矩阵分析与应用作业 6

潘澳旋

Exercise3.Evaluate the Frobenius matrix norm, 1-norm, 2-norm and ∞-norm for each matrix below.

$$A=\begin{pmatrix} 1 & -2 \\ -1 & 2 \end{pmatrix}, B=\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, C=\begin{pmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{pmatrix} \tag{1}$$

解　对于矩阵 A，我们首先计算 A^*A 的特征值，

$$A^*A-\lambda I=\begin{pmatrix} 2-\lambda & -4 \\ -4 & 8-\lambda \end{pmatrix} \tag{2}$$

解得 $\lambda_1=10, \lambda_2=0$
因此，

$$\begin{aligned} ||A||_F &= (\sum_{i,j}|a_{i,j}^2|)^{\frac{1}{2}}=\sqrt{10} \\ ||A||_2 &= \sqrt{\lambda_{max}}=\sqrt{10} \\ ||A||_1 &= \max_j \sum_i |a_{i,j}|=4 \\ ||A||_\infty &= \max_i \sum_j |a_{i,j}|=3 \end{aligned} \tag{3}$$

对于矩阵 B，我们首先计算 B^*B 的特征值，

$$B^*B-\lambda I=\begin{pmatrix} 1-\lambda & 0 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{pmatrix} \tag{4}$$

解得 $\lambda=1$
因此，

$$\begin{aligned} ||B||_F &= (\sum_{i,j}|b_{i,j}^2|)^{\frac{1}{2}}=\sqrt{3} \\ ||B||_2 &= \sqrt{\lambda_{max}}=1 \\ ||B||_1 &= \max_j \sum_i |b_{i,j}|=1 \\ ||B||_\infty &= \max_i \sum_j |b_{i,j}|=1 \end{aligned} \tag{5}$$

对于矩阵 C，我们首先计算 C^*C 的特征值，

$$C^*C-\lambda I=\begin{pmatrix} 36-\lambda & -18 & 36 \\ -18 & 9-\lambda & -18 \\ 36 & -18 & 36-\lambda \end{pmatrix} \tag{6}$$

解得 $\lambda_1=81, \lambda_2=\lambda_3=0$

因此，

$$
\begin{aligned}
&||C||_F = (\sum_{i,j} |c_{i,j}^2|)^{\frac{1}{2}} = 9 \\
&||C||_2 = \sqrt{\lambda_{max}} = 9 \\
&||C||_1 = \max_j \sum_i |c_{i,j}| = 10 \\
&||C||_\infty = \max_i \sum_j |c_{i,j}| = 10
\end{aligned} \tag{7}
$$

Exercise12.

$$
A = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 2 & 1 \\ 1 & 1 & -3 \\ 0 & 1 & 1 \end{pmatrix} \; and \quad b = (1 \quad 1 \quad 1 \quad 1) \tag{8}
$$

(a)Determine the rectangular QR factorization of A.

(b)Use the QR factor from part(a) to determine the least squares solution to Ax=b.

解　(a)QR 分解过程：

$$
\begin{aligned}
k = 1 : r_{11} \leftarrow ||\mathbf{a}_1|| = \sqrt{3}, \mathbf{q}_1 \leftarrow \frac{\mathbf{a}_1}{r_{11}} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \end{pmatrix} \\
k = 2 : r_{12} \leftarrow \mathbf{q}_1^T \mathbf{a}_2 = \sqrt{3}, \mathbf{q}_2 \leftarrow \mathbf{a}_2 - r_{12}\mathbf{q}_1 = \mathbf{a}_2 - \sqrt{3}\mathbf{q}_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\
r_{22} = ||\mathbf{q}_2|| = \sqrt{3}, \mathbf{q}_2 \leftarrow \frac{\mathbf{q}_2}{r_{22}} = \frac{1}{\sqrt{3}} \begin{pmatrix} -1 \\ 1 \\ 0 \\ 1 \end{pmatrix} \\
k = 3 : r_{13} \leftarrow \mathbf{q}_1^T \mathbf{a}_3 = -\sqrt{3}, r_{23} \leftarrow \mathbf{q}_2^T \mathbf{a}_3 = \sqrt{3} \\
\mathbf{q}_3 \leftarrow \mathbf{a}_3 - r_{13}\mathbf{q}_1 - r_{23}\mathbf{q}_2 = \mathbf{a}_3 + \sqrt{3}\mathbf{q}_1 - \sqrt{3}\mathbf{q}_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix} \\
r_{33} = ||\mathbf{q}_3|| = \sqrt{6}, \mathbf{q}_3 \leftarrow \frac{\mathbf{q}_3}{r_{33}} = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \\ 0 \end{pmatrix}
\end{aligned} \tag{9}
$$

所以，有：

$$
Q = \begin{pmatrix} \frac{\sqrt{3}}{3} & -\frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{6}}{6} \\ \frac{\sqrt{3}}{3} & 0 & -\frac{\sqrt{6}}{3} \\ 0 & \frac{\sqrt{3}}{3} & 0 \end{pmatrix}, R = \begin{pmatrix} \sqrt{3} & \sqrt{3} & -\sqrt{3} \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{6} \end{pmatrix} \tag{10}
$$

(b) 由 (a) 知，A=QR，则有：$Ax = b \Leftrightarrow QRx = b \Leftrightarrow Q^TQRX = Q^Tb \Leftrightarrow Rx = Q^Tb$
即：

$$\begin{pmatrix} \sqrt{3} & \sqrt{3} & -\sqrt{3} \\ 0 & \sqrt{3} & \sqrt{3} \\ 0 & 0 & \sqrt{6} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 \\ -\frac{\sqrt{3}}{3} & \frac{\sqrt{3}}{3} & 0 & \frac{\sqrt{3}}{3} \\ \frac{\sqrt{6}}{6} & \frac{\sqrt{6}}{6} & -\frac{\sqrt{6}}{3} & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3} \\ \frac{\sqrt{3}}{3} \\ 0 \end{pmatrix} \tag{11}$$

解得

$$x = \begin{pmatrix} \frac{2}{3} \\ \frac{1}{3} \\ 0 \end{pmatrix} \tag{12}$$

Exercise16.

$$u = \begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix}, v = \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} \tag{13}$$

(a)Determine the orthogonal projection of **u** onto spanv.
(b)Determine the orthogonal projection of **v** onto spanu.
(a)Determine the orthogonal projection of **u** onto $v^\perp$.
(a)Determine the orthogonal projection of **v** onto $u^\perp$.

解 (a). 由

$$v = \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} \text{知：} P_v = \frac{vv^*}{v^*v} = \frac{1}{18} \begin{pmatrix} 1 & 4 & 0 & -1 \\ 4 & 16 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & 0 & 1 \end{pmatrix} \tag{14}$$

$$\text{因此，u 在 v 张成的空间上的正交投影} = P_v u = \frac{1}{18} \begin{pmatrix} 1 & 4 & 0 & -1 \\ 4 & 16 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & 0 & 1 \end{pmatrix} \begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix} = \frac{1}{6} \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} \tag{15}$$

(b). 由

$$u = \begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \text{知：} P_u = \frac{uu^*}{u^*u} = \frac{1}{15} \begin{pmatrix} 4 & -2 & -6 & 2 \\ -2 & 1 & 3 & -1 \\ -6 & 3 & 9 & -3 \\ 2 & -1 & -3 & 1 \end{pmatrix} \tag{16}$$

$$\text{因此，v 在 u 张成的空间上的正交投影} = P_u v = \frac{1}{15} \begin{pmatrix} 4 & -2 & -6 & 2 \\ -2 & 1 & 3 & -1 \\ -6 & 3 & 9 & -3 \\ 2 & -1 & -3 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{5} \begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix} \tag{17}$$

(c). 由

$$v = \begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} \text{知：} P_{v^\perp} = I - \frac{vv^*}{v^*v} = I - \frac{1}{18} \begin{pmatrix} 1 & 4 & 0 & -1 \\ 4 & 16 & 0 & -4 \\ 0 & 0 & 0 & 0 \\ -1 & -4 & 0 & 1 \end{pmatrix} = \frac{1}{18} \begin{pmatrix} 17 & -4 & 0 & 1 \\ -4 & 2 & 0 & 4 \\ 0 & 0 & 18 & 0 \\ 1 & 4 & 0 & 17 \end{pmatrix} \tag{18}$$

$$\text{因此，u 在}v^{\perp}\text{张成的空间上的正交投影} = P_{v^{\perp}}u = \frac{1}{18}\begin{pmatrix} 17 & -4 & 0 & 1 \\ -4 & 2 & 0 & 4 \\ 0 & 0 & 18 & 0 \\ 1 & 4 & 0 & 17 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{13}{6} \\ \frac{1}{3} \\ 3 \\ -\frac{5}{6} \end{pmatrix} \quad (19)$$

(d). 由

$$u = \begin{pmatrix} -2 \\ 1 \\ 3 \\ -1 \end{pmatrix}, \text{知：} P_{u^{\perp}} = I - \frac{uu^*}{u^*u} = I - \frac{1}{15}\begin{pmatrix} 4 & -2 & -6 & 2 \\ -2 & 1 & 3 & -1 \\ -6 & 3 & 9 & -3 \\ 2 & -1 & -3 & 1 \end{pmatrix} = \frac{1}{15}\begin{pmatrix} 11 & 2 & 6 & -2 \\ 2 & 14 & -3 & 1 \\ 6 & -3 & 6 & 3 \\ -2 & 1 & 3 & 14 \end{pmatrix} \quad (20)$$

$$\text{因此，v 在}v^{\perp}\text{张成的空间上的正交投影} = P_{u^{\perp}}v = \frac{1}{15}\begin{pmatrix} 11 & 2 & 6 & -2 \\ 2 & 14 & -3 & 1 \\ 6 & -3 & 6 & 3 \\ -2 & 1 & 3 & 14 \end{pmatrix}\begin{pmatrix} 1 \\ 4 \\ 0 \\ -1 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 7 \\ 19 \\ -3 \\ -4 \end{pmatrix} \quad (21)$$