Algorithmic Operation Research

Homework 2

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Problem 1

- Our goal is to find a differentiable function $f : \mathbb{R} \to \mathbb{R}$ such that f does not have an extremum at its critical point, i.e. that there are criticals points which are not local extrema.

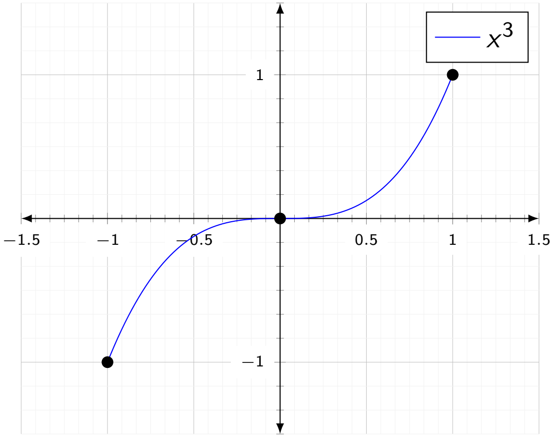
Function we are going to use

Let us consider the function $f : [-1, 1] \to \mathbb{R}$, such as $\mathbf{f(x) = x^3}$

- As we can see at the following $x_0 = 0$ is a critical point, but there is no extremum. The minimum and maximum values of the function occur for $x_1 = -1$ and $x_2 = 1$, neither of which are critical points.

Problem 1(cont'd)

Function Plot



Problem 2

Given S we want to maximize $a_1 \cdots a_n$ such that $a_1 + \cdots + a_n = S$, $a_i > 0$, $i \in N^*$. Let examine the below cases:

For $i = 1$

The problem is trivial as the positive integer S cannot be decomposed further more.

For $i = 1 \ldots 2$

The integer need to break in two parts. Then, to maximize their product, these parts should be equal.

For $i = 1 \ldots n$

The intenger need to break in n parts. Using the same concept their maximum product should be $n^{S/n}$.

Problem 2(cont'd)

So now, our goal is to find this value of n which maximizes the product.

Solution

This can be done by solving the equation $\frac{\partial n^{S/n}}{\partial n} = 0$.

Then, we have:

$$-Sn^{S/n-2}(lnn - 1) = 0 \Rightarrow lnn - 1 = 0 \Rightarrow lnn = 1 \Rightarrow n = e$$

As we know that $2 < e < 3$, each integer need to break into 2 or 3 only for maximum product.

Problem 3

The Diet Problem - Mathematical Model

Parameters

- F = set of foods
- N = set of nutrients
- a_{ij} = amount of nutrient i in food j, $\forall i \in N, \forall j \in F$
- c_j = cost per serving of food, $\forall j \in F$
- $F_{min,j}$ = minimum number of required servings of food, $\forall j \in F$
- $F_{max,j}$ = maximum number of required servings of food, $\forall j \in F$
- $N_{min,i}$ = minimum required level of nutrient, $\forall j \in F$
- $N_{max,i}$ = maximum required level of nutrient, $\forall j \in F$

Problem 3 (cont'd)

The Diet Problem - Mathematical Model

Decision Variables

x_j = number of servings of food i to purchase/consume, $\forall j \in F$

Objective Function

$$\text{Minimize Cost}(x_j) = \sum_{j \in F} c_j \cdot x_j$$

Problem 3 (cont'd)

The Diet Problem - Mathematical Model

Constraints Set 1

For each nutrient $i \in N$, at least meet the minimum required level

$$\sum_{j \in F} a_{ij} x_j \geq N_{min_i}, \forall i \in N$$

For each nutrient $i \in N$, do not exceed the maximum allowable level

$$\sum_{j \in F} a_{ij} x_j \leq N_{maxi}, \forall i \in N$$

Problem 3 (cont'd)

The Diet Problem - Mathematical Model

Constraints Set 2

For each food $j \in F$, select at least the minimum required number of servings

$$x_j \geq F_{min_j}, \forall j \in F$$

For each food $j \in F$, do not exceed the maximum allowable number of servings

$$x_j \leq F_{max_j}, \forall j \in F$$

Problem 3 (cont'd)

The Diet Problem - The Syntax of a Linear Programming Problem

$$c = \begin{bmatrix} c_1 \\ c_2 \\ . \\ . \\ . \\ c_n \end{bmatrix} \quad A = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1m} \\ a_{21} & a_{22} & . & . & a_{2m} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{(n-1)1} & a_{(n-1)2} & . & . & a_{(n-1)m} \\ a_{n1} & a_{n2} & . & . & a_{nm} \end{bmatrix} \quad x = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ b_2 \\ . \\ . \\ . \\ b_n \end{bmatrix}$$

Syntax of Problem

- objective function: min $c^T x, \quad c \in R^n$
- constraints: $Ax \lesseqgtr b, \quad A \in R^{m \times n}, \quad b \in R^m$,
- $x \geq 0, \quad x \in R^n, \quad 0 \in R^n$

Problem 3 (cont'd)

The Diet Problem - The Syntax of a Linear Programming Problem

- Any vector $x \in R^n$ satisfying all constraints is a **feasible solution**.
- Each $x^* \in R^n$ that gives the best possible value for $c^T x$ among all feasible x is an **optimal solution**.
- The value $c^T x^*$ is the **optimum value**.

Problem 3 (cont'd)

The Diet Problem - A Simple Example

Let's try to decide on lowest cost diet that provides sufficient amount of protein, with two choices:

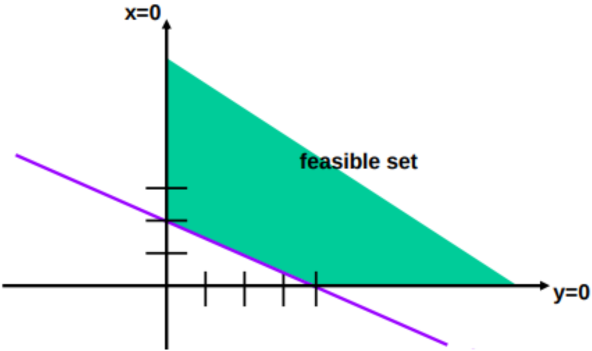
- steak (x): 2 units of protein/pound, \$3/pound.
- chocolate (y): 1 units of protein/pound, \$2/pound

In a proper diet, we need 4 units protein/day.

We want to minimize the total **cost** : $3x + 2y$, subject with **constraint**: $2x + y \geq 4 \quad \forall x, y \geq 0$

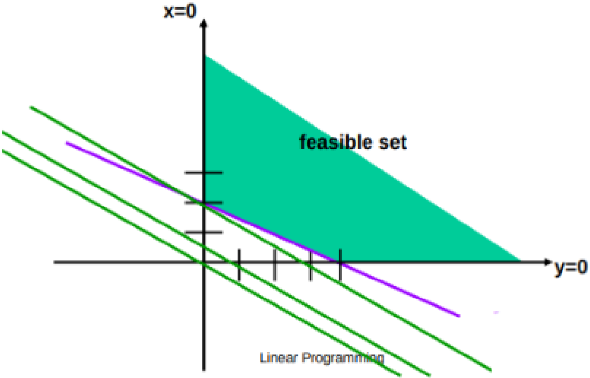
Problem 3 (cont'd)

The Diet Problem - A Simple Example



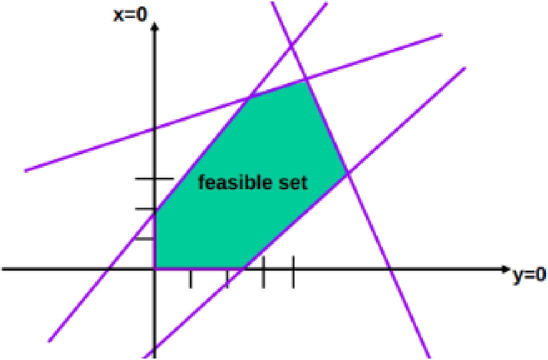
Problem 3 (cont'd)

The Diet Problem - A Simple Example



Problem 3 (cont'd)

The Diet Problem - A Simple Example



Problem 4

Definition of the matrix product

If A is an n x n matrix and B is an n x n matrix, the matrix product $C = AB$ is defined to be the n x n matrix, such that $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$.

So, the total number of operations for one element:

- n multiplications
- $n - 1$ additions

Since there are n^2 elements, we have:

Total number of operations

$$n^2(n + (n-1)) \Rightarrow n^2(2n-1)$$

Problem 5

Complexity for the Gaussian elimination

Goal

Our goal is to find the computational complexity for the Gaussian elimination of a matrix $A \in \mathbb{R}^{n*n}$

The complexity of the Gaussian elimination can be found based on the total number of operations of this kind:

$$r_i - (\frac{a_{ij}}{a_{jj}})r_j \rightarrow r_i, (j = 1, ..n-1, i = j+1, ..n)$$

Problem 5 (cont'd)

The division $\frac{a_{ij}}{a_{jj}}$ is carried for each of the components below the diagonal $a_{ij}\ \forall i > j$.

Number of divisions

$$(n-1)+(n-2)+...+1=\frac{n(n-1)}{2}=O(n^2)$$

Problem 5 (cont'd)

- Having found the scaling factor $\frac{a_{ij}}{a_{jj}}$, we need to use it to multiply through out the entire row.
- Therefore the number of multiplications will be $O(n^3)$.

Overall complexity of Gaussian elimination

$$O(n^3)$$

Problem 6

Linear Combination of vectors - Mathematical Approach

- What we are trying to find: *a vector of coefficients* $\vec{a}$.
- What we have: *n basis vectors* $\vec{b}_1 \ldots \vec{b}_n$.

Solution

Let's impose the constraints $\sum_{i=1}^{n} a_i b_j^i = y_i$. We can simply minimize a constant function subject to these constraints in order to find the desired coefficients.

Problem 6

Linear Combination of vectors - Example

Let:

$$b_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad b_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} \quad b_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix} \quad y = \begin{bmatrix} 6 \\ 4 \\ 3 \end{bmatrix}$$

We want to find real numbers a_1, a_2, and a_3 such that:

$$y = a_1 b_1 + a_2 b_2 + a_3 b_3$$

Problem 6

Linear Combination of vectors - Example

That leaves us with the following equations:

$$\begin{cases} \mathrm{a}_1 + 3a_2 + 3a_3 = 6 \\ 2\mathrm{a}_1 + a_2 + 2a_3 = 4 \\ \text{-}\mathrm{a}_1 + 2a_2 + -a_3 = 3 \end{cases}$$

Solving the above system we get that $\vec{a} = \{\frac{3}{2}, 2, -\frac{1}{2}\}$