Algorithmic Operation Research

Homework 1

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Problem 1

- Our goal is to show than given a convex set $C \subseteq \mathbb{R}^n$ with $x_1, ...x_k \in C$, and $\theta_1, ..., \theta_k \in \mathbb{R}$, such as $\theta_i \geq 0$ and $\theta_1 + ... + \theta_k = 1$, we have that $\theta_1 x_1 + ... + \theta_k x_k \in C$.
- We prove the result by induction.
- **Base Case:** Since C is convex the statement holds for $k = 1$, trivially, and by definition for $k = 2$
- **Induction hypothesis:** Suppose that the proposition is true for $k = n$

Problem 1(cont'd)

Induction Proof

- Consider the convex combination $\theta_1 x_1 + \cdots + \theta_{n+1} x_{n+1}$. Define

$$\Theta \sum_{i=1}^{n} \theta_i.$$

- We also have that

$$\theta_{n+1} = \sum_{i=1}^{n+1} \theta_i - \sum_{i=1}^{n} \theta_i = 1 - \Theta$$

- Then, we have:

$$\sum_{i=1}^{n} (\theta_i x_i) + \theta_{n+1} x_{n+1} = \Theta \sum_{i=1}^{n} (\frac{\theta_i}{\Theta} x_i) + (1 - \Theta) x_{n+1}$$

Problem 1(cont'd)

Note that $\sum_{i=1}^{n}(\frac{\theta_i}{\Theta}) = 1$ and so, by the induction hypothesis, $\sum_{i=1}^{n}(\frac{\theta_i}{\Theta}x_i) \in C$. Since $x_{n+1} \in C$ it follows that $\Theta \sum_{i=1}^{n}(\frac{\theta_i}{\Theta}x_i) + (1-\Theta)x_{n+1}$ is the convex combination of two points of C and hence lies in C.

Problem 2

We want to prove the following statement:
"C is convex $\leftrightarrow$ its intersection with any line (L) is convex".

- For the $\rightarrow$ we have:
  - **If** $C \cap L = \emptyset$**:**, C is convex by its definition.
  - **If** $C \cap L \neq \emptyset$**:** Let $x_1, x_2 \in C \cap L$.
    The two points also $\in C$, so, we have $y = \theta x_1 + (1-\theta)x_2 \in C$ by the definition of the convex set, and also their line segment $\in L$, so $y = \theta x_1 + (1-\theta)x_2 \in C \cap L$, which is the definition of a convex set. So $C \cap L$ is convex.

Problem 2(cont'd)

- For the $\leftarrow$ we have:
  - Let $x_1, x_2 \in C$, we want to prove that $\theta x_1 + (1-\theta)x_2 \in C$.
  - Let L be the line between x_1 and x_2.
  - The intersection of C with that line is a convex set, which means that $\theta x_1 + (1-\theta)x_2 \in C \cap L$, and therefore also to C.

Problem 3

By its definition, every affine set is convex. We have previously showed that the statement is true for convex sets, hence, it is true for affine sets as well.

Problem 4

- A set C is midpoint convex, if whenever two points $a, b \in C$, the average or midpoint $\frac{(a+b)}{2}$ is in C.
- Obviously, a convex set is midpoint convex.
- We want to prove that if C is closed and midpoint convex, then C is convex.

Problem 4(cont'd)

Solution

- We have to show that $\theta x_1 + (1-\theta)x_2 \in C$, $\forall x_1, x_2 \in C$ and $\forall \theta \in [0,1]$.
- By midpoint convexity we know

$$\frac{x_1 + x_2}{2} \in C \implies \frac{x_1 + \frac{x_1+x_2}{2}}{2} = \frac{3}{4}x_1 + \frac{1}{4}x_2 \in C$$

- By applying midpoint convexity k times we can show that

$$2^{-k}x_1 + (1-2^{-k})x_2 \in C.$$

Problem 4(cont'd)

Solution

- In this way, we showed that the proposition only for θ values that take the form of 2^{-k} where $k \in N$.
- To prove the proposition for all $\theta \in [0, 1]$ let $\theta^{(k)}$ be the binary number of length k.
- By midpoint convexity (applied k times, recursively)

$$\theta^{(k)} x_1 + (1 - \theta^{(k)}) x_2 \in C$$

- Because C is closed, it contains limits of sequences of its elements. This proves $\theta x_1 + (1 - \theta) x_2 \in C$. Hence, C is convex.

Problem 5

- Our goal is to show that the convex hull of a set S is the intersection of all convex sets that contain S.
- (The same method can be used to show that the conic, or affine, or linear hull of a set S is the intersection of all conic sets, or affine sets, or subspaces that contain S)
- Let H be the convex hull of S and let I be the intersection of all convex sets that contain S.

Problem 5(cont'd)

Solution

First we show that $H \subseteq I$.

- Suppose $x \in H$, a convex combination of some points $x_1, \ldots, x_n \in S$.
- Now let I be any convex set such that $S \subseteq I$.
- So, $x_1, \ldots, x_n \in I$. Since I is convex, and x is a convex combination of $x_1, \ldots, x_n$ it follows that $x \in I$.
- Hence, $H \subseteq I$.

Problem 5(cont'd)

Solution

For the second part, we show that $I \subseteq H$.

- By definition, H is convex and contains S.
- So, for some I in the construction of I we must have $H = I$.
- Hence $I \subseteq H$.

We have shown that $H \subseteq I$ and $I \subseteq H$. Hence, $H = I$

Problem 6

- Our goal is to find the distance between two parrallel hyperplanes.
- We can assume that the distance between two hyperplanes is also the distance between two vectors x_1 and x_2, where the hyperplanes intersect the extension of the vector a.
- We use a single vector a, because we know that the two hyperplanes are parallel.

Problem 6(cont'd)

- By combining the two equations, we get $a^T(x_1 - x_2) = (b_2 - b_1)$.
- To calculate the distance, we must take the euclidean norms of the above relation.
- So we have: $|a^T|(|x_1 - x_2)| = |b_1 - b_2|$.
- We know that $|a^T| = |a|$, so, finally, $|x_1 - x_2| = \frac{|b_1 - b_2|}{|a|}$.

So the distance between two hyperplanes is $\frac{|b_1 - b_2|}{|a|}$

Problem 7

- Supposing 2 distinct points $a, b \in \mathbb{R}^n$, we want to show that the set of all points that are closer (in Euclidean norm) to a than b is a halfspace.
- Because all norms of the vectors are positive numbers, we can assume: $|x - a|^2 < |x - b|^2$.
- From there, we can proceed as following.

Problem 7(cont'd)

$|x-a|^2 < |x-b|^2 \leftrightarrow (x-a)(x-a)^T \leq (x-b)(x-b)^T \leftrightarrow$
$x^Tx - 2a^Tx + a^Ta \leftrightarrow x^Tx - 2b^Tx + b^Tb \leftrightarrow 2(b-a)^Tx \leq b^Tb - a^Ta$.

Lets now suppose that $c = 2(b-a)$ and $d = b^Tb - a^Ta$.

So for every x, we have $c^Tx \leq d$. So the set of all points that are closer to a than b is the halfspace $c^Tx \leq d$, where $c = 2(b-a)$ and $d = b^Tb - a^Ta$.