矩阵分析与应用作业 7

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Exercise1.Using Housholder reduction and Givens reduction, compute the QR factors of

$$\begin{pmatrix} 1 & 19 & -34 \\ -2 & -5 & 20 \\ 2 & 8 & 37 \end{pmatrix} \tag{1}$$

解 Housholder reduction: 先消 A_{*1}：

$$u_1 = A_{*1} - ||A_{*1}||e_1 = A_{*1} - 3e_1 = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix} - \begin{pmatrix} 3 \\ 0 \\ 0 \end{pmatrix} = 2\begin{pmatrix} -1 \\ -1 \\ 1 \end{pmatrix} \tag{2}$$

$$R_1 = I - 2\frac{u_1u_1^T}{u_1^Tu_1} = I - \frac{2}{3}\begin{pmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 1 \end{pmatrix} = \frac{1}{3}\begin{pmatrix} 1 & -2 & 2 \\ -2 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix} \tag{3}$$

$$R_1A = \begin{pmatrix} 3 & 15 & 0 \\ 0 & -9 & 54 \\ 0 & 12 & 3 \end{pmatrix} \tag{4}$$

接着消 $(A_2)_{*1}$：

$$A_2 = \begin{pmatrix} -9 & 54 \\ 12 & 3 \end{pmatrix} \tag{5}$$

$$u_2 = (A_2)_{*1} - ||(A_2)_{*1}||e_1 = \begin{pmatrix} -9 \\ 12 \end{pmatrix} - 15\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 12\begin{pmatrix} -2 \\ 1 \end{pmatrix} \tag{6}$$

$$\hat{R}_2 = I - 2\frac{u_2u_2^T}{u_2^Tu_2} = \frac{1}{5}\begin{pmatrix} -3 & 4 \\ 4 & 3 \end{pmatrix}, \hat{R}_2A_2 = \begin{pmatrix} 15 & -30 \\ 0 & 45 \end{pmatrix} \tag{7}$$

则

$$R_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -\frac{3}{5} & \frac{4}{5} \\ 0 & \frac{4}{5} & \frac{3}{5} \end{pmatrix}, P = R_2R_1 = \begin{pmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{14}{15} & \frac{1}{3} & -\frac{2}{15} \\ -\frac{2}{15} & \frac{2}{3} & \frac{11}{15} \end{pmatrix} \tag{8}$$

所以

$$PA = R_2R_1A = \begin{pmatrix} 3 & 15 & 0 \\ 0 & 15 & -30 \\ 0 & 0 & 45 \end{pmatrix} = T \tag{9}$$

因此得到了 $A=P^TT=QR$，其中 $Q=P^T$,R=T 为上三角阵。
Givens reduction: 先消去 A_{21},

$$P_{12}=\frac{1}{\sqrt{5}}\begin{pmatrix}1&-2&0\\2&1&0\\0&0&\sqrt{5}\end{pmatrix}, P_{12}A=\frac{1}{\sqrt{5}}\begin{pmatrix}5&29&-74\\0&33&-48\\2\sqrt{5}&8\sqrt{5}&37\sqrt{5}\end{pmatrix} \tag{10}$$

再消 (3,1)，

$$P_{13}=\frac{1}{3}\begin{pmatrix}\sqrt{5}&0&2\\0&3&0\\-2&0&\sqrt{5}\end{pmatrix}, P_{13}P_{12}A=\frac{1}{\sqrt{5}}\begin{pmatrix}3\sqrt{5}&15\sqrt{5}&0\\0&33&-48\\0&-6&111\end{pmatrix} \tag{11}$$

再消 (3,2),

$$P_{23}=\frac{1}{5\sqrt{5}}\begin{pmatrix}5\sqrt{5}&0&0\\0&11&-2\\0&2&11\end{pmatrix}, P_{23}P_{13}P_{12}A=\begin{pmatrix}3&15&0\\0&15&-30\\0&0&45\end{pmatrix} \tag{12}$$

所以

$$P=P_{23}P_{13}P_{12}=\begin{pmatrix}\frac{1}{3}&-\frac{2}{3}&\frac{2}{3}\\\frac{14}{15}&\frac{1}{3}&-\frac{2}{15}\\-\frac{2}{15}&\frac{2}{3}&\frac{11}{15}\end{pmatrix}, T=P_{23}P_{13}P_{12}A=\begin{pmatrix}3&15&0\\0&15&-30\\0&0&45\end{pmatrix} \tag{13}$$

即 PA=T，所以有 $A=P^TT=QR$ 其中 $Q=P^T$，R=T 为上三角阵。

Exercise2.By using Housholder reduction, find an orthonormal basis for R(A),where

$$A=\begin{pmatrix}4&-3&4\\2&-14&-3\\-2&14&0\\1&-7&15\end{pmatrix} \tag{14}$$

解 Housholder: 先消 A_{*1} :

$$u_1=A_{*1}-||A_{*1}||e_1=A_{*1}-5e_1=\begin{pmatrix}4\\2\\-2\\1\end{pmatrix}-\begin{pmatrix}5\\0\\0\\0\end{pmatrix}=\begin{pmatrix}-1\\2\\-2\\1\end{pmatrix} \tag{15}$$

$$R_1=I-2\frac{u_1u_1^T}{u_1^Tu_1}=I-\frac{1}{5}\begin{pmatrix}1&-2&2&-1\\-2&4&-4&2\\2&-4&4&-2\\-1&2&-2&1\end{pmatrix}=\frac{1}{5}\begin{pmatrix}4&2&-2&1\\2&1&4&-2\\-2&4&1&2\\1&-2&2&4\end{pmatrix} \tag{16}$$

$$R_1A=\begin{pmatrix}5&-15&5\\0&10&-5\\0&-10&2\\0&5&14\end{pmatrix} \tag{17}$$

接着消 $(A_2)_{*1}$：

$$A_2 = \begin{pmatrix} -10 & -5 \\ -10 & 2 \\ 5 & 14 \end{pmatrix} \tag{18}$$

$$u_2 = (A_2)_{*1} - ||(A_2)_{*1}||e_1 = \begin{pmatrix} 10 \\ -10 \\ 5 \end{pmatrix} - 15 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = -5 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} \tag{19}$$

$$\hat{R_2} = I - 2\frac{u_2 u_2^T}{u_2^T u_2} = \frac{1}{3} \begin{pmatrix} 2 & -2 & 1 \\ -2 & -1 & 2 \\ 1 & 2 & 2 \end{pmatrix}, \hat{R}_2 A_2 = \begin{pmatrix} 15 & 0 \\ 0 & 12 \\ 0 & 9 \end{pmatrix} \tag{20}$$

则

$$R_2 = \frac{1}{3} \begin{pmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & -2 & 1 \\ 0 & -2 & -1 & 2 \\ 0 & 1 & 2 & 2 \end{pmatrix}, P = R_2 R_1 = \frac{1}{15} \begin{pmatrix} 12 & 6 & -6 & 3 \\ 9 & -8 & 8 & -4 \\ 0 & -10 & -5 & 10 \\ 0 & 5 & 10 & 10 \end{pmatrix} \tag{21}$$

所以

$$PA = R_2 R_1 A = \begin{pmatrix} 5 & -15 & 5 \\ 0 & 15 & 0 \\ 0 & 0 & 12 \\ 0 & 0 & 9 \end{pmatrix} = T \tag{22}$$

因此得到了 $A = P^T T = QR$，其中

$$Q = P^T = \begin{pmatrix} 12 & 9 & 0 & 0 \\ 6 & -8 & -10 & 5 \\ -6 & 8 & -5 & 10 \\ 3 & 4 & 10 & 10 \end{pmatrix} \tag{23}$$

R=T 为上三角阵。则取 Q 的前三列则为满足题意的一组正交基：

$$\begin{pmatrix} 12 \\ 6 \\ -6 \\ 3 \end{pmatrix}, \begin{pmatrix} 9 \\ -8 \\ 8 \\ 4 \end{pmatrix}, \begin{pmatrix} 0 \\ -10 \\ -5 \\ 10 \end{pmatrix} \tag{24}$$

Exercise7.Let $\mathcal{X}$ and $\mathcal{Y}$ be subspaces of $\mathcal{R}^3$ whose respective bases are

$$\mathcal{B}_{\mathcal{X}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \right\} \ and \quad \mathcal{B}_{\mathcal{Y}} = \left\{ \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \tag{25}$$

(a)Explain why $\mathcal{X}$ and $\mathcal{Y}$ are complementary subspaces of $\mathcal{R}^3$.

(b)Determine the projector P onto $\mathcal{X}$ along $\mathcal{Y}$ as well as the comeplementary projector Q onto $\mathcal{Y}$ along $\mathcal{X}$.
(c)Determine the projection of

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}$$

onto $\mathcal{Y}$ along $\mathcal{X}$.

解 (a) 由题知：$\mathcal{B}_{\mathcal{X}} \cap \mathcal{B}_{\mathcal{Y}} = \varnothing$

$$\mathcal{B} = \mathcal{B}_{\mathcal{X}} \cup \mathcal{B}_{\mathcal{Y}} = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \right\} \tag{26}$$

令 A=

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

, 而 rank(A)=3，所以 $\mathcal{X}$ 和 $\mathcal{Y}$ 构成了 $\mathcal{R}^3$ 空间的互补子空间。
(b) 沿着 $\mathcal{Y}$ 方向到 $\mathcal{X}$ 的投影算子 P 为：

$$P = [X|0][X|Y]^{-1} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 1 & 2 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & -1 \\ 0 & 3 & -2 \\ 0 & 3 & -2 \end{pmatrix} \tag{27}$$

沿着 $\mathcal{X}$ 方向到 $\mathcal{Y}$ 的投影算子 Q 为：

$$Q = I - P = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} \tag{28}$$

(c) 由 (b) 知，v 沿着 $\mathcal{X}$ 方向到 $\mathcal{Y}$ 的投影为：

$$Qv = \begin{pmatrix} 0 & -1 & 1 \\ 0 & -2 & 2 \\ 0 & -3 & 3 \end{pmatrix} \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \tag{29}$$