Solve Nonlinear Least-Squares Problem with the Gauss-Newton Methods.

What is linear/nonlinear?

- Linear: A polynomial of degree 1.
- Nonlinear: A function cannot be expressed as a polynomial of degree 1.

What is Least-Squares Problem?

f(x) is the objective function, x is the parameter vector.
The least-squares problem tries to find optimal parameters to minimize the overall cost.

$$cost = \sum_{i=0}^{n} f(x)^2$$

What is Gauss-Newton Methods?

The gauss–newton methods is used to solve nonlinear least-squares problems.
Unlike Newton's method, gauss-newton methods are not necessary to calculate second derivatves, which may difficult to compute in some cases.
Gauss–newton methods update x using iterative method, the update amount is Δx.

$$\Delta x = -H^{-1} g$$

here: g is the gradient vector; H is the hessian matrix.

$$g = J^T e$$

$$H \approx J^T J$$

J is the jacobian matrix, e is the residual vector.

The jacabian matrix of 3d rotation or 3d transform

If we define the increment of SO3/SE3 as:

$$T(x_0 \oplus \delta) \triangleq T(x_0) \exp(\delta)$$

The $\delta \in \mathfrak{so}(3)$ or $\delta \in \mathfrak{se}(3)$

We use a first-order Taylor expansion to approximate the original equation:

$$T(x_0 \oplus \delta) = T_0 \exp(\delta) \cong \ T_0 + T_0 \hat{\delta}$$

The f(x) is the objective function. We want to find the optimal parameters (x) that minimize the result of the objective function.

$$f(x) = T(x)a - b$$

a is the target point: b is the reference point.

We can use gauss-newton method to solve this problem. According to gauss-newton method, we need to find the Jacobian matrix for f(x).

$$\begin{aligned}\dot{f} &= \frac{T_0 \exp(\delta)\, a - T_0 a}{\delta} \\ &\cong \frac{T_0 a + T_0 \hat{\delta} a - T_0 a}{\delta} \\ &= \frac{T_0 \hat{\delta} a}{\delta} \\ &= -\frac{T_0 \delta \widehat{a}}{\delta} \\ &= -T_0 \widehat{a}\end{aligned}$$

When $\delta \in \mathfrak{so}(3)$

T_0 is a 3d rotation matrix(R_0), and $\widehat{a}$ is defined as a skew symmetric matrix for vector a

$$\dot{f} = -R_0 [a]_\times$$

When $\delta \in \mathfrak{se}(3)$

$$\delta = [\, v, \omega]$$

ω: the parameter of rotation.

v: the parameter of translation.

$$\hat{\delta} = \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix}$$

$$\dot{f} = \frac{R_0 \hat{\delta} a}{\delta}$$

$$= \frac{T_0 \begin{bmatrix} [\omega]_\times & v \\ 0 & 0 \end{bmatrix} \begin{bmatrix} a \\ 1 \end{bmatrix}}{[\omega,\ v]}$$

$$= \frac{T_0 \begin{bmatrix} [-a]_\times & I \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \omega \\ v \end{bmatrix}}{[\omega,\ v]}$$

$$= T_0 \begin{bmatrix} [-a]_\times & I \\ 0 & 0 \end{bmatrix}$$