

# Imu preintegration.

## Predicting navigation state by IMU

Suppose we know the navigation state of the robot at time  $i$ , as well as the IMU measurements from the  $i$  time to  $j$  time. We want predict the state of robot at time  $j$ .

$$s_j^* = \mathcal{R}(s_i, \mathcal{D}(\xi(\zeta, b))) \quad (1)$$

The navigation state combined by attitude  $R(\theta)$ , position  $p$  and velocity  $v$ .

$$\begin{aligned} s_i &= (R_{nb}, p_{nb}, v_{nb}) \\ s_j &= (R_{nc}, p_{nc}, v_{nc}) \end{aligned} \quad (2)$$

- $n$  denotes navigation state frame.
- $b$  denotes body frame in time  $i$ .
- $c$  denotes current frame in time  $j$ .
- $\theta$  is the lie algebra of  $R$ .

The retract action  $\mathcal{R}$  which defined on navigation state takes 2 parameters:  $s_i$  and  $\mathcal{D}$  to predict  $s_j$ .

The  $\mathcal{D}$  represents the difference between  $s_i$  and  $s_j$ .

$$d(\xi, s_i) = (R_{nc}, p_{nc}, v_{nc}) \quad (3)$$

$\xi$  represents bias corrected preintegration measurement (PIM), which take 2 parameters, the PIM  $\zeta$  and IMU bias  $b$ .

### The Jacobian of $s_i$

$$J_{s_i}^{s_j^*} = J_{s_i}^{\mathcal{R}} + J_{\mathcal{D}}^{\mathcal{R}} J_{s_i}^{\mathcal{D}} \quad (4)$$

### The Jacobian of $b$

$$J_b^{s_j^*} = J_{\mathcal{D}}^{\mathcal{R}} J_{\xi}^{\mathcal{D}} J_b^{\xi} \quad (5)$$

## Preintegration measurement (PIM)

The PIM  $\zeta(R(\theta), p, v)$  integrates all the IMU measurements without considering the IMU bias and the gravity.

$\omega_k^b, a_k^b$  are the acceleration and angular velocity measured by IMU (accelerometer + gyroscope) respectively.

$$\begin{aligned} R_{k+1} &= R_k \exp(\omega_k^b \Delta t) \\ p_{k+1} &= p_k + v_k \Delta t + R_k a_k^b \frac{\Delta t^2}{2} \\ v_{k+1} &= v_k + R_k a_k^b \Delta t \end{aligned} \quad (7)$$

$n$ : navigation frame,  $b$ : body frame.

### A: Derivative of old $\zeta$

$$\begin{aligned} A &= \frac{\partial \zeta_{k+1}}{\partial \zeta_k} \\ &= \begin{bmatrix} \frac{\partial R_{k+1}}{\partial R_k} & \frac{\partial R_{k+1}}{\partial p_k} & \frac{\partial R_{k+1}}{\partial v_k} \\ \frac{\partial p_{k+1}}{\partial R_k} & \frac{\partial p_{k+1}}{\partial p_k} & \frac{\partial p_{k+1}}{\partial v_k} \\ \frac{\partial v_{k+1}}{\partial R_k} & \frac{\partial v_{k+1}}{\partial p_k} & \frac{\partial v_{k+1}}{\partial v_k} \end{bmatrix} \\ &= \begin{bmatrix} I_{3 \times 3} - \Delta t \widehat{\omega_k^b} & 0_{3 \times 3} & 0_{3 \times 3} \\ -R_k \widehat{a_k^b} \frac{\Delta t^2}{2} & I_{3 \times 3} & I_{3 \times 3} \Delta t \\ -R_k \widehat{a_k^b} \Delta t & 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \end{aligned} \quad (8)$$

### B: Derivative of input $a$

$$B = \frac{\partial \zeta_{k+1}}{\partial a_k^b} = \begin{bmatrix} \frac{\partial R_{k+1}}{\partial a_k^b} \\ \frac{\partial p_{k+1}}{\partial a_k^b} \\ \frac{\partial v_{k+1}}{\partial a_k^b} \end{bmatrix} = \begin{bmatrix} 0_{3 \times 3} \\ R_k \frac{\Delta t^2}{2} \\ R_k \Delta t \end{bmatrix} \quad (9)$$

### C: Derivative of input $\omega$

$$C = \frac{\partial \zeta_{k+1}}{\partial \omega_k^b} = \begin{bmatrix} \frac{\partial R_{k+1}}{\partial \omega_k^b} \\ \frac{\partial p_{k+1}}{\partial \omega_k^b} \\ \frac{\partial v_{k+1}}{\partial \omega_k^b} \end{bmatrix} = \begin{bmatrix} I_{3 \times 3} \Delta t \\ 0_{3 \times 3} \\ 0_{3 \times 3} \end{bmatrix} \quad (10)$$

### Bias correct

We want correct  $\zeta$  by a given accelerometer and gyroscope bias.

$$\xi(b + \Delta b) = \zeta \oplus (\Delta b_{acc} \frac{\partial \zeta}{\partial b_{acc}} + \Delta b_{\omega} \frac{\partial \zeta}{\partial b_{\omega}}) \quad (11)$$

- $b_{acc}$  is bias for accelerometer.
- $b_{\omega}$  is bias for gyroscope.
- Because the parameter  $\theta$  cannot be added directly, we define the combination of  $\zeta$  with the symbol  $\oplus$ .

$$a \oplus b = [\log(\exp(\theta_a) \exp(\theta_b)), p_a + p_b, v_a + v_b] \quad (12)$$

## The jacobian of bias for corrected PIM.

$$J_b^{\xi} = [\frac{\partial \zeta}{\partial b_{acc}}, \frac{\partial \zeta}{\partial b_{\omega}}] \quad (13)$$

## Find the partial derivatives of accelerometer's bias

The bias model for accelerometer.

$$\tilde{a}_k^b = a_k^b - b_{acc} \quad (14)$$

$$\begin{aligned} \frac{\partial \zeta_{k+1}}{\partial b_{acc}} &= \frac{\partial \zeta_{k+1}}{\partial \zeta_k} \frac{\partial \zeta_k}{\partial b_{acc}} + \frac{\partial \zeta_{k+1}}{\partial \tilde{a}_k^b} \frac{\partial \tilde{a}_k^b}{\partial b_{acc}} \\ &= A \frac{\partial \zeta_k}{\partial b_{acc}} - B \end{aligned} \quad (15)$$

## Find the partial derivatives of gyroscope's bias

$$\tilde{\omega}_k^b = \omega_k^b - b_{\omega} \quad (16)$$

$$\begin{aligned} \frac{\partial \zeta_{k+1}}{\partial b_{\omega}} &= \frac{\partial \zeta_{k+1}}{\partial \zeta_k} \frac{\partial \zeta_k}{\partial b_{\omega}} + \frac{\partial \zeta_{k+1}}{\partial \tilde{\omega}_k^b} \frac{\partial \tilde{\omega}_k^b}{\partial b_{\omega}} \\ &= A \frac{\partial \zeta_k}{\partial b_{\omega}} - C \end{aligned} \quad (17)$$

- $\sim$  denotes the corrected measurement.

## Delta between two states

The  $\mathcal{D}$  represents the difference between two  $s_i$  and  $s_j$ .

$$\mathcal{D} = (R_{bc}, p_{bc}, v_{bc}) \quad (18)$$

We can calculate  $\mathcal{D}$  from corrected PIM  $\xi(R_{bc}^\xi, p_{bc}^\xi, v_{bc}^\xi)$  and velocity, which is included in  $s_i$ .

$$\mathcal{D}(\xi, s_i) = \begin{bmatrix} R_{bc}^\xi \\ p_{bc}^\xi + R_{nb}^{-1} v_{nb} \Delta t + R_{nb}^{-1} g \frac{\Delta t^2}{2} \\ v_{bc}^\xi + R_{nb}^{-1} g \Delta t \end{bmatrix} \quad (19)$$

- $g$  is the gravity vector.
- $*$  denotes the predicted navigation state.

## The jacobian matrix of navigation state

$$\begin{aligned} J_{s_i}^{\mathcal{D}} &= \begin{bmatrix} 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ \frac{\partial p_{bc}}{\partial R_{nb}} & 0_{3 \times 3} & \frac{\partial p_{bc}}{\partial v_{nb}} \\ \frac{\partial v_{bc}}{\partial R_{nb}} & 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \\ &= \begin{bmatrix} 0_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ \widehat{R_{nb}^{-1} v_{nb} \Delta t + R_{nb}^{-1} g \frac{\Delta t^2}{2}} & 0_{3 \times 3} & I_{3 \times 3} \Delta t \\ \widehat{R_{nb}^{-1} g \Delta t} & 0_{3 \times 3} & 0_{3 \times 3} \end{bmatrix} \end{aligned} \quad (20)$$

## The jacobian matrix of $\xi$

$$J_{\xi}^{\mathcal{D}} = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & I_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} & I_{3 \times 3} \end{bmatrix} \quad (21)$$

## Retraction $\mathcal{R}$

The retract action  $\mathcal{R}$  which defined on navigation state takes 2 parameters:  $s_i$  and  $\mathcal{D}$  to predict  $s_j$ .

- $s_j^*$  is the predicted  $s_j$ .

$$\begin{aligned} R_{nc}^* &= R_{nb} R_{bc} \\ p_{nc}^* &= p_{nb} + R_{nb} p_{bc} \\ v_{nc}^* &= v_{nb} + R_{nb} v_{bc} \end{aligned} \quad (22)$$

## Derivative of $s_i$

$$J_{s_i}^{\mathcal{R}} = \begin{bmatrix} R_{bc}^{-1} & 0_{3 \times 3} & 0_{3 \times 3} \\ -R_{bc}^{-1} \widehat{p_{bc}} & R_{bc}^{-1} & 0_{3 \times 3} \\ -R_{bc}^{-1} \widehat{v_{bc}} & 0_{3 \times 3} & R_{bc}^{-1} \end{bmatrix} \quad (23)$$

## Derivative of $d$

$$J_d^{\mathcal{R}} = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & R_{bc}^{-1} & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} & R_{bc}^{-1} \end{bmatrix} \quad (24)$$

## Navigation state prediction error (residual function)

If navigation  $state_j$  is measured by sensors, we can calculate the error between  $state_j$  and  $state_j^*$ .

$$r_{jj^*} = \mathcal{L}(s_j, s_j^*) = \begin{bmatrix} \Delta R \\ \Delta p \\ \Delta v \end{bmatrix} = \begin{bmatrix} R_j^{-1} R_j^* \\ R_j^{-1} (p_j^* - p_j) \\ R_j^{-1} (v_j^* - v_j) \end{bmatrix} \quad (25)$$

Local  $\mathcal{L}$  is the inverse function of  $\mathcal{R}$ , which takes two navigation states, and get the delta between the two states in tangent vector space

## Derivative of an $s_j$

$$J_{s_j}^{\mathcal{L}} = \begin{bmatrix} -\Delta R^{-1} & 0_{3 \times 3} & 0_{3 \times 3} \\ \widehat{\Delta p} & -I_{3 \times 3} & 0_{3 \times 3} \\ \widehat{\Delta v} & 0_{3 \times 3} & -I_{3 \times 3} \end{bmatrix} \quad (26)$$

## Derivative of an $s_j^*$

$$J_{s_j^*}^{\mathcal{L}} = \begin{bmatrix} I_{3 \times 3} & 0_{3 \times 3} & 0_{3 \times 3} \\ 0_{3 \times 3} & \Delta R & 0_{3 \times 3} \\ 0_{3 \times 3} & 0_{3 \times 3} & \Delta R \end{bmatrix} \quad (27)$$

## Overall Jaccobian for prediction error

To summarize, The prediction error  $r$  takes 3 parameters  $s_i$ ,  $s_j$  and  $b$ . According to the chain rule, their Jaccobian can be written in the following form.

$$J_{s_j}^r = J_{s_j}^{\mathcal{L}} \quad (28)$$

$$J_{s_i}^r = J_{s_j^*}^{\mathcal{L}} J_{s_i}^{s_j^*} = J_{s_j^*}^{\mathcal{L}} (J_{s_i}^{\mathcal{R}} + J_{\mathcal{D}}^{\mathcal{R}} J_{s_i}^{\mathcal{D}}) \quad (29)$$

$$J_b^r = J_{s_j^*}^{\mathcal{L}} J_b^{s_j^*} = J_{s_j^*}^{\mathcal{L}} J_{\mathcal{D}}^{\mathcal{R}} J_{\xi}^{\mathcal{D}} J_b^{\xi} \quad (30)$$

# Appendix

## A-1. Proof of (8)(9)(10)

$A$  and  $B$  are the two lie groups:  $\varphi(A, B) = AB$

$$\begin{aligned} \exp(\widehat{J_A \delta}) &= (AB)^{-1} (A \exp(\hat{\delta}) B) \\ &= B^{-1} A^{-1} A \exp(\hat{\delta}) B \\ &= \exp(B^{-1} \hat{\delta} B) \\ &= \exp(\widehat{B^{-1} \delta}) \end{aligned}$$

$$\begin{aligned} \exp(\widehat{J_B \delta}) &= (AB)^{-1} (AB \exp(\hat{\delta})) \\ &= B^{-1} A^{-1} AB \exp(\hat{\delta}) \\ &= \exp(\hat{\delta}) \end{aligned}$$

Hence:

$$J_A = B^{-1} \quad (\text{A1-1})$$

$$J_B = I \quad (\text{A1-2})$$

Proof of (8):

According to A1-1:

$$\begin{aligned} \frac{\partial R_{k+1}}{\partial R_k} &= -\exp(\omega_k^b \Delta t) \\ &= I_{3 \times 3} - \Delta t \widehat{\omega_k^b} \end{aligned}$$

$A$  is a lie group,  $p$  is a vector:  $\varphi(A, p) = Ap$

$$\begin{aligned}
J_A &= \frac{A \exp(\delta) p - Ap}{\delta} \\
&\cong \frac{Aa + A\hat{\delta}p - Ap}{\delta} \\
&= \frac{A\hat{\delta}p}{\delta} \\
&= -\frac{A\delta\hat{p}}{\delta} \\
&= -A\hat{p}
\end{aligned} \tag{A1-3}$$

$$\begin{aligned}
J_p &= \frac{A(p + \delta) - Ap}{\delta} \\
&= A
\end{aligned} \tag{A1-4}$$

According to A1-3:

$$\begin{aligned}
\frac{\partial p_{k+1}}{\partial R_k} &= -R_k \widehat{a_k^b} \frac{\Delta t^2}{2} \\
\frac{\partial v_{k+1}}{\partial R_k} &= -R_k \widehat{a_k^b} \Delta t
\end{aligned}$$

Proof of (9):

According to A1-4:

$$\begin{aligned}
\frac{\partial p_{k+1}}{\partial a_k^b} &= R_k \frac{\Delta t^2}{2} \\
\frac{\partial v_{k+1}}{\partial a_k^b} &= R_k \Delta t
\end{aligned}$$

Proof of (10):

According to A1-2:

$$\frac{\partial R_{k+1}}{\partial \omega_k^b} = I_{3 \times 3} \Delta t$$

## A-2. Proof of (20)

$A$  is a lie group,  $p$  is a vector:  $\varphi(A, p) = A^{-1}p$

$$\begin{aligned}
J_A &= \frac{(A \exp(\hat{\delta}))^{-1} p - A^{-1} p}{\delta} \\
&= \frac{\exp(\widehat{-\delta}) A^{-1} p - A^{-1} p}{\delta} \\
&= \frac{(I - \hat{\delta}) A^{-1} p - A^{-1} p}{\delta} \\
&= \frac{-\hat{\delta} A^{-1} p}{\delta} \\
&= \frac{\delta x \widehat{A^{-1} p}}{\delta} \\
&= \widehat{A^{-1} p}
\end{aligned} \tag{A2-1}$$

$$\begin{aligned}
J_p &= \frac{T^{-1}(p + \delta) - T^{-1} p}{\delta} \\
&= \frac{T^{-1} \delta}{\delta} \\
&= T^{-1}
\end{aligned} \tag{A2-2}$$

According to A2-1:

$$\begin{aligned}
\frac{\partial p_{bc}}{\partial R_{nb}} &= \widehat{R_{nb}^{-1} v_{nb}} \Delta t + \widehat{R_{nb}^{-1} g} \frac{\Delta t^2}{2} \\
\frac{\partial v_{bc}}{\partial R_{nb}} &= \widehat{R_{nb}^{-1} g} \Delta t
\end{aligned}$$

According to A2-2 and (22):

$$\begin{aligned}
\frac{\partial p_{bc}}{\partial v_{nb}} &= \frac{R_{nb}^{-1}(v_{nb} + R_{nb} \delta) - R_{nb}^{-1} v_{nb}}{\delta} \\
&= I_{3 \times 3} \Delta t
\end{aligned}$$

### A-3. Proof of (23)

According to A1-1:

$$\begin{aligned}
\frac{\partial R_{nc}^*}{\partial R_{nb}} &= R_{bc}^{-1} \\
\frac{\partial p_{bc}^*}{\partial R_{nb}} &= \widehat{R_{nb}^{-1} g} \Delta t
\end{aligned}$$



According to A2-2 and (25):

$$\begin{aligned}\frac{\partial p_{bc}^*}{\partial R_{nb}} &= \frac{R_{nc}^{-1}(R_{nb} \exp(\widehat{\delta\theta_b})p_{bc} - R_{nb}p_{bc})}{\delta\theta_b} \\ &= -R_{bc}^{-1}\widehat{p_{bc}}\end{aligned}$$

$$\begin{aligned}\frac{\partial v_{bc}^*}{\partial R_{nb}} &= \frac{R_{nc}^{-1}(R_{nb} \exp(\widehat{\delta\theta_b})v_{bc} - R_{nb}v_{bc})}{\delta\theta_b} \\ &= -R_{bc}^{-1}\widehat{v_{bc}}\end{aligned}$$

According to A1-3 and (22)(25):

$$\begin{aligned}\frac{\partial p_{bc}^*}{\partial p_{nb}} &= \frac{R_{nc}^{-1}(p_{nb} + R_{nb}\delta p_b - p_{nb})}{\delta p_b} \\ &= R_{bc}^{-1}\end{aligned}$$

$$\begin{aligned}\frac{\partial v_{bc}^*}{\partial v_{nb}} &= \frac{R_{nc}^{-1}(v_{nb} + R_{nb}\delta v_b - v_{nb})}{\delta v_b} \\ &= R_{bc}^{-1}\end{aligned}$$

## A-4. Proof of (24)

According to A2-2 and (25):

$$\begin{aligned}\frac{\partial p_{nc}^*}{\partial p_{bc}} &= \frac{R_{nc}^{-1}(R_{nb}(p_{bc} + \delta p_b) - R_{nb}p_{bc})}{\delta p_b} \\ &= R_{bc}^{-1}\end{aligned}$$

$$\begin{aligned}\frac{\partial v_{nc}^*}{\partial v_b} &= \frac{R_{nc}^{-1}(R_{nb}(v_{bc} + \delta v_b) - R_{nb}v_{bc})}{\delta v_b} \\ &= R_{bc}^{-1}\end{aligned}$$