Solve Nonlinear Least-Squares Problem with the Gauss-Newton Methods.

What is linear/nonlinear?

- Linear: A polynomial of degree 1.
- Nonlinear: A function cannot be expressed as a polynomial of degree 1.

What is Least-Squares Problem?

f(x) is the objective function, x is the parameter vector.
The least-squares problem tries to find optimal parameters to minimize the overall cost. $$ cost = \sum_{i=0}^{n}f(x)^2 \quad $$

What is Gauss-Newton Methods?

The gauss–newton methods is used to solve nonlinear least-squares problems.
Unlike Newton's method, gauss-newton methods are not necessary to calculate second derivatves, which may difficult to compute in some cases.
Gauss–newton methods update x using iterative method, the update amount is Δx.

$$ \Delta x = -H^{-1}g $$ here: g is the gradient vector; H is the hessian matrix. $$ g = J^Te $$ $$ H \approx J^TJ $$ J is the jacobian matrix, e is the residual vector.

The jacabian matrix of 3d rotation or 3d transform

If we define the increment of SO3/SE3 as:

$$T(x_{0}\oplus\delta) \triangleq T(x_{0})\exp(\delta)$$

The $\delta \in \mathfrak{so}(3)$ or $\delta \in \mathfrak{se}(3)$

We use a first-order Taylor expansion to approximate the original equation:

$$T(x_{0}\oplus\delta) = T_{0}\exp(\delta) \cong T_{0} + T_{0}\widehat{\delta}$$

The f(x) is the objective function. We want to find the optimal parameters (x) that minimize the result of the objective function. $$f(x) = T(x)a - b$$

a is the target point: b is the reference point.

We can use gauss-newton method to solve this problem. According to gauss-newton method, we need to find the Jacobian matrix for f(x).

$$\dot{f} = \frac{T_{0}\exp\left(\delta \right)a - T_{0}a}{\delta}$$

$$\cong \frac{T_{0}a + T_{0}\widehat{\delta}a - T_{0}a}{\delta}$$

$$= \frac{T_{0}\widehat{\delta}a}{\delta}$$

$$= - \frac{T_{0}\delta\widehat{a}}{\delta}$$

$$= - T_{0}\widehat{a} $$

When $\delta \in \mathfrak{so}(3)$

T_0 is a 3d rotation matrix(R_0), and $\widehat{a}$ is defined as a skew symmetric matrix for vector a

$$\dot{f} = - R_{0}[a]_{\times}$$

When $\delta \in \mathfrak{se}(3)$

$$\delta = [\ v, \omega]$$

ω: the parameter of rotation.

v: the parameter of translation.

$$\widehat{\delta} = \begin{bmatrix} [\omega]_{\times} & v \ 0 & 0 \ \end{bmatrix}$$

$$\dot{f} = \frac{R_{0}\widehat{\delta}a}{\delta}$$

$$= \frac{T_{0}\begin{bmatrix} [\omega]_{\times} & v \ 0 & 0 \ \end{bmatrix}\begin{bmatrix} a \ 1 \ \end{bmatrix}}{[\omega,\ v]}$$

$$= \frac{T_{0}\begin{bmatrix} [- a]_{\times} & I \ 0 & 0 \ \end{bmatrix}\begin{bmatrix} \omega \ v \ \end{bmatrix}}{[\omega,\ v]}$$

$$= T_{0}\begin{bmatrix} [- a]_{\times} & I \ 0 & 0 \ \end{bmatrix}$$