Design and Analysis of Algorithms

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Divide and Conquer

Revisit: Merge Sort

Merge Sort

$m = \lfloor (|A| - 1)/2 \rfloor;$

$B = A[0, ..., m];$

$C = A[m + 1, ..., |A| - 1];$

$B =$MergeSort$(B);$

$C =$MergeSort$(C);$

$A =$Merge$(B,\ C);$

Merge Sort

$$\begin{array}{cccc} 0 & 1 & 2 & 3 \\ & \uparrow & & \\ & m & & \end{array} \qquad \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \\ & & \uparrow & & \\ & & m & & \end{array}$$

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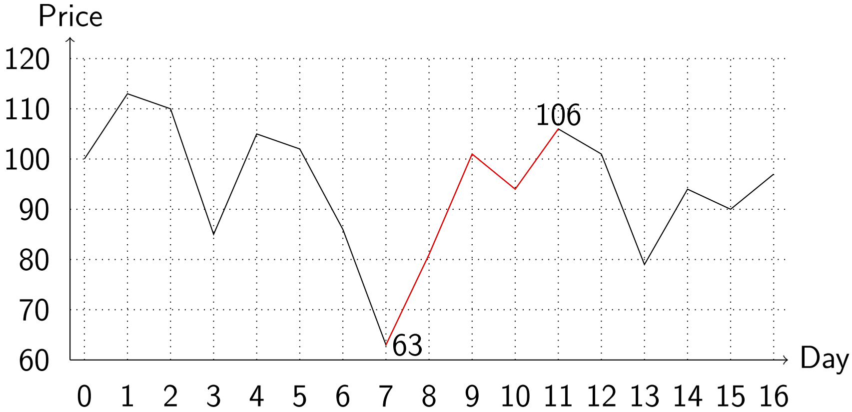
$$\boxed{T(n) = 2T(n/2) + n}$$

Example: The Maximum-Subarray Problem

Buy Low and Sell High

Buy Low and Sell High



Buy Low and Sell High

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Buy Low and Sell High

Buy Low and Sell High

Brute Force: $O(n^2)$

Algorithm: BuySell(P)

```
n = |P|;
b = s = −1;
max = 0;
for i = 0 to n − 2 do
    for j = i + 1 to n − 1 do
        p = P[j] − P[i];
        if p > max then
            max = p;
            b = i;
            s = j;
        end
    end
end
Return b, s;
```

Sum of The Changes

Day	0	1	2	3	4	5	6	7
Price	100	110	90	103	115	108	95	101

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$$
\begin{aligned}
-90 + 108 &= -90 + 103 - 103 + 115 - 115 + 108 \\
&= (103 - 90) + (115 - 103) + (108 - 115)
\end{aligned}
$$

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$$\begin{aligned} -90 + 108 &= -90 + 103 - 103 + 115 - 115 + 108 \\ &= (103 - 90) + (115 - 103) + (108 - 115) \end{aligned}$$

$$\arg\max_{(i,j)} P[j] - P[i] \iff \arg\max_{(i,j)} \sum_{k=i+1}^{j} C[k]$$

Find The Maximum-Subarray

Problem: Given an array, find the continuous subarray of the maximum sum.

$L = [0, -2, 3, 1, -2, 5, -6, 2]$
maximum-subarray:
$3 + 1 - 2 + 5 = 7$

Algorithm: MaxSub(C)

$b = s = -1$;
$max = 0$;
for i, j **do**
    $sum = \sum_{k=i+1}^{j} C[k]$;
    if $sum > max$ **then**
        $max = p$;
        $b = i$;
        $s = j$;
    end
end
Return b, s;

Find The Maximum-Subarray

10	12	-30	5	20	-25	18	3

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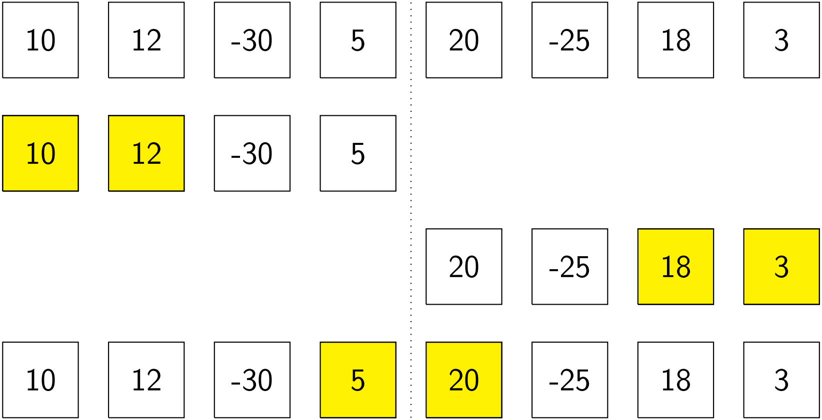
Find The Maximum-Subarray



Maximum-Subarray Crossing The Separator

Algorithm: MaxCross(C, m)

```
s = 0;
if m < |C| - 1 then
    a = 0;
    for i = m to 0 do
        a = max(a, sum(C[i, m]));
    end
    b = 0;
    for i = m + 1 to |C| - 1 do
        b = max(b, sum(C[m + 1, i]));
    end
    s = a + b;
end
Return s;
```

Find The Maximum-Subarray

- Divide C into two parts:
  - A: the first half
  - B: the second half
- Find the maximum subarray in each part: $s1$ and $s2$.
- Find the maximum subarray crossing the separator: $s3$.
- Compare $s1, s2$ and $s3$ to find the maximum one.

Algorithm: MaxSub2(C)

```
s = C[0];
if |C| > 1 then
    m = ⌊(|C| − 1)/2⌋;
    A = C[0, m];
    B = C[m + 1, |C| − 1];
    s1 = MaxSub2(A);
    s2 = MaxSub2(B);
    s3 = MaxCross(C, m);
    s = max(s1, s2, s3);
end
Return s;
```

Find The Maximum-Subarray: $O(n \log n)$

$$
\begin{array}{lll}
1: & T(n) = 2T(n/2) + n & \longrightarrow \quad n \\
2: & \quad = 4T(n/4) + 2(n/2) + n & \longrightarrow \quad 2n \\
3: & \quad = 8T(n/8) + 4(n/4) + 2(n/2) + n & \longrightarrow \quad 3n \\
i: & \quad = 2^i T(n/2^i) + 2^{i-1}(n/2^{i-1}) + \cdots + n & \longrightarrow i \cdot n \\
k: & \quad = 2^k T(n/2^k) + 2^{k-1}(n/2^{k-1}) + \cdots + n & \longrightarrow k \cdot n
\end{array}
$$

$T(n) = O(n \log n)$.

The Paradigm of Divide and Conquer

Divide and Conquer

1. **Divide** the problem instance into two/several smaller instances of the same problem.
2. **Conquer** the smaller problems.
3. **Combine** the results of the smaller problems to get the result of the original (large) instance.

Recall that in MergeSort,

1. **Divide** the numbers into two subsets.
2. **Conquer** the sorting problem on each of the subsets.
3. **Combine** the results, by Merge.

Recursion tree of $T(n) = 2T(n/2) + n$

$T(n)$ $\cdots$ n

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$$T(n) \qquad \cdots \qquad n$$

$$T(\tfrac{n}{2}) \qquad T(\tfrac{n}{2}) \qquad \cdots \qquad n$$

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$T(\frac{n}{4})$ $T(\frac{n}{4})$ $T(\frac{n}{4})$ $T(\frac{n}{4})$ $\cdots$ n

---

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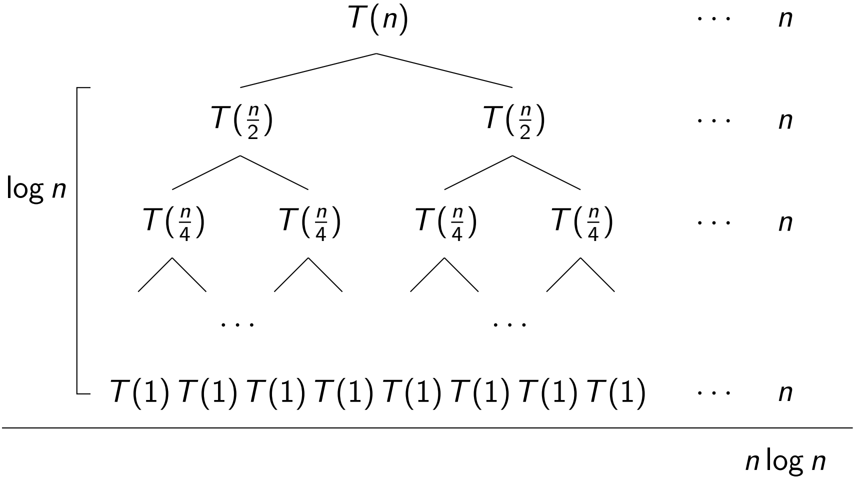
$$\cdots \qquad \cdots$$

$$T(1)\,T(1)\,T(1)\,T(1)\,T(1)\,T(1)\,T(1)\,T(1) \qquad \cdots \quad n$$

Recursion tree of $T(n) = 2T(n/2) + n$



Recursion tree of $T(n) = 3T(n/4) + n^2$

$T(n)$ n^2

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$$T\left(\frac{n}{4}\right) \qquad T\left(\frac{n}{4}\right) \qquad T\left(\frac{n}{4}\right) \qquad\qquad 3\left(\frac{n}{4}\right)^2$$

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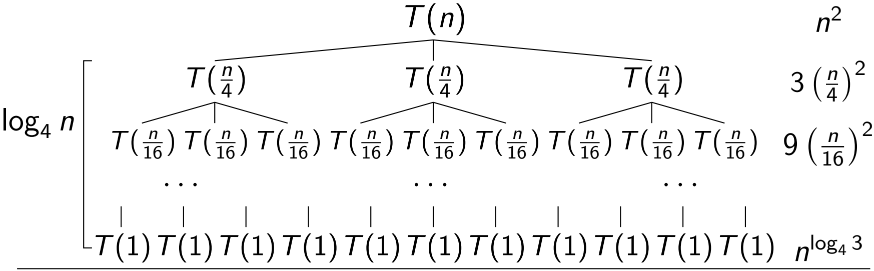
$$T\left(\tfrac{n}{16}\right)\ T\left(\tfrac{n}{16}\right)\ T\left(\tfrac{n}{16}\right)\ T\left(\tfrac{n}{16}\right)\ T\left(\tfrac{n}{16}\right)\ T\left(\tfrac{n}{16}\right)\ T\left(\tfrac{n}{16}\right)\ T\left(\tfrac{n}{16}\right)\ T\left(\tfrac{n}{16}\right) \qquad 9\left(\tfrac{n}{16}\right)^2$$

Recursion tree of $T(n) = 3T(n/4) + n^2$

$$
\begin{array}{c c}
T(n) & n^2 \\
T\left(\frac{n}{4}\right) \quad T\left(\frac{n}{4}\right) \quad T\left(\frac{n}{4}\right) & 3\left(\frac{n}{4}\right)^2 \\
T\left(\frac{n}{16}\right)\ T\left(\frac{n}{16}\right)\ T\left(\frac{n}{16}\right)\ T\left(\frac{n}{16}\right)\ T\left(\frac{n}{16}\right)\ T\left(\frac{n}{16}\right)\ T\left(\frac{n}{16}\right)\ T\left(\frac{n}{16}\right)\ T\left(\frac{n}{16}\right) & 9\left(\frac{n}{16}\right)^2 \\
\cdots \qquad \cdots \qquad \cdots & \\
T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1)\ T(1) & n^{\log_4 3} \\
\hline
\end{array}
$$

Recursion tree of $T(n) = 3T(n/4) + n^2$



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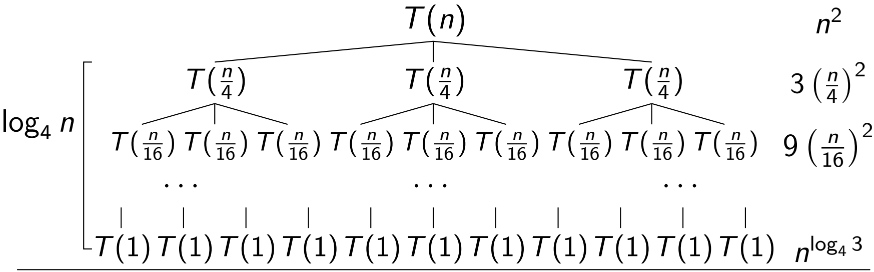
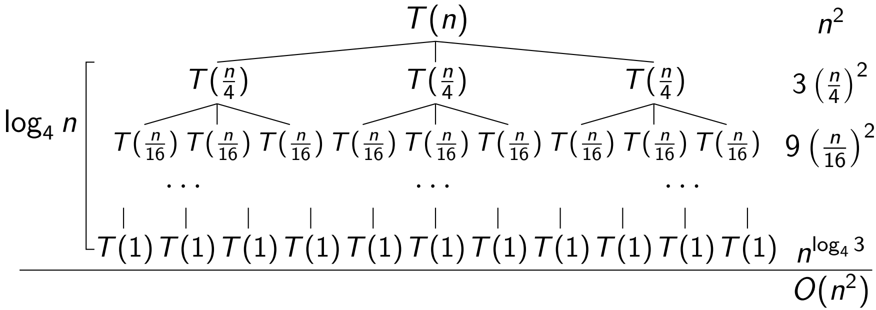


$$1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \cdots = \Theta(1)$$

Recursion tree of $T(n) = 3T(n/4) + n^2$



$$1 + \frac{3}{16} + \left(\frac{3}{16}\right)^2 + \cdots = \Theta(1)$$

Master Theorem

$T(n) = aT(n/b) + f(n)$, where $a \geq 1$ and $b > 1$

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- if $af(n/b) = Kf(n)$ for some constant $K > 1$, then

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- if $af(n/b) = f(n)$, then

$$T(n) = \Theta(f(n) \log n)$$

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- if $af(n/b) = f(n)$, then

$$T(n) = \Theta(f(n) \log n)$$

- if $af(n/b) = \kappa f(n)$ for some constant $\kappa < 1$, then

$$T(n) = \Theta(f(n))$$

Master Theorem for $f(n) = O(n^d)$

$T(n) = aT(n/b) + O(n^d)$, where $a \geq 1$, $b > 1$ and $d \geq 0$,

- if $d < log_b a$ then $T(n) = O(n^{\log_b a})$;
- if $d = log_b a$, then $T(n) = O(n^d \log n)$;
- if $d > log_b a$, then $T(n) = O(n^d)$.

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Recall that $af(n/b) = af(n)/b^d$. Then

- $d < log_b a \Leftrightarrow b^d < b^{\log_b a} = a \Leftrightarrow a/b^d > 1$;
- $d = log_b a \Leftrightarrow b^d = b^{\log_b a} = a \Leftrightarrow a/b^d = 1$;
- $d > log_b a \Leftrightarrow b^d > b^{\log_b a} = a \Leftrightarrow a/b^d < 1$.

Examples

$$T(n) = T(3n/4) + n$$

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- $a = 1$, $b = 4/3$, $d = 1$

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- $a = 1$, $b = 4/3$, $d = 1$
- $d = 1 > 0 = \log_{4/3} 1 = \log_b a \Rightarrow T(n) = O(n)$

Examples

$$T(n) = 3T(n/2) + n$$

Examples

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- $a = 3$, $b = 2$, $d = 1$

Examples

$T(n) = 3T(n/2) + n$

- $a = 3$, $b = 2$, $d = 1$
- $d = 1 < \log_2 3 = \log_b a \Rightarrow T(n) = O(n^{\log_2 3})$

Examples

$$T(n) = 4T(n/2) + n \log n$$

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- $f(n) = n \log n$.

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- $a = 4$, $b = 2$
- $f(n) = n \log n$.
- $af(n/b) = 4(n/2)\log(n/2) = 2n \log n - 2n$.

Examples

$T(n) = 4T(n/2) + n \log n$

- $a = 4,\ b = 2$
- $f(n) = n \log n$.
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- $2f(n) > af(n/b) > 1.9f(n)$ for **sufficient large** n.

Examples

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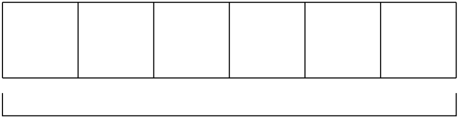
- $f(n) = n \log n$.
- $af(n/b) = 4(n/2)\log(n/2) = 2n\log n - 2n$.
- $2f(n) > af(n/b) > 1.9f(n)$ for **sufficient large** n.
- $T(n) = \Theta(n^{\log_b a}) = \Theta(n^2)$

Find The Lost Number

Find The Lost Number

Problem: Given $2^d - 1$ distinct numbers from $[0, 1, \ldots, 2^d - 1]$, find the lost one.



d bits

2^d numbers represented by d bits

Find The Lost Number

1	1	1

1	1	0

1	0	0

0	1	1

0	1	0

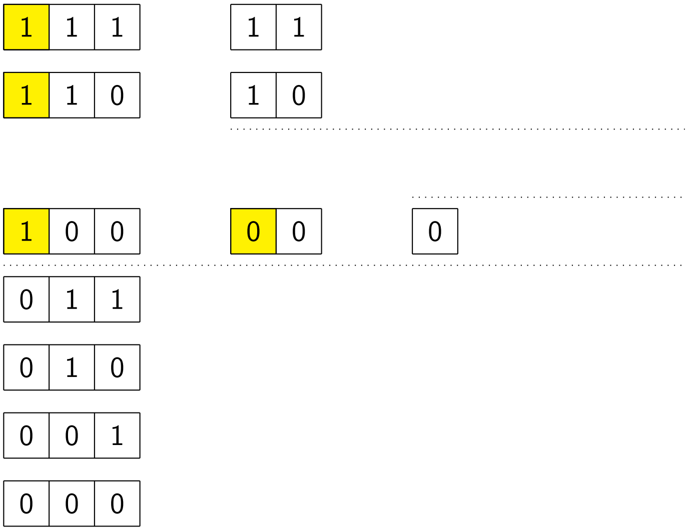
0	0	1

0	0	0

Find The Lost Number

1	1	1		1	1
1	1	0		1	0
1	0	0		0	0
0	1	1			
0	1	0			
0	0	1			
0	0	0			

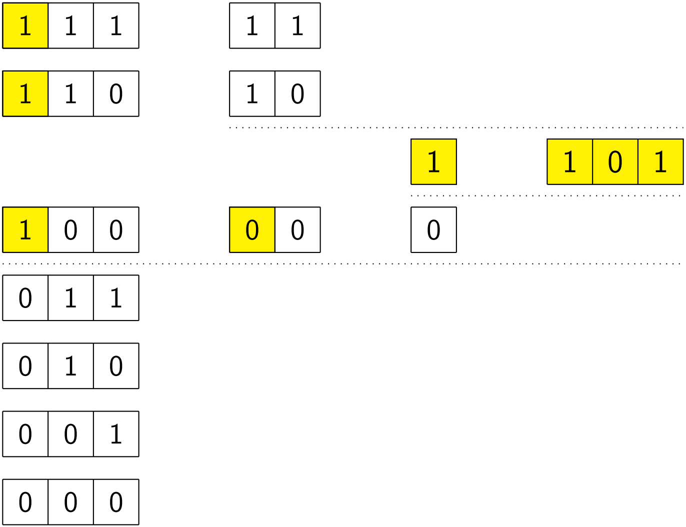
Find The Lost Number



Find The Lost Number

1 1 1	1 1	
1 1 0	1 0	
		1
1 0 0	0 0	0
0 1 1		
0 1 0		
0 0 1		
0 0 0		

Find The Lost Number



Find The Lost Number: $O(n)$

$$n = 2^d - 1$$

$$\begin{aligned} T(n) &= T(n/2) + n \\ &= T(n/4) + n/2 + n \\ &= ... + n/4 + n/2 + n \\ &= O(n) \end{aligned}$$

Matrix Multiplication

Matrix Multiplication

A $n \times m$ matrix

$$A = \begin{bmatrix} a_{0,0} & a_{0,1} & \cdots & a_{0,m-1} \\ a_{1,0} & a_{1,1} & \cdots & a_{0,m-1} \\ \cdots & \cdots & \cdots & \cdots \\ a_{n-1,0} & a_{n-1,1} & \cdots & a_{n-1,m-1} \end{bmatrix}$$

When $n = m$, A is called a square matrix. For two $n \times n$ matrices A and B, the multiplication is defined as $C = A \times B$, where C is also a $n \times n$ matrix, and

$$c_{i,j} = \sum_{k=0}^{n-1} a_{i,k} b_{k,j}$$

Matrix Multiplication

$a_{0,0}$	$a_{0,1}$	$a_{0,2}$
$a_{1,0}$	$a_{1,1}$	$a_{1,2}$
$a_{2,0}$	$a_{2,1}$	$a_{2,2}$

$b_{0,0}$	$b_{0,1}$	$b_{0,2}$
$b_{1,0}$	$b_{1,1}$	$b_{1,2}$
$b_{2,0}$	$b_{2,1}$	$b_{2,2}$

Matrix Multiplication

$$\begin{array}{|c|c|c|}\hline a_{0,0} & a_{0,1} & a_{0,2} \\ \hline a_{1,0} & a_{1,1} & a_{1,2} \\ \hline a_{2,0} & a_{2,1} & a_{2,2} \\ \hline\end{array} \qquad \begin{array}{|c|c|c|}\hline b_{0,0} & b_{0,1} & b_{0,2} \\ \hline b_{1,0} & b_{1,1} & b_{1,2} \\ \hline b_{2,0} & b_{2,1} & b_{2,2} \\ \hline\end{array}$$

$$\begin{array}{|c|c|c|}\hline c_{0,0} & \phantom{c_{0,0}} & \phantom{c_{0,0}} \\ \hline & & \\ \hline & & \\ \hline\end{array} \qquad c_{0,0} = \sum_{k=0}^{2} a_{0,k} b_{k,0}$$

Matrix Multiplication

$a_{0,0}$	$a_{0,1}$	$a_{0,2}$
$a_{1,0}$	$a_{1,1}$	$a_{1,2}$
$a_{2,0}$	$a_{2,1}$	$a_{2,2}$

$b_{0,0}$	$b_{0,1}$	$b_{0,2}$
$b_{1,0}$	$b_{1,1}$	$b_{1,2}$
$b_{2,0}$	$b_{2,1}$	$b_{2,2}$

$c_{0,0}$	$c_{0,1}$	

$$c_{0,0} = \sum_{k=0}^{2} a_{0,k} b_{k,0}$$

$$c_{0,1} = \sum_{k=0}^{2} a_{0,k} b_{k,1}$$

Matrix Multiplication

$a_{0,0}$	$a_{0,1}$	$a_{0,2}$
$a_{1,0}$	$a_{1,1}$	$a_{1,2}$
$a_{2,0}$	$a_{2,1}$	$a_{2,2}$

$b_{0,0}$	$b_{0,1}$	$b_{0,2}$
$b_{1,0}$	$b_{1,1}$	$b_{1,2}$
$b_{2,0}$	$b_{2,1}$	$b_{2,2}$

$c_{0,0}$	$c_{0,1}$	$c_{0,2}$
$c_{1,0}$	$c_{1,1}$	$c_{1,2}$
$c_{2,0}$	$c_{2,1}$	$c_{2,2}$

$$c_{0,0} = \sum_{k=0}^{2} a_{0,k} b_{k,0}$$

$$c_{0,1} = \sum_{k=0}^{2} a_{0,k} b_{k,1}$$

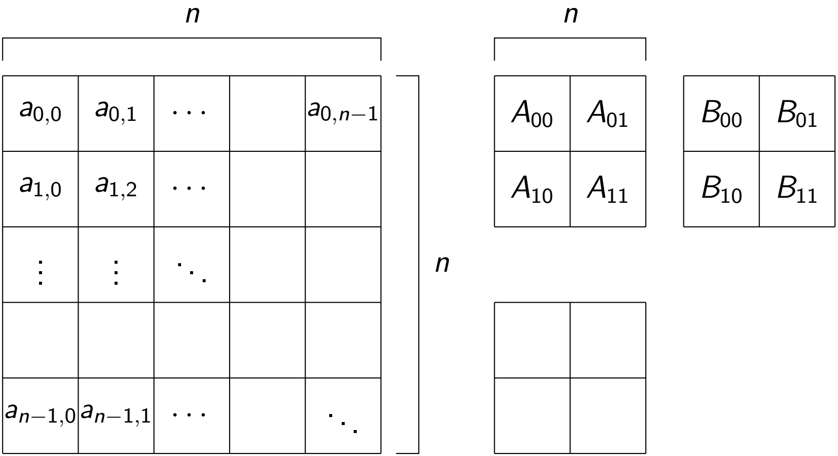
$$c_{2,2} = \sum_{k=0}^{2} a_{2,k} b_{k,2}$$

Follow The Definition: $O(n^3)$

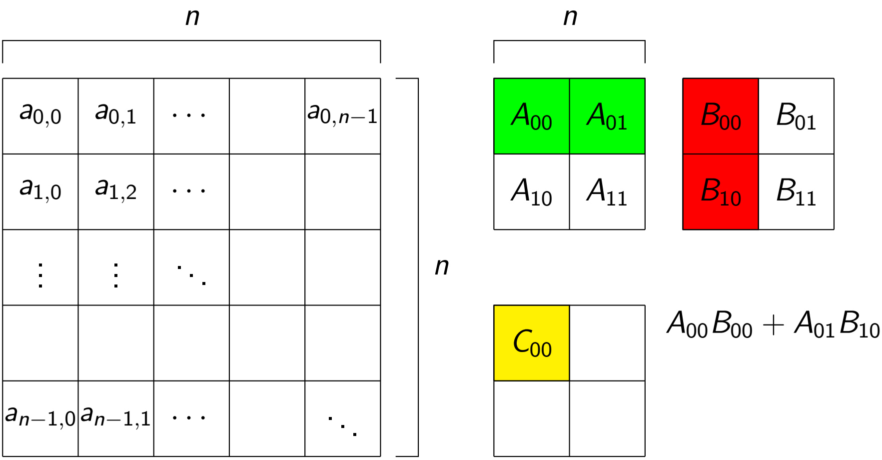
Algorithm: MatrixMul(A, B)

$C = \mathbf{0}^{n \times n}$;
for $i = 0$ *to* $n - 1$ **do**
    for $j = 0$ *to* $n - 1$ **do**
        for $k = 0$ *to* $n - 1$ **do**
            $c_{i,j} = c_{i,j} + a_{i,k} b_{k,j}$;
        end
    end
end
Return C;

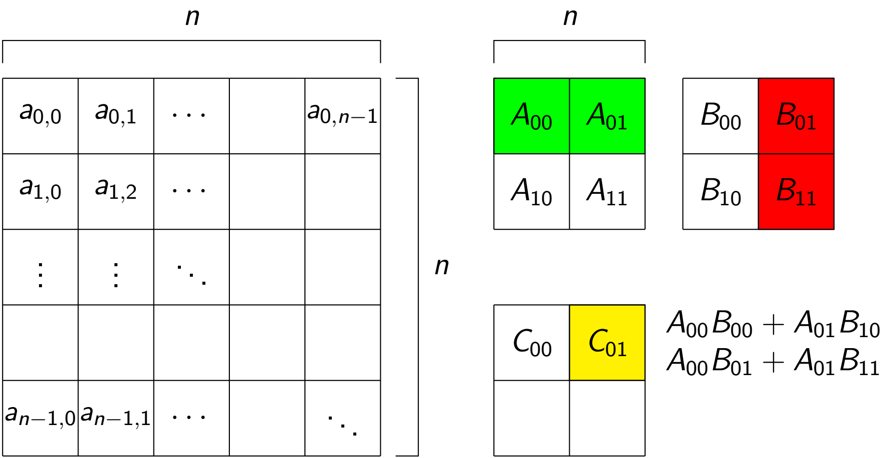
Simple Divide and Conquer Algorithm



Simple Divide and Conquer Algorithm



Simple Divide and Conquer Algorithm



Simple Divide and Conquer Algorithm

$$
\overbrace{\begin{array}{|c|c|c|c|c|}
\hline
a_{0,0} & a_{0,1} & \cdots & & a_{0,n-1} \\
\hline
a_{1,0} & a_{1,2} & \cdots & & \\
\hline
\vdots & \vdots & \ddots & & \\
\hline
 & & & & \\
\hline
a_{n-1,0} & a_{n-1,1} & \cdots & & \ddots \\
\hline
\end{array}}^{n}
\left.\vphantom{\begin{array}{c} a \\ a \\ a \\ a \\ a \end{array}}\right\} n
$$

$$
\overbrace{\begin{array}{|c|c|}
\hline
A_{00} & A_{01} \\
\hline
A_{10} & A_{11} \\
\hline
\end{array}}^{n}
\qquad
\begin{array}{|c|c|}
\hline
B_{00} & B_{01} \\
\hline
B_{10} & B_{11} \\
\hline
\end{array}
$$

$$
\begin{array}{|c|c|}
\hline
C_{00} & C_{01} \\
\hline
C_{10} & \\
\hline
\end{array}
\qquad
\begin{array}{l}
A_{00}B_{00} + A_{01}B_{10} \\
A_{00}B_{01} + A_{01}B_{11} \\
A_{10}B_{00} + A_{11}B_{10}
\end{array}
$$

Simple Divide and Conquer Algorithm

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 & & & & \\
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Simple Divide and Conquer Algorithm: $O(n^3)$

- $T(n) = 8T(n/2) + n^2$

Algorithm: MatrixMul2(A, B)

$C = \mathbf{0}^{n \times n}$;
for $i, j = 0, 1$ **do**
    $M_0 = MatrixMul2(A_{i0}, B_{0j})$;
    $M_1 = MatrixMul2(A_{i1}, B_{1j})$;
    $C_{ij} = M_0 + M_1$;
end
Return C;

Simple Divide and Conquer Algorithm: $O(n^3)$

- $T(n) = 8T(n/2) + n^2$
- $a = 8$
- $b = 2$
- $d = 2$

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Better Divide and Conquer Algorithm

$T(n) = 8T(n/2) + n^2$
$\Rightarrow T(n) = O(n^3)$

What about

- $T(n) = \color{red}{7}T(n/2) + n^2$

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What about

- $T(n) = 7T(n/2) + n^2$
- $a = 7$
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- $d = 2$
- $\log_b a \approx 2.807 > d$

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- $a = 7$
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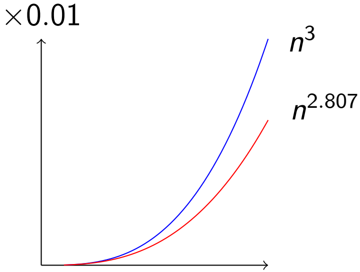
- $d = 2$
- $\log_b a \approx 2.807 > d$
- $T(n) = O(n^{\log_b a}) = O(n^{2.807})$



Better Divide and Conquer Algorithm

$$\begin{aligned}
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C_{01} &= A_{00}B_{01} + A_{01}B_{11} \\
C_{10} &= A_{10}B_{00} + A_{11}B_{10} \\
C_{11} &= A_{10}B_{01} + A_{11}B_{11}
\end{aligned}$$

- 8 multiplications; each cost $O(n^3)$ operations
- 4 additions; each cost $O(n^2)$ operations

Better Divide and Conquer Algorithm

$$
\begin{aligned}
S_1 &= B_{01} - B_{11} & P_1 = A_{00}S_1 &= A_{00}B_{01} - A_{00}B_{11} \\
S_2 &= A_{00} + A_{01} & P_2 = S_2B_{11} &= A_{00}B_{11} + A_{01}B_{11} \\
S_3 &= A_{10} + A_{11} & P_3 = S_3B_{00} &= A_{10}B_{00} + A_{11}B_{00} \\
S_4 &= B_{10} - B_{00} & P_4 = A_{11}S_4 &= A_{11}B_{10} - A_{11}B_{00} \\
S_5 &= A_{00} + A_{11} & P_5 = S_5S_6 &= A_{00}B_{00} + A_{00}B_{11} + \\
S_6 &= B_{00} + B_{11} & & \quad A_{11}B_{00} + A_{11}B_{11} \\
S_7 &= A_{01} - A_{11} & P_6 = S_7S_8 &= A_{01}B_{10} + A_{01}B_{11} - \\
S_8 &= B_{10} + B_{11} & & \quad A_{11}B_{10} - A_{11}B_{11} \\
S_9 &= A_{00} - A_{10} & P_7 = S_9S_{10} &= A_{00}B_{00} + A_{00}B_{01} - \\
S_{10} &= B_{00} + B_{01} & & \quad A_{10}B_{00} - A_{10}B_{01}
\end{aligned}
$$

Better Divide and Conquer Algorithm: $O(n^{2.807})$

$$
\begin{aligned}
C_{00} &= P_5 + P_4 - P_2 + P_6 \\
C_{01} &= P_1 + P_2 \\
C_{10} &= P_3 + P_4 \\
C_{11} &= P_5 + P_1 - P_3 - P_7
\end{aligned}
$$

- $T(n) = 7T(n/2) + O(n^2)$

- S: 10 additions
- P: 7 multiplications
- C: 8 additions

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\end{aligned}
$$

- $T(n) = 7T(n/2) + O(n^2)$
- $a = 7$
- $b = 2$
- $d = 2$
- $\log_b a = 2.807$
- $T(n) = n^{2.807}$

- S: 10 additions
- P: 7 multiplications
- C: 8 additions

Fibonacci Number

The n-th Fibonacci number is defined by

$$F_n = \begin{cases} F_{n-1} + F_{n-2}, & n \geq 2 \\ 1, & n = 1 \\ 0, & n = 0 \end{cases}$$

$$\begin{bmatrix} F_{n+2} & F_{n+1} \\ F_{n+1} & F_n \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}^n$$

Find The Medians

Median of An Array

Problem: Given an array $A = [a_0, a_1, \cdots, a_{n-1}]$ of n numbers, find the $(\lfloor n/2 \rfloor)$-th smallest element.

Problem: Given an array $A = [a_0, a_1, \cdots, a_{n-1}]$ of n numbers and $1 \le i \le n$, find the i-th smallest element.

Find The i-th Smallest Element

3	-5	7	2	-1	1	$n = 6$

Find The i-th Smallest Element

3	-5	7	2	-1	1	$i = 6$

Find The i-th Smallest Element

3	-5	7	2	-1	1	$i = 6$

Find The i-th Smallest Element

3	-5	7	2	-1	1	$i = 1$

Find The i-th Smallest Element

3	-5	7	2	-1	1	$i = 1$

Find The i-th Smallest Element

3	-5	7	2	-1	1	$i = 3$

Find The i-th Smallest Element

3	-5	7	2	-1	1	$i = 3$

Find The i-th Smallest Element

3	-5	7	2	-1	1	$i = 5$

Find The i-th Smallest Element

3	-5	7	2	-1	1	$i = 5$

Find The i-th Smallest Element

3	-5	7	2	-1	1	$n = 6$

-5	-1	1	2	3	7	$n = 6$

Find The i-th Smallest Element

3	-5	7	2	-1	1	$n = 6$
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Find The i-th Smallest Element

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-5	-1	1	2	**3**	7	$i = 5$

Sort at First: $O(n \log n)$

Finding the i-th smallest element is trivial if A is sorted.

- Sort A in the increasing order;
- Return the i-th number.

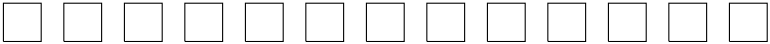
Special Cases

- The smallest element ($i = 1$) can be found in $O(n)$ comparisons.
- The largest element ($i = n$) can be found in $O(n)$ comparisons.

There is a lower bound on the number of necessary comparisons to find the i-th smallest element: $\Omega(n)$.

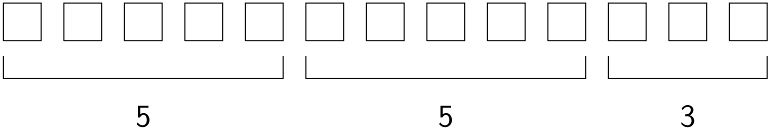
What is the upper bound?

Try with Divide and Conquer



find the i-th smallest one of n distinct elements

Try with Divide and Conquer



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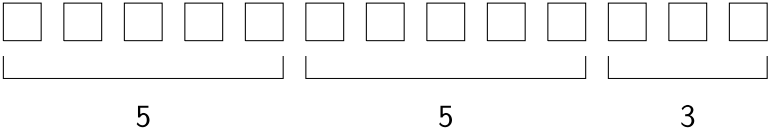
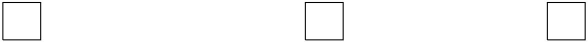


find the medians of subarrays



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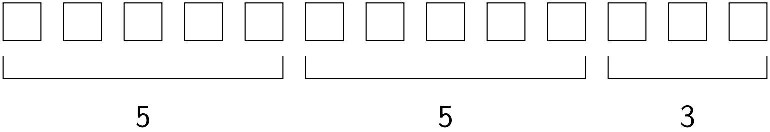

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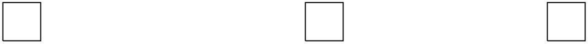
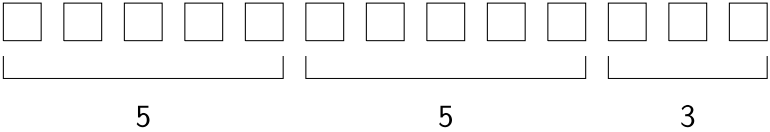


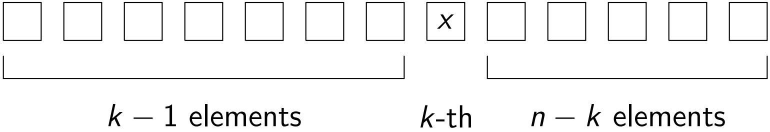
find the median of medians

find the i-th smallest one of n distinct elements

Try with Divide and Conquer



divide the elements



find the i-th smallest one of n distinct elements

Try with Divide and Conquer

5 5 3

divide the elements

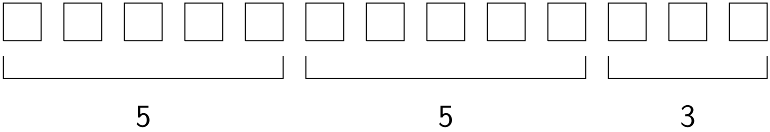
x

$k-1$ elements k-th $n-k$ elements

$i = k$

find the i-th smallest one of n distinct elements

Try with Divide and Conquer



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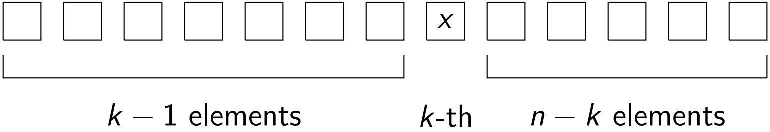
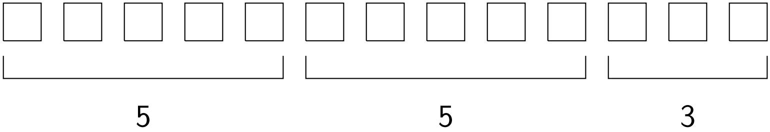


$i < k$

find the i-th smallest one of n distinct elements

Try with Divide and Conquer



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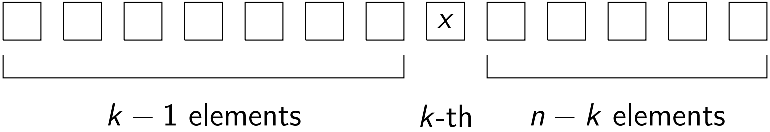


$i > k$

find the i-th smallest one of n distinct elements

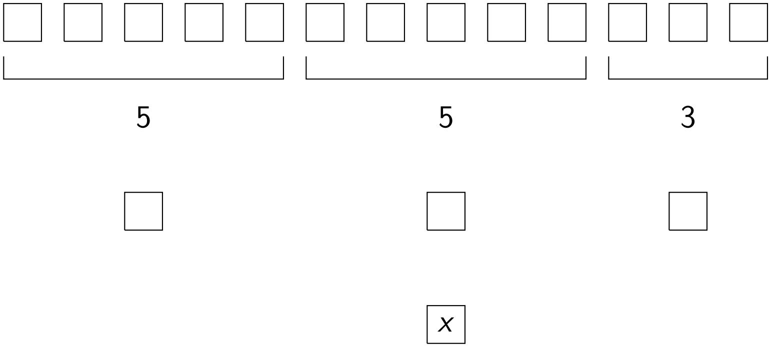
Find The i-th Smallest Element

Algorithm: Median(A, i)

if $|A| \leq 5$ **then** **return** The i-th smallest element of A ;
Divide A into groups of size 5;
$M =$ medians of the groups;
$x =$Median(M, $\lceil |M|/2 \rceil$);
$B =$ elements in A smaller than x;
$C =$ elements in A larger than x;
$k = |B| + 1$;
if $i < k$ **then** **return** Median(B, i) ;
else if $i > k$ **then** **return** Median(C, $i - k$) ;
else **return** x ;

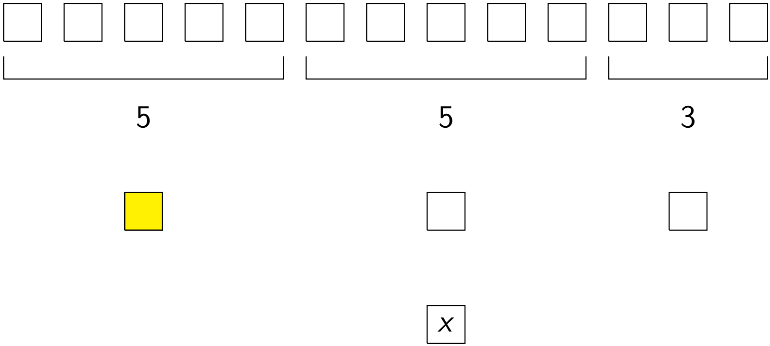
$$T(n) \leq T(max(k-1, n-k)) + T(\lceil n/5 \rceil) + O(n)$$

Efficiency of The Pivot



$$max(k-1, n-k) \leq 7n/10$$

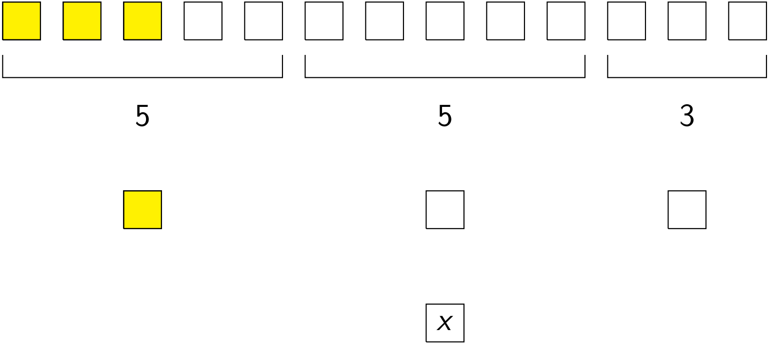
Efficiency of The Pivot



at least half of the group medians $\leq x$: $n/10$

$max(k-1, n-k) \leq 7n/10$

Efficiency of The Pivot



at least half of the group medians $\leq x$: $n/10$

3 numbers inside each group $\leq$ the median: $3n/10$

$max(k-1, n-k) \leq 7n/10$

Find The i-th Smallest Element: $O(n)$

$$T(n) \leq T(7n/10) + T(n/5) + O(n)$$

If $T(n) \leq T(a \cdot n) + T(b \cdot n) + c \cdot n$ and $a + b < 1$, then $T(n) = O(n)$.

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- Induction on n: $T(n) \le C \cdot n$ for some $C > 1$.

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- Base case $n = 1$: $T(1) = 1 < C$.

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- Assume $T(k) \leq C \cdot k$ for all $k < n$.

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- $T(n) \le C \cdot a \cdot n + C \cdot b \cdot n + c \cdot n = (C(a+b) + c) \cdot n$.
- $C(a+b) + c \le C$ as long as $C \ge c/(1-a-b)$.

Divide The Elements into Groups of Size k

$k = 5 \Rightarrow O(n)$. What about $k = 3$?

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- half of the medians $\leq$ than x: $n/6$.

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- 2 numbers $\leq$ than the median: $n/3$.
- $T(n) \leq T(2n/3) + T(n/3) + O(n)$.
- $T(n) = O(n \log n)$.

Divide The Elements into Groups of Size k

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- n elements are divided into $n/3$ groups, each of size 3.
- x be the median of the medians.
- half of the medians $\leq$ than x: $n/6$.
- 2 numbers $\leq$ than the median: $n/3$.
- $T(n) \leq T(2n/3) + T(n/3) + O(n)$.
- $T(n) = O(n \log n)$.
- Other k?

Random Pivot: $O(n)$ in Expectation

Algorithm: RandMedian(A, i)

if $|A|$ *is small* **then** **return** The i-th smallest element of A ;
$x =$ random element in A;
$B =$ elements in A smaller than x;
$C =$ elements in A larger than x;
$k = |B| + 1$;
if $i < k$ **then** **return** RandMedian(B, i) ;
else if $i > k$ **then** **return** RandMedian(C, $i - k$) ;
else **return** x ;

The Fast Fourier Transform

Value Representation of Polynomials

Fact: A degree-d polynomial

$$A(x) = a_0 + a_1 x + \cdots + a_d x^d$$

is uniquely characterized by its values at $d + 1$ distinct points.

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The polynomial given by the coefficients $a_0, a_1, \ldots, a_d$ can be alternatively represented by values

$$A(x_0), A(x_1), \ldots, A(x_d)$$

at points $x_0, x_1, \ldots, x_d$.

Value Representation of Polynomials

Why the value representations? Consider the multiplication of two polynomials: $C(x) = A(x) \times B(x)$.

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- If $A(x) = \sum_{i=0}^{d} a_i x^i$ and $B(x) = \sum_{i=0}^{d} b_i x^i$, then

$$C(x) = \sum_{i=0}^{2d} c_i x^i \text{ and } c_i = \sum_{k=0}^{i} a_k b_{i-k}$$

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- Given $A(x)$ as $A(x_0)$, $A(x_1)$, …, $A(x_{2d})$, and $B(x)$ is given by $B(x_0)$, $B(x_1)$, …, $B(x_{2d})$, then $C(x)$ can be represented by

$$C(x_0) = A(x_0)B(x_0), \ldots, C(x_{2d}) = A(x_{2d})B(x_{2d}).$$

Transform between Representations

$$a_0, a_1, \ldots, a_d \Rightarrow A(x_0), A(x_1), \ldots, A(x_d)$$

For each point x_i, calculate

$$A(x_i) = \sum_{j=0}^{d} a_i x_i^j$$

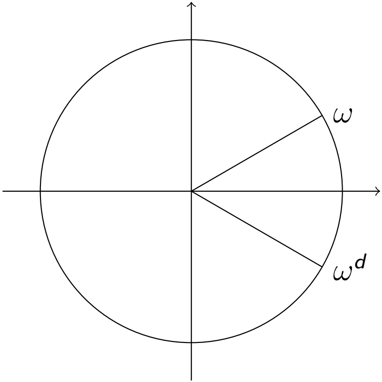
It requires $O(d \times d)$ operations.

Careful Selection of The Points

$x_0 = 1, x_1 = \omega, x_2 = \omega^2, \ldots, x_d = \omega^d$

ω: $(d+1)$-th (complex) root of 1

$\omega = e^{i\theta} = \cos\theta + i\sin\theta$
$\theta = 2\pi/(d+1)$



Divide The Calculations

For $x_i = \omega^i$, $i = 0, 1, \ldots, d$ with $d = 2k + 1$,

$$\sum_{j=0}^{d} a_i x_i^j \quad = \quad \sum_{j=0}^{k} a_{2j} x_i^{2j} + \sum_{j=0}^{k} a_{2j+1} x_i^{2j+1}$$

where $A_e(x) = \sum_{j=0}^{k} a_{2j} x^j$, $A_o(x) = \sum_{j=0}^{k} a_{2j+1} x^j$.

Divide The Calculations

For $x_i = \omega^i$, $i = 0, 1, \ldots, d$ with $d = 2k + 1$,

$$\begin{aligned} \sum_{j=0}^{d} a_i x_i^j &= \sum_{j=0}^{k} a_{2j} x_i^{2j} + \sum_{j=0}^{k} a_{2j+1} x_i^{2j+1} \\ &= \sum_{j=0}^{k} a_{2j} x_i^{2j} + x_i \sum_{j=0}^{k} a_{2j+1} x_i^{2j} \end{aligned}$$

where $A_e(x) = \sum_{j=0}^{k} a_{2j} x^j$, $A_o(x) = \sum_{j=0}^{k} a_{2j+1} x^j$.

Divide The Calculations

For $x_i = \omega^i$, $i = 0, 1, \ldots, d$ with $d = 2k+1$,

$$\begin{aligned}
\sum_{j=0}^{d} a_i x_i^j &= \sum_{j=0}^{k} a_{2j} x_i^{2j} + \sum_{j=0}^{k} a_{2j+1} x_i^{2j+1} \\
&= \sum_{j=0}^{k} a_{2j} x_i^{2j} + x_i \sum_{j=0}^{k} a_{2j+1} x_i^{2j} \\
&= A_e(x_i^2) + x_i A_o(x_i^2).
\end{aligned}$$

where $A_e(x) = \sum_{j=0}^{k} a_{2j} x^j$, $A_o(x) = \sum_{j=0}^{k} a_{2j+1} x^j$.

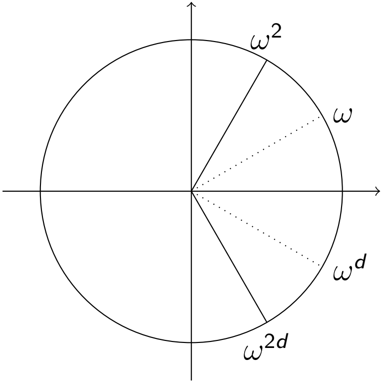
Careful Selection of The Points

$(k+1) = (d+1)/2$. Thus the $x_0, x_2, x_4, \ldots, x_{2k}$ are the $(k+1)$-th roots of 1.

$$x_0 = 1, x_1 = \omega, x_2 = \omega^2, \ldots, x_d = \omega^d$$

$\omega^d = e^{-i \cdot \theta}$

$\omega^{2d} = e^{-i \cdot 2\theta}$



The Fast Fourier Transform

- $a = [a_0, a_1, \ldots, a_d]$
- $d = 2k + 1$
- $(d+1)$-th roots: $1, \omega, \omega^2, \ldots, \omega^d$
- $(k+1)$-th roots: $1, \omega^2, \omega^4, \ldots, \omega^{2k}$

Algorithm: FFT(a, ω)

---

if $\omega = 1$ **then return** a_0;
$a_e = [a_0, a_2, \ldots, a_{2k}]$;
$a_0 = [a_1, a_3, \ldots, a_{2k+1}]$;
$re =$FFT(a_e, ω^2);
$ro =$FFT(a_o, ω^2);
for $i = 0$ *to* d **do**
    // $A(\omega_i) = \sum_{j=0}^{d} a_j \omega_i^j$;
    $r_i = re_i + \omega^i ro_i$;
end
return $[r_0, r_1, \ldots, r_d]$;

The Fast Fourier Transform

- $a = [a_0, a_1, \ldots, a_d]$
- $d = 2k + 1$
- $(d+1)$-th roots: $1, \omega, \omega^2, \ldots, \omega^d$
- $(k+1)$-th roots: $1, \omega^2, \omega^4, \ldots, \omega^{2k}$
- $T(d) = 2T(d/2) + O(d)$
- $T(d) = O(d \log d)$

Algorithm: FFT(a, ω)

if $\omega = 1$ **then return** a_0;
$a_e = [a_0, a_2, \ldots, a_{2k}]$;
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for $i = 0$ *to* d **do**
  // $A(\omega_i) = \sum_{j=0}^{d} a_j \omega_i^j$;
  $r_i = re_i + \omega^i ro_i$;
end
return $[r_0, r_1, \ldots, r_d]$;

Transform between The Representations

$$x_0 = 1, x_1 = \omega, x_2 = \omega^2, \ldots, x_d = \omega^d$$

$$\begin{bmatrix} A(1) \\ A(\omega) \\ \vdots \\ A(\omega^d) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega & \omega^2 & \cdots & \omega^d \\ & & \vdots & & \\ 1 & \omega^d & \omega^{2d} & \cdots & \omega^{d^2} \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_d \end{bmatrix}$$

$$M(\omega)^{-1} = \frac{1}{n} M(\omega^{-1})$$

Multiplication of Polynomials

Given $A(x) = \sum_{i=0}^{d} a_i x^i$ and $B(x) = \sum_{i=0}^{d} b_i x^i$,

- calculate the value representation of $A(x)$ and $B(x)$: $O(d \log d)$
- calculate the value representation of $C(x) = A(x)B(x)$: $O(d)$
- calculate the c_i's for $C(x) = \sum_{i=0}^{d} c_i x^i$: $O(d \log d)$

In total, it costs $O(d \log d)$.

THANK YOU

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