

Solve Nonlinear Least-Squares Problem with the Gauss-Newton Methods.

What is linear/nonlinear?

- Linear: A polynomial of degree 1.
- Nonlinear: A function cannot be expressed as a polynomial of degree 1.

What is Least-Squares Problem?

$r(x)$ is the residual function, x is the parameter vector.

The least-squares problem tries to find optimal parameters to minimize the sum of squared residuals.

$$cost = \sum_{i=0}^n r(x)^2 \quad (1)$$

What is Gauss-Newton Methods?

The gauss–newton methods is used to solve nonlinear least-squares problems.

Unlike Newton's method, gauss-newton methods are not necessary to calculate second derivatives of residual function, which may difficult to compute in some cases.

Gauss–newton methods update x using iterative method, the update amount is Δx .

$$\Delta x = -H^{-1}g \quad (2)$$

here: g is the gradient vector; H is the hessian matrix.

$$g = \sum g_i = \sum J_i^T r_i \quad (3)$$

$$H = \sum H_i \approx \sum J_i^T J_i \quad (4)$$

- r is the residual vector.
- J is the jacobian matrix of r .

The problem of 3D points matching

If we define the increment of SO3/SE3 as:

$$T(x_0 \oplus \delta) \triangleq T(x_0) \exp(\delta) \quad (5)$$

The $\delta \in \mathfrak{so}(3)$ or $\delta \in \mathfrak{se}(3)$

We use a first-order Taylor expansion to approximate the original equation:

$$T(x_0 \oplus \delta) = T_0 \exp(\delta) \cong T_0 + T_0 \hat{\delta} \quad (6)$$

The the residual function of 3D points matching problem can be defined as:

$$r(x) = T(x)a - b \quad (7)$$

a is the target point:

b is the reference point.

We can use gauss-newton method to solve this problem.

According to gauss-newton method, we need to find the Jacobian matrix of r

$$\begin{aligned} \dot{r} &= \frac{T_0 \exp(\delta) a - T_0 a}{\delta} \\ &\cong \frac{T_0 a + T_0 \hat{\delta} a - T_0 a}{\delta} \\ &= \frac{T_0 \hat{\delta} a}{\delta} \\ &= -\frac{T_0 \delta \hat{a}}{\delta} \\ &= -T_0 \hat{a} \end{aligned} \quad (8)$$

When $\delta \in \mathfrak{so}(3)$

T_0 is a 3d rotation matrix(R_0),

and \hat{a} is defined as a skew symmetric matrix for vector a

$$\dot{r} = -R_0[a]_{\times} \quad (9)$$

When $\delta \in \mathfrak{se}(3)$

$$\delta = [v, \omega] \quad (10)$$

ω : the parameters of rotation.

v : the parameters of translation.

$$\hat{\delta} = \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix} \tag{11}$$

$$\dot{r} = \frac{R_0 \hat{\delta} a}{\delta}$$

$$\begin{aligned} & T_0 \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix} \\ = & \frac{\begin{bmatrix} v, \omega \end{bmatrix}}{T_0 \begin{bmatrix} I & [-a]_{\times} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} v \\ \omega \end{bmatrix}} \\ = & \frac{\begin{bmatrix} v, \omega \end{bmatrix}}{\begin{bmatrix} I & [-a]_{\times} \\ 0 & 0 \end{bmatrix}} \\ = & T_0 \begin{bmatrix} I & [-a]_{\times} \\ 0 & 0 \end{bmatrix} \end{aligned} \tag{12}$$