The Backward Process of 3D Gaussian Splatting

This document describes the **backward process** of 3D Gaussian Splatting, which involves training 3D Gaussians using a set of given 2D images. The training process can generally be treated as an optimization problem, aiming to find a set of parameters that minimize an overall loss function $\mathcal{L}$ (or objective function).

$$\underset{x}{\operatorname{argmin}} \quad \mathcal{L} = \mathcal{L}(\gamma, \gamma_{gt}) \tag{1}$$

where γ is the output image from the forward process, and γ_{gt} is the given ground truth image.

The loss function $\mathcal{L}$ for 3D Gaussian Splatting is defined as a combination of L1 loss ($\mathcal{L}1$) and D-SSIM loss ($\mathcal{L}_{D-SSIM}$).

$$\mathcal{L} = (1-\lambda)\mathcal{L}_1 + \lambda\mathcal{L}_{D-SSIM} \tag{2}$$

The value of λ ranges between 0 and 1. When λ is close to 0, the loss function $\mathcal{L}$ is more similar to L1 loss, whereas when λ is close to 1, $\mathcal{L}$ is more similar to D-SSIM loss.

To solve this optimization problem, it is necessary to find the Jacobians of the loss function with respect to each input parameter. These Jacobians provide information about how the loss changes as each input parameter is varied.

In the following sections, we will describe how to calculate these Jacobians.

Jacobians

The computation of γ in (2) has already been described in `forward.md` , so the Jacobian for each parameter can be computed using the chain rule.

The Jacobian of Rotation

$$\frac{\partial \mathcal{L}}{\partial q_i} = \sum_j \frac{\partial \mathcal{L}}{\partial \gamma_j} \frac{\partial \gamma_j}{\partial \sigma'_i} \frac{\partial \sigma'_i}{\partial \sigma_i} \frac{\partial \sigma_i}{\partial q_i} \tag{3}$$

The Jacobian of Scale

$$\frac{\partial \mathcal{L}}{\partial s_i} = \sum_j \frac{\partial \mathcal{L}}{\partial \gamma_j} \frac{\partial \gamma_j}{\partial \sigma'_i} \frac{\partial \sigma'_i}{\partial \sigma_i} \frac{\partial \sigma_i}{\partial s_i} \tag{4}$$

The Jacobian of Spherical Harmonics Parameters

$$\frac{\partial \mathcal{L}}{\partial h_i} = \sum_j \frac{\partial \mathcal{L}}{\partial \gamma_j} \frac{\partial \gamma_j}{\partial c_i} \frac{\partial c_i}{\partial h_i} \tag{5}$$

The Jacobian of Alpha

$$\frac{\partial \mathcal{L}}{\partial \alpha_i} = \sum_j \frac{\partial \mathcal{L}}{\partial \gamma_j} \frac{\partial \gamma_j}{\partial \alpha'_{ij}} \frac{\partial \alpha'_{ij}}{\partial \alpha_i} \tag{6}$$

The Jacobian of Location

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial p_{w,i}} &= \sum_j \frac{\partial \mathcal{L}}{\partial \gamma_j} \frac{\partial \gamma_j}{\partial \alpha'_{ij}} \frac{\partial \alpha'_{ij}}{\partial u_i} \frac{\partial u_i}{\partial p_{c,i}} \frac{\partial p_{c,i}}{\partial p_{w,i}} \\ &+ \sum_j \frac{\partial \mathcal{L}}{\partial \gamma_j} \frac{\partial \gamma_j}{\partial c_i} \frac{\partial c_i}{\partial p_{w,i}} \\ &+ \sum_j \frac{\partial \mathcal{L}}{\partial \gamma_j} \frac{\partial \gamma_j}{\partial \alpha'_{ij}} \frac{\partial \alpha'_{ij}}{\partial \sigma'_i} \frac{\partial \sigma'_i}{\partial p_{c,i}} \frac{\partial p_{c,i}}{\partial p_{w,i}} \end{aligned} \tag{7}$$

Next, let's discuss all the partial derivatives mentioned above. The computation of $\frac{\partial \mathcal{L}}{\partial \gamma_j}$ can be performed using PyTorch's automatic differentiation, so we will not discuss it here.

1. Derivatives of Transform and Projection Function

The partial derivatives of transform function.

$$\frac{\partial p_{c,i}}{\partial p_{w,i}} = R_{cw} \tag{B.1.1}$$

The partial derivatives of projection function.

$$\frac{\partial u_i}{\partial p_{c,i}} = \begin{bmatrix} f_x/z & 0 & -f_x x/z^2 \\ 0 & f_y/z & -f_y y/z^2 \end{bmatrix} \tag{B.1.2}$$

x, y, z are the elements of p_c.

2. Derivatives of 3D Covariances

We can rewrite equation (F.2) as follows:

$$\sigma = \text{upper\_triangular}(MM^T)$$

where:

$$M = M(q, s) = R(q)S(s)$$

The partial derivatives of 3D Covariances with respect to q are given by:

$$\frac{\partial \sigma}{\partial q} = \frac{\partial \sigma}{\partial M} \frac{\partial M}{\partial q} \tag{B.2a}$$

The partial derivatives of 3D Covariances with respect to s are given by:

$$\frac{\partial \sigma}{\partial s} = \frac{\partial \sigma}{\partial M} \frac{\partial M}{\partial s} \tag{B.2b}$$

The matrices $\frac{\partial \sigma}{\partial M}$, $\frac{\partial M}{\partial q}$, and $\frac{\partial M}{\partial s}$ are as follows:

$$\frac{\partial \sigma}{\partial M} = \begin{bmatrix} 2M_{r0} & & \\ M_{r1} & M_{r0} & \\ M_{r2} & & M_{r0} \\ & 2M_{r1} & \\ & M_{r2} & M_{r1} \\ & & 2M_{r2} \end{bmatrix}$$

Here, M_{r0}, M_{r1}, and M_{r2} represent the first, second, and third rows of M.

$$\frac{\partial M}{\partial q} = \begin{bmatrix} 0 & 0 & -4s_0q_y & -4s_0q_z \\ -2s_1q_z & 2s_1q_y & 2s_1q_x & -2s_1q_w \\ 2s_2q_y & 2s_2q_z & 2s_2q_w & 2s_2q_x \\ 2s_0q_z & 2s_0q_y & 2s_0q_x & 2s_0q_w \\ 0 & -4s_1q_x & 0 & -4s_1q_z \\ -2s_2q_x & -2s_2w & 2s_2q_z & 2s_2q_y \\ -2s_0q_y & 2s_0z & -2s_0q_w & 2s_0q_x \\ 2s_1q_x & 2s_1w & 2s_1q_z & 2s_1q_y \\ 0 & -4s_2q_x & -4s_2q_y & 0 \end{bmatrix}$$

$$\frac{\partial M}{\partial s} = \begin{bmatrix} \mathrm{diag}(r_0) \\ \mathrm{diag}(r_1) \\ \mathrm{diag}(r_2) \end{bmatrix}$$

Here, r_0, r_1, and r_2 represent the first, second, and third rows of R, and $\mathrm{diag}(a)$ creates a diagonal matrix from vector a.

3. Derivatives of 2D Covariances

The equation (F.3) can be expressed as:

$$\sigma'(\sigma, p_c) = \mathrm{upper\_triangular}(M\Sigma(\sigma)M^T)$$

where:

$$M = M(p_c) = J(p_c)R_{cw}$$

The partial derivatives of 2D covariances with respect to 3D covariances are given by:

$$\frac{\partial \sigma'}{\partial \sigma} = \begin{bmatrix} m_{00}^2 & 2m_{00}m_{01} & 2m_{00}m_{02} & m_{01}^2 & 2m_{01}m_{02} & m_{02}^2 \\ m_{00}m_{10} & m_{00}m_{11} + m_{01}m_{10} & m_{00}m_{12} + m_{02}m_{10} & m_{01}m_{11} & m_{01}m_{12} + m_{02}m_{11} & m_{02}m_{12} \\ m_{10}^2 & 2m_{10}m_{11} & 2m_{10}m_{12} & m_{11}^2 & 2m_{11}m_{12} & m_{12}^2 \end{bmatrix} \quad \text{(B.3a)}$$

The partial derivatives of 2D covariances with respect to p_c are given by:

$$\frac{\partial \sigma'}{\partial p_c} = \frac{\partial \sigma'}{\partial M}\frac{\partial M}{\partial p_c} \quad \text{(B.3b)}$$

where:

$$\frac{\partial \sigma'}{\partial M} = \begin{bmatrix} 2\Sigma_{r0}M_{r0}^T & 2\Sigma_{r1}M_{r0}^T & 2\Sigma_{r2}M_{r0}^T & 0 & 0 & 0 \\ \Sigma_{r0}M_{r1}^T & \Sigma_{r1}M_{r1}^T & \Sigma_{r2}M_{r1}^T & \Sigma_{r0}M_{r0}^T & \Sigma_{r1}M_{r0}^T & \Sigma_{r2}M_{r0}^T \\ 0 & 0 & 0 & 2\Sigma_{r0}M_{r2}^T & 2\Sigma_{r1}M_{r2}^T & 2\Sigma_{r2}M_{r2}^T \end{bmatrix}$$

Here, M_{ri}, and Σ_{ri} represent the i-th row of M and Σ.

$$\frac{\partial M}{\partial p_c} = \begin{bmatrix} -f_x R_{c2}/z^2 & 0_{3\times 1} & -f_x R_{c0}/z^2 + 2f_x R_{c2}x/z^3 \\ 0_{3\times 1} & -f_y R_{c2}/z^2 & -f_y R_{c1}/z^2 + 2f_y R_{c2}y/z^3 \end{bmatrix}$$

Here, r_{c0}, r_{c1}, r_{c2} represent the first, second, and third columns of R_{cw}.

The elements a, b, c, d, e, f represent the elements of σ.

4. Derivatives of Spherical Harmonics

5. Derivatives of Final Colors

We compute γ_j using the following equation (Refer to F.5):

$$\gamma_j = \sum_{i \in N} \alpha'_{ij} c_i \tau_{ij}$$

$$\tau_{ij} = \prod_{k=1}^{i-1} (1 - \alpha'_{kj})$$

The above equation can be rewritten in the following iterative way.

Where N represents the number of 3D Gaussians and $\gamma_{i,j}$ represents the current color by considering 3D Gaussians from i to N (the farthest one).

$$\begin{aligned} &\gamma_{N+1,j} = 0 \\ &\gamma_{N,j} = \alpha'_{N,j} c_{N,j} + (1 - \alpha'_{N,j})\gamma_{N+1,j} \\ &\dots \\ &\gamma_{2,j} = \alpha'_{2,j} c_2 + (1 - \alpha'_{2,j})\gamma_{3,j} \\ &\gamma_{1,j} = \alpha'_{1,j} c_1 + (1 - \alpha'_{1,j})\gamma_{2,j} \\ &\gamma_j = \gamma_{1,j} \end{aligned}$$

Therefore, we can calculate the partial derivatives of γ_j with respect to each $\gamma_{i,j}$ iteratively.

$$\frac{\partial \gamma_j}{\partial \alpha'_{i,j}} = \tau_{i,j}(c_i - \gamma_{i+1,j}) \tag{B.5a}$$

Similarly, the partial derivatives of γ_j with respect to c_i can be calculated as follows.

$$\frac{\partial \gamma_j}{\partial c_i} = \tau_{i,j}\alpha'_{i,j} \tag{B.5b}$$

The α'_{ij} is calculated using the following equation (Refer to F.5.1):

$$\alpha'_{ij}(\alpha_i, \sigma'_i, u_i) = \exp\left(-0.5(u_i - x_j)\Sigma_i'^{-1}(u_i - x_j)^T\right)\alpha_i$$

We define $\exp(\dots)$ as g, and rewrite F.5.1 as follows.

$$\alpha'_{ij}(\alpha_i, \sigma'_i, u_i) = g\alpha_i$$

where:

$$g = g(u_i, \sigma'_i) \equiv \exp\left(-0.5(u_i - x_j)\Sigma_i'^{-1}(u_i - x_j)^T\right)$$

$$= \exp(-0.5\omega'_0 d_0^2 - 0.5\omega'_2 d_1^2 - \omega'_1 d_0 d_1)$$

ω'_0, ω'_1, ω'_2 are the upper triangular elements of the inverse of 2D covariance, and d_0, d_1 are the 2 elements of $u_i - x_j$.

Therefore, the partial derivatives of α'_{ij} with respect to α can be easily computed as follows.

$$\frac{\partial \alpha'_{ij}}{\partial \alpha_i} = g \tag{B.5.1a}$$

Since g is a function with respect to u_i and σ_i, the partial derivatives of α'_{ij} with respect to u_i and σ_i can be written in the following form.

$$\frac{\partial \alpha'_{ij}}{\partial u_i} = a\frac{\partial g}{\partial u_i} \tag{B.5.2b}$$

$$\frac{\partial \alpha'_{ij}}{\partial \sigma'_i} = a\frac{\partial g}{\partial \omega'_i}\frac{\partial \omega'_i}{\partial \sigma'_i} \tag{B.5.2c}$$

where:

$$\frac{\partial g}{\partial \omega'_i} = \begin{bmatrix} -0.5d_0^2 \\ -d_0 d_1 \\ -0.5d_1^2 \end{bmatrix} g$$

$$\frac{\partial g}{\partial u_i} = \begin{bmatrix} -\omega'_0 d_0 - \omega'_1 d_1 \\ -\omega'_2 d_1 - \omega'_1 d_0 \end{bmatrix} g$$

The partial derivatives of ω_i with respect to σ_i.

$$\frac{\partial \omega'_i}{\partial \sigma'_i} = \det(\sigma'_i) = \begin{bmatrix} -c^2\det^{-2} & 2bc\det^{-2} & -ac\det^{-2} + \det^{-1} \\ bc\det^{-2} & -2b^2\det^{-2} - \det^{-1} & ab\det^{-2} \\ -ac\det^{-2} + \det^{-1} & 2ab\det^{-2} & -a^2\det^{-2} \end{bmatrix} \tag{B.5.3}$$

where a, b, and c are the upper triangular elements of the 2D covariance, and $\det$ is the determinant of the 2D covariance.