Solve Nonlinear Least-Squares Problem with the Gauss-Newton Methods.

What is linear/nonlinear?

- Linear: A polynomial of degree 1.
- Nonlinear: A function cannot be expressed as a polynomial of degree 1.

What is Least-Squares Problem?

r(x) is the residual function, x is the parameter vector.
The least-squares problem tries to find optimal parameters to minimize the sum of squared residuals.

$$cost = \sum_{i=0}^{n}(\frac{r^T \Sigma r}{2}) \tag{1}$$

Where Σ is the information matrix for the measurement. In the simple case, it can be the identity matrix.

What is Gauss-Newton Methods?

The gauss–newton methods is used to solve nonlinear least-squares problems.
Unlike Newton's method, gauss-newton methods are not necessary to calculate second derivatves of residual function, which may difficult to compute in some cases.
Gauss–newton methods update x using iterative method, the update amount is Δx.

$$\Delta x = -H^{-1} g \tag{2}$$

here: g is the gradient vector; H is the hessian matrix.

$$g = \sum g_i = \sum J_i^T \Sigma r_i \tag{3}$$

$$H = \sum H_i \approx J_i^T \Sigma J_i \tag{4}$$

- r is the residual vector.

- J is the jacobian matrix of r.

The problem of 3D points matching

If we define the increment of SO3/SE3 as:

$$T(x_0 \oplus \delta) \triangleq T(x_0)\exp(\delta) \tag{5}$$

The $\delta \in \mathfrak{so}(3)$ or $\delta \in \mathfrak{se}(3)$

We use a first-order Taylor expansion to approximate the original equation:

$$T(x_0 \oplus \delta) = T_0 \exp(\delta) \cong T_0 + T_0\hat{\delta} \tag{6}$$

The the residual function of 3D points matching problem can be defined as:

$$r(x) = T(x)a - b \tag{7}$$

a is the target point:
b is the reference point.

We can use gauss-newton method to solve this problem.
According to gauss-newton method, we need to find the Jacobian matrix of r

$$\begin{aligned}
\dot{r} &= \frac{T_0 \exp{(\delta)}\, a - T_0 a}{\delta} \\
&\cong \frac{T_0 a + T_0 \hat{\delta} a - T_0 a}{\delta} \\
&= \frac{T_0 \hat{\delta} a}{\delta} \\
&= -\frac{T_0 \delta \widehat{a}}{\delta} \\
&= -T_0 \widehat{a}
\end{aligned} \tag{8}$$

When $\delta \in \mathfrak{so}(3)$

T_0 is a 3d rotation matrix(R_0),
and $\widehat{a}$ is defined as a skew symmetric matrix for vector a

$$\dot{r} = -R_0 [a]_{\times} \tag{9}$$

When $\delta \in \mathfrak{se}(3)$

$$\delta = [\, v, \omega] \tag{10}$$

ω: the parameters of rotation.

v: the parameters of translation.

$$\hat{\delta} = \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \tag{11}$$

$$\begin{aligned} \dot{r} &= \frac{R_0 \hat{\delta} a}{\delta} \\ &= \frac{T_0 \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix}}{[v, \omega]} \\ &= \frac{T_0 \begin{bmatrix} I & [-a]_\times \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}}{[v, \omega]} \\ &= T_0 \begin{bmatrix} I & [-a]_\times \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{12}$$