Kalman filter

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June 2021

State estimation

Consider a linear time-invariant state-space model given by:

$$\dot{x} = Ax + Bu + v$$
$$y = Cx + w$$

where x is the state (vector), u is the input or control signal and y is the output signal, v is the process disturbance and w is measurement noise. The disturbance v and the noise w are zero mean and Gaussian.

$$\mathbb{E}(v(s)v^T(t)) = R_v\delta(t-s)$$
$$\mathbb{E}(w(s)w^T(t)) = R_w\delta(t-s)$$

where δ is the unit impulse function (dirac function).

State estimation

The state estimator (observer) is given as:

$$\dot{\hat{x}} = A\hat{x} + Bu + L(y - C\hat{x})$$

The estimation error $\tilde{x} = x - \hat{x}$ can be computed as

$$\begin{aligned}\dot{\tilde{x}} = \dot{x} - \dot{\hat{x}} &= Ax + Bu + v - A\hat{x} - Bu - L(Cx + w - C\hat{x}) \\ &= (A - LC)\tilde{x} + v - Lw\end{aligned}$$

If $A - LC$ is stable, then the estimation error $\tilde{x}$ is a stationary stochastic process.

The covariance of the estimation error, $P_{\tilde{x}} = \mathbb{E}\left(\tilde{x}(t)\tilde{x}^T(t)\right)$, is given by the following equation:

$$0 = (A - LC)P_{\tilde{x}} + P_{\tilde{x}}(A - LC)^T + R_v + LR_wL^T$$

The optimal observer minimizes $P_{\tilde{x}}$.

Continuous Kalman Filter

State estimation

The optimal observer gain, if the system is *observable*, is:

$$L = P_{\tilde{x}} C^T R_w^{-1}$$

where $P_{\tilde{x}} = P_{\tilde{x}}^T \geq 0$ is the solution to the Riccati equation:

$$0 = AP_{\tilde{x}} + P_{\tilde{x}} A^T + R_v - P_{\tilde{x}} C^T R_w^{-1} C P_{\tilde{x}}$$

The observer is called the ***Kalman-Bucy filter.***

Similarities with LQR:

LQR $\leftrightarrow$ Kalman

$A \leftrightarrow A^T$ $\quad$ $B \leftrightarrow C^T$

$S \leftrightarrow P$ $\quad$ $K \leftrightarrow L^T$

$Q_x \leftrightarrow R_v$ $\quad$ $Q_u \leftrightarrow R_w$

The Kalman-Bucy filter is:

- always stable.
- *the optimal linear filter* for state estimation.
- R_v and R_w are regarded as the design parameters.

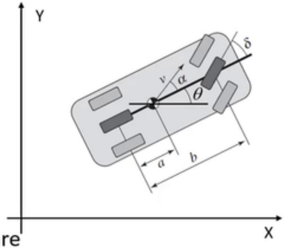
Revisit Example - Vehicle steering (Ex 7.4)

Consider the following system:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 12 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 6 \\ 3 \end{bmatrix} u + \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + w$$

where x_1 is the lateral position Y , x_2 is the heading orientation θ and u is the steering angle δ.



The process disturbance and the measurement noise are zero mean with covariance

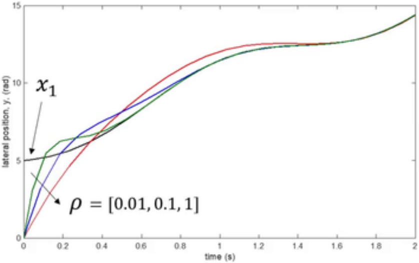
$$R_v = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad R_w = \rho$$

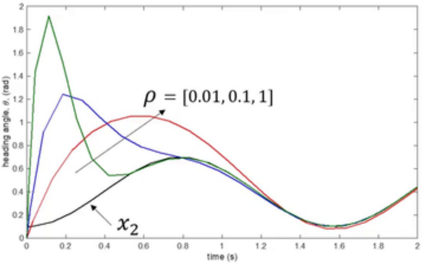
Vehicle data: $v_0 = 12\ m/s$
$a = 2\ m$
$b = 4\ m$

Design a Kalman filter to estimate the vehicle's states, from measurement of the lateral position.

Revisit Example - Vehicle steering (Ex 7.4)

Simulations using a sinusiodal input, with $x(0)$ = (5, 0.1) and $\hat{x}(0)$ = (0, 0):





State estimation – discrete time case

The covariance of the estimation error, $P_{\tilde{x}}[k] = \mathbb{E}\left(\tilde{x}[k]\tilde{x}^T[k]\right)$, is given by:

$$P_{\tilde{x}}[k+1] = (A - L[k]C)P_{\tilde{x}}[k](A - L[k]C)^T + R_v + L[k]R_w L^T[k]$$

The observer gain that minimizes $P_{\tilde{x}}[k]$ is given by

$$L[k] = AP_{\tilde{x}}[k]\, C^T\left(R_w + CP_{\tilde{x}}[k]\, C^T\right)^{-1}$$

This is the ***discrete time Kalman filter***.

Note, that the Kalman filter is a recursive filter.

If $P_{\tilde{x}}[k]$ converges, then L is constant.

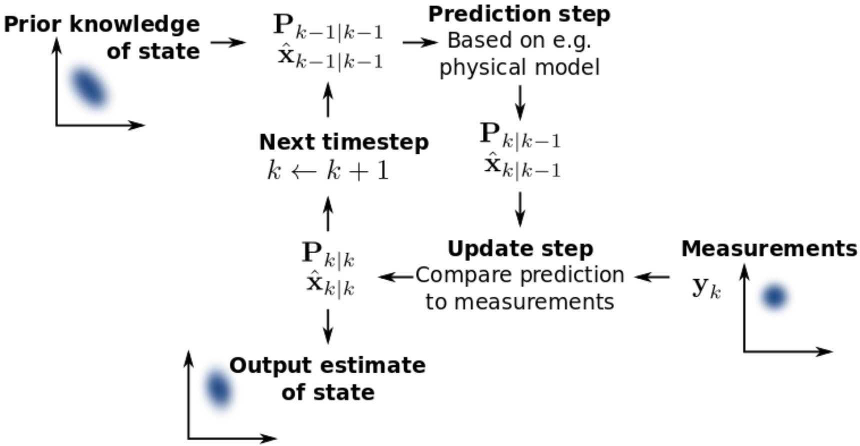
Discrete Kalman Filter Algorithm

Predict

Predicted (*a priori*) state estimate	$\hat{\mathbf{x}}_{k\mid k-1} = \mathbf{F}_k\hat{\mathbf{x}}_{k-1\mid k-1} + \mathbf{B}_k\mathbf{u}_k$
Predicted (*a priori*) estimate covariance	$\mathbf{P}_{k\mid k-1} = \mathbf{F}_k\mathbf{P}_{k-1\mid k-1}\mathbf{F}_k^{\mathsf{T}} + \mathbf{Q}_k$

Update

Innovation or measurement pre-fit residual	$\tilde{\mathbf{y}}_k = \mathbf{z}_k - \mathbf{H}_k\hat{\mathbf{x}}_{k\mid k-1}$
Innovation (or pre-fit residual) covariance	$\mathbf{S}_k = \mathbf{H}_k\mathbf{P}_{k\mid k-1}\mathbf{H}_k^{\mathsf{T}} + \mathbf{R}_k$
Optimal Kalman gain	$\mathbf{K}_k = \mathbf{P}_{k\mid k-1}\mathbf{H}_k^{\mathsf{T}}\mathbf{S}_k^{-1}$
Updated (*a posteriori*) state estimate	$\hat{\mathbf{x}}_{k\mid k} = \hat{\mathbf{x}}_{k\mid k-1} + \mathbf{K}_k\tilde{\mathbf{y}}_k$
Updated (*a posteriori*) estimate covariance	$\mathbf{P}_{k\mid k} = (\mathbf{I} - \mathbf{K}_k\mathbf{H}_k)\mathbf{P}_{k\mid k-1}$


Prior knowledge
of state
$\mathbf{P}_{k-1|k-1}$
$\hat{\mathbf{x}}_{k-1|k-1}$
Prediction step
Based on e.g.
physical model
Next timestep
$k \leftarrow k+1$
$\mathbf{P}_{k|k-1}$
$\hat{\mathbf{x}}_{k|k-1}$
$\mathbf{P}_{k|k}$
$\hat{\mathbf{x}}_{k|k}$
Update step
Compare prediction
to measurements
Measurements
$\mathbf{y}_k$
Output estimate
of state

Bibliography

- Karl J. Astrom and Richard M. Murray *Feedback Systems*. Version v3.0i. Princeton University Press. September 2018. Chapter 8.
- Wikipedia. Kalman filter. https://en.wikipedia.org/wiki/Kalman_filter.