Imu preintegration.

Predicting navigation state by IMU

Suppose we know the navigation state of the robot at time i ,as well as the IMU measurements from the i time to j time. We want predict the state of robot at time j.

$$s_j^* = \mathscr{R}(s_i, \mathscr{D}(\xi(\zeta, b))) \tag{1}$$

The navigation state combined by attitude $R(\theta)$, position p and velocity v.

$$\begin{aligned} s_i &= (R_{nb}, p_{nb}, v_{nb}) \\ s_j &= (R_{nc}, p_{nc}, v_{nc}) \end{aligned} \tag{2}$$

- n denotes navigation state frame.
- b denotes body frame in time i.
- c denotes current frame in time j.
- θ is the lie algebra of R.

The retract action $\mathscr{R}$ which defined on navigation state takes 2 parameters: s_i and $\mathscr{D}$ to predict s_j. The $\mathscr{D}$ represents the difference between s_i and s_j.

$$d(\xi, s_i) = (R_{nc}, p_{nc}, v_{nc}) \tag{3}$$

ξ represents bias corrected preintegration measurement (PIM), which take 2 parameters, the PIM ζ and IMU bias b.

The Jacobian of s_i

$$J_{s_i}^{s_j^*} = J_{s_i}^{\mathscr{R}} + J_{\mathscr{D}}^{\mathscr{R}} J_{s_i}^{\mathscr{D}} \tag{4}$$

The Jacobian of b

$$J_b^{s_j^*} = J_{\mathscr{D}}^{\mathscr{R}} J_{\xi}^{\mathscr{D}} J_b^{\xi} \tag{5}$$

Preintegration measurement (PIM)

The PIM $\zeta(R(\theta), p, v)$ integrates all the IMU measurements without considering the IMU bias and the gravity.
ω_k^b,a_k^b are the acceleration and angular velocity measured by IMU (accelerometer + gyroscope) respectively.

$$\begin{aligned} R_{k+1} &= R_k \exp(\omega_k^b \Delta t) \\ p_{k+1} &= p_k + v_k \Delta t + R_k a_k^b \frac{\Delta t^2}{2} \\ v_{k+1} &= v_k + R_k a_k^b \Delta t \end{aligned} \tag{7}$$

n: navigation frame, b: body frame.

A:Derivative of old ζ

$$\begin{aligned} A &= \frac{\partial \zeta_{k+1}}{\partial \zeta_k} \\ &= \begin{bmatrix} \frac{\partial R_{k+1}}{\partial R_k} & \frac{\partial R_{k+1}}{\partial p_k} & \frac{\partial R_{k+1}}{\partial v_k} \\ \frac{\partial p_{k+1}}{\partial R_k} & \frac{\partial p_{k+1}}{\partial p_k} & \frac{\partial p_{k+1}}{\partial v_k} \\ \frac{\partial v_{k+1}}{\partial R_k} & \frac{\partial v_{k+1}}{\partial p_k} & \frac{\partial v_{k+1}}{\partial v_k} \end{bmatrix} \\ &= \begin{bmatrix} I_{3\times3} - \Delta t \widehat{\omega_k^b} & 0_{3\times3} & 0_{3\times3} \\ -R_k \widehat{a_k^b} \frac{\Delta t}{2}^2 & I_{3\times3} & I_{3\times3} \Delta t \\ -R_k \widehat{a_k^b} \Delta t & 0_{3\times3} & I_{3\times3} \end{bmatrix} \end{aligned} \tag{8}$$

B:Derivative of input a

$$B = \frac{\partial \zeta_{k+1}}{\partial a_k^b} = \begin{bmatrix} \frac{\partial R_{k+1}}{\partial a_k^b} \\ \frac{\partial p_{k+1}}{\partial a_k^b} \\ \frac{\partial v_{k+1}}{\partial a_k^b} \end{bmatrix} = \begin{bmatrix} 0_{3\times3} \\ R_k \frac{\Delta t^2}{2} \\ R_k \Delta t \end{bmatrix} \tag{9}$$

C:Derivative of input ω

$$C = \frac{\partial \zeta_{k+1}}{\partial \omega_k^b} = \begin{bmatrix} \frac{\partial R_{k+1}}{\partial \omega_k^b} \\ \frac{\partial p_{k+1}}{\partial \omega_k^b} \\ \frac{\partial v_{k+1}}{\partial \omega_k^b} \end{bmatrix} = \begin{bmatrix} I_{3\times3} \Delta t \\ 0_{3\times3} \\ 0_{3\times3} \end{bmatrix} \tag{10}$$

Bias correct

We want correct ζ by a given accelerometer and gyroscope bias.

$$\xi(b+\Delta b)=\zeta \oplus (\Delta b_{acc}\frac{\partial \zeta}{\partial b_{acc}}+\Delta b_{\omega}\frac{\partial \zeta}{\partial b_{\omega}}) \tag{11}$$

- b_{acc} is bias for accelerometer.
- b_{ω} is bias for gyroscope.
- Because the parameter θ cannot be added directly, we define the combination of ζ with the symbol $\oplus$.

$$a \oplus b=[\log(\exp(\theta_a)\exp(\theta_b)), p_a+p_b, v_a+v_b] \tag{12}$$

The jocabian of bias for corrected PIM.

$$J_b^{\xi}=[\frac{\partial \zeta}{\partial b_{acc}},\frac{\partial \zeta}{\partial b_{\omega}}] \tag{13}$$

Find the partial derivatives of accelerometer's bias

The bias model for accelerometer.

$$\tilde{a_k^b}=a_k^b-b_{acc} \tag{14}$$

$$\begin{aligned}\frac{\partial \zeta_{k+1}}{\partial b_{acc}}&=\frac{\partial \zeta_{k+1}}{\partial \zeta_k}\frac{\partial \zeta_k}{\partial b_{acc}}+\frac{\partial \zeta_{k+1}}{\partial \tilde{a_k^b}}\frac{\partial \tilde{a_k^b}}{\partial b_{acc}}\\&=A\frac{\partial \zeta_k}{\partial b_{acc}}-B\end{aligned} \tag{15}$$

Find the partial derivatives of gyroscope's bias

$$\tilde{\omega_k^b}=\omega_k^b-b_{\omega} \tag{16}$$

$$\begin{aligned}\frac{\partial \zeta_{k+1}}{\partial b_{\omega}}&=\frac{\partial \zeta_{k+1}}{\partial \zeta_k}\frac{\partial \zeta_k}{\partial b_{\omega}}+\frac{\partial \zeta_{k+1}}{\partial \tilde{\omega_k^b}}\frac{\partial \tilde{\omega_k^b}}{\partial b_{\omega}}\\&=A\frac{\partial \zeta_k}{\partial b_{\omega}}-C\end{aligned} \tag{17}$$

- ~ denotes the corrected measurement.

Delta between two states

The $\mathscr{D}$ represents the difference between two s_i and s_j.

$$\mathscr{D}=(R_{bc}, p_{bc}, v_{bc}) \tag{18}$$

We can calculate $\mathscr{D}$ from corrected PIM $\xi(R^{\xi}_{bc}, p^{\xi}_{bc}, v^{\xi}_{bc})$ and velocity, which is included in s_i.

$$\mathscr{D}(\xi, s_i) = \begin{bmatrix} R^{\xi}_{bc} \\ p^{\xi}_{bc} + R^{-1}_{nb} v_{nb} \Delta t + R^{-1}_{nb} g \frac{\Delta t^2}{2} \\ v^{\xi}_{bc} + R^{-1}_{nb} g \Delta t \end{bmatrix} \tag{19}$$

- g is the gravity vector.
- $*$ denotes the predicted navigation state.

The jocabian matrix of navigation state

$$\begin{aligned} J^{\mathscr{D}}_{s_i} &= \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ \frac{\partial p_{bc}}{\partial R_{nb}} & 0_{3\times3} & \frac{\partial p_{bc}}{\partial v_{nb}} \\ \frac{\partial v_{bc}}{\partial R_{nb}} & 0_{3\times3} & 0_{3\times3} \end{bmatrix} \\ &= \begin{bmatrix} 0_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ \widehat{R^{-1}_{nb} v_{nb}} \Delta t + \widehat{R^{-1}_{nb} g} \frac{\Delta t^2}{2} & 0_{3\times3} & I_{3\times3} \Delta t \\ \widehat{R^{-1}_{nb} g} \Delta t & 0_{3\times3} & 0_{3\times3} \end{bmatrix} \end{aligned} \tag{20}$$

The jocabian matrix of ξ

$$J^{\mathscr{D}}_{\xi} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & I_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & I_{3\times3} \end{bmatrix} \tag{21}$$

Retraction $\mathscr{R}$

The retract action $\mathscr{R}$ which defined on navigation state takes 2 parameters: s_i and $\mathscr{D}$ to predict s_j.

- s^*_j is the predicted s_j.

$$\begin{aligned} R^*_{nc} &= R_{nb} R_{bc} \\ p^*_{nc} &= p_{nb} + R_{nb} p_{bc} \\ v^*_{nc} &= v_{nb} + R_{nb} v_{bc} \end{aligned} \tag{22}$$

Derivative of s_i

$$J^{\mathscr{R}}_{s_i} = \begin{bmatrix} R^{-1}_{bc} & 0_{3\times3} & 0_{3\times3} \\ -R^{-1}_{bc} \widehat{p_{bc}} & R^{-1}_{bc} & 0_{3\times3} \\ -R^{-1}_{bc} \widehat{v_{bc}} & 0_{3\times3} & R^{-1}_{bc} \end{bmatrix} \tag{23}$$

Derivative of d

$$J_d^{\mathscr{R}} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & R_{bc}^{-1} & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & R_{bc}^{-1} \end{bmatrix} \tag{24}$$

Navigation state prediction error (residual function)

If navigtion $state_j$ is measured by sensors, we can calculate the error between $state_j$ and $state_j^*$.

$$r_{jj^*} = \mathscr{L}(s_j, s_j^*) = \begin{bmatrix} \Delta R \\ \Delta p \\ \Delta v \end{bmatrix} = \begin{bmatrix} R_j^{-1} R_j^* \\ R_j^{-1}(p_j^* - p_j) \\ R_j^{-1}(v_j^* - v_j) \end{bmatrix} \tag{25}$$

Local $\mathscr{L}$ is the inverse function of $\mathscr{R}$, which takes two navigation states, and get the delta between the two states in tangent vector space

Derivative of an s_j

$$J_{s_j}^{\mathscr{L}} = \begin{bmatrix} -\Delta R^{-1} & 0_{3\times3} & 0_{3\times3} \\ \widehat{\Delta p} & -I_{3\times3} & 0_{3\times3} \\ \widehat{\Delta v} & 0_{3\times3} & -I_{3\times3} \end{bmatrix} \tag{26}$$

Derivative of an s_j^*

$$J_{s_j^*}^{\mathscr{L}} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} & 0_{3\times3} \\ 0_{3\times3} & \Delta R & 0_{3\times3} \\ 0_{3\times3} & 0_{3\times3} & \Delta R \end{bmatrix} \tag{27}$$

Overall Jaccobian for prediction error

To summarize, The prediction error r takes 3 parameters s_i, s_j and b. According to the chain rule, their Jaccobian can be written in the following form.

$$J_{s_j}^r = J_{s_j}^{\mathscr{L}} \tag{28}$$

$$J_{s_i}^r = J_{s_j^*}^{\mathscr{L}} J_{s_i}^{s_j^*} = J_{s_j^*}^{\mathscr{L}} (J_{s_i}^{\mathscr{R}} + J_{\mathscr{D}}^{\mathscr{R}} J_{s_i}^{\mathscr{D}}) \tag{29}$$

$$J_b^r = J_{s_j^*}^{\mathscr{L}} J_b^{s_j^*} = J_{s_j^*}^{\mathscr{L}} J_{\mathscr{D}}^{\mathscr{R}} J_{\xi}^{\mathscr{D}} J_b^{\xi} \tag{30}$$

Appendix

A-1. Proof of [Preintegration measurement (PIM)] (8)(9)(10)

A and B are the two lie groups: $\varphi(A, B) = AB$

$$\begin{aligned}
\exp(\widehat{J_A \delta}) &= (AB)^{-1}(A \exp(\hat{\delta}) B) \\
&= B^{-1} A^{-1} A \exp(\hat{\delta}) B \\
&= \exp(B^{-1} \hat{\delta} B) \\
&= \exp(\widehat{B^{-1} \delta})
\end{aligned}$$

$$\begin{aligned}
\exp(\widehat{J_B \delta}) &= (AB)^{-1}(AB \exp(\hat{\delta})) \\
&= B^{-1} A^{-1} AB \exp(\hat{\delta}) \\
&= \exp(\hat{\delta})
\end{aligned}$$

Hence:

$$J_A = B^{-1} \tag{A1-1}$$

$$J_B = I \tag{A1-2}$$

Proof of (8):

According to A1-1:

$$\begin{aligned}
\frac{\partial R_{k+1}}{\partial R_k} &= \exp(-\omega_k^b \Delta t) \\
&= I_{3\times 3} - \Delta t \widehat{\omega_k^b}
\end{aligned}$$

A is a lie group, p is a vector: $\varphi(A, p) = Ap$

$$
\begin{aligned}
J_A &= \frac{A\exp(\delta)\,p - Ap}{\delta} \\
&\cong \frac{Aa + A\hat{\delta}p - Ap}{\delta} \\
&= \frac{A\hat{\delta}p}{\delta} \\
&= -\frac{A\delta\hat{p}}{\delta} \\
&= -A\hat{p}
\end{aligned} \tag{A1-3}
$$

$$
\begin{aligned}
J_p &= \frac{A(p+\delta) - Ap}{\delta} \\
&= A
\end{aligned} \tag{A1-4}
$$

According to A1-3:

$$
\frac{\partial p_{k+1}}{\partial R_k} = -R_k\widehat{a_k^b}\frac{\Delta t^2}{2}
$$

$$
\frac{\partial v_{k+1}}{\partial R_k} = -R_k\widehat{a_k^b}\Delta t
$$

Proof of (9):

According to A1-4:

$$
\frac{\partial p_{k+1}}{\partial a_k^b} = R_k\frac{\Delta t^2}{2}
$$

$$
\frac{\partial v_{k+1}}{\partial a_k^b} = R_k\Delta t
$$

Proof of (10):

According to A1-2:

$$
\frac{\partial R_{k+1}}{\partial \omega_k^b} = I_{3\times 3}\Delta t
$$

A-2. Proof of [Delta between two states] (20)

A is a lie group, p is a vector: $\varphi(A,p) = A^{-1}p$

$$
\begin{aligned}
J_A &= \frac{(A\exp(\hat{\delta}))^{-1}p - A^{-1}p}{\delta} \\
&= \frac{\exp(\widehat{-\delta})A^{-1}p - A^{-1}p}{\delta} \\
&= \frac{(I-\hat{\delta})A^{-1}p - A^{-1}p}{\delta} \\
&= \frac{-\hat{\delta}A^{-1}p}{\delta} \\
&= \frac{\delta x \widehat{A^{-1}p}}{\delta} \\
&= \widehat{A^{-1}p}
\end{aligned}
\tag{A2-1}
$$

$$
\begin{aligned}
J_p &= \frac{T^{-1}(p+\delta) - T^{-1}p}{\delta} \\
&= \frac{T^{-1}\delta}{\delta} \\
&= T^{-1}
\end{aligned}
\tag{A2-2}
$$

Proof of (20)

The $\mathscr{D}$ function:

$$
\mathscr{D}(\xi, s_i) = \begin{bmatrix} R_{bc}^{\xi} \\ p_{bc}^{\xi} + R_{nb}^{-1} v_{nb}\Delta t + R_{nb}^{-1} g \frac{\Delta t^2}{2} \\ v_{bc}^{\xi} + R_{nb}^{-1} g \Delta t \end{bmatrix}
$$

According to A2-1:

$$
\frac{\partial p_{bc}}{\partial R_{nb}} = \widehat{R_{nb}^{-1} v_{nb}}\Delta t + \widehat{R_{nb}^{-1} g}\frac{\Delta t^2}{2}
$$

$$
\frac{\partial v_{bc}}{\partial R_{nb}} = \widehat{R_{nb}^{-1} g}\Delta t
$$

According to A2-2 and (22):

$$
\begin{aligned}
\frac{\partial p_{bc}}{\partial v_{nb}} &= \frac{R_{nb}^{-1}(v_{nb} + R_{nb}\delta v_b) - R_{nb}^{-1} v_{nb}}{\delta v_b} \\
&= I_{3\times 3}\Delta t
\end{aligned}
$$

A-3. Proof of Retraction $\mathscr{R}$ (23)(24)

The $\mathscr{R}$ function:

$$\begin{aligned} R_{nc}^* &= R_{nb}R_{bc} \\ p_{nc}^* &= p_{nb} + R_{nb}p_{bc} \\ v_{nc}^* &= v_{nb} + R_{nb}v_{bc} \end{aligned}$$

The Jacobian of x for F:

$$J_x^F = \frac{\mathscr{L}(F(x), F(\mathscr{R}(x, \delta x)))}{\delta x} \tag{A3-1}$$

Proof of $J_{s_i}^{\mathscr{R}}$(23):

According to A1-1:

$$\frac{\partial R_{nc}^*}{\partial R_{nb}} = R_{bc}^{-1}$$

According to A2-2 and A3-1:

$$\begin{aligned} \frac{\partial p_{nc}^*}{\partial R_{nb}} &= \frac{R_{nc}^{-1}(R_{nb}\exp(\widehat{\delta\theta_b})p_{bc} - R_{nb}p_{bc})}{\delta\theta_b} \\ &= -R_{bc}^{-1}\widehat{p_{bc}} \end{aligned}$$

$$\begin{aligned} \frac{\partial v_{nc}^*}{\partial R_{nb}} &= \frac{R_{nc}^{-1}(R_{nb}\exp(\widehat{\delta\theta_b})v_{bc} - R_{nb}v_{bc})}{\delta\theta_b} \\ &= -R_{bc}^{-1}\widehat{v_{bc}} \end{aligned}$$

According to A1-3 and (22)(25):

$$\begin{aligned} \frac{\partial p_{bc}^*}{\partial p_{nb}} &= \frac{R_{nc}^{-1}(p_{nb} + R_{nb}\delta p_b - p_{nb})}{\delta p_b} \\ &= R_{bc}^{-1} \end{aligned}$$

$$\begin{aligned} \frac{\partial v_{bc}^*}{\partial v_{nb}} &= \frac{R_{nc}^{-1}(v_{nb} + R_{nb}\delta v_b - v_{nb})}{\delta v_b} \\ &= R_{bc}^{-1} \end{aligned}$$

Proof of $J_d^{\mathscr{R}}$(24)

According to A2-2 and A3-1:

$$\frac{\partial p_{nc}^*}{\partial p_{bc}} = \frac{R_{nc}^{-1}(R_{nb}(p_{bc} + \delta p_b) - R_{nb} p_{bc})}{\delta p_b}$$
$$= R_{bc}^{-1}$$

$$\frac{\partial v_{nc}^*}{\partial v_b} = \frac{R_{nc}^{-1}(R_{nb}(v_{bc} + \delta v_b) - R_{nb} v_{bc})}{\delta v_b}$$
$$= R_{bc}^{-1}$$

A-4. Proof of Local $\mathscr{L}$ (23)(24)

The $\mathscr{L}$ function:

$$r_{jj^*} = \mathscr{L}(s_j, s_j^*) = \begin{bmatrix} \Delta R \\ \Delta p \\ \Delta v \end{bmatrix} = \begin{bmatrix} R_j^{-1} R_j^* \\ R_j^{-1}(p_j^* - p_j) \\ R_j^{-1}(v_j^* - v_j) \end{bmatrix}$$