Homework 3 Fisherface

Pana Wanitchollakit 6532136721

[1]:
```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt
import matplotlib.image as mpimg
import scipy
import scipy.io
from scipy import stats
```

1 Hello Soft Clustering (GMM)

[2]:
```
class GaussianMixtureModel:
    def __init__(self, n_mixtures = 3, mode=None):
        if mode == "T1":
            n_mixtures = 3
            self.means = np.array([[3, 3], [2, 2], [-3, -3]], dtype=np.float64)
            self.verbose = True

        elif mode == "T3":
            n_mixtures = 2
            self.means = np.array([[3, 3],[-3, -3]], dtype=np.float64)
            self.verbose = True

        elif mode == "OT1":
            n_mixtures = 2
            self.means = np.array([[0,0],[10_000, 10_000]], dtype=np.float64)
            self.verbose = True

        self.n_mixtures = n_mixtures
        self.weights = np.full(n_mixtures, 1 / n_mixtures, dtype=np.float64)

    def _expectation_step(self, X):
        W = np.vstack([ stats.multivariate_normal(self.means[j], self.covs[j]).
↪pdf(X) * self.weights[j]
                        for j in range(self.n_mixtures) ]).T
        return W / W.sum(axis=1, keepdims=True) #normalize

    def _maximization_step(self, X, W):
```

```
        self.weights = W.sum(axis=0) / W.shape[0]
        self.means = W.T @ X / W.sum(axis=0).reshape(-1, 1)

        self.covs = np.stack([
            ( (X - self.means[j]).T * W[:, j] ) @ (X - self.means[j]) / W[:, j].
↪sum()
            for j in range(self.n_mixtures)
        ])
        self.covs[:, np.eye(X.shape[1])==0] = 0 # Σ_(i,j) = 0, for i != j.

    def _calculate_likelihood(self, X):
        W = np.vstack([ stats.multivariate_normal(self.means[j], self.covs[j]).
↪pdf(X) * self.weights[j]
                        for j in range(self.n_mixtures) ]).T

        self.log_likelihoods.append( np.log( W.sum(axis=1) ).sum() )


    def fit(self, X, epochs=3):

        num_samples, num_features = X.shape
        self.covs = np.stack( [ np.eye(num_features, dtype=np.float64) for _ in␣
↪range(self.n_mixtures) ] )
        self.log_likelihoods = []

        for epoch in range(epochs):
            if self.verbose:
                print(f"epoch {epoch+1}:")
                print(127 * '*')

            W = self._expectation_step(X)
            self._maximization_step(X, W)

            if self.verbose:
                self.describe(X, W, num_features)

            self._calculate_likelihood(X)


        return self

    def plot_likelihood(self):
        plt.title('log likelihood')
        plt.xlabel('epochs')
        plt.xticks([1,2,3])
        plt.plot(range(1, len(self.log_likelihoods)+1), self.log_likelihoods)
        plt.show()
```

```python
    def plot_gauss_iteration(self):
        pass

    def describe(self, X, W, num_features):
        print(60*'-', 'W_n,j', 60*'-')
        for i, w in enumerate(W):
            print(f"({X[i, 0]}, {X[i, 1]}): ", *w, sep='\t\t\t')
        print(127 * '-')

        print(61*'-', 'm_j', 61*'-')
        print('', *self.weights, sep='\t\t\t')
        print(127 * '-')

        print(61*'-', ' _j', 61*'-')
        print('', *[f'({x:.7f}, {y:.7f})' for x,y in self.means], sep='\t\t\t')
        print(127 * '-')

        print(61*'-', 'Σ_j', 61*'-')
        for row in range(num_features):
            print('', *self.covs[:, row, :] , sep='\t\t\t')
        print(127 * '*')
```

1.1 T1, T2

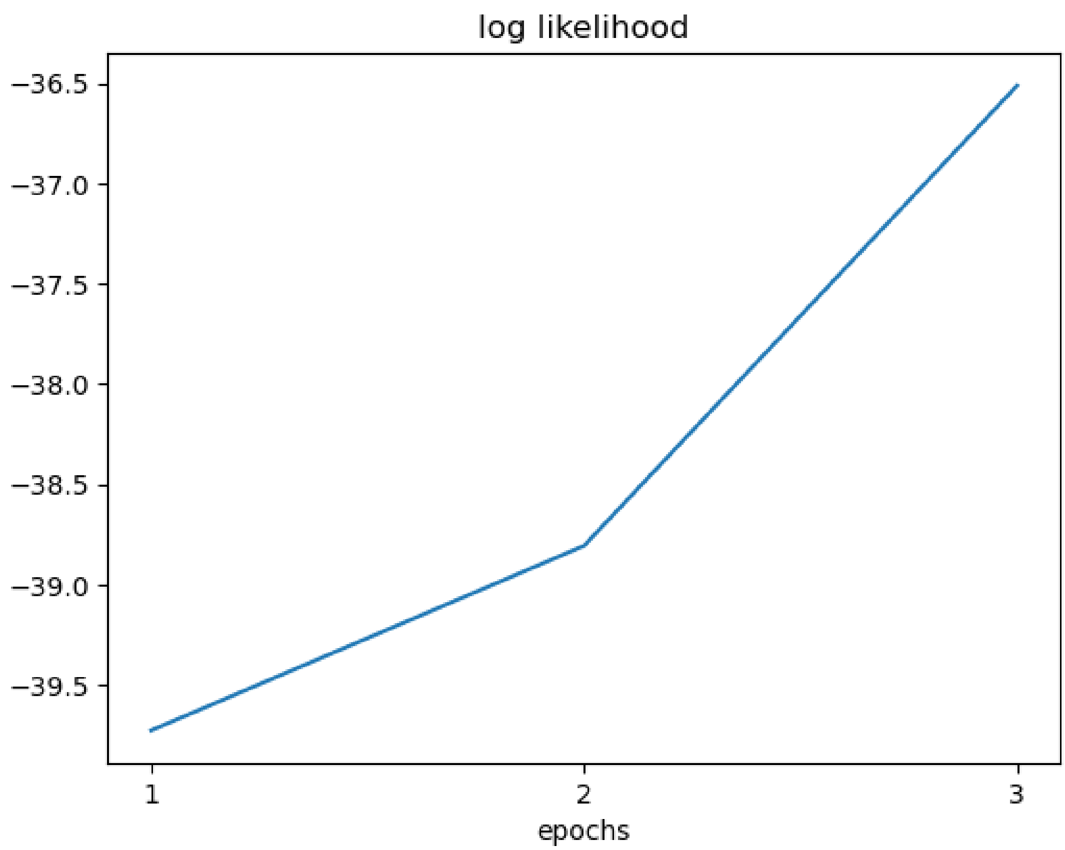
[3]:
```python
data = np.array([[1, 2], [3, 3], [2, 2], [8, 8], [6, 6], [7, 7], [-3, -3], [-2,␣
↪-4], [-7, -7]], dtype=np.float64)
gmm_t1 = GaussianMixtureModel(mode="T1").fit(data)
gmm_t1.plot_likelihood()
```

```
epoch 1:
********************************************************************************
***********************************************
------------------------------------------------------------ W_n,j
------------------------------------------------------------
(1.0, 2.0):                     0.11920292180570964
0.8807970763788322                      1.8154580846115228e-09
(3.0, 3.0):                     0.7310585786300048
0.2689414213699951                      1.6957070633777055e-16
(2.0, 2.0):                     0.2689414213672646
0.7310585786225826                      1.0152900501910824e-11
(8.0, 8.0):                     0.9999832985781519
1.6701421848095123e-05                  2.0310587405994306e-42
(6.0, 6.0):                     0.9990889488055994
0.0009110511944006453                   5.375284534993934e-32
(7.0, 7.0):                     0.9998766054240137
0.00012339457598623172                  3.3052927212335458e-37
```

```
(-3.0, -3.0):                   2.3195228302113496e-16
1.3887943864771155e-11                  0.9999999999861118
(-2.0, -4.0):                   2.319522830211349e-16
1.3887943864771157e-11                  0.9999999999861118
(-7.0, -7.0):                   3.3057006267607222e-37
5.900090541597041e-29                   1.0
--------------------------------------------------------------------------------
-----------------------------------------------
-------------------------------------------------------------- m_j
--------------------------------------------------------------
                        0.45757241940119386
0.20909424706571342                     0.33333333353309275
--------------------------------------------------------------------------------
-----------------------------------------------
--------------------------------------------------------------  _j
--------------------------------------------------------------
                        (5.7899269, 5.8188727)                  (1.6771821,
2.1452311)                  (-4.0000000, -4.6666667)
--------------------------------------------------------------------------------
-----------------------------------------------
-------------------------------------------------------------- Σ_j
--------------------------------------------------------------
                        [4.53619412 0.        ]                 [0.51645579 0.
]                 [4.66666668 0.        ]
                        [0.         4.28700611]                 [0.
0.13152618]                 [0.         2.88888891]
********************************************************************************
***********************************************
epoch 2:
********************************************************************************
***********************************************
------------------------------------------------------------- W_n,j
-------------------------------------------------------------
(1.0, 2.0):                     0.003169328213704367
0.9968247024298843                      5.969356411214957e-06
(3.0, 3.0):                     0.6551012066408073
0.34489810910850743                     6.842506851718458e-07
(2.0, 2.0):                     0.005775035369053813
0.9942236646081284                      1.3000228178827438e-06
(8.0, 8.0):                     1.0                     9.145017595274126e-73
4.530983722859346e-19
(6.0, 6.0):                     0.9999999999999449
3.182410750856766e-32                   5.499606551919197e-14
(7.0, 7.0):                     0.9999999999999999
1.6028239105455066e-50                  1.6650896648066796e-16
(-3.0, -3.0):                   4.73612484079033e-08
1.9798170363582452e-52                  0.9999999526387516
(-2.0, -4.0):                   3.0850269354184714e-08
```

```
1.358747162145948e-67                  0.9999999691497307
(-7.0, -7.0):                  5.395094425793122e-16
1.0875862279323009e-168                0.9999999999999994
--------------------------------------------------------------------------------
-----------------------------------------------
-------------------------------------------------------------- m_j
--------------------------------------------------------------
                       0.40711618315944764
0.2595496084607245                     0.3333342083798279
--------------------------------------------------------------------------------
-----------------------------------------------
--------------------------------------------------------------  _j
--------------------------------------------------------------
                       (6.2717622, 6.2726271)                 (1.7209154,
2.1476481)                 (-3.9999859, -4.6666488)
--------------------------------------------------------------------------------
-----------------------------------------------
-------------------------------------------------------------- Σ_j
--------------------------------------------------------------
                       [2.94672736 0.        ]                [0.49649261 0.
]                 [4.66673088 0.        ]
                       [0.          2.93847196]               [0.
0.12584815]                [0.          2.88900236]
********************************************************************************
***********************************************
epoch 3:
********************************************************************************
***********************************************
------------------------------------------------------------- W_n,j
-------------------------------------------------------------
(1.0, 2.0):                    9.82897442932805e-05
0.9998966672314759                     5.043024230876107e-06
(3.0, 3.0):                    0.24596547395001012
0.7540332951078749                     1.2309421149600602e-06
(2.0, 2.0):                    0.0003180138032810278
0.9996809918945071                     9.943022117740655e-07
(8.0, 8.0):                    1.0                9.434305079669937e-76
3.1452799372305255e-19
(6.0, 6.0):                    0.999999999999958
1.8659272195507823e-33                 4.197327501307116e-14
(7.0, 7.0):                    0.9999999999999998
1.3741875458751898e-52                 1.0824837720547552e-16
(-3.0, -3.0):                  5.617488669800274e-13
6.984609247796788e-55                  0.9999999999994382
(-2.0, -4.0):                  3.649212585635183e-13
1.0251745557304669e-70                 0.9999999999996352
(-7.0, -7.0):                  1.0304459263418196e-25
1.7242469482395505e-176                1.0
```

```
------------------------------------------------------------------------------------
--------------------------------------------------
---------------------------------------------------------------- m_j
----------------------------------------------------------------
                         0.3607090863887188
0.30595677269265087                          0.33333414091863034
------------------------------------------------------------------------------------
--------------------------------------------------
----------------------------------------------------------------  _j
----------------------------------------------------------------
                         (6.6962644, 6.6962947)                 (1.9107124,
2.2738344)                   (-3.9999867, -4.6666501)
------------------------------------------------------------------------------------
--------------------------------------------------
---------------------------------------------------------------- Σ_j
----------------------------------------------------------------
                         [1.73961067 0.        ]                   [0.62898406 0.
]                  [4.66672942 0.        ]
                         [0.         1.73929602]                   [0.
0.1988491]                   [0.         2.88899545]
************************************************************************************
**************************************************
```



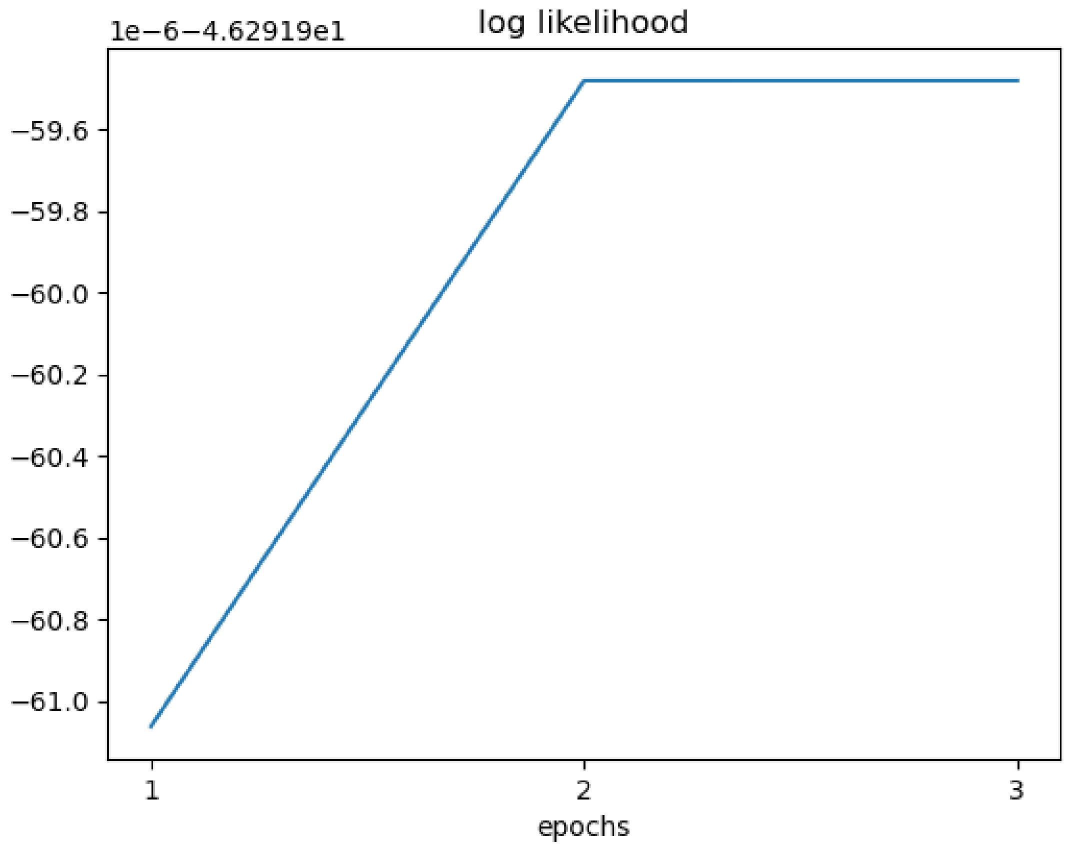
1.2 T3, T4

```
[4]: gmm_t3 = GaussianMixtureModel(mode="T3").fit(data)
     gmm_t3.plot_likelihood()
```

```
epoch 1:
********************************************************************************
***********************************************
------------------------------------------------------------ W_n,j
------------------------------------------------------------
(1.0, 2.0):                     0.9999999847700205
1.5229979512760363e-08
(3.0, 3.0):                     0.9999999999999998
2.319522830243563e-16
(2.0, 2.0):                     0.9999999999622486
3.775134544136584e-11
(8.0, 8.0):                     1.0                     2.031092662734804e-42
(6.0, 6.0):                     1.0                     5.380186160021119e-32
(7.0, 7.0):                     1.0                     3.3057006267607226e-37
(-3.0, -3.0):                   2.319522830243563e-16
0.9999999999999998
(-2.0, -4.0):                   2.3195228302435627e-16
0.9999999999999998
(-7.0, -7.0):                   3.3057006267607226e-37                 1.0
--------------------------------------------------------------------------------
-----------------------------------------------
------------------------------------------------------------- m_j
-------------------------------------------------------------
                        0.6666666649702522
0.3333333350297478
--------------------------------------------------------------------------------
-----------------------------------------------
-------------------------------------------------------------  _j
-------------------------------------------------------------
                        (4.5000000, 4.6666667)                  (-4.0000000,
-4.6666666)
--------------------------------------------------------------------------------
-----------------------------------------------
------------------------------------------------------------- Σ_j
-------------------------------------------------------------
                        [6.91666665 0.        ]                 [4.66666677 0.
]
                        [0.         5.88888889]                 [0.
2.8888891]
********************************************************************************
```

```
***********************************************
epoch 2:
********************************************************************************
***********************************************
------------------------------------------------------------ W_n,j
------------------------------------------------------------
(1.0, 2.0):                     0.9998792741684416
0.00012072583155842144
(3.0, 3.0):                     0.9999997405966382
2.594033618073692e-07
(2.0, 2.0):                     0.9999759216659174
2.4078334082538396e-05
(8.0, 8.0):                     1.0                     9.392866067809247e-19
(6.0, 6.0):                     0.999999999999926
7.410431536481986e-14
(7.0, 7.0):                     0.9999999999999998
2.9836636999584706e-16
(-3.0, -3.0):                   0.00024144822344170674
0.9997585517765584
(-2.0, -4.0):                   0.00015286907539070388
0.9998471309246092
(-7.0, -7.0):                   5.2242929952336704e-09
0.999999994775707
--------------------------------------------------------------------------------
-----------------------------------------------
------------------------------------------------------------- m_j
-------------------------------------------------------------
                        0.6666943621060054
0.3333056378939946
--------------------------------------------------------------------------------
-----------------------------------------------
-------------------------------------------------------------  _j
-------------------------------------------------------------
                        (4.4996131, 4.6662018)                  (-3.9999324,
-4.6665123)
--------------------------------------------------------------------------------
-----------------------------------------------
------------------------------------------------------------- Σ_j
-------------------------------------------------------------
                        [6.91944755 0.        ]                 [4.66806942 0.
]
                        [0.         5.89275124]                 [0.
2.89103318]
********************************************************************************
***********************************************
epoch 3:
********************************************************************************
***********************************************
```

```
------------------------------------------------------------ W_n,j
------------------------------------------------------------
(1.0, 2.0):                      0.9998785886169532
0.00012141138304683007
(3.0, 3.0):                      0.9999997382304249
2.6176957511039935e-07
(2.0, 2.0):                      0.9999757704169572
2.4229583042876973e-05
(8.0, 8.0):                      1.0                     9.634960207192914e-19
(6.0, 6.0):                      0.9999999999999246
7.548390466442755e-14
(7.0, 7.0):                      0.9999999999999998
3.049637759856525e-16
(-3.0, -3.0):                    0.0002427926968658732
0.999757207303134
(-2.0, -4.0):                    0.0001538383186326602
0.9998461616813673
(-7.0, -7.0):                    5.2888255427156375e-09
0.9999999947111744
--------------------------------------------------------------------------------
-----------------------------------------------
------------------------------------------------------------- m_j
-------------------------------------------------------------
                        0.6666945259520648
0.3333054740479351
--------------------------------------------------------------------------------
-----------------------------------------------
-------------------------------------------------------------  _j
-------------------------------------------------------------
                        (4.4996108, 4.6661990)               (-3.9999321,
-4.6665114)
--------------------------------------------------------------------------------
-----------------------------------------------
------------------------------------------------------------- Σ_j
-------------------------------------------------------------
                        [6.91946372 0.        ]              [4.66807754 0.
]
                        [0.         5.8927741]               [0.
2.89104566]
********************************************************************************
***********************************************
```



1.2.1 T4 ans

The three mixture model has higher log likelihood

1.3 OT1

[5]:
```
try:
    gmm_ot1 = GaussianMixtureModel(mode="OT1").fit(data)
except:
    print('Got Nan. Very Sad')
```

```
epoch 1:
********************************************************************************
***********************************************
------------------------------------------------------------ W_n,j
------------------------------------------------------------
(1.0, 2.0):                     1.0                     0.0
(3.0, 3.0):                     1.0                     0.0
(2.0, 2.0):                     1.0                     0.0
(8.0, 8.0):                     1.0                     0.0
(6.0, 6.0):                     1.0                     0.0
```

```
(7.0, 7.0):                     1.0                     0.0
(-3.0, -3.0):                   1.0                     0.0
(-2.0, -4.0):                   1.0                     0.0
(-7.0, -7.0):                   1.0                     0.0
--------------------------------------------------------------------------------
-----------------------------------------------
-------------------------------------------------------------- m_j
--------------------------------------------------------------
                        1.0                     0.0
--------------------------------------------------------------------------------
-----------------------------------------------
--------------------------------------------------------------  _j
--------------------------------------------------------------
                        (1.6666667, 1.5555556)                  (nan, nan)
--------------------------------------------------------------------------------
-----------------------------------------------
-------------------------------------------------------------- Σ_j
--------------------------------------------------------------
                        [22.22222222  0.        ]                       [nan
0.]
                        [ 0.         24.24691358]                       [ 0.
nan]
********************************************************************************
***********************************************
Got Nan. Very Sad
```

```
/var/folders/wh/91lwmxs51lb2njldckd6jd440000gp/T/ipykernel_37408/838807285.py:28
: RuntimeWarning: invalid value encountered in divide
  self.means = W.T @ X / W.sum(axis=0).reshape(-1, 1)
```

1.3.1 OT1 ans

The m_2 is equal to 0 which implies that the model intepret that data are distributed only from 1 mixture. The reason is that initial point is very far from the data points. To prevent this situation, set the initial point on the sample data like the K-mean does.

2 The face database

[6]:
```python
data = scipy.io.loadmat('facedata.mat')
num_persons, num_images = data['facedata'].shape
img_h, img_w = data['facedata'][0, 0].shape
```

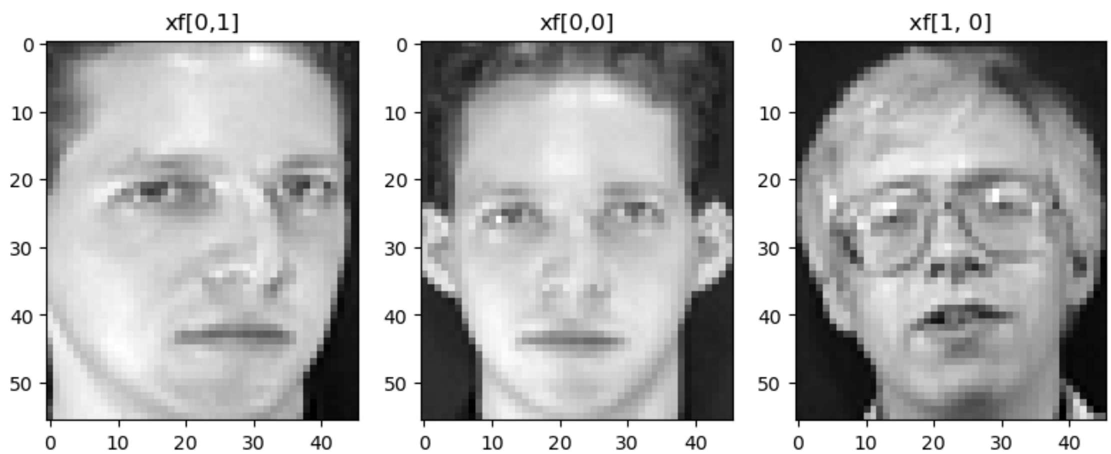
[7]:
```python
from skimage import img_as_float

xf = { (i, j) : img_as_float(data['facedata'][i, j])
            for i in range(num_persons)
            for j in range(num_images) }
```

2.1 T5

[8]:
```
print('Euclidean distance between xf[0, 0] and xf[0, 1]', np.sqrt( np.
    power(xf[0, 0] - xf[0, 1], 2).sum() ) )
print('Euclidean distance between xf[0, 0] and xf[1, 0]', np.sqrt( np.
    power(xf[0, 0] - xf[1, 0], 2).sum() ) )
```

```
Euclidean distance between xf[0, 0] and xf[0, 1] 10.037616294165492
Euclidean distance between xf[0, 0] and xf[1, 0] 8.173295099737281
```

[9]:
```
fig, ax = plt.subplots(1, 3, figsize=(10, 10))
ax[0].imshow(xf[0, 1], cmap='gray')
ax[0].set_title('xf[0,1]')

ax[1].imshow(xf[0, 0], cmap='gray')
ax[1].set_title('xf[0,0]')

ax[2].imshow(xf[1, 0], cmap='gray')
ax[2].set_title('xf[1, 0]')
plt.show()
```



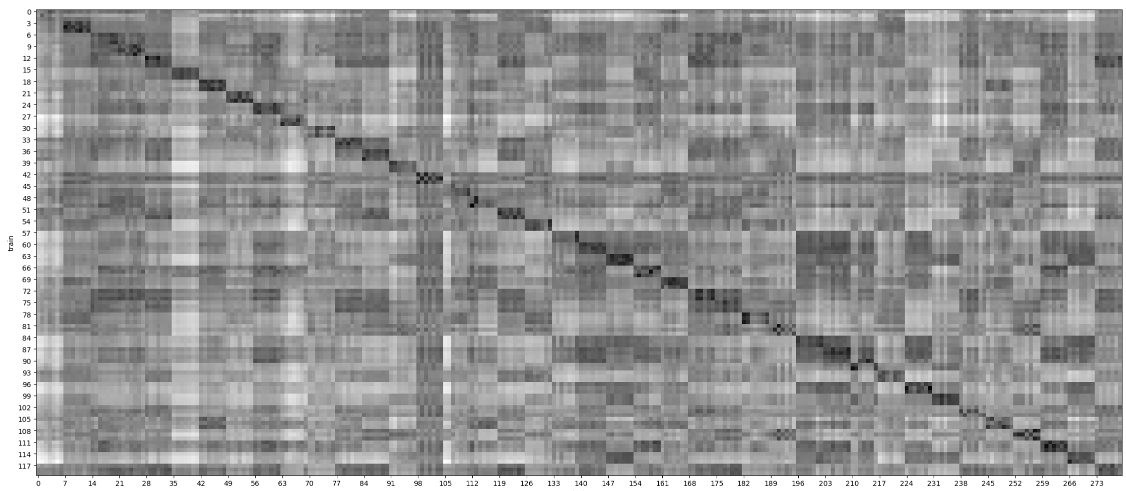
2.1.1 T5 ans:

The euclidean distance seem doesn't make sense because the same person must has higher similarity than other person. The main reason is $xf[0,0]$ and $xf[1,0]$ has share many black spots which yield small distance values

2.2 T6

[10]:
```
def compute_similarity_matrix(T, D):
    return np.sqrt( np.power(T[:, np.newaxis, :] - D, 2).sum(axis=2) )
```

```
[11]: T = np.array([xf[i, j].flatten() for i in range(num_persons) for j in range(3)])
D = np.array([xf[i, j].flatten() for i in range(num_persons) for j in range(3,␣
  ↪num_images)])

sim_mat = compute_similarity_matrix(T, D)
plt.figure(figsize=(28, 12))

plt.xlabel('test')
plt.ylabel('train')
plt.xticks(np.arange(0, 280, 7))
plt.yticks(np.arange(0, 120, 3))

plt.imshow(sim_mat, cmap='gray')

plt.show()
print('person 1')
pd.DataFrame(sim_mat).head()
```



person 1

[11]:

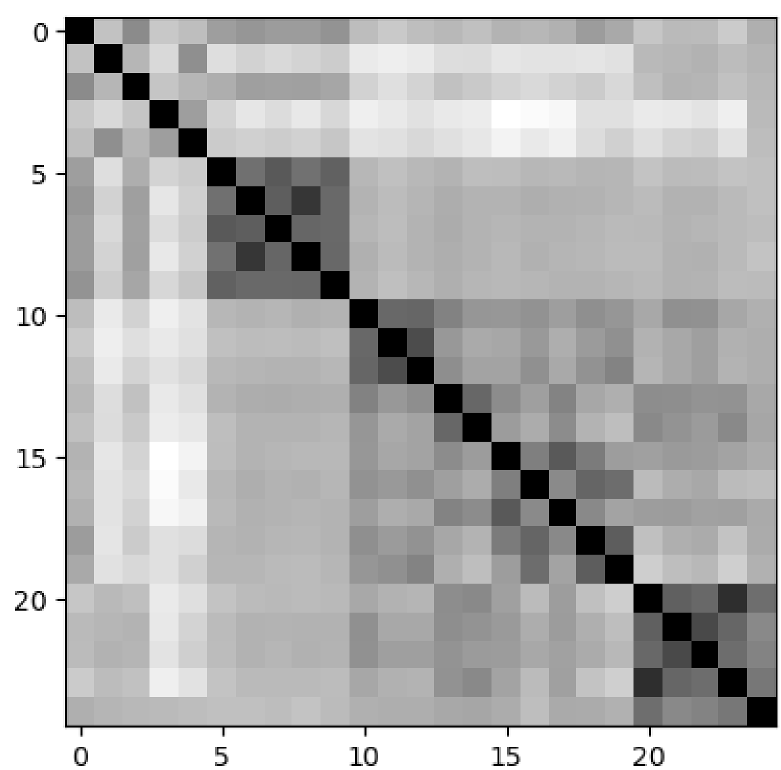
	0	1	2	3	4	5	6	\
0	10.369606	9.848695	8.996228	6.742481	7.975400	9.991797	8.618199	
1	11.249875	7.417211	9.880670	9.015946	11.040278	10.901092	9.585383	
2	10.222093	9.413216	9.299877	7.861811	8.883991	10.678250	9.084197	
3	10.945727	10.499381	10.369089	9.118717	9.401359	11.073716	9.223274	
4	11.921229	10.748505	10.642385	8.981316	9.789288	11.375868	9.280341	

	7	8	9	…	270	271	272	\
0	8.070297	7.671553	8.310970	…	12.896797	14.025518	13.461977	
1	10.959881	10.583454	10.958266	…	14.965500	15.837001	15.333624	

```
2   8.274936   8.595986   8.968311  …  14.082668  14.632209  14.351621
3   5.842049   5.063729   7.251770  …  11.343069  12.112799  11.426320
4   2.796472   5.497650   4.775989  …  11.404157  12.068349  11.176825

          273        274        275        276        277        278        279
0  11.063111   8.861481   9.816839   9.219246   9.896388   9.361519  10.660626
1  11.240393  10.691312  10.954066  10.470241  10.969427  10.902390  10.936306
2  10.621731   9.610012   9.826431   9.814474   9.996895   9.944995  10.410081
3  10.784801   9.800406   9.963266   9.910681  10.045208   9.384494  10.432161
4  10.123507   9.754289   9.758180   9.786770   9.966323   9.928984  10.053046

[5 rows x 280 columns]
```

2.3 T7

[12]:
```python
T7 = np.array([xf[i, j].flatten() for i in range(5) for j in range(5)])

A = compute_similarity_matrix(T7, T7)
plt.imshow(A, cmap='gray')
plt.show()
```



[13]:
```python
pd.DataFrame(A[5:10, 5:10])
```

[13]:

	0	1	2	3	4
0	0.000000	5.824298	4.620959	5.842049	5.063729
1	5.824298	0.000000	4.876667	2.796472	5.497650
2	4.620959	4.876667	0.000000	5.309403	5.494179
3	5.842049	2.796472	5.309403	0.000000	5.415199
4	5.063729	5.497650	5.494179	5.415199	0.000000

[14]:
```
pd.DataFrame( A[:5, :5] )
```

[14]:

	0	1	2	3	4
0	0.000000	10.037616	7.187433	10.369606	9.848695
1	10.037616	0.000000	9.419973	11.249875	7.417211
2	7.187433	9.419973	0.000000	10.222093	9.413216
3	10.369606	11.249875	10.222093	0.000000	8.204364
4	9.848695	7.417211	9.413216	8.204364	0.000000

2.3.1 T7 ans

- The black square show that images of person number 2 have high similarity to each other images of himself.
- The images of person number 1 has low to medium similarity to himself because the pattern color is in light gray range.

3 A simple face verification system

3.1 T8

[15]:
```
def recall_score(y_test, y_pred):
    if y_test.sum():
        return np.sum((y_test == 1) & (y_pred == 1)) / y_test.sum()
    return 0

def fpr_rate(y_test, y_pred):
    if (1 - y_test).sum():
        return np.sum((y_test == 0) & (y_pred == 1)) / (1 - y_test).sum()
    return 0

def get_eval(train, test, train_sz, test_sz, n_persons, threshold):
    y_pred = _get_prob(train, test, train_sz, test_sz, n_persons) < threshold
    y_test = np.repeat(np.eye(n_persons), test_sz, axis=1)
    return recall_score(y_test, y_pred), fpr_rate(y_test, y_pred)

def _get_prob(train, test, train_sz, test_sz, n_persons):
    sim_mat = compute_similarity_matrix(train, test)
    prob = sim_mat.reshape(n_persons, train_sz, n_persons, test_sz).min(axis=1).
↪reshape(n_persons, -1)
    return prob
```

```
[16]: tpr, fpr = get_eval(T, D, 3, 7, num_persons, 10)
      print(f"tpr: {tpr}", f"fpr: {fpr}", sep='\n')
```

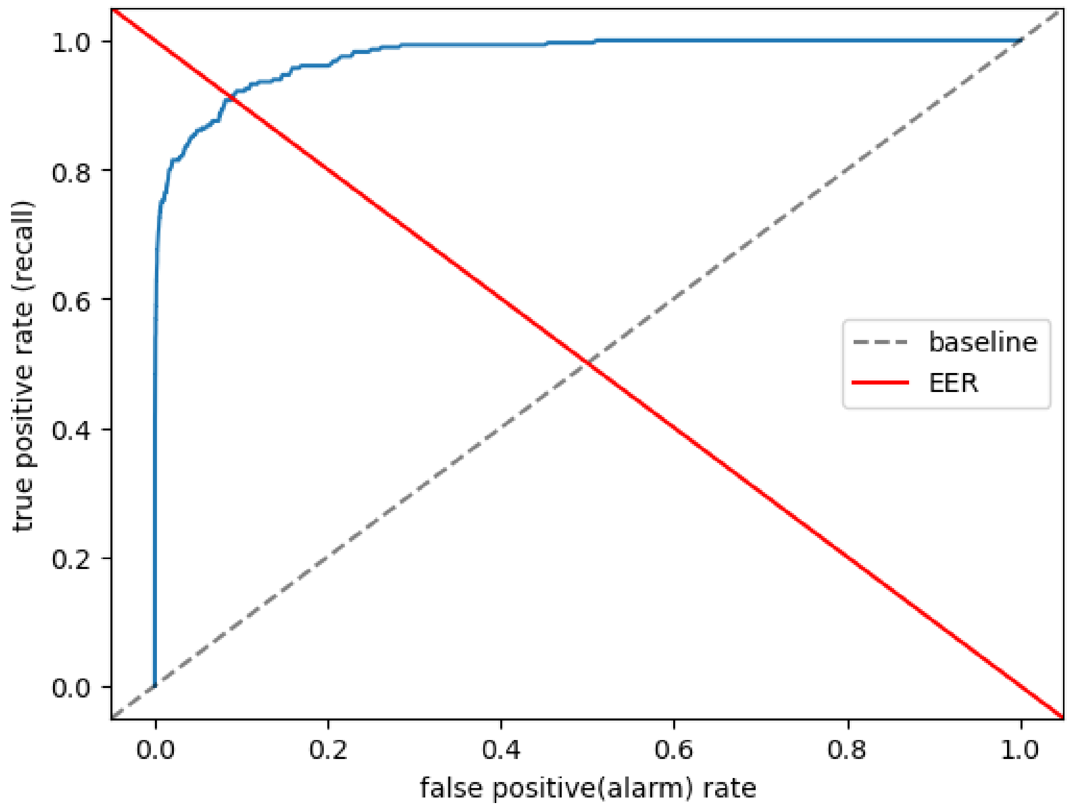
```
tpr: 0.9964285714285714
fpr: 0.4564102564102564
```

3.2 T9

```
[17]: def roc_curve(train, test, train_sz, test_sz, n_persons, print_threshold=False):
          prob = _get_prob(train, test, train_sz, test_sz, n_persons)
          mn, mx = prob.min(), prob.max()
          if print_threshold:
              print('(min, max) threshold ->', f"({mn} , {mx})" )
          thresholds = np.linspace(mn, mx, 1000)
          y = np.repeat(np.eye(n_persons), test_sz, axis=1)
          fpr_s, tpr_s = zip(*[ (fpr_rate(y, y_pred:=prob<threshold), recall_score(y,␣
       ↪y_pred))
                                for threshold in thresholds ])
          return np.array(fpr_s), np.array(tpr_s), thresholds

      def plot_roc(train, test, train_sz, test_sz, n_persons, print_threshold=False):
          fpr, tpr, thresholds = roc_curve(train, test, train_sz, test_sz, n_persons,␣
       ↪print_threshold=print_threshold)
          plt.plot(fpr, tpr)
          plt.axline((0, 0), slope=1, c='k', linestyle = '--', alpha=0.5,␣
       ↪label='baseline')
          plt.axline((1, 0), slope=-1, c='r', label='EER')

          plt.xlabel('false positive(alarm) rate')
          plt.ylabel('true positive rate (recall)')

          plt.legend()
          plt.show()
          return fpr, tpr, thresholds
```

```
[18]: fpr, tpr, thresholds = plot_roc(T, D, 3, 7, num_persons, print_threshold=True)
```

```
(min, max) threshold -> (1.7420153428787784 , 16.434561906764714)
```



3.3 T10

Equal Error rate(EER) $\Rightarrow$ False Alarm Rate = False Negative Rate

$$FAR = FRR$$
$$FAR = 1 - TPR$$
$$FAR + TPR - 1 = 0$$

use this to find FAR and TPR, and

$$EER = \frac{FAR + FRR}{2} = \frac{FAR + (1 - TPR)}{2}$$

[19]:
```
def compute_eer(tpr, fpr, thresholds=None, return_eer=False):
    mn_idx = np.argmin(np.abs(tpr + fpr - 1))
    far=fpr[mn_idx]
    frr=1-tpr[mn_idx]
    eer=np.mean((far, frr))

    if return_eer:
        return eer
```

```
    print('FAR:', far)
    print('FRR:', frr)
    print(30*'-')
    print('EER:', eer)
    print('EER Threshold', thresholds[mn_idx])
```

[20]:
```
compute_eer(tpr, fpr, thresholds)
```

```
FAR: 0.08864468864468865
FRR: 0.09285714285714286
------------------------------
EER: 0.09075091575091576
EER Threshold 8.080841738309047
```

3.3.1 False Alarm Rate at reall 0.1%

[21]:
```
def recall_at_far_rate(tpr, fpr, rate, thresholds=None, return_recall=False):
    fpr_idx = np.argmin(np.abs(fpr - rate))
    if return_recall:
        return tpr[fpr_idx]
    print('recall rate at far 0.1%:', tpr[fpr_idx])
    print('At threshold:', thresholds[fpr_idx])
```

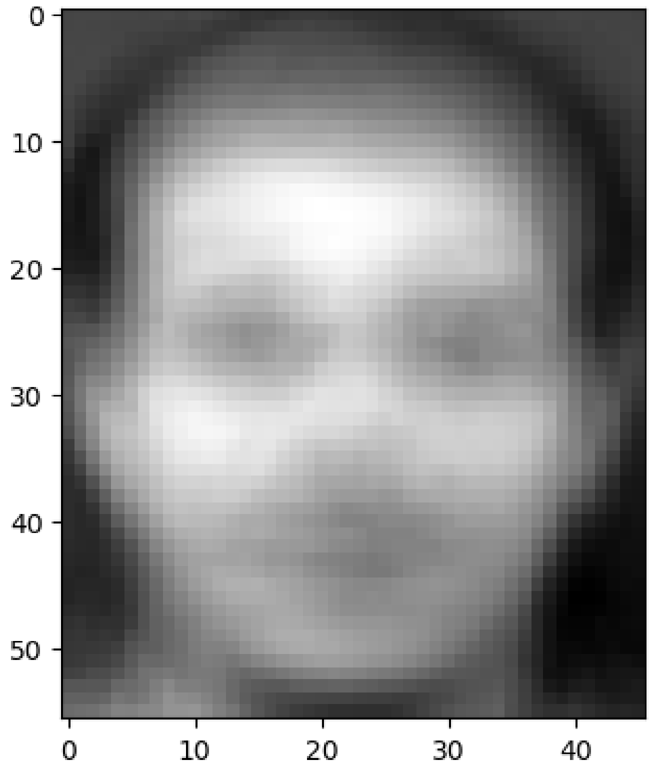
[22]:
```
recall_at_far_rate(tpr, fpr, .001, thresholds)
```

```
recall rate at far 0.1%: 0.5428571428571428
At threshold: 5.948289934742019
```

4 Principle Component Analysis (PCA)

4.1 T11

[23]:
```
mean_face = T.mean(axis = 0)
plt.imshow(mean_face.reshape(img_h, img_w) ,cmap='gray')
plt.show()
```



4.2 T12

[24]:
```
X = T.T
X_hat = X - mean_face.reshape(-1, 1)
cov_mat = np.cov(X)
print('Covariance Matrix size =', cov_mat.shape)
print('Covariance Matrix rank =', np.linalg.matrix_rank(cov_mat))
```

```
Covariance Matrix size = (2576, 2576)
Covariance Matrix rank = 119
```

4.3 T13

[25]:
```
gram_mat = X_hat.T @ X_hat
print('Gram Matrix size =', gram_mat.shape)
print('Gram Matrix rank =', np.linalg.matrix_rank(gram_mat))

eigen_vals, eigen_vecs = np.linalg.eigh(gram_mat)

# filter out 0 eigen vals
eigen_vecs = eigen_vecs[:, eigen_vals > 1e-6]
eigen_vals = eigen_vals[eigen_vals > 1e-6]
```

```
print('Gram Matrix non-zero eigen value =', len(eigen_vals) )
```

```
Gram Matrix size = (120, 120)
Gram Matrix rank = 119
Gram Matrix non-zero eigen value = 119
```

4.4 T14

Yes, The gram matrix is symmetric.

$$\hat{X}^T\hat{X} = \begin{bmatrix} — & x_1 & — \\ — & x_2 & — \\ & \vdots & \\ — & x_N & — \end{bmatrix} \begin{bmatrix} | & | & & | \\ x_1 & x_2 & \dots & x_N \\ | & | & & | \end{bmatrix}$$

$$= \begin{bmatrix} x_1^T x_1 & x_1^T x_2 & x_1^T x_N \\ x_2^T x_1 & x_2^T x_1 & x_2^T x_N \\ & \vdots & \\ x_N^T x_1 & x_N^T x_1 & x_N^T x_N \end{bmatrix}$$

and we know that $x_i^T x_j = x_j^T x_i$

4.5 T15

[26]:
```
sorted_idx = np.argsort(eigen_vals)[::-1]

eigen_vals = eigen_vals[sorted_idx]
eigen_vecs = eigen_vecs[:, sorted_idx]

print('number of non-zero eigen value:', len(eigen_vals) )
print('highest eigen val:', eigen_vals[0])
```

```
number of non-zero eigen value: 119
highest eigen val: 1423.9297148381547
```

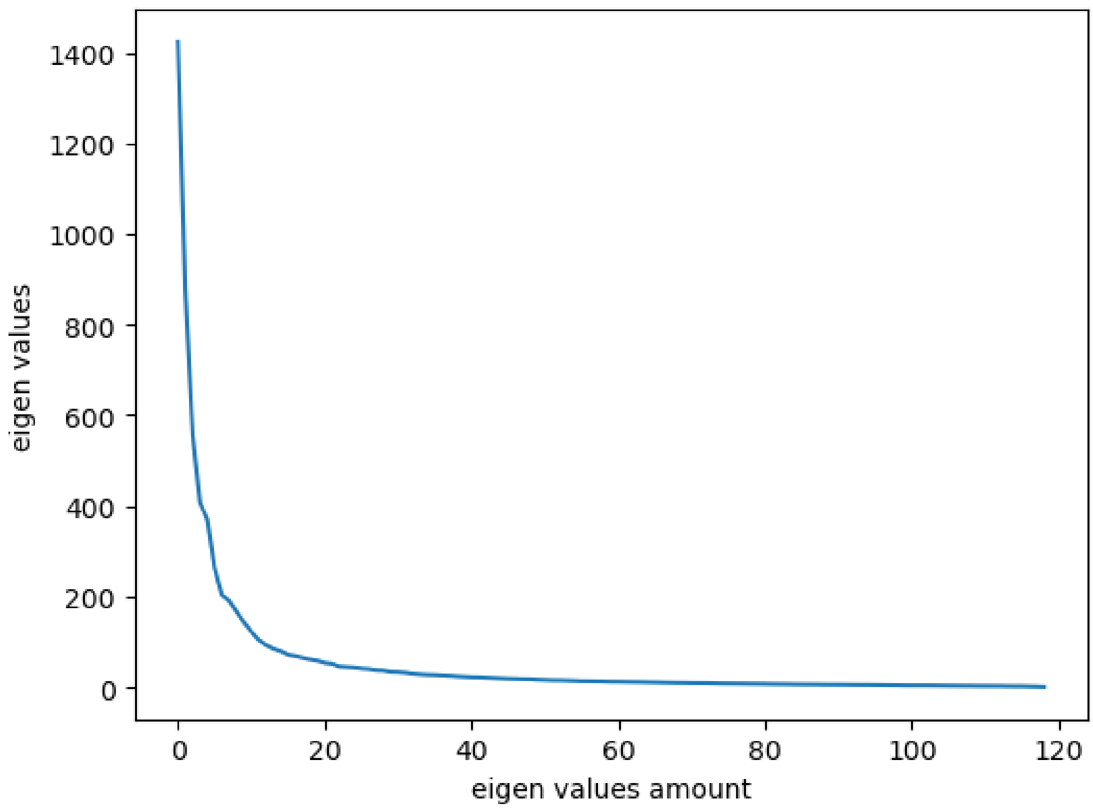
4.6 T16

[27]:
```
def search_eigen_amt(eigen_vals, var):
    l, r = 0, len(eigen_vals)+1
    while l < r:
        m = (l+r)>>1
        if _compute_variance(eigen_vals, m) >= var:
            r = m
        else:
            l = m+1
    return l
def _compute_variance(eigen_vals, amt):
    return eigen_vals[:amt].sum() / eigen_vals.sum()
```

[28]:

```
plt.plot(eigen_vals)
plt.ylabel('eigen values')
plt.xlabel('eigen values amount')

print('Amount of eigen vectors =', amt:=search_eigen_amt(eigen_vals, .95) )
print('Var=', _compute_variance(eigen_vals, amt))
plt.show()
```

```
Amount of eigen vectors = 64
Var= 0.9514558774601827
```

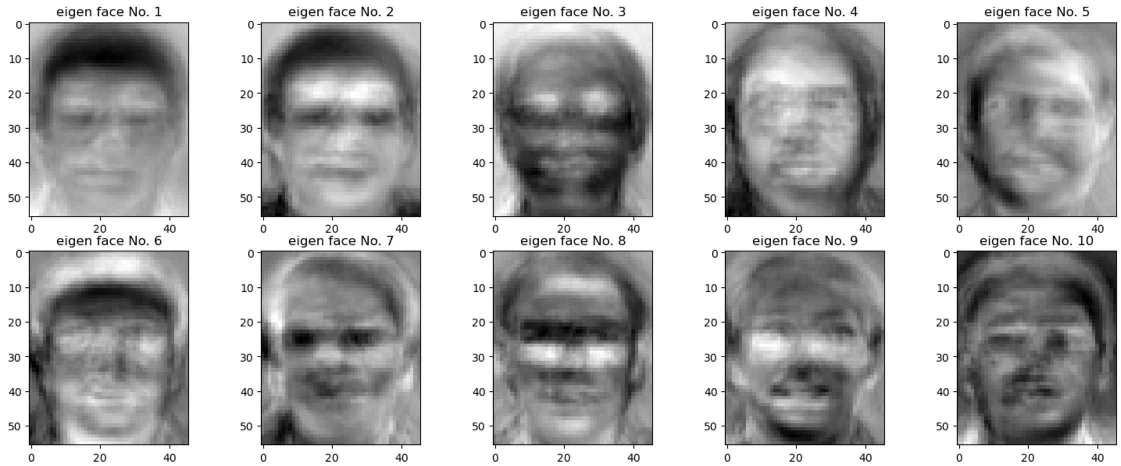


4.7 T17

[29]:

```
v = X_hat @ eigen_vecs
v = v / np.linalg.norm(v, axis=0)

fig, ax = plt.subplots(2, 5, figsize=(15, 6))
fig.tight_layout()

for i in range(2):
    for j in range(5):
```

```
        ax[i, j].set_title(f"eigen face No. {5*i+j+1}")
        ax[i, j].imshow(v[:, 5*i+j].reshape(img_h, img_w), cmap='gray_r')
plt.show()
```



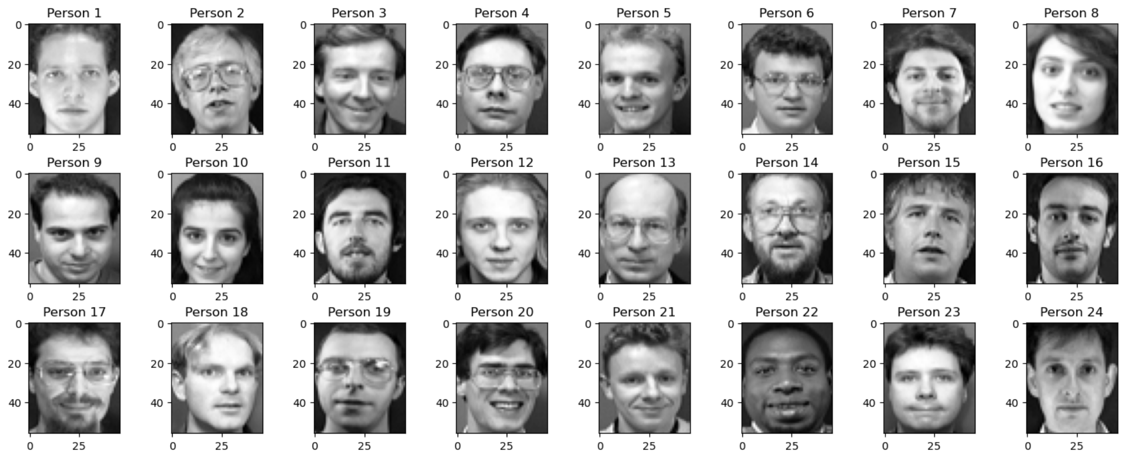
4.8 T18

The darker pixel show that there is high value in eigen face.

- Eigenface No.1
  - The dark part is hair
- Eigenface No.2
  - The dark part is also hair but there also have eye and collar that have dark part

let see the sample of people

[30]:
```
fig, ax = plt.subplots(3, 8, figsize=(15, 6))
fig.tight_layout()

for i in range(3):
    for j in range(8):
        ax[i, j].set_title(f'Person {8*i+j+1}')
        ax[i, j].imshow(xf[8*i+j, 0], cmap='gray')

plt.show()
```



The first and second eigen face are capture the biggest variance of the images because many people have different hair color and style and size of eyes. There also images of people have collar in its so the eigen face will capture it too.

5 PCA subspace and the face verification system

5.1 T19

[31]:
```
def reduce_dimension(X, v, mean_face, k):
    '''
        X is matrix that have vector in column
    '''
    V = v[:, :k]
    projection = V.T @ (X - mean_face.reshape(-1, 1))
    return projection

def face_verification_pca(train, test, train_sz, test_sz, n_persons, ks,␣
↪return_roc=False):
    mean_face = train.mean(axis = 0)
    X_hat = (train - mean_face).T
    gram_mat = X_hat.T @ X_hat

    # Compute eigen value
    eigen_vals, eigen_vecs = np.linalg.eigh(gram_mat)
    eigen_vecs = eigen_vecs[:, eigen_vals > 1e-6]
    eigen_vals = eigen_vals[eigen_vals > 1e-6]
    # Sort by eigen value
    sorted_idx = np.argsort(eigen_vals)[::-1]
    eigen_vals = eigen_vals[sorted_idx]
    eigen_vecs = eigen_vecs[:, sorted_idx]
```

```
    # compute and normalize v
    v = X_hat @ eigen_vecs
    v = v / np.linalg.norm(v, axis=0)

    eers = np.empty(len(ks))
    recall_at_far = np.empty(len(ks))

    for i, k in enumerate(ks):
        train_reduced = reduce_dimension(train.T, v, mean_face, k).T
        test_reduced = reduce_dimension(test.T, v, mean_face, k).T
        fpr, tpr, _ = roc_curve(train_reduced, test_reduced, train_sz, test_sz,␣
↪n_persons)

        eers[i] = compute_eer(tpr, fpr, return_eer=True)
        recall_at_far[i] = recall_at_far_rate(tpr, fpr, .001, return_recall =␣
↪True)
    if return_roc:
        return fpr, tpr
    return eers, recall_at_far
```

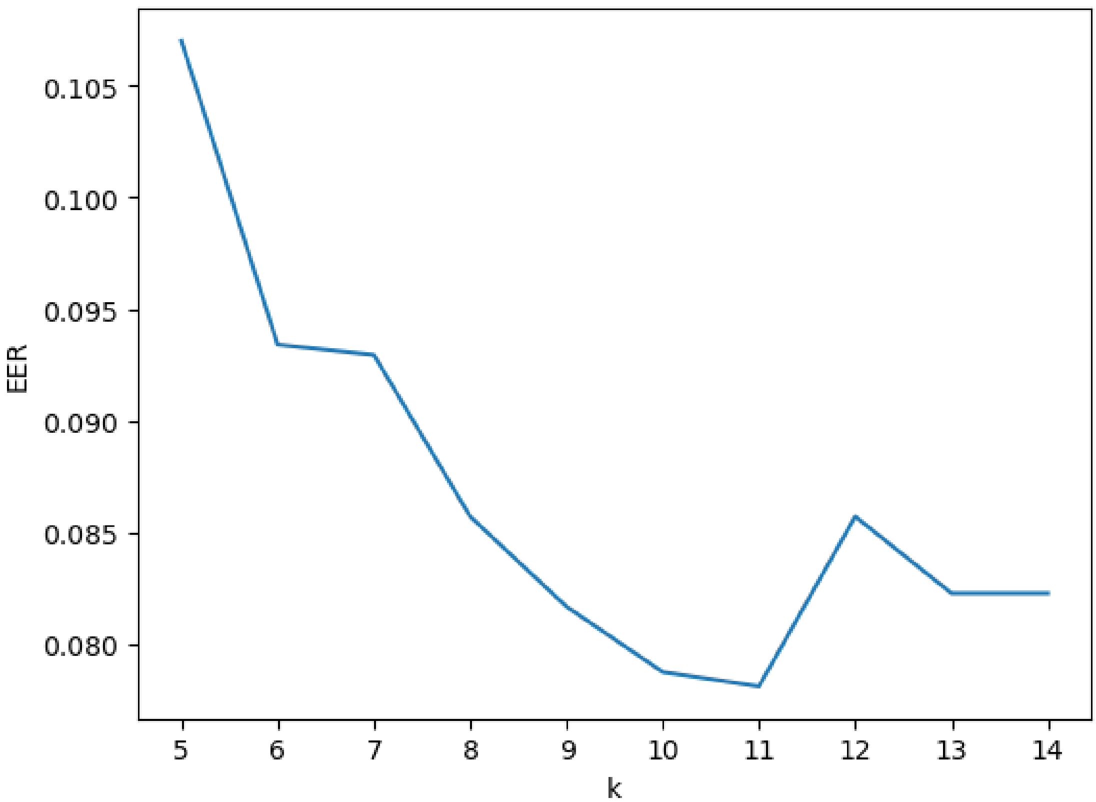
[32]:
```
eers, recall_001 = face_verification_pca(T, D, 3, 7, num_persons, [10] )
print("EER at k=10:", *eers)
print("Recall at FAR 0.1%:", *recall_001)
```

```
EER at k=10: 0.07875457875457878
Recall at FAR 0.1%: 0.5142857142857142
```

5.2 T20

[33]:
```
ks = np.arange(5, 15)
eers, _ = face_verification_pca(T, D, 3, 7, num_persons, ks)
print('Min eer at', eers.min(), 'at k =', ks[eers.argmin()])
pd.DataFrame([eers], columns=ks)

plt.plot(ks, eers)
plt.xticks(ks)
plt.xlabel('k')
plt.ylabel('EER')
plt.show()
pd.DataFrame([eers], columns=ks)
```

```
Min eer at 0.07811355311355314 at k = 11
```



[33]:

```
         5         6         7         8         9        10        11  \
0  0.107005  0.093407  0.092949  0.085714  0.081685  0.078755  0.078114

         12       13       14
0  0.085714  0.08228  0.08228
```

[34]:

```
fpr_pca, tpr_pca = face_verification_pca(T, D, 3, 7, num_persons, [11],␣
 ↪return_roc = True)
```

6 (Optional) PCA reconstruction

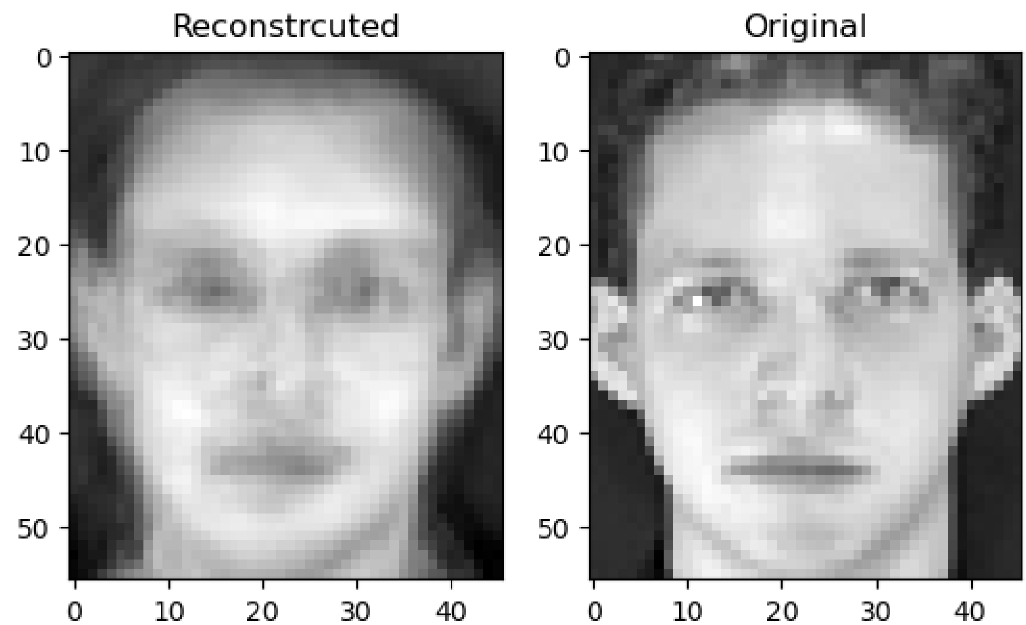
6.1 OT2

[35]:

```
def mean_squarred_error(a, b):
    return np.mean((a-b)**2)

def reconstruct_image(mean_face, v, k, projected):
    return mean_face + v[:, :k] @ projected[:k]

k = 10
projected_image = v.T @ (T[0] - mean_face)
```

```
reconstructed = reconstruct_image(mean_face, v, k, projected_image)

fig, ax = plt.subplots(1, 2)
ax[0].imshow( reconstructed.reshape(img_h, img_w), cmap='gray')
ax[0].set_title('Reconstrcuted')
ax[1].imshow( T[0].reshape(img_h, img_w), cmap='gray')
ax[1].set_title('Original')

print('MSE:', mean_squarred_error(reconstructed, T[0]))
plt.show()
```

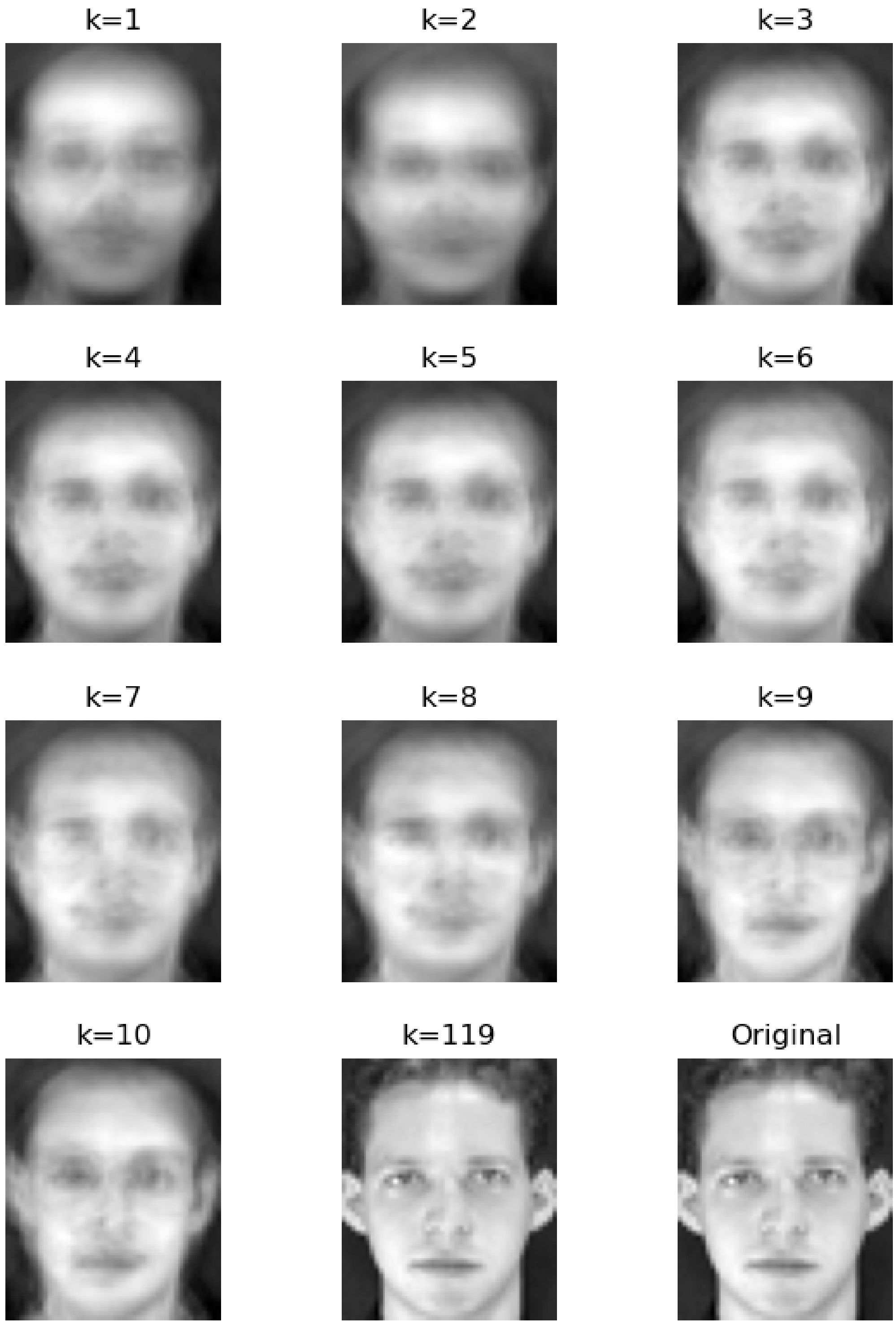
```
MSE: 0.006148335016488302
```

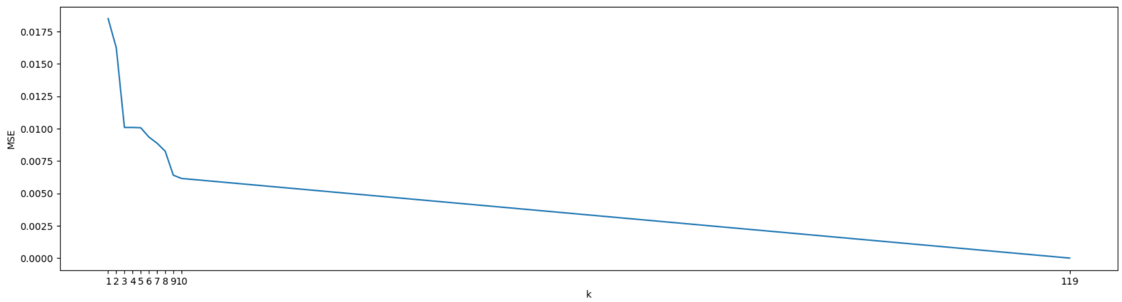


6.2 OT3

[36]:
```
ks = np.block([np.arange(1, 11), 119])
fig, ax = plt.subplots(4, 3, figsize=(6, 8))
fig.tight_layout()

mse_list = np.empty_like(ks, dtype=np.float64)
for i, k in enumerate(ks):
    projected_image = v[:, :k].T @ (T[0] - mean_face)
    reconstructed_img = reconstruct_image(mean_face, v, k, projected_image)
    ax[i//3, i%3].imshow(reconstructed_img.reshape(img_h, img_w), cmap='gray')
    ax[i//3, i%3].set_title(f"k={k}")
    ax[i//3, i%3].set_axis_off()
```

```
    mse_list[i] = mean_squarred_error(reconstructed_img, T[0])

ax[3, 2].set_title('Original')
ax[3, 2].imshow(T[0].reshape(img_h, img_w), cmap='gray')
ax[3, 2].set_axis_off()
plt.show()

plt.figure(figsize=(20, 5))
plt.plot(ks, mse_list)
plt.ylabel('MSE')
plt.xlabel('k')
plt.xticks(ks)

plt.show()
```


k=1
k=2
k=3
k=4
k=5
k=6
k=7
k=8
k=9
k=10
k=119
Original



6.3 OT4

6.3.1 Original database

total image pixel is 2576 pixels each.
1 pixel use 32 bits float datatype.
1 image use $32 \times 2576 = 82,432$ bits $= 10,304$ bytes per image
we have total $1,000,000$ images, therefore we use $10,304 \times 10^6$ bytes $= 9.5963$ Gibibytes (10.304 Gigabytes)

6.3.2 PCA database

After projected 2576 features(pixels)/image $\rightarrow$ 10 features/image
All images data $(10 \times 10^6) \times 4$ bytes = 40,000,000 bytes
Store total 10 eigen faces and 1 mean face $= ((10 + 1) \times 2576) \times 4$ btyes = 113,344 bytes
$\therefore$ total memory usage $= 40,113,344$ bytes $= 38.2551$ Mebibytes (40.1133 Megabytes)

7 Linear Discriminant Analysis (LDA)

7.1 T21

S_W has $N-C$ rank due to we compute C mean for each class. For our example $N-C$ is $120-40 = 80$ (120 training images with 40 different people)

7.2 T22

[38]:
```
def compute_sw(projected_train, num_classes, data_per_class):
    projected_by_class = projected_train.T.reshape(num_classes, data_per_class,␣
 ↪-1)
    mean_face_by_class = projected_by_class.mean(axis=1, keepdims=True)
    deviation = (projected_by_class - mean_face_by_class).reshape(num_classes *␣
 ↪data_per_class, -1).T # turn back to 80 * 120
    return deviation @ deviation.T

def compute_sb(projected_train, num_classes, data_per_class):
    global_mean_faces  = projected_train.mean(axis=1)
```

```
    projected_by_class = projected_train.T.reshape(num_classes, data_per_class,␣
↪-1)
    mean_face_by_class = projected_by_class.mean(axis=1)
    mean_deviation = (mean_face_by_class - global_mean_faces).T
    return mean_deviation @ mean_deviation.T
```

[39]:
```
sw_rank = T.shape[0] - num_persons
projected_train = v[:, :sw_rank].T @ (T - mean_face).T

sw = compute_sw(projected_train, num_persons, 3)
sb = compute_sb(projected_train, num_persons, 3)

lda_proj = np.linalg.inv(sw) @ sb
is_symmetric = (lda_proj-lda_proj.T < 1e-6).all()

print("lda proj is", "symmetric." if is_symmetric else "asymmetric.", end=' ')
print("Therefore, we", "can" if is_symmetric else "can not", "use np.linalg.
↪eigh.")

lda_eigen_vals, lda_eigen_vecs = np.linalg.eig(lda_proj)
lda_eigen_vecs = lda_eigen_vecs.real # only real part

lda_eigen_vecs = lda_eigen_vecs[:, lda_eigen_vals > 1e-6]
lda_eigen_vals = lda_eigen_vals[lda_eigen_vals > 1e-6]
sorted_idx = np.argsort(lda_eigen_vals)[::-1]

lda_eigen_vecs = lda_eigen_vecs[:, sorted_idx]
lda_eigen_vals = lda_eigen_vals[sorted_idx]


print("sw rank", np.linalg.matrix_rank(sw))
print("sb rank", np.linalg.matrix_rank(sb))
print("lda proj rank", np.linalg.matrix_rank(lda_proj))

print("number of non zero eigen values", len(lda_eigen_vals))
```

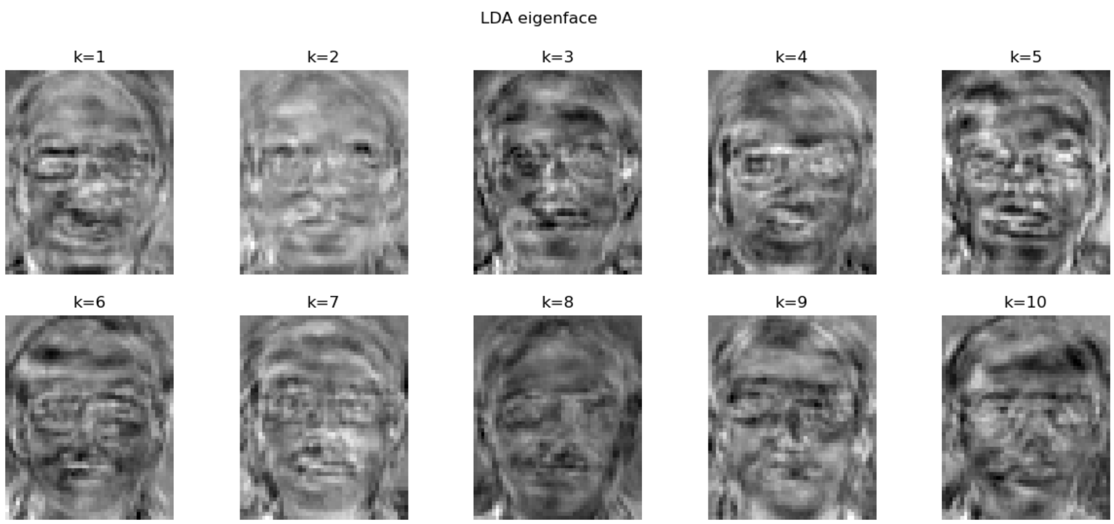
```
lda proj is asymmetric. Therefore, we can not use np.linalg.eigh.
sw rank 80
sb rank 39
lda proj rank 39
number of non zero eigen values 39
```

7.3 T23

[69]:
```
# lda eigen -> pca -> image
lda_eigen_constructed = v[:, :sw_rank] @ lda_eigen_vecs
print(lda_eigen_constructed.shape)
```

```
fig, ax = plt.subplots(2, 5, figsize=(15, 6))
fig.suptitle('LDA eigenface')

for k in range(10):
    i, j = k // 5, k % 5
    ax[i, j].imshow(lda_eigen_constructed[:, k].reshape(img_h, img_w),␣
 ↪cmap='gray_r')
    ax[i, j].set_title(f"k={k+1}")
    ax[i, j].set_axis_off()

plt.show()
```

```
(2576, 39)
```



LDA eigenface tell what different between each people

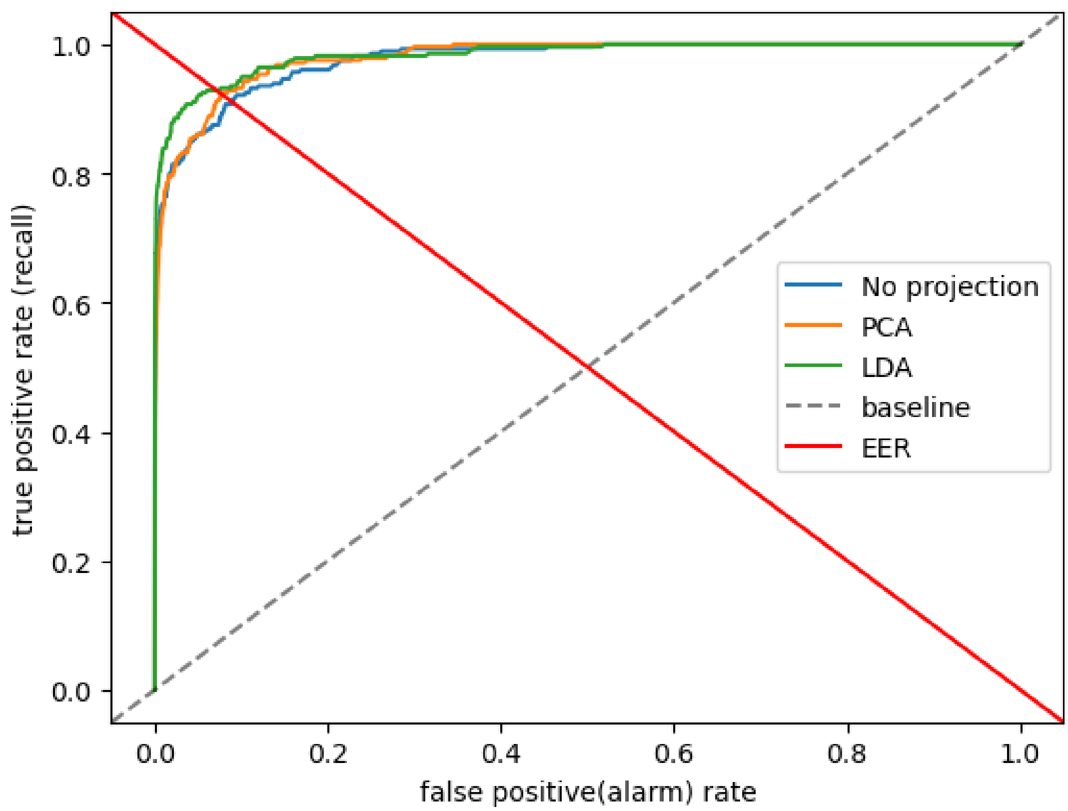
7.4 T24

[41]:
```
projected_train_lda = lda_eigen_constructed.T @ T.T
projected_test_lda = lda_eigen_constructed.T @ D.T

fpr_lda, tpr_lda, _ = roc_curve(projected_train_lda.T, projected_test_lda.T, 3,␣
 ↪7, num_persons)
eer = compute_eer(tpr_lda, fpr_lda, return_eer=True)
recall = recall_at_far_rate(tpr_lda, fpr_lda, .001, return_recall = True)

print("EER", eer)
print("Recall at FAR 0.1%:", recall)
```

```
EER 0.07170329670329668
Recall at FAR 0.1%: 0.6821428571428572
```

7.5 T25

[42]:
```
plt.plot(fpr, tpr, label='No projection')
plt.plot(fpr_pca, tpr_pca, label='PCA')
plt.plot(fpr_lda, tpr_lda, label='LDA')

plt.axline((0, 0), slope=1, c='k', linestyle = '--', alpha=0.5,␣
 ↪label='baseline')
plt.axline((1, 0), slope=-1, c='r', label='EER')


plt.xlabel('false positive(alarm) rate')
plt.ylabel('true positive rate (recall)')

plt.legend()
plt.show()
```



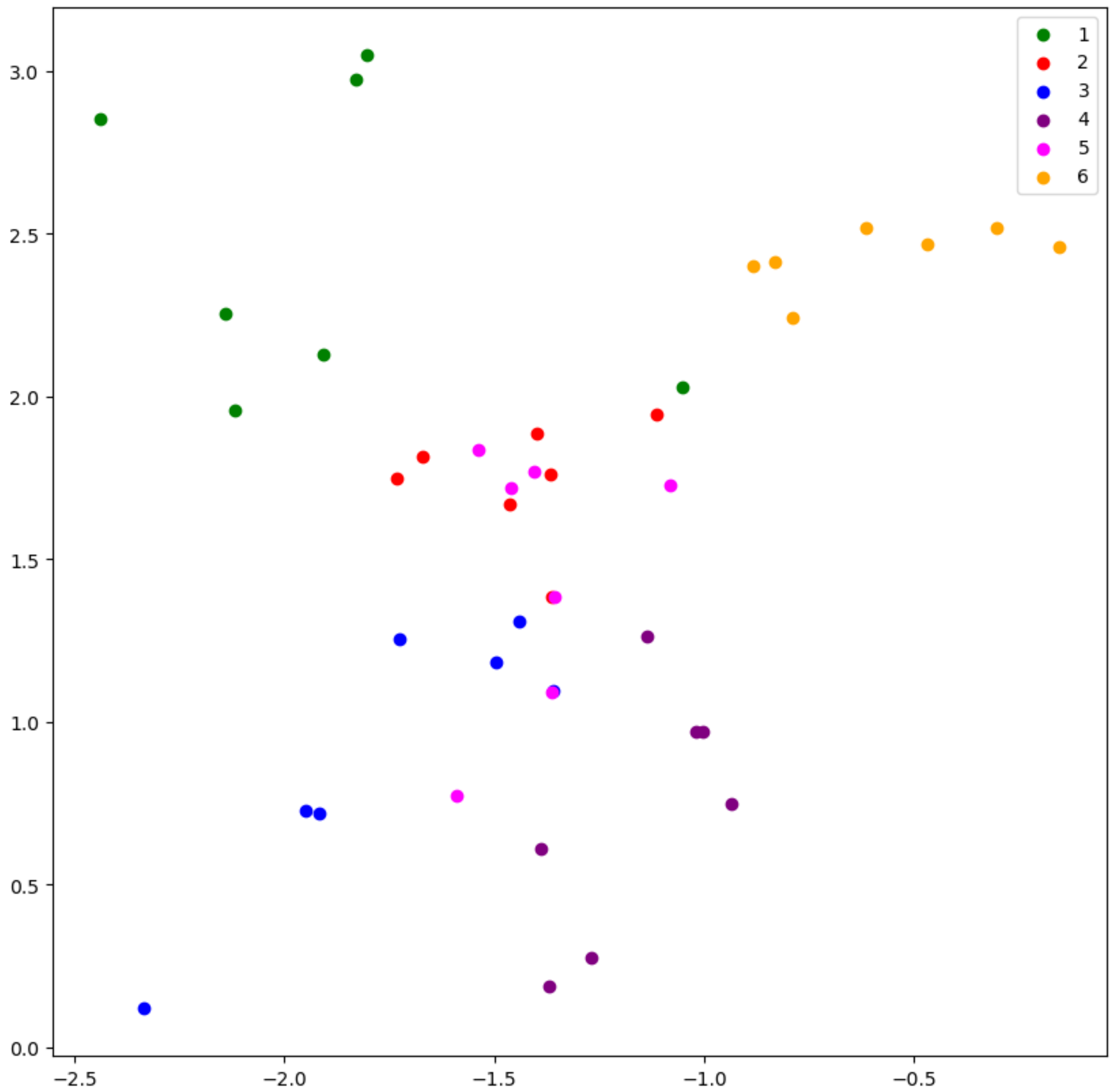
LDA has the best performance. :)

7.6 OT5

[43]:
```
def plot_cluster(projected_data, samples_size, img_per_sample):
    plt_dim = projected_data[:2, :samples_size * img_per_sample].T
    plt_x, plt_y = plt_dim[:, 0], plt_dim[:, 1]
    plt_x = plt_x.reshape(samples_size, img_per_sample)
    plt_y = plt_y.reshape(samples_size, img_per_sample)
    color = ['g', 'r', 'b', 'purple', 'magenta', 'orange', 'cyan']
    plt.figure(figsize=(10, 10))
    for i in range(samples_size):
        plt.scatter(plt_x[i], plt_y[i], c=color[i], label=str(i+1))
    plt.legend()
    plt.show()
```

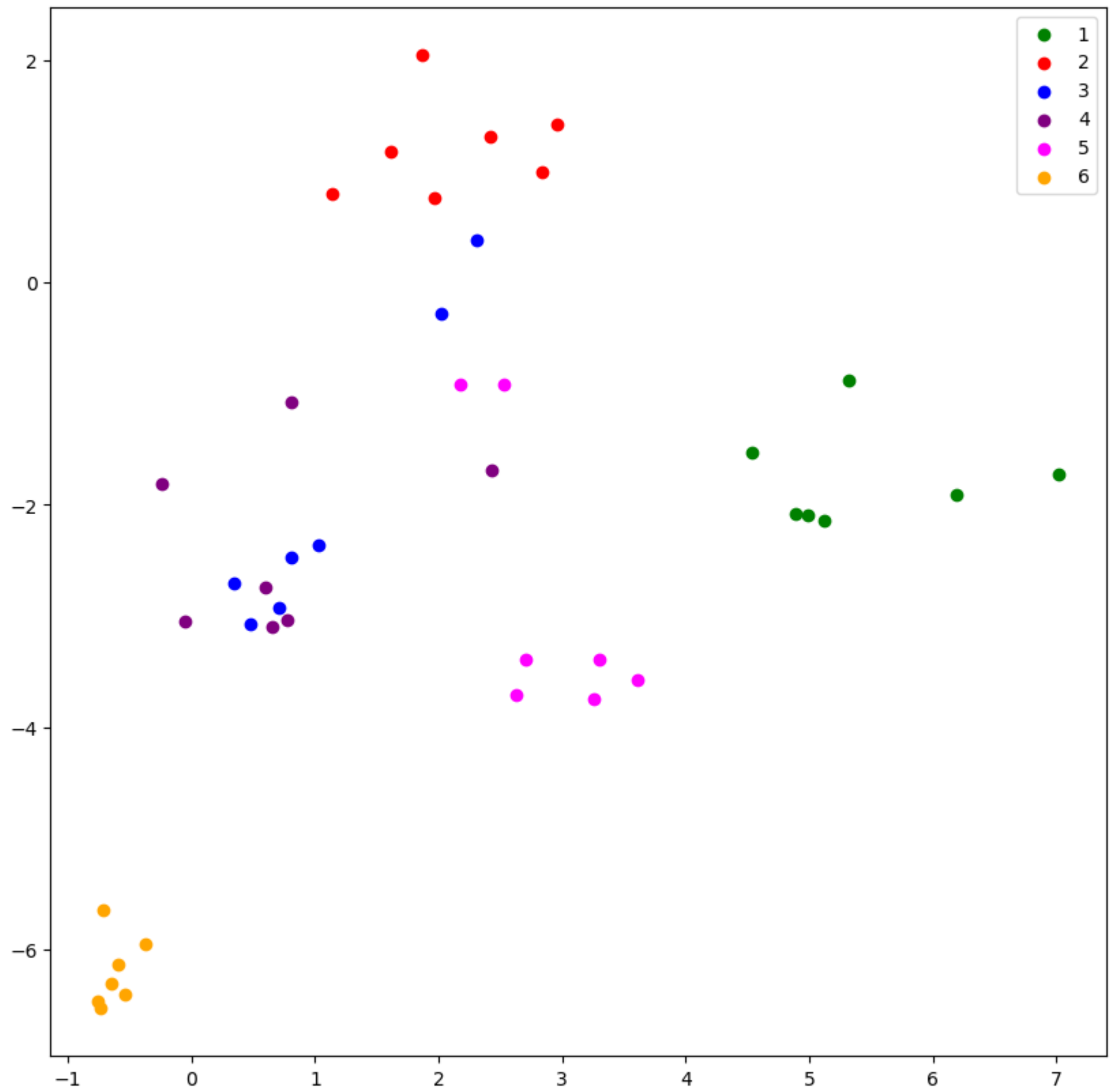
7.6.1 LDA

[66]:
```
plot_cluster(projected_test_lda, 6, 7)
```



7.6.2 PCA

[76]:
```
projected_pca = reduce_dimension(D.T, v, mean_face, 2)
plot_cluster(projected_pca, 6, 7)
```



The result seem that PCA is better cluster than LDA (not as my expected). I think the main reason is our LDA is compressed from the PCA (to make S_w invertible, non-singular).