

# *A gentle* introduction to MCTDH

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- 1 Nuclear Quantum Dynamics
  - The Standard method
  - The Time-Dependent Hartree method
- 2 The Multiconfiguration Time-Dependent Hartree method
  - EOMs
  - Relaxation and block-improved relaxation
- 3 The problem of the PES
  - Tensor decomposition algorithms
- 4 Example applications

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# Nuclear Quantum Dynamics

## What

The subfield of Theoretical Chemistry in which both the **electrons** and the **nuclei** of a molecular system are treated in a **quantum-mechanical** manner.

# Nuclear Quantum Dynamics

## What

The subfield of Theoretical Chemistry in which both the **electrons** and the **nuclei** of a molecular system are treated in a **quantum-mechanical** manner.

## When

- Spectroscopy (e.g. IR transitions)
- Quantum tunneling
- Strong vibronic coupling
- Accurate ZPE determination

# Nuclear Quantum Dynamics

Find the numerical solution of the TDSE truncating the Hilbert space to a finite dimension (Galerkin's method):

$$i\hbar \frac{\partial \Psi}{\partial t} = \hat{H} \Psi \quad (1)$$

Given a parametric representation of the WF ( $\Psi$ ), the optimal solution can be found using the Dirac-Frenkel Variational Principle (DF-VP):

$$\langle \delta \Psi | \hat{H} - i \frac{\partial}{\partial t} | \Psi \rangle = 0 \quad (2)$$

# The Standard method

Simplest representation of the WF:

$$\Psi(q_i, \dots, q_f, t) = \sum_{j_1=1}^{N_1} \cdots \sum_{j_f=1}^{N_f} C_{j_1 \dots j_f}(t) \prod_{\kappa=1}^f \varphi_{j_\kappa}^{(\kappa)}(q_\kappa) \quad (3)$$

After plugging this WF into the DF-VP, and performing the corresponding algebra we obtain the following EOMs:

$$i\dot{C}_L = \sum_J \langle \varphi_L | \hat{H} | \varphi_J \rangle C_J$$

$$C(t) = e^{-iHt} C(0) \quad (4)$$

where we have introduced the composite indexes  $J = (j_1, \dots, j_f)$ .

## A slightly more clever *ansatz*:

If we now consider time-dependent single-particle functions (SPFs):

$$\Psi(q_1, \dots, q_f, t) = A(t) \prod_{\kappa=1}^f \underbrace{\sum_{\mu=1}^{N_{\kappa}} c_{\mu}^{(\kappa, j_{\kappa})}(t) \cdot \chi_{\mu}^{(\kappa)}(q_{\kappa})}_{\varphi_{\kappa}(q_{\kappa}, t)} \quad (5)$$

and use the DF-VP with arbitrary real constraints  $g_{\kappa} = i \langle \varphi_{\kappa}(t) | \dot{\varphi}_{\kappa}(t) \rangle$ , we get the EOMs:

$$\begin{aligned} A(t) &= A(0) \cdot e^{-i \int_0^t E(t') dt'} \\ i \dot{\varphi}_{\kappa} &= (\mathcal{H}^{(\kappa)} - E) \varphi_{\kappa} \end{aligned} \quad (6)$$



# Limitations of SM and TDH

## Standard method

Its application is largely limited due to the **curse of dimensionality**. Only systems up to four atoms (6D) can be addressed in practice.

## Time-Dependent Hartree

A better approach, but physically inaccurate. The **nuclear correlation** is harder to retrieve than the electronic correlation due to the nuclei's larger mass. The character of the nuclear WF is inherently **multiconfigurational**.

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# Ansätze comparison

## Standard Method (CI)

$$\Psi(q_i, \dots, q_f, t) = \sum_{j_1=1}^{N_1} \cdots \sum_{j_f=1}^{N_f} C_{j_1 \dots j_f}(t) \prod_{\kappa=1}^f \varphi_{j_\kappa}^{(\kappa)}(q_\kappa) \quad (7)$$

# Ansätze comparison

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## Time-Dependent Hartree (HM)

$$\Psi(q_1, \dots, q_f, t) = A(t) \prod_{\kappa=1}^f \varphi_\kappa(q_\kappa, t) \quad (8)$$

# Ansätze comparison

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## Time-Dependent Hartree (HM)

$$\Psi(q_1, \dots, q_f, t) = A(t) \prod_{\kappa=1}^f \varphi_\kappa(q_\kappa, t) \quad (8)$$

## Multiconfiguration Time-Dependent Hartree (MCSCF)

$$\Psi(q_1, \dots, q_f, t) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_f=1}^{n_f} A_{j_1, \dots, j_f}(t) \prod_{\kappa=1}^f \varphi_{j_\kappa}^{(\kappa)}(q_\kappa, t) \quad (9)$$

# MCTDH origins and distribution

MCTDH was conceived by **Hans-Dieter Meyer** and coworkers from the University of Heidelberg, in the early nineties:

## THE MULTI-CONFIGURATIONAL TIME-DEPENDENT HARTREE APPROACH

H.-D. MEYER, U. MANTHE and L.S. CEDERBAUM

*Theoretische Chemie, Physikalisch-Chemisches Institut, Universität Heidelberg, D-6900 Heidelberg, Federal Republic of Germany*

Received 11 October 1989

There are currently three major implementations of the algorithm:



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# The MCTDH EOMs

The MCTDH ansatz has a very flexible Sum-of-Products (SOP) form:

$$\Psi(q_1, \dots, q_f, t) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_f=1}^{n_f} A_{j_1, \dots, j_f}(t) \prod_{\kappa=1}^f \varphi_{j_\kappa}^{(\kappa)}(q_\kappa, t) \quad (10)$$

with time dependent SPFs

$$\varphi_{j_\kappa}^{(\kappa)}(q_\kappa, t) = \sum_{\mu=1}^{N_\kappa} c_\mu^{(\kappa, j_\kappa)}(t) \cdot \chi_\mu^{(\kappa)}(q_\kappa) \quad (11)$$

The  $\chi_\mu^{(\kappa)}(q_\kappa)$  are typically DVR functions.

# The MCTDH EOMs

The *ansatz* WF is determined up to a multiplicative constant. To derive the EOMs arbitrary constraint operators ( $\hat{g}^{(\kappa)}$ ) are introduced:

$$i \langle \varphi_l^{(\kappa)} | \dot{\varphi}_j^{(\kappa)} \rangle = \langle \varphi_l^{(\kappa)} | \hat{g}^{(\kappa)} | \varphi_l^{(\kappa)} \rangle \quad (12)$$

Using once again the DF-VP we get (for  $\hat{g}^{(\kappa)} \equiv 0$ ):

$$i \dot{A}_J = \sum_L \langle \Phi_J | \hat{H} | \Phi_L \rangle A_L$$

$$i \dot{\varphi}_j^{(\kappa)} = (1 - \hat{P}^{(\kappa)}) \sum_{k,l=1}^{n_\kappa} (\rho^{(\kappa)^{-1}})_{jk} \langle \hat{H} \rangle_{kl}^{(\kappa)} \varphi_l^{(\kappa)} \quad (13)$$



# The MCTDH EOMs

$$i\dot{A}_J = \sum_L \langle \Phi_J | \hat{H} | \Phi_L \rangle A_L$$

$$i\dot{\varphi}_j^{(\kappa)} = (1 - \hat{\rho}^{(\kappa)}) \sum_{k,l=1}^{n_\kappa} (\rho_{jk}^{(\kappa)})^{-1} \langle \hat{H} \rangle_{kl}^{(\kappa)} \varphi_l^{(\kappa)} \quad (14)$$

$$\Phi_J = \prod_{\kappa=1}^f \varphi_{j_\kappa}^{(\kappa)} \quad (15)$$

$$\rho_{kl}^{(\kappa)} = \langle \Psi_k^{(\kappa)} | \Psi_l^{(\kappa)} \rangle = \sum_{J^\kappa} A_{j_k^\kappa}^* A_{j_l^\kappa}^* \quad \langle \hat{H} \rangle_{kl}^{(\kappa)} = \langle \Psi_k^{(\kappa)} | \hat{H} | \Psi_l^{(\kappa)} \rangle \quad (16)$$

$$\hat{\rho}^{(\kappa)} = \sum_{j=1}^{n_\kappa} |\varphi_j^{(\kappa)}\rangle \langle \varphi_j^{(\kappa)}| \quad (17)$$

# MCTDH integration scheme

The MCTDH-EOMs solution is costly due to the large amount of multidimensional integrals to solve. Since the **mean fields** are not strongly oscillating we can consider (CMF integration):

$$\begin{aligned}
 i\dot{A}_J &= \sum_L \bar{\mathcal{K}}_{JL} A_J \\
 i\dot{\varphi}_j^{(1)} &= (1 - \hat{P}^{(1)}) \{ \hat{h}^{(1)} \varphi_j^{(1)} + \sum_{k,l=1}^{n_1} (\rho^{(1)})_{jk}^{-1} \langle \bar{H}_R \rangle_{kl}^{(1)} \varphi_l^{(1)} \} \\
 &\vdots \\
 i\dot{\varphi}_j^{(f)} &= (1 - \hat{P}^{(f)}) \{ \hat{h}^{(f)} \varphi_j^{(f)} + \sum_{k,l=1}^{n_f} (\rho^{(f)})_{jk}^{-1} \langle \bar{H}_R \rangle_{kl}^{(f)} \varphi_l^{(f)} \}
 \end{aligned} \tag{18}$$

# Mode combination

Nothing stop us to group physical coordinates into logical particles:

$$Q_\kappa \equiv (q_{\kappa,1}, q_{\kappa,1}, \dots, q_{\kappa,d})$$

$$\varphi_j^{(\kappa)}(Q_\kappa, t) = \varphi_j^{(\kappa)}(q_{\kappa,1}, q_{\kappa,1}, \dots, q_{\kappa,d}, t) \quad (19)$$

Under these conditions, the MCTDH *ansatz* will take the form:

$$\Psi(Q_1, \dots, Q_p, t) = \sum_{j_1=1}^{n_1} \cdots \sum_{j_p=1}^{n_p} A_{j_1, \dots, j_p}(t) \prod_{\kappa=1}^p \varphi_{j_\kappa}^{(\kappa)}(Q_\kappa, t) \quad (20)$$

$$\varphi_j^{(\kappa)}(Q_\kappa, t) = \sum_{i_1 \dots i_d} c_{i_1 \dots i_d}^{(\kappa, j)}(t) \prod_{\nu=1}^d \chi^{(\kappa, \nu)}(q_{\kappa, \nu})$$

# Multilayer MCTDH (3-layer case)

We can propagate the multidimensional SPFs with MCTDH itself!

$$\Psi(q_1, q_2, q_3, t) = \sum_{j_{12}=1}^{n_{12}} \sum_{j_3=1}^{n_3} A_{j_{12}j_3}(t) \varphi_{j_{12}}^{(12)}(q_1, q_2, t) \varphi_{j_3}^{(3)}(q_3, t) \quad (21)$$

where we have introduced:

$$\varphi_{j_{12}}^{(12)}(q_1, q_2, t) = \sum_{k_1=1}^{n_1} \sum_{k_2=1}^{n_2} B_{k_1, k_2}^{(12, j_{12})} \prod_{\mu=1}^2 \xi_{k_\mu}^{(\mu)}(q_\mu, t) \quad (22)$$

and:

$$\xi_{k_\mu}^{(\mu)}(q_\mu, t) = \sum_{i_\mu=1}^{N_\mu} c_{i_\mu}^{(\mu, k_\mu)} \chi_{i_\mu}^{(\mu)}(q_\mu) \quad (23)$$

# Obtaining vibrational orbitals

MCTDH can be also used to solve the TISE. The GS distribution of the system can be obtained by propagation in negative imaginary time  $\tau = -it$ :

$$\dot{\Psi} = -\hat{H}\Psi \quad (24)$$

The new algorithm can be derived by applying the time-independent variational principle with Lagrange multipliers:

$$\delta\{\langle\Psi|\hat{H}|\Psi\rangle - E(\sum_J A_J^* A_J - 1) - \sum_{\kappa=1}^f \sum_{j,l=1}^{n_{\kappa}} \epsilon_{jl}^{(\kappa)} [\langle\varphi_j^{(\kappa)}|\varphi_l^{(\kappa)}\rangle - \delta_{jl}]\} = 0 \quad (25)$$

# Obtaining vibrational orbitals

Taking the variations with respect to the complex conjugate of both the A-vector and the SPFs independently we get:

$$\sum_K H_{JK} A_K = E A_J$$
$$\frac{\partial \varphi_j^{(\kappa)}}{\partial \tau} = -(1 - \hat{P}^{(\kappa)}) \sum_{k,l} (\rho^{(\kappa)^{-1}})_{jk} \langle \hat{H} \rangle_{kl}^{(\kappa)} \varphi_l^{(\kappa)} = 0 \quad (26)$$

The second of these equations implies that we can obtain the updated SPFs simply by relaxation. The A-vector in the first equation can be obtained by using the Davidson diagonalization algorithm.

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# The importance of the SOP form

The multidimensional integrals arising from the MCTDH-EOMs are the bottleneck of the propagation. To address this issue, we enforce SOP form to **all quantities**:

$$\begin{aligned}\hat{O} &= \sum_{\alpha=1}^S c_{\alpha} \prod_{\kappa=1}^f \hat{o}_{\alpha}^{(\kappa)} \\ \langle \Phi_J | \hat{O} | \Phi_L \rangle &= \sum_{\alpha=1}^S c_{\alpha} \prod_{\kappa=1}^f \langle \varphi_{j_{\kappa}}^{(\kappa)} | \hat{o}_{\alpha}^{(\kappa)} | \varphi_{l_{\kappa}}^{(\kappa)} \rangle\end{aligned}\tag{27}$$

- KEO already in the desired form
- PES might be challenging to transform



# Transforming the PES

Usually the PES needs to be refitted (**tensor decomposed**) before using it. The POTFIT algorithm is an elegant way of achieving this in **Tucker form**:

$$V_{i_1, \dots, i_f} = V(q_{i_1}^{(1)}, \dots, q_{i_f}^{(f)})$$

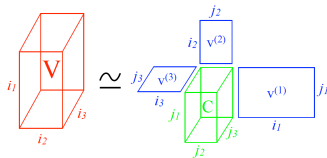
$$V_{i_1, \dots, i_f} = \sum_{j_1=1}^{m_1} \cdots \sum_{j_f=1}^{m_f} C_{j_1 \dots j_f} \prod_{\kappa=1}^f u_{i_\kappa j_\kappa}^{(\kappa)} \quad (28)$$

with the core tensor coefficients given by the overlap with the potential:

$$C_{j_1 \dots j_f} = \sum_{i_1 \dots i_f} V_{i_1 \dots i_f} u_{i_1 j_1}^{(1)} \cdots u_{i_f j_f}^{(f)} \quad (29)$$

# The Tucker form

The tucker decomposition of a 3D tensor can be represented graphically as<sup>1</sup>



which can be contrasted with the algebraic and tensor forms:

$$V_{i_1, \dots, i_f} = \sum_{j_1=1}^{m_1} \cdots \sum_{j_f=1}^{m_f} C_{j_1 \cdots j_f} \prod_{\kappa=1}^f u_{i_\kappa j_\kappa}^{(\kappa)} \quad (30)$$

$$\mathcal{V} = \mathcal{C} \times_1 \mathbf{U}_1 \cdots \times_n \mathbf{U}_n$$

<sup>1</sup>Panadés-Barrueta R., and Peláez D. JCP (in review)

# Tensor decomposition algorithms

There is a number of tensor decomposition algorithms currently in use (e.g. POTFIT, MGPF, MCPF, MLPF), however, they are all limited to product grids. The **SOP-FBR** method was developed as an alternative to the former:

$$V(q_1, \dots, q_f) = \sum_{j_1=1}^{m_1} \cdots \sum_{j_f=1}^{m_f} C_{j_1 \dots j_f} \prod_{\kappa=1}^f \Phi_{j_\kappa}^{(\kappa)}(q_\kappa) \quad (31)$$

$$\Phi_{j_\kappa}(q_\kappa) = \sum_{\nu_\kappa=1}^{t_\kappa} B_{\nu_\kappa j_\kappa}^{(\kappa)} T_{\nu_\kappa}(q_\kappa)$$

This is a fully analytical SOP form, differentiable *ad infinitum*, and that can be directly interfaced with MCTDH.

# POTFIT compared to HOOI

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**Algorithm 1: POTFIT**


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**Result:**  $\mathcal{C}, U_1, \dots, U_n$ 

 Input:  $\mathcal{V}$ ;

 for  $k \leftarrow 1$  to  $n$  do

    $U_k \leftarrow \text{EVD}(V_{(k)}^\dagger \cdot V_{(k)})$ 

end

 $\mathcal{C} \leftarrow \mathcal{V} \times_1 U_1^{-1} \dots \times_n U_n^{-1}$ 


---



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**Algorithm 2: HOSVD HOOI**


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**Result:**  $\mathcal{C}, U_1, \dots, U_n$ 

 Input:  $\mathcal{V}$ ;

repeat

   for  $k \leftarrow 1$  to  $n$  do

      $\mathcal{Y} \leftarrow \mathcal{V} \times_1 U_1^{-1} \dots \times_{k-1} U_{k-1}^{-1} \times_{k+1}$ 

      $U_{k+1}^{-1} \dots \times_n U_n^{-1}$ ;

      $U_k \leftarrow \text{SVD}(V_{(k)})$  SVD( $\mathcal{Y}_{(k)}$ )

end

 $\mathcal{C} \leftarrow \mathcal{V} \times_1 U_1^{-1} \dots \times_n U_n^{-1}$ 

 until  $\|\mathcal{V}_{app} - \mathcal{V}\| < \epsilon$ ;

---

- $\text{EVD}(V_{(k)}^\dagger \cdot V_{(k)}) \equiv \text{SVD}(V_{(k)})$  !
- POTFIT optimizes the SPPs in a slightly different manner:

$$\tilde{v}_j^{(\kappa)} = v_j^{(\kappa)} + \sum_{l=n_\kappa+1}^{N_\kappa} \mu_{jl}^{(\kappa)} v_l^{(\kappa)}$$

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# The Heidelberg implementation of MCTDH

The actual implementation is written mainly in `f(ortran)`, with some small `C` contributions. The program has a modular structure with a very intuitive and consistent input syntax. Some sections of a POTFIT input file:

```

RUN-SECTION
  name = h2o.pfit
end-run-section

OPERATOR-SECTION
  pes = pjt2{binding}
  vcut < 0.5
end-operator-section

PRIMITIVE-BASIS-SECTION
  r1    sin    34    1.0    3.475
  r2    sin    34    1.0    3.475
  theta Leg/R  50    0    all 0.5 3.2
end-primitive-basis-section
  
```

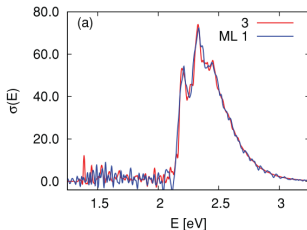
# System declaration  
 # The file extension only  
 # suggests a POTFIT calculation

# Define Hamiltonian

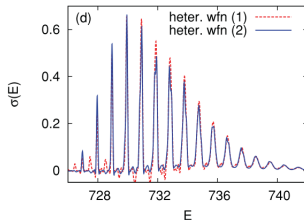
# Define coordinates  
 # and basis functions

# Applications

Some interesting applications that showcase the power of MCTDH are <sup>2</sup>:



(a) 24D



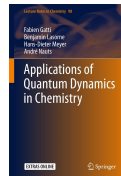
(b) 1458D

**Figure:** Power spectrum obtained with ML-MCTDH for (a) pyrazine (b) the Henon-Heiles Hamiltonian

<sup>2</sup>Vendrell, O., and Meyer, H.D., JCP 134.4 (2011): 044135.

# Bibliography

Gatti, F., *et al.* Applications of quantum dynamics in chemistry. Vol. 98. Springer, 2017.



Beck, M.H., *et al.* The multiconfiguration time-dependent Hartree (MCTDH) method: a highly efficient algorithm for propagating wavepackets. Physics reports 324.1 (2000): 1–105.





Thanks for your attention!

Questions?