Representation Theory of Compact Lie Groups

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MA5211 Lie Theory Presentation

Outline

Representation Theory of Groups

The general setting:

A **representation** of a group G is a pair (V, ϕ), where V is a vector space over a field $\mathbb{F}$ and $\phi : G \to \mathrm{GL}_{\mathbb{F}}(V)$ is a group homomorphism.

Our setting:

A **Lie group representation** of a **compact Lie group** G is a pair (V, ϕ), where V is a **finite-dimensional** vector space over $\mathbb{C}$ and $\phi : G \to \mathrm{GL}_{\mathbb{C}}(V)$ is a **Lie group homomorphism**.

Some Simple Examples

1. **Adjoint representation:**

$$\mathrm{Ad} : G \to \mathrm{GL}_{\mathbb{C}}(\mathrm{Lie}(G)), \qquad g \mapsto dc(g)|_e;$$

2. **Finite group representations:**

$$\chi : \mathbb{Z}/n\mathbb{Z} \to \mathbb{C}^*, \qquad k \mapsto \exp\left(\frac{2\pi ik}{n}\right);$$

3. **Standard representations:**

$$\iota : \mathrm{O}(n), \mathrm{SO}(n), \mathrm{U}(n), \mathrm{SU}(n) \hookrightarrow \mathrm{GL}_{\mathbb{C}}(\mathbb{C}^n);$$

4. **Rotations in $\mathbb{R}^2$:**

$$R : S^1 \to \mathrm{SO}(2), \qquad \exp(it) \mapsto \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}.$$

Origins of Representation Theory

Finite representation theory:

A **representation** of a **finite group** G is a pair (V, ϕ), where V is a **finite-dimensional** vector space over $\mathbb{C}$ and $\phi : V \to \mathrm{GL}_{\mathbb{C}}(V)$ is a group homomorphism.

Maschke's Theorem (1898)

Every representation V of a finite group G over $\mathbb{C}$ is a direct sum of irreducible representations.

The irreducible representations of G do not **interact**.

Guiding principle: To understand the representations of G, it suffices to understand the **irreducible representations**.

Key Idea Behind Maschke's Theorem

Maschke's Theorem (1898)

Every representation V of a finite group G over $\mathbb{C}$ is a direct sum of irreducible representations.

Step 1: Let $U \subseteq V$ be a subrepresentation. We seek to find a **complementary subrepresentation** W such that $V = U \oplus W$.

Step 2: A Hermitian inner product $\langle -, - \rangle$ on V is ***G*-invariant** if for all $g \in G$ and $v_1, v_2 \in V$, we have

$$\langle g v_1, g v_2 \rangle = \langle v_1, v_2 \rangle.$$

If V has a G-invariant inner product, then choose $W = U^\perp$.

Step 3: Starting with any inner product $\langle -, - \rangle$, a G-invariant inner product is given by

$$\langle v, w \rangle_G := \frac{1}{|G|} \sum_{g \in G} \langle g v, g w \rangle.$$

The entire proof **breaks down** if $|G|$ is infinite or if the characteristic of $\mathbb{F}$ divides $|G|$.

Are there any other ways to perform "averaging" on groups?

Approach 1: Invariant integral

Lie groups are **orientable manifolds**. Any volume form defines an integral. But we can do even better:

Hurwitz (1897)

Every compact Lie group G has a unique integral $\int : C(G) \to \mathbb{R}$ satisfying

1. **Normalised:** $\displaystyle\int_G 1 \; dg = 1$;
2. **Left-invariance:** $\displaystyle\int_G f \circ \ell_h \; dg = \int_G f \; dg$ for any $h \in G$.

Analogue of left-invariance in the finite group setting:

$$\sum_{g \in G} f(g) = \sum_{g \in G} f(hg).$$

Approach 2: Haar measure

First define a **measure** on the group G and then use this **measure** to define the integral.

Haar (1933)

Every compact group G has a unique regular measure μ satisfying

1. **Normalised:** $\mu(G) = 1$;
2. **Left-invariance:** $\mu(h \cdot E) = \mu(E)$ for any $h \in G$ and measurable subset E of G.

In both cases, the **compactness** of G is required to ensure that the integral is finite.

Semisimplicity of Compact Lie Groups

$$\textbf{Finite groups: } \frac{1}{|G|}\sum_{g\in G} f(g), \quad \textbf{Unit circle } S^1\textbf{: } \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta$$

If V is a Lie group representation of G, then a ***G*-invariant inner product** is given by

$$\langle v, w\rangle_G := \int_G \langle gv, gw\rangle\, d\mu(g).$$

Weyl (1925)

Any Lie group representation of a compact Lie group G is a direct sum of irreducible representations.

Character Theory

If (V, ϕ) is a representation of G, the **character** of the representation is the map $\chi_V : G \to \mathbb{C}$ defined by

$$\chi_V(g) := \mathrm{Tr}_V(g).$$

In the theory of finite groups, characters are powerful tools to **simplify computations** without **sacrificing vital information**.

How much can be generalised to compact Lie groups?

$$\langle \chi_V, \chi_W \rangle_{L^2(G)} := \int_G \chi_V(g)\overline{\chi_W(g)}\, d\mu(g).$$

Schur Orthogonality Relations

If V and W are **irreducible** Lie group representations,

$$\langle \chi_V, \chi_W \rangle_{L^2(G)} = \begin{cases} 1 & \text{if } V \cong W \\ 0 & \text{otherwise.} \end{cases}$$

Hence a representation is determined up to isomorphism by its character.

Character Theory

Characters are **class functions**, i.e. functions satisfying

$$\psi(g) = \psi(hgh^{-1}) \quad \text{for all } h \in G.$$

Finite Groups:

Schur Orthogonality Relations

The set of irreducible characters $\{\chi_i\}_{i \in I}$ form an orthonormal basis for the space of class functions on G.

Corollary

The number of irreducible representations of G is equal to the number of conjugacy classes of G.

The Bridge to Harmonic Analysis

The inner product naturally defines a **Hilbert space structure** on the set of square-integrable (L^2) class functions on G.

$$\langle \chi_V, \chi_W \rangle_{L^2(G)} := \int_G \chi_V(g) \overline{\chi_W(g)} \, d\mu(g).$$

Compact groups:

Peter-Weyl Theorem (1927)

The set of irreducible characters $\{\chi_i\}_{i \in I}$ form an orthonormal (Hilbert) basis for the space of square-integrable (i.e. L^2) class functions on G.

Corollary

There is a (at most) countable number of irreducible representations of G.

Representations of SU(2)

$$\mathrm{SU}(2) := \left\{ \boldsymbol{A} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \;:\; \boldsymbol{A}^\dagger \boldsymbol{A} = \boldsymbol{I_2}, \;\; \det(\boldsymbol{A}) = 1 \right\}.$$

What are its **irreducible representations**?

For any $n \in \mathbb{Z}_{\geq 0}$, let V_n be the $\mathbb{C}$-vector space of homogeneous polynomials in two variables x and y of degree n. A basis for V_n is

$$\left\{ x^n, x^{n-1}y, \cdots, xy^{n-1}, y^n \right\}.$$

Each polynomial in V_n defines a **polynomial function** $P : \mathbb{C}^2 \to \mathbb{C}$. The group SU(2) acts on P by

$$(\boldsymbol{A} \cdot P)(\boldsymbol{v}) := P(\boldsymbol{v}\boldsymbol{A}).$$

Goal: Show that $\{V_n\}_{n=0}^{\infty}$ forms a **complete set of irreducible representations** of SU(2).

First step: Show that each V_n is irreducible i.e. the only SU(2)-equivariant map $T : V_n \to V_n$ is of the form $\lambda \cdot \mathbf{Id}$.

Irreducibility of V_n

Let $P_k = x^k y^{n-k}$. The corresponding basis is $\{P_n, P_{n-1}, \cdots, P_0\}$.

$$(\boldsymbol{A} \cdot P)(\boldsymbol{v}) := P(\boldsymbol{v}\boldsymbol{A}).$$

The action of the subgroup $D := \left\{ \boldsymbol{A} \in \mathrm{SU}(2) \;:\; \boldsymbol{A} = \operatorname{diag}\begin{pmatrix} a & a^{-1} \end{pmatrix} \right\}$ is:

$$\boldsymbol{A} \cdot P_k = a^{2k-n} P_k$$

for $P_k = x^k y^{n-k}$. Thus

$$V_n = \mathbb{C} \cdot P_n \oplus \mathbb{C} \cdot P_{n-1} \oplus \cdots \oplus \mathbb{C} \cdot P_0$$

is an **eigenspace decomposition** for D. Every $\mathrm{SU}(2)$-equivariant map $T : V_n \to V_n$ satisfies

$$\boldsymbol{A} \cdot T(P_k) = T(\boldsymbol{A} \cdot P_k) = a^{2k-n} T(P_k).$$

Hence T must preserve each eigenspace.

Irreducibility of V_n

What we have shown:

$$T(P_k) = \lambda_k P_k \quad \text{for some } \lambda_k \in \mathbb{C}.$$

Actual computations are now much more **tractable**.

By considering the rotations $r_t = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \in \mathrm{SU}(2)$, we get

$$\boldsymbol{A}(r_t \cdot P_n) = \sum_{k=0}^{n} \binom{n}{k} \cos^k t \cdot \sin^{n-k} t \cdot \lambda_k P_k$$

and

$$r_t \cdot \boldsymbol{A}(P_n) = \sum_{k=0}^{n} \binom{n}{k} \cos^k t \cdot \sin^{n-k} t \cdot \lambda_n P_k.$$

Hence $\lambda_k = \lambda_n$ for all n.

Completeness of the V_n's

To show completeness, we turn to **character theory**.

Every matrix in SU(2) is **conjugate** to a diagonal matrix. Let's analyse the diagonal matrices more closely:

$$\boldsymbol{D}(\theta) := \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}, \quad \theta \in [-\pi, \pi].$$

Two diagonal matrices $\boldsymbol{D}(\theta)$ and $\boldsymbol{D}(\psi)$ are conjugate iff $\theta = -\psi$.

The conjugacy classes of SU(2) is in bijection with $[0, \pi]$. The class functions of SU(2) can be identified with the **even functions** on $[-\pi, \pi]$.

The eigenvalues of $\boldsymbol{D}(\theta)$ are $\left\{e^{in\theta}, e^{i(n-2)\theta}, \cdots, e^{-in\theta}\right\}$. After some computation:

$$\chi_{V_n}(\theta) = \cos n\theta + \chi_{V_{n-1}}(\theta)\cos\theta$$

hence

$$\mathrm{span}_{\mathbb{C}}\left\{\chi_{V_0}(\theta), \cdots, \chi_{V_n}(\theta)\right\} = \mathrm{span}_{\mathbb{C}}\left\{1, \cos\theta, \cdots, \cos n\theta\right\}.$$

Completeness of the V_n's

In conclusion:

$$\mathrm{span}_{\mathbb{C}}\left\{\chi_{V_n}(\theta) \ : \ n \in \mathbb{Z}_{\geq 0}\right\} = \mathrm{span}_{\mathbb{C}}\left\{ \cos n\theta \ : \ n \in \mathbb{Z}_{\geq 0}\right\}.$$

From harmonic analysis: The cosine functions are **dense** in the space of **even** L^2 functions on $[-\pi, \pi]$.

If W is another irreducible representation of $\mathrm{SU}(2)$, then a density argument forces

$$\left\langle \chi_W, \chi_{V_k} \right\rangle_{L^2(G)} = 1$$

for some $k \in \mathbb{Z}_{\geq 0}$.

Even deeper connection: The **matrix coefficients** of the irreducible representations are precisely the **spherical harmonics** on $S^3 \cong \mathrm{SU}(2)$.

Maximal Torus Subgroups

We used the diagonal subgroup

$$D := \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \;:\; \theta \in [-\pi, \pi] \right\}$$

of SU(2) in our computations. The diagonal action is **simple to understand** yet still encodes **key information**.

This is a **maximal connected abelian** Lie subgroup of SU(2). Subgroups with these properties are known as **maximal torus subgroups**.

Conjugacy Theorem (Weyl 1925)

Let G be a compact connected Lie group and let T be a maximal torus subgroup of G. Then

$$G = \bigcup_{g \in G} gTg^{-1}.$$

Lie Algebra Representations of Lie Groups

How does this relate to what we have covered in this course?

If $\phi : \mathrm{SU}(2) \to \mathrm{GL}_{\mathbb{C}}(V)$ is a Lie group representation, then the Jacobian

$$d\phi|_e : \mathrm{Lie}(\mathrm{SU}(2)) \to \mathrm{End}(V)$$

is a **Lie algebra representation**.

Lie Group-Lie Algebra Correspondence

Let G and H be Lie groups, with G simply connected. If $\phi : \mathrm{Lie}(G) \to \mathrm{Lie}(H)$ is a Lie algebra homomorphism, then there exists a unique Lie group homomorphism f such that $\phi = df|_e$.

$$\mathrm{Lie}(\mathrm{SU}(2)) = \mathfrak{su}(2) := \left\{ \boldsymbol{X} \in \mathrm{Mat}_{2\times 2}(\mathbb{C}) \mid \boldsymbol{X}^\dagger = -\boldsymbol{X},\ \mathrm{Tr}(\boldsymbol{X}) = 0 \right\}.$$

We use the **passageway:**

$$\mathrm{SU}(2) \xrightarrow{\text{differentiation}} \mathfrak{su}(2) \xrightarrow{\text{complexification}} \mathfrak{sl}(2, \mathbb{C})$$

to obtain a **bijective correspondence**

$$\mathrm{SU}(2) \text{ action on } V_k \longleftrightarrow \mathfrak{sl}(2, \mathbb{C}) \text{ action on } V_k.$$

The End

Thank you for your attention.