Algorithmic Operational Research Project

Discrete Bidding Games

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Introduction

What is a Discrete Bidding Game?

Playing Discrete Bidding Games, is like playing any well known 2-player game, but instead of alternating moves, each player bids for the privilege to move. Each one has a pre-determined number of coins at the start of the game. The rules are the following:

- Each player has a number of chips, that may differ from one player to another.
- There is a tie-breaking advantage (symbolized as *), which is held by player A in the beginning. If the bids are equal, this advantage is used to determine which player gains the right to play.
- Every time the advantage is used, it is given on the opposite player.
- The game ends as every other game: the winner is the one that outplays their opponent.
- Money does not have any value at the end of the game.

Similar Game Theories

Similar bidding games were studied by David Richman in the late 1980s. The theory on which Discrete Bidding Games are based, has many similarities with Richman's theory. Thus, we will also examine this theory.

Difference with Continuous Bidding Games

A key difference from the continuous-bidding model is that there, the issue of how to break ties was largely ignored, by only considering cases where the initial budget does not equal the money in a vertex V of the game graph. In discrete-bidding, however, ties are a central part of the game. A discrete-bidding game is characterized explicitly by a tie-breaking mechanism in addition to the standard components, i.e., an arena, the players' budgets, and an objective.

Richman Games

Introduction

There are two known game theories: Matrix game theory, and combinatorial theory.

In matrix games, two players make simultaneous moves and a payment is made from one player to the other depending on the chosen moves. Optimal strategies often involve randomness and concealment of information.

On the other hand, in combinatorial games, two players move alternately. We may assume that each move consists of sliding a token from one vertex to another along an arc in a directed graph. A player who cannot move loses. There is no hidden information and there exist deterministic optimal strategies.

David Richman suggested a class of games that share some aspects of both sorts of game theory. The rules of those game where originally as listed:

- The game is played by two players , each of whom has some money.
- The two players repeatedly bid for the right to make the next move, by writing the amount of the bid in a simultaneously revealed piece of paper.
- Whoever makes the higher bid is eligible to play. Should the two bids be equal, the tie is broken by a toss of a coin.
- The sole objective of each player is to win the game: at the game's end, money loses all value.

Despite the fact that this theory seems similar to our objective, there are indeed some differences.

- If the bids are tied, then a coin flip determines which player wins the bid.

- The Richman theory requires that the games be symmetric, with all legal moves available to both players, to avoid the possibility of zugzwang, positions where neither player wants to make the next move.

However, we are going to dig deeper to the Richman theory, in order to construct strategies and techniques that are similar to discrete bidding games.

Strategy

Basic Richman Calculus

From now on, we will denote player 1 as B and player 2 as R.

Let the game be represented by a finite directed graph. There is an underlying combinatorial game in which a token rests on a vertex of this graph. There are two special vertices, denoted by b and r; the first player's goal is to bring the token to b and the other's goal is to bring the token to r.

A winning strategy is a policy for bidding and moving that guarantees a player the victory, given fixed initial data.

It is proven, that we can use the following facts in order to construct a winning strategy:

- Given a starting vertex v in the graph, there exists a critical ratio R(v) such that B has a winning strategy if B's share of the money, expressed as a fraction of the total money supply, is greater than R(v), and R has a winning strategy if B's share of the money is less than R(v).

- There exists a strategy such that if a player has more than R(v) and applies the strategy, the player will win with probability 1, without needing to know how much money the opponent has.

It is also shown that the critical (and in many cases optimal) bid for B is $R(v) - R(u)$ times the total money supply, where v is the current vertex and u is a successor of v for which R(u) is as small as possible.

A player who cannot bid this amount has already lost, in the sense that there is no winning strategy for that player. On the other hand, a player who has a winning strategy of any kind and bids $R(v) - R(u)$ will still have a winning strategy one move later, regardless of who wins the bid, as long as the player is careful to move to u if he or she does win the bid.

Thus, $R(v) - R(u)$ is a "fair price" that B should be willing to pay for the privilege of trading the position v for the position u. We define $1 - R(v)$ as the Richman value of the position v, so that the fair price of a move exactly equals the difference in values of the two positions. However, it is more convenient to work with R(v) than with $1 - R(v)$. We call $R(v)$ the **Richman cost** of the position v.

The Richman Cost Function

Let D be a finite directed graph (V,E) with a blue vertex b and a red vertex r such that from every vertex there is a path to at least one of the vertices. Also, for every $v \in V$, let $S(v)$ be the set of successors of v in D.

Given any function $f : V \to [0, 1]$, we define:

$$f^{+}(v) = \max_{w \in S(v)} f(w) \quad \text{and} \quad f^{-}(v) = \min_{w \in S(v)} f(w)$$

To successfully construct a strategy for a Richman game given a graph D, is to attribute costs to the vertices of D such that the cost of every vertex (except r and b) is the average of the lowest and highest costs of its successors. Thus, the **Richman Cost Function** is:

$$R(v) = \begin{cases} 1 & \text{if } v == r \\ 0 & \text{if } v == b \\ \frac{1}{2}(R^{+}(v) + R^{-}(v)) & \text{otherwise} \end{cases}$$

Playing the Game

It is proven that the Richman cost function of the digraph D of a Richman Game is unique.

So, let's suppose that R and B are playing a Richman Game. The game graph is D, and the token is initially located at vertex v. If B's share of the total money exceeds $R(v) = \lim_{t \to \infty} R(v, t)$, B has a winning strategy. Indeed, if his share of the money exceeds $R(v, t)$, his victory requires at most t moves.

Considering the need to move, we denote the difference $(R^{+}(v) - R^{-}(v))$ as a measure. If this difference is positive, then both players want to move, and if the difference is negative then both players want to force the other to move.

Incomplete Knowledge

It is common that while playing a game, we may not have all the required information for the opponent. In this case, this information is the money amount of the other player. Surprisingly, it is often possible to implement a winning strategy without knowing it.

We define B's **safety ratio** at v as the fraction of the total money that he has in his possession, divided by $R(v)$ (the fraction that he needs in order to win). Note that B will not know the value of his safety ratio, since we are assuming that he has no idea how much money Red has.

If B's safety ratio is grater than 1, then he has a strategy that wins with probability 1 and does not require knowledge of R's money supply. If , moreover , the digraph D is acyclic, his strategy wins regardless of tiebreaks.

The Economist's View of Combinatorial Games

A Combinatorial Game

A game is defined recursively as a divided set $G = \{G^L|G^R\}$ where G^L and G^R are sets of games. The base case is the game $\{\emptyset|\emptyset\}$, called the Endgame. The line | is just a partition line, and has no connection to the similar line in set theory. The game is viewed as a position from where both players can move. The nodes are positions and the branches are moves. Branches going left are player Left's possible moves, and branches going right are Right's moves.

Examples

1. The simplest game is the Endgame. We draw the tree corresponding to the Endgame $\{\emptyset|\emptyset\}$ as a root without any branches: In this game neither player has any moves, so the game consists of but one position.

2. Using the Endgame as the Left option G^L, and the empty set as the Right option G^R, we get the game $\{\{\emptyset|\emptyset\}|\emptyset\}\}$. In this game Left has one move, namely to the Endgame, and Right has no moves. The corresponding tree must have one branch going left and none going right: Left will always win this game: If Right starts, she loses (because she has no legal moves). If Left starts, he can move to the Endgame; and Right, who is next, loses.

3. We have labelled the nodes, such that we can refer to them. If Right starts, she can move to the Endgame (D), and Left, who is next, loses. If Left starts, he can move to the position that we have labelled (B). From there Right, who is then next, has no moves; whereas Left could move to the Endgame (C). The game A is thus $\{B|D\}$, where the game B is $\{C|\emptyset\}$, and both C and D are $\{\emptyset|\emptyset\}$. So writing it all out with empty sets as the definition prescribes, we get A = $\{\{\emptyset|\emptyset\}|\emptyset\}|\{\emptyset|\emptyset\}$.

Figure 1: Example 1

Figure 2: Example 2

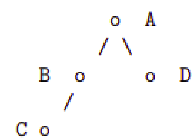


Figure 3: Example 3

An Introduction

At every position of such games there two important questions that need to be answered: **Who is ahead and by how much?** and **How big is the next move?**. Classical abstract combinatorial game theorists answer these questions with a value and an incentive. These answers are precisely correct when the objective of the game is to get the last legal move. Values and incentives are themselves games, and can quickly become complicated.

Berlekamp takes a different view. He views the game as a contest to accumulate points, which can eventually be converted into cash. He also, monetarizes the answers to our opening two questions into prices-real numbers that can be determined by competitive, free-market auctions. Specifically, after two gurus have completed their studies of a position, the economist might ask each to submit a sealed bid, representing the amount the guru is willing to pay in order to play Black. The bid may be any real number, including a negative ones (if the position favors White). Assuming the bids differ, the referee computes their average μ. The player with the higher bid plays Left, in return for which he pays μ dollars to his opponent. The player with the lower bid plays Right. Each player is necessarily happy with this assignment, since it is no worse for him than the bid he submitted.

Berlekamp also uses competitive auctions to determine the sizes of the moves. Throughout the game, the economist requires that each move made by either player must be accompanied by a "tax", which the mover must pay to his opponent. At the beginning of the game, the tax rate is determined by a competitive auction. Each player submits a sealed bid. The bids are opened, and the higher one becomes the new tax rate. The player making this bid makes the first play at the new tax rate. Then, at each turn, a player may elect to either

- pay the current tax (to his opponent) and play a move on the board, or
- pass at the current tax rate.

After two consecutive passes at the current tax rate, a new auction is held to restart the game at a new and lower tax rate. Legal bids must be strictly less than the prior tax rate. There is also a minimum legal bid, t_{min}. A player unwilling to submit a bid $\geq t_{min}$ may pass the auction. The game ceases when both players pass the auction.

If the players submit equal bids, the referee may let an arbitrator decide who moves next. The arbitrator is disreputable: Each player suspects that the arbitrator may be working in collusion with his opponent. Alternatively, when there are equal bids for the tax rate, the referee may require that the next move be made by the player who did not make the prior move. In this way, the referee, at his sole discretion, can implement a "preference for alternating moves".

In some traditional games, the minimum tax rate t_{min} is defined as 0. When the game stops, the score is tallied according to the position of the board, and the corresponding final payment of "score" is made. This "score" payment is added to the tax payments that the players have paid each other, and to the payments to buy the more desirable colors initially. Since all payments are zero-sum, one player's net gain is his opponent's net loss. The player who realizes an overall net gain is the winner. In Economic games, a tied outcome occurs whenever each player's total net cumulative payment to his opponent is zero.

For Winning Ways-style games, which include mathematical play on "numbers" beyond the conventional mathematical "stopping positions", t_{min} is defined to be -1.

When such a game ceases, no score remains on the board: the outcome depends entirely on the net cumulative total of payments made between the players.

Erdoos' Book

The mathematician Paul Erdoos frequently mentions "The Book", a supernatural reference document that contains all mathematical truths in their most revealing forms. The Book Strategy is as follows:

1. Initially, bid μ for the privilege of playing.
2. At every stage of play, define the global temperature of the game as the maximum of the temperatures of its regions. If it is your turn, proceed as follows:
   (a) If opponent's prior move has created a region whose temperature exceeds the current tax rate, respond by playing in the same region as the opponent's prior move.
   (b) Otherwise, if the temperature of the game is at least as large as the current tax rate, play in a region of maximum temperature.
   (c) If the tax rate exceeds the maximum regional temperature, pass. If you have any canonical move(s), bid the maximum regional temperature.

The strategy is designed to answer directly this fundamental question:

- Should I respond locally to my opponent's prior move?
- Does my opponent's prior move have sente?

Book Play Ensures No Monetary Loss

After the initial auction has determined who plays Left and who plays Right, then either player may elect to use the Book Strategy to select his subsequent tax rate bids and plays. The following theorem asserts that this strategy is optimal for games played according to the Economist's Rules: If Left player, plays Book Strategy, she ensures a total net gain of at least μ dollars. If Right player plays Book Strategy, he ensures that Left's total net gain will be at most μ dollars.

Thermography

Introduction

Thermography may be viewed as a methodology for deriving the Book values of μ and t. Classical thermography for loopfree games is presented in Winning Ways. Thermography always gives the unique value for the count, μ(G), and a usable value for the size of the move, t(G). This Thermostatic Strategy provides a very good estimation of the best move, and takes just a little calculation.

Scaffolds

The final stages of the construction of a thermograph begin with another pair of trajectories LS and RS, called scaffolds. In the most general case, the Left and Right scaffolds might cross several times.

Hills

A temperature interval in which the Left scaffold exceeds the Right scaffold is a hill, or solid region. Within a hill, the Left wall equals the Left scaffold and the Right wall equals the Right scaffold.

Caves

A temperature interval in which the Right scaffold exceeds the Left scaffold is a cave, or gaseous region. Within a cave, the Left and Right walls coincide in a local trajectory called the mast.

Hot Games and Confusion

First, we need to introduce the temperature of a game. A game is hot if having the move in it is an advantage. So a game where $G^L >> G^R$ for some options is a very hot game. On the other hand, numbers are very cold, because $G^L < G < G^R$ for all the options, and moving in a number will only make our position worse; as the Number Avoidance Theorem states: **Don't move in a Number, unless there's Nothing else to do!**

Numbers

Numbers represent the number of free moves, or the move advantage of a particular player. By convention positive numbers represent an advantage for Left, while negative numbers represent an advantage for Right. They are defined recursively with 0 being the base case.

Translation Principle

If x is a number and the game $G = \{G^L|G^R\}$ is not a number, then $G = \{G^L + x|G^R + x\}$. We realize that the numerical options are dominated; and that is the reasoning behind the Number Avoidance Theorem. Since numbers are not interesting to play, sensible players will just stop the game and sum up the score, when all the components of the game have become numbers. Using this stopping criterion, we can define the Left and Right stops of the game G.

The Stops of the Game

The value that the game G will stop at, if Left moves first, is called the Left stop of G, and is denoted L(G). Similarly, the Right stop R(G) is the value that the game will stop at, if Right moves first. The result of a game lies between the two stops, with the players fighting to pull it in their direction. The interval of the possible outcomes is called the confusion interval of the game.

The Confusion Interval

The interval between the Left and the Right stops contains the possible1 outcomes of the game and is called the confusion interval.

The larger the confusion interval, the hotter the game, and the more eager the players are to move in it.

We want to decrease the confusion in order to find the mean value of a hot game; so we cool the game down to a number.

To cool the game, we will lower the tension by imposing a tax t on moves. Left, who wants the game to be as great (positive) as possible, will be less anxious to move the smaller his options are, so we subtract a positive number t from all the Left options. Right wants the game to be as negative as possible, so we add the positive number t to all her options, making her less anxious to move.

When the tax we impose is large enough to cool the game down infinitely close to a number, then neither player has any incentive to move in it anymore, and there is no need for a further taxation.

Cooling of a Game

The game G cooled by the temperature $t \in R$ is:

$$G(t) = \begin{cases} \{G(t)^L - t | G(t)^R + t\} & \text{if} \quad \forall \quad t' < t: \quad G(t') - \epsilon \quad \text{is not a number} \\ \mu & \text{if} \quad \exists \quad t' < t: \quad G(t') - \epsilon = \mu \quad \text{is a number} \end{cases}$$

Where ϵ is an arbitrary infinitesimal. So the first line of the definition is in effect, as long as G(t) is not infinitely close to a number.

Another way of viewing taxation is the following: In a hot game the players are eager to move, therefore the players are interested in paying for the privilege of moving. The players are willing to pay t (victory points), so this is the price.

Example

Let us look at the game $G = \{4|0\}$ as an example. This is not a number, since $4 - 3$. If we cool this game by one degree, we get:

$$G_1 = \{4 - 1|0 + 1\} = \{3|1\}$$

This game is not infinitely close to a number, so we impose some more tax. If we cool the game down another degree, we get:

$$G_2 = \{4 - 2|0 + 2\} = \{2|2\}$$

This is infinitely close to the number 2; and imposing more tax will still yield 2. We conclude that the game $G = \{4|0\}$ can be cooled down to 2, the mean value of the game. When cooling a game, there will always be a taxation for which the game G(t) is a number.

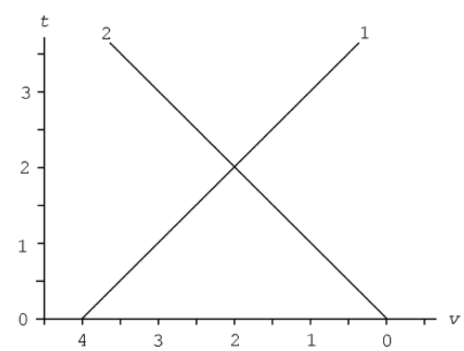
The Temperature of a Game

The temperature T of a game G is the smallest taxation t, for which the game G(t) is infinitely close to a number. The temperature of the game in the example above is 2.

Drawing Thermographs

The coordinate system used in thermography is rotated by $\frac{\pi}{2}$ about the origin with respect to the usual one. This implies that the tax level t (the independent variable) is on the vertical axis, and the game value G(t) is on the (reversed) horizontal one. The reason behind this rotated coordinate system is our definition that Left plays for positive values and Right plays for negative ones. This way Left's options are mapped to the left, and Right's options to the right.

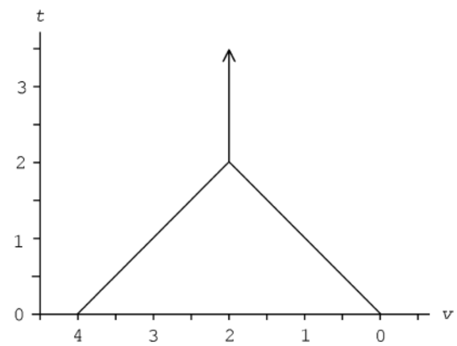
Let us reuse the game G = $\{4|0\}$. To draw its Thermograph we begin by marking the uncooled (t = 0) Left and Right options on the horizontal axis, $G^L = 4$, $G^R = 0$. Then we plot the game values that we calculated in the example: $G_1 = \{3|1\}$, $G_2 = \{2|2\}$.



For $t > 2$ we do not impose more tax, so $G(t \geq 2) = 2$. If we calculated G(t) for all taxes $t \quad \exists \quad [0, 2]$, we would get the following graph:

Sentestrat and Thermostrat

If the current tax rate is t, and there are several current positions whose thermographs are hills (or summits) at temperature t, then these two strategies choose among those options in slightly different ways:



1. Thermostrat: Move in a region whose hill is widest at temperature t.
2. Sentestrat: If your opponent has just played in one of these hill regions, respond directly by playing in the same region in which he just moved.

Both strategies essentially agree under other circumstances. If all thermographs are caves at temperature t, then pick one of them whose mast touches your own scaffold at temperature t, if any such exist. When you are about to play a move of this type, you should decline any tax cut proposal that may be pending. If the only hill move at the current temperature is banned, then pass but do not lower the bid tax rate. If there are no moves of any of the previously mentioned types, then you should pass and propose a lower tax rate. Sentestrat proposes the tax rate equal to the next lowest value at which you would be willing to move, namely, the drain of a cave or the point at which a vertical mast touches your (necessarily jagged) scaffold inside the cave. Thermostrat calculates other trajectories in hopes of getting a stronger result. Thermostrat computes the sum of all of your opponent's walls from Sentestrat's proposed tax rate on downwards, and to this sum it adds the width of the widest cave at each temperature. It then picks the temperature at which this trajectory is most favorable. Often, that is the same temperature as Sentestrat's, but on some examples, Thermostrat takes advantage of an opponent's recent mistake to outperform Sentestrat, even though both do equally well against a good opponent.

An Economic Guru Kibitzes a Conventional Game

Sentestrat and Thermostrat both ensure perfect play according to economic rules. Either strategy can also be used in conventional games, in which there are no taxes and alternating play is required. In loopfree games, these strategies ensure that the first player will attain a stopping position at least as good as the mean. When there are many summands and the temperature is t, a good player can often improve that score by about 0.5t.

Playing a Discrete Bidding Game

Rules

Let's suppose A and B are playing a game. Given the previously mentioned Richman calculus, we have a game graph G, which we suppose is bounded. We compute the critical thresholds $R(v)$ for all positions $v \in G$ by working backwards from the end. positions.

$$R(v) = \begin{cases} 1 & \text{if } v == r \\ 0 & \text{if } v == b \\ \frac{1}{2}(R^{+}(v) + R^{-}(v)) & \text{otherwise} \end{cases}$$

Generally, if v is an ending position, then $R(v)$ is either 1 or 0, and $\frac{1}{2}(R^{+}(v) + R^{-}(v))$ if not. So the optimal bid, given those circumstances, for each player is $\Delta_v = |\frac{1}{2}(R^{+}(v) - R^{-}(v))|$.

Bidding TIc-Tac-Toe

Bidding Tic-Tac-Toe has been implemented in python3 programming language. The aim of this project was to get a hands-on experience in a simple yet very famous strategy game, the Tic-Tac-Toe.

However when it comes to bidding games, Tic-Tac-Toe seems to behave differently from its naive gaming approach. In other words, a computer can't play bidding Tic-Tac-Toe just by using a simple minimax, or even better an alpha-beta prunning algorithm. It would be catastrophic for the computer to evaluate all possible states as a combination of move and bidding.

For this reason, bidding games like Tic-Tac-Toe can be solved by following Richman's theory. The source code for the agent player implements exactly the

idea of the paper Richman Games. Specifically, we first assign discrete-Richman values to all terminal states.
In order to run the program , make sure you have python3 installed in your computer, navigate to src directory, and execute the following command:

```
$ python3 main,py
```

If you desire to run with command line arguments, get helped just by typing:

```
$ python3 python main.py --help
```

Readings

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