AM5640 Turbulence Modeling
Assignment 1

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1 Problem definition

In the present assignment, we have written a C++ code to solve a fully-developed turbulent channel flow using a $k-\omega$ model. We have used the Finite Volume Method to discretize the governing equations and compared our results with DNS data in the subsequent section.

2 Governing Equations

The continuity and Navier-Stokes equation in generalized, 2D Cartesian framework is as follows:

$$\frac{\partial U}{\partial X}+\frac{\partial V}{\partial Y}=0 \tag{1}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{1}{\rho}\frac{\partial P}{\partial X}+\frac{\partial}{\partial X}\left[(\nu+\nu_t)\frac{\partial U}{\partial X}\right]+\frac{\partial}{\partial Y}\left[(\nu+\nu_t)\frac{\partial U}{\partial Y}\right] \tag{2}$$

$$U\frac{\partial V}{\partial X}+V\frac{\partial V}{\partial Y}=-\frac{1}{\rho}\frac{\partial P}{\partial Y}+\frac{\partial}{\partial X}\left[(\nu+\nu_t)\frac{\partial V}{\partial X}\right]+\frac{\partial}{\partial Y}\left[(\nu+\nu_t)\frac{\partial V}{\partial Y}\right] \tag{3}$$

Since the flow is fully developed, $V=0$, $\frac{\partial U}{\partial X}=0$, $\frac{\partial k}{\partial X}=0$. This leads to simplified NSE,

$$0=-\frac{1}{\rho}\frac{\partial P}{\partial X}+\frac{\partial}{\partial Y}\left[(\nu+\nu_t)\frac{\partial U}{\partial Y}\right] \tag{4}$$

Here, we have used wilcox's $k-\omega$ model, and the simplified model equations are as follows:

$$0=\frac{\partial}{\partial Y}\left[\left(\nu+\frac{\nu_t}{\sigma_k}\right)\frac{\partial K}{\partial Y}\right]+P_k+\beta^*\omega k \tag{5}$$

$$0=\frac{\partial}{\partial Y}\left[\left(\nu+\frac{\nu_t}{\sigma_\omega}\right)\frac{\partial \omega}{\partial Y}\right]+\frac{\omega}{k}C_1P_k-C_2\omega^2 \tag{6}$$

$$\nu_t=\frac{k}{\omega}, P_k=\nu_t\left(\frac{\partial U}{\partial Y}\right)^2 \tag{7}$$

3 Numerical Methodology

3.1 Grid generation

We have used a nonuniform grid that is available from the DNS data.

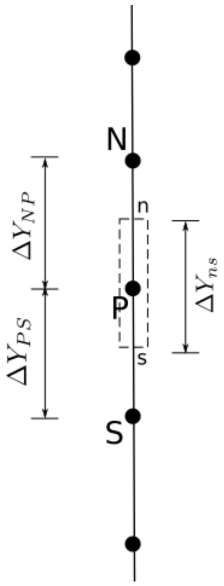

Figure 1: Grid configuration

3.2 Discritezation of Governing Equations

3.2.1 Discretization of U

The control volume configuration is shown in figure 1. Integrating equation (4) leads to

$$0 = \left[\nu \frac{\partial U}{\partial Y} + \nu_t \frac{\partial U}{\partial Y}\right]_n - \left[\nu \frac{\partial U}{\partial Y} + \nu_t \frac{\partial U}{\partial Y}\right]_s - \frac{1}{\rho}\frac{\partial P}{\partial X}\delta ns$$
$$\left[\nu \frac{\partial U}{\partial Y} + \nu_t \frac{\partial U}{\partial Y}\right]_n = \nu\left[\frac{U_N - U_P}{\Delta Y_{PN}}\right] + \left[\frac{\nu_{t,N}U_N - \nu_{t,P}U_P}{\Delta Y_{PN}}\right]$$
$$\left[\nu \frac{\partial U}{\partial Y} + \nu_t \frac{\partial U}{\partial Y}\right]_s = \nu\left[\frac{U_P - U_S}{\Delta Y_{PS}}\right] + \left[\frac{\nu_{t,P}U_P - \nu_{t,S}U_S}{\Delta Y_{PS}}\right]$$

here, we have taken $\rho = 1, and \frac{\partial P}{\partial X} = 1$. and general form of source terms is $S = S_u + S_p U_p$. The simplified algebraic equations derived from the U equation are as follows.

$$A_P U_P = A_N U_N + A_S U_S + S_U$$
$$A_P = A_N + A_S - S_P$$
$$A_N = \frac{(\nu + \nu_{t,N})}{\Delta Y_{PN}}$$
$$A_S = \frac{(\nu + \nu_{t,N})}{\Delta Y_{PS}}$$
$$S_p = 0, S_U = -1 * \Delta Y_{ns}$$

3.2.2 Discretization of k

Similar to U, k is discretized by integrating the equation (5). the discretization procedure is as follows:
Integrating equation

$$0 = \left[\nu \frac{\partial k}{\partial Y} + \frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial Y}\right]_n - \left[\nu \frac{\partial k}{\partial Y} + \frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial Y}\right]_s + P_k * \Delta Y_{ns} - \beta^* \omega k \Delta Y_{ns}$$
$$\left[\nu \frac{\partial k}{\partial Y} + \frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial Y}\right]_n = \nu\left[\frac{k_N - k_P}{\Delta Y_{PN}}\right] + \left[\frac{\nu_{t,N}k_N - \nu_{t,P}k_P}{\Delta Y_{PN}}\right]$$
$$\left[\nu \frac{\partial k}{\partial Y} + \frac{\nu_t}{\sigma_k}\frac{\partial k}{\partial Y}\right]_s = \nu\left[\frac{k_P - k_S}{\Delta Y_{PS}}\right] + \left[\frac{\nu_{t,P}k_P - \nu_{t,S}k_S}{\Delta Y_{PS}}\right]$$

Here, $+P_k * \Delta Y_{ns} - \beta^* \omega k \Delta Y_{ns}$ acts as a source term, and we can compare it with a model form of a source term $S = S_U + S_P k_P$. The final algebraic equations for the k - equation are as follows.

$$\begin{aligned} A_P k_P &= A_N k_N + A_S k_S + S_U \\ A_P &= A_N + A_S - S_P \\ A_N &= (\nu + \frac{\nu_{t,N}}{\sigma_k}) \frac{1}{\Delta Y_{PN}} \\ A_S &= (\nu + \frac{\nu_{t,S}}{\sigma_k}) \frac{1}{\Delta Y_{PS}} \\ S_p &= -\beta^* \omega * \Delta Y_{ns}, S_U = P_k * \Delta Y_{ns} \end{aligned}$$

Here, P_k is a source term, which is calculated before solving the k equation

$$P_k = \nu_t \frac{U_n - U_s}{\Delta Y_{ns}}^2$$

Where U_n and U_s are face velocities and are interpolated from the values available at nodes.

3.2.3 Discretization of ω

Similar to k, ω is discretized by integrating the equation (5). the discretization procedure is as follows:

$$\begin{aligned} 0 = \left[\nu \frac{\partial \omega}{\partial Y} + \frac{\nu_t}{\sigma_\omega} \frac{\partial \omega}{\partial Y}\right]_n - \left[\nu \frac{\partial \omega}{\partial Y} + \frac{\nu_t}{\sigma_\omega} \frac{\partial \omega}{\partial Y}\right]_s + \frac{\omega}{k} C_1 * P_k * \Delta Y_{ns} - C_2 * \omega^2 \Delta Y_{ns} \\ \left[\nu \frac{\partial \omega}{\partial Y} + \frac{\nu_t}{\sigma_\omega} \frac{\partial \omega}{\partial Y}\right]_n = \nu \left[\frac{\omega_N - \omega_P}{\Delta Y_{PN}}\right] + \left[\frac{\nu_{t,N} \omega_N - \nu_{t,P} \omega_P}{\Delta Y_{PN}}\right] \\ \left[\nu \frac{\partial \omega}{\partial Y} + \frac{\nu_t}{\sigma_\omega} \frac{\partial \omega}{\partial Y}\right]_s = \nu \left[\frac{\omega_P - \omega_S}{\Delta Y_{PS}}\right] + \left[\frac{\nu_{t,P} \omega_P - \nu_{t,S} \omega_S}{\Delta Y_{PS}}\right] \end{aligned}$$

Similar to k equation, here in ω equation the term $\frac{\omega}{k} C_1 * P_k * \Delta Y_{ns} - C_2 * \omega^2 \Delta Y_{ns}$ is a source term, and it can be represented in terms of $S = S_U + S_P \omega_P$. The final algebraic equations for the ω - equation are as follows.

$$\begin{aligned} A_P \omega_P &= A_N \omega_N + A_S \omega_S + S_U \\ A_P &= A_N + A_S - S_P \\ A_N &= (\nu + \frac{\nu_{t,N}}{\sigma_\omega}) \frac{1}{\Delta Y_{PN}} \\ A_S &= (\nu + \frac{\nu_{t,S}}{\sigma_\omega}) \frac{1}{\Delta Y_{PS}} \\ S_p &= \left(\frac{C_1 * P_k}{k} - (C_2 * \omega_{old})\right) * \Delta Y_{ns}, S_U = 0 \end{aligned}$$

3.3 Boundary Conditions

At the wall, we have used the Dirichlet condition for U and k, i.e., at the wall $U = k = 0$. As mentioned in lectures, the boundary condition for ω is taken in the near wall region, and not on the wall. The boundary condition for ω is also a dirichlet condition, and its value on the 2nd node(nearest to the wall) is calculated as $\omega = \frac{6\nu}{Y^2 * C_2}$. Since the problem is symmetric, we have calculated in the half region and applied symmetric conditions at the top.

4 Results

Proof of $\tau_w = -\frac{\partial P}{\partial X}$

$$\begin{aligned} 0 = -\frac{1}{\rho} \frac{\partial P}{\partial X} + \frac{\partial}{\partial Y} \left[(\nu + \nu_t) \frac{\partial U}{\partial Y}\right] \\ 0 = \int_0^\delta -\frac{1}{\rho} \frac{\partial P}{\partial X} dy + \int_0^\delta \frac{\partial}{\partial Y} \left[(\nu + \nu_t) \frac{\partial U}{\partial Y}\right] dy \qquad (8) \\ 0 = -\left[\frac{1}{\rho} \frac{\partial P}{\partial X} * \Delta Y\right]_\delta + \left[\frac{1}{\rho} \frac{\partial P}{\partial X} * \Delta Y\right]_{wall} + \left[\nu \frac{\partial U}{\partial Y}\right]_\delta - \left[\nu \frac{\partial U}{\partial Y}\right]_{wall} + \left[\nu_t \frac{\partial U}{\partial Y}\right]_\delta - \left[\nu_t \frac{\partial U}{\partial Y}\right]_{wall} \end{aligned}$$

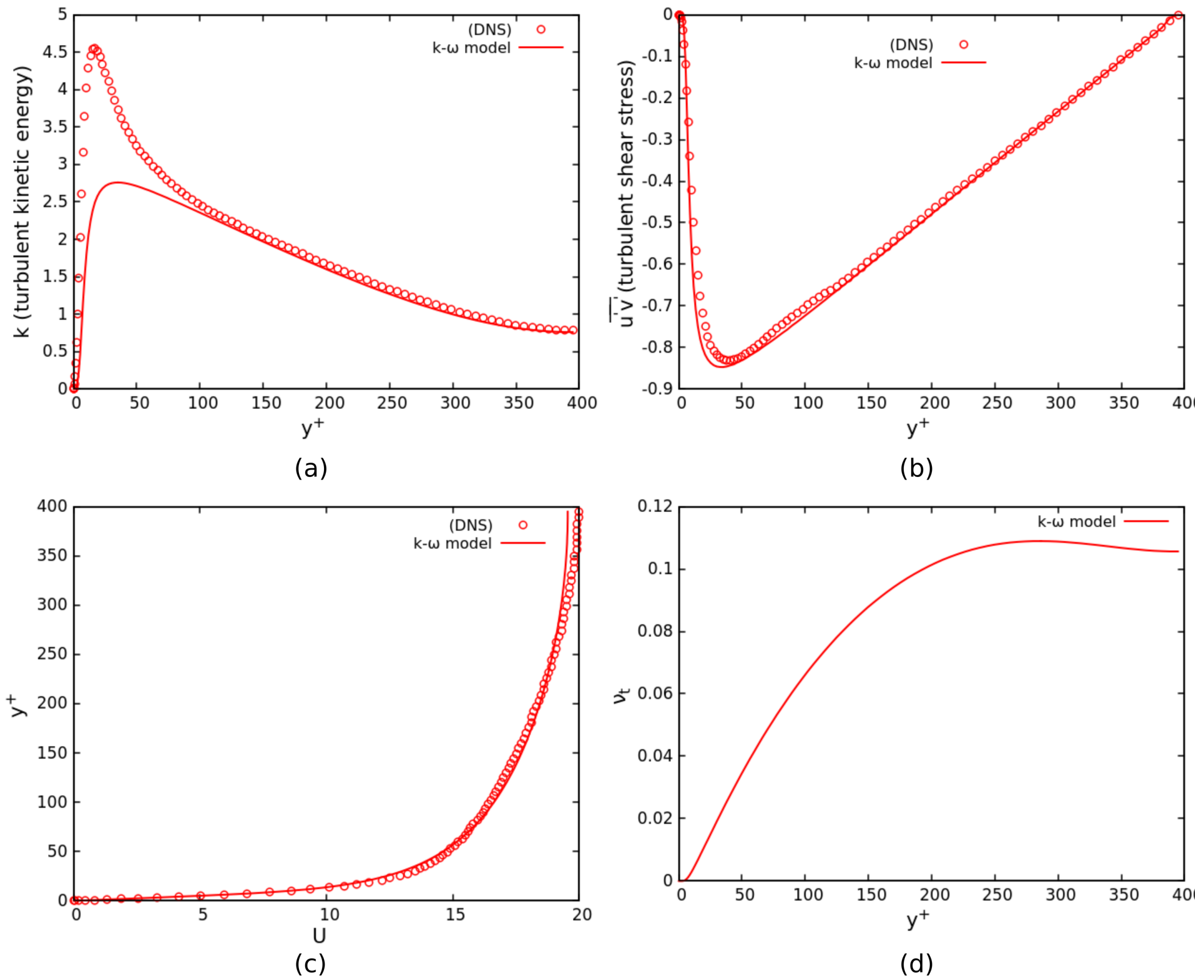

Figure 2: Comparison of model quantities with DNS data, for a TKE, (b) cross-correlation, and (c) average velocity U

In the above equation, the terms $\frac{\partial U}{\partial Y}$ is zero at $y = \delta$ because of the symmetric boundary condition. The term $\nu_t \frac{\partial U}{\partial Y}$ is zero at wall since ν_t is zero at the wall. Hence, the equation is simplified to

$$0 = -\frac{1}{\rho}\frac{\partial P}{\partial X} * \Delta Y - \left[\nu \frac{\partial U}{\partial Y}\right]_{wall}$$
$$\tau_w = -\frac{\partial P}{\partial X}$$

The mean velocity profile shows good agreement with the DNS data as observed in figure 2. It shows little variation near the center line of the channel. The $k-\omega$ model quietly well predicts the nature of turbulent kinetic energy near the wall and towards the center line of the channel as from the DNS data. But, as from figure 2, $k-\omega$ model is unable to predict the maximum value of turbulent kinetic energy in the buffer layer. The modeling assumption of business to model Reynolds stress in the eddy viscosity model could be the reason for the error. Figure 3 indicates the turbulent shear stress increases as we move away from the inertial sub-layer and enters the buffer layer (y^+ >= 3 and y^+ <=30). The turbulent viscosity gradually increases as we move away from the wall (y+>=3), as observed in figure 3. The turbulent viscosity reaches a nearly constant value almost after y^+ >=200.

In Figure 3, the budget of turbulent kinetic energy (k) such as viscous and turbulent diffusion, production and dissipation term is analyzed and compared with the DNS data. From figure 3, it is observed that viscous diffusion is higher near the wall and reduced to almost 1e-04 order after y+>=30 and the same can be confirmed from the DNS result. The turbulent diffusion is negligible near the wall and maximum in the buffer zone; further, this behavior is supported by higher turbulent production (P_k) in the buffer zone (y^+ >= 3 and y^+<=30) and very lower near the wall. The specific dissipation rate ω is higher near the wall (10^7 order) and reduced as it moves toward the channel's center line.

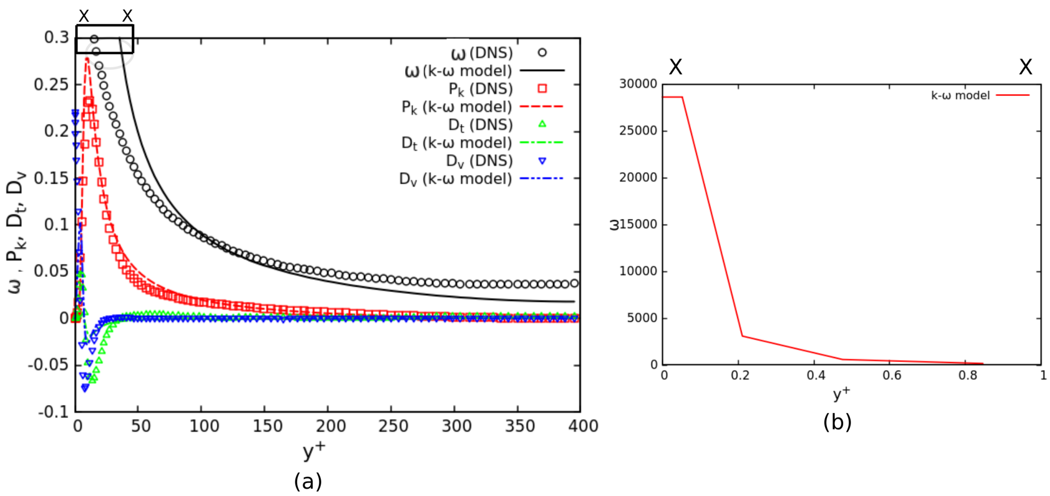

Figure 3: (a) Turbulence budgeting, (b) behavior of ω near wall