Graph Optimization

What is graph?

A graph is a pair $G = (V, E)$,
where V is a set of nodes, each of which contains some parameters to be optimized. E is a set of connected information, whose elements are denotes the constraint relationship between two nodes. Many robotics and computer vision problems can be represented by a graph problem.

How to solve graph problem?

A graph problem can be defined as a nonlinear least squares problems.
$f_{ij}(v_i, v_j; e_{ij})$ shows the constraint relationship between node v_i and v_j
e_{ij} is the prior error of v_i and v_j.

$$F(V) = \sum_{\{i,j\}\in E} f_{ij}(v_i, v_j; e_{ij})^2 \tag{1}$$

We need to find a optimal set of nodes (i.e. V) to minimize the overall cost.
According to guass_newton_method.md,
as soon as we can compute the hessian matrix H and gradient g, we can solve this graph optimization problem.

The hessian matrix H

We note that the size of the hessian matrix will be very large,
since there are many parameters for F.
The hessian matrix of f_{ij} can be show as:

$$H_{ij} = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & J_i^T J_i & \dots & J_i^T J_j & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & J_j^T J_i & \dots & J_j^T J_j & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \tag{2}$$

The $J_i^T J_i$ is located in row i column i of H_{ij}
The $J_j^T J_j$ is located in row j column j of H_{ij}
The $J_i^T J_j$ is located in row i column j of H_{ij}
The $J_j^T J_i$ is located in row j column i of H_{ij}

The overall hessian matrix of F is:

$$H = \sum_{\{i,j\} \in E} H_{ij} \tag{3}$$

The gradient g

The gradient vector of f_{ij} can be show as:

$$g_{ij} = \begin{bmatrix} \cdots \\ J_i^T r_i \\ \cdots \\ J_j^T r_n \\ \cdots \end{bmatrix} \tag{4}$$

The $J_i^T r_i$ is located in row i of g_{ij}
The $J_j^T r_j$ is located in row j of g_{ij}

The overall gradient vector of F is:

$$g = \sum_{\{i,j\} \in E} g_{ij} \tag{5}$$

Derivative of edge between two lie groups

Suppose φ is an smooth mapping between two lie groups,
we can define the derivative of φ as J:

$$\exp(\widehat{J\delta}) = \varphi(x)^{-1} \varphi(x \oplus \delta) \tag{6}$$

x is a the parameter of φ, and δ is a small increment to x.

The the transfrom error of two lie groups can define as:

$$\varphi(A, B) = Z^{-1} A^{-1} B \tag{7}$$

Where A and B are the two lie groups, which represent the poses of two nodes. The Z represents the relative pose of A nad B, which usually measured by odometry or loop-closing.

If A and B are SO3

$$\exp(\widehat{J_A \delta}) = (Z^{-1} A^{-1} B)^{-1} (Z^{-1} (A \exp(\hat{\delta}))^{-1} B)$$

$$
\begin{aligned}
&= B^{-1}AZZ^{-1}\exp(-\hat{\delta})A^{-1}B \\
&= B^{-1}A\exp(-\hat{\delta})A^{-1}B \\
&= -\exp(B^{-1}A\hat{\delta}A^{-1}B) \\
&= -\exp(\widehat{B^{-1}A\delta})
\end{aligned} \tag{8}
$$

Hence:

$$
J_A = -B^{-1}A \tag{9}
$$

$$
\begin{aligned}
\exp(\widehat{J_B\delta}) &= (Z^{-1}A^{-1}B)^{-1}(Z^{-1}AB\exp(\hat{\delta})) \\
&= B^{-1}AZZ^{-1}AB\exp(\hat{\delta}) \\
&= \exp(\hat{\delta})
\end{aligned} \tag{10}
$$

Hence:

$$
J_B = I \tag{11}
$$

If A and B are SE2

The small incremental matrix of SE2 can be shown as follow:

$$
\hat{\delta} = \begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix} \tag{12}
$$

Where $\delta = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathfrak{se}(2)$

ω: the parameter of rotation (is a scalar). $[w]_+ = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}$

v: the parameters of translation (is a 2d vector).

We rewrite the $B^{-1}A$ as T_{BA}.

$$
T_{BA} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \tag{13}
$$

We substitute (12) and (13) into (8), we get:

$$
\begin{aligned}
\exp(\widehat{J_A\delta}) &= -\exp(T_{BA}\hat{\delta}T_{BA}^{-1}) \\
&= -\exp(T_{BA}\begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix}T_{BA}^{-1}) \\
&= -\exp(\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix}\begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix})
\end{aligned}
$$

$$= -\exp(\begin{bmatrix} R[\omega]_+ & Rv \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix})$$
$$= -\exp(\begin{bmatrix} R[\omega]_+ R^T & R[\omega]_+(-R^T t) + Rv \\ 0 & 0 \end{bmatrix})$$
$$= -\exp(\begin{bmatrix} [\omega]_+ & -[\omega]_+ t + Rv \\ 0 & 0 \end{bmatrix})$$
$$= -\exp(\begin{bmatrix} [\omega]_+ & -[\omega]_+ t + Rv \\ 0 & 0 \end{bmatrix}) \tag{14}$$

According to (12), we can rewrite (14) as:

$$\exp(\widehat{J_A \delta}) = -\exp(\widehat{\begin{bmatrix} -[\omega]_+ t + Rv \\ w \end{bmatrix}})$$
$$= -\exp(\widehat{\begin{bmatrix} -\omega t^\perp + Rv \\ w \end{bmatrix}})$$
$$= -\exp(\widehat{\begin{bmatrix} R & -t^\perp \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}})$$

Where $t^\perp = [1]_+ t = \begin{bmatrix} -t_2 \\ t_1 \end{bmatrix}$

Hence:

$$J_A = -\begin{bmatrix} R & -t^\perp \\ 0 & 1 \end{bmatrix} = -\begin{bmatrix} R_{BA} & -t^\perp_{BA} \\ 0 & 1 \end{bmatrix} \tag{15}$$

similer with (11):

$$J_B = I \tag{16}$$

If A and B are SE3

The small incremental matrix of SE3 can be shown as follow:

$$\hat{\delta} = \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \tag{17}$$

Where $\delta = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathfrak{se}(3)$

ω: the parameters of rotation (is a 3d vector). $[w]_\times$ is the skew symmetric matrix of w.

v: the parameters of translation (is a 3d vector).

Similar to (14), we get:

$$\begin{aligned}
\exp(\widehat{J_A\delta}) &= -\exp(T_{BA}\hat{\delta}T_{BA}^{-1}) \\
&= -\exp(T_{BA}\begin{bmatrix}[\omega]_\times & v \\ 0 & 0\end{bmatrix}T_{BA}^{-1}) \\
&= -\exp(\begin{bmatrix}R & t \\ 0 & 1\end{bmatrix}\begin{bmatrix}[\omega]_\times & v \\ 0 & 0\end{bmatrix}\begin{bmatrix}R^T & -R^Tt \\ 0 & 1\end{bmatrix}) \\
&= -\exp(\begin{bmatrix}R[\omega]_\times & Rv \\ 0 & 0\end{bmatrix}\begin{bmatrix}R^T & -R^Tt \\ 0 & 1\end{bmatrix}) \\
&= -\exp(\begin{bmatrix}R[\omega]_\times R^T & -R[\omega]_\times R^Tt + Rv \\ 0 & 0\end{bmatrix}) \\
&= -\exp(\begin{bmatrix}[R\omega]_\times & -[R\omega]_\times t + Rv \\ 0 & 0\end{bmatrix})
\end{aligned} \tag{18}$$

According to (12), we can rewrite (18) as:

$$\begin{aligned}
\exp(\widehat{J_A\delta}) &= -\exp(\widehat{\begin{bmatrix}-[R\omega]_\times t + Rv \\ Rw\end{bmatrix}}) \\
&= -\exp(\widehat{\begin{bmatrix}[t]_\times R\omega + Rv \\ Rw\end{bmatrix}}) \\
&= -\exp(\widehat{\begin{bmatrix}R & [t]_\times R \\ 0 & R\end{bmatrix}\begin{bmatrix}v \\ w\end{bmatrix}})
\end{aligned} \tag{19}$$

Hence:

$$J_A = -\begin{bmatrix}R_{BA} & [t_{BA}]_\times R_{BA} \\ 0 & R_{BA}\end{bmatrix} \tag{20}$$

similer with (11):

$$J_B = I \tag{21}$$