$I^n$: <u>Min / Max</u> $F(x_1, x_2)$ pa $x_1, x_2 \in [a_i, b_i]$

n.x max $F(x) = x_1^2 + x_2^2 - x_1 - 2x_2$, $x_1, x_2 \in [-1, 1]$

$1^0$: $\frac{\partial F}{\partial x_1} = 2x_1 - 1 = 0 \Rightarrow \boxed{x_1 = \frac{1}{2}}$

$\frac{\partial F}{\partial x_2} = 2x_2 - 2 = 0 \Rightarrow \boxed{x_2 = 1}$

- $F(1/2, 1) = 1/4 + 1 - \frac{1}{2} - 2 = \frac{1}{4} - \frac{1}{2} - 1 = \frac{1}{4} - \frac{3}{2} = \frac{1}{4} - \frac{6}{4} = -\frac{5}{4}$
- $F(-1, -1) = 1 + 1 + 1 + 2 = 5$
- $F(1, -1) = 1 + \cancel{1} - \cancel{1} + 2 = 3$
- $F(1, 1) = 1 + \cancel{1} - \cancel{1} - 2 = -1$
- $F(-1, 1) = 1 + 1 + 1 - 2 = 1$

$x^* = x_1, x_2 = (-1, -1)$:

Για $x_1$: $\frac{\partial F}{\partial x_1} = 2x_1 - 1 = -3 \leq 0$ και $(\overbrace{-1 \cancel{+1}}^{0}) \cdot (-3) = 0$

Για $x_2$: $\frac{\partial F}{\partial x_2} = 2x_2 - 2 = -4 \leq 0$ και $(\overbrace{\cancel{-1} + 1}^{0})(-4) = 0$

Εφόσον το σημείο $x^* = (x_1, x_2) = (-1, -1)$

Πληροί τις αναγκαίες προϋποθέσεις

για max $\Rightarrow$ ότι στο $(-1, -1)$ τ.μ

2] $\underline{\max}$ $F(x) = 5x_1 - 10x_2$, $x_1, x_2 \in [0, 20]$

1°: $\frac{\partial F}{\partial x_1} = 5 \geq 0$ ↑ αυξανουσα ως προς x1

$\Rightarrow x_1 = 20$

$\frac{\partial F}{\partial x_2} = -10 \leq 0$ ↓ φθίνουσα ως προς x2

$\Rightarrow x_2 = 0$

$x^* = (x_1, x_2) = (20, 0)$: Για x1: $\frac{\partial F}{\partial x_1} = 5 \geq 0$ και $(20 - 20) \cdot 5 = 0$

Για x2: $\frac{\partial F}{\partial x_2} = -10 \leq 0$ και $(0 - 0)(-10) = 0$

Εφόσον το σημείο τηρεί τις αναγκαίες προϋποθέσεις για max $\Rightarrow$ το max βρίσκεται στο $(20,0)$

3) $\underline{\min}\ F(x) = x_1 - 5x_2,\ x_{1,2} \in [0,20]$

1ο: $\frac{\partial F}{\partial x_1} = 1 \geq 0$ $\downarrow$ ως προς $x_1 = 0$

$\frac{\partial F}{\partial x_2} = -5 \leq 0$ $\uparrow$ ως προς $x_2 = 20$

$x^* = x_1, x_2 = (0,20)$: Για $x_1$: $\frac{\partial F}{\partial x_1} = 1 \geq 0$ και $(0-0)\cdot 1 = 0$

Για $x_2$: $\frac{\partial F}{\partial x_2} = -5 \leq 0$ και $(20-20)(-5) = 0$

Συνεπώς, το σημείο (0, 20) πληροί τις αναγκαίες προϋποθέσεις για min

$\Rightarrow$ Ελάχιστο στο (0, 20)

$$L = F(x_1, x_2) + \lambda \cdot g(x_1, x_2)$$

$F_1 = \dfrac{\partial F}{\partial x_1}$, $F_{11} = \dfrac{\partial F_1}{\partial x_1}$

$F_{12} = \dfrac{\partial F_1}{\partial x_2}$

Brifa 1°: $\dfrac{\partial L}{\partial x_1} = 0$, $\dfrac{\partial L}{\partial x_2} = 0$, $\dfrac{\partial L}{\partial \lambda} = 0$

$H_1$ $H_2$

Brifa 2°: $$H^* = \begin{bmatrix} F_{11} + \lambda \cdot g_{11} & F_{12} + \lambda g_{12} & g_1 \\ F_{21} + \lambda g_{21} & F_{22} + \lambda g_{22} & g_2 \\ g_1 & g_2 & 0 \end{bmatrix}$$

1) $F(x) = x_1^2 + 2x_2^2$ υ.π $1 - x_1 - x_2 = 0$

Βήμα 1°: $L = x_1^2 + 2x_2^2 + \lambda \cdot (1 - x_1 - x_2)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 - \lambda = 0 \Rightarrow 2x_1 = \lambda \Rightarrow \boxed{\lambda = \frac{4}{3}}$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4x_2 - \lambda = 0 \Rightarrow 4x_2 - 2x_1 = 0 \Rightarrow 4x_2 - 2 + 2x_2 = 0$

$\Rightarrow 6x_2 = 2 \Rightarrow \boxed{x_2 = 1/3}$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow 1 - x_1 - x_2 = 0 \Rightarrow 1 - x_2 = x_1 \Rightarrow \boxed{x_1 = 1 - \frac{1}{3} = 2/3}$

Βήμα 2°: $g_1 = \frac{\partial g}{\partial x_1} = -1$ $\quad g_{11} = \frac{\partial g_1}{\partial x_1} = 0$, $g_{12} = \frac{\partial g_1}{\partial x_2} = 0$

$g_2 = \frac{\partial g}{\partial x_2} = -1$ $\quad g_{21} = \frac{\partial g_2}{\partial x_1} = 0$, $g_{22} = \frac{\partial g_2}{\partial x_2} = 0$

$F_1 = \frac{\partial F}{\partial x_1} = 2x_1 \qquad F_{11} = \frac{\partial F_1}{\partial x_1} = 2 \quad , \quad F_{12} = \frac{\partial F_1}{\partial x_1} = 0$

$F_2 = \frac{\partial F}{\partial x_2} = 4x_2 \qquad F_{21} = \frac{\partial F_2}{\partial x_1} = 0 \quad , \quad F_{22} = \frac{\partial F_2}{\partial x_2} = 4$

$$H^* = \begin{array}{c} \begin{matrix} a & b & \lambda \end{matrix} \\ \begin{bmatrix} 2 & 0 & -1 \\ 0 & 4 & -1 \\ -1 & -1 & 0 \end{bmatrix} \end{array}$$

$$|H| = 2 \begin{vmatrix} 4 & -1 \\ -1 & 0 \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & -1 \\ -1 & 0 \end{vmatrix} + (-1) \begin{vmatrix} 0 & 4 \\ -1 & -1 \end{vmatrix}$$

$$= 2\left(4 \cdot 0 - (-1)\cdot(-1)\right) - 0 - 1 \cdot \left(0 \cdot (-1) - (-1) \cdot 4\right)$$

$$= -2 - 4 = -6 < 0$$ άρα τ.ε στο $\left(2/3, \frac{1}{2}\right)$

2] $F(x_1, x_2) = x_1^2 + \frac{1}{10} x_2^2 + x_2$ υ.π $\overbrace{x_1 - x_2 = 5 \Rightarrow x_1 - x_2 - 5 = 0}^{g(x)}$

Βήμα 1°: $L = x_1^2 + \frac{1}{10} x_2^2 + x_2 + \lambda(x_1 - x_2 - 5)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 + \lambda = 0 \Rightarrow -2x_1 = \lambda \Rightarrow \lambda = 0$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow \frac{1}{5} x_2 + 1 - \lambda = 0 \Rightarrow \frac{1}{5} x_2 + 1 + 2(x_2 + 5) = 0 \Rightarrow \frac{1}{5} x_2 + 1 + 2x_2 + 10 = 0$

$\Rightarrow x_2 + 5 + 10x_2 + 50 = 0 \Rightarrow \boxed{x_2 = -\frac{55}{11} = -5}$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1 - x_2 - 5 = 0 \Rightarrow x_1 = x_2 + 5 \Rightarrow \boxed{x_1 = 0}$

$$H^* = \begin{bmatrix} F_{11} + \lambda g_{11} & F_{12} + \lambda g_{12} & g_1 \\ F_{21} + \lambda g_{21} & F_{22} + \lambda g_{22} & g_2 \\ g_1 & g_2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1/5 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$|H| = 2\begin{vmatrix} 1/5 & 1 \\ 1 & 0 \end{vmatrix} - 0 + 1\begin{vmatrix} 0 & 1/5 \\ 1 & 1 \end{vmatrix}$$

$$= 2\left(1/5 \cdot 0 - 1 \cdot 1\right) + 1\left(0 \cdot 1 - 1 \cdot 1/5\right)$$

$$= -2 - \frac{1}{5} = -\frac{10}{5} - \frac{1}{5} = -11/5$$

Άρα επειδή $-11/5 < 0$ η συνάρτηση ελαχιστοποιείται στο $(0, -5)$

3] $F(x) = x_1^2 + 2(x_2-3)^2$ υ.π. $g(x) = x_1^2 + x_2^2 - 1$

Βήμα 1°: $L = x_1^2 + 2(x_2-3)^2 + \lambda(x_1^2 + x_2^2 - 1)$

$\frac{\partial L}{\partial x_1} = 0 \Rightarrow 2x_1 + 2\lambda x_1 = 0 \Rightarrow 2x_1(1+\lambda) = 0 <$ $x_1 = 0$ ή $\lambda = -1$

$\frac{\partial L}{\partial x_2} = 0 \Rightarrow 4(x_2 - 3) + 2\lambda x_2 = 0$

$\frac{\partial L}{\partial \lambda} = 0 \Rightarrow x_1^2 + x_2^2 - 1 = 0$

Για $x_1 = 0$:

$x_2 = \pm 1$

$x_2 = 1: \lambda = 4$

$x_2 = -1: \lambda = -8$

$\lambda = -1$:

$4x_2 - 12 - 2x_2 = 0$

$2x_2 = 12$

$x_2 = 6$

$x_1^2 = -35$ αδύνατο

Βήμα 2°: $g_1 = 2x_1$, $g_{11} = 2$, $g_{12} = 0$   $F_1 = 2x_1$, $F_{11} = 2$, $F_{12} = 0$

$g_2 = 2x_2$, $g_{21} = 0$, $g_{22} = 2$   $F_2 = 4(x_2 - 3)$, $F_{21} = 0$, $F_{22} = 4$

$$H^* = \begin{bmatrix} 2+2\lambda & 0 & 2x_1 \\ 0 & 4+2\lambda & 2x_2 \\ 2x_1 & 2x_2 & 0 \end{bmatrix}$$

Για $x_1=0, x_2=1$, $\lambda=4$: $H = \begin{bmatrix} 10 & 0 & 0 \\ 0 & 12 & 2 \\ 0 & 2 & 0 \end{bmatrix}$, $|H| = 10 \begin{vmatrix} 12 & 2 \\ 2 & 0 \end{vmatrix} = \underline{-40<0}$

Για $x_1=0$, $x_2=-1$, $\lambda=-8$: $H = \begin{bmatrix} -14 & 0 & 0 \\ 0 & -12 & -2 \\ 0 & -2 & 0 \end{bmatrix}$, $|H| = -14 \begin{vmatrix} -12 & -2 \\ -2 & 0 \end{vmatrix} = 56>0$

Η συνάρτηση ελαχιστοποιείται στο $(0,1)$

και μεγιστοποιείται στο $(0,-1)$.

Αν ζητάει βελτιστοποίηση στην $F(x_n)$ αλλά χωρίς περιορισμούς έχω την εξής μεθοδολογία.

1°: $\frac{\partial F}{\partial x_i} = 0, \frac{\partial F}{\partial x_{i+1}} = 0, \dots \frac{\partial F}{\partial x_n} = 0$ Εύρεση στάσιμων σημείων

2°: $H = \begin{bmatrix} F_{11} & F_{12} & F_{1n} \\ F_{21} & F_{22} & F_{2n} \\ F_{n1} & F_{n2} & F_{nn} \end{bmatrix}$

$|H_1| = F_{11}$

$|H_2| = \begin{vmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{vmatrix}$

$|H_3| = |H|$

$(+ + +) \rightarrow$ Τ.ε

$(- + -) \rightarrow$ Τ.μ

- Να βρεθούν τα τοα της

$$F(x) = 2x_1^2 + 4x_2^2 + x_3^2 - 2x_1 - x_3$$