:

- β_0 and β_1 are *unknown regression coefficients*
- Estimation = calculating our "best" guess: $\widehat{\beta}_0$, and $\widehat{\beta}_1$

Defining the SLR Model

- Two variables: the explanatory variable, X, explains—in part or wholly—the response variable Y
- You may encounter other terminology when referring to explanatory and/or response variables:

Y	X
Dependent variable	Independent variable
Explained variable	Explanatory variable
Response variable	Control variable
Predicted variable	Predictor variable
Regressand	Regressor variable

Table: Terminology for Simple Linear Regression

Why *Simple*? Why *Linear*? Why *Regression*?

- Simple: only dealing with two variables, an *explanatory* variable and a *response* variable
- Linear: linear relationship between X and y
- Regression: developed by Sir Francis Galton in the late 1800s:
  - when studying the relationship between the heights of parents and their children, he noted "children of both shorter and taller parents *regressed* to the group mean height"

Features

- Functional relationship, e.g., $y = f(X) = 3X - 9$
  - for each X, function returns a corresponding value of y
- Statistical relationships are imperfect
  - observations for a statistical relationship do not typically fall directly on the curve (line) of the relationship

Before Regression... Exploratory Data Analysis

directly look at the graph to observe if there is any kind of trend
-use bar chart instead of pie chart, we are not good at compare areas

- Analyze individual variables
  - **numerically**: mean, median, mode, variance, quantiles, skewness, kurtosis, etc.
  - **graphically**: histograms, box plots, etc.
- Analyze relationship(s) between variables
  - **numerically**: covariance, correlation
  - **graphically**: scatter plots

Quantifying Relationships Between X and y

Suppose that (X, Y) are jointly distributed random variables

Covariance between X and Y

Let X and Y be two random variables. Then the covariance of the two random variables is

if independent, Cov(x,y) = E[XY]

$$\text{Cov}(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] = \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$$

Idea: Covariance gives idea of the relationship between X and Y.

Linearity of expectation makes this possible
one of the most important property in statistics

Properties of Covariance

Basic Properties

- $\text{Cov}(X, Y) = \text{Cov}(Y, X)$
- $\text{Cov}(X, X) = \text{Var}(X)$
- $\text{Cov}(aX, Y) = a\text{Cov}(X, Y)$
- If X and Y are indepedendent, then $\text{Cov}(X, Y) = 0$

Important property

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ be a collection of random variables.

Then,

$$\text{Cov}(\sum_{i=1}^{n} X_i, \sum_{j=1}^{n} Y_j) = \sum_{i=1}^{n} \sum_{j=1}^{n} \text{Cov}(X_i, Y_j)$$

Properties of Covariance

Important consequence 1

$$\text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i) + 2\sum_{i<j} \text{Cov}(X_i, X_j)$$

Important consequence 2

If X_i's are *pairwise independent*, then

$$\text{Var}(\sum_{i=1}^{n} X_i) = \sum_{i=1}^{n} \text{Var}(X_i)$$

Notes:

- These properties can be extended to multivariate X and Y
- Covariances can take on *any* value on the real line..

make us hard to grasp the difference or magnitude of this measure

Correlation

Correlation between X and Y

$$\rho(X, Y) = \text{Corr}(X, Y) = \frac{\text{Cov}(X, Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

Fact 1

$$-1 \leq \rho(X, Y) \leq 1$$

Terminology: X and Y are called uncorrelated when $\rho(X, Y) = 0$.

relation with the independence

Properties of Correlation

The strongest correlations (r = 1.0 and r = -1.0) occur when data points fall exactly on a straight line.

Fact 2

Suppose that $\rho(X, Y) = -1$, then $Y = -aX + b$ with $a > 0$.

... what does this suggest? ...

Fact 3

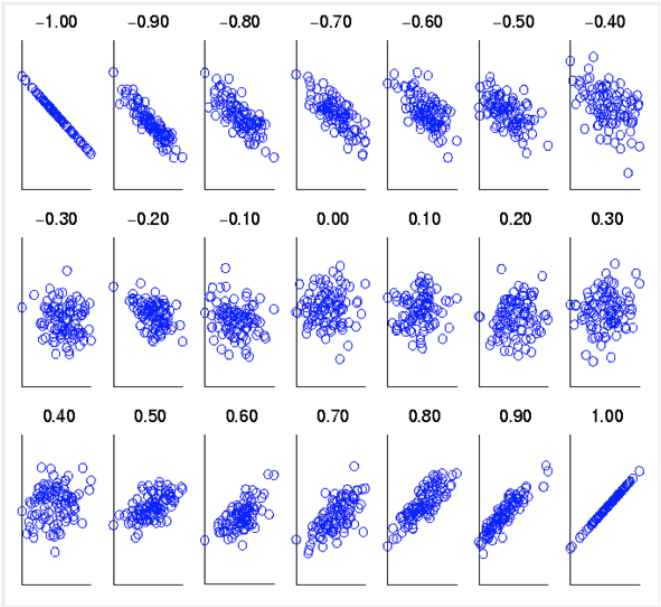
For $a > 0$,

$$\rho(aX + b, Y) = \rho(X, Y)$$

apply linear function on variable won't change the correlation

Properties of Correlation

visual obs can be confused by the scaling of this thing



Important Takeaway:

$\rho(X, Y)$ measures the strength and direction of a linear relationship between X and Y (close to 1: strong linear, positive slope; close to -1: strong linear, negative slope).

Empirical Correlation

Setting: Observe n values $(x_1, \ldots, x_n)$ and $(y_1, \ldots, y_n)$

actual obs that we need to calculate, strong law of large numbers enforce below statement to be same as true Cov

- Empirical Correlation r_{xy}:

$$r_{xy} = \frac{s_{xy}}{s_x s_y} = \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{(n-1)s_x s_y}$$

$$= \frac{\sum_{i=1}^{n}(x_i - \bar{x})(y_i - \bar{y})}{\sqrt{\sum_{i=1}^{n}(x_i - \bar{x})^2 \sum_{i=1}^{n}(y_i - \bar{y})^2}}$$

where $\bar{x}, \bar{y}$ are sample means, s_x, s_y are sample standard deviations and s_{xy} is the sample covariance between x and y

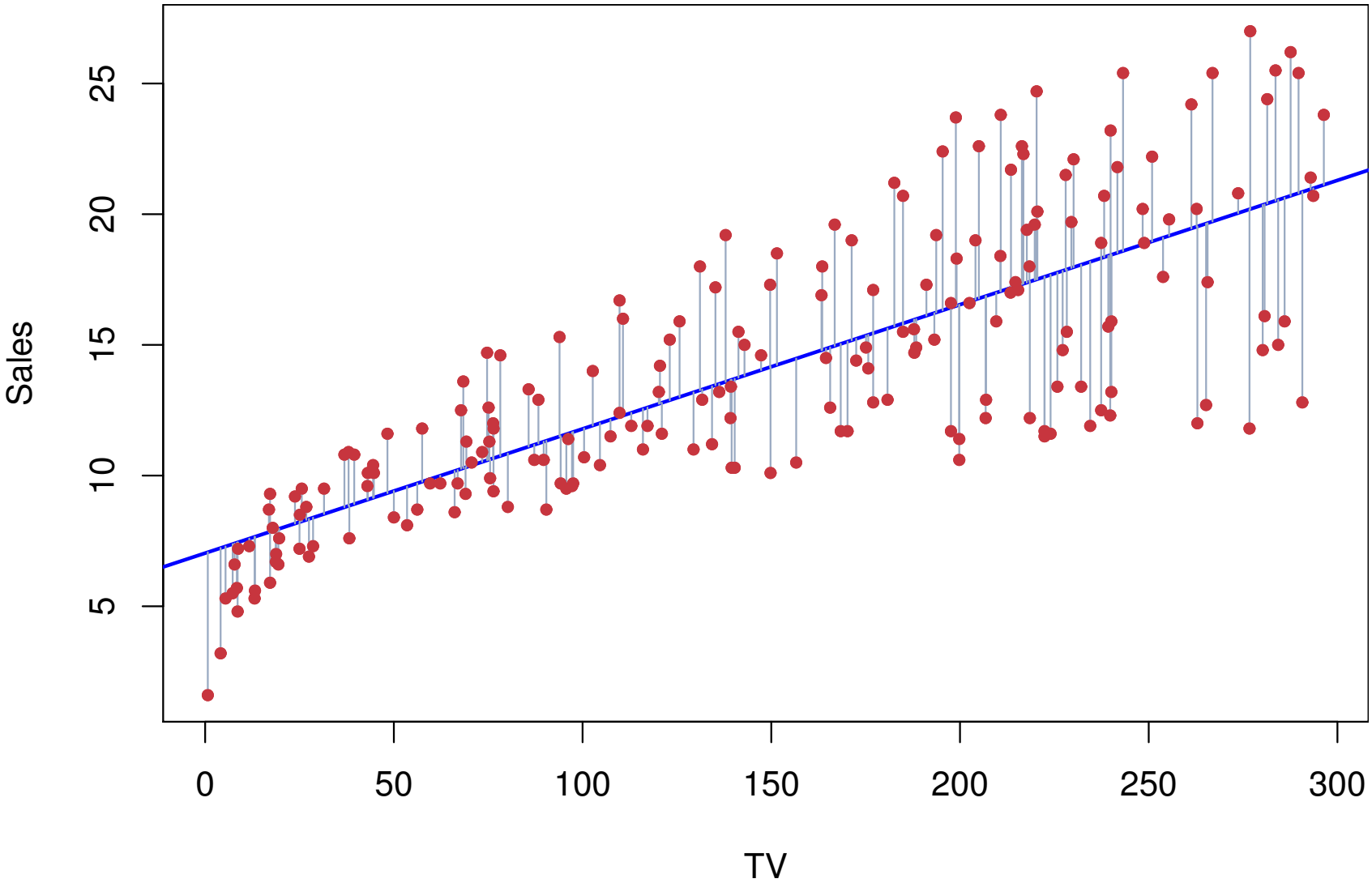
- This is how we *quantify* the linear relationship between two variables!

Regression Models: Basic Concepts

Once we have an idea that a regression model seems reasonable (from correlation values), we can then proceed to fit the model.

- Regression models postulate that:
  - There is a probability distribution of Y for each level of X
  - The means of these probability distributions vary in some systematic fashion with X

Example



not seems like a constant error

Caution: Regression and Causality

- The existence of a statistical relationship between the response variable Y and the explanatory variable X does not imply in any way that Y depends *causally* on X
- No matter how strong the relationship between Y and X, no cause and effect pattern is implied by a regression model

The SLR Model

- **Data**:
  - Explanatory variable: $(x_1, \ldots, x_n)$
  - Response: $(y_1, \ldots, y_n)$

  SLR Model:

$$Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i \qquad i = 1, \ldots, n$$

  - Coefficients: β_0 = y-intercept and β_1 = slope of the regression function
  - Error: ε_i is a random error term for observation i

Possible Assumptions on ε

- ε is a random error term with
  - $\mathbb{E}[\varepsilon_i] = 0$
  - $\text{Var}(\varepsilon_i) = \sigma^2 \geq 0$
  - $\text{Cov}(\varepsilon_i, \varepsilon_j) = 0 \quad \forall i \neq j$

  or
  - $\varepsilon_1, \ldots, \varepsilon_n \overset{i.i.d}{\sim} N(0, \sigma^2)$

The second assumption is *stronger* than the first. It is required for *inference* but not *estimation*.

difference get

Why Simple? Why Linear? [REVISITED]

- Simple: only one explanatory variable, X
- Linear:

  a wired case: as long as beta is scalar. it's linear function

  - linear in the parameters because no parameters appear as exponents or are multiplied or divided by another parameter
  - linear in the explanatory variable because this variable appears only in the first power
  - a model that linear in both the parameters and the variables is called a *first-order model*

Important Features of the Model

Expectation

Taking expectations we obtain

prove these two statement for the quiz

$$\mathbb{E}[Y_i] = \mathbb{E}[\beta_0 + \beta_1 x_i + \varepsilon_i] = \beta_0 + \beta_1 x_i$$

Variance

Taking the variance of Y_i, we obtain:

$$\text{Var}(Y_i) = \text{Var}(\beta_0 + \beta_1 x_i + \varepsilon_i) = \sigma^2$$

X is fixed thus var(x) is zero

Why are the above two statements true?

Conclusion: The probability distribution of Y have the same variance, regardless of value X

Important Features of the Model

Correlation

Given the assumption that the error terms are *uncorrelated*, i.e.,

Linearity of EXPECTATION!!!!!!!!!!!!

$$\text{Corr}(\varepsilon_i, \varepsilon_j) = 0 \quad \forall \quad i, j; i \neq j$$

it follows that the responses Y_i and Y_j are also *uncorrelated*

$$\text{Corr}(Y_i, Y_j) = 0 \quad \forall i \neq j$$

Example

- An electrical distributor is studying the relationship between the number of bids requested by construction contractors for basic lighting equipment during a week (x) and the time required to prepare those bids (Y), and the regression model is:

$$Y_i = 9.5 + 2.1x_i + \varepsilon_i$$

and the regression function is

$$\mathbb{E}[Y] = 9.5 + 2.1x_i$$

- Suppose in the i^{th} week, $x_i = 45$ bids are prepared and the actual number of hours required is $Y_i = 108$, which implies that the error term value is $\varepsilon_i = 4$, as $9.5 + 2.1(45) = 104$

Example

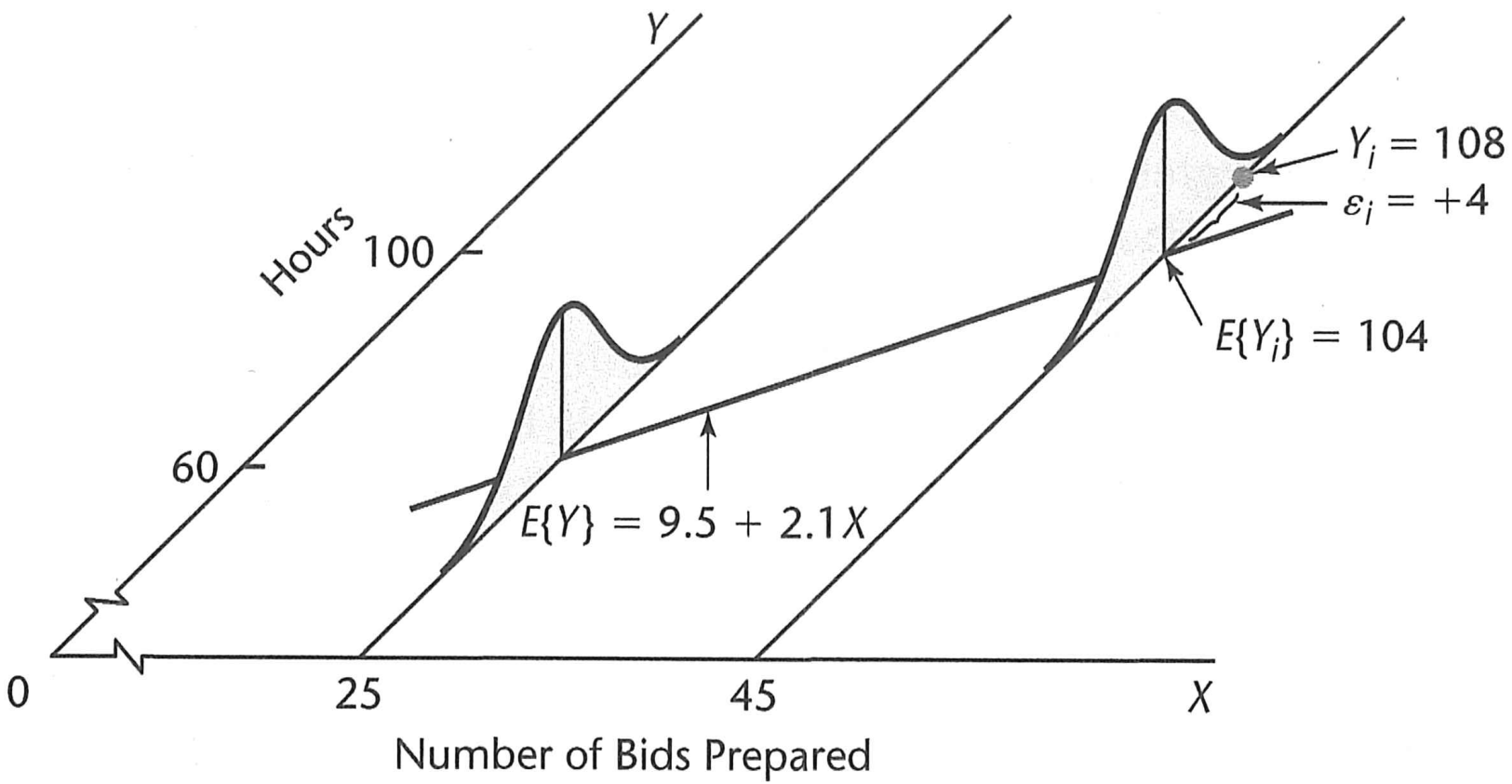


Figure: Illustration of an SLR Model

Example

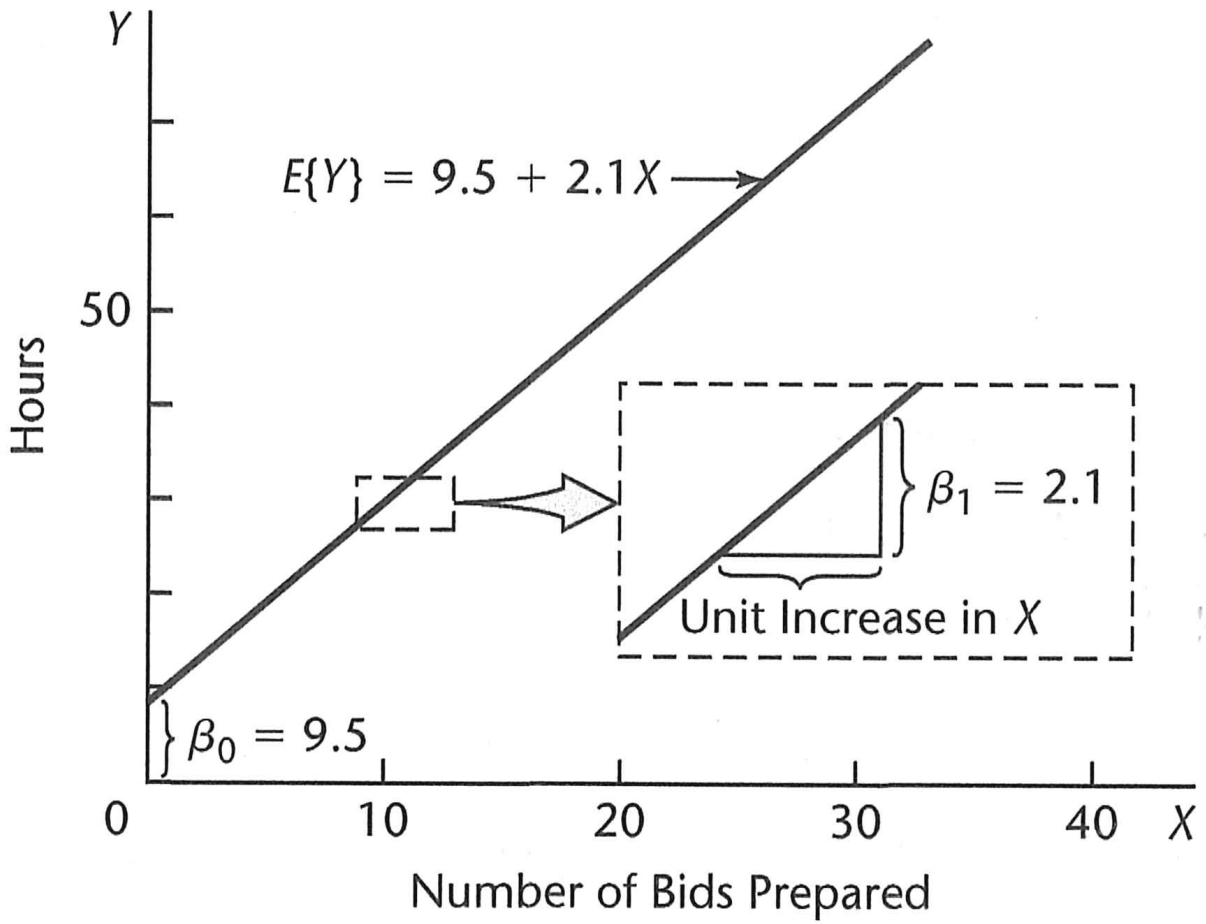


Figure: Meaning of the SLR Parameters

Population Regression Function

The population regression function expresses the mean (but unknown) value of Y given the data x:

$$\mathbb{E}[Y|x] = \beta_0 + \beta_1 x$$

Properties

- Linear function of x
- A one-unit increase in explanatory variable x changes the *expected* value of the response variable y by the amount β_1

Estimating β_0 and β_1

- Recall in general, we would like to minimize the mean squared error:

$$MSE(f) = \frac{1}{n}\sum_{i=1}^{n}(Y_i - f(x_i))^2$$

- In the SLR setting, we have

$$f(x_i) = \mathbb{E}[Y_i|x_i] = \beta_0 + \beta_1 x_i$$

- Thus, to estimate β_0 and β_1 we find $\widehat{\beta}_0$ and $\widehat{\beta}_1$ that minimize:

Quiz!!!!!

$$S(\beta) = \sum_{i=1}^{n}(Y_i - (\beta_0 + \beta_1 x_i))^2$$

- This is called the method of least squares

Method of Least Squares

Goal: Minimize $S(\beta) = \sum_{i=1}^{n} (Y_i - (\beta_0 + \beta_1 x_i))^2$ with respect to β

- Setting $\beta = (\beta_0, \beta_1)^T$ and

$$X = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$$

we can write

Capital Y here

$$S(\beta) = (Y - X\beta)^T (y - X\beta)$$

- In general, $S(\beta)$ can always be written in the matrix form above

Method of Least Squares

Facts: (test in the 1-dimensional case)

- $\frac{\partial}{\partial \beta} X\beta = X$
- $\frac{\partial}{\partial \beta} \beta^T (X^T X)\beta = 2X^T X\beta$

We can now estimate β by solving the following:

$$\frac{\partial}{\partial \beta} S(\beta) = 0$$

photo on procedure

Doing so gives the normal equations:

$$X^T y = X^T X\beta$$

Method of Least Squares

- If $X^T X$ is invertible, then the normal equations yield the least squares estimates:

quiz!!!!!!!!!!!!

$$\widehat{\beta} = (X^T X)^{-1} X^T y$$

- $\widehat{\beta}$ is often referred to as the ordinary least squares (OLS) estimates

what if x is not invertible, then we run into new ways of estimate it

- In the simple linear regression model, $\widehat{\beta}$ simplifies to:
  - $\widehat{\beta}_0 = \overline{y} - \widehat{\beta}_1 \overline{x}$
  - $\widehat{\beta}_1 = \dfrac{\sum_{i=1}^{n}(x_i - \overline{x})(y_i - \overline{y})}{\sum_{i=1}^{n}(x_i - \overline{x}_i)^2} = \dfrac{s_{xy}}{s_x^2}$

The Gauss-Markov Theorem

Theorem

Under either assumption of the linear regression model, the least-squares estimators are unbiased and have a minimum variance among all unbiased linear estimators. That is,

- $\mathbb{E}[\widehat{\beta}] = \beta$
- *For all linear estimators b such that* $\mathbb{E}[b] = \beta$*, we have:*

$$Var(\widehat{\beta}) \leq Var(b)$$

We say that $\widehat{\beta}$ is the best linear unbiased estimator (BLUE) of β.

Properties of $\widehat{\beta}$

- Among other unbiased linear estimators, $\widehat{y} = X\widehat{\beta}$ will have the smallest MSE (at least for the training set!)
- $\mathbb{E}[\widehat{\beta}] = \beta$ linearity of expectation
- $\text{Var}(\widehat{\beta}) = \sigma^2(X^TX)^{-1}$ mutilvariate variance

Question: How do we prove the above 2 statements?

Fitted Values and Residuals

- With $\widehat{\beta}$ in hand, we can now estimate y with the fitted value $\widehat{y} = X\widehat{\beta}$
- In SLR, the ith fitted value is given by:

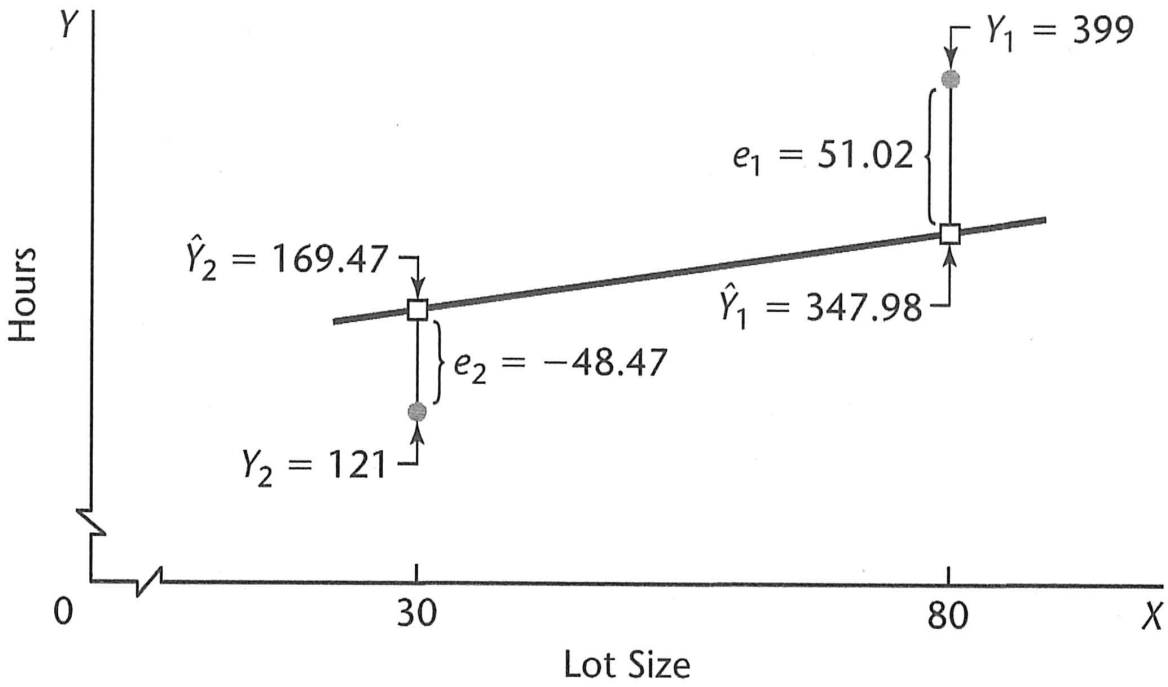
$$\widehat{y}_i = \widehat{\beta}_0 + \widehat{\beta}_1 x_i$$

- The residual for the ith observation is defined as:

$$e_i = y_i - \widehat{y}_i$$

Residuals

The residual e_i is the vertical deviation of y_i from the fitted value $\hat{y}_i$ on the estimated regression line.



Comments on SLR

- Evaluating features of the residuals $e_1, \ldots, e_n$ is how we formally *test* the assumptions of the regression model (more on this next time!)
- Estimation *did not* depend on any assumptions of ε!
- Estimation *did* depend on $X^T X$ being invertible, which requires-
  - X must be of rank p
  - This means that the columns of X must be linearly independent (i.e. no multicollinearity!)

Next Time

- Properties of the residuals
- Point estimation for σ^2
- Distributional theory for Normal random variables
- Hypothesis testing and confidence intervals for estimated coefficients