

Graph Optimization

What is graph?

A graph is a pair $G = (V, E)$, where V is a set of nodes, each of which contains some parameters to be optimized. E is a set of connected information, whose elements are denotes the constraint relationship between two nodes. Many robotics and computer vision problems can be represented by a graph problem.

How to solve graph problem?

A graph problem can be defined as a nonlinear least squares problems.

$f_{ij}(v_i, v_j; e_{ij})$ shows the constraint relationship between node v_i and v_j
 e_{ij} is the prior error of v_i and v_j .

$$F(V) = \sum_{\{i,j\} \in E} f_{ij}(v_i, v_j; e_{ij})^2 \quad (1)$$

We need to find a optimal set of nodes (i.e. V) to minimize the overall cost.

According to [guass_newton_method.md](#),

as soon as we can compute the hessian matrix H and gradient g , we can solve this graph optimization problem.

The hessian matrix H

We note that the size of the hessian matrix will be very large, since there are many parameters for F .

The hessian matrix of f_{ij} can be show as:

$$H_{ij} = \begin{bmatrix} \dots & \dots & \dots & \dots & \dots \\ \dots & J_i^T J_i & \dots & J_i^T J_j & \dots \\ \dots & \dots & \dots & \dots & \dots \\ \dots & J_j^T J_i & \dots & J_j^T J_j & \dots \\ \dots & \dots & \dots & \dots & \dots \end{bmatrix} \quad (2)$$

The $J_i^T J_i$ is located in row i column i of H_{ij}

The $J_j^T J_j$ is located in row j column j of H_{ij}

The $J_i^T J_j$ is located in row i column j of H_{ij}

The $J_j^T J_i$ is located in row j column i of H_{ij}

The overall hessian matrix of F is:

$$H = \sum_{\{i,j\} \in E} H_{ij} \quad (3)$$

The gradient g

The gradient vector of f_{ij} can be show as:

$$g_{ij} = \begin{bmatrix} \dots \\ J_i^T r_i \\ \dots \\ J_j^T r_n \\ \dots \end{bmatrix} \quad (4)$$

The $J_i^T r_i$ is located in row i of g_{ij}

The $J_j^T r_j$ is located in row j of g_{ij}

The overall gradient vector of F is:

$$g = \sum_{\{i,j\} \in E} g_{ij} \quad (5)$$

Derivative of edge between two lie groups

Suppose φ is an smooth mapping between two lie groups,
we can define the derivative of φ as J :

$$\exp(\widehat{J\delta}) = \varphi(x)^{-1} \varphi(x \oplus \delta) \quad (6)$$

x is a the parameter of φ , and δ is a small increment to x .

The the transfrom error of two lie groups can define as:

$$\varphi(A, B) = Z^{-1} A^{-1} B \quad (7)$$

Where A and B are the two lie groups, which represent the poses of two nodes. The Z represents the relative pose of A nad B , which usually measured by odometry or loop-closing.

If A and B are SO3

$$\begin{aligned}
\exp(\widehat{J_A \delta}) &= (Z^{-1} A^{-1} B)^{-1} (Z^{-1} (A \exp(\hat{\delta}))^{-1} B) \\
&= B^{-1} A Z Z^{-1} \exp(-\hat{\delta}) A^{-1} B \\
&= B^{-1} A \exp(-\hat{\delta}) A^{-1} B \\
&= -\exp(B^{-1} A \hat{\delta} A^{-1} B) \\
&= -\exp(\widehat{B^{-1} A \delta})
\end{aligned} \tag{8}$$

Hence:

$$J_A = -B^{-1} A \tag{9}$$

$$\begin{aligned}
\exp(\widehat{J_B \delta}) &= (Z^{-1} A^{-1} B)^{-1} (Z^{-1} A^{-1} B \exp(\hat{\delta})) \\
&= B^{-1} A Z Z^{-1} A^{-1} B \exp(\hat{\delta}) \\
&= \exp(\hat{\delta})
\end{aligned} \tag{10}$$

Hence:

$$J_B = I \tag{11}$$

If A and B are SE2

The small incremental matrix of SE2 can be shown as follow:

$$\hat{\delta} = \begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix} \tag{12}$$

Where $\delta = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathfrak{se}(2)$

ω : the parameter of rotation (is a scalar). $[w]_+ = \begin{bmatrix} 0 & -w \\ w & 0 \end{bmatrix}$

v : the parameters of translation (is a 2d vector).

We rewrite the $B^{-1} A$ as T_{BA} .

$$T_{BA} = \begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \tag{13}$$

We substitute (12) and (13) into (8), we get:

$$\begin{aligned}
\exp(\widehat{J_A \delta}) &= -\exp(T_{BA} \hat{\delta} T_{BA}^{-1}) \\
&= -\exp(T_{BA} \begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix} T_{BA}^{-1}) \\
&= -\exp\left(\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\omega]_+ & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}\right) \\
&= -\exp\left(\begin{bmatrix} R[\omega]_+ & Rv \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}\right) \\
&= -\exp\left(\begin{bmatrix} R[\omega]_+ R^T & R[\omega]_+ (-R^T t) + Rv \\ 0 & 0 \end{bmatrix}\right) \\
&= -\exp\left(\begin{bmatrix} [\omega]_+ & -[\omega]_+ t + Rv \\ 0 & 0 \end{bmatrix}\right) \\
&= -\exp\left(\begin{bmatrix} [\omega]_+ & -[\omega]_+ t + Rv \\ 0 & 0 \end{bmatrix}\right)
\end{aligned} \tag{14}$$

According to (12), we can rewrite (14) as:

$$\begin{aligned}
\exp(\widehat{J_A \delta}) &= -\exp\left(\begin{bmatrix} -[\omega]_+ t + Rv \\ w \end{bmatrix}\right) \\
&= -\exp\left(\begin{bmatrix} -\omega t^\perp + Rv \\ w \end{bmatrix}\right) \\
&= -\exp\left(\begin{bmatrix} R & -t^\perp \\ 0 & 1 \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}\right)
\end{aligned}$$

Where $t^\perp = [1]_+ t = \begin{bmatrix} -t_2 \\ t_1 \end{bmatrix}$

Hence:

$$J_A = -\begin{bmatrix} R & -t^\perp \\ 0 & 1 \end{bmatrix} = -\begin{bmatrix} R_{BA} & -t_{BA}^\perp \\ 0 & 1 \end{bmatrix} \tag{15}$$

similer with (11):

$$J_B = I \tag{16}$$

If A and B are SE3

The small incremental matrix of SE3 can be shown as follow:

$$\hat{\delta} = \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix} \quad (17)$$

Where $\delta = \begin{bmatrix} v \\ w \end{bmatrix} \in \mathfrak{se}(3)$

ω : the parameters of rotation (is a 3d vector). $[w]_{\times}$ is the skew symmetric matrix of w .

v : the parameters of translation (is a 3d vector).

Similar to (14), we get:

$$\begin{aligned} \exp(\widehat{J_A \delta}) &= -\exp(T_{BA} \hat{\delta} T_{BA}^{-1}) \\ &= -\exp(T_{BA} \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix} T_{BA}^{-1}) \\ &= -\exp\left(\begin{bmatrix} R & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} [\omega]_{\times} & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}\right) \\ &= -\exp\left(\begin{bmatrix} R[\omega]_{\times} & Rv \\ 0 & 0 \end{bmatrix} \begin{bmatrix} R^T & -R^T t \\ 0 & 1 \end{bmatrix}\right) \\ &= -\exp\left(\begin{bmatrix} R[\omega]_{\times} R^T & -R[\omega]_{\times} R^T t + Rv \\ 0 & 0 \end{bmatrix}\right) \\ &= -\exp\left(\begin{bmatrix} [R\omega]_{\times} & -[R\omega]_{\times} t + Rv \\ 0 & 0 \end{bmatrix}\right) \end{aligned} \quad (18)$$

According to (12), we can rewrite (18) as:

$$\begin{aligned} \exp(\widehat{J_A \delta}) &= -\exp\left(\overline{\begin{bmatrix} -[R\omega]_{\times} t + Rv \\ R\omega \end{bmatrix}}\right) \\ &= -\exp\left(\overline{\begin{bmatrix} [t]_{\times} R\omega + Rv \\ R\omega \end{bmatrix}}\right) \\ &= -\exp\left(\overline{\begin{bmatrix} R & [t]_{\times} R \\ 0 & R \end{bmatrix} \begin{bmatrix} v \\ w \end{bmatrix}}\right) \end{aligned} \quad (19)$$

Hence:

$$J_A = -\begin{bmatrix} R_{BA} & [t_{BA}]_{\times} R_{BA} \\ 0 & R_{BA} \end{bmatrix} \quad (20)$$

similer with (11):

$$J_B = I \quad (21)$$