$$riangleq a' = rac{a+a_1}{2}, \ b' = rac{b+b_1}{2}$$

构造

$$M=\max\left\{\left|rac{f(a')-f(a_1)}{a'-a_1}
ight|,\left|rac{f(b_1)-f(b')}{b_1-b'}
ight|
ight\}$$

下面证 M 满足题意

当 x = y 时,

$$|f(x) - f(y)| = 0 = M|x - y|$$

显然成立

当 $x \neq y$ 时,

不失一般性,我们假设 x < y,

则有

$$a < a' < a_1 \le x < y \le b_1 < b' < b$$

由于 $f:(a,b)\to\mathbb{R}$ 是凸函数

故

$$\frac{f(a') - f(a_1)}{a' - a_1} \leq \frac{f(a') - f(y)}{a' - y} \leq \frac{f(x) - f(y)}{x - y} \leq \frac{f(x) - f(b')}{x - b'} \leq \frac{f(b_1) - f(b')}{b_1 - b'}$$

则

$$\left\{ egin{aligned} rac{f(x) - f(y)}{x - y} & \leq rac{f(b_1) - f(b')}{b_1 - b'} \leq \left| rac{f(b_1) - f(b')}{b_1 - b'}
ight| \ - rac{f(x) - f(y)}{x - y} & \leq -rac{f(a') - f(a_1)}{a' - a_1} \leq \left| rac{f(a') - f(a_1)}{a' - a_1}
ight| \end{aligned}$$

所以

$$egin{split} rac{|f(x)-f(y)|}{|x-y|} &= \left|rac{f(x)-f(y)}{x-y}
ight| \ &= \max\left\{rac{f(x)-f(y)}{x-y}, -rac{f(x)-f(y)}{x-y}
ight\} \ &\leq \max\left\{\left|rac{f(a')-f(a_1)}{a'-a_1}
ight|, \left|rac{f(b_1)-f(b')}{b_1-b'}
ight|
ight\} = M \end{split}$$

即

$$|f(x)-f(y)| \leq M|x-y|$$