Algorithmic Operation Research

Discrete Bidding Games

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Introduction

What is a Discrete Bidding Game?

Definition

Playing Discrete Bidding Games, is like playing any well known 2-player game, but instead of alternating moves, each player bids for the privilege to move. Each one has a pre-determined number of coins at the start of the game.

Introduction

Rules

- Each player has a number of chips, that may differ from one player to another.
- There is a tie-breaking advantage (symbolized as *), which is held by player A in the beginning. If the bids are equal, this advantage is used to determine which player gains the right to play.
- Every time the advantage is used, it is given on the opposite player.
- The game ends as every other game: the winner is the one that outplays their opponent.
- Money does not have any value at the end of the game.

Introduction

Similar Game Theories

- Similar bidding games were studied by **David Richman** in the late 1980s.
- The theory on which Discrete Bidding Games are based, has many similarities with Richman's theory.
- Thus, we will also examine this theory.

Introduction

Continuous Bidding Games

Differences

A key difference from the continuous-bidding model is that there, the issue of how to break ties was largely ignored, by only considering cases where the initial budget does not equal the money in a vertex V of the game graph.

Similar theory: Richman Games

Introduction

There are two known game theories: Matrix game theory, and combinatorial theory.

Matrix Games

Two players make simultaneous moves and a payment is made from one player to the other depending on the chosen moves. Optimal strategies often involve randomness and concealment of information.

Combinatorial Games

Two players move alternately. A player who cannot move loses. There is no hidden information and there exist deterministic optimal strategies.

David Richman suggested a class of games that share some aspects of both sorts of game theory.

Richman Games

Rules

- The game is played by two players , each of whom has some money.
- The two players repeatedly bid for the right to make the next move, by writing the amount of the bid in a simultaneously revealed piece of paper.
- Whoever makes the higher bid is eligible to play. Should the two bids be equal, the tie is broken by a toss of a coin.
- The sole objective of each player is to win the game: at the game's end, money loses all value.

Richman Games

Differences with Discrete Bidding

Despite the fact that this theory seems similar to our objective, there are indeed some differences.

- If the bids are tied, then a coin flip determines which player wins the bid.
- The Richman theory requires that the games be symmetric, with all legal moves available to both players, to avoid the possibility of zugzwang, positions where neither player wants to make the next move.

Moving forward...

However, we are going to dig deeper to the Richman theory, in order to construct strategies and techniques that are similar to our objective: discrete bidding games.

Richman Games

Richman Calculus

Basics

- The game is represented by a finite directed graph.
- There is an underlying combinatorial game in which a token rests on a vertex of this graph.
- There are two special vertices, denoted by b and r.
- The first player's goal is to bring the token to b and the other's goal is to bring the token to r.

Winning strategy

A policy for bidding and moving that guarantees a player the victory, given fixed initial data.

Richman Calculus

Constructing a winning strategy

Critical ratio

- Given a starting vertex v in the graph, there exists a critical ratio R(v) such that B has a winning strategy if B's share of the money, expressed as a fraction of the total money supply, is greater than R(v), and R has a winning strategy if B's share of the money is less than R(v).
- There exists a strategy such that if a player has more than R(v) and applies the strategy, the player will win with probability 1, without needing to know how much money the opponent has.

Richman Calculus

Constructing a winning strategy

- The critical (and in many cases optimal) bid for B is $R(v) - R(u)$ times the total money supply, where v is the current vertex and u is a successor of v for which R(u) is as small as possible.
- A player who cannot bid this amount has already lost, in the sense that there is no winning strategy for that player.
- A player who has a winning strategy of any kind and bids $R(v) - R(u)$ will still have a winning strategy one move later, regardless of who wins the bid, as long as the player is careful to move to u if he or she does win the bid.

Richman Calculus

Constructing a winning strategy

Recommended amount of money to bid

$R(v) - R(u)$ is a "fair price" that B should be willing to pay for the privilege of trading the position v for the position u.

Richman value

We define $1 - R(v)$ as the Richman value of the position v, so that the fair price of a move exactly equals the difference in values of the two positions.

However, it is more convenient to work with R(v) than with $1 - R(v)$.

Richman cost

We call $R(v)$ the **Richman cost** of the position v.

Richman Calculus

The Richman Cost Function

Definitions

Let D be a finite directed graph (V,E) with a blue vertex b and a red vertex r such that from every vertex there is a path to at least one of the vertices. Also, for every $v \in V$, let $S(v)$ be the set of successors of v in D.

Given any function $f : V \to [0,1]$, we define:

$$f^{+}(v) = \max_{w \in S(v)} f(w) \quad \text{and} \quad f^{-}(v) = \min_{w \in S(v)} f(w)$$

Richman Calculus

The Richman Cost Function

To successfully construct a strategy for a Richman game given a graph D, is to attribute costs to the vertices of D such that the cost of every vertex (except r and b) is the average of the lowest and highest costs of its successors.

Cost function definition

The **Richman Cost Function** is:

$$R(v) = \begin{cases} 1 & \text{if } v == r \\ 0 & \text{if } v == b \\ \frac{1}{2}(R^+(v) + R^-(v)) & \text{otherwise} \end{cases}$$

Richman Calculus

Playing a Richman Game

Let's suppose that R and B are playing a Richman Game. The game graph is D, and the token is initially located at vertex v.

Having a winning strategy

If B's share of the total money exceeds $R(v) = \lim_{t \to \infty} R(v, t)$, B has a winning strategy. Indeed, if his share of the money exceeds $R(v, t)$, his victory requires at most t moves.

Richman Calculus

Playing a Richman Game

Having incomplete knowledge

While playing a game, we may not have all the required information for the opponent. In this case, this information is the money amount of the other player. Surprisingly, it is often possible to implement a winning strategy without knowing it.

Safety ratio

We define B's **safety ratio** at v as the fraction of the total money that he has in his possession, divided by $R(v)$ (the fraction that he needs in order to win). Note that B will not know the value of his safety ratio, since we are assuming that he has no idea how much money Red has.

If B's safety ratio is grater than 1, then he has a strategy that wins with probability 1 and does not require knowledge of R's money supply.

If , moreover , the digraph D is acyclic, his strategy wins regardless of tiebreaks.

Richman Calculus

Constructing a winning strategy

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The Economist's View of Combinatorial Games

Game Notation

Game $G = \{G^L | G^R\}$

- Game defined by Left and Right options.
- Positive values favour Left player, negative favour Right.
- Base case: $\{\emptyset | \emptyset\}$, called the Endgame.
- Zero game is Right's win.
- Fuzzy game is Black's win.

The Economist's View of Combinatorial Games

Extension of Numbers

Numeric Games: $G^L < G < G^R$ (Conway's Sureal Numbers)

- $\{|\} = 0$. (any second-player win is equivalent)
- $\{0|\} = 1$ ... $\{n-1|\} =$ n.
- Reversed Game: $-G = \{-G^L| - G^R\}$
- $\{0|1\} = \frac{1}{2}, \{0|\frac{1}{2}\} = \frac{1}{4}$

The Economist's View of Combinatorial Games

Major Questions

Who is ahead and by how much ?

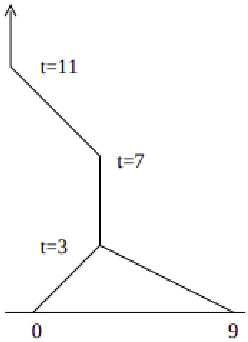
How big is the next move ?

The Economist's View of Combinatorial Games

Thermography

Thermography always gives the unique value for the count, $\mu(G)$, and a usable value for the size of the move, t(G).

- Very good estimation of the best move.
- Just a Little calculation.



The Economist's View of Combinatorial Games

Berlekamp's view

Berlekamp takes a different view:

- Game is a contest to accumulate cash.
- Answers previous questions with prices-real numbers that can be determined by competitive, free-market auctions.
- Uses competitive auctions to determine the sizes of the moves: Each move made by either player must be accompanied by a "tax", which the mover must pay to his opponent

The Economist's View of Combinatorial Games

The Game

Initial Setup:

- Bid for roles in each subgame
- Bid for privilege of first move
- Taxation - payment to opponent for making move

Player's Move:

- Pay and Play or
- Pass

The Economist's View of Combinatorial Games

The Game

Game Transitions:

- New decreased bid on taxation when both players pass.
- Higher bid plays next.
- Game ends when neither willing to $bid \geq t_{min}$
- Wealth determines victory in a zero sum game.
- $t_{min} = -1$, opponent pays you to play.

The Economist's View of Combinatorial Games

Sentestrat

Bidding:

- Use mean for Left player bid.
- Use temperature for tax/first move bid.

Playing:

- If opponent increases temp, respond locally.
- If *temp* $\geq$ *tax*, play in subgame of max temp.
- If *temp* $<$ *tax*, pass.

Sentestrat is optimal under Economist's Rules

The Economist's View of Combinatorial Games

Hot Games

A game is hot if havingthe move in it is an advantage.

- A very Hot Game: $G^L >> G^R$.
- Numbers are very cold.
- Moving in a number will only make our position worse.

Number Avoidance Theorem

Don't move in a Number, unless there's Nothing else to do!

In a hot game the players are eager to move and willing to pay t (victory points), as a price.

The Economist's View of Combinatorial Games

Cooling the Game

The game G cooled by the temperature $t \in R$ is:

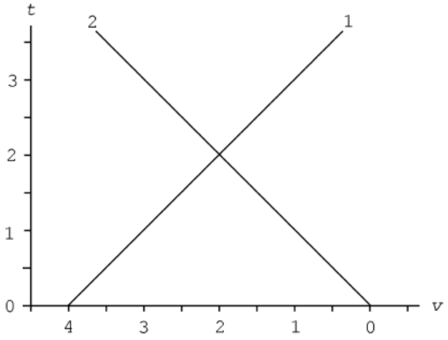
$$G(t) = \begin{cases} \{G(t)^L - t | G(t)^R + t\} & \text{if } \forall t' < t : G(t') - \epsilon \text{ is not a number} \\ \mu & \text{if } \exists t' < t : G(t') - \epsilon = \mu \text{ is a number} \end{cases}$$

In a hot game the players are eager to move and willing to pay t (victory points), as a price.

The Economist's View of Combinatorial Games

Example

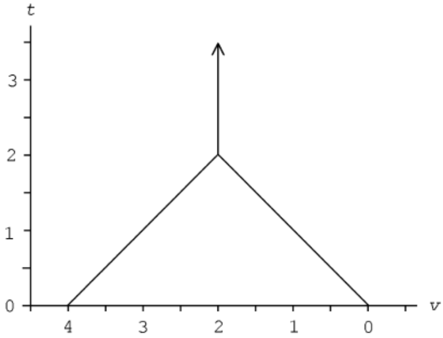
Game G $= \{4|0\}$.
Cooling the game by calculatinh: $G_1 = \{3|1\}$, $G_2 = \{2|2\}$.



The Economist's View of Combinatorial Games

Example

For $t > 2$ we do not impose more tax, so $G(t \geq 2) = 2$:



Bidding Tic-Tac-Toe

Implementing the game

- Bidding Tic-Tac-Toe has been implemented in python3 programming language.
- The aim of this project was to get a hands-on experience in a simple yet very famous strategy game, the Tic-Tac-Toe.

Bidding Tic-Tac-Toe

Implementing the game

Difficulties

- When it comes to bidding games, Tic-Tac-Toe seems to behave differently from its naive gaming approach.
- A computer can't play bidding Tic-Tac-Toe just by using a simple minimax, or even better an alpha-beta prunning algorithm.
- It would be catastrophic for the computer to evaluate all possible states as a combination of move and bidding.

Bidding Tic-Tac-Toe

Solution

- For this reason, bidding games like Tic-Tac-Toe can be solved by following Richman's theory.
- The source code for the agent player implements exactly the idea of the paper Richman Games.
- As the theory implies, we first assign discrete-Richman values to all terminal states.

Readings

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