Algorithmic Operation Research

Homework 5

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Problem 1

Problem-Solving Approach

Our goal

Considering the following primal problem, form the revised primal problem to standard form and then convert its dual equivalent maximization problem.

$$
\begin{aligned}
\min \quad & x_1 - x_2 \\
\text{s.t.} \quad & 2x_1 + 3x_2 - x_3 + x_4 \leq 0 \\
& 3x_1 + x_2 + 4x_3 - 2x_4 \geq 3 \\
& -x_1 - x_2 + 2x_3 + x_4 = 6 \\
& x_1 \leq 0,\ x_2, x_3 \geq 0,\ x_4 \in R
\end{aligned}
$$

Problem 1 cont'd

Convert primal to standard form

Our goal

- The signs in all the constraints in primal must be in matching with its objective
- All primal's variables x_i must be ≥ 0

$$
\begin{aligned}
\min \quad & -x_1' - x_2 \\
\text{s.t.} \quad & 2x_1' - 3x_2 + x_3 - x_4' + x_4'' \geq 0 \\
& -3x_1' + x_2 + 4x_3 - 2x_4' + 2x_4'' \geq 3 \\
& x_1' - x_2 + 2x_3 + x_4' - x_4'' \geq 6 \\
& -x_1' + x_2 - 2x_3 - x_4' + x_4'' - 6 \\
& x_1', x_2, x_3, x_4', x_4'' \geq 0
\end{aligned}
$$

Problem 1 cont'd

Format the dual problem

The equivalent maximization problem is:

$$
\begin{aligned}
\max \quad & 3w_2 + 6w_3 \\
\text{s.t.} \quad & 2w_1 - 3w_2 + w_3 \le -1 \\
& -3w_1 + w_2 - w_3 \le -1 \\
& w_1 + 4w_2 + 2w_3 \le 0 \\
& w_1 + 2w_2 - w_3 = 0 \\
& w_1, w_2 \ge 0, \;\; w3 \in R
\end{aligned}
$$

Problem 2

Dual problems

Our goal

- Considering the following primal problem, form the dual problem and convert it into an equivalent minimization problem.
- Derive a set of conditions on the matrix A and the vectors b,c under which the dual is identical to the primal.

$$
\begin{aligned}
\min c'x \\
s.t\ Ax \geq b \\
x \geq 0
\end{aligned}
$$

Problem 2 cont'd

Conversion to dual

At first we have to convert the LP problem into a dual one:

L.P. problem

$$\min c'x$$
$$s.t \; Ax \geq b$$
$$x \geq 0$$

$$\downarrow$$

Dual problem

$$\max p'c$$
$$s.t \; p'A \leq c$$
$$p \geq 0$$

Problem 2 cont'd

Conversion to equivalent minimization problem

Then, we have to convert the dual problem into an equivalent minimization problem:

Dual problem

$$\max p'c$$
$$s.t \;\; p'A \leq c$$
$$p \geq 0$$

$$\downarrow$$

E.M. problem

$$\min -p'b$$
$$s.t \;\; -p'A \geq -c$$
$$p \geq 0$$

Problem 2 cont'd

A set of conditions for problems to be equal

Restrictions

- A is a $n \times n$ matrix, in order for the problems to be of the same dimensions.
- $b = -c$, in order to satisfy the constraints.

As a result of the first restriction, we have

$$-p'A \geq -c' \leftrightarrow -A'p \geq -c \leftrightarrow A$$

Then, by applying the second restriction, we can ensure that

$$-p'b = c'x$$

So now we have a dual problem identical to the original one.

Problem 3

Our goal

Show that solving linear programming problems is no harder than solving systems of linear inequalities.

We are going to use:

- A **subroutine** which, given a system of linear inequalities either produces a solution or decides that no solution exists.
- An **LP problem** that has an optimal solution.

How we're going to prove it

Create an algorithm that given a system of inequalities, calls the subroutine to provide us a solution equal to the LP problem's solution.

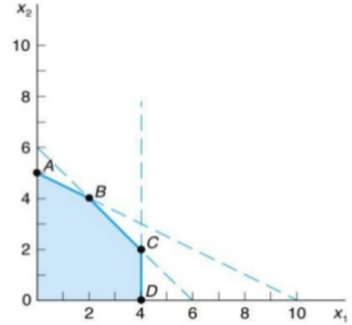
Problem 3

The algorithm

1. We change the LP Problem, thus it only contains inequalities as constraints (by adding a bias on the one side of a possible equality).
2. We call the subroutine, by passing all the constraints of the LP problem as arguments.
3. The subroutine returns the optimal solution for all of its arguments, so we have the vector of solutions for the LP problem.
4. We then create the halfspace by the function of the variables of the LP problem, and we find the intersection with the previously mentioned vector.
5. Thus, we have an optimal solution to our LP problem, aka the feasible solution space.

Graphical example

$$\textbf{min } 4x_1 + 5x_2$$
$$s.t \; x_1 + 2x_2 < 10$$
$$6x_1 + 6x_2 < 36 \quad x_1 < 4$$
$$x_1, x_2 \geq 0$$



Problem 4

Our goal

Let A be a symmetric matrix. Considering the following primal problem, prove that if x^* satisfies:

- $Ax^* = c$
- $x^* \geq 0$

then x^* is an optimal solution.

$$\begin{aligned} \min c'x \\ s.t\ Ax \geq c \\ x \geq 0 \end{aligned}$$

Problem 4 cont'd

Form the dual problem

Problem-Solving Approach

- Suppose $Ax^* = c$ and $x^* \geq 0$
- Form the dual of the above linear programming problem

Conversion to dual

$$\begin{aligned} \max p'c \\ s.t\ p'A \geq c \\ p \geq 0 \end{aligned}$$

Problem 4 cont'd

Conclusion

Let $p^* = x^*$, then clearly $p^* = x^* \geq 0$

Goal

We have to show that objective functions of the primal and the dual problem take on equivalent values for solutions x^*, p^* respectively.

Because A is symmetric and square we have:

$$p'^* A = A' p^* = A p^* = A x^* = c$$

Thus p^* is a feasible solution and because:

$$c' x^* = x'^* c = p'^* c$$

we have reached our goal.

Problem 5

Complementary Slackness

Theorem

Consider an x_0 and y_0, feasible in the primal and dual respectively. That is, $\mathrm{A}x_0 \cdot \mathrm{b}$ and $A^T y_0 = \mathrm{c}$; $y_0 \geq 0$. Then $c^T x_0 = y_0^T b$ if and only if $(y_0)_i > 0$ then $A_i x_0 = b_i$

In other words if a dual variable is greater than zero (slack) then the corresponding primal constraint must be an equality (tight). Also, if the primal constraint is slack then the corresponding dual variable is tight (or zero).

Problem 5, cont'd

Proof

First assume that the complementary slackness condition holds. We need to prove that the costs of the points are equal.

$$y_0^T = \sum_{j=1}^{m} y_{0j} b_j = \sum_{j=1}^{m} y_{0j}(A_j x_0)$$

$$= \sum_{j=1}^{m} y_{0j}(\sum_{i=1}^{n} A_{ji} x_{0i})$$

$$= \sum_{i=1}^{n} x_{0i}(\sum_{j=1}^{m} A_{ji} y_{0j})$$

$$= x_0^T c = c^T x_0$$

Problem 5, cont'd

Proof

Now assume that the costs are equal.

$$c^T x_0 = x_0^T c$$

$$= \sum_{i=1}^{n} x_{0i} (\sum_{j=1}^{m} A_{ji} y_{0j})$$

$$= \sum_{j=1}^{m} y_{0j} (\sum_{i=1}^{n} A_{ji} x_{0i})$$

$$= \sum_{j=1}^{m} y_{0j} (A_j x_0) \leq \sum_{j=1}^{m} y_{0j} b_j = y_0^T$$

But we know that $c^T x_0 = y_0^T b$. Hence, $y_0 > 0$ then $A_i x_0 = b_i$