

# 3D rotation and Lie group

## Euler angles

The Euler angles are three angles introduced by Leonhard Euler to describe the orientation of a rigid body.

- $\alpha$ : Often be called as *Roll*, rotation around the x axis.
- $\beta$ : Often be called as *Pitch*, rotation around the y axis.
- $\gamma$ : Often be called as *Yaw*, rotation around the z axis.

Rotation around the x axis.

$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \quad (1)$$

Rotation around the y axis.

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \quad (2)$$

Rotation around the z axis.

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad (3)$$

Any rotation can be composed by above three elemental rotation matrices.

## Rotation order of Euler angles

But since the multiplication of matrices does not satisfy the commutative law, different orders of x,y and z rotation generate different final rotation matrices.

$$\begin{aligned}
R_{xyz}(\alpha, \beta, \gamma) &= R_z(\gamma)R_y(\beta)R_x(\alpha) \\
&= \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \\
&= \begin{bmatrix} c_\gamma c_\beta & c_\gamma s_\beta s_\alpha - c_\alpha s_\gamma & s_\gamma s_\alpha + c_\gamma c_\alpha s_\beta \\ c_\beta s_\gamma & c_\gamma c_\alpha + s_\gamma s_\beta s_\alpha & c_\alpha s_\gamma s_\beta - c_\gamma s_\alpha \\ -s_\beta & c_\beta s_\alpha & c_\beta c_\alpha \end{bmatrix}
\end{aligned} \tag{4}$$

$$\begin{aligned}
R_{zxy}(\alpha, \beta, \gamma) &= R_y(\beta)R_x(\alpha)R_z(\gamma) \\
&= \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} c_\beta c_\gamma + s_\beta s_\alpha s_\gamma & c_\gamma s_\beta s_\alpha - c_\beta s_\gamma & c_\alpha s_\beta \\ c_\alpha s_\gamma & c_\alpha c_\gamma & -s_\alpha \\ c_\beta s_\alpha s_\gamma - s_\beta c_\gamma & s_\beta s_\gamma + c_\beta c_\gamma s_\alpha & c_\beta c_\alpha \end{bmatrix}
\end{aligned} \tag{5}$$

Therefore it is very important to determine the order of rotation when using Euler angles.

Euler angles are frequently used, however, they are not continuous because of the gimbal lock problem. And they can be very tricky to handle in many mathematical problems due to the non-commutative nature.

## Infinitesimal Rotation

If the angles are small enough, the following equations are true.

- $\cos(a) \approx 1$
- $\sin(a) \approx a$
- $\sin(a)\sin(b) \approx 0$

If we substitute the above equations into (8) and (9), we can yield the same results.

$$\begin{aligned}
R_{xyz} &\approx R_{zxy} \approx \begin{bmatrix} 1 & -\gamma & \beta \\ \gamma & 1 & -\alpha \\ -\beta & \alpha & 1 \end{bmatrix} \\
&= I + \begin{bmatrix} 0 & -\gamma & \beta \\ \gamma & 0 & -\alpha \\ -\beta & \alpha & 0 \end{bmatrix} \quad \text{if } \alpha, \beta, \gamma \ll 1 \\
&= I + [\omega]_{\times}
\end{aligned} \tag{6}$$

$[\omega]_{\times}$  is a [skew-symmetric matrix](#), which is composed of  $\alpha$ ,  $\beta$  and  $\gamma$ .

Now that we can obtain a commutative representation of a small rotation by a 3d vector. The remaining question is how to represent a larger rotation?

If we want a larger rotation, we can simply split 3d vector  $\omega$  into  $n$  pieces, and compose the as follows:

$$R(\omega) = \underbrace{\left(I + \frac{[\omega]_{\times}}{n}\right) \times \dots \left(I + \frac{[\omega]_{\times}}{n}\right)}_{n \text{ factors}} = \left(I + \frac{[\omega]_{\times}}{n}\right)^n \quad (7)$$

For real numbers, this series is very famous, shows a way to compute the exponential function.

Similarly, we can extend the definition of exponential function to skew-symmetric matrix.

$$R(\omega) = \left(I + \frac{[\omega]_{\times}}{n}\right)^n = e^{[\omega]_{\times}} \quad (8)$$

The exponential sum formula is also applicable to skew-symmetric matrix. Now, the exponential map (8) or (9) can transform a 3d vector into a rotation matrix.

$$R(\omega) = e^{[\omega]_{\times}} = \sum_{k=0}^{\infty} \frac{[\omega]_{\times}^k}{k!} \quad (9)$$

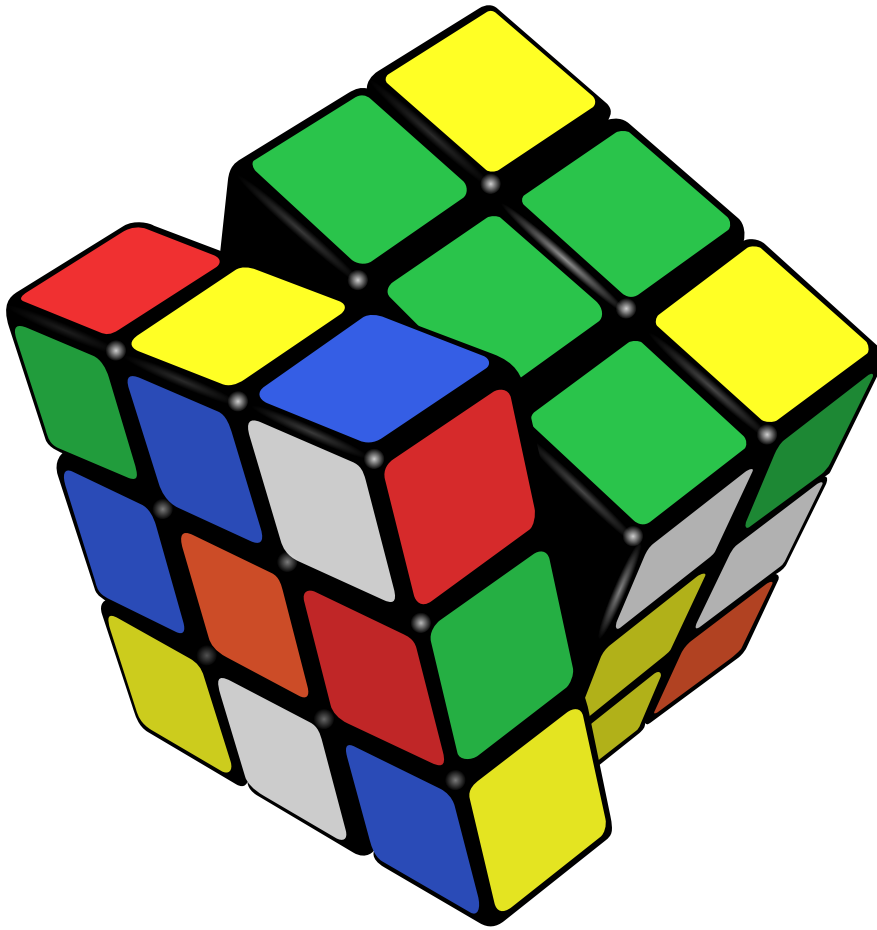
In actually, some part of Lie group theories have been described in above. The 3D rotation space  $R$  is called as *special orthogonal group*  $SO(3)$ . The 3d vector  $\omega$  is called the Lie algebra  $\mathfrak{so}(3)$  associated with  $SO(3)$  by the exponential map.

## Group

A [Group](#) satisfied following requirements, known as group axioms.

- Closure:  
if  $a_1, a_2 \in G$ , then  $a_1 \cdot a_2 \in G$
- Associativity:  
if  $a_1, a_2, a_3 \in G$ , then  $(a_1 \cdot a_2) \cdot a_3 = a_1 \cdot (a_2 \cdot a_3)$
- Identity:  
For every  $a \in G$ , there exists a  $a_0 \in G$ , such that  $a_0 \cdot a = a \cdot a_0 = a$
- Invertibility:  
For every  $a \in G$ , there exists a  $a^{-1} \in G$ , such that  $a \cdot a^{-1} = a_0$

For example, the Rubik's Cube group is a group, we can simply verify that the group axioms are satisfied for it.



## Lie group

A Lie group is a continuous group, which means a Lie group is infinitely differentiable (smooth).

Therefore, the Rubik's Cube group is a group, but not a Lie group. 3D rotation space is a group as well as a Lie group.

Because of the following advantages, Lie group and Lie algebra are often used to represent rotations in the latest SLAM studies.

- Lie algebra only use 3 values to represent a rotation.
- A Lie group and Lie algebra is differentiable.
- There is no gimbal lock problem in Lie group or Lie algebra.
- For small rotation, Lie group is easy to be linearized (6).

## Special orthogonal group $SO(3)$

### Exponential map

We can map a  $\mathfrak{so}(3)$  to  $SO(3)$  by (8) or (9), But the calculation is too complicated. Here we will try to simplify it.

We define  $\omega = \theta r$

- $r$  is a unit vector of  $\omega$ ,  $r = \frac{\omega}{\|\omega\|}$
- $\theta$  is the norm of a  $\omega$ ,  $\theta = \|\omega\|$

$$\begin{aligned}
 \exp([\omega]_{\times}) &= \exp(\theta[r]_{\times}) \\
 &= \sum_{k=0}^{\infty} \frac{1}{k!} (\theta[r]_{\times})^k \\
 &= I + \theta[r]_{\times} + \frac{1}{2!}\theta^2[r]_{\times}^2 + \frac{1}{3!}\theta^3[r]_{\times}^3 + \frac{1}{4!}\theta^4[r]_{\times}^4 + \dots \\
 &= r^T r - [r]_{\times}^2 + \theta[r]_{\times} + \frac{1}{2!}\theta^2[r]_{\times}^2 + \frac{1}{3!}\theta^3[r]_{\times}^3 + \frac{1}{4!}\theta^4[r]_{\times}^4 + \dots \\
 &= r^T r - ([r]_{\times}^2 - \frac{1}{2!}\theta^2[r]_{\times}^2 - \frac{1}{4!}\theta^4[r]_{\times}^4 - \dots) + (\theta[r]_{\times} + \frac{1}{3!}\theta^3[r]_{\times}^3 + \dots) \tag{10} \\
 &= r^T r - (1 - \frac{1}{2!}\theta^2 + \frac{1}{4!}\theta^4 - \dots)[r]_{\times}^2 + (\theta - \frac{1}{3!}\theta^3 + \dots)[r]_{\times} \\
 &= r^T r - \cos\theta[r]_{\times}^2 + \sin\theta[r]_{\times} \\
 &= [r]_{\times}^2 + I - \cos\theta[r]_{\times}^2 + \sin\theta[r]_{\times} \\
 &= I + (1 - \cos\theta)[r]_{\times}^2 + \sin\theta[r]_{\times}
 \end{aligned}$$

In (10) two properties of skew symmetric matrices are used.

If  $r$  is a unit vector:

- $[r]_{\times}[r]_{\times}[r]_{\times} = -[r]_{\times}$
- $r^T r = [r]_{\times}^2 + I$

The formula (10) show a fast way to calculate the exponential map from  $\mathfrak{so}(3)$  to  $SO(3)$ , it is known as [Rodrigues' Formula](#).

## Logarithm map

In contrast to Exponential map, using the Logarithm map can correspond a Lie algebra to a Lie group.

The formula (10) can be wrtten as matrix:

$$\begin{aligned}
R(\theta, r) &= I + (1 - \cos\theta) \begin{bmatrix} 1 - r_1^2 & r_1 r_2 & r_1 r_3 \\ r_1 r_2 & 1 - r_1^2 & -r_2 r_3 \\ -r_1 r_3 & r_2 r_3 & 1 - r_1^2 \end{bmatrix} + \sin\theta \begin{bmatrix} 0 & -r_3 & r_2 \\ r_3 & 0 & -r_1 \\ -r_2 & r_1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} r_1^2(1 - \cos\theta) + \cos\theta & r_1 r_2(1 - \cos\theta) - r_3 \sin\theta & r_1 r_3(1 - \cos\theta) + r_2 \sin\theta \\ r_1 r_2(1 - \cos\theta) + r_3 \sin\theta & r_2^2(1 - \cos\theta) + \cos\theta & -r_2 r_3(1 - \cos\theta) - r_1 \sin\theta \\ -r_1 r_3(1 - \cos\theta) - r_2 \sin\theta & r_2 r_3(1 - \cos\theta) + r_1 \sin\theta & r_3^2(1 - \cos\theta) + \cos\theta \end{bmatrix} \quad (11)
\end{aligned}$$

From (11) we can see:

$$\begin{aligned}
\theta &= \arccos\left(\frac{1}{2}(R_{11} + R_{22} + R_{33} - 1)\right) \\
&= \arccos\left(\frac{1}{2}(\text{tr}(r)) - 1\right) \quad (12)
\end{aligned}$$

$$r = [R_{32} - R_{23}, R_{13} - R_{31}, R_{12} - R_{21}]/2\sin\theta \quad (13)$$

from (12) (13), Logarithm map can be implmented as follows:

$$\log(R)^\vee = \omega = \frac{\theta[R_{32} - R_{23}, R_{13} - R_{31}, R_{12} - R_{21}]}{2\sin\theta} \quad (14)$$