Design and Analysis of Algorithms

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Greedy Algorithm

Climb The Hill

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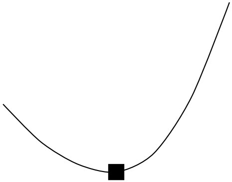
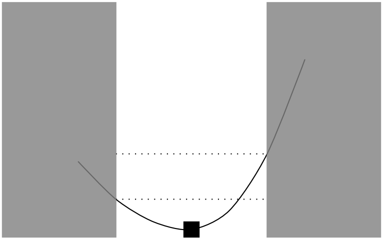


Climb The Hill

$$x = 0$$

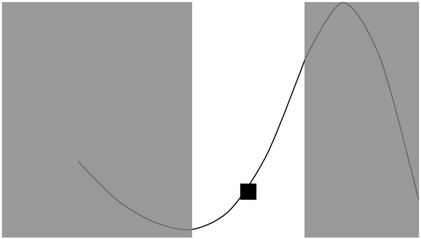
$$y \uparrow 0.6 \qquad y \uparrow 1$$



Climb The Hill

$$x = 1$$

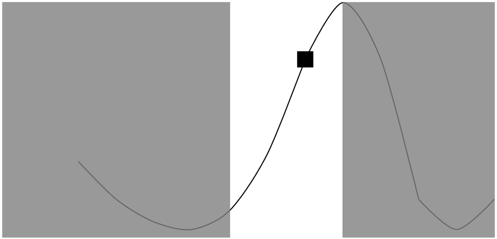
$$y \downarrow 1 \qquad y \uparrow 2$$



Climb The Hill

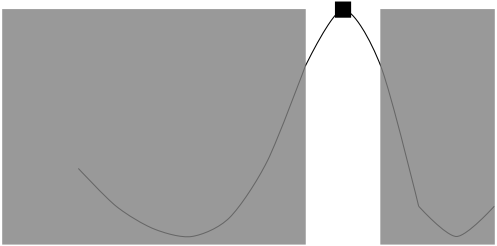
$$x = 2$$

$$y \downarrow 2 \qquad y \uparrow 0.5$$



Climb The Hill

$$x = 3$$



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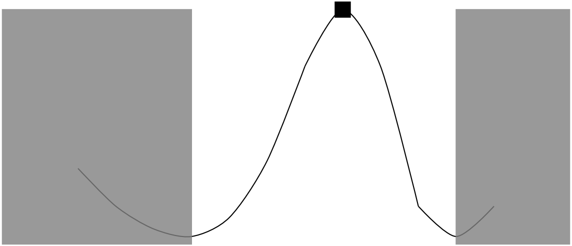

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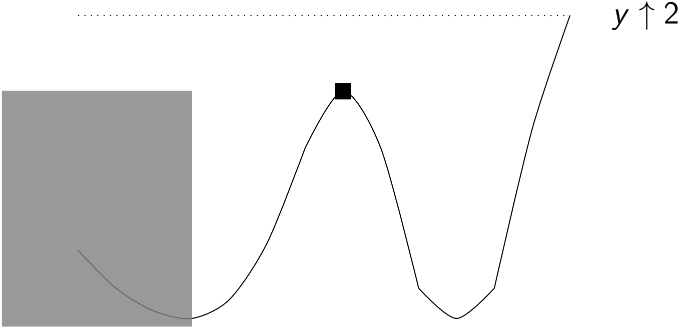

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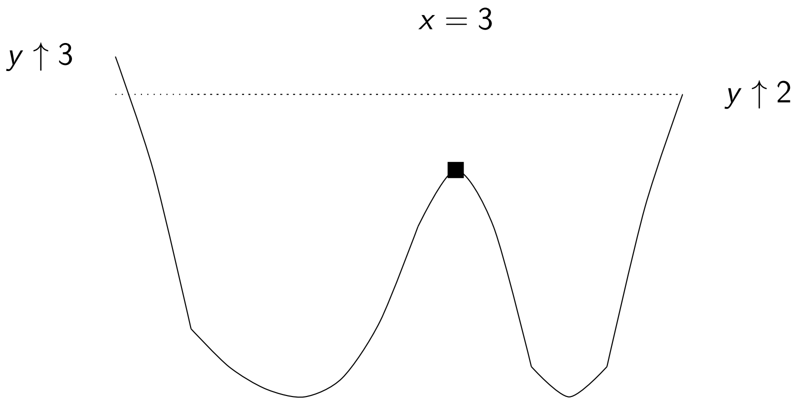

Climb The Hill

Local search: Optimize the action within the observable local area.

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- y increases by 1 if $x = x + 1$;
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Climb The Hill

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- go with $x = x + 1$, if the **maximum** is desired.

How far can you see?

- narrow (local) area:
  - less choice
  - less calculation
- wide (local) area
  - more choice
  - more calculation

Revisit: Knapsack

The Knapsack Problem: 0/1 Version

Given

- a container $\mathcal{K}$ of capacity W
- n items $\{x_0, x_1, \ldots, x_{n-1}\}$
  - integral weight $w_i > 0$
  - value $v_i > 0$

Fill the knapsack so as to maximize the total value.

The Knapsack Problem: Fractional Version

When filling the knapsack, you can take part of an item.

Consider an item of weight 2 and value 4. Taking $1/2$ of the item results in

- weight 1
- value 2

The Knapsack Problem: Example

$\mathcal{K}$ of capacity $W = 10$

- x_0 : $w_0 = 1$, $v_0 = 1$
- x_1 : $w_1 = 2$, $v_1 = 6$
- x_2 : $w_2 = 5$, $v_2 = 18$
- x_3 : $w_3 = 6$, $v_3 = 22$
- x_4 : $w_4 = 7$, $v_4 = 28$

The Knapsack Problem: Example

0/1 version

- x_0, x_1, x_4
- total weight: 10
- total value: 35

Fractional version

- $\frac{1}{2}x_3, x_4$
- total weight: 10
- total value: $28 + \frac{1}{2}22 = 39$

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The Knapsack Problem: Greedy

Select the item of the maximum v_i/w_i

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- take $\frac{3}{w_3} = \frac{1}{2}$ of x_3
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  - value: $\frac{1}{2}22 = 11$

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Is 39 optimal?

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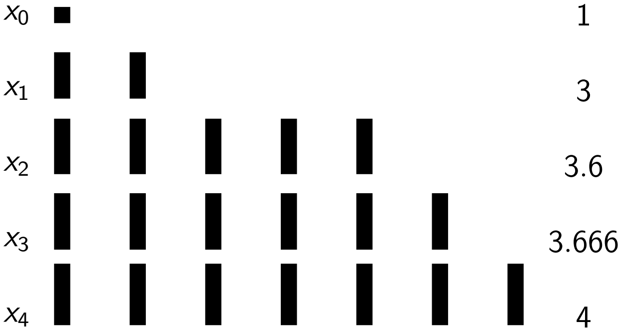
The Knapsack Problem: Greedy

Item x_i can be divided into w_i pieces. Thus each piece is of weight 1, and value v_i/w_i.



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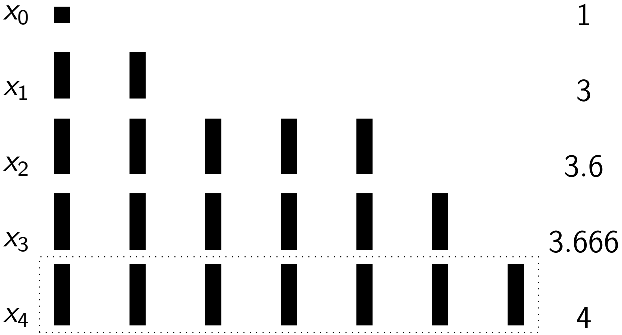

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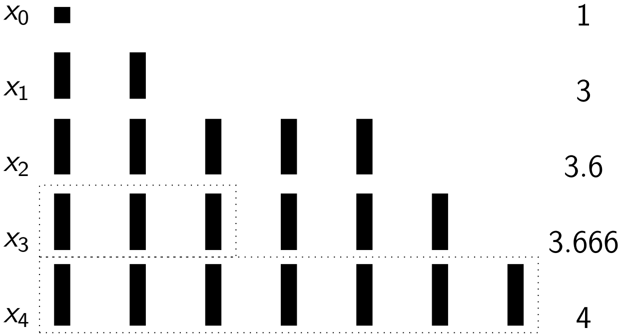
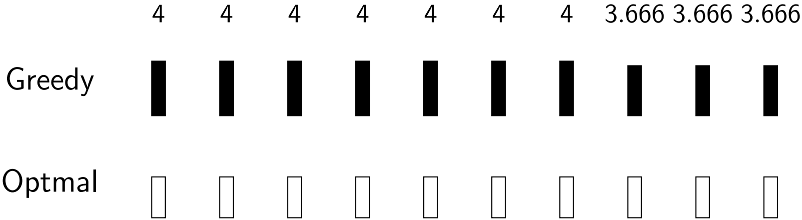


The Knapsack Problem: Transform



The Knapsack Problem: Transform

4 4 4 4 4 4 4 3.666 3.666 3.666

Greedy

Optmal $\square \geq \square \geq \square \geq \square \geq \square \geq \square \geq \square \geq \square \geq \square \geq \square$

≤ 4

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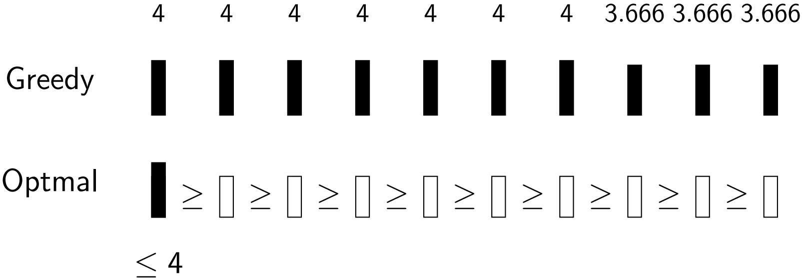

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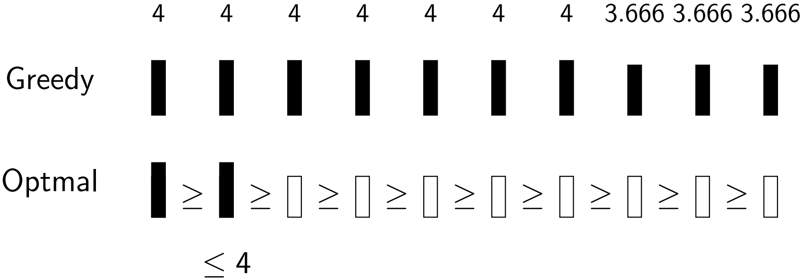

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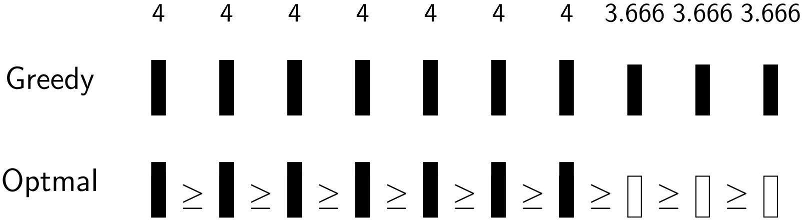

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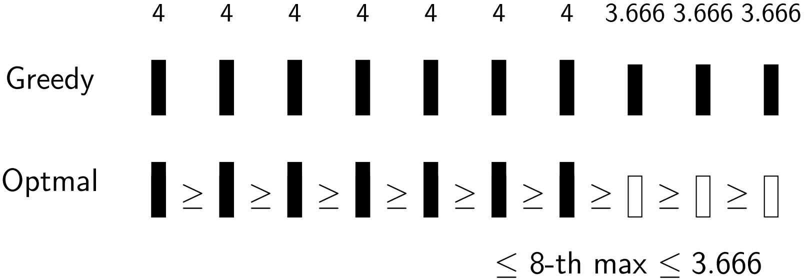

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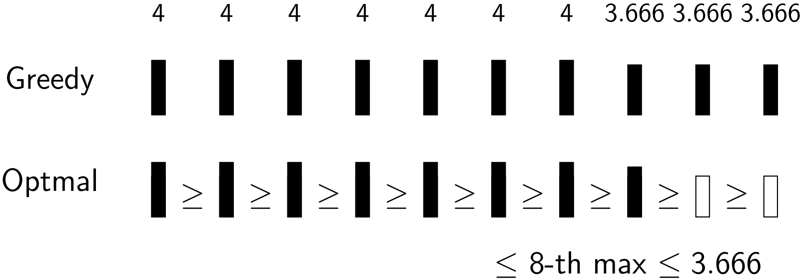

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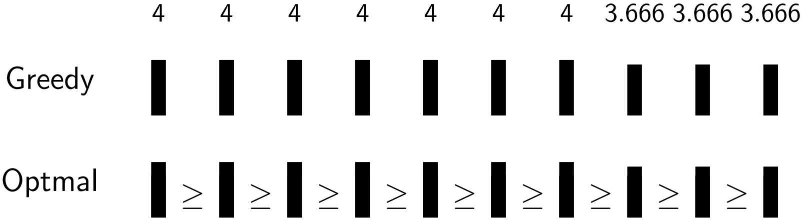

Paradigm of The Greedy Algorithms

Greedy Algorithms

Algorithm is **greedy** if

- it builds up a solution in small steps.
- it optimizes the action at each step according to the local observation.

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Algorithm is **greedy** if

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The optimality of the greedy algorithm is shown by

- in every step, it is **not worse** than **any other** algorithm.
- every algorithm can be gradually transformed to the greedy one **without loosing any quality**.

Cash Only

Pay with Cash

Pay $B \in \mathbb{Z}^+$ with coins of value $v_0 > v_1 > \ldots > v_{n-1}$, where $v_i \in \mathbb{Z}^+$ and $v_{n-1} = 1$. Try to minimize the number of coins.

For $i = 0$ to $n - 1$

- find max n_i such that $n_i \cdot v_i \leq B$
- $B = B - n_i \cdot v_i$

Pay with n_i coins of value v_i.

Interval Scheduling

Scheduling

Given jobs

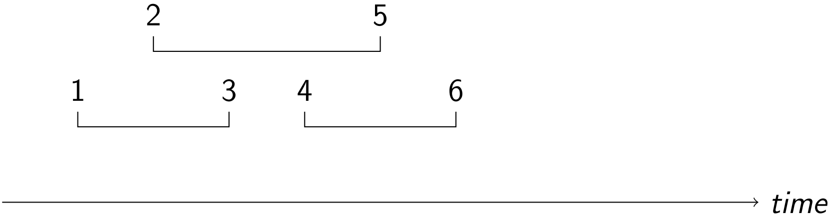
- Job 1: must be started at time 1 and finished at time 3
- Job 2: must be started at time 2 and finished at time 5
- Job 3: must be started at time 4 and finished at time 6

Restriction: two jobs can not be processed at the same time.

Problem: Try to process as many jobs as possible.

- All three jobs: overlap!
- Job 1 and Job 2: overlap!
- Job 2 and Job 3: overlap!
- Job 1 and Job 3: great!

Interval Scheduling



Problem: Find the largest subset of the given intervals, such that none two of them overlap.

Interval Scheduling: Greedy

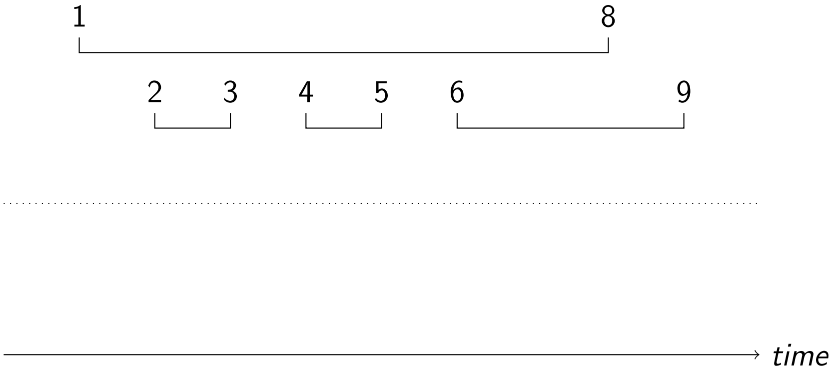
- Initialize a set as empty

$$S = \emptyset$$

- Select the intervals one by one **according to a specific rule**, and put it into S.
- Stop when no interval can be added.

The First Try

Rule: select the interval (among the remaining ones) that starts earliest, but not overlapping the already selected ones.



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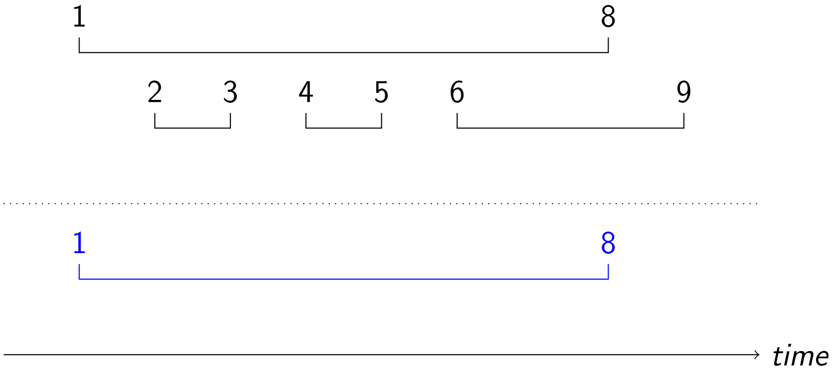

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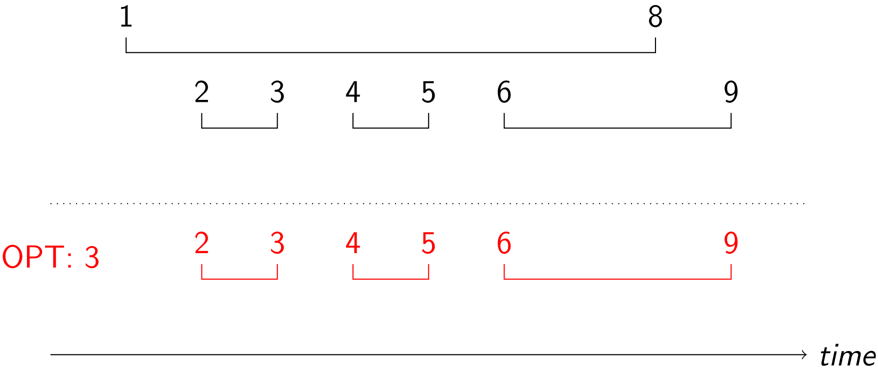
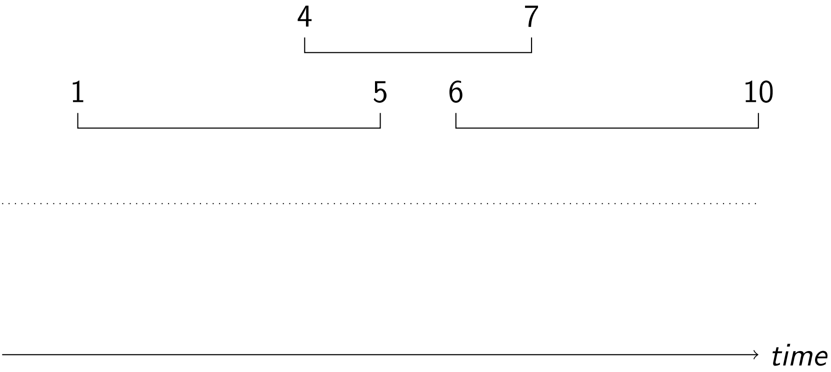


The Second Try

Rule: select the interval (among the remaining ones) that is the shortest, but not overlapping the already selected ones.



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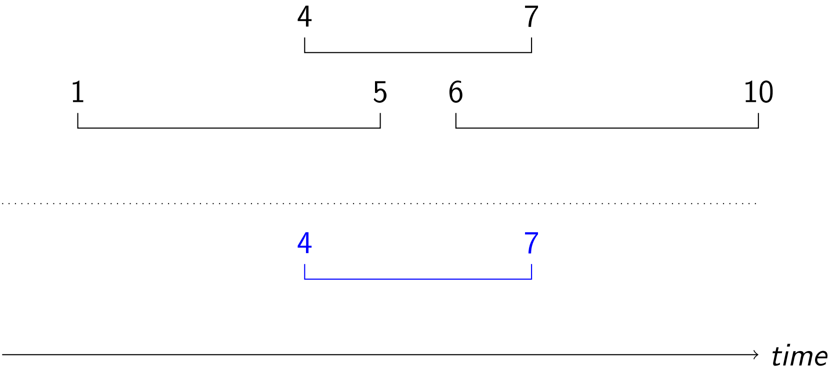

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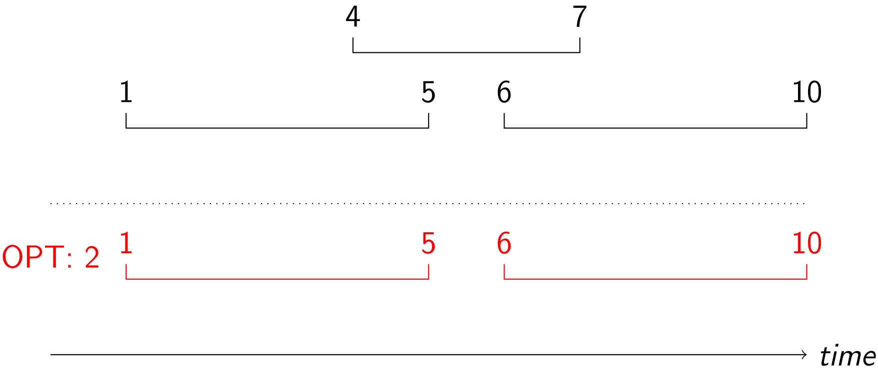

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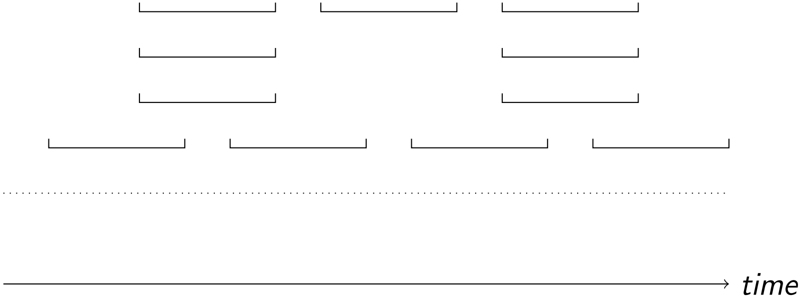

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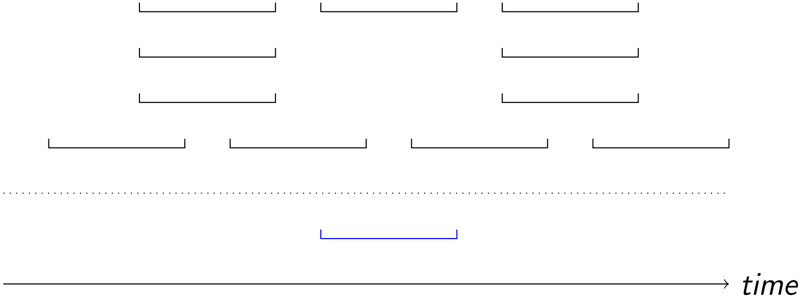

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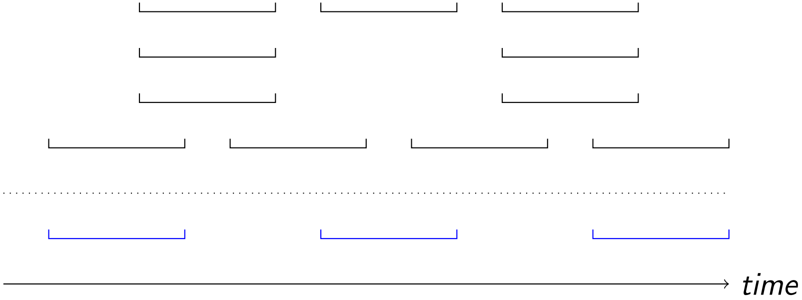

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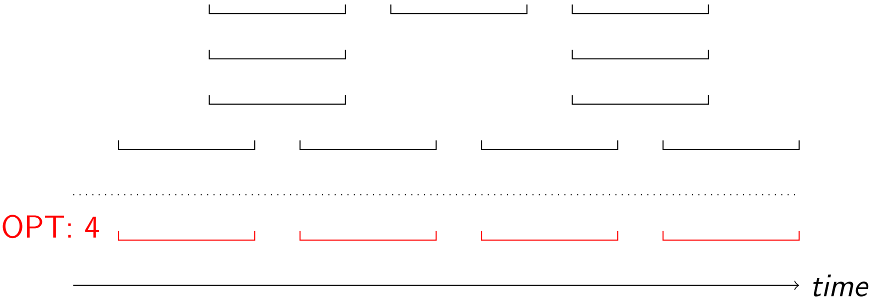

The Fourth Try

Rule: select the interval (among the remaining ones) that ends first, but not overlapping the already selected ones.

Idea: after processing a job, we want as much remaining time as possible, and thus we have more chance to process the other jobs.

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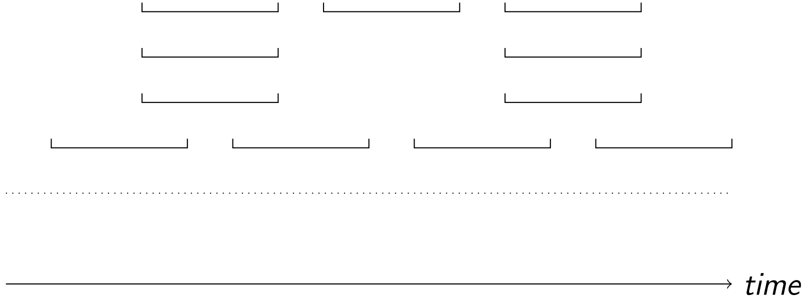

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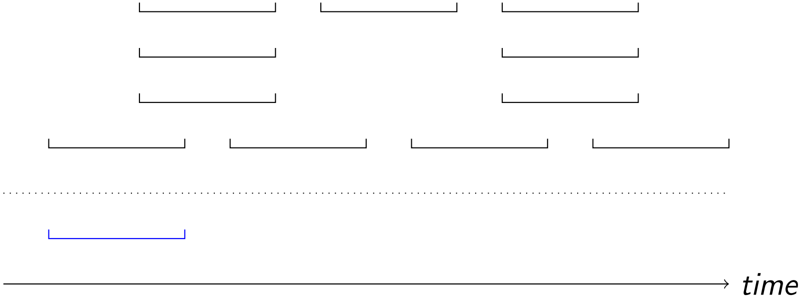

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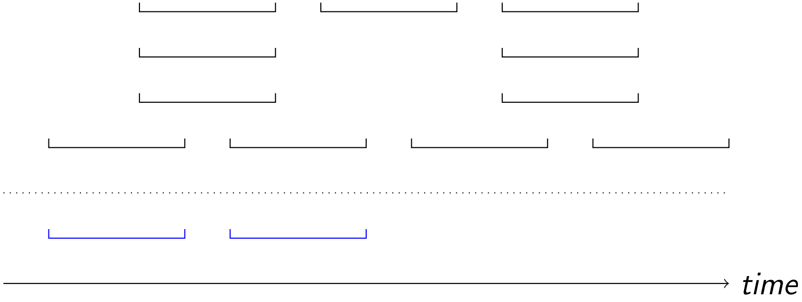

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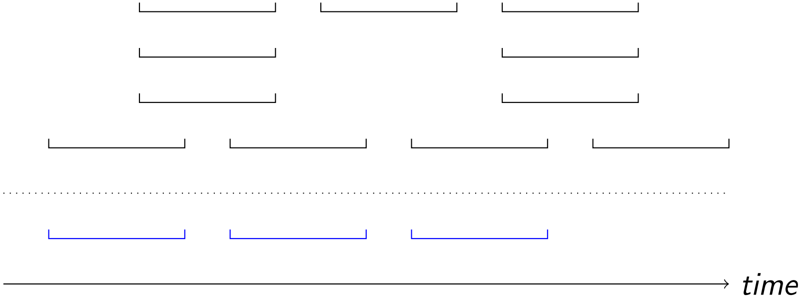

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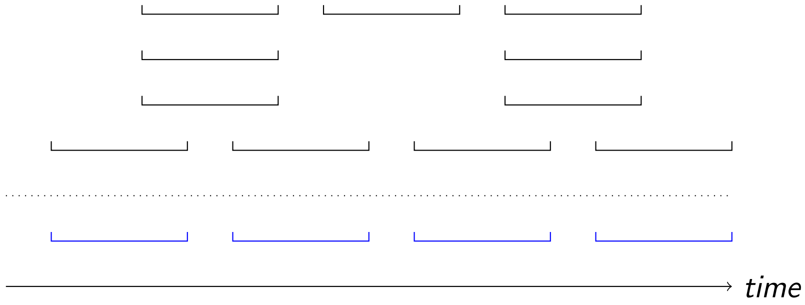

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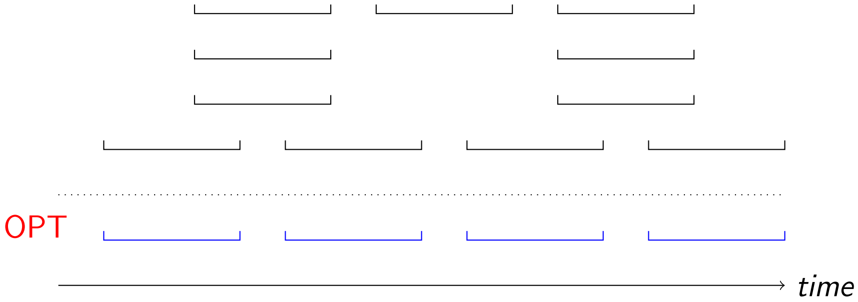

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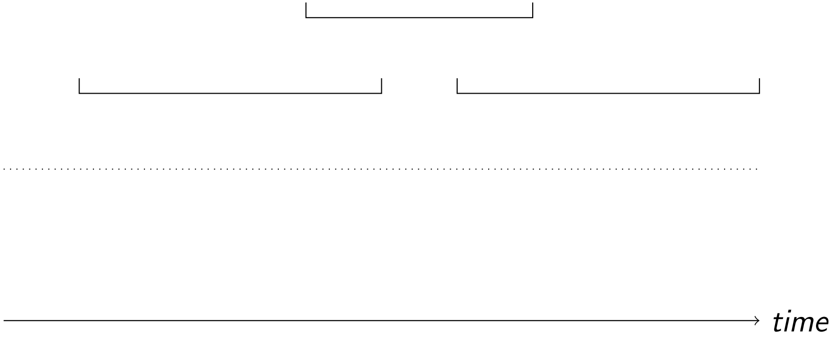

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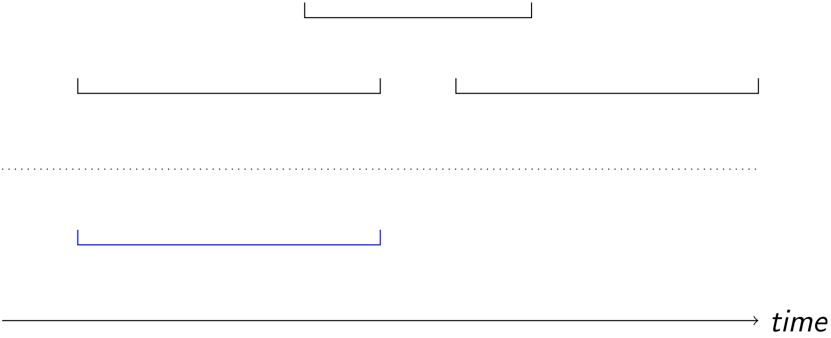

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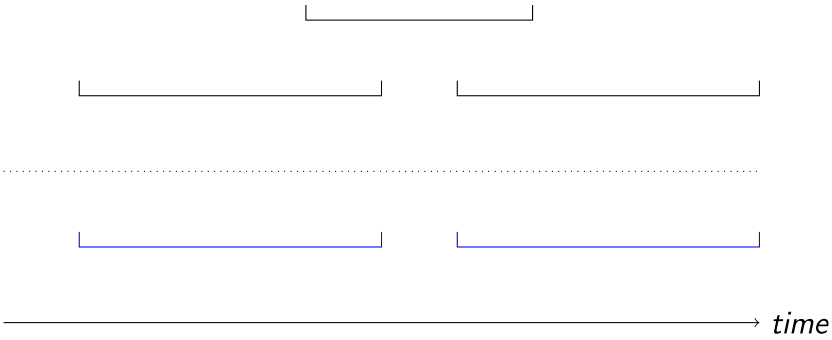

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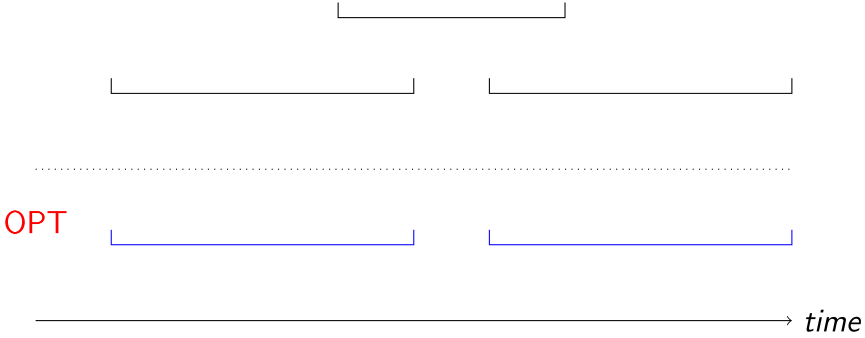

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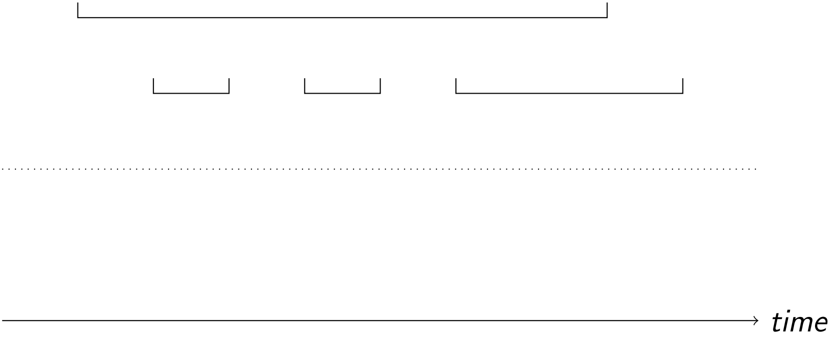

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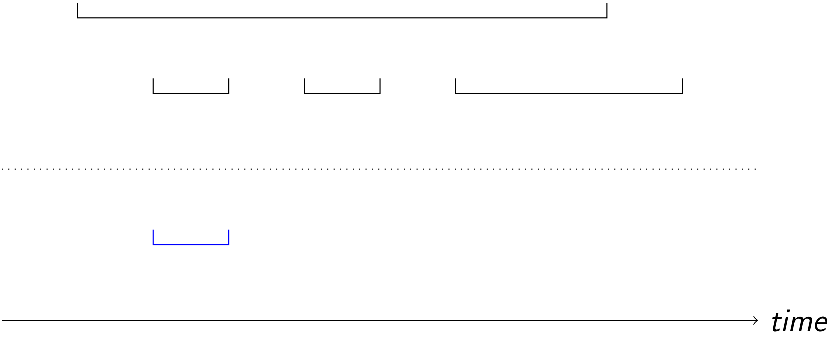

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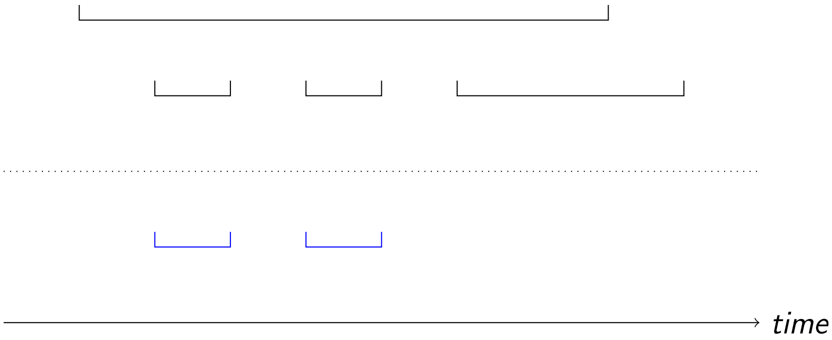

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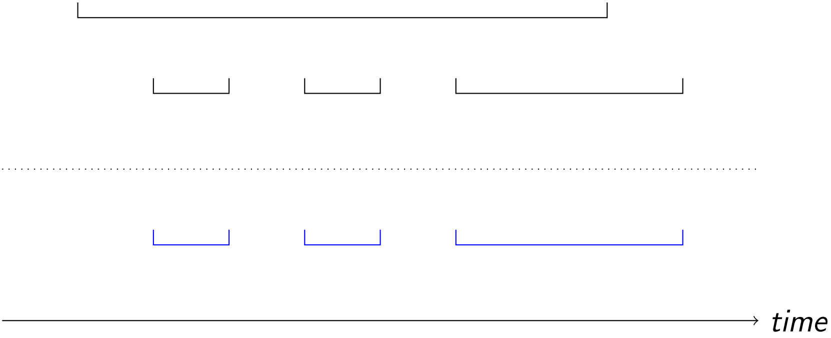

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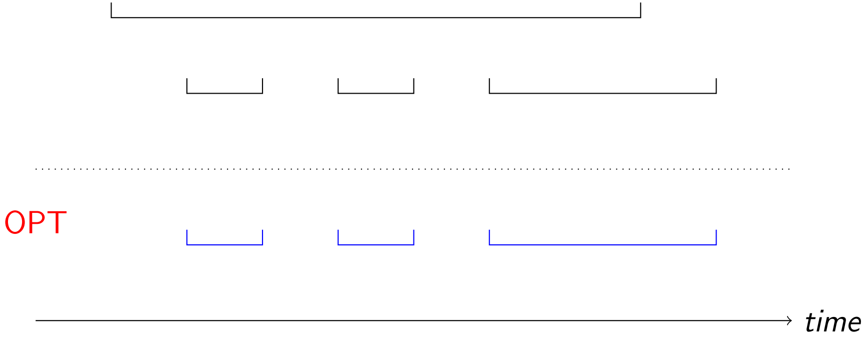

The Fourth Try: Analysis

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  - one of the largest set of non-overlapping intervals.
- we are going to show: $|S| = |P|$.

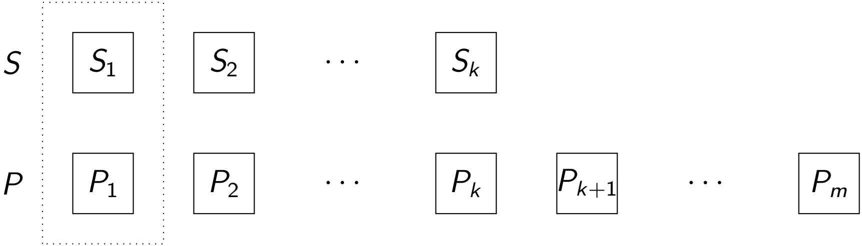
The Fourth Try: $|S| = |P|$, i.e. $k = m$

S $\boxed{S_1}$ $\boxed{S_2}$ $\cdots$ $\boxed{S_k}$

P $\boxed{P_1}$ $\boxed{P_2}$ $\cdots$ $\boxed{P_k}$ $\boxed{P_{k+1}}$ $\cdots$ $\boxed{P_m}$

Intervals in S and P are sorted according to the ending time.

The Fourth Try: $|S| = |P|$, i.e. $k = m$



S_1 ends before or at the same time with P_1

The Fourth Try: $|S| = |P|$, i.e. $k = m$

$$\begin{array}{cccccccc} S & \boxed{S_1} & \boxed{S_2} & \cdots & \boxed{S_k} & & & \\ P & \boxed{S_1} & \boxed{P_2} & \cdots & \boxed{P_k} & \boxed{P_{k+1}} & \cdots & \boxed{P_m} \end{array}$$

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S	S_1	S_2	$\cdots$	S_k			
P	S_1	P_2	$\cdots$	P_k	P_{k+1}	$\cdots$	P_m

- P_2 starts after S_1 ends
- S_2 ends before or at the same time with P_2

The Fourth Try: $|S| = |P|$, i.e. $k = m$

S	S_1	S_2	$\cdots$	S_k			
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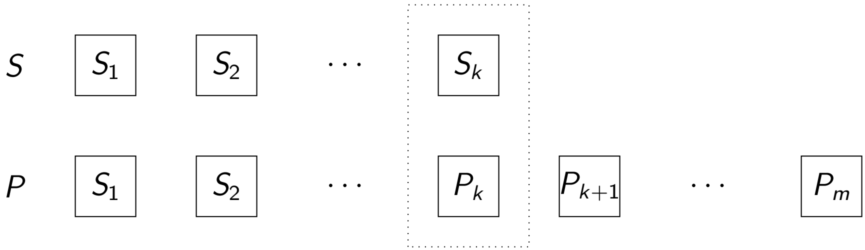

- P_k starts after S_{k-1} ends
- S_k ends before or at the same time with P_k

The Fourth Try: $|S| = |P|$, i.e. $k = m$

S	$\boxed{S_1}$	$\boxed{S_2}$	$\cdots$	$\boxed{S_k}$			
P	$\boxed{S_1}$	$\boxed{S_2}$	$\cdots$	$\boxed{S_k}$	$\boxed{P_{k+1}}$	$\cdots$	$\boxed{P_m}$

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P	$\boxed{S_1}$	$\boxed{S_2}$	$\cdots$	$\boxed{S_k}$	$\boxed{P_{k+1}}$	$\cdots$	$\boxed{P_m}$

- P_{k+1} starts after S_k ends
- $k = m$; otherwise, S can be extended.

The Fourth Try: $|S| = |P|$, i.e. $k = m$

$$\begin{array}{ccccc} S & \boxed{S_1} & \boxed{S_2} & \cdots & \boxed{S_k} \\ & & & & \\ P & \boxed{S_1} & \boxed{S_2} & \cdots & \boxed{S_k} \end{array}$$

- P_{k+1} starts after S_k ends
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The Fourth Try: Optimal

Lemma: For $1 \leq i \leq k+1$, P_i starts after the ending time of S_{i-1}.

Induction on i

- **Base case**: P_2 starts after the ending time of S_1.
  - P_2 starts after the ending time of P_1
  - P_1 ends after the ending time of S_1
- **Assumption**: for $i \geq 1$, P_i starts after the ending time of S_{i-1}.
- P_{i+1} starts after the ending time of S_i.

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  - P_i must ends after or at the same time with S_i; otherwise, P_i is added to S, following S_{i-1}.
  - P_{i+1} starts after the ending time of S_i.

The Fourth Try: Optimal

Lemma: For $1 \leq i \leq k+1$, P_i starts after the ending time of S_{i-1}.

Theorem: $k = m$.

If $k < m$, then P_{k+1} starts after the ending time of S_k. Thus P_{k+1} can be added to S. **Contradiction!**

The Fourth Try: $O(n \log n)$

Algorithm: IntervalSchedule(I)

$S = \emptyset$;
$e = -1$;
Sort intervals in I in the ascending order of the ending time;
for $i = 0$ *to* $|I| - 1$ **do**
    if $I[i]$ *starts after* e **then**
        add $I[i]$ to S;
        $e =$ ending time of $I[i]$;
    end
end
Return S;

Complexity: $O(n \log n) + O(n) = O(n \log n)$

Matroid Optimization

Matroid

Matroid: $M(S, \mathcal{I})$, where

- S is a finite and nonempty set (of elements)
- $\mathcal{I} \subseteq 2^S$ is **hereditary**:

$$B \in \mathcal{I}, \; and \; A \subseteq B \Rightarrow A \in \mathcal{I}$$

- M satisfies the **exchange property**:

$$A, B \in \mathcal{I}, \; and \; |A| < |B| \Rightarrow \exists x \in B \setminus A, A \cup \{x\} \in \mathcal{I}$$

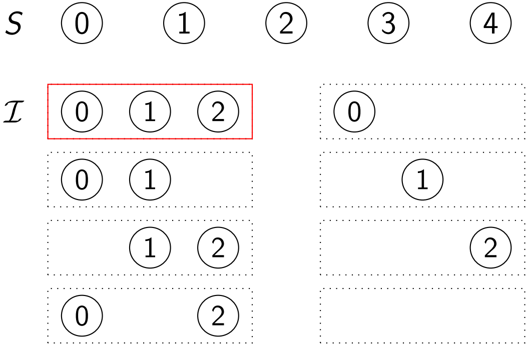
Matroid

S 0 1 2 3 4

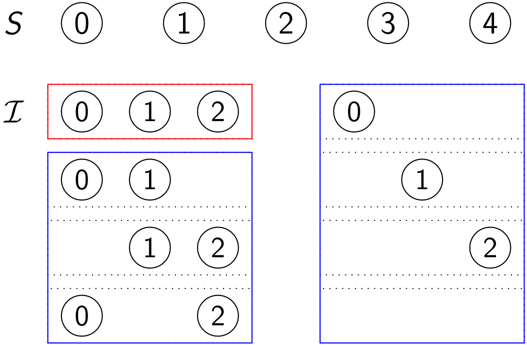
$\mathcal{I}$

0 1 2	0
0 1	1
1 2	2
0 2	

Matroid



Matroid



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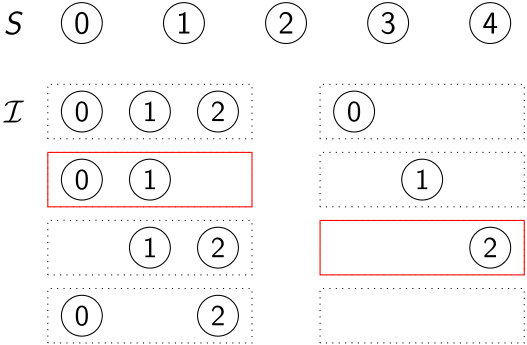

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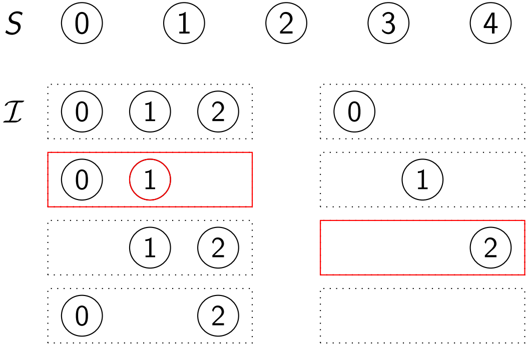

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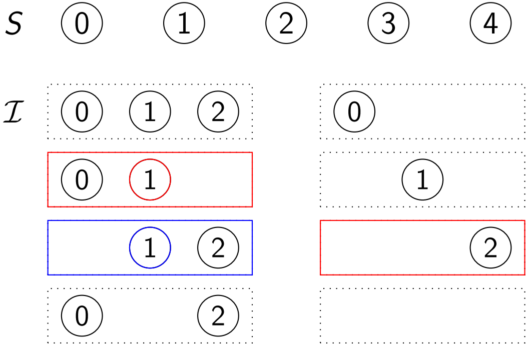

Maximal Independent Subsets

Theorem: For $A, B \in \mathcal{I}$, if A and B are maximal (within $\mathcal{I}$), then $|A| = |B|$.

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A subset $A \subseteq 2^S$ is maximal in $\mathcal{I} \Leftrightarrow$

$$\forall B \in \mathcal{I} \setminus \{A\},\ s.t.\ A \setminus B \neq \emptyset$$

$\mathcal{I} = \{[0,1,2],[0,1],[1,2],[0,2],[0],[1],[2],\emptyset\}$

- $[0]$ is not maximal: $[0] \setminus [0,1] = \emptyset$
- $[1,2]$ is not maximal: $[1,2] \setminus [0,1,2] = \emptyset$
- $[0,1,2]$ is maximal

Maximal Independent Subsets

Theorem: Given $M(S, \mathcal{I})$, for $A, B \in \mathcal{I}$, if A and B are maximal (within $\mathcal{I}$), then $|A| = |B|$.

Proof: Suppose to the contrary that there are

- A is maximal
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- A is not maximal
- **Controdiction!**

Weighted Matroid

Weighted Matroid: $M(S, \mathcal{I})$ is associated with a weighted function w

- $w(x) > 0, \ \forall x \in S$
- $w(A) = \sum_{x \in A} w(x), \ \forall A \subseteq S$

Maximum-Weight Subset

Problem: Given a weighted matroid $M(S, \mathcal{I})$, with weight function w, find the subset $A \in \mathcal{I}$ such that $w(A)$ is maximized.

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S	$\boxed{0}$	$\boxed{1}$	$\boxed{2}$
w	2	1	3

$\mathcal{I}$ $\quad \boxed{0}\ \boxed{1} \qquad \boxed{1}\ \boxed{2}$

$\boxed{0}\ \boxed{2}$

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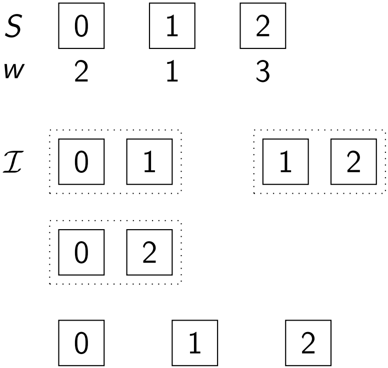

- $w(0) = 2$
- $w(1) = 1$
- $w(2) = 3$

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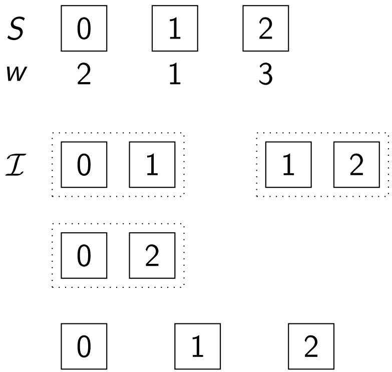

- $w(0) = 2$
- $w(1) = 1$
- $w(2) = 3$
- $w([0, 1]) = 3$
- $w([1, 2]) = 4$
- $w([0, 2]) = 5$

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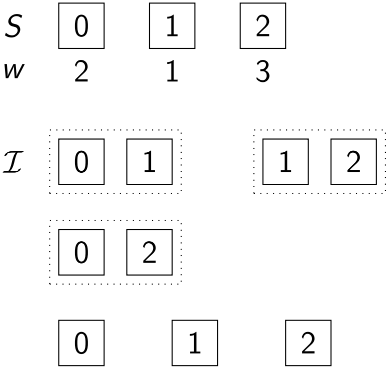

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- $w([0, 2]) = 5$
- $A = [0, 2]$

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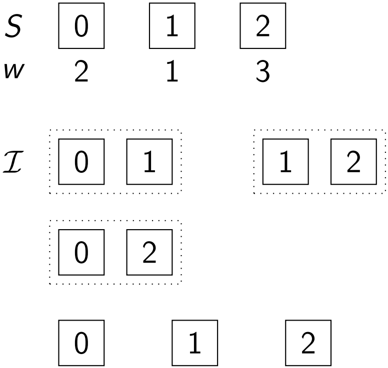

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- $w(2) = 3$
- $w([0, 1]) = 3$
- $w([1, 2]) = 4$
- $w([0, 2]) = 5$
- $A = [0, 2]$
- A is maximal.

Maximum-Weight Subset: Greedy

Algorithm: Greedy(M, w)

$A = \emptyset$;
Sort S in the decreasing order of $w[s]$;
for $i = 0$ *to* $|S| - 1$ **do**
    if $A \cup \{S[i]\} \in \mathcal{I}$ **then** add $S[i]$ to A;
end
Return A;

Maximum-Weight Subset: Greedy

Lemma: Given $M(S, \mathcal{I})$ and w, sort S in the decreasing order of $w[s]$. Let x be the first element in S such that $\{x\} \in \mathcal{I}$, then $\exists A \in \mathcal{I}$, s.t.

- $x \in A$, and
- $w(A)$ is maximized

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Assume $x = S[i]$. Thus

- $\{S[i]\} \in \mathcal{I}$
- $\{S[j]\} \notin \mathcal{I}, \forall j < i$

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Lemma: Given $M(S, \mathcal{I})$ and w, sort S in the decreasing order of $w[s]$. Let x be the first element in S such that $\{x\} \in \mathcal{I}$, then $\exists A \in \mathcal{I}$, s.t.

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Proof: If $\{S[i]\} \notin \mathcal{I},\ \forall i$, then $\mathcal{I} = \{\emptyset\}$. Thus $A = \emptyset$.

When $\exists i, \{x\} = \{S[i]\} \in \mathcal{I}$ and $\{S[j]\} \notin \mathcal{I}, \forall j < i$, assume there is nonempty $B \in \mathcal{I}$, such that

- $w(B)$ is maximized
- $x \notin B$

Maximum-Weight Subset: Greedy

Then $w(y) \leq w(x), \forall y \in B$. Otherwise

- $y = S[j], j < i$
- $\{y\} = \{S[j]\} \in \mathcal{I}$
- **Contradiction**.

Maximum-Weight Subset: Greedy

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- **Contradiction**.

Now, we try to construct A, such that $x \in A$ and $w(A) = w(B)$. As M satisfies the exchange property, we can do

- $A = \{x\}$
- **while** $|A| < |B|$
  - let $y \in B \setminus A$
  - $A = A \cup \{y\}$
- let z be the last element in $B \setminus A$

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  - $A = A \cup \{y\}$
- let z be the last element in $B \setminus A$
- $w(A) = w(B) - w(z) + w(x) \ge w(B)$

Maximum-Weight Subset: Greedy

Lemma: Given $M(S, \mathcal{I})$, if $x \in S$, $A \in \mathcal{I}$ and $A \cup \{x\} \in \mathcal{I}$, then $\{x\} \in \mathcal{I}$.

Corollary: If $\{x\} \notin \mathcal{I}$, then

$$x \notin A, \forall A \in \mathcal{I}$$

Maximum-Weight Subset: Greedy

Algorithm: Greedy(M, w)

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Sort S in the decreasing order of $w[s]$;
for $i = 0$ *to* $|S| - 1$ **do**
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end
Return A;

$S[0]$	$S[1]$	$S[2]$	$\cdots$	$S[k]$	$\cdots$

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$\boxed{S[0]}$ $\boxed{S[1]}$ $\boxed{S[2]}$ $\cdots$ $\boxed{S[k]}$ $\cdots$

$\{S[0]\} \notin \mathcal{I}$

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$S[0]$	$S[1]$	$S[2]$	$\cdots$	$S[k]$	$\cdots$

$\{S[1]\} \notin \mathcal{I}$

Maximum-Weight Subset: Greedy

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$\{S[2]\} \in \mathcal{I}$

Maximum-Weight Subset: Greedy

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$S[0]$ $S[1]$ $S[2]$ $\cdots$ $S[k]$ $\cdots$

$\{S[2]\} \in \mathcal{I}$

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Maximum-Weight Subset: Greedy

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- $A = \{x\} \cup A'$

Maximum-Weight Subset: Greedy

x is the first element, such that $\{x\} = \{S[i]\} \in \mathcal{I}$.

- $S' = \{y \in S : \{x, y\} \in \mathcal{I}\}$
- $\mathcal{I}' = \{B \subseteq S \setminus \{x\} : B \cup \{x\} \in \mathcal{I}\}$
- $A' = \text{Greedy}(M'(S', \mathcal{I}'), w)$
- $A = \{x\} \cup A'$
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- A' is optimal to M' $\Rightarrow$ A is optimal to M
- Thus, we can reduce M to smaller M', step by step, until $\mathcal{I}' = \emptyset$, which implies $A' = \emptyset$.

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- $S[j] \notin B, \forall j < i$
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  - $\{S[j]\} \notin \mathcal{I} \Rightarrow x \notin C, \forall C \in \mathcal{I}$
- Let $B' = B \setminus \{x\}$. Then
  - $B' \in \mathcal{I}'$
  - $w(B') = w(B) - w(x) > w(A) - w(x) = w(A')$
  - **Contradiction**.

Maximum-Weight Subset: Greedy

Algorithm: Greedy(M, w)

$A = \emptyset$;
Sort S in the decreasing order of $w[s]$;
for $i = 0$ *to* $|S| - 1$ **do**
    if $A \cup \{S[i]\} \in \mathcal{I}$ **then** add $S[i]$ to A;
end
return A;

Theorem: Given matroid $M(S, \mathcal{I})$ and weight function w, then Greedy(M, w) returns an optimal subset $A \in \mathcal{I}$.

The Knapsack Problem: Greedy

Given

- a container $\mathcal{K}$ of capacity $W \in \mathbb{Z}^+$
- n items $\{x_0, x_1, \ldots, x_{n-1}\}$
  - integral weight $w_i \in \mathbb{Z}^+$
  - value $v_i > 0$

Fill the knapsack so as to maximize the total value.

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Fill the knapsack so as to maximize the total value.

- Divide item x_i into w_i pieces. Each piece is of weight 1 and value v_i/w_i.
- Define
  - S: the set of all the pieces
  - $\mathcal{I}$: all subsets of size $\leq W$
  - for a piece x from item x_i, $w(x) = v_i/w_i$

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Matroid $M(S, \mathcal{I})$ satisfies

- **hereditary**: For $A \in \mathcal{I}$,

$$A' \subseteq A \Rightarrow |A'| \leq |A| \leq W$$

- **exchange property**: For $A, B \in \mathcal{I}$, and $|A| < |B| \leq W$. Let x be any element in $B \setminus A$. Then

$$|A \cup \{x\}| = |A| + 1 \leq W$$

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Thus, with the optimal subset for matroid $M(S, \mathcal{I})$ we know how to fill the knapsack to maximize the total value.

Set Cover

Set and Cover

Problem: Given

- B: a set of elements
- k sets
  $S_0, S_1, \ldots, S_{k-1} \in 2^B$

find a cover

- a selection
  $S_{i_0}, S_{i_1}, \ldots, S_{i_{h-1}}$
- $\cup_{i_j} S_{i_j} = B$

such that h is minimized.

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- Let P be the optimal solution.
- $|S| \leq |P| \ln |B|$

Set and Cover: Greedy

$|S| \leq |P| \ln |B|$

Proof: Let n_t be the number of uncovered elements after step t.

- $n_0 = n$
- at step 1, select a set S_1
- $n_1 = n - |S_1|$

The remaining n_t elements are covered by optimal selection P. Thus, there exists a not-selected set S' with $|S'| \geq n_t/|P|$.

- $|S_{t+1}| \geq n_t/|P|$
- $n_{t+1} \leq n_t - n_t/|P| = n_t(1 - 1/|P|)$
- $n_t \leq n_0(1 - 1/|P|)^t$

Set and Cover: Greedy

With $n_t \leq n_0(1 - 1/|P|)^t$

- recall that $1 - x < e^{-x}$ for all $x \neq 0$
- $n_t \leq n_0(1 - 1/|P|)^t < n \cdot e^{-t/|P|}$
- let $t^* = |P| \ln |B|$
- $n_{t^*} < n \cdot e^{-\ln |B|} = 1$
- $|S| \leq t^* = |P| \ln |B|$

Submodularity-Based Optimization

Matroid and Subset-Weight

Matroid: $M(S, \ell)$, where

- S is a finite and nonempty set (of elements)
- $\mathcal{I} \subseteq 2^S$ is **hereditary**:

$$B \in \ell, \text{ and } A \subseteq B \Rightarrow A \in \mathcal{I}$$

- M satisfies the **exchange property**:

$$A, B \in \mathcal{I}, \text{ and } |A| < |B| \Rightarrow \exists x \in B \setminus A, A \cup \{x\} \in \mathcal{I}$$

Subset-Weight: $w : 2^S \to \mathcal{R}^+$

Monotone and Submodular

w is **monotone** if

$$A \subseteq B \Rightarrow w(A) \leq w(B)$$

w is **submodular** if

$$\forall A \subseteq B \textit{ and } x \in S, w(A \cup \{x\}) - w(A) \geq w(B \cup \{x\}) - w(B)$$

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$$\forall A \subseteq B \text{ and } x \in S, w(A \cup \{x\}) - w(A) \geq w(B \cup \{x\}) - w(B)$$

or, equivalently

$$w(A) + w(B) \geq w(A \cup B) + w(A \cap B)$$

Notice that

$$A \cap B \subseteq B,\ x = A \setminus B$$

Thus

$$A \cap B \cup x = A,\ B \cup x = A \cup B$$

Climb The Hill: Submodularity

Algorithm: HillClimb(M, w)

$A = \emptyset$;

while true do

  let $X = \{x | x \in S \setminus A, A \cup \{x\} \in \mathcal{I}\}$

  if $|X| < 1$ **then break**;

  let $x^* = \arg\max_{x \in X} w(A \cup \{x\}) - w(A)$

  add x to A;

end

return A;

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$$w(A) \geq w(P) \cdot (1 - 1/e)$$

THANK YOU

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